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The Banach Space of Quasinorms on a Finite-Dimensional Space

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Abstract
We show that the set of continuous quasinorms on a finite-dimensional linear space, after quotienting by the dilations, has a natural structure of Banach space. Our main result states that, given a finite-dimensional vector space E, the pseudometric defined in the set of continuous quasinorms $\mathcal{Q}_0 = \{\|\cdot\| : E \to \mathbb{R}\}$ as

$$\mathrm{d}(\|\cdot\|_X, \|\cdot\|_Y) = \min\{\mu : \|\cdot\|_X \leq \lambda\|\cdot\|_Y \leq \mu\|\cdot\|_X \text{ for some } \lambda\}$$

induces, in fact, a complete norm when we take the obvious quotient $\mathcal{Q} = \mathcal{Q}_0/\sim$ and define the appropriate operations on $\mathcal{Q}$. We finish the paper with a little explanation of how this space and the Banach–Mazur compactum are related.

Keywords Quasinorms · Finite-dimensional spaces · Banach spaces · Banach–Mazur compactum

Mathematics Subject Classification 46B20 · 47A30

1 Introduction

Our main goal in this short paper is to show that the set of continuous quasinorms defined on $\mathbb{R}^n$ for some $n \geq 2$ has a, somehow, canonical structure of Banach space after quotienting by the proportional quasinorms.

For this to make sense, we first need to endow this set with a vector space structure, this will be done by means of something that everyone can expect to represent the mean

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of two quasinorms: $\sqrt{\|\cdot\|_X \|\cdot\|_Y}$ for each pair of quasinorms $\|\cdot\|_X, \|\cdot\|_Y$. Once the mean is given, we just need to choose the element of the space which will play the rôle of the origin in order to determine a vector space structure, in the present paper we have chosen $(\mathbb{R}^n, \|\cdot\|_2)$. Of course, this may seem anything but canonical. On the bright side, the choice of an origin will not affect any property of the newly defined vector (or Banach) space. For example, we may consider $C[0, 1]$ endowed with the scalar multiplication $(\lambda \star f)(x) = \lambda(f(x)-1)$ and the addition $(f \oplus g)(x) = f(x)+g(x)-1$, of course, the same can be done with any other function in $C[0, 1]$ instead of 1. Now, if we define a norm in $(C[0, 1], \oplus, \star)$ as $\|f\| = \max\{|f(x) - 1|\}$ then we have a Banach space structure $(C[0, 1], \oplus, \star, \|\cdot\|)$ that is indistinguishable from the usual $(C[0, 1], +, \cdot, \|\cdot\|_\infty)$, in the sense that the map

$$(C[0, 1], +, \cdot, \|\cdot\|_\infty) \to (C[0, 1], \oplus, \star, \|\cdot\|), \quad f \mapsto f + 1$$

is a linear isometry. What we have done is equivalent to considering the affine structure of $C[0, 1]$ and taking two different choices for the origin. This is doable because every norm gives a translation invariant metric.

Once the operations are given, we have to define the norm. This idea is not ours, but taken from A. Khare's preprint [9]. Given two continuous quasinorms $\|\cdot\|_X, \|\cdot\|_Y$, the distance between them is defined as

$$\mathrm{d}(\|\cdot\|_X, \|\cdot\|_Y) = \min\{\mu : \|\cdot\|_X \le \lambda\|\cdot\|_Y \le \mu\|\cdot\|_X \text{ for some } \lambda > 0\},$$

where the order relation is the pointwise order: $\|\cdot\|_X \le \lambda\|\cdot\|_Y$ means $\|x\|_X \le \lambda\|x\|_Y$ for every $x \in \mathbb{R}^n$. Of course, two quasinorms are proportional if and only if the distance between them is 1, so we must take the reasonable quotient

$$\spadesuit \quad \|\cdot\|_X \sim \|\cdot\|_Y \text{ if and only if } \|\cdot\|_X = \lambda\|\cdot\|_Y \text{ for some } \lambda \in (0, \infty)$$

to make d an actual (multiplicative) metric. So, defining

$$\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y]) = \min\{\mu : \|\cdot\|_X \le \lambda\|\cdot\|_Y \le \mu\|\cdot\|_X \text{ for some } \lambda > 0\} \quad (1)$$

we have a distance between the equivalence classes of quasinorms that turns out to induce a norm when we endow $\{\text{Continuous quasinorms on } \mathbb{R}^n\}/\sim$ with the above-explained operations.

This paper is far from being the first one in which the sets of (quasi) norms are endowed with some structure. The best known structure given to the set of norms on a finite-dimensional space is the Banach–Mazur pseudometric defined as

$$\mathrm{d}(\|\cdot\|_X, \|\cdot\|_Y) = \inf\{\|T\|\|T^{-1}\|\}, \quad (2)$$

where T runs over the set of linear isomorphisms $T : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_Y)$. It is well known that, after taking the appropriate quotient, this pseudometric turns out to be a metric that makes the space to be compact, there is still significant interest on this topic, see, e.g. [1,18,19].

The present paper is neither the first one about, say, mixing pairs of norms to obtain something new. In this setting, interpolation of (quasi) normed spaces—or even more general spaces—has been the main topic for at least half a century, see [8,12,17]. For the reader interested in interpolation, we suggest [3] and the very interesting [15]. A nice paper on interpolation in quasinormed spaces is [16]. To the best of our knowledge, this paper is the first where someone considers the kind of interpolation that we have in Definition 3.3, that is

$$\|\cdot\|_{(X,Y)_\theta} = \|\cdot\|_X^\theta \|\cdot\|_Y^{1-\theta}.$$

There is a very good reason to avoid this kind of interpolation in the *normed* space literature. Namely, in Remark 3.8 we provide an example to show that the mean of a pair of norms on $\mathbb{R}^2$ does not need to be a norm but a quasinorm.

2 Notations and Preliminary Results

We will consider some positive integer n fixed throughout the paper. Every vector space will be over $\mathbb{R}$; observe that any $\mathbb{C}^n$ can be seen as $\mathbb{R}^{2n}$. Moreover, we will consider from now on the vector space $\mathbb{R}^n$ endowed with its only topological vector space structure, i.e. the one given by $\|\cdot\|_2$.

Definition 2.1 A map $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ is a quasinorm if the following conditions hold:

(1) $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$.
(3) There exists $k > 0$ such that $\|x+y\| \le k(\|x\| + \|y\|)$ for every $x, y \in \mathbb{R}^n$.

If the map $\|\cdot\|$ is continuous then we say that it is a continuous quasinorm. If k can be chosen to be 1, then $\|\cdot\|$ is a norm and it is continuous.

2.2 As is customary, given a quasinormed space $(\mathbb{R}^n, \|\cdot\|_X)$, we will denote its unit (closed) ball as B_X, its unit sphere as S_X.

Definition 2.3 Some subset $B \subset \mathbb{R}^n$ is bounded if, for every neighbourhood U of 0 there is $M \in (0,\infty)$ such that $B \subset MU$. $B \subset \mathbb{R}^n$ is balanced when $\lambda B \subset B$ for every $\lambda \in [-1, 1]$.

Definition 2.4 For $B \subset \mathbb{R}^n$, the Minkowski functional of B is $\rho_B(x) = \inf\{\lambda \in (0,\infty) : x \in \lambda B\}$.

It is quite well known that the quasinorms on a topological vector space are in correspondence with the bounded, balanced neighbourhoods of the origin, see the beginning of Section 2 in [7], and for a proof of such a key result the reader may check [6, Theorem 4]. The version that we will use is the following, where we use that $\|\cdot\|_2$ gives the only topological vector space structure to $\mathbb{R}^n$ and B_2 denotes the Euclidean unit ball of $\mathbb{R}^n$:

Theorem 2.5 *The Minkowski functional ρ_B of a given subset $B \subset \mathbb{R}^n$ is a quasinorm if and only if B fulfils the following:*

- *B contains εB_2 for some $\varepsilon > 0$.*
- *For every $\lambda \in [-1, 1]$ one has $\lambda B \subset B$, i.e. B is balanced.*
- *B is contained in $M B_2$ for some $M > 0$.*

In this case, ρ_B is a continuous quasinorm if and only if B is closed. Moreover, ρ_B is a norm if and only if the above hold and $\frac{x+y}{2} \in B$ for any pair $x, y \in B$.

We could even replace the first and third items in 2.5 by
"If B' is the unit ball of some quasinorm on $\mathbb{R}^n$ then there are $\varepsilon, M > 0$ such that $\varepsilon B \subset B' \subset MB$". Observe that this implies that the constants λ, μ in (1) actually exist.

We will deal in this note with $\mathcal{Q}_0 = \{\text{Continuous quasinorms defined on } \mathbb{R}^n\}$ and $\mathcal{Q} = \mathcal{Q}_0/\sim$, where two quasinorms are equivalent if and only if they are proportional, endowed with the multiplicative distance on $\mathcal{Q}$ defined in [9] by A. Khare and given by

$$d([\|\cdot\|_X], [\|\cdot\|_Y]) = \min\{\mu : \|\cdot\|_X \leq \lambda\|\cdot\|_Y \leq \mu\|\cdot\|_X \text{ for some } \lambda > 0\}. \quad (3)$$

In the same paper, it is shown that d endows $\mathcal{N} = \{\text{Norms defined on } \mathbb{R}^n\}/\sim$ with a complete metric space structure. To keep the notations consistent, we will write $\mathcal{N}_0$ for $\{\text{Norms defined on } \mathbb{R}^n\}$. The infimum in (3) exists because in $\mathbb{R}^n$ every pair of continuous quasinorms are equivalent and, moreover, by the continuity of the quasinorms, it is pretty clear that the minimum is attained. A nice feature of Khare's distance is that, in $\mathbb{R}^2$, it distinguishes the max-norm from $\|\cdot\|_1$. In some sense, these norms are as different as two norms can be, but the usual distances between norms, such as the Banach–Mazur or the Gromov–Hausdorff, make them indistinguishable.

3 The Main Result

Throughout this section, we will only consider continuous quasinorms.

Our first goal is to show that d is actually a multiplicative distance on $\mathcal{Q}$. For this, the following lemma will be useful.

Lemma 3.1 *Take any pair of quasinorms $\|\cdot\|_X, \|\cdot\|_Y$, $\lambda > 0$ and $\mu \geq 1$ such that $\|\cdot\|_X \leq \lambda\|\cdot\|_Y \leq \mu\|\cdot\|_X$. Then, μ is minimal if and only if both $S_X \cap \lambda^{-1} S_Y$ and $\lambda^{-1} S_Y \cap \mu^{-1} S_X$ are non-empty.*

Moreover, the distance between $[\|\cdot\|_X]$ and $[\|\cdot\|_Y]$ is μ if and only if there are representatives $\|\cdot\|_X$ and $\|\cdot\|_Y$ such that

(1) *One has $\|\cdot\|_X \leq \|\cdot\|_Y \leq \mu\|\cdot\|_X$.*
(2) *There are $x \in S_X \cap S_Y$ and $y \in S_Y \cap \mu^{-1} S_X$.*

In particular, the distance μ is always attained.

Proof The chain of inequalities in the statement is equivalent to the chain of inclusions $\mu^{-1}B_X \subset \lambda^{-1}B_Y \subset B_X$, so suppose that $S_X \cap \lambda^{-1}S_Y = \emptyset$. The distance between the compact sets $\lambda^{-1}B_Y$ and S_X is attained, so if they do not meet, then the distance between them is strictly positive and we can multiply the sets $\lambda^{-1}B_Y$ and $\mu^{-1}B_X$ by $1+\varepsilon$ for some $\varepsilon > 0$ and the contentions are still fulfilled. So, if we define $\mu' = \dfrac{\mu}{1+\varepsilon}$ we obtain $\mu'^{-1}B_X \subset (1+\varepsilon)\lambda^{-1}B_Y \subset B_X$. So, μ would not be minimal because $\mu' < \mu$. The case $\lambda^{-1}S_Y \cap \mu^{-1}S_X = \emptyset$ is analogous.

The other implication is clear. □

Proposition 3.2 *The function d defined in* (3) *is a multiplicative distance.*

Proof We need to show that d fulfils the following:

(1) $\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y]) = 1$ if and only if $\|\cdot\|_X$ and $\|\cdot\|_Y$ are proportional.
(2) $\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y]) = \mathrm{d}([\|\cdot\|_Y], [\|\cdot\|_X])$.
(3) $\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y]) \le \mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Z])\mathrm{d}([\|\cdot\|_Z], [\|\cdot\|_Y])$.

The first item is obvious since we have taken the quotient exactly for this.
It is clear that

$$\begin{aligned}\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y]) &= \min\{\mu : \|\cdot\|_X \le \|\cdot\|_Y \le \mu\|\cdot\|_X\}\\ &= \min\{\mu : \|\cdot\|_X \le \|\cdot\|_Y \le \mu\|\cdot\|_X \le \mu\|\cdot\|_Y\}\\ &= \min\{\mu : \|\cdot\|_Y \le \mu\|\cdot\|_X \le \mu\|\cdot\|_Y\}\\ &= \mathrm{d}([\|\cdot\|_Y], [\|\cdot\|_X]),\end{aligned} \tag{4}$$

so the second item also holds.

For the third item, let $\mu = \mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Z])$, $\mu' = \mathrm{d}([\|\cdot\|_Z], [\|\cdot\|_Y])$. There exist λ, λ' such that

$$\|\cdot\|_X \le \lambda\|\cdot\|_Z \le \mu\|\cdot\|_X \quad \text{and} \quad \|\cdot\|_Z \le \lambda'\|\cdot\|_Y \le \mu'\|\cdot\|_Z.$$

Joining these inequalities, we obtain $\|\cdot\|_X \le \lambda\|\cdot\|_Z \le \lambda\lambda'\|\cdot\|_Y \le \lambda\mu'\|\cdot\|_Z \le \mu\mu'\|\cdot\|_X$. This readily implies that $\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y]) \le \mu\mu' = \mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Z])\mathrm{d}([\|\cdot\|_Z], [\|\cdot\|_Y])$. □

In order to define the operations in $\mathcal{Q}$, we need the following:

Definition 3.3 Let us denote $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^n, \|\cdot\|_Y)$ and let $\theta \in [0, 1]$. We will call the space $\mathbb{R}^n$ endowed with the quasinorm

$$\|\cdot\|_{(X,Y)_\theta} = \|\cdot\|_X^\theta \|\cdot\|_Y^{1-\theta}$$

the interpolated space between X and Y at θ and will denote it as $(X, Y)_\theta$.

3.4 Observe that this kind of interpolation cannot be applied directly to infinite-dimensional spaces unless we consider only equivalent quasinorms on a given space.

3.5 When dealing with vector spaces, it is customary to have clear which vector is the origin of the space, in function spaces it is the 0 function, in spaces of sequences it is the sequence $(0, 0, \ldots)$. But we are giving a vector space structure to a set without a clear 0, so we need to choose it. The idea behind this work is that we have been given a kind of *mean* of two norms in a quite intuitive way, for our purposes, the most suitable candidate to be the mean of $\|\cdot\|_X$ and $\|\cdot\|_Y$ is

$$\|\cdot\|_{(X,Y)_{1/2}} = \|\cdot\|_X^{1/2}\|\cdot\|_Y^{1/2}.$$

Of course, this means that when we choose the origin of our space, we are given the opposite $\|\cdot\|_{\tilde{X}}$ for each $\|\cdot\|_X$. The central rúle that the Euclidean norm plays in the classical analysis could be enough for it to be our origin, but there is another reason for choosing it. When we think of a non-strictly convex norm, it seems that it is, in some faint sense, an extreme point of a segment. A visual way to explain this is the *curve* $\{[\|\cdot\|_p] : p \in [1, \infty]\}$. If you reach a non-strictly convex norm like $[\|\cdot\|_1]$ or $[\|\cdot\|_\infty]$ and you keep going in the same direction you will find that what you are dealing with is not convex any more. In this sense, the Euclidean norm is the most convex norm and it deserves to be the centre of our vector space. The space $(\mathbb{R}^n, \|\cdot\|_2)$ is, up to isometric isomorphism, the only homogeneous n-dimensional space and so, the one with the greatest group of isometries. So, we have defined our vector space as follows:

Definition 3.6 Let $n \in \mathbb{N}$ and consider $\mathcal{Q}$ as the quotient of the set of quasinorms on $\mathbb{R}^n$ by the equivalence relation of dilating quasinorms. We consider $[\|\cdot\|_2]$ as the origin of our space and the mean of two classes of quasinorms as

$$([\|\cdot\|_X], [\|\cdot\|_Y])_{1/2} = \left[\|\cdot\|_X^{1/2}\|\cdot\|_Y^{1/2}\right],$$

so the opposite of some $[\|\cdot\|_X]$ is $[\|\cdot\|_{\tilde{X}}]$, where

$$\|\cdot\|_{\tilde{X}} = \frac{\|\cdot\|_2^2}{\|\cdot\|_X}$$

on $\mathbb{R}^n \setminus \{0\}$ and $\|0\|_{\tilde{X}} = 0$; the *scalar multiplication* is given by

$$\theta \star [\|\cdot\|_X] = \left[\|\cdot\|_X^\theta \|\cdot\|_2^{1-\theta}\right], \quad -\theta \star [\|\cdot\|_X] = \left[\|\cdot\|_{\tilde{X}}^\theta \|\cdot\|_2^{1-\theta}\right]$$

for $\theta \in [0, \infty)$; and the *addition* of two classes of quasinorms by

$$[\|\cdot\|_X] \oplus [\|\cdot\|_Y] = 2\star\left[\|\cdot\|_{(X,Y)_{1/2}}\right].$$

Theorem 3.7 *With the above operations, $\mathcal{Q}$ is a linear space. If we, moreover, define*

$$\left|\!\left|\!\left| \|\cdot\|_X \right|\!\right|\!\right| = \log_2(\mathrm{d}(\|\cdot\|_X, \|\cdot\|_2)),$$

then $\left(\mathcal{Q}, \left|\!\left|\!\left|\cdot\right|\!\right|\!\right|\right)$ is a Banach space where the set of equivalence classes of norms in $\mathbb{R}^n$ is closed.

Proof It is easy to see that, whenever $\|\cdot\|_X$ and $\|\cdot\|_Y$ are quasinorms over a finite-dimensional space $\mathbb{R}^n$ and $\theta > 0$, the subset

$$B_\theta = \{x \in \mathbb{R}^n : \|x\|_X^\theta \|x\|_Y^{1-\theta} \leq 1\}$$

is bounded, absorbing, and balanced, and its boundary is bounded away from 0, so Theorem 2.5 implies that $\|\cdot\|_X^\theta \|\cdot\|_Y^{1-\theta}$ is a quasinorm, and it is clear that it is continuous. So, this kind of *extrapolation* of quasinorms is well defined. In order to show that $\mathcal{Q}$ is a linear space we need to show that the scalar multiplication and the addition are well defined. On the one hand, it is clear that the operations do not depend on the representative of any class of quasinorms. On the other hand, all the expressions in Definition 3.6 give rise to a continuous quasinorm.

In [9, Theorem 1.18] it is seen that the distance we are dealing with is complete on $\mathcal{N}$, and this implies that $\mathcal{N}$ is closed in any metric space where it is isometrically embedded, in particular in $\mathcal{Q}$. Anyway, it is not hard to see that its complement $\mathcal{Q} \setminus \mathcal{N}$ is open.

Now, we need to show that d is absolutely homogeneous and additively invariant.

For the homogeneity, let $\theta \in (0, \infty)$ and take any $\|\cdot\|_X$ such that $\|\cdot\|_X \geq \|\cdot\|_2$ and $S_X \cap S_2 \neq \emptyset$. Then, $(\|\cdot\|_X, \|\cdot\|_2)_\theta$ fulfils the same, i.e. $(\|\cdot\|_X, \|\cdot\|_2)_\theta \geq \|\cdot\|_2$ and $S_{(\|\cdot\|_X, \|\cdot\|_2)_\theta} \cap S_2 \neq \emptyset$. Moreover, if we take $y \in S_2$ such that

$$\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_2]) = \|y\|_X,$$

then it is quite clear that

$$\mathrm{d}([(\|\cdot\|_X, \|\cdot\|_2)_\theta], [\|\cdot\|_2]) = \|y\|_X^\theta.$$

For negative values of θ we only need to see what happens when $\theta = -1$, but it is easily seen that $\mathrm{d}([\|\cdot\|_X], [\|\cdot\|_2]) = \mathrm{d}([\|\cdot\|_{\tilde{X}}], [\|\cdot\|_2])$.

To see that d is additively invariant, take $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$. We just need to show that

$$\mathrm{d}([(\|\cdot\|_X, \|\cdot\|_Z)_{1/2}], [(\|\cdot\|_Y, \|\cdot\|_Z)_{1/2}]) = \mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y])^{1/2}.$$

For any $z \in \mathbb{R}^n$, $z \neq 0$ one has

$$\frac{\|z\|_{(X,Z)_{1/2}}}{\|z\|_{(Y,Z)_{1/2}}} = \frac{\|z\|_X^{1/2}\|z\|_Z^{1/2}}{\|z\|_Y^{1/2}\|z\|_Z^{1/2}} = \left(\frac{\|z\|_X}{\|z\|_Y}\right)^{1/2}. \tag{5}$$

Let $\mu = \mathrm{d}([\|\cdot\|_X], [\|\cdot\|_Y])$. Applying Lemma 3.1 we may suppose that $\|\cdot\|_X \leq \|\cdot\|_Y \leq \mu\|\cdot\|_X$ and choose x, y such that $\|x\|_X = \|x\|_Y = 1$, $\|y\|_X = 1/\mu$ and

$\|y\|_Y = 1$. Now (5) implies that

$$\mathrm{d}([\|\cdot\|_X],[\|\cdot\|_Y])^{1/2} = \mu^{1/2} = \left(\frac{\|x\|_X}{\|x\|_Y}\frac{\|y\|_Y}{\|y\|_X}\right)^{1/2} = \frac{\|x\|_{(X,Z)_{1/2}}}{\|x\|_{(Y,Z)_{1/2}}}\frac{\|y\|_{(Y,Z)_{1/2}}}{\|y\|_{(X,Z)_{1/2}}}$$
$$\leq \mathrm{d}([(\|\cdot\|_X,\|\cdot\|_Z)_{1/2}],[(\|\cdot\|_Y,\|\cdot\|_Z)_{1/2}])$$

Applying Lemma 3.1 to $(\|\cdot\|_X,\|\cdot\|_Z)_{1/2}$ and $(\|\cdot\|_Y,\|\cdot\|_Z)_{1/2}$ we see that the symmetric inequality also holds.

It remains to show the completeness of our norm. Take a Cauchy sequence

$$\left([\|\cdot\|^1],[\|\cdot\|^2],\ldots,[\|\cdot\|^k],\ldots\right)\subset \mathcal{Q}.$$

We may choose a representative of each class, so we may suppose that $\|\cdot\|^k(e_1) = 1$ for every $k \in \mathbb{N}$. Every Cauchy sequence is bounded, so we may take $\varepsilon, M > 0$ such that

$$\varepsilon\|\cdot\|_2 \leq \|\cdot\|^k \leq M\|\cdot\|_2 \text{ for every } k. \tag{6}$$

With this in mind, the very definition of $|||\cdot|||$ implies that for every $x \in \mathbb{R}^n$ the sequence $\|x\|^k$ is also Cauchy, so we may define $\|x\|_X$ as the limit of $\|x\|^k$ as $k \to \infty$. By (6) we have that B_X is a bounded, balanced, neighbourhood of 0, so $\|\cdot\|_X$ is a quasinorm and it is continuous because, locally, it is the uniform limit of continuous quasinorms. It is easy see that it is the limit of the sequence, and this implies that $|||\cdot|||$ is complete on $\mathcal{Q}$. □

Remark 3.8 The set of norms is not convex in $\mathcal{Q}$. In fact, if we define $\|(a,b)\|_X = 2|a| + |b|/2$, $\|(a,b)\|_Y = 2|b| + |a|/2$ then we have

$$\|(1,0)\|_X = 2 = \|(0,1)\|_Y, \quad \|(0,1)\|_X = 1/2 = \|(1,0)\|_Y,$$

but $\|(1,1)\|_X = 5/2 = \|(1,1)\|_Y$, which implies that

$$\|(1,1)\|_{(X,Y)_{1/2}} > \|(1,0)\|_{(X,Y)_{1/2}} + \|(0,1)\|_{(X,Y)_{1/2}}.$$

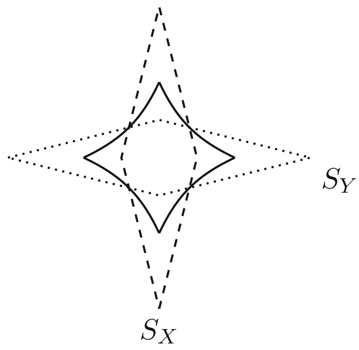
3.9 We can describe the space $\mathcal{Q}$ as some $C(K)$. Namely, let $\mathbb{P}_{n-1}$ be the projective space of dimension $n-1$, i.e.

$$\mathbb{P}_{n-1} = \left(\mathbb{R}^n \setminus \{0\}\right)/\sim, \text{ with } x \sim y \text{ if and only if } x = \lambda y \text{ for some } \lambda \in \mathbb{R}\setminus\{0\},$$

endowed with the quotient topology relative to the projection $\mathbb{R}^n \setminus \{0\} \to \left(\mathbb{R}^n\setminus\{0\}\right)/\sim$, $x \mapsto [x]$. In the sequel, we will think the projective space of dimension $n-1$ as the quotient $S^{n-1}/\sim$, where $x \sim y$ if and only if $x = \pm y$ and S^{n-1} denotes the sphere of $(\mathbb{R}^n, \|\cdot\|_2)$.

This is essentially the same idea as that reflected in Subsection 3.2 (Proof of the main result) in [9].

Fig. 1 The spheres of the three quasinorms of Remark 3.8



As every quasinorm is absolutely homogeneous, i.e. $\|\lambda x\| = |\lambda|\|x\|$ for any $\lambda \in \mathbb{R}$, $x \in X$, $\|\cdot\|$ is always determined by its value at every point of, say, the Euclidean sphere S^{n-1}. Take into account now the universal property of the quotient, that assures that any continuous function $f : S_X \to \mathbb{R}$ such that $f(x) = f(-x)$ gives rise to a well-defined and continuous $\tilde{f} : \mathbb{P}_{n-1} \to \mathbb{R}$, $\tilde{f}([x]) = f(x)$. With this in mind, it is clear that each continuous quasinorm $\|\cdot\|_X$ defines a continuous function $f_X : \mathbb{P}_{n-1} \to (0, \infty)$.

Recall that $\mathbb{P}_{n-1}$ is compact—it is the continuous image of a compact space—so every continuous $f : \mathbb{P}_{n-1} \to (0, \infty)$ is bounded from above and bounded away from 0, and we can define a quasinorm on $\mathbb{R}^n$ as $\|\lambda x\|_f = |\lambda| f([x])$ for every $\lambda \in \mathbb{R}$, $x \in S_2$.

It is clear that this is a one-to-one correspondence between the space of continuous quasinorms $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ and the space of positive continuous functions $\mathbb{P}_{n-1} \to (0, \infty)$. If we consider again the equivalence relation $\|\cdot\|_X \sim \|\cdot\|_Y \iff \|\cdot\|_X = \lambda\|\cdot\|_Y$, $\lambda \in \mathbb{R} \setminus \{0\}$, then the correspondence still holds if we consider $C(\mathbb{P}_{n-1})$ endowed with the equivalence relation $f \equiv g \iff f = \lambda g$, $\lambda \in \mathbb{R} \setminus \{0\}$. So, we have a bijection $\mathcal{Q} \longleftrightarrow C(\mathbb{P}_{n-1}, (0, \infty))/\equiv$. To end the description of $\mathcal{Q}$ we just need to consider $\log : C(\mathbb{P}_{n-1}, (0, \infty)) \to C(\mathbb{P}_{n-1})$, endow this space with the equivalence relation $f \sim g \iff f = \lambda + g$ for some $\lambda \in \mathbb{R}$—to preserve the bijection with the former space—and, for any $[f], [g] \in C(\mathbb{P}_{n-1})/\sim$ define the metric

$$d([f], [g]) = \max_{x \in \mathbb{P}_{n-1}} \{f(x) - g(x)\} - \min_{x \in \mathbb{P}_{n-1}} \{f(x) - g(x)\}, \text{ where } f, g \text{ are any representatives of } [f], [g].$$

Observe that this value is the range of $f - g$ and that we can rewrite this as

$$d([f], [g]) = \max_{x \in \mathbb{P}_{n-1}} \{f(x) - g(x)\} + \max_{x \in \mathbb{P}_{n-1}} \{g(x) - f(x)\}, \text{ where } f, g \text{ are any representatives of } [f], [g].$$

With this, we have $[\|\cdot\|_2] \in \mathcal{Q} \mapsto [0] \in C(\mathbb{P}_{n-1})/\sim$ (see, again, [9], Subsection 3.2). We also have that the map between $(\mathcal{Q}, \left|\!\left|\!\left| \cdot \right|\!\right|\!\right|)$ and $C(\mathbb{P}_{n-1})/\sim$ is an onto isometry. Moreover, if e_1 is the first vector of the usual basis of $\mathbb{R}^n$, then we can see the latter space as

$$C_0(\mathbb{P}_{n-1}) = \{f \in C(\mathbb{P}_{n-1}) : f([e_1]) = 0\},$$

whose bijection with some space of quasinorms arises from considering only the quasinorms in $\mathcal{Q}_0$ that take value 1 at e_1.

The reader interested in Projective Geometry can check out [4,13].

4 The Banach–Mazur Compactum

4.1 The space of $n \times n$ real matrices will be denoted as $\mathcal{M}_n$.

Every time we write *isometry* we will mean *linear isometry*. This, by the Mazur–Ulam Theorem, means just that we will consider only isometries sending 0 to 0.

As we will deal just with finite-dimensional spaces, we can fix the standard basis of $\mathbb{R}^n$, so that each operator $T : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_Y)$ can be seen as a matrix $A \in \mathcal{M}_n$. We will use $AB_X = \{Ax : x \in B_X\}$ and TB_X indistinctly.

We will intertwine operators and norms and will need some notation for *the norm whose value at each* x is $\|Ax\|_X$ (respectively, $\|Tx\|_X$), where $A \in \mathrm{GL}(n)$ (respectively, T is a linear isomorphism). This will be written as $A^*\|\cdot\|_X$ (respectively, $T^*\|\cdot\|_X$).

Now that we have determined the structure of $\mathcal{Q}$, we may relate it to the well-known Banach–Mazur compactum. This compactum is obtained by endowing the set $\mathcal{N}_0$ of norms defined on $\mathbb{R}^n$ with the pseudometric

$$\mathrm{d}_{BM}(\|\cdot\|_X, \|\cdot\|_Y) = \min\left\{\|T\|\|T^{-1}\|\right\},$$

where the minimum is taken in

$$\{T : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_Y) \text{ is a linear isomorphism}\}.$$

This pseudometric does not distinguish between isometric norms, so the quotient needed to turn it into a metric is by the equivalence relation

$$\|\cdot\|_X \equiv \|\cdot\|_Y \text{ when there is a linear isometry } T : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_Y).$$

As we are dealing with finite-dimensional spaces, the isomorphism T can be seen as an invertible matrix of order n, i.e. T is associated to some $A \in \mathrm{GL}(n)$. Conversely, every invertible matrix gives an isomorphism, so the Banach–Mazur distance can be seen as

$$\mathrm{d}_{BM}(\|\cdot\|_X, \|\cdot\|_Y) = \min\{\mu : B_X \subset AB_Y \subset \mu B_X \text{ for some } A \in \mathrm{GL}(n)\}$$

and the quotient as

$$\|\cdot\|_X \equiv \|\cdot\|_Y \text{ if and only if there is } A \in \mathrm{GL}(n) \text{ such that } AB_X = B_Y.$$

As the equivalence relation $\sim$ that defines $\mathcal{Q}$ can obviously be seen as

$$\|\cdot\|_X \sim \|\cdot\|_Y \text{ if and only if there is } \lambda \neq 0 \text{ such that } \lambda B_X = B_Y,$$

if we denote $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, then the relation between both spaces seems to be given by $\mathrm{PGL}(n, \mathbb{R}) = \mathrm{GL}(n)/\mathbb{R}^*$. It is, however, a little more complex.

Let us study the fibres in $\mathcal{N} = \{\text{Norms defined on } \mathbb{R}^n\}/\sim$ of each element of $BM = (\mathcal{N}_0/\equiv) = (\mathcal{N}/\equiv)$. Suppose we are given a norm $\|\cdot\|_X$ whose group of autoisometries is trivial, i.e. the only (linear) isometries $(\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_X)$ are the identity and its opposite. Then, $AB_X = CB_X$ implies $A = \pm C$ and this means that the fibre of $[\|\cdot\|_X] \in BM$ in $\mathcal{N}$ is indeed $\{A^*\|\cdot\|_X : A \in \mathrm{GL}(n)\}/\mathbb{R}^*$. However, if $\|\cdot\|_X$ has non-trivial group of autoisometries then $AB_X = AGB_X$ whenever $G : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_X)$ is an isometry. Denoting as Iso_X this group of autoisometries for each $\|\cdot\|_X$ we obtain a one-to-one relation

$$\mathcal{N} \longleftrightarrow \{(\{[\|\cdot\|_X]\} \times \mathrm{PGL}(n, \mathbb{R}))/\,\mathrm{Iso}_X : [\|\cdot\|_X] \in BM\}.$$

Before we proceed with the main result in this section we need a couple of results about the group Iso_X. As is customary, the distance between two linear operators $F, G : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_Y)$ is defined as the operator norm of its difference:

$$\mathrm{d}(F, G) = \|F - G\|_Y = \max\{\|F(x) - G(x)\|_Y : x \in B_X\}.$$

The first result we need is as follows:

Lemma 4.2 *Let $F : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_X)$ be an isometry. Then there are linearly independent $u, v \in \mathbb{R}^n$ such that the plane $\langle u, v\rangle$ is invariant for F and such that the matrix of the restriction of F to $\langle u, v\rangle$ with respect to the basis $\{u, v\}$ is one of the following:*

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

where $\alpha \in (-\pi, \pi]$.

Proof If $F = \pm\,\mathrm{Id}$ then the results holds with $\alpha = 0, \pi$, so we assume henceforth that this is not the case. It is well known that every linear endomorphism $F : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_X)$ has at least one complex eigenvalue λ (see, e.g. [2, 9.8]) and non-real eigenvalues occur in conjugate pairs. If $\lambda = a + bi \notin \mathbb{R}$, then there are $u, v \in X \setminus \{0\}$ such that $F(u) = au - bv$ and $F(v) = av + bu$. Let $|\lambda| = \sqrt{a^2 + b^2}$ and H be the plane generated by u and v endowed by the basis $\{u, v\}$, observe that $F(H) = H$.

Then, there is $\alpha \in (-\pi, \pi)$ such that the matrix of the restriction of F to H is

$$|\lambda| \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

As F is an isometry, one has $\|F^k(u)\|_X = 1$ for every $k \in \mathbb{N}$, so the sequence $(F^k(u))_k$ is bounded (with respect to every norm) and it is clear that this implies that $|\lambda| = 1$.

Suppose, now, that every eigenvalue is real, and let $\lambda \in \mathbb{R}$ and $u \in S_X$ be such that $F(u) = \lambda u$. It is obvious that, again, $|\lambda| = 1$. If F has at least two different eigenvalues then we may suppose $\lambda = 1$ and the other eigenvalue must be -1, so let v be such that $F(v) = -v$. With respect to the basis $\{u, v\}$ the matrix of the restriction of F to $\langle u, v\rangle$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This leaves the case where all the eigenvalues of F are real and are the same. We may suppose that all of them equal 1. As we are assuming that $F \neq \mathrm{Id}$, it is clear that the dimension of $\ker(F - \mathrm{Id})$ is at most $n - 1$ and the Cayley–Hamilton Theorem implies that $\dim(\ker(F - \mathrm{Id})^2) > \dim(\ker(F - \mathrm{Id}))$. This means that we may find $v \in S_X$, $u \in X \setminus \{0\}$, such that $u = F(v) - v \neq 0$, and $F(u) - u = (F - \mathrm{Id})(u) = (F - \mathrm{Id})^2(v) = 0$. Thus, we have $F(v) = u + v$ and $F(u) = u$ and this implies that in the plane $\langle u, v\rangle$ endowed with the basis $\{u, v\}$, the matrix of the restriction of F is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{8}$$

which leads to the matrix of F^k:

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \tag{9}$$

So, the sequence $(F^k(v))_k = (ku + v)_k$ is unbounded and we are done. □

Remark 4.3 It is easy to see that the same computation as the one at the end of the proof of Lemma 4.2 rules out the option that F has some Jordan block of the form:

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 1 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 1 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \tag{10}$$

so every isometry is diagonalizable over $\mathbb{C}$.

Remark 4.4 We have not used the fact that F is an isometry, we merely needed that the sequence $(\|F^k\|_X)_k$ is bounded and bounded away from 0.

Proposition 4.5 *Whenever* $\|\cdot\|_X$ *has non-trivial group of isometries, there is some autoisometry* $F : (\mathbb{R}^n, \|\cdot\|_X) \to (\mathbb{R}^n, \|\cdot\|_X)$ *such that* $\min\{\|F+\mathrm{Id}\|_X, \|F-\mathrm{Id}\|_X\} \geq 1$.

Proof Let $F \in \mathrm{Iso}_X$, $F \neq \pm\mathrm{Id}$. Then $\max\{\|F(x)+x\|_X, \|F(x)-x\|_X\} \leq 2$ for every $x \in S_X$, so $\max\{\|F+\mathrm{Id}\|_X, \|F-\mathrm{Id}\|_X\} \leq 2$. If all the eigenvalues of F are real, then the proof of Lemma 4.2 implies that there are $u, v \in S_X$ such that $\|F(u)+u\|_X = 2$, $\|F(v)-v\|_X = 2$, so we actually have $\|F+\mathrm{Id}\|_X = \|F-\mathrm{Id}\|_X = 2$.

If some eigenvalue is not real, say $\lambda = a+bi$, $b \neq 0$, then we know by Lemma 4.2 that $|\lambda| = 1$. Let $u \in S_X$, $v \in X \setminus \{0\}$ be such that the matrix of the restriction of F to $H = \langle u, v\rangle$ is, with respect to the basis $\{u, v\}$, the rotation of angle α

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

for some $\alpha \in (-\pi, \pi]$, the existence of such a basis is outlined in the proof of Lemma 4.2. If $\alpha > \pi/2$ (respectively, $\alpha \leq -\pi/2$) then we may compose with $-\mathrm{Id}$ and get the rotation of angle $-\pi+\alpha$ (respectively, $\pi+\alpha$), so we may suppose $\alpha \in (-\pi/2, \pi/2]$. If $\alpha < 0$ then the inverse of $F_{|H}$ is the rotation of angle $-\alpha$, so we only need to deal with $\alpha \in [0, \pi/2]$. As $\alpha = 0$ gives the identity, what we have is $\alpha \in (0, \pi/2]$. Now we have to break down the different options.

If $\alpha = \pi/m$ for some $m \in \mathbb{N}$, then $F^m_{|H} = -\mathrm{Id}_{|H}$. Consider the *half-orbit* of u, $\{x_0 = u, x_1 = F(u), \ldots, x_m = F^m(u) = -u\}$.

If $m \in 2\mathbb{Z}$, then $x_{m/2} = v$ is at the same distance from u and $-u$ because $F^{m/2}(u) = v$ and $F^{m/2}(v) = -u$. Indeed,

$$\|v-u\|_X = \|F^{m/2}(u)-u\|_X = \|F^{m/2}(F^{m/2}(u)-u)\|_X = \|-u-v\|_X.$$

This readily implies that

$$\min\{\|F^{m/2}-\mathrm{Id}\|_X, \|F^{m/2}+\mathrm{Id}\|_X\} \geq \|v-u\|_X = \|v+u\|_X \geq 1, \tag{11}$$

where the last inequality holds because of the triangular inequality:

$$2 = \|v+v\|_X \leq \|v-u\|_X + \|v+u\|_X = 2\|v-u\|_X.$$

For $m \notin 2\mathbb{N}$ we are going to restrict every coordinate-wise computation to the plane H, the difference would be a certain amount of zeroes after the two first coordinates. Taking coordinates with respect to $\{u, v\}$, we have

$$x_{(m-1)/2} = \left(\cos\left(\frac{(m-1)\pi}{2m}\right), \sin\left(\frac{(m-1)\pi}{2m}\right)\right),$$
$$x_{(m+1)/2} = \left(\cos\left(\frac{(m+1)\pi}{2m}\right), \sin\left(\frac{(m+1)\pi}{2m}\right)\right).$$

We are going to show that $\|x_{(m-1)/2}-u\|_X \geq 1$. For this, we first need to check that the segment whose endpoints are $x_{(m\pm1)/2}$ equals the intersection of the line that contains both of them with the unit ball B_X. Observe that the first coordinate of $x_{(m+1)/2}$ is the opposite of the first coordinate of $x_{(m-1)/2}$ and that the second coordinates of both points agree. So, if we denote by r the horizontal line whose height is $\sin((m+1)\pi/2m)$, we have $x_{(m\pm1)/2} \in r \cap S_X$. The convexity of B_X implies that if there are three collinear points in S_X, then the segment determined by them is included in S_X, too. In particular, if there is some $y \in (S_X \cap r) \setminus \{x_{(m\pm1)/2}\}$, then the segment whose endpoints are $x_{(m\pm1)/2}$ is included in S_X. On the one hand, this means that the *Euclidean* regular $2m$-gon with vertices in every x_k is included in S_X because each segment of the $2m$-gon is the image of this segment by some F^k. On the other hand, under these circumstances it is clear that this $2m$-gon is $S_X \cap H$, so in any case, $(t, \sin((m-1)\pi/2m)) \in B_X$ if and only if

$$t \in [-\cos((m-1)\pi/2m)), \cos((m-1)\pi/2m))]. \tag{12}$$

If $m = 3$, then $x_1 = (1/2, \sqrt{3}/2)$ and $x_2 = (-1/2, \sqrt{3}/2)$, so $x_1 - u = x_2$. This implies that $\|x_1 - u\|_X = \|x_2 + u\|_X = 1$, and also $\|x_1 + u\|_X = \|x_2 - u\|_X > 1$. So, $\min\{\|F + \mathrm{Id}\|_X, \|F - \mathrm{Id}\|_X\} \geq 1$.

If $m \geq 5$, then $0 < \cos((m-1)\pi/2m)) < \cos(\pi/3) = 1/2$ and this, along with (12), implies that

$$x_{(m-1)/2} - u = (\cos((m-1)\pi/2m)) - 1, \sin((m-1)\pi/2m)))$$

lies outside the unit ball, so $\|F^{(m-1)/2} \pm \mathrm{Id}\|_X > 1$.

If $\alpha = \frac{p}{q}\pi$ for some coprime $p, q \in \mathbb{N}$, then the Chinese Remainder Theorem implies that the rotation of angle π/q is also an isometry and we are in the previous case.

If $\alpha \neq \frac{p}{q}\pi$ for any $p, q \in \mathbb{N}$, then the orbit of u is dense in S_H and, actually, in the sphere of $\|\cdot\|_2$, i.e. in $\{\lambda u + \mu v \in H : \lambda^2 + \mu^2 = 1\}$. The continuity of $\|\cdot\|_X$ with respect to any norm defined over H implies that $\|\cdot\|_X$ restricted to H is $\|\cdot\|_2$ and so, any map F^k that sends u close enough to v has distance to $\pm\,\mathrm{Id}$ close to $\sqrt{2} > 1$. $\square$

Definition 4.6 Let $\|\cdot\|_X$ be a norm defined over $\mathbb{R}^n$. We say that $\|\cdot\|_X$ is a polyhedral norm or, equivalently, that $(\mathbb{R}^n, \|\cdot\|_X)$ is a polyhedral space, if its closed unit ball is a polytope.

Definition 4.7 Given a normed space $(X, \|\cdot\|_X)$, we say that $x \in S_X$ is an exposed point if there is $f \in X^*$ such that $f(x) = 1$ and $f(y) < 1$ for every $y \in S_X$, $y \neq x$. We say that $x \in S_X$ is an extreme point if it does not lie in the interior of a segment included in S_X.

4.8 It is clear that if B_X is a polytope, then $x \in B_X$ is extreme if and only if it is exposed.

We will need the following weak version of the Krein–Milman Theorem, see [10]:

Theorem 4.9 *(Krein–Milman) The unit ball of every finite-dimensional normed space is the convex hull of its subset of extreme points.*

In the proof of Theorem 4.11 we will also use the Brouwer fixed-point Theorem, see [11, Theorem 6] or directly, [14]:

Theorem 4.10 *If C is a closed convex subset of a Banach space, then every compact continuous map $f : C \to C$ has a fixed point. In particular, if C is convex and compact, then every continuous map $f : C \to C$ has a fixed point.*

Now we can proceed with the main result in this section.

Theorem 4.11 *Let $U = \{[\|\cdot\|_X] \in \mathcal{N} : \mathrm{Iso}_X = \{\mathrm{Id}, -\mathrm{Id}\}\}$. Then, U is a dense open subset of $\mathcal{N}$.*

Proof To see that U is dense we need the following fact:
The subset of equivalence classes of polyhedral norms is dense in $\mathcal{N}$. This is clear from [5, Theorem 1.1].

With this fact in mind, and given some polyhedral norm $\|\cdot\|_X$, we are going to sketch how to construct a norm with trivial group of isometries and whose distance to $\|\cdot\|_X$ is as small as we want. The Krein–Milman Theorem implies that there is a basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ such that every x_i is an exposed point of B_X. Given $\delta > 0$ we may consider

$$x_{n+1} = \frac{1+\delta}{\|(1, \ldots, 1)\|_X}(1, \ldots, 1)$$

and the norm $\|\cdot\|_{X'}$ whose unit ball is the convex hull of $B_X \cup \{\pm x_{n+1}\}$. On the one hand, this norm is as close as we want to $\|\cdot\|_X$, so we just need to approximate $\|\cdot\|_{X'}$. On the other hand, every x_i with $i = 1, \ldots, n+1$ is exposed in $B_{X'}$. Indeed, for each $i \in \{1, \ldots, n\}$, consider some linear $f_i : \mathbb{R}^n \to \mathbb{R}$ such that $f_i(x_i) = 1$ and $f_i(y) < 1$ for every $y \in S_X$, $y \neq x_i$. It is clear that there exist $\alpha_1, \ldots, \alpha_n \in (0, \infty)$ such that $f_i(x_{n+1}) < (1 - \alpha_i)$. As there are finitely many α_i we can choose $\delta > 0$ so that $f_i(x_{n+1}) < (1-\alpha_i)(1+\delta) < 1$ for every $i = 1, \ldots, n$. This means that, when $\delta > 0$ is small enough, $f_i(x_{n+1}) < 1$. The only points in $B_{X'}$ that do not belong to B_X are x_{n+1} and convex combinations $\lambda x_{n+1} + (1-\lambda)z$ with $z \in B_X$ and $\lambda \in (0, 1)$. This clearly implies that $f_i(y) < 1$ for every $y \in B_{X'}$, $y \neq x_i$, with $i = 1, \ldots, n$.

Now, consider some linear $f_{n+1} : \mathbb{R}^n \to \mathbb{R}$ such that $f_{n+1}(x_{n+1}) = 1$ and $f_{n+1}(y) < 1$ for every $y \in S_X$, $y \neq x_{n+1}$. Choose a basis $\mathcal{B}^i = \{u_1^i, \ldots, u_n^i\} \subset S_X$ with $u_1^i = x_i$ and $u_j^i \in \ker(f_i)$ when $j \neq i$. Given some $M_i > 0$ and $1 > \varepsilon_i > 0$ we may define $\|\cdot\|_i$ as

$$\begin{aligned}\|\lambda_1 u_1^i + \cdots + \lambda_n u_n^i\|_i = \big(&|(1+\varepsilon_i)\lambda_1|^{2i+2} \\ &+(|\lambda_2|/M_i)^{2i+2} + \cdots + (|\lambda_n|/M_i)^{2i+2}\big)^{1/(2i+2)}.\end{aligned}$$

If we take ε_i small enough and M_i big enough, then the norm $\|\cdot\|_Y = \max\{\|\cdot\|_{X'}, \|\cdot\|_1, \ldots, \|\cdot\|_n\}$ equals $\|\cdot\|_{X'}$ in every point of $S_{X'}$ except for small neighbourhoods of $(1-\varepsilon)x_1, \ldots, (1-\varepsilon)x_{n+1}$, say $V_1, \ldots, V_{n+1}$, where the sphere takes the form of a variant of the p-norm with $p = 2i + 2$. Observe that we may take each e_{i+1} small enough and M_{i+1} big enough to make the diameter of V_{i+1} strictly smaller than that of V_i and, moreover, we may suppose that the diameter of every V_i is strictly smaller than the distance between any pair of V_j, V_k, with $i, j, k \in 1, \ldots, n+1$.

Claim *Reducing ε and increasing M if necessary, we may suppose that every collection $y_1 \in V_1, \ldots, y_{n+1} \in V_{n+1}$ are in general position, i.e. no hyperplane contains n of them.*

Proof This is clear from the following facts:

(1) $x_1, \ldots, x_{n+1}$ are in general position.
(2) An n-tuple $\{u_1, \ldots, u_n\}$ lies in the same hyperplane if and only if every skew-symmetric linear n-form vanishes when applied to it, i.e. $\omega(u_1, \ldots, u_n) = 0$ for every (some) $\omega : (\mathbb{R}^n)^n :\to \mathbb{R}, \omega \neq 0$.
(3) Any skew-symmetric linear n-form is continuous.

□

This new norm $\|\cdot\|_Y$ has trivial autoisometry group. Indeed, the points in $V_1, \ldots, V_{n+1}$ are the only exposed points where S_Y is smooth, besides $-V_1, \ldots, -V_{n+1}$. So, as being exposed and being smooth are properties preserved by linear isometries, $(\bigcup V_i) \bigcup (\bigcup -V_i)$ is invariant for any autoisometry $F : (\mathbb{R}^n, \|\cdot\|_Y) \to (\mathbb{R}^n, \|\cdot\|_Y)$. It is clear that every V_i is connected, so its image by F (or any other continuous function) is also connected. This implies that for each i there is some j such that $F(V_i) \subset V_j$. This is also true for F^{-1}, so $F^{-1}(V_j) \subset V_i$ and we have the equality $F(V_i) = V_j$. There is no way that $(V_i, \|\cdot\|_Y)$ is isometric to $(V_j, \|\cdot\|_Y)$, $j \neq i$, because their diameters are different, so every $V_i \bigcup (-V_i)$ is invariant for F.

Let us denote by $\mathrm{ch}(V_i)$ the convex hull of V_i, analogously $\mathrm{ch}(-V_i)$. As F is linear, $\mathrm{ch}(V_i) \bigcup \mathrm{ch}(-V_i)$ is invariant for F, too. Now, either F or $-F$ sends V_i onto itself. Thus, the Brouwer fixed-point Theorem implies that either F or $-F$ has some fixed point $y_i \in \mathrm{ch}(V_i)$. In any case, $\{F(y_i), F(-y_i)\} = \{y_i, -y_i\}$ for every $i = 1, \ldots, n+1$.

So, in the basis $\{y_1, \ldots, y_n\}$, the matrix of F is diagonal, and all the diagonal entries are $\{\pm 1\}$, say the k-th is δ_k. In this basis, we have $y_{n+1} = (\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \cdots \lambda_n \neq 0$—recall that $\{y_1, \ldots, y_{n+1}\}$ are in general position—and $\{\pm y_{n+1}\}$ is also invariant, say $F(y_{n+1}) = \delta_{n+1} y_{n+1}$. As F is linear we have

$$\delta_{n+1}(\lambda_1, \ldots, \lambda_n) = \delta_{n+1} y_{n+1} = F(\lambda_1, \ldots, \lambda_n) = (\delta_1 \lambda_1, \ldots, \delta_n \lambda_n).$$

So, $\delta_1 = \cdots = \delta_n = \delta_{n+1}$ and this implies that F is either the identity or $-\mathrm{Id}$.

With ε close enough to 0 and M great enough, $\|\cdot\|'_X$ is as close to $\|\cdot\|_X$ as we want, so U is dense.

To show that U is open, let $([\|\cdot\|^k])_k \subset U^c$ be a convergent sequence. We need to show that $[\|\cdot\|] = \lim([\|\cdot\|^k])$ has non-trivial group of isometries, i.e. that U^c is closed. As the sequence of norms converges, in particular it is bounded, so there exists $R \in (1, \infty)$ such that $\mathrm{d}([\|\cdot\|]^k, [\|\cdot\|]_2) \leq R$ for every $k \in \mathbb{N}$. So, for each k, we may take representatives $\|\cdot\|^k$ such that

$$\|\cdot\|_2 \leq \|\cdot\|^k \leq R\|\cdot\|_2, \tag{13}$$

and also $\|\cdot\|_2 \leq \|\cdot\| \leq R\|\cdot\|_2$. Suppose that for every $\|\cdot\|^k$ there exists $T_k \in \mathrm{Iso}_{X_k} \setminus \{\mathrm{Id}, -\mathrm{Id}\}$. By (13), $1 \leq \|T_k x\|_2 \leq R$ for every $k \in \mathbb{N}$ and $x \in S_{X_k}$, so (T_k) is uniformly bounded in $\mathcal{M}_n$ endowed with the Euclidean matrix norm. This implies that $(T_k)_k$ must have some accumulation point T; we will suppose that T is the limit of the sequence. We need to see that T is an autoisometry for $\|\cdot\|$ and that it can be chosen to be neither Id nor $-$Id.

For the first part, applying the triangle inequality to $|||\cdot|||$ gives

$$\begin{aligned} |||T^*\|\cdot\| - \|\cdot\|||| \leq |||T^*\|\cdot\| - T_k^*\|\cdot\|||| + |||T_k^*\|\cdot\| - T_k^*\|\cdot\|^k||| \\ + |||T_k^*\|\cdot\|^k - \|\cdot\|^k||| + |||\|\cdot\|^k - \|\cdot\|||| . \end{aligned} \tag{14}$$

The third term in the sum is 0 for every k and the fourth term tends to 0 when $k \to \infty$, so we need to show that it is also the case for the first two terms, or, equivalently, that the map

$$(T, \|\cdot\|) \mapsto T^*\|\cdot\|$$

—that assigns to each linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and each norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ the norm $T^*\|\cdot\|$ defined as $T^*\|x\| = \|Tx\|$—is continuous. For the sake of clarity, we will denote $S = S_{X_k}$ for the remainder of the proof. We need to show that

$$\lim_k \max_{y\in S} \left\{ \frac{T_k^*\|y\|}{T^*\|y\|} \right\} \max_{y\in S} \left\{ \frac{T^*\|y\|}{T_k^*\|y\|} \right\} = 1.$$

Given $k \in \mathbb{N}$, Lemma 3.1, implies that we may take y_k, z_k such that

$$\max_{y\in S} \left\{ \frac{T_k^*\|y\|}{T^*\|y\|} \right\} \max_{y\in S} \left\{ \frac{T^*\|y\|}{T_k^*\|y\|} \right\} = \frac{T_k^*\|y_k\|}{T^*\|y_k\|} \frac{T^*\|z_k\|}{T_k^*\|z_k\|}$$

and one has

$$\lim_k \max_{y\in S} \left\{ \frac{T_k^*\|y\|}{T^*\|y\|} \right\} \max_{y\in S} \left\{ \frac{T^*\|y\|}{T_k^*\|y\|} \right\} = \lim_k \frac{T_k^*\|y_k\|}{T^*\|y_k\|} \frac{T^*\|z_k\|}{T_k^*\|z_k\|} = \lim_k \frac{\|T_k y_k\|}{\|T y_k\|} \frac{\|T z_k\|}{\|T_k z_k\|} = 1$$

since $\|\cdot\|$ is continuous. Analogously we see that

$$\lim_k \max_{y\in S} \left\{ \frac{T_k^*\|y\|^k}{T_k^*\|y\|} \right\} \max_{y\in S} \left\{ \frac{T_k^*\|y\|}{T_k^*\|y\|^k} \right\} = 1.$$

So, taking logarithms, the right-hand side of the inequality (14) converges to 0 and this implies that $|||T^*\|\cdot\| - \|\cdot\|||| = 0$, so $T^*\|\cdot\| = \|\cdot\|$ and T is an isometry.

Proposition 4.5 implies that we can choose every T_k at distance at least 1 from $\pm$Id, so $T \neq \pm$Id and we are done. □

Remark 4.12 In the previous proof we have seen that $(T, \|\cdot\|) \mapsto T^*\|\cdot\|$ is continuous when $\|\cdot\|$ is a norm. This is not always true when $\|\cdot\|$ is a quasinorm. Indeed, we just need to consider $\mathbb{R}^2$ endowed with the quasinorm

$$\|(x, y)\| = \begin{cases} \|(x, y)\|_2 & \text{if } (x, y) \notin \{(\lambda, 0) : \lambda \in \mathbb{R}^*\} \\ \frac{1}{2}\|(x, y)\|_2 & \text{if } (x, y) \in \{(\lambda, 0) : \lambda \in \mathbb{R}\} \end{cases},$$

define the operators

$$T_k(x, y) = \begin{pmatrix} \cos(\pi/k) & -\sin(\pi/k) \\ \sin(\pi/k) & \cos(\pi/k) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and observe that $\left\|\left| T_k^*\|\cdot\| - T_l^*\|\cdot\| \right|\right\|$ does not depend on $k, l \in \mathbb{N}$ as long as they are different. Indeed, the operator T_k is the rotation of angle π/k and the only points in the sphere of $\|\cdot\|$ outside the Euclidean sphere are $\pm(2, 0)$. So, $T_k^*\|x\| = \|x\|_2$ for every $k \in \mathbb{N}$ unless $T_k(x) = (\lambda, 0)$, in which case $T_k^*\|x\| = \|x\|_2/2$. So, if k and l are different then one has

$$\begin{aligned} &\left\|\left| T_k^*\|\cdot\| - T_l^*\|\cdot\| \right|\right\| \\ &= \log_2(\mathrm{d}(T_k^*\|\cdot\|, T_l^*\|\cdot\|)) = \log_2\left(\max_{x\in S} \frac{\|T^k(x)\|}{\|T^l(x)\|} \max_{x\in S} \frac{\|T^l(x)\|}{\|T^k(x)\|} \right) \\ &= \log_2(4) = 2, \end{aligned}$$

where S denotes the unit sphere of the norm $\|\cdot\|$.

In spite of this, it is quite clear that the proof of the continuity of $(T, \|\cdot\|) \mapsto T^*\|\cdot\|$ still works when we deal with continuous quasinorms.

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