The Journal of Geometric Analysis (2021) 31:9597–9632
https://doi.org/10.1007/s12220-021-00621-4

(p, q)-John Ellipsoids

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Received: 22 February 2020 / Accepted: 4 February 2021 / Published online: 25 February 2021

Abstract

As an extension of the classical John ellipsoid and the L_p-John ellipsoids due to Lutwak–Yang–Zhang, this paper studies (p, q)-John ellipsoids. We consider an optimization problem about the (p, q)-mixed volumes, whose solution is uniquely existed for all $0 < p \le q$. The solution allows us to introduce the concept of (p, q)-John ellipsoids. As applications, we established an analog of the John's inclusion theorem and Ball's volume-ratio inequality for (p, q)-John ellipsoids. Moreover, the connection between the isotropy of measures and the characterization of (p, q)-John ellipsoids is demonstrated.

Keywords L_p Brunn–Minkowski theory · L_p dual curvature measures · (p, q)-John ellipsoid · Extremal problems

AMS Subject Classification 52A30 · 52A40

1 Introduction

The concept of John ellipsoid, introduced by Fritz John [20], is extremely useful in convex geometry and Banach space geometry. For each convex body (compact convex

This work was supported by the National Natural Science Foundation of China (Grant No. 11561020) and Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2020JQ-236).

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set with nonempty interior) K in the n-dimensional Euclidean $\mathbb{R}^n$, its John ellipsoid JK is defined as the unique ellipsoid of maximal volume contained in K.

Two fundamental results concerning the John ellipsoid are John's inclusion and Ball's volume-ratio inequality. Let K be an origin-symmetric convex body K in $\mathbb{R}^n$. John's inclusion shows that

$$K \subseteq \sqrt{n} JK. \tag{1.1}$$

As an application of John's inclusion, the best upper bound of the Banach–Mazur distance is $\sqrt{n}$, for an n-dimensional normed space to n-dimensional Euclidean space. Ball's volume-ratio inequality states that

$$\frac{|K|}{|JK|} \le \frac{2^n}{\omega_n}, \tag{1.2}$$

with equality if and only if K is a parallelotope. Here $|\cdot|$ denotes n-dimensional volume and $\omega_n = |B| = \pi^{n/2}/\Gamma\left(1+\frac{n}{2}\right)$ denotes the volume of the unit ball B in $\mathbb{R}^n$. The fact that there is equality in (1.2) only for parallelotopes was established by Barthe [3]. For more information about the John ellipsoid, one can refer to [1,2,12,14,15,21,22,44] and the references within.

In 2005, Lutwak, Yang and Zhang [30] extend the John ellipsoid to L_p John ellipsoids, which is an important concept in the L_p Brunn–Minkowski theory initiated by Lutwak [27,28]. During the last two decades, the L_p Brunn–Minkowski theory has achieved great developments and expanded rapidly, see, e.g., [4–6,8,9,17–19,24–26,29,31–34,37,38,47–51]. Moreover, the Orlicz Brunn–Minkowski theory, as an extension of the L_p Brunn–Minkowski theory, emerged in [16,35,36]. In these papers, the fundamental notions of the L_p projection body and the L_p centroid body were extended to an Orlicz setting, see also [7,53,55]. For more information, please refer to the literature [11,23,39–41,54,56–60]. In particular, the classical John ellipsoid is extended to the L_p setting by Lutwak, Yang and Zhang [30] and to the Orlicz setting by Zou and Xiong [58].

Suppose $p \in (0,\infty]$ and K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Among all origin-symmetric ellipsoids E, the unique ellipsoid that solves the constrained maximization problem

$$\max_E \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}, \quad \text{subject to} \quad \overline{V}_p(K,E) \le 1, \tag{1.3}$$

is called the L_p John ellipsoid [30] of K and denoted by E_pK. Clearly, $E_pB = B$. Here

$$\overline{V}_p(K,E) = \left(\frac{1}{n|K|}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)\mathrm{d}S(K,u)\right)^{\frac{1}{p}}, \quad 0<p<\infty,$$

is the normalized L_p mixed volume of K and E; S^{n-1} is the unit sphere in $\mathbb{R}^n$; h_K and h_E are the support functions (see Sect. 2) of K and E, respectively. In the case

$p = \infty$, we define

$$\overline{V}_\infty(K, E) = \sup\left\{\frac{h_E(u)}{h_K(u)} : u \in \operatorname{supp} S(K, \cdot)\right\}.$$

Therefore, when the John point of K, i.e., the center of JK, is at the origin, $E_\infty K$ is precisely the classical John ellipsoid JK. In the case $p = 2$, the L_2 John ellipsoid $E_2 K$ is the new ellipsoid $\Gamma_{-2}K$ found by Lutwak, Yang and Zhang in [32], which is now called the LYZ ellipsoid and is in some sense dual to the Legendre ellipsoid of inertia in classical mechanics [42]. In the case $p = 1$, $E_1 K$ is the so-called Petty ellipsoid, see [13,43]. The volume-normalized Petty ellipsoid is obtained by minimizing the surface area of K under SL(n)-transformations.

In general, the L_p John ellipsoid $E_p K$ is not contained in K (except when $p = \infty$). However, when $1 \le p \le \infty$, it has $|E_p K| \le |K|$. In reverse, for $0 < p \le \infty$, the L_p version of Ball's volume-ratio inequality [30] states that

$$\frac{|K|}{|E_p K|} \le \frac{2^n}{\omega_n}$$

with equality if and only if K is a parallelotope.

By L_p dual curvature measures, Lutwak, Yang and Zhang [31] introduced the notion of L_p dual mixed volumes which unifies L_p mixed volumes of convex bodies in the L_p Brunn–Minkowski theory and dual mixed volumes of star bodies in the dual Brunn–Minkowski theory. Therefore, L_p dual mixed volumes become to be a core concept in convex geometry with unifying some contents of the L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory.

Let $\mathcal{K}_o^n$ denote the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. And let $\mathcal{S}_o^n$ denote the set of star bodies (compact star-shaped set about the origin) in $\mathbb{R}^n$.

Suppose K is a convex body in $\mathbb{R}^n$. For each $v \in \mathbb{R}^n \backslash \{o\}$, the hyperplane

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}$$

is called the supporting hyperplane to K with outer normal v.

The spherical image (Gauss image) of $\sigma \subset \partial K$ is defined by

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}.$$

Let $\sigma_K \subset \partial K$ be the set consisting of boundary points $x \in \partial K$, for which the set $\boldsymbol{\nu}_K(\{x\})$ contains more than a single element. It is well known that the spherical Lebesgue measure of σ_K is $\mathcal{H}^{n-1}(\sigma_K) = 0$ (see, e.g., [46, p. 84]). On precisely the functions

$$\nu_K : \partial K \backslash \sigma_K \to S^{n-1},$$

is called the spherical image map (Gauss map) of K and is continuous (see, e.g., [46, Lemma 2.2.12]). The set $\partial K \backslash \sigma_K$ is usually abbreviated by $\partial' K$. Since $\mathcal{H}^{n-1}(\sigma_K) = 0$, the integrals over subsets of $\partial' K$ and ∂K are equal with respect to $\mathcal{H}^{n-1}$.

For $\omega \subset S^{n-1}$, the radial Gauss image of ω is denoted by

$$\boldsymbol{\alpha}_K(\omega) = \{v \in S^{n-1} : \rho_K(u)u \in H_K(v) \text{ for some } u \in \omega\}.$$

For a subset $\eta \subset S^{n-1}$, the reverse radial Gauss image of η is denoted by

$$\boldsymbol{\alpha}^*_K(\eta) = \{u \in S^{n-1} : \rho_K(u)u \in H_K(v) \text{ for some } v \in \eta\}.$$

For $K \in \mathcal{K}^n_o$, the radial map of K, $r_K : S^{n-1} \to \partial K$, is defined by

$$r_K(u) = \rho_K(u)u \in \partial K,$$

for $u \in S^{n-1}$. Here, $\rho_K(u) = \max\{\lambda > 0 : \lambda u \in K\}$ is the radial function of K for $u \in S^{n-1}$. Note that $r_K^{-1} : \partial K \to S^{n-1}$ is given by $r_K^{-1}(x) = x/|x|$ for $x \in \partial K$. Let $\omega_K = \overline{\sigma_K} = r_K^{-1}(\sigma_K)$. Observe that ω_K has spherical Lebesgue measure 0, and the integrals over subsets of $S^{n-1}\backslash\omega_K$ and S^{n-1} are equal with respect to the spherical Lebesgue measure.

The radial Gauss map of $K \in \mathcal{K}^n_o$, $\alpha_K : S^{n-1}\backslash\omega_K \to S^{n-1}$, is given by

$$\alpha_K = \nu_K \circ r_K.$$

Obviously, for any $\lambda > 0$ and any $u \in S^{n-1}$,

$$\alpha_{\lambda K}(u) = \alpha_K(u). \tag{1.4}$$

For $p, q \in \mathbb{R}$, $K \in \mathcal{K}^n_o$, and $Q \in \mathcal{S}^n_o$, the L_p dual curvature measures $\widetilde{C}_{p,q}(K, Q)$ are Borel measures on S^{n-1} given by

$$\int_{S^{n-1}} g(v)\mathrm{d}\widetilde{C}_{p,q}(K, Q, v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_Q(u)^{n-q}\mathrm{d}u, \tag{1.5}$$

for each continuous function $g : S^{n-1} \to \mathbb{R}$. For each Borel set $\eta \subseteq S^{n-1}$, we have

$$\widetilde{C}_{p,q}(K, Q, \eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(\eta)} h_K(\alpha_K(u))^{-p}\rho^q_K(u)\rho^{n-q}_Q(u)\mathrm{d}u. \tag{1.6}$$

It has shown that [31, Proposition 5.4] that the L_p surface area measure, the dual curvature measure and the integral measure are all special cases of the L_p dual curvature measure. In particular, for $p, q \in \mathbb{R}$, and $K \in \mathcal{K}^n_o$,

$$\widetilde{C}_{p,q}(K, K, \cdot) = \frac{1}{n}S_p(K, \cdot), \tag{1.7}$$

$$\widetilde{C}_{p,n}(K, B, \cdot) = \frac{1}{n} S_p(K, \cdot), \tag{1.8}$$

where $S_p(K, \cdot)$ is the L_p-surface area measure of K.

Using L_p dual curvature measures, Lutwak, Yang and Zhang [31] introduced the concept of (p, q)-mixed volume volumes. For $p, q \in \mathbb{R}$, and convex bodies $K, L \in \mathcal{K}_o^n$, and a star body $Q \in \mathcal{S}_o^n$, the (p, q)-mixed volume $\widetilde{V}_{p,q}(K, L, Q)$ is defined by

$$\begin{aligned}\widetilde{V}_{p,q}(K, L, Q) &= \int_{S^{n-1}} h_L^p(v) \mathrm{d}\widetilde{C}_{p,q}(K, Q, v) \\ &= \frac{1}{n} \int_{S^{n-1}} h_L(\alpha_K(u))^p h_K(\alpha_K(u))^{-p} \rho_K(u)^q \rho_Q(u)^{n-q} \mathrm{d}u \quad (1.9) \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^q \rho_Q(u)^n \mathrm{d}u. \quad (1.10)\end{aligned}$$

The concept of the (p, q)-mixed volume unifies the L_p mixed volume and the dual mixed volume in the sense that

$$\widetilde{V}_{p,q}(K, L, K) = V_p(K, L), \quad \widetilde{V}_{p,q}(K, K, Q) = \widetilde{V}_q(K, Q). \tag{1.11}$$

In this paper we will consider the problem of **minimizing total L_p dual curvature measures under SL(n)-transformations**. Let K be a smooth convex body in $\mathbb{R}^n$ with the origin in its interior, and let Q be a smooth star body in $\mathbb{R}^n$. For real number p, q, find

$$\min_{\phi \in \mathrm{SL}(n)} \int_{S^{n-1}} \mathrm{d}\widetilde{C}_{p,q}(\phi K, \phi Q, u).$$

From (1.9) and [31, Proposition 7.3] (see also Lemma 2.3 of our paper), it follows that the original problem of minimizing total L_p dual curvature under SL(n)-transformations can be rewritten as

$$\begin{aligned}\min_{\phi \in \mathrm{SL}(n)} \int_{S^{n-1}} \mathrm{d}\widetilde{C}_{p,q}(\phi K, \phi Q, u) &= \min_{\phi \in \mathrm{SL}(n)} \widetilde{V}_{p,q}(\phi K, B, \phi Q) \\ &= \min_{\phi \in \mathrm{SL}(n)} \widetilde{V}_{p,q}(K, \phi^{-1} B, Q) \\ &= \min_{|E| = \omega_n} \widetilde{V}_{p,q}(K, E, Q),\end{aligned}$$

where the last minimum is taken over all origin-symmetric ellipsoids with volume ω_n. A $\phi_{p,q} \in \mathrm{SL}(n)$ at which this minimum is attained defines an ellipsoid $\bar{E}_{p,q}(K, Q)$ which $\phi_{p,q}$ maps into the unit ball B, i.e., $\bar{E}_{p,q}(K, Q) = \phi_{p,q}^{-1} B$. This ellipsoid is unique and will be called the volume-normalized (p, q)-John ellipsoid of K and Q. For $p = \infty$, define

$$\bar{E}_{\infty,q}(K, Q) = \lim_{p \to \infty} \bar{E}_{p,q}(K, Q).$$

For $r \in [0,+\infty)$, the normalized r-th dual area measure of $K, Q \in \mathcal{S}_o^n$, $\overline{\widetilde{V}}_r(K, Q; \cdot)$, is defined by

$$\mathrm{d}\overline{\widetilde{V}}_r(K, Q; u) = \frac{1}{n\widetilde{V}_r(K, Q)} \rho_K^r(u)\rho_Q^{n-r}(u)\mathrm{d}u, \quad \text{for } u \in S^{n-1}, \tag{1.12}$$

where $\widetilde{V}_r(K, Q)$ is the r-th dual mixed volume of $K, Q \in \mathcal{S}_o^n$. Clearly, $\mathrm{d}\overline{\widetilde{V}}_r(K, Q; \cdot)$ is a probability measure on S^{n-1}. In the case $Q = K$, $\mathrm{d}\overline{\widetilde{V}}_r(K, K; u) = \mathrm{d}\overline{\widetilde{V}}_K(u) = \frac{1}{n|K|}\rho_K^n \mathrm{d}u$, for $u \in S^{n-1}$, is the normalized dual area measure of $K \in \mathcal{S}_o^n$. And for the cases $r = 0, n$, we have $\mathrm{d}\overline{\widetilde{V}}_0(K, Q; \cdot) = \mathrm{d}\overline{\widetilde{V}}_Q(\cdot)$ and $\mathrm{d}\overline{\widetilde{V}}_n(K, Q; \cdot) = \mathrm{d}\overline{\widetilde{V}}_K(\cdot)$.

In order to rewrite the formulation of our problem for the case $p = \infty$, we next introduce a normalized version of (p, q)-dual mixed volumes. If $K, L \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$ and $q \geq p > 0$ with $r = q - p \geq 0$, then we define the normalized (p, q)-dual mixed volume by

$$\begin{aligned}\overline{\widetilde{V}}_{p,q}(K, L, Q) &= \left(\frac{\widetilde{V}_{p,q}(K, L, Q)}{\widetilde{V}_r(K, Q)}\right)^{\frac{1}{p}} \\ &= \left(\int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))\rho_K(u)}{h_K(\alpha_K(u))\rho_Q(u)}\right)^p \mathrm{d}\overline{\widetilde{V}}_r(K, Q; u)\right)^{\frac{1}{p}}. \end{aligned}\tag{1.13}$$

In the case $p = \infty$ (then $q = \infty$), define

$$\overline{\widetilde{V}}_{\infty,\infty}(K, L, Q) = \max\left\{\frac{h_L(\alpha_K(u))\rho_K(u)}{h_K(\alpha_K(u))\rho_Q(u)} : u \in \operatorname{supp}\widetilde{V}_r(K, Q; \cdot)\right\}. \tag{1.14}$$

Unless $\frac{h_L(\alpha_K(u))\rho_K(u)}{h_K(\alpha_K(u))\rho_Q(u)}$ is constant on $\operatorname{supp}\widetilde{V}_r(K, Q; \cdot)$, it follows from (1.13) and Jensen's inequality that

$$\overline{\widetilde{V}}_{p_1,q_1}(K, L, Q) < \overline{\widetilde{V}}_{p_2,q_2}(K, L, Q), \tag{1.15}$$

for $0 < p_1 < p_2 \leq \infty$, $0 < q_1 = p_1 + r \leq p_2 + r = q_2 \leq \infty$, and

$$\lim_{p\to\infty} \overline{\widetilde{V}}_{p,q}(K, L, Q) = \overline{\widetilde{V}}_{\infty,\infty}(K, L, Q).$$

We shall require the fact that, for $p_0 \in (0, \infty]$, $q_0 = p_0 + r \in (0, \infty]$ and $r \in [0, \infty)$,

$$\lim_{p\to p_0} \overline{\widetilde{V}}_{p,q}(K, L, Q) = \overline{\widetilde{V}}_{p_0,q_0}(K, L, Q). \tag{1.16}$$

In fact, we have already proved a more general conclusion, see Theorem 3.1 in subsequent. By (1.14), we have

$$\overline{\widetilde{V}}_{\infty,\infty}(K, L, Q) \leq 1 \quad \text{if and only if} \quad L \subseteq \left(\frac{\rho_Q}{\rho_K}\right) K. \tag{1.17}$$

In the sequel, we use $\mathcal{E}^n$ to denote the class of origin-symmetric ellipsoids in $\mathbb{R}^n$.

Inspired by the constrained maximization problem (1.3) posed by Lutwak, Yang and Zhang [30], this paper will consider a (p, q)-version of the problem:

Optimization Problems 1.1 *Let $0 < p \le q$ with $q = p + r$, $r \ge 0$. For $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, find an ellipsoid, among all origin-symmetric ellipsoids, which solves the following constrained maximization problem:*

$$\max_{E \in \mathcal{E}^n} \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}} \quad \text{subject to} \quad \overline{\widetilde{V}}_{p,q}(K, E, Q) \le 1. \tag{$S_{p,q}$}$$

An ellipsoid that solves the constrained maximization problem will be called a $S_{p,q}$ solution for K and Q. The dual problem is

$$\min_{E \in \mathcal{E}^n} \overline{\widetilde{V}}_{p,q}(K, E, Q) \quad \text{subject to} \quad \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}} \ge 1. \tag{$\bar{S}_{p,q}$}$$

An ellipsoid that solves the dual problem will be called a $\bar{S}_{p.q}$ solution for K and Q.

We will prove in Sect. 4 there is a unique solution to the constrained maximization problem, which will be called the (p, q)-John ellipsoid $E_{p,q}(K, Q)$ in Definition 4.6. The dual problem is equivalent to the problem of minimizing total L_p dual curvature measures under $\mathrm{SL}(n)$-transformations. The dual problem has a unique solution with volume ω_n, which differs by only a scale factor to the $S_{p,q}$ solution. Therefore, it is called the normalized (p, q)-John ellipsoid $\overline{E}_{p,q}(K, Q)$.

In the case of $Q = K$, $E_{p,q}(K, Q) = E_p(K)$ is the L_p John ellipsoid studied by Lutwak, Yang and Zhang [30]. In the case that $Q = B$ and $p = n$, one also has $E_{p,q}(K, Q) = E_p(K)$.

This paper is organized as follows. In Sect. 2 we recall some basic results in convex geometry. Section 3 proves the continuity of $\widetilde{V}_{p,q}$ and $\overline{\widetilde{V}}_{p,q}$. We prove in Sect. 4 the existence, uniqueness and geometric characterization of the (p, q)-John ellipsoid which solves Problem 1.1. Using the continuity of $\widetilde{V}_{p,q}$ and $\overline{\widetilde{V}}_{p,q}$, we study continuity of (p, q)-John ellipsoids in Sect. 5 In Sect. 6, we discuss generalizations of John's inclusion for (p, q)-John ellipsoids. In the last section, the inequality for the volume ratio is established.

2 Preliminaries

For quick reference we recall some basic results of convex geometry. We refer the reader to [10,46] for details.

The setting will be the n-dimensional Euclidean space $\mathbb{R}^n$. As usual $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of x. For $x \in \mathbb{R}^n \backslash \{o\}$, we use both $\bar{x}$ and $\langle x \rangle$ to denote $x/|x|$.

In addition to its denoting absolute value, without confusion we will use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$, often to denote n-dimensional volume, and on occasion to denote the absolute value of the determinant of an $n \times n$ matrix.

For $K \in \mathcal{K}_o^n$, its support function, $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by $h_K(x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$. Obviously, for real $\lambda > 0$,

$$h_{\lambda K}(x) = \lambda h_K(x), \text{ for } x \in \mathbb{R}^n. \tag{2.1}$$

More generally, for $\phi \in \mathrm{GL}(n)$ the image $\phi K = \{\phi x : x \in K\}$ have that

$$h_{\phi K}(x) = h_K(\phi^t x), \tag{2.2}$$

where ϕ^t denotes the transpose of ϕ.

The Hausdorff distance between convex bodies K and L is given by

$$\delta_H(K, L) := |h_K - h_L|_\infty = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

If $K, L \in \mathcal{K}_o^n$, then for real $p > 0$, the L_p-mixed volume of K and L is defined by

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) \mathrm{d}S_p(K, u). \tag{2.3}$$

If K contains the origin in its interior, then its polar body K^* is given by $K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \text{ for all } y \in K\}$. Obviously, for $\phi \in \mathrm{GL}(n)$,

$$(\phi K)^* = \phi^{-t} K^*, \tag{2.4}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

A star body $K \subset \mathbb{R}^n$ is a compact star-shaped set about the origin whose radial function $\rho_K : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$, defined for $x \in \mathbb{R}^n \backslash \{o\}$ by $\rho_Q(x) = \max\{\lambda > 0 : \lambda x \in Q\}$, is continuous. We call two star bodies K and L in $\mathcal{S}_o^n$ are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. If $\lambda > 0$, we have

$$\rho_{\lambda K}(x) = \lambda \rho_K(x), \quad \text{for all } x \in \mathbb{R}^n \backslash \{o\}. \tag{2.5}$$

More generally, for $\phi \in \mathrm{GL}(n)$, the image $\phi K = \{\phi x : x \in K\}$ of K have the property

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1} x), \tag{2.6}$$

for all $x \in \mathbb{R}^n \backslash \{o\}$.

The radial distance between $K, L \in \mathcal{S}_o^n$ is

$$\widetilde{\delta}_H(K, L) := |\rho_K - \rho_L|_\infty = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

The dual Brunn–Minkowski theory is a theory of dual mixed volumes of star bodies. For $q \in \mathbb{R}$, the q-th dual mixed volume of $K, Q \in \mathcal{S}_o^n$, is defined by (see [31])

$$\widetilde{V}_q(K, Q) = \frac{1}{n}\int_{S^{n-1}} \rho_K^q(u)\rho_Q^{n-q}(u)\mathrm{d}u, \tag{2.7}$$

where the integral is with respect to spherical Lebesgue measure. It is well know that for $\phi \in \mathrm{GL}(n)$,

$$\widetilde{V}_q(\phi K, \phi Q) = |\phi|\widetilde{V}_q(K, Q), \quad q \in \mathbb{R}\backslash\{0\}. \tag{2.8}$$

Dual Minkowski inequality can be expressed as follows: If $0 \le q \le n$ and $K, Q \in \mathcal{S}_o^n$, then

$$\widetilde{V}_q(K, Q)^n \le |K|^q|Q|^{n-q}, \tag{2.9}$$

with equality if and only if K and Q are dilates when $0 < q < n$.

If $K \in \mathcal{K}_o^n$, then it is easy to see that the radial function and the support function of K are related by

$$h_K(v) = \max_{u\in S^{n-1}} (u \cdot v)\rho_K(u), \quad \text{for } v \in S^{n-1}, \tag{2.10}$$

$$\frac{1}{\rho_K(u)} = \max_{v\in S^{n-1}} \frac{u \cdot v}{h_K(v)}, \quad \text{for } u \in S^{n-1}. \tag{2.11}$$

From definitions of $\widetilde{V}_{p,q}$ and the radial Gauss map, the support function and the radial function imply that

Lemma 2.1 *Let* $\lambda > 0$, *then*

$$\widetilde{V}_{p,q}(\lambda K, L, Q) = \lambda^{q-p}\widetilde{V}_{p,q}(K, L, Q), \tag{2.12}$$

$$\widetilde{V}_{p,q}(K, \lambda L, Q) = \lambda^{p}\widetilde{V}_{p,q}(K, L, Q), \tag{2.13}$$

$$\widetilde{V}_{p,q}(K, L, \lambda Q) = \lambda^{n-q}\widetilde{V}_{p,q}(K, L, Q). \tag{2.14}$$

For $\lambda > 0$ and $p \in (0, \infty]$, $q = p + r$, $r \in [0, \infty)$, based on the (1.13), (2.1) and (2.5), we can immediately obtain the results,

Lemma 2.2 *Let* $\lambda > 0$, *then*

$$\overline{\widetilde{V}}_{p,q}(\lambda K, L, Q) = \overline{\widetilde{V}}_{p,q}(K, L, Q), \tag{2.15}$$

$$\overline{\widetilde{V}}_{p,q}(K, \lambda L, Q) = \lambda\overline{\widetilde{V}}_{p,q}(K, L, Q), \tag{2.16}$$

$$\overline{\widetilde{V}}_{p,q}(K, L, \lambda Q) = \lambda^{-1}\overline{\widetilde{V}}_{p,q}(K, L, Q). \tag{2.17}$$

We shall need the following fact.

Lemma 2.3 (cf. [31]) *The* (p,q)*-mixed volume is* $\mathrm{SL}(n)$*-invariant, in that for* $p,q\in\mathbb{R}$*, and* $K,L\in\mathcal{K}_o^n$*, with* $Q\in\mathcal{S}_o^n$*,*

$$\widetilde{V}_{p,q}(\phi K,\phi L,\phi Q)=\widetilde{V}_{p,q}(K,L,Q), \tag{2.18}$$

for each $\phi\in\mathrm{SL}(n)$*.*

Lemma 2.1, together with Lemma 2.3, shows that for $\phi\in\mathrm{GL}(n)$,

$$\widetilde{V}_{p,q}(\phi K,\phi L,\phi Q)=|\phi|\widetilde{V}_{p,q}(K,L,Q). \tag{2.19}$$

We will also need the fact that for $\phi\in\mathrm{GL}(n)$ and $p\in(0,\infty]$, $q=p+r$, $r\in[0,\infty)$,

$$\overline{\widetilde{V}}_{p,q}(\phi K,\phi L,\phi Q)=\overline{\widetilde{V}}_{p,q}(K,L,Q). \tag{2.20}$$

This follows immediately from (2.8) and (2.19) for all $p\in(0,\infty]$, $q=p+r$ and $r\in[0,\infty)$.

The following inequality for (p,q)-mixed volume is a generalization of the L_p Minkowski inequality for mixed volume (see [31]).

Lemma 2.4 *Suppose* p,q *are such that* $1\le\frac{q}{n}\le p$*. If* $K,L\in\mathcal{K}_o^n$ *and* $Q\in\mathcal{S}_o^n$*, then*

$$\widetilde{V}_{p,q}(K,L,Q)^n\ge|K|^{q-p}|L|^p|Q|^{n-q}, \tag{2.21}$$

with equality if and only if K,L,Q *are dilates when* $1<\frac{q}{n}<p$*, while only* K *and* L *need be dilates when* $q=n$ *and* $p>1$*, and* K *and* L *are homothets when* $q=n$ *and* $p=1$*.*

We shall require the following definition.

Definition 2.5 (cf. [31]) Suppose $p\in\mathbb{R}$. If μ is a Borel measure on S^{n-1} and $\phi\in\mathrm{SL}(n)$ then, $\phi_p\dashv\mu$, the L_p image of μ under ϕ, is a Borel measure such that

$$\int_{S^{n-1}}f(u)\mathrm{d}\phi_p\dashv\mu(u)=\int_{S^{n-1}}|\phi^{-1}u|^pf(\langle\phi^{-1}u\rangle)\mathrm{d}\mu(u)$$

for each Borel $f:S^{n-1}\to\mathbb{R}$.

Lemma 2.6 (cf. [31]) *Suppose* $p\ne0$ *and* $q\ne0$*. Then for all* $Q\in\mathcal{S}_o^n$ *and* $K,L\in\mathcal{K}_o^n$*, and* $\phi\in\mathrm{SL}(n)$*,*

$$\widetilde{C}_{p,q}(\phi K,\phi Q,\cdot)=\phi_p^t\dashv\widetilde{C}_{p,q}(K,L,\cdot). \tag{2.22}$$

We also need the following lemma:

Lemma 2.7 (cf. [19]) *Suppose $K_i \in \mathcal{K}_o^n$ with $\lim_{i\to\infty} K_i = K_0$. Let $\omega = \cup_{i=0}^{\infty}\omega_{K_i}$, be the set (of $\mathcal{H}^{n-1}$-measure 0) off of which all of the α_{K_i} are defined. Then if $u_i \in S^{n-1}\backslash\omega$ are such that $\lim_{i\to\infty} u_i = u_0 \in S^{n-1}\backslash\omega$, then $\lim_{i\to\infty} \alpha_{K_i}(u_i) = \alpha_{K_0}(u_0)$.*

Let $K \in \mathcal{K}_o^n$. The classical projection body ΠK of K is given by (see [10])

$$h_{\Pi K}(u) = \mathrm{vol}_{n-1}(K|u^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |u\cdot v| \mathrm{d}S(K,v), \quad \forall u \in S^{n-1}.$$

We will use the concept of a L_p-projection body (see [28,29,45,52]). For $p \geq 1$, the L_p-projection body $\Pi_p K$ is given by

$$h_{\Pi_p K}(u) = \left(\frac{1}{2n}\int_{S^{n-1}} |u\cdot v|^p \mathrm{d}S_p(K,v)\right)^{\frac{1}{p}}, \quad u \in S^{n-1},$$

where $S_p(K,\cdot)$ is the L_p-surface area measure. Clearly, $\Pi_1 K = \frac{1}{n}\Pi K$.

We shall use the concepts of (p,q)-mixed projection body and (p,q)-mixed polar projection body. For each $K \in \mathcal{K}_o^n$ with a star body $Q \in \mathcal{S}_o^n$, and $p > 0, q > 0$, the (p,q)-mixed projection body, $\Pi_{p,q}(K,Q)$, of K and Q is the origin-symmetric convex body whose support function is defined by

$$h_{\Pi_{p,q}(K,Q)}(u) = \left(\frac{1}{2}\int_{S^{n-1}} |u\cdot v|^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,v)\right)^{\frac{1}{p}}, \quad \text{for all } u \in S^{n-1}. \tag{2.23}$$

In particular, we have $\Pi_{p,n}(K,B) = \Pi_{p,q}(K,K) = \Pi_p K$ for $p > 1$, and $\Pi_{1,n}(K,B) = \Pi_{1,q}(K,K) = \Pi_1(K) = \frac{1}{n}\Pi K$.

If $K \in \mathcal{K}_o^n$ and real $p > 0$, the star body $\Gamma_{-p}K$ (called by L_p-polar projection body, see [30]) is defined as, for $u \in S^{n-1}$:

$$\rho_{\Gamma_{-p}K}(u)^{-1} = \left(\frac{1}{|K|}\int_{S^{n-1}} |u\cdot v|^p \mathrm{d}S_p(K,v)\right)^{\frac{1}{p}}.$$

If $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, and real $p > 0, q > 0$ and $q = p + r, r \in [0,+\infty)$, the star body $\Gamma_{-p,-q}(K,Q)$ is defined by, for $x \in \mathbb{R}^n$,

$$\rho^{-1}_{\Gamma_{-p,-q}(K,Q)}(x) = \left(\frac{n}{\widetilde{V}_r(K,Q)}\int_{S^{n-1}} |x\cdot v|^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,v)\right)^{\frac{1}{p}}. \tag{2.24}$$

The star body $\Gamma_{-p,-q}(K,Q)$ is called the (p,q)-mixed polar projection body of K and Q. It is easy to know that $\Gamma_{-p,-q}(K,K) = \Gamma_{-p}K$.

Note that for $q \geq p \geq 1$, the body $\Gamma_{-p,-q}(K,Q)$ is a convex body. Define $\Gamma_{-\infty,-\infty}(K,Q)$ by

$$\Gamma_{-\infty,-\infty}(K,Q) = \lim_{p\to\infty}\Gamma_{-p,-q}(K,Q). \tag{2.25}$$

For real $p > 0, q = p + r, r \in [0, +\infty)$, and using(1.5), we can rewrite (2.25) as

$$n^{-\frac{1}{p}} \rho_{\Gamma_{-p,-q}(K,Q)}(u)^{-1} = \left(\int_{S^{n-1}} \left(\frac{|u \cdot v| \rho_K(v)}{h_K(\alpha_K(v)) \rho_Q(v)} \right)^p \mathrm{d}\widetilde{V}_r(K, Q; v) \right)^{\frac{1}{p}} \tag{2.26}$$

for $u \in S^{n-1}$. Thus, from (2.25) and (2.26),

$$\rho_{\Gamma_{-\infty,-\infty}(K,Q)}(u)^{-1} = \max \left\{ \frac{|u \cdot v| \rho_K(v)}{h_K(\alpha_K(v)) \rho_Q(v)} : v \in \operatorname{supp} \widetilde{V}_r(K, Q; \cdot) \right\}, \quad u \in S^{n-1}. \tag{2.27}$$

3 The Continuity of $\widetilde{V}_{p,q}$ and $\overline{\widetilde{V}}_{p,q}$

In this section, we consider the continuity of $\widetilde{V}_{p,q}$ and $\overline{\widetilde{V}}_{p,q}$.

Theorem 3.1 *Suppose $K, K_i, L, L_j \in \mathcal{K}_o^n$, $Q, Q_k \in \mathcal{S}_o^n$ and $p_l, p, q_m, q \in (0, \infty]$, where $i, j, k, l, m \in \mathbb{N}$. Let $r \in [0, +\infty)$. If $K_i \to K$, $L_j \to L$, $Q_k \to Q$, $p_l \to p$, and $q_m \to q$ as $i, j, k, l, m \to \infty$, then*

$$\lim_{i,j,k,l,m \to \infty} \widetilde{V}_{p_l,q_m}(K_i, L_j, Q_k) = \widetilde{V}_{p,q}(K, L, Q), \tag{3.1}$$

and

$$\lim_{i,j,k,l \to \infty} \overline{\widetilde{V}}_{p_l,p_l+r}(K_i, L_j, Q_k) = \overline{\widetilde{V}}_{p,p+r}(K, L, Q). \tag{3.2}$$

Proof Let

$$c_m = \min\{c_1, c_2\}, \ c_M = \max\{c_3, c_4\},$$

where

$$c_1 = \frac{\inf \left(\left\{ \min_{S^{n-1}} h_L \right\} \cup \left\{ \min_{S^{n-1}} h_{L_j} : j \in \mathbb{N} \right\} \right)}{\sup \left(\left\{ \max_{S^{n-1}} h_K \right\} \cup \left\{ \max_{S^{n-1}} h_{K_i} : i \in \mathbb{N} \right\} \right)},$$

$$c_2 = \frac{\inf \left(\left\{ \min_{S^{n-1}} \rho_K \right\} \cup \left\{ \min_{S^{n-1}} \rho_{K_i} : i \in \mathbb{N} \right\} \right)}{\sup \left(\left\{ \max_{S^{n-1}} \rho_Q \right\} \cup \left\{ \max_{S^{n-1}} \rho_{Q_k} : k \in \mathbb{N} \right\} \right)},$$

$$c_3 = \frac{\sup \left(\left\{ \max_{S^{n-1}} h_L \right\} \cup \left\{ \max_{S^{n-1}} h_{L_j} : j \in \mathbb{N} \right\} \right)}{\inf \left(\left\{ \min_{S^{n-1}} h_K \right\} \cup \left\{ \min_{S^{n-1}} h_{K_i} : i \in \mathbb{N} \right\} \right)},$$

and

$$c_4 = \frac{\sup\left(\left\{\max\limits_{S^{n-1}} \rho_K\right\} \cup \left\{\max\limits_{S^{n-1}} \rho_{K_i} : i \in \mathbb{N}\right\}\right)}{\inf\left(\left\{\min\limits_{S^{n-1}} \rho_Q\right\} \cup \left\{\min\limits_{S^{n-1}} \rho_{Q_k} : k \in \mathbb{N}\right\}\right)}.$$

We first claim $0 < c_m \leq c_M < \infty$. Since $K_i \to K$, $L_j \to L$ and $Q_k \to Q$, $p_l \to p$ as $i, j, k \to \infty$, we have $h_{K_i} \to h_K$, $h_{L_j} \to h_L$ and $h_{L_k} \to h_L$ uniformly on S^{n-1}, respectively. From $K, K_i, L, L_j \in \mathcal{K}_o^n$, $Q, Q_k \in \mathcal{S}_o^n$, it follows that there exists an $N_0 \in N$, such that for all $i, j, k > N_0$ and $u \in S^{n-1}$,

$$\begin{aligned} &\min_{S^{n-1}} h_{\frac{1}{2}K} \leq h_{K_i}(u) \leq \max_{S^{n-1}} h_{2K} \text{ and } \min_{S^{n-1}} h_{\frac{1}{2}L} \leq h_{L_j}(u) \leq \max_{S^{n-1}} h_{2L}, \\ &\min_{S^{n-1}} \rho_{\frac{1}{2}K} \leq \rho_{K_i}(u) \leq \max_{S^{n-1}} \rho_{2K} \text{ and } \min_{S^{n-1}} \rho_{\frac{1}{2}Q} \leq \rho_{Q_k}(u) \leq \max_{S^{n-1}} \rho_{2Q}. \end{aligned}$$

For brevity, we write

$$a_m = \min\{a : a \in A_1 \cup A_2\}, \quad a_M = \max\{a : a \in A_3 \cup A_4\},$$

where

$$\begin{aligned} A_1 &= \bigcup_{u \in S^{n-1}} \left\{h_{\frac{1}{2}K}(u), h_{\frac{1}{2}L}(u), \rho_{\frac{1}{2}K}(u), \rho_{\frac{1}{2}Q}(u)\right\}, \\ A_2 &= \bigcup_{1 \leq i \leq N_0} \bigcup_{u \in S^{n-1}} \left\{h_{K_i}(u), h_{L_j}(u), \rho_{K_i}(u), \rho_{Q_k}(u)\right\}, \\ A_3 &= \bigcup_{u \in S^{n-1}} \left\{h_{2K}(u), h_{2L}(u), \rho_{2K}(u), \rho_{2Q}(u)\right\}, \end{aligned}$$

and

$$A_4 = \bigcup_{1 \leq i \leq N_0} \bigcup_{u \in S^{n-1}} \left\{h_{K_i}(u), h_{L_j}(u), \rho_{K_i}(u), \rho_{Q_k}(u)\right\}.$$

Then we have $0 < a_m \leq a_M < \infty$, and

$$\begin{aligned} &a_m B \subseteq K \subseteq a_M B, \quad a_m B \subseteq K_i \subseteq a_M B \quad \text{for } i \in \mathbb{N}, \\ &a_m B \subseteq L \subseteq a_M B, \quad a_m B \subseteq L_j \subseteq a_M B \quad \text{for } j \in \mathbb{N}, \\ &a_m B \subseteq Q \subseteq a_M B, \quad a_m B \subseteq Q_k \subseteq a_M B \quad \text{for } k \in \mathbb{N}. \end{aligned}$$

Thus, by the definitions of c_m and c_M, it yields

$$0 < \frac{a_m}{a_M} \leq c_m \leq c_M \leq \frac{a_M}{a_m} < \infty.$$

Next, we prove

$$\lim_{i,j,k,l,m\to\infty} \widetilde{V}_{p_l,q_m}(K_i, L_j, Q_k) = \widetilde{V}_{p,q}(K, L, Q).$$

For any $\varepsilon > 0$, three observations are in order. Firstly, let $f(t) = t^p$, $f_l(t) = t^{p_l}$, $l = 1, 2, \cdots$, defined on $[c_m, c_M]$, then the sequence of $\{f_l\}$ converges uniformly to f on $[c_m, c_M]$. And let $g(t) = t^p$, $g_m(t) = t^{p_m}$, $m = 1, 2, \cdots$, defined on $[c_m, c_M]$, then the sequence of $\{g_m\}$ converges uniformly to g on $[c_m, c_M]$. For all $u \in S^{n-1}$,

$$c_m \le \frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))} \le c_M, \quad c_m \le \frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)} \le c_M,$$

there exists an $N_1 \in \mathbb{N}$, such that for all $l, m \ge N_1$,

$$\left| \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))}\right)^{p_l} \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q_m} - \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))}\right)^{p} \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q} \right| < \frac{\varepsilon}{3}, \quad (3.3)$$

independently of i and j and uniformly on $u \in S^{n-1}$.

Secondly, since $K_i \to K$, $L_j \to L$ and $Q_k \to Q$, $p_l \to p$ as $i, j, k \to \infty$, and Lemma 2.7, there exists an $N_2 \in \mathbb{N}$ such that for all $i, j, k > N_2$ and for all $u \in S^{n-1}$,

$$\left| \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))}\right)^{p} \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q} - \left(\frac{h_{L}(\alpha_{K}(u))}{h_{K}(\alpha_{K}(u))}\right)^{p} \left(\frac{\rho_{K}(u)}{\rho_{Q}(u)}\right)^{q} \right| < \frac{\varepsilon}{3}. \quad (3.4)$$

Indeed, since functions f and g are all Lipschitzian on $[c_m, c_M]$, there exist constants $C_1, C_2 > 0$, such that for all $u \in S^{n-1}$,

$$\begin{aligned}
&\left| \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))}\right)^{p} \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q} - \left(\frac{h_{L}(\alpha_{K}(u))}{h_{K}(\alpha_{K}(u))}\right)^{p} \left(\frac{\rho_{K}(u)}{\rho_{Q}(u)}\right)^{q} \right| \\
&\le \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q} \left| \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))}\right)^{p} - \left(\frac{h_{L}(\alpha_{K}(u))}{h_{K}(\alpha_{K}(u))}\right)^{p} \right| \\
&\quad + \left(\frac{h_{L}(\alpha_{K})(u)}{h_{K}(\alpha_{K})(u)}\right)^{p} \left| \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q} - \left(\frac{\rho_{K}(u)}{\rho_{Q}(u)}\right)^{q} \right| \\
&\le C_1 \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)}\right)^{q} \left| \frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))} - \frac{h_{L}(\alpha_{K}(u))}{h_{K}(\alpha_{K}(u))} \right| \\
&\quad + C_2 \left(\frac{h_{L}(\alpha_{K}(u))}{h_{K}(\alpha_{K}(u))}\right)^{p} \left| \frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)} - \frac{\rho_{K}(u)}{\rho_{Q}(u)} \right| \\
&\le C_M^q C_1 \cdot \frac{\delta_H(L_j, L) \max_{S^{n-1}} h_K + \delta_H(K_i, K) \max_{S^{n-1}} h_L}{\min_{S^{n-1}} h_{K_i} \min_{S^{n-1}} h_K} \\
&\quad + C_M^P C_2 \cdot \frac{\widetilde{\delta}_H(K_i, K) \max_{S^{n-1}} \rho_Q + \widetilde{\delta}_H(Q_k, Q) \max_{S^{n-1}} \rho_K}{\min_{S^{n-1}} \rho_{Q_k} \min_{S^{n-1}} \rho_Q}.
\end{aligned}$$

Thirdly, since the measure sequence $\{\overline{\widetilde{V}}_{Q_k}\}$ weakly converges to $\overline{\widetilde{V}}_Q$, there exists an $N_3 \in \mathbb{N}$, such that for all $k \geq N_3$,

$$\left| \int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^q \mathrm{d}\overline{\widetilde{V}}_{Q_k}(u) - \int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^q \mathrm{d}\overline{\widetilde{V}}_Q(u) \right| < \frac{\varepsilon}{3}. \tag{3.5}$$

From (3.3), (3.4) and (3.5), it follows that for all $i, j, k, l, m \geq \max\{N_1, N_2, N_3\}$,

$$\begin{aligned}
&\left| \int_{S^{n-1}} \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))} \right)^{p_l} \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)} \right)^{q_m} \mathrm{d}\overline{\widetilde{V}}_{Q_k}(u) - \int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^q \mathrm{d}\overline{\widetilde{V}}_Q(u) \right| \\
&\leq \int_{S^{n-1}} \left| \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))} \right)^{p_l} \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)} \right)^{q_m} - \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))} \right)^p \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)} \right)^q \right| \mathrm{d}\overline{\widetilde{V}}_{Q_k}(u) \\
&\quad + \int_{S^{n-1}} \left| \left(\frac{h_{L_j}(\alpha_{K_i}(u))}{h_{K_i}(\alpha_{K_i}(u))} \right)^p \left(\frac{\rho_{K_i}(u)}{\rho_{Q_k}(u)} \right)^q - \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^q \right| \mathrm{d}\overline{\widetilde{V}}_{Q_k}(u) \\
&\quad + \left| \int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^q \mathrm{d}\overline{\widetilde{V}}_{Q_k}(u) - \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^q \mathrm{d}\overline{\widetilde{V}}_Q(u) \right| \\
&< \varepsilon.
\end{aligned}$$

Namely,

$$\lim_{i,j,k,l,m \to \infty} \frac{\widetilde{V}_{p_l,q_m}(K_i, L_j, Q_k)}{|Q_k|} = \frac{\widetilde{V}_{p,q}(K, L, Q)}{|Q|}.$$

The first conclusion follows from the fact $|Q_k| \to |Q|$ by sending k to infinity.

Finally, we proceed to prove

$$\lim_{i,j,k,l \to \infty} \overline{\widetilde{V}}_{p_l, p_l+r}(K_i, L_j, Q_k) = \overline{\widetilde{V}}_{p,p+r}(K, L, Q).$$

Fix $\delta > 0$. For $0 \leq r < \infty$, we note that

$$\frac{\widetilde{V}_{p_l,p_l+r}(K_i, L_j, Q_k)}{\widetilde{V}_r(K_i, Q_k)}, \frac{\widetilde{V}_{p,p+r}(K, L, Q)}{\widetilde{V}_r(K, Q)} \in [c_1, c_3], \quad \text{for each } i, j, k, l \in \mathbb{N}.$$

The continuity of $t^{\frac{1}{p}}$ on $[c_1, c_3]$ implies there exists an $N_4 > 0$ such that for all $l \geq N_4$,

$$\left| \left(\frac{\widetilde{V}_{p_l,p_l+r}(K_i, L_j, Q_k)}{\widetilde{V}_r(K_i, Q_k)} \right)^{\frac{1}{p_l}} - \left(\frac{\widetilde{V}_{p_l,p_l+r}(K_i, L_j, Q_k)}{\widetilde{V}_r(K_i, Q_k)} \right)^{\frac{1}{p}} \right| < \frac{\delta}{2} \tag{3.6}$$

holds independently of i, j and k.

From (1.11) and (3.1), it follows $\lim_{i,k\to\infty}\widetilde{V}_r(K_i,Q_k)=\widetilde{V}_r(K,Q)$. Combining this with (3.1), the continuity of $t^{\frac{1}{p}}$ on $[c_1,c_3]$ shows there exists an $N_5>0$, such that for all $i,j,k,l>N_5$,

$$\left|\left(\frac{\widetilde{V}_{p_l,p_l+r}(K_i,L_j,Q_k)}{\widetilde{V}_r(K_i,Q_k)}\right)^{\frac{1}{p}}-\left(\frac{\widetilde{V}_{p,p+r}(K,L,Q)}{\widetilde{V}_r(K,Q)}\right)^{\frac{1}{p}}\right|<\frac{\delta}{2}. \tag{3.7}$$

In terms of (3.6) and (3.7), it follows that for $i,j,k,l\geq\max\{N_4,N_5\}$,

$$\left|\left(\frac{\widetilde{V}_{p_l,q_l}(K_i,L_j,Q_k)}{\widetilde{V}_r(K_i,Q_k)}\right)^{\frac{1}{p_l}}-\left(\frac{\widetilde{V}_{p,q}(K,L,Q)}{\widetilde{V}_r(K,Q)}\right)^{\frac{1}{p}}\right|<\delta.$$

That is,

$$\lim_{i,j,k,l\to\infty}\overline{\widetilde{V}}_{p_l,p_l+r}(K_i,L_j,Q_k)=\overline{\widetilde{V}}_{p,p+r}(K,L,Q).$$

□

4 (*p*, *q*)-John Ellipsoids

In this section, we focus on the main Problem 1.1 proposed in Sect. 1.

Optimization Problems. Let $0<p\leq q$ with $q=p+r$, $r\geq0$. For $K\in\mathcal{K}_o^n$ and $Q\in\mathcal{S}_o^n$, find an ellipsoid, among all origin-symmetric ellipsoids, which solves the following constrained maximization problem:

$$\max_{E\in\mathcal{E}^n}\left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}\quad\text{subject to}\quad\overline{\widetilde{V}}_{p,q}(K,E,Q)\leq1. \tag{$S_{p,q}$}$$

An ellipsoid that solves the constrained maximization problem will be called a $S_{p,q}$ solution for K and Q. The dual problem is

$$\min_{E\in\mathcal{E}^n}\overline{\widetilde{V}}_{p,q}(K,E,Q)\quad\text{subject to}\quad\left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}\geq1. \tag{$\bar{S}_{p,q}$}$$

An ellipsoid that solves the dual problem will be called a $\bar{S}_{p.q}$ solution for K and Q.

The following theorem gives the existence of Problem $S_{p,q}$ when $0<p\leq q$, and proves its uniqueness when $1\leq p\leq q$.

Theorem 4.1 *For any $0<p\leq q$, there exists an ellipsoid which solves Problem $S_{p,q}$. The solution is unique for $1\leq p\leq q$.*

Proof For an ellipsoid $E \in \mathcal{E}^n$ (the class of origin-symmetric ellipsoids in $\mathbb{R}^n$), we use d_E to denote its maximal principal radius. There exists a $v_E \in S^{n-1}$ such that $d_E|v_E \cdot u| \le h_E(u)$, for all $u \in S^{n-1}$. From definitions of the (p, q)-mixed projection body and the L_p-dual mixed volume, it yields

$$
\begin{aligned}
&\left(\frac{2}{\widetilde{V}_r(K,Q)}\right)^{\frac{1}{p}} d_E \min_{S^{n-1}} h_{\Pi_{p,q}(K,Q)}(v_E) \\
&\le \left(\frac{2}{\widetilde{V}_r(K,Q)}\right)^{\frac{1}{p}} d_E h_{\Pi_{p,q}(K,Q)}(v_E) \\
&= \left(\frac{1}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} (d_E|u \cdot v_E|)^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,u)\right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} h_E^p(u) \mathrm{d}\widetilde{C}_{p,q}(K,Q,u)\right)^{\frac{1}{p}} \\
&= \overline{\widetilde{V}}_{p,q}(K,E,Q).
\end{aligned}
\tag{4.1}
$$

Let $\mathcal{E}_{p,q} = \left\{E \in \mathcal{E}^n : \overline{\widetilde{V}}_{p,q}(K,E,Q) \le 1\right\}$. Then, the above inequality yields that

$$
\begin{aligned}
d_E &\le \left(\frac{\widetilde{V}_r(K,Q)}{2}\right)^{\frac{1}{p}} \frac{\overline{\widetilde{V}}_{p,q}(K,E,Q)}{\min_{S^{n-1}} h_{\Pi_{p,q}(K,Q)}} \\
&\le \left(\frac{\widetilde{V}_r(K,Q)}{2}\right)^{\frac{1}{p}} \frac{1}{\min_{S^{n-1}} h_{\Pi_{p,q}(K,Q)}}, \quad \text{for all } E \in \mathcal{E}_{p,q}.
\end{aligned}
\tag{4.2}
$$

Thus, the set $\mathcal{E}_{p,q}$ is bounded in the metric space $(\mathcal{E}^n, \delta_H)$. Using Theorem 3.1, the functional $\overline{\widetilde{V}}_{p,q}(K, \cdot, Q)$ is continuous, then $\mathcal{E}_{p,q}$ is also closed. According to the Blaschke selection theorem, each maximizing sequence of ellipsoids for Problem $S_{p,q}$ has a convergent subsequence whose limit is still in $\mathcal{E}_{p,q}$. Therefore, a solution to Problem $S_{p,q}$ exists.

We next prove the uniqueness by contradiction. We assume that the ellipsoids E_1 and E_2 are two different solutions to Problem $S_{p,q}$. Let $E_1 = T_1 B$ and $E_2 = T_2 B$, where $T_1, T_2 \in \mathrm{GL}(n)$. Then $\det(T_1) = \det(T_2)$ and $\overline{\widetilde{V}}_{p,q}(K, E_i, Q) \le 1$, for $i = 1, 2$.

Since each symmetric matrices $T \in \mathrm{GL}(n)$ could be represented in the form $T = PQ$, where P is symmetric, positive definite and Q is orthogonal. Then we may assume that T_1 and T_2 are symmetric and positive definite. Then $T_1 \neq \lambda T_2$, for all $\lambda > 0$. The Minkowski inequality for positive definite matrices implies

$$
\det\left(\frac{1}{2}T_1 + \frac{1}{2}T_2\right)^{\frac{1}{n}} > \frac{1}{2}\det(T_1)^{\frac{1}{n}} + \frac{1}{2}\det(T_2)^{\frac{1}{n}}.
$$

Let $E_3 = \frac{1}{2}(T_1 + T_2)B$. Then we have

$$|E_3| > |E_1| = |E_2|. \tag{4.3}$$

From (2.2) and the triangle inequality, one has for all $u \in S^{n-1}$,

$$h_{E_3}(u) = \left| \frac{T_1^t + T_2^t}{2} u \right| \le \frac{|T_1^t u| + |T_2^t u|}{2} = \frac{h_{E_1}(u) + h_{E_2}(u)}{2}. \tag{4.4}$$

Now, from Definition (1.13), the monotonicity of $f(t) = t^p,\ p \ge 1$, (4.4), and the convexity of $f(t) = t^p$, it follows that

$$\begin{aligned}
&\overline{\widetilde{V}}_{p,q}(K, E_3, Q)^p \\
&= \int_{S^{n-1}} \left(\frac{h_{E_3}(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^p \mathrm{d}\widetilde{V}_r(K, Q; u) \\
&\le \int_{S^{n-1}} \left(\frac{h_{E_1}(\alpha_K(u)) + h_{E_2}(\alpha_K(u))}{2h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^p \mathrm{d}\widetilde{V}_r(K, Q; u) \\
&\le \int_{S^{n-1}} \left[\frac{1}{2} \left(\frac{h_{E_1}(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^p + \frac{1}{2} \left(\frac{h_{E_2}(\alpha_K(u))}{h_K(\alpha_K(u))} \right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)} \right)^p \right] \\
&\quad \mathrm{d}\widetilde{V}_r(K, Q; u) \\
&= \frac{1}{2} \overline{\widetilde{V}}_{p,q}(K, E_1, Q)^p + \frac{1}{2} \overline{\widetilde{V}}_{p,q}(K, E_2, Q)^p \le 1.
\end{aligned}$$

Then $E_3 \in \mathcal{E}_{p,q}$. That is, E_3 satisfies the constraint $\overline{\widetilde{V}}_{p,q}(K, E_3, Q) \le 1$. Then, it will result in $|E_3| \le |E_1| = |E_2|$, which contradicts (4.3). □

Our main problems $S_{p,q}$ and $\overline{S}_{p,q}$ are two equivalent description. The solutions to $S_{p,q}$ and $\bar{S}_{p,q}$ differ by only a scale factor. To prove this conclusion, we need the next lemma.

Lemma 4.2 *Let $p, q > 0$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$. Then*

$$\max_{\{E \in \mathcal{E}^n : \overline{\widetilde{V}}_{p,q}(K,E,Q) \le 1\}} |E| = \max_{\{E \in \mathcal{E}^n : \overline{\widetilde{V}}_{p,q}(K,E,Q) = 1\}} |E|; \tag{4.5}$$

and

$$\min_{\{E \in \mathcal{E}^n : |E| \ge \omega_n\}} \overline{\widetilde{V}}_{p,q}(K, E, Q) = \min_{\{E \in \mathcal{E}^n : |E| = \omega_n\}} \overline{\widetilde{V}}_{p,q}(K, E, Q). \tag{4.6}$$

Proof We first prove that the ellipsoid E_1 with $\overline{\widetilde{V}}_{p,q}(K, E_1, Q) < 1$ cannot be the maximizer of $\max_{\{E \in \mathcal{E}^n : \overline{\widetilde{V}}_{p,q}(K,E,Q) \le 1\}} |E|$. In fact, for the ellipsoid $\overline{\widetilde{V}}_{p,q}(K, E_1, Q)^{-1} E_1$, its volume is greater than the volume of E_1, i.e.,

$$\left| \overline{\widetilde{V}}_{p,q}(K, E_1, Q)^{-1} E_1 \right| > |E_1|.$$

And one has from (2.16),

$$\widetilde{\overline{V}}_{p,q}\left(K, \widetilde{\overline{V}}_{p,q}(K, E_1, Q)^{-1}E_1, Q\right) = 1,$$

as required.

We next prove (4.6). For any ellipsoid E_2 with $|E_2| > \omega_n$, the ellipsoid $\left(\frac{\omega_n}{|E_2|}\right)^{\frac{1}{n}} E_2$ satisfies $\left|\left(\frac{\omega_n}{|E_2|}\right)^{\frac{1}{n}} E_2\right| = \omega_n$. And from (2.16), it follows that

$$\widetilde{\overline{V}}_{p,q}\left(K, \left(\frac{\omega_n}{|E_2|}\right)^{\frac{1}{n}} E_2, Q\right) = \left(\frac{\omega_n}{|E_2|}\right)^{\frac{1}{n}} \widetilde{\overline{V}}_{p,q}(K, E_2, Q) < \widetilde{\overline{V}}_{p,q}(K, E_2, Q).$$

□

Theorem 4.3 *Suppose $p, q > 0$ and K is an origin-symmetric convex body in $\mathbb{R}^n$, and Q is a star body in $\mathbb{R}^n$ about the origin.*

(1) *If E_M is an origin-symmetric ellipsoid that is a $S_{p,q}$ solution for K and Q, then*

$$\left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} E_M \tag{4.7}$$

is a solution to Problem $\bar{S}_{p,q}$.

(2) *If E_m is an origin-symmetric ellipsoid that is a $\bar{S}_{p,q}$ solution for K and Q, then*

$$\widetilde{\overline{V}}_{p,q}(K, E_m, Q)^{-1} E_m \tag{4.8}$$

is a solution to Problem $S_{p,q}$.

Proof (1) Let $E \in \{E \in \mathcal{E}^n : |E| \geq \omega_n\}$. It follows from (2.16) that

$$\widetilde{\overline{V}}_{p,q}\left(K, \widetilde{\overline{V}}_{p,q}(K, E, Q)^{-1}E, Q\right) = 1.$$

Then, from the assumption that E_M is a $S_{p,q}$ solution, it follows

$$|E_M| \geq \left|\widetilde{\overline{V}}_{p,q}(K, E, Q)^{-1}E\right| = \widetilde{\overline{V}}_{p,q}(K, E, Q)^{-n}\,|E|\,.$$

Therefore,

$$\widetilde{\overline{V}}_{p,q}(K, E, Q) \geq \left(\frac{|E|}{|E_M|}\right)^{\frac{1}{n}} \geq \left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} = \widetilde{\overline{V}}_{p,q}\left(K, \left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} E_M, Q\right),$$

where the last equality uses the fact $\overline{\widetilde{V}}_{p,q}(K, E_M, Q) = 1$ by (4.5). Added that $\left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} E_M \in \{E \in \mathcal{E}^n : |E| \geq \omega_n\}$, it implies that the ellipsoid $\left(\frac{\omega_n}{|E_M|}\right)^{\frac{1}{n}} E_M$ is a solution to Problem $\bar{S}_{p,q}$.

(2) Let $E \in \left\{E \in \mathcal{E}^n : \overline{\widetilde{V}}_{p,q}(K, E, Q) \leq 1\right\}$. Since E_m is an $\overline{S}_{p,q}$ solution, and $\left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E \in \{E \in \mathcal{E}^n : |E| = \omega_n\}$, it follows from (2.16) that

$$\left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} \overline{\widetilde{V}}_{p,q}(K, E, Q) = \overline{\widetilde{V}}_{p,q}\left(K, \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E, Q\right) \geq \overline{\widetilde{V}}_{p,q}(K, E_m, Q).$$

Using (4.6), we have $|E_m| = \omega_n$. Then $\overline{\widetilde{V}}_{p,q}(K, E_m, Q)^{-1}|E_m|^{\frac{1}{n}} \geq \overline{\widetilde{V}}_{p,q}(K, E, Q)^{-1}$ $|E|^{\frac{1}{n}}$. Thus, it results in

$$\left(\frac{\left|\overline{\widetilde{V}}_{p,q}(K, E_m, Q)^{-1} E_m\right|}{\omega_n}\right)^{\frac{1}{n}} \geq \left(\frac{\left|\overline{\widetilde{V}}_{p,q}(K, E, Q)^{-1} E\right|}{\omega_n}\right)^{\frac{1}{n}} \geq \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}.$$

Then the proof is completed by observing $\overline{\widetilde{V}}_{p,q}\left(K, \overline{\widetilde{V}}_{p,q}(K, E_m, Q)^{-1} E_m, Q\right) = 1.$ □

In Theorem 4.1, we proved the existence for all cases of $0 < p \leq q$, and the uniqueness for the cases of $1 < p \leq q$. In order to show the uniqueness of for all cases of $0 < p \leq q$, we need the next lemma that shows that, without loss of generality, we may assume that the ellipsoid E is the unit ball B in $\mathbb{R}^n$.

Lemma 4.4 *Suppose real $p, q \neq 0$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$. If $\phi \in \mathrm{GL}(n)$, then*

$$\begin{aligned} &\widetilde{V}_{p,q}(\phi^{-1}K, B, \phi^{-1}Q)|x|^2 \\ &\quad = n \int_{S^{n-1}} |x \cdot v|^2 \mathrm{d}\widetilde{C}_{p,q}(\phi^{-1}K, \phi^{-1}Q, v), \;\; \text{for all } x \in \mathbb{R}^n, \end{aligned} \tag{4.9}$$

if and only if

$$\begin{aligned} &\widetilde{V}_{p,q}(K, \phi B, Q)h^2_{(\phi B)^*}(x) \\ &\quad = n \int_{S^{n-1}} |x \cdot v|^2 h^{p-2}_{\phi B}(v) \mathrm{d}\widetilde{C}_{p,q}(K, Q, v), \;\; \text{for all } x \in \mathbb{R}^n. \end{aligned} \tag{4.10}$$

Proof In light of Lemma 2.1, it suffices to prove the statement for $\mathrm{SL}(n)$. In terms of (2.2), (2.4) and Lemma 2.3, we have, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} \widetilde{V}_{p,q}(K, \phi B, Q)h^2_{(\phi B)^*}(x) &= \widetilde{V}_{p,q}(K, \phi B, Q)h^2_{\phi^{-t}B^*}(x) \\ &= \widetilde{V}_{p,q}(\phi^{-1}K, B, \phi^{-1}Q)h^2_{B^*}(\phi^{-1}x). \end{aligned}$$

Then, using Definition 2.5, (4.10) is equivalent to, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\widetilde{V}_{p,q}(\phi^{-1}K, B, \phi^{-1}Q)h_{B^*}^2(x) &= n\int_{S^{n-1}} |\phi x \cdot v|^2 h_B^{p-2}(\phi^t v)\mathrm{d}\widetilde{C}_{p,q}(K, Q, v)\\ &= n\int_{S^{n-1}} |x \cdot \phi^t v|^2 |\phi^t v|^{p-2}\mathrm{d}\widetilde{C}_{p,q}(K, Q, v)\\ &= n\int_{S^{n-1}} |x \cdot \langle \phi^t v\rangle|^2 |\phi^t v|^{p}\mathrm{d}\widetilde{C}_{p,q}(K, Q, v)\\ &= n\int_{S^{n-1}} |x \cdot v|^2 \mathrm{d}\phi_p^{-t} \dashv \widetilde{C}_{p,q}(K, Q, v),\end{aligned}$$

which by Lemma 2.6 is in turn equivalent to

$$\begin{aligned}&\widetilde{V}_{p,q}(\phi^{-1}K, B, \phi^{-1}Q)|x|^2\\ &\quad= n\int_{S^{n-1}} |x \cdot v|^2 \mathrm{d}\widetilde{C}_{p,q}(\phi^{-1}K, \phi^{-1}Q, v), \quad \text{for all } x \in \mathbb{R}^n.\end{aligned}$$

□

Now we show the existence and uniqueness of solution $S_{p,q}$ and $\bar{S}_{p,q}$ for all cases $0 < p \le q$.

Theorem 4.5 *Suppose that* $0 < p \le q = p + r, r \in [0, \infty)$, $K \in \mathcal{K}_o^n$ *and* $Q \in \mathcal{S}_o^n$. *Then* $S_{p,q}$ *as well as* $\bar{S}_{p,q}$ *has a unique solution. Moreover, an ellipsoid* $E \in \mathcal{E}^n$ *solves* $\bar{S}_{p,q}$ *if and only if it satisfies*

$$\widetilde{V}_{p,q}(K, E, Q)h_{E^*}^2(x) = n\int_{S^{n-1}} |x \cdot u|^2 h_E^{p-2}(u)\mathrm{d}\widetilde{C}_{p,q}(K, Q, u), \quad \textit{for all } x \in \mathbb{R}^n, \tag{4.11}$$

and an ellipsoid $E \in \mathcal{E}^n$ *solves* $S_{p,q}$ *if and only if it satisfies*

$$\widetilde{V}_r(K, Q)h_{E^*}^2(x) = n\int_{S^{n-1}} |x \cdot u|^2 h_E^{p-2}(u)\mathrm{d}\widetilde{C}_{p,q}(K, Q, u), \quad \textit{for all } x \in \mathbb{R}^n. \tag{4.12}$$

Proof We first show that an ellipsoid $E \in \mathcal{E}^n$ solves $\bar{S}_{p,q}$ if and only if it satisfies (4.11). Without loss of generality, we may assume $E = B$ by using Lemma 4.4. Namely, we will show that B is a $\bar{S}_{p,q}$ solution for K and Q if and only if

$$\widetilde{V}_{p,q}(K, B, Q)|x|^2 = n\int_{S^{n-1}} |x \cdot u|^2 \mathrm{d}\widetilde{C}_{p,q}(K, Q, u), \quad \text{for all } x \in \mathbb{R}^n. \tag{4.13}$$

Firstly, we show if $B \in \mathcal{E}^n$ solves $\bar{S}_{p,q}$, then (4.13) holds. Indeed, suppose that $T \in \mathrm{SL}(n)$. Choose $\varepsilon_0 > 0$ sufficiently small so that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $I_n + \epsilon T$ is invertible, where I_n is identity matrix. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, define $T_\varepsilon \in \mathrm{SL}(n)$ by

$$T_\varepsilon = |I_n + \varepsilon T|^{-\frac{1}{n}}(I_n + \varepsilon T).$$

Since $|T_\varepsilon|=1$, the ellipsoid $E_\varepsilon=T_\varepsilon^t B$ clearly has volume ω_n. The support function of E_ε is given by

$$h_{E_\varepsilon}(u)=h_{T_\varepsilon^t B}(u)=|T_\varepsilon u|.$$

Since $E_0=B$ is a $\bar{S}_{p,q}$ solution, we have

$$\widetilde{V}_{p,q}(K,E_0,Q)\le \widetilde{V}_{p,q}(K,E_\varepsilon,Q),\quad \text{for all } \varepsilon,$$

and hence using (1.9), it is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\int_{S^{n-1}}|T_\varepsilon u|^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,u)=0. \tag{4.14}$$

Note that

$$|I_n+\varepsilon T|^{\frac{1}{n}}=1+\frac{\varepsilon}{n}\mathrm{tr}T+O(\varepsilon^2)$$

and

$$|u+\varepsilon Tu|=[1+2\epsilon\cdot Tu+\epsilon^2(Tu\cdot Tu)]^{\frac{1}{2}}=1+\varepsilon(u\cdot Tu)+O(\varepsilon^2),$$

then (4.14) implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\int_{S^{n-1}}\left(\frac{1+\varepsilon(u\cdot Tu)+O(\varepsilon^2)}{1+\frac{\varepsilon}{n}\mathrm{tr}T+O(\varepsilon^2)}\right)^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,u)\\ &=p\int_{S^{n-1}}\left(u\cdot Tu-\frac{1}{n}\mathrm{tr}T\right)\mathrm{d}\widetilde{C}_{p,q}(K,Q,u)\\ &=0.\end{aligned} \tag{4.15}$$

Let $T=x\otimes x$ for nonzero $x\in\mathbb{R}^n$, where the notation $x\otimes x$ represents the rank 1 linear operator on $\mathbb{R}^n$ that takes y to $(x\cdot y)x$. It immediately gives that $\mathrm{tr}(x\otimes x)=|x|^2$. Using the facts $\mathrm{tr}(x\otimes x)=|x|^2$ and $u\cdot(x\otimes x)u=(u\cdot x)^2$, (4.15) is

$$\int_{S^{n-1}}|u\cdot x|^2\mathrm{d}\widetilde{C}_{p,q}(K,Q,u)=\frac{\widetilde{V}_{p,q}(K,B,Q)}{n}|x|^2,\quad \text{for all } x\in\mathbb{R}^n.$$

Secondly, we show if

$$\widetilde{V}_{p,q}(K,B,Q)|x|^2=n\int_{S^{n-1}}|x\cdot u|^2\mathrm{d}\widetilde{C}_{p,q}(K,Q,u),\quad \text{for all } x\in\mathbb{R}^n, \tag{4.16}$$

then B is a solution to Problem $\bar{S}_{p,q}$. Moreover, B is a unique $\bar{S}_{p,q}$ solution.

To prove that B is a $\bar{S}_{p,q}$ solution for K, Q, we show that for any ellipsoid E with $|E| = \omega_n$, one has

$$\widetilde{V}_{p,q}(K, E, Q) \geq \widetilde{V}_{p,q}(K, B, Q), \tag{4.17}$$

with equality if and only if $E = B$. It is equivalent to show that for any ellipsoid E with $E = P^t B$, $P \in \mathrm{SL}(n)$, one has

$$\left(\frac{1}{\widetilde{V}_{p,q}(K, B, Q)} \int_{S^{n-1}} |Pu|^p \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) \right)^{\frac{1}{p}} \geq 1, \tag{4.18}$$

with equality if and only if $Pu = 1$ for all $u \in S^{n-1}$. From Jensen's inequality,

$$\begin{aligned} &\left(\frac{1}{\widetilde{V}_{p,q}(K, B, Q)} \int_{S^{n-1}} |Pu|^p \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) \right)^{\frac{1}{p}} \\ &\quad \geq \exp \left(\frac{1}{\widetilde{V}_{p,q}(K, B, Q)} \int_{S^{n-1}} \log |Pu| \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) \right), \end{aligned}$$

with equality if and only if there exists $c > 0$ such that $|Pu| = c$ for all $u \in \mathrm{supp}\widetilde{C}_{p,q}(K, Q, \cdot)$. Hence, we need show

$$\int_{S^{n-1}} \log |Pu| \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) \geq 0, \tag{4.19}$$

We write P as $P = O^t D O$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, and O is orthogonal.

From Definition 2.5 and Lemma 2.6, it follows that

$$\begin{aligned} \int_{S^{n-1}} \log |Pu| \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) &= \int_{S^{n-1}} |Ou|^p \log |O^t D O u| \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) \\ &= \int_{S^{n-1}} \log |O^t D v| \mathrm{d}O^t_p \dashv \widetilde{C}_{p,q}(K, Q, v) \\ &= \int_{S^{n-1}} \log |D v| \mathrm{d}\widetilde{C}_{p,q}(OK, OQ, v). \end{aligned}$$

Then by the concavity of the log function and (4.16),

$$\begin{aligned} \int_{S^{n-1}} \log |Pu| \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) &= \frac{1}{2} \int_{S^{n-1}} \log \left(\sum_{i=1}^{n} \lambda_i^2 u_i^2 \right) \mathrm{d}\widetilde{C}_{p,q}(OK, OQ, v) \\ &\geq \sum_{i=1}^{n} \int_{S^{n-1}} u_i^2 \log(\lambda_i) \mathrm{d}\widetilde{C}_{p,q}(OK, OQ, v) \end{aligned}$$

$$= \sum_{i=1}^{n} \log(\lambda_i) \int_{S^{n-1}} |u \cdot e_i|^2 \mathrm{d}\widetilde{C}_{p,q}(OK, OQ, v)$$
$$= \frac{1}{n}\widetilde{V}_{p,q}(K, B, Q) \sum_{i=1}^{n} \log(\lambda_i),$$

where u_i denotes $u \cdot e_i$ for $i = 1, \cdots, n$. Since $|D| = 1$, we have $\sum_{i=1}^{n} \log(\lambda_i) = \log(\prod_{i=1}^{n} \lambda_i) = 0$ Thus (4.19) holds. And then we have (4.16), namely B is a solution to Problem $\bar{S}_{p,q}$.

For the uniqueness of Problem $\bar{S}_{p,q}$, we only need consider the equality condition. Note that the strict concavity of log function implies that equality in (4.16) holds only if $u_{i_1}, \cdots, u_{i_N} \neq 0$ implies $\lambda_{i_1} = \cdots = \lambda_{i_N}$, for $u \in \mathrm{supp}\widetilde{C}_{p,q}(OK, OQ, \cdot)$. Thus $|Du| = \lambda_i$ when $u_i \neq 0$ for $u \in \mathrm{supp}\widetilde{C}_{p,q}(OK, OQ, \cdot)$. Equality in (4.18) forces $|Pu| = c$ for all $u \in \mathrm{supp}\widetilde{C}_{p,q}(OK, OQ, \cdot)$. Since $\mathrm{supp}\widetilde{C}_{p,q}(OK, OQ, \cdot)$ is not contained in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, we have $\lambda_i = c$ for all i. Combining with $|D| = \lambda_1 \cdots \lambda_n = 1$, we have $\lambda_i = 1$ for all i. Thus $D = I_n$, and $P = I_n$.

Note that Theorems 4.1 and 4.3 get the existence of the solution to Problems $S_{p,q}$ and $\bar{S}_{p,q}$. And their uniqueness is proved from the above proof and Theorem 4.3.

Finally, we let the ellipsoid $E \in \mathcal{E}^n$ solve Problem $S_{p,q}$. Using Theorem 4.3, it is equivalent to that $c_0 E$ is a solution to Problem $\bar{S}_{p,q}$, where $c_0 = (\frac{\omega_n}{|E|})^{\frac{1}{n}}$. It holds if and only if (4.11) holds, i.e.,

$$\widetilde{V}_{p,q}(K, E, Q) h_{E^*}^2(x) = n \int_{S^{n-1}} |x \cdot u|^2 h_E^{p-2}(u) \mathrm{d}\widetilde{C}_{p,q}(K, Q, u), \quad \text{for all } x \in \mathbb{R}^n.$$

This completes the result by noticing that $\overline{\widetilde{V}}_{p,q}(K, E, Q) = \left(\frac{\widetilde{V}_{p,q}(K,E,Q)}{\widetilde{V}_r(K,Q)}\right)^{\frac{1}{p}} = 1$ from Lemma 4.2. □

Let $0 < p \le q \le \infty$. Theorem 4.5 shows that problem $(S_{p,q})$ has a unique solution. In the case $Q = K$, the $S_{p,q}$ problem had been considered by Lutwak, Yang and Zhang in [30].

In the case $p = \infty$, with the aid of (1.16), we may rephrase $(S_{\infty,q})$ as: Among all origin-symmetric ellipsoids, find an ellipsoid which solves the following constrained maximization problem:

$$\max \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}} \quad \text{subject to } E \subseteq \left(\frac{\rho_Q}{\rho_K}\right) K. \qquad (S_{\infty,\infty})$$

When $Q = K$, the problem is the classical John-ellipsoid problem (see, e.g., Giannopoulos and Milman [12]).

In light of Theorem 4.1, Theorem 4.3 and Theorem 4.5, we introduce a family of ellipsoids, which is an extension of LYZ's L_p John ellipsoids.

Definition 4.6 Let $0 < p \le q = p + r \le \infty, r \in [0, \infty)$. Suppose K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and Q is a star body (about the origin) in $\mathbb{R}^n$. Among all origin-symmetric ellipsoids, the unique ellipsoid that solves the constrained maximization problem

$$\max_{E \in \mathcal{E}^n} |E| \quad \text{subject to} \quad \overline{\widetilde{V}}_{p,q}(K, E, Q) \le 1$$

will be called the (p, q)-John ellipsoid of K and Q, and will be denoted by $E_{p,q}(K, Q)$.

Among all origin-symmetric ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min_{E \in \mathcal{E}^n} \overline{\widetilde{V}}_{p,q}(K, E, Q) \quad \text{subject to} \quad |E| = \omega_n$$

will be called the normalized (p, q)-John ellipsoid of K and Q, and will be denoted by $\bar{E}_{p,q}(K, L)$.

Note that in the case $Q = K$, $E_{p,q}(K, K) = E_p(K)$ is the L_p-John ellipsoid. In the case that $q = n$ and $Q = B$, $E_{p,n}(K, B) = E_p(K)$ is also the L_p-John ellipsoid. In the case that $p = \infty$ and $Q = K$, $E_{\infty,\infty}(K, K) = J(K)$ is also the classic John ellipsoid.

From Definition 4.6 and (2.20), we immediately obtain

Lemma 4.7 *Suppose $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, and $0 < p \le q \le \infty$. Then for $\phi \in$ GL(n),*

$$E_{p,q}(\phi K, \phi Q) = \phi E_{p,q}(K, Q).$$

From $E_{p,q}(B, B) = E_p B = B$ and Lemma 4.7, we see that if $E \in \mathcal{E}^n$, then

$$E_{p,q}(E, E) = E. \tag{4.20}$$

Note that if the John point of K is at the origin (e.g., if K is origin-symmetric), then

$$E_{\infty,\infty}(K, Q) \subseteq \left(\frac{\rho_Q}{\rho_K}\right) K.$$

From (2.24), (4.12) of Theorem 4.5, we immediately obtain

Lemma 4.8 *Suppose $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$ and $2 \le q \le \infty$. Then*

$$E_{2,q}(K, Q) = \Gamma_{-2,-q}(K, Q).$$

A finite positive Borel measure μ on S^{n-1} is said to be isotropic if (see [12])

$$\int_{S^{n-1}} |u \cdot v|^2 \mathrm{d}\mu(u) = \frac{|\mu|}{n},$$

for all $v \in S^{n-1}$, where $|\mu|$ denotes the total mass of μ. For nonzero $x \in \mathbb{R}^n$, the notation $x \otimes x$ represents the rank 1 linear operator on $\mathbb{R}^n$ that takes y to $(x \cdot y)x$. It immediately gives that $\mathrm{tr} x \otimes x = |x|^2$. Equivalently, μ is isotropic if

$$\int_{S^{n-1}} u \otimes u \mathrm{d}\mu(u) = \frac{|\mu|}{n} I_n.$$

From definition (1.6) and (1.9), we see that

$$\begin{aligned}\widetilde{V}_{p,q}(K, B, Q) &= \int_{S^{n-1}} \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) \\ &= \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(S^{n-1})} h_K(\alpha_K(u))^{-p} \rho_K^q(u) \rho_Q^{n-q}(u) \mathrm{d}u = \widetilde{C}_{p,q}(K, Q, S^{n-1}).\end{aligned}$$

Therefore, the condition (4.11) is equivalent to

$$\int_{S^{n-1}} |x \cdot u|^2 \mathrm{d}\widetilde{C}_{p,q}(K, Q, u) = \frac{\widetilde{C}_{p,q}(K, Q, S^{n-1})}{n} |x|^2, \quad \text{for all } x \in \mathbb{R}^n.$$

Then an immediate consequence of Theorem 4.5 is

Corollary 4.9 *Suppose $K \in \mathcal{K}_o^n$ with $Q \in \mathcal{S}_o^n$, and $0 < p \leq q \in (0, \infty]$. Then there exists a unique solution to the following constrained minimization problem:*

$$\min\{\widetilde{V}_{p,q}(K, TB, Q) : T \in \mathrm{SL}(n)\}.$$

Moreover, the identity operator I_n is the solution if and only if L_p dual curvature measures $\widetilde{C}_{p,q}(K, Q, \cdot)$ are isotropic on S^{n-1}.

Corollary 4.10 *Suppose $K \in \mathcal{K}_o^n$ with $Q \in \mathcal{S}_o^n$, and $0 < p \leq q \in (0, \infty]$.*

(1) *There exists an $\mathrm{SL}(n)$ transformation T, such that $\widetilde{C}_{p,q}(TK, TQ, \cdot)$ is isotropic on S^{n-1}.*
(2) *If $T_1, T_2 \in \mathrm{SL}(n)$ such that $\widetilde{C}_{p,q}(T_1K, T_1Q, \cdot)$, $\widetilde{C}_{p,q}(T_2K, T_2Q, \cdot)$ are both isotropic on S^{n-1}, then there exists an orthogonal $O \in \mathrm{O}(n)$ such that $T_2 = OT_1$.*

5 Continuity of (p, q)-John Ellipsoids

In this section, we show that the family of (p, q)-John ellipsoids associated with a convex body and a star body in $\mathbb{R}^n$ is continuous in $p \in (0, \infty]$.

We assume that $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$ are two fixed bodies in this section.

Lemma 5.1 *Suppose $0 < p \leq q \leq \infty$. If $aB \subseteq K \subseteq bB$ and $aB \subseteq Q \subseteq bB$ for $a, b > 0$, then*

$$\bar{E}_{p,q}(K, Q) \subseteq \left(\frac{b}{a}\right)^{\frac{p+2q+n}{p}} (c_{n-2,p})^{-\frac{1}{p}} B,$$

where

$$c_{n-2,p}=\frac{(n+p)\omega_{n+p}}{n\omega_2\omega_n\omega_{p-1}},\quad \omega_n=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}.$$

Proof From (4.1) and the definition of $\bar{E}_{p,q}(K,Q)$, we have

$$d_E\le\left(\frac{\widetilde{V}_r(K,Q)}{2}\right)^{\frac{1}{p}}\frac{\overline{\widetilde{V}}_{p,q}(K,B,Q)}{h_{\Pi_{p,q}}(K,Q)},\tag{5.1}$$

Now, we estimate the value of $\overline{\widetilde{V}}_{p,q}(K,B,Q)$. By the definition of $\overline{\widetilde{V}}_{p,q}(K,L,Q)$, we have

$$\begin{aligned}\overline{\widetilde{V}}_{p,q}(K,B,Q)&=\left(\frac{1}{\widetilde{V}_r(K,Q)}\int_{S^{n-1}}\mathrm{d}\widetilde{C}_{p,q}(K,Q,v)\right)^{\frac{1}{p}}\\&=\left(\frac{1}{\widetilde{V}_r(K,Q)}\int_{S^{n-1}}\left(\frac{\rho_K(v)}{h_K(\alpha_K(v))\rho_Q(v)}\right)^p\mathrm{d}\widetilde{V}_r(K,Q;v)\right)^{\frac{1}{p}}\\&\le\frac{b}{a^2}\left(\frac{1}{\widetilde{V}_r(K,Q)}\int_{S^{n-1}}\mathrm{d}\widetilde{V}_r(K,Q;v)\right)^{\frac{1}{p}}=\frac{b}{a^2}.\end{aligned}\tag{5.2}$$

Note that

$$\int_{S^{n-1}}|u\cdot v|^p\mathrm{d}u=\frac{(n+p)\omega_{n+p}}{\omega_2\omega_{p-1}}.\tag{5.3}$$

By the definition of (p,q)-mixed projection body and (5.3), we have

$$\begin{aligned}h_{\Pi_{p,q}(K,Q)}(v_E)&=\left(\frac{1}{2}\int_{S^{n-1}}|u\cdot v_E|^p\mathrm{d}\widetilde{C}_{p,q}(K,Q,u)\right)^{\frac{1}{p}}\\&=\left(\frac{1}{2n}\int_{S^{n-1}}|u\cdot v_E|^p h_K(\alpha_K(u))^{-p}\left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^q\rho_Q^n(u)\mathrm{d}u\right)^{\frac{1}{p}}\\&\ge\left(\frac{a^{q+n}}{2nb^{p+q}}\int_{S^{n-1}}|u\cdot v_E|^p\mathrm{d}u\right)^{\frac{1}{p}}\\&=\left(\frac{(n+p)\omega_{n+p}a^{q+n}}{2n\omega_2\omega_{p-1}b^{p+q}}\right)^{\frac{1}{p}}.\end{aligned}\tag{5.4}$$

Together with (5.1), (5.2) and (5.4), and note that $\widetilde{V}_r(K,Q)\le\frac{\omega_n b^{n+r}}{a^r}$, we have

$$d_{\bar{E}_{p,q}(K,Q)}\le\left(\frac{b}{a}\right)^{\frac{p+2q+n}{p}}(c_{n-2,p})^{-\frac{1}{p}}.$$

Therefore,

$$\bar{E}_{p,q}(K,Q) \subseteq \left(\frac{b}{a}\right)^{\frac{p+2q+n}{p}} (c_{n-2,p})^{-\frac{1}{p}} B.$$

Note that $\lim\limits_{p\to\infty}(c_{n-2,p})^{\frac{1}{p}} = 1$, then $\bar{E}_{\infty,\infty}(K,L) \subseteq \frac{b}{a}B$. □

From Definition 4.6, we recall that for each $p \in (0,\infty]$ and $q = p+r$, $r \in (0,\infty)$, the ellipsoid $\bar{E}_{p,q}(K,Q)$ is the unique ellipsoid that satisfies

$$\overline{\widetilde{V}}_{p,q}\left(K,\bar{E}_{p,q}(K,Q),Q\right) = \min_{|E|=\omega_n} \overline{\widetilde{V}}_{p,q}(K,E,Q). \tag{5.5}$$

Lemma 5.2 *If $p, p_0 \in (0,\infty]$, $q = p+r$, $r \in (0,\infty)$, $p \to p_0$, $q \to p_0 + r = q_0$, $K \in \mathcal{K}_o^n$, and $Q \in \mathcal{S}_o^n$, then*

$$\lim_{p\to p_0} \overline{\widetilde{V}}_{p,q}(K,\bar{E}_{p,q}(K,Q),Q) = \overline{\widetilde{V}}_{p_0,q_0}(K,\bar{E}_{p_0,q_0}(K,Q),Q).$$

Proof Using the Definition $\bar{E}_{p,q}(K,Q)$, Theorem 3.1, (5.5), and again the definition of $\bar{E}_{p,q}(K,Q)$, we have

$$\begin{aligned}\lim_{p\to p_0} \overline{\widetilde{V}}_{p,q}(K,\bar{E}_{p,q}(K,Q),Q) &= \lim_{p\to p_0} \min_{|E|=\omega_n} \overline{\widetilde{V}}_{p,q}(K,E,Q)\\ &= \min_{|E|=\omega_n} \overline{\widetilde{V}}_{p_0,q_0}(K,E,Q)\\ &= \overline{\widetilde{V}}_{p_0,q_0}(K,\bar{E}_{p_0,q_0}(K,Q),Q).\end{aligned}$$

□

Lemma 5.3 *Suppose that $p, p_0 \in (0,\infty]$, $q = p+r$, $r \in (0,\infty)$, $p \to p_0$, $q \to p_0 + r = q_0$, and $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$. If $aB \subseteq Q \subseteq K \subseteq bB$ or $aB \subseteq K \subseteq Q \subseteq bB$, for $a, b > 0$, then*

$$\lim_{p\to p_0} \bar{E}_{p,q}(K,Q) = \bar{E}_{p_0,q_0}(K,Q).$$

Proof We argue by contradiction and assume the conclusion to be false. Lemma 5.1, the Blaschke selection theorem, and our assumption, give a sequence $p_i \to p_0$, as $i \to \infty$, such that $\lim\limits_{i\to\infty} \bar{E}_{p_i,q_i}(K,Q) = E' \neq \bar{E}_{p_0,q_0}(K,Q)$. Since the solution to Problem $(\bar{S}_{p,q})$ is unique, and by the uniform convergence established in Theorem 3.1, we get

$$\begin{aligned}\overline{\widetilde{V}}_{p_0,q_0}\left(K,\bar{E}_{p_0,q_0}(K,Q),Q\right) &< \overline{\widetilde{V}}_{p_0,q_0}(K,E',Q)\\ &= \lim_{i\to\infty} \overline{\widetilde{V}}_{p_0,q_0}\left(K,\bar{E}_{p_i,q_i}(K,Q),Q\right)\end{aligned}$$

$$= \lim_{i\to\infty} \overline{\widetilde{V}}_{p_i,q_i}\left(K, \bar{E}_{p_i,q_i}(K,Q), Q\right).$$

This contradicts to Lemma 5.2. □

Since, by Theorem 4.3, $E_{p,q}(K,Q) = \overline{\widetilde{V}}_{p,q}\left(K, \bar{E}_{p,q}(K,Q), Q\right)^{-1} \bar{E}_{p,q}(K,Q)$, the above gives

Theorem 5.4 *If $p, p_0 \in (0,\infty]$, $q = p + r$, $r \in (0,\infty)$, $p \to p_0$, $q \to p_0 + r = q_0$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, then*

$$\lim_{p\to p_0} E_{p,q}(K,Q) = E_{p_0,q_0}(K,Q).$$

6 Generalizations of John's Inclusion

John's inclusion states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$E_\infty K \subseteq K \subseteq \sqrt{n} E_\infty K. \tag{6.1}$$

L_p version of John's inclusion is (see [30]): If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then

$$E_p K \supseteq \Gamma_{-p} K \supseteq n^{\frac{1}{2}-\frac{1}{p}} \quad \text{when} \quad 0 < p \le 2,$$
$$E_p K \subseteq \Gamma_{-p} K \subseteq n^{\frac{1}{2}-\frac{1}{p}} \quad \text{when} \quad 2 \le p \le \infty.$$

In this section, we shall prove a (p,q)-version of John's inclusion.

From (1.4), (2.1), (2.5) and Definition (2.26), we see immediately that if $\lambda > 0$, then

$$\Gamma_{-p,-q}(\lambda K, \lambda Q) = \lambda \Gamma_{-p,-q}(K,Q). \tag{6.2}$$

Lemma 6.1 *If $p \in (0,\infty]$, $q = p + r$, $r \in [0,\infty)$ and $K \in \mathcal{K}_o^n$, as well as $Q \in \mathcal{S}_o^n$, then for $\phi \in \mathrm{GL}(n)$*

$$\Gamma_{-p,-q}(\phi K, \phi Q) = \phi \Gamma_{-p,-q}(K,Q).$$

Proof From (6.2) it is sufficient to prove the formula when $\phi \in \mathrm{SL}(n)$. For real $p > 0$, it follows from Definition (2.24), Lemma 2.6, Definition 2.5, Definition (2.24) again, and (2.6) that for $u \in S^{n-1}$,

$$\begin{aligned}
\rho_{\Gamma_{-p,-q}(\phi K,\phi Q)}(u)^{-p} &= \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |u\cdot v|^p \mathrm{d}\widetilde{C}_{p,q}(\phi K, \phi Q, v) \\
&= \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |u\cdot v|^p \mathrm{d}\phi_p^t \dashv \widetilde{C}_{p,q}(K, Q, v)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |u \cdot \langle \phi^{-t} v\rangle|^p |\phi^{-t} v|^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,v) \\
&= \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |u \cdot \phi^{-t} v|^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,v) \\
&= \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |\phi^{-1} u \cdot v|^p \mathrm{d}\widetilde{C}_{p,q}(K,Q,v) \\
&= \rho_{\Gamma_{-p,-q}(K,Q)}(\phi^{-1}u)^{-p}.
\end{aligned}
$$

The $p = \infty$ case is now a direct consequence of the real case and Definition (2.25). □

Lemma 6.2 *If* $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$, $p \in (0, \infty]$ *and* $q = p + r$, $r \in [0, \infty)$, *then*

$$
\begin{aligned}
E_{p,q}(K,Q) &\supseteq \Gamma_{-p,-q}(K,Q) \quad \text{when} \quad 0 < p < 2, \\
E_{p,q}(K,Q) &\subseteq \Gamma_{-p,-q}(K,Q) \quad \text{when} \quad 2 \le p \le \infty.
\end{aligned}
$$

Proof Lemmas 4.7 and 6.1 show that it suffices to prove the inclusions when $E_{p,q}(K,Q) = B$. For $0 < p < 2$, Definition (2.24) and Theorem 4.5 show that for each $u \in S^{n-1}$,

$$
\begin{aligned}
\rho_{\Gamma_{-p,-q}(K,Q)}(u)^{-p} &= \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}\tilde{C}_{p,q}(K,Q,v) \\
&\ge \frac{n}{\widetilde{V}_r(K,Q)} \int_{S^{n-1}} |u \cdot v|^2 \mathrm{d}\tilde{C}_{p,q}(K,Q,v) \\
&= 1.
\end{aligned}
$$

This gives $\Gamma_{-p,-q}(K,Q) \subseteq B = E_{p,q}(K,Q)$ when $0 < p < 2$.

When $2 \le p < \infty$, the inequality is reversed. Thus $E_{p,q}(K,Q) \subseteq \Gamma_{-p,-q}(K,Q)$ for $2 \le p < \infty$. The case $p = \infty$ follows from the real case together with Theorem 5.4 and Definition (2.25). □

Of course the case $p = 2$ of Lemma 6.2 is known from Lemma 4.8: $E_{2,q}(K,Q) = \Gamma_{-2,-q}(K,Q)$.

Our general L_p version of John's inclusion will be a corollary of

Theorem 6.3 *If* $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$, $p_i \in (0, \infty]$, $q_i = p_i + r$, $r \in [0, \infty)$, $i = 1, 2$, *then*

$$
\begin{aligned}
\Gamma_{-p_1,-q_1}(K,Q) &\supseteq n^{\frac{1}{2} - \frac{1}{p_1}} E_{p_2,q_2}(K,Q) \quad \text{when} \quad 0 < p_1 \le p_2 \le 2, \\
\Gamma_{-p_1,-q_1}(K,Q) &\subseteq n^{\frac{1}{2} - \frac{1}{p_1}} E_{p_2,q_2}(K,Q) \quad \text{when} \quad 2 \le p_2 \le p_1 \le \infty.
\end{aligned}
$$

Proof Note that $q_i = p_i + r$, $i = 1, 2$ and $0 \le r < \infty$. Lemmas 4.7 and 6.1 show that it suffices to prove the inclusions when $E_{p_2,q_2}(K,Q)$ is the unit ball B. Since $E_{p_2,q_2}(K,Q) = B$, Definition 4.6 gives

$$
\widetilde{V}_{p_2,q_2}(K,B,Q) = \widetilde{V}_r(K,Q). \tag{6.3}
$$

Suppose $0 < p_2 \le 2$. Now Definition (2.26), Definition (1.6), Jensen's inequality, Definition (1.6) again, (6.3), Jensen's inequality again, (6.3) again, and finally Theorem 4.5 show that for each $u \in S^{n-1}$,

$$\begin{aligned}
\rho_{\Gamma_{-p_1,-q_1}(K,Q)}(u)^{-1} &= n^{\frac{1}{p_1}} \left[\int_{S^{n-1}} \left(\frac{|u \cdot v| \rho_K(v)}{h_K(\alpha_K(v)) \rho_Q(v)} \right)^{p_1} \mathrm{d}\widetilde{\overline{V}}_r(K, Q; v) \right]^{\frac{1}{p_1}} \\
&\le n^{\frac{1}{p_1}} \left[\int_{S^{n-1}} \left(\frac{|u \cdot v| \rho_K(v)}{h_K(\alpha_K(v)) \rho_Q(v)} \right)^{p_2} \mathrm{d}\widetilde{\overline{V}}_r(K, Q; v) \right]^{\frac{1}{p_2}} \\
&= n^{\frac{1}{p_1}} \left[\frac{1}{\widetilde{V}_r(K, Q)} |u \cdot v|^{p_2} \mathrm{d}\widetilde{C}_{p_2,q_2}(K, Q, v) \right]^{\frac{1}{p_2}} \\
&= n^{\frac{1}{p_1}} \left[\frac{1}{\widetilde{V}_{p_2,q_2}(K, B, Q)} \int_{S^{n-1}} |u \cdot v|^{p_2} \mathrm{d}\widetilde{C}_{p_2,q_2}(K, Q, v) \right]^{\frac{1}{p_2}} \\
&\le n^{\frac{1}{p_1}} \left[\frac{1}{\widetilde{V}_{p_2,q_2}(K, B, Q)} \int_{S^{n-1}} |u \cdot v|^{2} \mathrm{d}\widetilde{C}_{p_2,q_2}(K, Q, v) \right]^{\frac{1}{2}} \\
&= n^{\frac{1}{p_1}} \left[\frac{1}{\widetilde{V}_r(K, Q)} \int_{S^{n-1}} |u \cdot v|^{2} \mathrm{d}\widetilde{C}_{p_2,q}(K, Q, v) \right]^{\frac{1}{2}} \\
&= n^{\frac{1}{p_1} - \frac{1}{2}}.
\end{aligned}$$

Thus, $n^{\frac{1}{2} - \frac{1}{p_1}} E_{p_2,q_2}(K, Q) \subseteq \Gamma_{-p_1,q_1}(K, Q)$.

When $2 \le p_1 \le p_2 < \infty$, the inequality above is reversed. Thus,

$$\Gamma_{-p_1,q_1}(K, Q) \subseteq n^{\frac{1}{2} - \frac{1}{p_1}} E_{p_2,q_2}(K, Q).$$

The case $p = \infty$ follows from the real case together with Theorem 5.4 and Definition (2.25). □

By taking $p_1 = p_2 = p$ in Theorem 6.3 and combining the inclusions with those of Lemma 6.2 we get the general L_p version of John's inclusion:

Corollary 6.4 *If $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$, $p, q \in (0, \infty]$ with $p \le q$, then*

$$\begin{aligned}
E_{p,q}(K, Q) &\supseteq \Gamma_{-p-q}(K, Q) \supseteq n^{\frac{1}{2} - \frac{1}{p}} E_{p,q}(K, Q) \quad \textit{when} \quad 0 < p \le 2, \\
E_{p,q}(K, Q) &\subseteq \Gamma_{-p,-q}(K, Q) \subseteq n^{\frac{1}{2} - \frac{1}{p}} E_{p,q}(K, Q) \quad \textit{when} \quad 2 \le p \le \infty.
\end{aligned}$$

7 Volume-Ratio Inequalities

We first established the following inequality.

Theorem 7.1 *If* $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$, $r \in [0, \infty)$ *and* $p_1, p_2, p_1, p_2 \in (0, +\infty]$ *with satisfying that* $p_1 < p_2$, $q_1 = p_1 + r$ *and* $q_2 = p_2 + r$, *then*

$$|E_{p_1,q_1}(K, Q)| \le |E_{p_2,q_2}(K, Q)|.$$

Proof From Definitions (1.10), together with Jensen's inequality, it follows that for $0 < p_1 \le p_2 \le \infty$,

$$\begin{aligned}\left(\frac{\widetilde{V}_{p_1,q_1}(K, L, Q)}{\widetilde{V}_r(K, Q)}\right)^{\frac{1}{p_1}} &= \left(\int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))\rho_K(u)}{h_K(\alpha_K(u))\rho_Q(u)}\right)^{p_1} \mathrm{d}\overline{\widetilde{V}}_r(K, Q; u)\right)^{\frac{1}{p_1}} \\ &\le \left(\int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))\rho_K(u)}{h_K(\alpha_K(u))\rho_Q(u)}\right)^{p_2} \mathrm{d}\overline{\widetilde{V}}_r(K, Q; u)\right)^{\frac{1}{p_2}} \\ &= \left(\frac{\widetilde{V}_{p_2,q_2}(K, L, Q)}{\widetilde{V}_r(K, Q)}\right)^{\frac{1}{p_2}}.\end{aligned}$$

This together with Definition 4.6 immediately gives the desired result for real p_2 and q_2. For the case $p_2 = \infty, q_2 = \infty$, the result follows from the real case and Theorem 5.4. □

In general, the (p, q)-John ellipsoid $E_{p,q}(K, Q)$ is not contained in K or Q. However when $1 \le \frac{q}{n} \le p \le q \le n + p \le \infty$, the volume of $E_{p,q}(K, Q)$ can be dominated by volume of Q.

Theorem 7.2 *If* $K \in \mathcal{K}_o^n$, $Q \in \mathcal{S}_o^n$ *and* $1 \le \frac{q}{n} \le p \le q \le n + p \le \infty$, *then*

$$|E_{p,q}(K, Q)| \le |Q|, \tag{7.1}$$

with equality if and only if K, Q *are origin-symmetric ellipsoids with dilates of each other when* $1 \le \frac{q}{n} < p$, *while* K, Q *are an ellipsoid with dilates of each other when* $p = 1, q = n$.

Proof First suppose $p < \infty$. From Definition (1.9), Definition 4.6 and the L_p-Minkowski inequality (see Lemma 2.4), we have

$$\begin{aligned}\widetilde{V}_r(K, Q) &= \widetilde{V}_{p,q}(K, E_{p,q}(K, Q), Q) \\ &\ge |K|^{\frac{q-p}{n}} |E_{p,q}(K, Q)|^{\frac{p}{n}} |Q|^{\frac{n-q}{n}}, r = q - p > 0,\end{aligned} \tag{7.2}$$

with equality if and only if K, Q and $E_{p,q}(K, Q)$ are dilates when $1 < \frac{q}{n} < p$, while K and $E_{p,q}(K, Q)$ are dilates when $q = n$ and $p > 1$, and K and $E_{p,q}(K, Q)$ are homothetic when $q = n$, $p = 1$.

From the dual L_p-Minkowski inequality (2.9), we have

$$\widetilde{V}_r(K, Q)^n \leq |K|^{\frac{r}{n}} |Q|^{\frac{n-r}{n}}, \tag{7.3}$$

with equality if and only if K and Q are dilates for $0 < r = q - p < n$.

Together with (7.2) and (7.3), we immediately get

$$|E_{p,q}(K, Q)| \leq |Q|.$$

The condition of equality follows from ones in (7.2) and (7.3).

For $p = \infty$ the results follows from the argument for the real case and Theorem 7.1. □

When $Q = K$, an immediate consequence of Theorem 7.2 is

Corollary 7.3 *If $K \in \mathcal{K}_o^n$ and $1 \leq p \leq \infty$, then*

$$|E_p(K)| \leq |K|, \tag{7.4}$$

with equality for $p > 1$, if and only if K is an origin-symmetric ellipsoid, and equality for $p = 1$ if and only if K is an ellipsoid.

Note that this inequality is about L_p John ellipsoid proved by Lutwak, Yang and Zhang [30].

If $p, q \in (0, \infty]$, K is an origin-symmetric convex body in $\mathbb{R}^n$, and Q is a star body (about the origin) in $\mathbb{R}^n$, then K is said to be (p, q)-isotropic with respect to Q, if there exists a $c > 0$, such that

$$c|x|^2 = n \int_{S^{n-1}} |x \cdot v|^2 \mathrm{d}\widetilde{C}_{p,q}(K, Q, v), \quad \text{for all } x \in \mathbb{R}^n.$$

For $Q = K$, then K is said to be L_p isotropic (see [30]).

Theorem 4.5 shows that K is (p, q)-isotropic with respect to Q if and only if there exists a $\lambda > 0$, such that

$$E_{p,q}(K, Q) = \lambda B.$$

Theorem 7.4 *If $0 \leq r \leq n$, K and Q are origin-symmetric convex body in $\mathbb{R}^n$, and K is $(1, 1 + r)$-isotropic with respect to Q, then for $u \in S^{n-1}$,*

$$h_{\Pi_{1,1+r}(K,Q)}(u) \leq \frac{1}{2\sqrt{n}} |K|^{\frac{r}{n}} |Q|^{\frac{n-r}{n}} \left(\frac{\omega_n}{|E_{1,1+r}(K, Q)|} \right)^{\frac{1}{n^2}}. \tag{7.5}$$

Proof If inequality (7.5) holds for bodies K and Q, then it obviously holds for all λK and λQ with $\lambda > 0$. Thus for K that is $(1, 1 + r)$-isotropic with respect to Q we may assume that $E_{1,1+r}(K, Q) = B$. It is necessary to show that

$$h_{\Pi_{1,1+r}(K,Q)}(u) \leq \frac{1}{2\sqrt{n}}|K|^{\frac{r}{n}}|Q|^{\frac{n-r}{n}}.$$

Definition 4.6 combined with Definition (1.13) gives

$$\widetilde{V}_{1,1+r}(K, B, Q) = \widetilde{V}_r(K, Q). \tag{7.6}$$

From Definition (2.23), (7.6), Jensen's inequality, (7.6) again, and finally Theorem 4.5, it follows

$$\begin{aligned} &\frac{2}{\widetilde{V}_r(K, Q)} h_{\Pi_{1,r+1}(K,Q)}(u) \\ &= \frac{1}{\widetilde{V}_r(K, Q)} \int_{S^{n-1}} |u \cdot v| \mathrm{d}\widetilde{C}_{1,r+1}(K, Q, v) \\ &= \frac{1}{\widetilde{V}_{1,1+r}(K, B, Q)} \int_{S^{n-1}} |u \cdot v| \mathrm{d}\widetilde{C}_{1,1+r}(K, Q, v) \\ &\leq \left[\frac{1}{\int_{S^{n-1}} \mathrm{d}\widetilde{C}_{1,1+r}(K, Q, v)} \int_{S^{n-1}} |u \cdot v|^2 \mathrm{d}\widetilde{C}_{1,1+r}(K, Q, v) \right]^{\frac{1}{2}} \\ &= \left[\frac{1}{\widetilde{V}_r(K, Q)} \int_{S^{n-1}} |u \cdot v|^2 \mathrm{d}\widetilde{C}_{1,1+r}(K, Q, v) \right]^{\frac{1}{2}} = \frac{1}{\sqrt{n}}. \end{aligned} \tag{7.7}$$

Then we have,

$$h_{\Pi_{1,r+1}(K,Q)}(u) \leq \frac{1}{2\sqrt{n}} \widetilde{V}_r(K, Q), \quad \text{for } u \in S^{n-1}. \tag{7.8}$$

Note that $0 \leq r \leq n$, by using dual Minkowski inequality (2.9), we have

$$h_{\Pi_{1,1+r}(K,Q)}(u) \leq \frac{1}{2\sqrt{n}}|K|^{\frac{r}{n}}|Q|^{\frac{n-r}{n}}.$$

□

In particular, by taking $Q = K$ in (7.5), and $h_{\Pi_{1,1+r}(K,Q)}(u) = \frac{1}{n} h_{\Pi(K)}(u) = \frac{1}{n}\mathrm{vol}_{n-1}(K|u^{\perp})$, we have (see [30])

$$\mathrm{vol}_{n-1}(K|u^{\perp}) \leq \frac{\sqrt{n}}{2}|K| \left(\frac{\omega_n}{|J(K)|} \right)^{\frac{1}{n^2}}. \tag{7.9}$$

References

1. Ball, K.: Ellipsoids of maximal volume in convex bodies. Geom. Dedicata **41**, 241–250 (1992)
2. Ball, K.: Volumes ratios and a reverse isoperimetric inequality. J. Lond. Math. Soc. **44**, 351–359 (1991)
3. Barthe, F.: On a reverse form of the Brascamp-Lieb inequality. Invent. Math. **134**, 335–361 (1998)
4. Böröczky, K., Lutwak, E., Yang, D., Zhang, G.: The log-Brunn-Minkowski inequality. Adv. Math. **231**, 1974–1997 (2012)
5. Böröczky, K., Lutwak, E., Yang, D., Zhang, G.: The logarithmic-Minkowski problem. J. Am. Math. Soc. **26**, 831–852 (2013)
6. Campi, S., Gronchi, P.: The L_p-Busemann-Petty centroid inequality. Adv. Math. **167**, 128–141 (2002)
7. Chen, F., Zhou, J., Yang, C.: On the reverse Orlicz Busemann-Petty centroid inequality. Adv. Appl. Math. **47**, 820–828 (2011)
8. Chou, K.S., Wang, X.J.: The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry. Adv. Math. **205**, 33–83 (2006)
9. Fleury, B., Guédon, O., Paouris, G.A.: A stability result for mean width of L_p-centroid bodies. Adv. Math. **214**, 865–877 (2007)
10. Gardner, R.J.: Geometric Tomography. Cambridge University Press, Cambridge (2006)
11. Gardner, R.J., Hug, D., Weil, W.: The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities. J. Differ. Geom. **97**, 427–476 (2014)
12. Giannopoulos, A.A., Milman, V.: Extremal problems and isotropic position of convex bodies. Israel J. Math. **117**, 29–60 (2000)
13. Giannopoulos, A.A., Papadimitrakis, M.: Isotropic surface area measures. Mathematika **46**, 1–13 (1999)
14. Gruber, P.M., Schuster, F.E.: An arithmetic proof of John's ellipsoid theorem. Arch. Math. **85**, 82–88 (2005)
15. Gruber, P.M.: John and Loewner ellipsoids. Discrete Comput. Geom. **46**, 776–788 (2011)
16. Haberl, C., Lutwak, E., Yang, D., Zhang, G.: The even Orlicz Minkowski problem. Adv. Math. **224**, 2485–2510 (2010)
17. Haberl, C., Schuster, F.: General L_p affine isoperimetric inequalities. J. Differ. Geom. **83**, 1–26 (2009)
18. He, B., Leng, G., Li, K.: Projection problems for symmetric polytopes. Adv. Math. **207**, 73–90 (2006)
19. Huang, Y., Lutwak, E., Yang, D., Zhang, G.: Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems. Acta Math. **216**, 325–388 (2016)
20. John, F.: Extremum problems with inequalities as subsidiary conditions. In: Studies and Essays Presented to R. Courant on His 60th Birthday, pp. 187–204. Interscience Publishers, Inc., New York (1948)
21. Klartag, B.: On John-type ellipsoids. In: Milman, V.D., Schechtman, G. (eds.) Geometric Aspects of Functional Analysis. Lecture Notes in Math, vol. 1850, pp. 149–158. Springer, Berlin (2004)
22. Lewis, D.: Ellipsoids defined by Banach ideal norms. Mathematika **26**, 18–29 (1979)
23. Lin, Y.: Affine Orlicz Pólya-Szegö principle for log-concave functions. J. Funct. Anal. **273**, 3295–3326 (2017)
24. Ludwig, M.: Ellipsoids and matrix-valued valuations. Duke Math. J. **119**, 159–188 (2003)
25. Ludwig, M.: General affine surface areas. Adv. Math. **224**, 2346–2360 (2010)
26. Ludwig, M., Reitzner, M.: A classification of SL(n) invariant valuations. Ann. Math. **172**, 1219–1267 (2010)
27. Lutwak, E.: The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. J. Differ. Geom. **38**, 131–150 (1993)
28. Lutwak, E.: The Brunn-Minkowski-Firey Theory. II. Adv. Math. **118**, 244–294 (1996)
29. Lutwak, E., Yang, D., Zhang, G.: L_p affine isoperimetric inequalities. J. Differ. Geom. **56**, 111–132 (2000)
30. Lutwak, E., Yang, D., Zhang, G.: L_p John ellipsoids. Proc. Lond. Math. Soc. **90**, 497–520 (2005)
31. Lutwak, E., Yang, D., Zhang, G.: L_p dual curvature measures. Adv. Math. **329**, 85–132 (2018)
32. Lutwak, E., Yang, D., Zhang, G.: A new ellipsoid associated with convex bodies. Duke Math. J. **104**, 375–390 (2000)
33. Lutwak, E., Yang, D., Zhang, G.: A new affine invariant for polytopes and Schneider's projection problem. Trans. Am. Math. Soc. **353**, 1767–1779 (2001)
34. Lutwak, E., Yang, D., Zhang, G.: On the L_p-Minkowski problem. Trans. Am. Math. Soc. **356**, 4359–4370 (2004)

35. Lutwak, E., Yang, D., Zhang, G.: Orlicz projection bodies. Adv. Math. **223**, 220–242 (2010)
36. Lutwak, E., Yang, D., Zhang, G.: Orlicz centroid bodies. J. Differ. Geom. **84**, 365–387 (2010)
37. Ma, T., Wang, W.: On the Analong of Shephard Problem for the L_p-projection Body. Math. Inequal. Appl. **14**(1), 181–192 (2011)
38. Ma, T.: The generalized L_p-Winternitz problem. J. Math. Inequal. **9**(2), 597–614 (2015)
39. Ma, T., Wang, W.: Dual Orlicz geominimal surface area. J. Ineq. Appl. (56), 1–13 (2016)
40. Ma, T.: The minimal dual Orlicz surface area. Taiwan. J. Math. **20**(2), 287–309 (2016)
41. Ma, T.: On the Reverse Orlicz Blaschke-Santaló inequality. Mediterr. J. Math. **15**(32), 1–11 (2018)
42. Milman, V.D., Pajor, A.: Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In: Lindenstrauss, J., Milman, V.D. (eds.) Geometric Aspects of Functional Analysis, Lecture Notes in Math., vol. 1376, pp. 64–104. Springer, New York (1989)
43. Petty, C.M.: Surface area of a convex body under affine transformations. Proc. Am. Math. Soc. **12**, 824–828 (1961)
44. Pisier, G.: The Volume of Convex Bodies and Banach Geometry. Cambridge University Press, Cambridge (1989)
45. Ryabogin, D., Zvavitch, A.: The Fourier transform and Firey projections of convex bodies. Indiana Univ. Math. J. **53**(3), 667–682 (2004)
46. Schneider, R.: Convex Bodies: The Brunn-Minkowski Theory. Encyclopedia Math. Appl., 2nd edn. Cambridge University Press, Cambridge (2014)
47. Schütt, C., Werner, E.: Surface bodies and p-affine surface area. Adv. Math. **187**, 98–145 (2004)
48. Stancu, A.: The discrete planar L_0-Minkowski problem. Adv. Math. **167**, 160–174 (2002)
49. Wang, W., Ma, T.: Asymmetric L_p-difference bodies. Proc. Am. Math. Soc. **142**(7), 2517–2527 (2014)
50. Werner, E.M.: Rényi divergence and L_p-affine surface area for convex bodies. Adv. Math. **230**, 1040–1059 (2012)
51. Werner, E.M., Ye, D.: New L_p affine isoperimetric inequalities. Adv. Math. **218**, 762–780 (2008)
52. Wu, D.: Firey-Shephard problems for homogeneous measures. J. Math. Anal. Appl. **458**, 43–57 (2018)
53. Wu, D., Zhou, J.: The LYZ centroid conjecture for star bodies. Sci. China Math. **61**(7), 1273–1286 (2018)
54. Xi, D., Jin, H., Leng, G.: The Orlicz Brunn-Minkowski inequality. Adv. Math. **260**, 350–374 (2014)
55. Zhu, G.: The Orlicz centroid inequality for star bodies. Adv. Appl. Math. **48**, 432–445 (2012)
56. Zhu, B., Zhou, J., Xu, W.: Dual Orlicz-Brunn-Minkowski theory. Adv. Math. **264**, 700–725 (2014)
57. Zhu, B., Hong, H., Ye, D.: The Orlicz-Petty bodies. Int. Math. Res. Not. **2018**, 4356–4403 (2018)
58. Zou, D., Xiong, G.: Orlicz-John ellipsoids. Adv. Math. **265**, 132–168 (2014)
59. Zou, D., Xiong, G.: Orlicz-Legendre ellipsoids. J Geom. Anal. **26**, 1–29 (2014)
60. Zou, D., Xiong, G.: The minimal Orlicz surface area. Adv. Appl. Math. **61**, 25–45 (2014)