



Remarks on ‘Norm Estimates of the Partial Derivatives for Harmonic Mappings and Harmonic Quasiregular Mappings’

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Abstract

Let $f = P[F]$ denote the Poisson integral of F in the unit disk \mathbb{D} with F being absolutely continuous in the unit circle \mathbb{T} and $\hat{F} \in L^p(\mathbb{T})$, where $\hat{F}(e^{it}) = \frac{d}{dt} F(e^{it})$ and $p \geq 1$. Recently, the author in Zhu (J Geom Anal, 2020) proved that (1) if f is a harmonic mapping and $1 \leq p < 2$, then f_z and $\overline{f_{\bar{z}}} \in \mathcal{B}^p(\mathbb{D})$, the classical Bergman spaces of \mathbb{D} [12, Theorem 1.2]; (2) if f is a harmonic quasiregular mapping and $1 \leq p \leq \infty$, then $f_z, \overline{f_{\bar{z}}} \in \mathcal{H}^p(\mathbb{D})$, the classical Hardy spaces of \mathbb{D} [12, Theorem 1.3]. These are the main results in Zhu (J Geom Anal, 2020). The purpose of this paper is to generalize these two results. First, we prove that, under the same assumptions, [12, Theorem 1.2] is true when $1 \leq p < \infty$. Also, we show that [12, Theorem 1.2] is not true when $p = \infty$. Second, we demonstrate that [12, Theorem 1.3] still holds true when the assumption f being a harmonic quasiregular mapping is replaced by the weaker one f being a harmonic elliptic mapping.

Keywords Poisson integral · Harmonic mapping · Elliptic mapping · Hardy-type space · Bergman-type space

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1 Preliminaries and the Statement of Main Results

For $a \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$. In particular, we use \mathbb{D}_r to denote the disk $\mathbb{D}(0, r)$ and \mathbb{D} to denote the unit disk \mathbb{D}_1 . Moreover, let $\mathbb{T} := \partial\mathbb{D}$ be the unit circle. For $z = x + iy \in \mathbb{C}$, the two complex differential operators are defined by

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

For $\alpha \in [0, 2\pi]$, the *directional derivative* of a harmonic mapping (i.e., a complex-valued harmonic function) f at $z \in \mathbb{D}$ is defined by

$$\partial_\alpha f(z) = \lim_{\rho \rightarrow 0^+} \frac{f(z + \rho e^{i\alpha}) - f(z)}{\rho} = f_z(z) e^{i\alpha} + f_{\bar{z}}(z) e^{-i\alpha},$$

where $z + \rho e^{i\alpha} \in \mathbb{D}$, $f_z := \partial f / \partial z$ and $f_{\bar{z}} := \partial f / \partial \bar{z}$. Then

$$\|D_f(z)\| := \max\{|\partial_\alpha f(z)| : \alpha \in [0, 2\pi]\} = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$l(D_f(z)) := \min\{|\partial_\alpha f(z)| : \alpha \in [0, 2\pi]\} = \left| |f_z(z)| - |f_{\bar{z}}(z)| \right|.$$

For a sense-preserving harmonic mapping f defined in \mathbb{D} , the Jacobian of f is given by

$$J_f = \|D_f\| l(D_f) = |f_z|^2 - |f_{\bar{z}}|^2,$$

and the *second complex dilatation* of f is given by $\omega = \overline{f_{\bar{z}}} / f_z$. It is well known that every harmonic mapping f defined in a simply connected domain Ω admits a decomposition $f = h + \bar{g}$, where h and g are analytic. Recall that f is sense-preserving in Ω if $J_f > 0$ in Ω . Thus f is locally univalent and sense-preserving in Ω if and only if $J_f > 0$ in Ω , which means that $h' \neq 0$ in Ω and the analytic function $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ on Ω (cf. [4,10]).

Hardy-Type Spaces

For $p \in (0, \infty]$, the *generalized Hardy space* $\mathcal{H}_G^p(\mathbb{D})$ consists of all measurable functions from \mathbb{D} to \mathbb{C} such that $M_p(r, f)$ exists for all $r \in (0, 1)$, and $\|f\|_p < \infty$,

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}$$

and

$$\|f\|_p = \begin{cases} \sup\{M_p(r, f) : 0 < r < 1\} & \text{if } p \in (0, \infty), \\ \sup\{|f(z)| : z \in \mathbb{D}\} & \text{if } p = \infty. \end{cases}$$

The classical *Hardy space* $\mathcal{H}^p(\mathbb{D})$, that is, all the elements are analytic, is a subspace of $\mathcal{H}_G^p(\mathbb{D})$ (cf. [3,5]).

Bergman-Type Spaces

For $p \in (0, \infty]$, the *generalized Bergman space* $\mathcal{B}_G^p(\mathbb{D})$ consists of all measurable functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\|f\|_{b^p} = \begin{cases} \left(\int_{\mathbb{D}} |f(z)|^p d\sigma(z) \right)^{\frac{1}{p}} & \text{if } p \in (0, \infty), \\ \text{ess sup}\{|f(z)| : z \in \mathbb{D}\} & \text{if } p = \infty, \end{cases}$$

where $d\sigma(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure on \mathbb{D} . The classical *Bergman space* $\mathcal{B}^p(\mathbb{D})$, that is, all the elements are analytic, is a subspace of $\mathcal{B}_G^p(\mathbb{D})$ (cf. [7]). Obviously, $\mathcal{H}^p(\mathbb{D}) \subset \mathcal{B}^p(\mathbb{D})$ for each $p \in (0, \infty]$.

Poisson Integrals

Denote by $L^p(\mathbb{T})$ ($p \in [1, \infty]$) the space of all measurable functions F of \mathbb{T} into \mathbb{C} with

$$\|F\|_{L^p} = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \text{ess sup}\{|F(e^{i\theta})| : \theta \in [0, 2\pi)\} & \text{if } p = \infty. \end{cases}$$

For $\theta \in [0, 2\pi]$ and $z \in \mathbb{D}$, let

$$P(z, e^{i\theta}) = \frac{1}{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2}$$

be the *Poisson kernel*. For a mapping $F \in L^1(\mathbb{T})$, the *Poisson integral* of F is defined by

$$f(z) = P[F](z) = \int_0^{2\pi} P(z, e^{i\theta}) F(e^{i\theta}) d\theta.$$

It is well known that if F is absolutely continuous, then it is of bounded variation. This implies that for almost all $e^{i\theta} \in \mathbb{T}$, the derivative $\dot{F}(e^{i\theta})$ exists, where

$$\dot{F}(e^{i\theta}) := \frac{dF(e^{i\theta})}{d\theta}.$$

In [12], the author posed the following problem.

Problem 1.1 What conditions on the boundary function F ensure that the partial derivatives of its harmonic extension $f = P[F]$, i.e., f_z and $\overline{f_{\bar{z}}}$, are in the space $\mathcal{B}^p(\mathbb{D})$ (or $\mathcal{H}^p(\mathbb{D})$), where $p \geq 1$?

In [12], the author discussed Problem 1.1 under the condition that F is absolutely continuous. First, he proved the following, which is one of the two main results in [12]. On the related discussion, we refer to the recent paper [9].

Theorem A ([12, Theorem 1.2]) *Suppose that $p \in [1, 2)$ and $f = P[F]$ is a harmonic mapping in \mathbb{D} with $\dot{F} \in L^p(\mathbb{T})$, where F is an absolutely continuous function. Then both f_z and $\overline{f_{\bar{z}}}$ are in $\mathcal{B}^p(\mathbb{D})$.*

Furthermore, by requiring the mappings $P[F]$ to be harmonic quasiregular, the interval of p is widened from $[1, 2)$ into $[1, \infty)$, as shown in the following result, which is the other main result in [12].

Theorem B ([12, Theorem 1.3]) *Suppose that $p \in [1, \infty]$ and $f = P[F]$ is a harmonic K -quasiregular mapping in \mathbb{D} with $\dot{F} \in L^p(\mathbb{T})$, where F is an absolutely continuous function and $K \geq 1$. Then both f_z and $\overline{f_{\bar{z}}}$ are in $\mathcal{H}^p(\mathbb{D})$.*

The purpose of this paper is to discuss these two results further. Regarding Theorem A, our result is as follows, which shows that Theorem A is true for $p \in [1, \infty)$, and also indicates that Theorem A is not true when $p = \infty$.

Theorem 1.1 *Suppose that $f = P[F]$ is a harmonic mapping in \mathbb{D} and $\dot{F} \in L^p(\mathbb{T})$, where F is an absolutely continuous function.*

- (1) *If $p \in [1, \infty)$, then both f_z and $\overline{f_{\bar{z}}}$ are in $\mathcal{B}^p(\mathbb{D})$.*
- (2) *If $p = \infty$, then there exists a harmonic mapping $f = P[F]$, where F is an absolutely continuous function with $\dot{F} \in L^\infty(\mathbb{T})$, such that neither f_z nor $\overline{f_{\bar{z}}}$ is in $\mathcal{B}^\infty(\mathbb{D})$.*

About Theorem B, we show that this result also holds true for harmonic elliptic mappings, which are more general than harmonic quasiregular mappings. In order to state our result, we need to introduce the definition of elliptic mappings.

A mapping $f : \Omega \rightarrow \mathbb{C}$ is said to be *absolutely continuous on lines*, *ACL* in brief, in the domain Ω if for every closed rectangle $R \subset \Omega$ with sides parallel to the axes x and y , f is absolutely continuous on almost every horizontal line and almost every vertical line in R . Such a mapping has, of course, partial derivatives f_x and f_y a.e. in Ω . Moreover, we say $f \in ACL^2$ if $f \in ACL$ and its partial derivatives are locally L^2 integrable in Ω .

A sense-preserving and continuous mapping f of \mathbb{D} into \mathbb{C} is said to be a (K, K') -elliptic mapping if

- (1) f is ACL^2 in \mathbb{D} and $J_f \neq 0$ a.e. in \mathbb{D} ;
- (2) there are constants $K \geq 1$ and $K' \geq 0$ such that

$$\|D_f\|^2 \leq K J_f + K' \text{ a.e. in } \mathbb{D}.$$

We remark that the unit disk \mathbb{D} in the definition of (K, K') -elliptic mapping can be replaced by a general domain in \mathbb{C} . In particular, if $K' \equiv 0$, then a (K, K') -elliptic mapping is said to be K -quasiregular. It is well known that every quasiregular mapping is an elliptic mapping. But the inverse of this statement is not true. This can be seen from the example: Let $f(z) = z + \bar{z}^2/2$ in \mathbb{D} which is indeed a univalent harmonic mapping of \mathbb{D} . Then elementary computations show that (a) $\sup_{z \in \mathbb{D}} \{|\omega(z)|\} = 1$, which implies that f is not K -quasiregular for any $K \geq 1$, and (b) f is a $(1, 4)$ -elliptic mapping. We refer to [1,2,6,8,11] for more details of elliptic mappings.

Now, we are ready to state our next result.

Theorem 1.2 *Suppose that $p \in [1, \infty]$ and $f = P[F]$ is a (K, K') -elliptic mapping in \mathbb{D} with $\dot{F} \in L^p(\mathbb{T})$, where F is an absolutely continuous function, $K \geq 1$ and $K' \geq 0$. Then both f_z and $\overline{f_{\bar{z}}}$ are in $\mathcal{H}^p(\mathbb{D})$.*

The proofs of Theorems 1.1 and 1.2 will be presented in Sect. 2.

2 Proofs of the Main Results

We start this section by recalling the following two lemmas from [12].

Lemma C ([12, Theorem 1.1]) *Suppose $p \in [1, \infty)$ and $f = P[F]$ is a harmonic mapping in \mathbb{D} with $\dot{F} \in L^p(\mathbb{T})$, where F is an absolutely continuous function. Then for $z = re^{it} \in \mathbb{D}$,*

$$\|f_r\|_{L^p} \leq (2C(p))^{\frac{1}{p}} \|\dot{F}\|_{L^p},$$

and thus, $f_r \in \mathcal{B}_G^p(\mathbb{D})$, where

$$C(p) = \int_0^1 \left(\frac{4 \tanh^{-1} r}{\pi r} \right)^p r dr \leq \frac{4^{p-1}}{\pi^p} (2^p + (2 - 2^{-p})\Gamma(1 + p))$$

and Γ denotes the usual Gamma function.

Lemma D ([12, Lemma 2.3]) *Assume the hypotheses of Lemma C. Then for $z = re^{it} \in \mathbb{D}$,*

$$\|f_t\|_p \leq \|\dot{F}\|_{L^p},$$

and thus, $f_t \in \mathcal{H}_G^p(\mathbb{D})$.

2.1 Proof of Theorem 1.1

For the proof of the first statement of the theorem, let $z = re^{it} \in \mathbb{D}$. Then we have

$$\begin{cases} f_t(z) := \frac{\partial f(z)}{\partial t} = i(zf_z(z) - \bar{z}f_{\bar{z}}(z)) \\ f_r(z) := \frac{\partial f(z)}{\partial r} = f_z(z)e^{it} + f_{\bar{z}}(z)e^{-it}, \end{cases} \tag{2.1}$$

which implies that

$$f_z(z) = \frac{e^{-it}}{2} \left(f_r(z) - \frac{i}{r} f_t(z) \right) \text{ and } \overline{f_{\bar{z}}(z)} = \frac{e^{-it}}{2} \left(\overline{f_r(z)} - \frac{i}{r} \overline{f_t(z)} \right).$$

It follows that for $p \in [1, \infty)$,

$$|f_z(z)|^p \leq \frac{1}{2^p} \left(|f_r(z)| + \left| \frac{f_t(z)}{r} \right| \right)^p \leq \frac{1}{2} \left(|f_r(z)|^p + \left| \frac{f_t(z)}{r} \right|^p \right)$$

and similarly,

$$|f_{\bar{z}}(z)|^p \leq \frac{1}{2} \left(|f_r(z)|^p + \left| \frac{f_t(z)}{r} \right|^p \right).$$

Obviously, to prove that f_z and $\overline{f_{\bar{z}}}$ are in $\mathcal{B}^p(\mathbb{D})$, it suffices to show the following:

$$\int_{\mathbb{D}} |f_r(z)|^p d\sigma(z) < \infty \text{ and } \int_{\mathbb{D}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z) < \infty.$$

We only need to check the boundedness of the integral

$$\int_{\mathbb{D}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z)$$

because the boundedness of the integral $\int_{\mathbb{D}} |f_r(z)|^p d\sigma(z)$ easily follows from Lemma C.

By Lemma D, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f_t(re^{it})|^p dt \leq \|\dot{F}\|_{L^p}^p,$$

which yields that

$$\int_{\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z) \leq \frac{2^{p-1}}{\pi} \int_{\frac{1}{2}}^1 \left(\int_0^{2\pi} |f_t(re^{it})|^p dt \right) dr \leq 2^{p-1} \|\dot{F}\|_{L^p}^p. \tag{2.2}$$

To demonstrate the boundedness of the integral

$$\int_{\mathbb{D}_{\frac{1}{2}}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z),$$

assume that $f = h + \bar{g}$ where both h and g are analytic in \mathbb{D} . Then $\|D_f\| = |h'| + |g'|$. This implies that $\|D_f\|$ is continuous in $\mathbb{D}_{\frac{1}{2}}$, and thus, $\|D_f\|$ is bounded in $\mathbb{D}_{\frac{1}{2}}$. Hence, by (2.1), we have

$$\begin{aligned} \int_{\mathbb{D}_{\frac{1}{2}}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z) &= \frac{1}{\pi} \int_0^{\frac{1}{2}} \int_0^{2\pi} r |e^{it} f_z(re^{it}) - e^{-it} f_{\bar{z}}(re^{it})|^p dt dr \tag{2.3} \\ &\leq \frac{1}{\pi} \int_0^{\frac{1}{2}} \int_0^{2\pi} r \|D_f(re^{it})\|^p dt dr \\ &= \int_{\mathbb{D}_{\frac{1}{2}}} \|D_f(z)\|^p d\sigma(z) < \infty. \end{aligned}$$

Combining (2.2) and (2.3) gives the final estimate

$$\int_{\mathbb{D}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z) = \int_{\mathbb{D}_{\frac{1}{2}}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z) + \int_{\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}}} \left| \frac{f_t(z)}{r} \right|^p d\sigma(z) < \infty,$$

which is what we need, and so, the statement (1) of the theorem is true.

To prove the second statement of the theorem, let $F(e^{i\theta}) = |\sin \theta|$, where $\theta \in [0, 2\pi]$. Then F is absolutely continuous and $\dot{F} \in L^\infty(\mathbb{T})$. Also, elementary computations guarantee that for $z = re^{it} \in \mathbb{D}$,

$$\begin{aligned} f(z) &= P[F](z) = \int_0^{2\pi} P(z, e^{i\theta}) |\sin \theta| d\theta \\ &= \frac{1}{2\pi r(r^2 - 1)} \left[(1 - r^2) \cos t \log \frac{1 + r^2 - 2r \cos t}{1 + r^2 + 2r \cos t} \right. \\ &\quad \left. + 2(1 + r^2) \sin t \left(\arctan \left(\frac{1+r}{r-1} \cot \frac{t}{2} \right) + \arctan \left(\frac{1+r}{r-1} \tan \frac{t}{2} \right) \right) \right]. \end{aligned}$$

Then

$$|f_z(z)| = \frac{1}{2} \left| f_r(z) - i \frac{f_t(z)}{r} \right| = \frac{1}{2} \sqrt{|f_r(z)|^2 + \frac{|f_t(z)|^2}{r^2}},$$

which implies that

$$|f_z(r)| = \frac{1}{2} \sqrt{|f_r(r)|^2 + \frac{|f_t(r)|^2}{r^2}}. \quad (2.4)$$

Since

$$f_r(r) = \frac{1}{\pi r^2} \log \left(\frac{1-r}{1+r} \right) + \frac{2}{\pi} \frac{1}{r(1-r^2)},$$

we see that

$$\lim_{r \rightarrow 1^-} f_r(r) = \infty. \quad (2.5)$$

Combining (2.4) and (2.5) gives

$$\lim_{r \rightarrow 1^-} |f_z(r)| = \infty,$$

which implies that f_z is not in $\mathcal{B}^\infty(\mathbb{D})$.

By the similar reasoning, we know that $\overline{f_z}$ is not in $\mathcal{B}^\infty(\mathbb{D})$ either, and hence, the theorem is proved. \square

2.2 Proof of Theorem 1.2

Assume that $f = P[F]$ is a (K, K') -elliptic mapping in \mathbb{D} , which means that for $z \in \mathbb{D}$,

$$\|D_f(z)\|^2 \leq K \|D_f(z)\| l(D_f(z)) + K'. \quad (2.6)$$

We divide the proof of this theorem into two cases.

Case 2.1 Suppose that $p \in [1, \infty)$.

It follows from (2.6) that

$$\begin{aligned} \|D_f(z)\|^p &\leq \left(\frac{Kl(D_f(z)) + \sqrt{(Kl(D_f(z)))^2 + 4K'}}{2} \right)^p \\ &\leq \left(Kl(D_f(z)) + \sqrt{K'} \right)^p \leq 2^{p-1} \left(K^p l^p(D_f(z)) + K'^{\frac{p}{2}} \right), \end{aligned}$$

and thus, we have

$$l^p(D_f(z)) \geq \frac{1}{2^{p-1}K^p} \|D_f(z)\|^p - \frac{K'^{\frac{p}{2}}}{K^p}. \tag{2.7}$$

By (2.1), (2.7) and Lemma D, we know that for $z = re^{it} \in \mathbb{D}$,

$$\begin{aligned} 2\pi \|\dot{F}\|_{L^p}^p &\geq \int_0^{2\pi} |f_t(re^{it})|^p dt \geq r^p \int_0^{2\pi} l^p(D_f(re^{it})) dt \\ &\geq \frac{r^p}{2^{p-1}K^p} \int_0^{2\pi} \|D_f(re^{it})\|^p dt - \frac{2\pi K'^{\frac{p}{2}}}{K^p}, \end{aligned}$$

which implies that

$$\sup_{r \in (0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} \|D_f(re^{it})\|^p dt \right)^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} \left(K^p \|\dot{F}\|_{L^p}^p + K'^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

Hence $f_z, \overline{f_z} \in \mathcal{H}^p(\mathbb{D})$.

Case 2.2 Suppose that $p = \infty$.

By (2.6), we have

$$\|D_f(z)\| \leq \frac{Kl(D_f(z)) + \sqrt{(Kl(D_f(z)))^2 + 4K'}}{2} \leq Kl(D_f(z)) + \sqrt{K'},$$

which, together with (2.1) and Lemma D, gives

$$\|\dot{F}\|_\infty \geq \|f_t\|_\infty \geq |f_t(re^{it})| \geq rl(D_f(re^{it})) \geq \frac{r}{K} \left(\|D_f(re^{it})\| - \sqrt{K'} \right).$$

Consequently,

$$\sup_{z \in \mathbb{D}} (|z| \|D_f(z)\|) \leq \sqrt{K'} + K \|\dot{F}\|_\infty,$$

from which we conclude that $f_z, \overline{f_z} \in \mathcal{H}^\infty(\mathbb{D})$, and hence the theorem is proved. \square

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