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Approximation and Extension of Functions of Vanishing Mean Oscillation

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Abstract

We consider various definitions of functions of vanishing mean oscillation on a domain $\Omega \subset \mathbb{R}^n$. If the domain is uniform, we show that there is a single extension operator which extends functions in these spaces to functions in the corresponding spaces on $\mathbb{R}^n$, and also extends $\mathrm{BMO}(\Omega)$ to $\mathrm{BMO}(\mathbb{R}^n)$, generalizing the result of Jones. Moreover, this extension maps Lipschitz functions to Lipschitz functions. Conversely, if there is a linear extension map taking Lipschitz functions with compact support in Ω to functions in $\mathrm{BMO}(\mathbb{R}^n)$, which is bounded in the BMO norm, then the domain must be uniform. In connection with these results we investigate the approximation of functions of vanishing mean oscillation by Lipschitz functions on unbounded domains.

Keywords Bounded mean oscillation · Vanishing mean oscillation · Continuous mean oscillation · Extension theorems · Uniform domains

Mathematics Subject Classification 42B35 · 46E30

Dedicated to the memory of E. M. Stein.

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1 Introduction

Let f be a real-valued function defined on some subset $\Omega \subset \mathbb{R}^n$. It is a natural question to ask how to extend f to a function on $\mathbb{R}^n$ while preserving some of its properties. Those properties can be described by requiring that f belong to some function space. Most trivially, bounded functions on any set Ω can be immediately extended to bounded functions on $\mathbb{R}^n$. Less trivially, Lipschitz continuous functions on any set Ω can be extended to Lipschitz functions on $\mathbb{R}^n$: the McShane–Whitney theorem states that an L-Lipschitz function on a nonempty set Ω can be extended to an L-Lipschitz function F on $\mathbb{R}^n$ (see [17]) or more generally on a metric measure space (see [18]), with $F = f$ on Ω, for example by setting

$$F(x) = \inf\{f(y) + L|x - y| : y \in \Omega\}.$$

Looking at smoother functions, the problem of extending C^m functions from closed sets to $\mathbb{R}^n$ crucially depends on how differentiable functions are defined on a *closed* set. Considering them as jets, Whitney proved the extension theorem introducing his famous decomposition (see e.g. Chapter VI in [29] or Chapter II in [5]). However, defining C^m functions on a closed set as traces of $C^m(\mathbb{R}^n)$ functions to that set makes the problem much harder. Only recently classic questions in this setting were settled by Fefferman [11–13].

In the category of Sobolev spaces $W^{s,p}$, if the domain Ω is regular enough then there is a *universal* operator extending $W^{s,p}(\Omega)$ functions to $W^{s,p}(\mathbb{R}^n)$ *simultaneously* for all $s > 0$ and $1 \le p \le \infty$. This is shown by Stein in [30], extending the results of Calderón (for extension in rougher domains see [21]).

In this paper, we want to go in the other direction, namely from Lipschitz functions to functions of zeroth order smoothness, specifically functions of vanishing mean oscillation. The space of functions of vanishing mean oscillation, VMO, was introduced by Sarason in [26] as a subspace of BMO, the functions of bounded mean oscillation defined by John and Nirenberg [19]. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, set

$$\omega(f, t) := \sup_{\substack{\ell(Q) \le t \\ Q \subset \mathbb{R}^n}} \fint_Q |f(x) - f_Q| \mathrm{d}x, \quad t > 0. \tag{1}$$

Here, the supremum is taken over all cubes with sides parallel to the axes, $\ell(Q)$ is the sidelength of the cube Q, $|Q|$ is its measure, and $f_Q := \fint_Q f := |Q|^{-1} \int_Q f$ is the average of f on Q. Following [1], we call $\omega(f, \cdot)$ the *modulus of mean oscillation*. We say $f \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}} := \sup_{t>0} \omega(f, t) < \infty,$$

and this defines a norm modulo constants. The space $\mathrm{VMO}(\mathbb{R}^n)$ can be defined using either one of the two characterizations in the following theorem, which was proved in [26] for the case $n = 1$.

Theorem 1 (Sarason) *For* $f \in \mathrm{BMO}(\mathbb{R}^n)$*, the following are equivalent:*

1. $\lim_{t\to 0^+} \omega(f,t) = 0$*;*
2. $f \in \overline{\mathrm{UC}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)}$*, the closure of the uniformly continuous functions in* BMO.

If instead we consider the closure in BMO of the continuous functions with compact support (or equivalently the C^∞ functions with compact support), we get a smaller space which is sometimes also called VMO (see [8]) or CMO (for "continuous mean oscillation" - see [25]), the notation we will use. Note that since the functions are considered modulo constants, "compact support" means the function is equal to a constant outside some compact set. Membership in CMO is equivalent to the function satisfying vanishing mean oscillation conditions not only as the size of the cube goes to zero (condition 1 in Theorem 1), but also as the size of the cube increases to ∞ and as the cube itself goes to ∞- see Theorem 6 below, which was proved by Uchiyama in [31], originally appears in [25], and is credited by Neri to Herz, Strichartz and Sarason. A comprehensive exposition of the properties of these spaces is given by Bourdaud in [2].

Let Ω be an open subset of $\mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\Omega)$. We can consider the modulus of mean oscillation restricted to cubes which are contained in Ω

$$\omega_\Omega(f,t) := \sup_{\substack{\ell(Q)\le t \\ Q\subset\Omega}} \fint_Q |f(x) - f_Q| \mathrm{d}x, \quad t > 0, \tag{2}$$

and define

$$f \in \mathrm{BMO}(\Omega) \ \text{ if } \ \|f\|_{\mathrm{BMO}(\Omega)} := \sup_{t>0} \omega_\Omega(f,t) < \infty.$$

This is again a norm modulo constants provided we also assume Ω is connected, i.e. is a domain. Jones [20] considered these spaces and proved the following extension theorem:

Theorem 2 (Jones) *There is a bounded linear extension from* $\mathrm{BMO}(\Omega)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$ *if and only if* Ω *is a uniform domain.*

In fact, Jones' condition on the domain was phrased differently, in terms of Whitney decompositions, as we will see in Sect. 2.3, but it was shown in [15] that this condition is equivalent to the domain being uniform.

We want to consider the same question for the spaces $\mathrm{VMO}(\Omega)$ and $\mathrm{CMO}(\Omega)$: for which domains is there a extension operator from these spaces to the corresponding spaces on $\mathbb{R}^n$? Our main result shows that there exists a linear operator T that *simultaneously* extends CMO, VMO and BMO functions on a uniform domain Ω to the corresponding functions on $\mathbb{R}^n$. Moreover, the same operator also extends Lipschitz functions.

As in the smooth case, this question turns out to be closely related to the definition of these spaces. In view of Sarason's result, Theorem 1, we define

$$\mathrm{VMO}_1(\Omega) := \{f \in \mathrm{BMO}(\Omega) : \lim_{t \to 0^+} \omega_\Omega(f, t) = 0\}$$

and

$$\mathrm{VMO}_2(\Omega) := \overline{\mathrm{UC}(\overline{\Omega}) \cap \mathrm{BMO}(\Omega)}.$$

That $\mathrm{VMO}_2 \subset \mathrm{VMO}_1$ is immediate from the definition of uniform continuity. In the case of a bounded domain the two conditions are known to be equivalent. In fact a stronger result is true: every function with vanishing mean oscillation can be approximated in $\mathrm{BMO}(\Omega)$ by smooth functions with compact support in Ω, i.e. $\mathrm{VMO}_1(\Omega) = \mathrm{CMO}(\Omega)$ for a bounded domain (see Theorem 5 below, which is proved and credited to Jones in [4]). Using similar techniques, namely approximation by bounded functions via truncations, we are able to prove an analogue of the Neri/Uchiyama result (Proposition 3) showing that $\mathrm{CMO}(\Omega)$, for any domain Ω, can also be characterized by three vanishing oscillation results. However, as we show in Example 8, Sarason's equivalence can fail when Ω is unbounded, that is, we can have a proper inclusion $\mathrm{VMO}_2 \subsetneq \mathrm{VMO}_1$. In [16], this question is considered in a more general context of a metric measure space with a general basis for BMO, and Sarason's equivalence is proved in the compact case and for subsets of $\mathbb{R}^n$ with a basis satisfying rather strong conditions, such as a cube or a ball. We will see that as a corollary of our extension result, we get the analogue of Sarason's theorem for a uniform domain.

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain. Then there exists a linear extension operator T such that*

(i) $T : \mathrm{BMO}(\Omega) \to \mathrm{BMO}(\mathbb{R}^n)$ *is bounded;*
(ii) $T : \mathrm{VMO}_1(\Omega) \to \mathrm{VMO}(\mathbb{R}^n)$ *is bounded;*
(iii) $T : \mathrm{VMO}_2(\Omega) \to \mathrm{VMO}(\mathbb{R}^n)$ *is bounded;*
(iv) $T : \mathrm{CMO}(\Omega) \to \mathrm{CMO}(\mathbb{R}^n)$ *is bounded;*
(v) $T : \mathrm{Lip}(\Omega) \to \mathrm{Lip}(\mathbb{R}^n)$ *is bounded.*

Boundedness in (i)-(iv) refers to the BMO *norm while in (v) the boundedness is with respect to the Lipschitz constant.*

Corollary 1 *If Ω is a uniform domain then* $\mathrm{VMO}_1(\Omega) = \mathrm{VMO}_2(\Omega)$.

We also show the analogue of the other direction in Jones' characterization of extension domains for BMO, but instead of assuming that we have an extension from $\mathrm{BMO}(\Omega)$ we only need to assume that the extension acts on the much smaller $\mathrm{CMO}(\Omega)$.

Theorem 4 *If there is a bounded linear extension operator $T : \mathrm{CMO}(\Omega) \to \mathrm{BMO}(\mathbb{R}^n)$, then Ω is uniform.*

The proofs of these results consist of a combination of Jones' construction in the proof of his extension theorem, Theorem 2, with a smoothing technique which goes back to Sarason's proof of Theorem 1. Both of these distill a function $f \in \text{BMO}$ into discrete data, namely its averages on cubes, thereby producing a step function. As observed by Sarason, in the case of $f \in \text{VMO}$, the step function approximates f in the BMO norm. The smoothing is just the basic way of turning a step function into a continuous function via averaging over balls, but as we show in Sect. 2, this actually produces a function which is not just continuous but locally Lipschitz. Thus, the proof allows us to go immediately from condition 1 in Theorem 1 to the density of Lipschitz functions in $\text{VMO}(\mathbb{R}^n)$. Depending on the modulus of mean oscillation, something much stronger may be true: as proved in [7,24] and [28] (see also [27]), functions in $\text{VMO}(\mathbb{R}^n)$ for which $\omega(f,t)$ satisfies a Dini condition are themselves uniformly continuous; the special case when $\omega(f,t) \le Ct^\alpha$, $0 \le \alpha \le 1$, gives Hölder or Lipschitz continuity.

2 Averaging and Approximation

In this section, we do not yet consider the extension problem for a function on a domain but rather a more general situation in which we have a countable collection of cubes $\{S_i\}$ with disjoint interiors, whose union forms an open set $\mathcal{O}$. We assume all cubes have sides parallel to the axes. We are given a real-valued function ϕ on $\mathcal{O}$ such that ϕ is constant on each of the cubes S_i, hence ϕ is a step function. We denote the value of ϕ on S_i by ϕ_{S_i}.

Definition 1 Let R be a measurable function on $\mathcal{O}$ with $0 < R(x) < \text{dist}(x, \partial\mathcal{O})$ for all $x \in \mathcal{O}$. We define the averaging of ϕ by

$$\tilde{\phi}(x) = \fint_{B(x,R(x))} \phi(y)\mathrm{d}y = \int \phi(y)\tilde{\chi}_{B(x,R(x))}(y)\mathrm{d}y,$$

where we use $\tilde{\chi}_B$ to denote the normalized characteristic function $\frac{\chi_B}{|B|}$ of the ball B. Denote by A the linear operator taking ϕ to $\tilde{\phi}$.

We note some basic properties of $\tilde{\phi}$. First of all, $\tilde{\phi}$ coincides with ϕ for those points $x \in S_i$ for which $B(x, R(x)) \subset S_i$, namely

$$\tilde{\phi}(x) = \phi(x) = \phi_{S_i} \text{ if } x \in S_i \text{ and } R(x) < \text{dist}(x, \partial S_i). \tag{3}$$

Furthermore, for each x, letting

$$\mathcal{N}(x) = \{S_j : S_j \cap B(x, R(x)) \neq \emptyset\}, \tag{4}$$

since the values of $\tilde{\phi}(x)$ are determined by the values of ϕ on the cubes in $\mathcal{N}(x)$, we have, for a given S_i,

$$\forall x \in S_i \quad |\phi(x) - \tilde{\phi}(x)| \le \fint_{B(x,R(x))} |\phi(x) - \phi(y)|\mathrm{d}y \le \sup_{S_j \in \mathcal{N}(x)} |\phi_{S_i} - \phi_{S_j}|. \tag{5}$$

Similarly,

$$|\tilde{\phi}(x) - \tilde{\phi}(y)| \leq \sup_{S_j \in \mathcal{N}(x), S_k \in \mathcal{N}(y)} |\phi_{S_j} - \phi_{S_k}|. \tag{6}$$

The goal of the following lemma is to refine this rough estimate and show that $\tilde{\phi}$ possesses certain smoothness, depending on the smoothness of R and the *discrete* smoothness of ϕ.

Lemma 1 *Let $x_1, x_2 \in \mathcal{O}$ and set*

$$d := \frac{|x_1 - x_2|}{\min_{i=1,2} R(x_i)}, \quad \rho := \frac{|R(x_1) - R(x_2)|}{\max_{i=1,2} R(x_i)}.$$

Then,

$$|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| \leq C \sup_{S_i, S_j \in \mathcal{N}(x_1) \cup \mathcal{N}(x_2)} \left|\phi_{S_i} - \phi_{S_j}\right| \min\{d + \rho, 1\}. \tag{7}$$

In particular, if we have $\rho \leq Cd$ then the minimum on the right-hand-side is bounded by a constant multiple of d.

Proof Denote $R(x_i)$ by R_i and $B(x_i, R(x_i))$ by B_i, $i = 1, 2$, and $\mathcal{N}(x_1) \cup \mathcal{N}(x_2)$ by $\mathcal{N}$. Without loss of generality, assume $R_2 \geq R_1$. We may also assume $\phi_{S_1} = 0$, where $x_1 \in S_1$. Then,

$$\begin{aligned}
|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| &= \left| \int \phi(y) \left[\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right] \, dy \right| \\
&= \left| \sum_{S_j \in \mathcal{N}} \int_{S_j} \phi_{S_j} \left[\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right] \, dy \right| \\
&= \left| \sum_{S_j \in \mathcal{N}} \int_{S_j} [\phi_{S_j} - \phi_{S_1}] \left[\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right] \, dy \right| \\
&\leq \sup_{S_i, S_j \in \mathcal{N}} \left|\phi_{S_i} - \phi_{S_j}\right| \sum_{S_j \in \mathcal{N}} \int_{S_j} \left|\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right| \, dy \\
&= \sup_{S_i, S_j \in \mathcal{N}} \left|\phi_{S_i} - \phi_{S_j}\right| \int \left|\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right| \, dy.
\end{aligned}$$

We can estimate the last integral on the right-hand-side by

$$\begin{aligned}
&\int \left|\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right| \, dy \\
&= \int_{B_1 \cap B_2} \left| \frac{1}{|B_1|} - \frac{1}{|B_2|} \right| \, dy + \int_{B_1 \triangle B_2} \left|\tilde{\chi}_{B_1}(y) - \tilde{\chi}_{B_2}(y)\right| \, dy \\
&= \frac{|B_1 \cap B_2|(|B_2| - |B_1|)}{|B_1||B_2|} + \frac{|B_1 \setminus B_2|}{|B_1|} + \frac{|B_2 \setminus B_1|}{|B_2|}. \qquad (8)
\end{aligned}$$

Note that the quantity above is bounded by 3. Thus, the case $d + \rho > 1$ is proved with $C = 3$ (it also follows from (6)).

So we restrict to the case $d + \rho \le 1$, hence $|x_1 - x_2| \le R_1 < R_1 + R_2$, which means $B_1 \cap B_2 \ne \emptyset$. Since $\rho = \frac{R_2 - R_1}{R_2} < 1$, we can use Bernoulli's inequality to bound the first term in (8) as follows:

$$\begin{aligned}
\left| \frac{1}{|B_1|} - \frac{1}{|B_2|} \right| |B_1 \cap B_2| &= \frac{|B_1 \cap B_2|(|B_2| - |B_1|)}{|B_1||B_2|} \le \frac{|B_2| - |B_1|}{|B_2|} \\
&= 1 - \frac{R_1^n}{R_2^n} = 1 - (1 - \rho)^n \le n\rho.
\end{aligned}$$

For the second term in (8), we need to only consider the case $B_1 \setminus B_2 \ne \emptyset$, i.e. $B_1 \not\subset B_2$. Since by the triangle inequality $B_1 \subset B(x_2, R_1 + |x_1 - x_2|)$, we must have $R_1 + |x_1 - x_2| > R_2$ and

$$\begin{aligned}
\frac{|B_1 \setminus B_2|}{|B_1|} &\le \frac{|B(x_2, R_1 + |x_1 - x_2|) \setminus B_2|}{|B_1|} = \frac{(R_1 + |x_1 - x_2|)^n - R_2^n}{R_1^n} \\
&\le (1 + d)^n - 1 = \sum_{k=1}^{n} \binom{n}{k} d^k \le (2^n - 1)d.
\end{aligned}$$

For the last term in (8), we can proceed similarly and use the two estimates above to get

$$\begin{aligned}
\frac{|B_2 \setminus B_1|}{|B_2|} &\le \frac{(R_2 + |x_1 - x_2|)^n - R_1^n}{R_2^n} \\
&= \left[\left(1 + \frac{|x_1 - x_2|}{R_2} \right)^n - \frac{R_1^n}{R_2^n} \right] \\
&= \left[1 - \frac{R_1^n}{R_2^n} \right] + \sum_{k=1}^{n} \binom{n}{k} \left(\frac{|x_1 - x_2|}{R_2} \right)^k \\
&\le n\rho + (2^n - 1)d.
\end{aligned}$$

Combining all the estimates, we get (7) with $C = 2^n - 1$. □

2.1 Approximation on $\mathbb{R}^n$

As an application of the averaging lemma Lemma 1, we prove the following improved version of the implication ($1 \implies 2$) in Sarason's result, Theorem 1. Of course, this result immediately follows from that theorem by the approximation of uniformly continuous functions by Lipschitz functions, but the lemma gives us the Lipschitz approximation directly from what is essentially Sarason's proof. We came up with this observation in answer to a question of M. Mitrea; the density of Lipschitz functions in $\mathrm{VMO}(\mathbb{R}^n)$ was also proved in an indirect way in [23].

Proposition 1 *If $f \in \mathrm{BMO}(\mathbb{R}^n)$ satisfies $\lim_{t\to 0} \omega(f,t) = 0$ then f can be approximated in the* BMO *norm by Lipschitz functions. That is, given $\epsilon > 0$, there exists a Lipschitz function f_ϵ on $\mathbb{R}^n$ with $f_\epsilon \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|f_\epsilon - f\|_{\mathrm{BMO}} < \epsilon$.*

Proof Fix f as in the hypothesis of the proposition, and $\epsilon > 0$. We want to find a Lipschitz function f_ϵ which is within ϵ of f in the BMO norm. We follow the steps in Sarason's proof for the case $n = 1$, with Lemma 1 allowing us to conclude that the approximation is not just uniformly continuous but actually Lipschitz.

We consider a grid of pairwise disjoint cubes of sidelength $\delta > 0$ and for each cube S in the grid we define ϕ_δ to be equal to the average of f on S, denoted f_S. This is the same set-up as in Sect. 2, with $\mathcal{O} = \mathbb{R}^n$. Continuing as in Sect. 2, we apply the averaging with a function R which is constant, i.e. for all x, $R(x) = r$ for some fixed $r > 0$. This gives us a convolution, namely

$$f_\epsilon = \widetilde{\phi_\delta} = \phi_\delta * \tilde{\chi}_{B_r},$$

where $\tilde{\chi}_{B_r} = \frac{1}{|B_r|}\chi_{B_r}$ is the normalized characteristic function of the ball $B_r = B(0,r)$. We will choose δ depending on ϵ and r depending on δ so that f_ϵ is the desired Lipschitz function.

Here, we break the argument into two parts: the approximation of f by the step function ϕ_δ in the BMO norm, and the approximation of the step function by the convolution in the L^∞ norm.

First, let us estimate $\|g\|_{\mathrm{BMO}}$, where $g = f - \phi_\delta$. If $\omega(t) = \omega(f,t)$ denotes the modulus of oscillation of f, we claim that the oscillation of g on a cube Q is bounded by a constant times $\omega(3\delta)$.

Note that g has average 0 on each of the cubes in the grid. Thus if $Q = \cup S_i$ is the union of cubes in the grid, then $g_Q = 0$ and the oscillation of g on Q is controlled by the oscillation of f on the cubes in the grid:

$$\fint_Q |g| = \sum_i \frac{|S_i|}{|Q|} \fint_{S_i} |g| = \sum_i \frac{|S_i|}{|Q|} \fint_{S_i} |f - f_{S_i}| \le \sup_{S_i} \fint_{S_i} |f - f_{S_i}|. \tag{9}$$

This is in turn bounded by $\omega(\delta) \le \omega(3\delta)$.

For a general cube Q, if $\ell(Q) > 2\delta$ then there are cubes Q' and Q'' which are unions of cubes of the grid such that $Q' \subset Q \subset Q''$ and $|Q'| \approx |Q| \approx |Q''|$ with constants depending only on n. This can be seen by first dilating to reduce to the case

$\delta = 1$ and assuming the grid is given by $\mathbb{Z}^n$. Write $Q = I_1 \times I_2 \times \ldots I_n$ where each I_j is an interval of length $\ell = \ell(Q)$. If $l \geq 2$, and we set $m = \lfloor \ell \rfloor$, the greatest integer in l, then each I_j contains an interval of length $m-1$ and is contained in an interval of length $m+1$ with integer endpoints. Letting Q' and Q'' be the products of these intervals, respectively, we get the desired inclusion with constants bounded above by $\sup_{m \geq 2} \left(\frac{m+1}{m-1}\right)^n = 3^n$ and below by 1.

Thus, we have that

$$|g_Q| \leq \fint_Q |g| \leq C_n \fint_{Q''} |g|$$

and therefore,

$$\fint_Q |g - g_Q| \leq C_n \fint_{Q''} |g - g_Q| \leq 2C_n \fint_{Q''} |g| = 2C_n \fint_{Q''} |g - g_{Q''}|,$$

and since Q'' is a union of grid cubes, we can apply (9) to conclude that the oscillation of g on Q is also bounded by a constant multiple of the oscillation of f on the cubes of the grid, which in turn is bounded by $\omega(3\delta)$.

The only remaining case is a cube of sidelength $\ell(Q) \leq 2\delta$. In that case, we can again find a cube $Q'' \supset Q$ with $Q'' = \cup S_i$ and $\ell(Q'') = 3\delta$. We can assume without loss of generality that $f_{S_i} = 0$ for one of the grid cubes S_i contained in Q''. For another grid cube $S_j \subset Q''$, we have

$$\begin{aligned} |f_{S_j}| = |f_{S_j} - f_{S_i}| &\leq \fint_{S_i} \fint_{S_j} |f(x) - f(y)| \mathrm{d}x \mathrm{d}y \\ &\leq 3^{2n} \fint_{Q''} \fint_{Q''} |f(x) - f(y)| \mathrm{d}x \mathrm{d}y \leq C_n \fint_{Q''} |f(x) - f_{Q''}| \mathrm{d}x \leq C_n \omega(3\delta). \end{aligned} \tag{10}$$

This means that on Q'', $|f - g| = |\phi_\delta|$ is bounded by $C_n \omega(3\delta)$. Thus

$$\fint_Q |g - g_Q| \leq \fint_Q |f - f_Q| + 2 \sup_{Q''} |f - g| \leq (2 + C_n)\omega(3\delta).$$

Now, we need to estimate $\|h\|_{\mathrm{BMO}}$, where $h = \phi_\delta - f_\epsilon$. We will show that for the given δ and $r \leq \delta$, h is a bounded function with

$$\|h\|_{\mathrm{BMO}} \leq 2\|h\|_\infty \leq C\omega(3\delta).$$

By the definition of f_ϵ as $\widetilde{\phi_\delta} = \phi_\delta * \tilde{\chi}_{B_r}$, we have, as in (5),

$$\forall x \in S \quad |h(x)| \leq \sup_{S' \in \mathcal{N}(x)} |f_S - f_{S'}|,$$

and since $r \le \delta$, $\mathcal{N}(x)$ consists only of those cubes in the grid which are adjacent to S. Just as we did above in (10), we can estimate $|f_S - f_{S'}| \le C_n\omega(3\delta)$. Thus, we have shown the desired estimate on $\|h\|_\infty$.

Combining, we have that

$$\|f - f_\epsilon\|_{\mathrm{BMO}} \le \|g\|_{\mathrm{BMO}} + \|h\|_{\mathrm{BMO}} \lesssim \omega(3\delta),$$

which is bounded by ϵ if we choose δ sufficiently small, by our hypothesis on f. Here and below we use the notation $A \lesssim B$ if there exists a constant C, independent of A and B, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$ we write $A \approx B$.

Finally, the fact that f_ϵ is Lipschitz will follows from Lemma 1. The Lipschitz constant will depend on the choice of δ and r and will blow up as $\delta \to 0$. We let $r = \delta/2$.

Fix $x_1, x_2 \in \mathbb{R}^n$ with $|x_1 - x_2| \le r$. Applying Lemma 1 to $f_\epsilon = \widetilde{\phi_\delta}$, with our choice of R, we have that $d = r^{-1}|x_1 - x_2|$ and $\rho = 0$, so that

$$|f_\epsilon(x_1) - f_\epsilon(x_2)| \le C \sup_{S_i, S_j \in \mathcal{N}(x_1)\cup\mathcal{N}(x_2)} \left|f_{S_i} - f_{S_j}\right| r^{-1}|x_1 - x_2|.$$

The restriction $|x_1 - x_2| \le r = \delta/2$ forces the set $B_r(x_1) \cup B_r(x_2)$ to have diameter bounded by $3r = 3\delta/2$, and therefore the grid cubes intersecting this set, namely the cubes in $\mathcal{N}(x_1) \cup \mathcal{N}(x_2)$, lie in a cube Q'' of sidelength 3δ. By (10), we get that

$$\sup_{S_i, S_j \in \mathcal{N}(x_1)\cup\mathcal{N}(x_2)} \le C_n\omega(3\delta).$$

Thus, we conclude that locally f_ϵ is Lipschitz with Lipschitz constant $L = C\omega(3\delta)\delta^{-1}$.

Going from the local to the global is standard: given any $x, y \in \mathbb{R}^n$, we take points x_i, $i = 1, \ldots k$, lying on the straight line from x to y, with $x_0 = x$, $x_k = y$ and $|x_i - x_{i-1}| \le r$ for all $1 \le i \le k$. Applying the local Lipschitz estimate, we have

$$|f_\epsilon(y) - f_\epsilon(x)| \le \sum_{i=1}^{k} |f_\epsilon(x_i) - f_\epsilon(x_{i-1})| \le \sum_{i=1}^{k} L|x_i - x_{i-1}| = L|y - x|.$$

□

2.2 Approximation on a Domain

Let $\Omega \subset \mathbb{R}^n$ be a domain. Recall that a function f is called *locally Lipschitz* if it is Lipschitz continuous in a neighborhood of every point $x \in \Omega$. However, the Lipschitz constant may vary from point. The following more restrictive condition, which requires the Lipschitz constant to be uniform over all points, was originally called *Lipschitz in the small* by Luukainen [22], hence the notation.

Definition 2 By $\mathrm{LS}(\Omega)$ we denote the set of uniformly locally Lipschitz functions, i.e. $f \in \mathrm{LS}(\Omega)$ if there exists $r > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in \Omega; |x - y| < r.$$

We denote by $\mathrm{Lip}(\Omega)$ the set of Lipschitz functions on Ω, i.e. $f \in \mathrm{Lip}(\Omega)$ if

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in \Omega.$$

Both $\mathrm{CMO}(\Omega)$ and $\mathrm{VMO}_2(\Omega)$ can equivalently be defined as the closures of smaller classes. Indeed by the locally compact version of the Stone-Weierstrass theorem, Lipschitz functions compactly supported in Ω are dense in $C_0(\Omega)$, the closure of the continuous functions of compact support, in the $\|.\|_\infty$ norm, hence in the BMO norm. Moreover, as is shown in [14], every $\mathrm{UC}(\Omega)$ function can be approximated by $\mathrm{LS}(\Omega)$ functions. Furthermore, if Ω is a quasi-convex set (i.e. for any $x, y \in \Omega$ there is a rectifiable curve $\gamma \subset \Omega$ of length $\lesssim |x - y|$), then $\mathrm{LS}(\Omega) = \mathrm{Lip}(\Omega)$ (see e.g. [14]). These observations are summarized in the following proposition

Proposition 2 *Let Ω be any domain in $\mathbb{R}^n$. Then,*

- *Compactly supported Lipschitz functions in Ω are dense in* $\mathrm{CMO}(\Omega)$.
- *Uniformly locally Lipschitz functions are dense in* $\mathrm{VMO}_2(\Omega)$
- *If Ω is a quasi-convex domain then Lipschitz functions are dense in* $\mathrm{VMO}(\Omega)$.

When a domain is bounded, the following stronger result, attributed to Jones, is Theorem 1 in [4]:

Theorem 5 ([4], d'après Jones) *If Ω is bounded and $f \in \mathrm{VMO}_1(\Omega)$ then f can be approximated in the* BMO *norm by smooth functions with compact support in Ω, i.e.* $\mathrm{VMO}_1(\Omega) = \mathrm{VMO}_2(\Omega) = \mathrm{CMO}(\Omega)$.

Recall the following characterization of $\mathrm{CMO}(\mathbb{R}^n)$ mentioned in the introduction, which originally appears in [25] (who credits Herz, Strichartz and Sarason), and is proved in [31].

Theorem 6 ([25,31]) *For $f \in \mathrm{BMO}(\mathbb{R}^n)$ we have $f \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if the following conditions hold*

1. $\lim_{t \to 0^+} \omega(f, t) = 0$;
2. $\lim_{\beta \to \infty} \sup_{\ell(Q) \geq \beta} \fint_Q |f(x) - f_Q| \mathrm{d}x = 0$;
3. $\lim_{\beta \to \infty} \sup_{\mathrm{dist}(Q,0) \geq \beta} \fint_Q |f(x) - f_Q| \mathrm{d}x = 0$.

The original formulation of condition 3, the "vanishing at infinity", is weaker in the statement of the Lemma in [31], where it is only assumed that for a fixed cube Q,

$$Q + x \to \infty \quad \text{as } x \to \infty, \tag{11}$$

but in the proof it is observed that conditions 1 and 2 together with (11) give the uniform condition 3. See also Theorems 6 and 7 in [2], where it is shown that for $n \geq 2$, condition 3 implies 2.

The analogue of Theorem 6 is true for an arbitrary domain Ω.

Proposition 3 *Let Ω be a domain. For $f \in \mathrm{BMO}(\Omega)$, $\beta > 0$, denote by $\mathcal{Q}_\beta$ the collection of all cubes $Q \subset \Omega$ with $\ell(Q) \geq \beta$ or $\mathrm{dist}(Q, 0) \geq \beta$, and set*

$$\gamma_\Omega(f, \beta) := \sup_{Q \in \mathcal{Q}_\beta} \fint_Q |f(x) - f_Q| \mathrm{d}x$$

if $\mathcal{Q}_\beta \neq \emptyset$, $\gamma_\Omega(f, \beta) = 0$ otherwise. Then,

$$\mathrm{CMO}(\Omega) = \{f \in \mathrm{VMO}_1(\Omega) : \lim_{\beta \to \infty} \gamma_\Omega(f, \beta) = 0\}.$$

One direction follows from the definition of $\mathrm{CMO}(\Omega)$ as the closure of the continuous functions with compact support in $\mathrm{BMO}(\Omega)$, and the fact that if $f \in \mathrm{BMO}(\Omega)$ has compact support, then $\gamma_\Omega(f, \beta) \to 0$ as $\beta \to \infty$.

In the case of a bounded Ω, $\mathcal{Q}_\beta = \emptyset$ and $\gamma_\Omega(f, \beta) = 0$ for all sufficiently large β so this is just Theorem 5. We follow the main steps of the proof of this theorem in [4] to get the result for the unbounded case. The first step is to approximate f in BMO by functions in $L^\infty(\mathcal{O})$. This is done via the truncations, as in Lemma A.17 in [3]:

Lemma 2 *Fix $f \in \mathrm{VMO}_1(\Omega)$ with $\lim_{\beta \to \infty} \gamma_\Omega(f, \beta) = 0$. Set*

$$f^k = \max(\min(f, k), -k), \quad k \in \mathbb{N}.$$

Then $f^k \in \mathrm{VMO}_1(\Omega)$, $\lim_{\beta \to \infty} \gamma_\Omega(f^k, \beta) = 0$ and $f^k \to f$ in $\mathrm{BMO}(\Omega)$.

Proof For every cube Q, we have

$$\fint_Q |f^k(x) - (f^k)_Q| \mathrm{d}x \leq \fint_Q |f(x) - f_Q| \mathrm{d}x \tag{12}$$

(see [9] for the constant 1 in this inequality). Thus, f^k satisfies the same vanishing mean oscillation conditions as f.

To estimate the distance of f^k to f in $\mathrm{BMO}(\Omega)$, apply the following equivalence of norms to $g = f^k - f$:

$$\|g\|_{\mathrm{BMO}(\Omega)} \approx \sup_{\substack{Q \subset \Omega \\ \ell(Q) \leq \mathrm{dist}(Q, \partial\Omega)}} \fint_Q |f(x) - f_Q| \mathrm{d}x. \tag{13}$$

See Theorem A1.1 in [4] (in the case of the ℓ^∞ norm in $\mathbb{R}^n$), whose proof takes place in a cube and is, therefore, valid for any domain, not just a bounded one.

Given $\epsilon > 0$, by the assumptions on f, there exist δ and β_0 such that $\fint_Q |f - f_Q| < \epsilon/2$ for $Q \subset \Omega$ in one of three cases: $\ell(Q) < \delta$, $\ell(Q) \geq \beta_0$ or $\operatorname{dist}(Q, 0) \geq \beta_0$. In each of these cases we have

$$\fint_Q |(f - f^k) - (f - f^k)_Q| \leq \fint_Q |f - f_Q| + \fint_Q |f^k - (f^k)_Q| \leq 2 \fint_Q |f - f_Q| < \epsilon.$$

Any cube that is not in one of these three cases much have $\ell(Q) \in [\delta, \beta_0]$ and $\operatorname{dist}(Q, 0) < \beta_0$, which means $Q \subset B(0, \beta_0 + \beta_0\sqrt{n})$. If we also assume $\ell(Q) \leq \operatorname{dist}(Q, \partial\Omega)$ we have that

$$Q \subset K_{\delta,\beta_0} := \{x \in \Omega : |x| \leq \beta_0 + \beta_0\sqrt{n}, \operatorname{dist}(x, \partial\Omega) \geq \delta\}$$

which is a compact subset of Ω. Thus

$$\fint_Q |(f - f^k) - (f - f^k)_Q| \leq \frac{2}{\delta^n} \int_{K_{\delta,\beta_0}} |f - f^k|.$$

Since f^k converge to f pointwise, $|f - f^k| \leq |f|$ for all k, and $f \in L^1_{\text{loc}}(\Omega)$, we can apply the Dominated Convergence Theorem to make this smaller than ϵ for sufficiently large k.

By (13), we have thus shown that $\|f - f^k\|_{\text{BMO}(\Omega)} \to 0$ as $k \to \infty$. □

Lemma 3 *If* $f \in L^\infty(\Omega) \cap \text{VMO}_1(\Omega)$ *then there is a sequence of functions* $f_j \in L^\infty(\Omega) \cap \text{VMO}_1(\Omega)$ *with* $\operatorname{dist}(\operatorname{supp}(f_j), \partial\Omega) > 0$ *for each* j *and such that* $f_j \to f$ *in* $\text{BMO}(\Omega)$. *Moreover, if* f *satisfies* $\lim_{\beta\to\infty} \gamma_\Omega(f, \beta) = 0$ *then we can take* f_j *with compact support in* Ω.

Proof The first statement is what is shown in proof of Theorem 1 in [4] for Ω bounded. It can be adapted to the unbounded case by setting the auxiliary functions h_j to be identically equal to 1 when $\operatorname{dist}(x, \partial\Omega) \geq 1$, that is, h_j is the truncation of the function $1 - \frac{1}{j}\varphi_\Omega$ below by 0 and above by 1, namely

$$h_j = \max\left(\min\left(1 - \frac{1}{j}\varphi_\Omega, 1\right), 0\right) \quad \text{for } \varphi_\Omega(x) := \log(\operatorname{dist}(x, \partial\Omega)^{-1}),$$

and letting

$$f_j = f h_j.$$

There is only one step in the proof of Theorem 1 in [4] which uses the boundedness of the domain, the last step at the top of p. 349, in order to show that, for some fixed $\epsilon_0 > 0$,

$$\sup\left\{\fint_B |h_j - 1| : B \subset \Omega, r(B) \geq \epsilon_0\right\} \to 0 \quad \text{as } j \to \infty.$$

By our definition of BMO(Ω) we want to show this for balls B in the ℓ^∞ norm in $\mathbb{R}^n$, namely cubes, but if we prove it for Euclidean balls it is equivalent.

To see that this holds for an unbounded Ω, fix $B \subset \Omega$ with radius $r_0 = r(B) \geq \epsilon_0$ and note that $\varphi_B(x) := \log(\operatorname{dist}(x, \partial B)^{-1}) \geq \varphi_\Omega$. Thus

$$
\begin{aligned}
\fint_B |h_j - 1| &= \frac{1}{|B|}\left\{\int_{\{x\in B:\varphi_\Omega(x)\geq j\}} 1\mathrm{d}x + \int_{\{x\in B: 0<\varphi_\Omega(x)<j\}} \frac{1}{j}\varphi_\Omega(x)\mathrm{d}x\right\} \\
&\leq \frac{|x\in B:\varphi_B(x)\geq j|}{|B|} + \frac{1}{j|B|}\int_{\{x\in B:\varphi_B(x)>0\}} \varphi_B(x)\mathrm{d}x \\
&= \frac{|x\in B:\operatorname{dist}(x,\partial B)\leq e^{-j}|}{|B|} \\
&\quad + \frac{1}{j|B|}\int_{\{x\in B:\operatorname{dist}(x,\partial B)<1\}} \log(\operatorname{dist}(x,\partial B)^{-1})\mathrm{d}x \\
&\leq \frac{1}{r_0{}^n}\int_{r_0-e^{-j}}^{r_0} r^{n-1}dr + \frac{1}{jr_0}\int_{r_0-1}^{r_0} -\log(r_0-r)dr \\
&\leq \int_{1-e^{-j}/\epsilon_0}^{1} r^{n-1}dr + \frac{1}{j\epsilon_0}\int_0^1 -\log(r)dr
\end{aligned}
$$

and the right-hand-side tends to 0 as $j \to \infty$, independently of B.

For the second statement, assuming $\lim_{\beta\to\infty}\gamma_\Omega(f,\beta) = 0$, we let $f_j = f\widetilde{h}_j$, where we define $\widetilde{h}_j$ by using $\Omega_j = \Omega \cap B(0, j)$ instead of Ω, i.e. $\widetilde{h}_j$ is defined on Ω_j as the truncation of $1 - \frac{1}{j}\varphi_{\Omega_j}$ below by 0 and above by 1, and extended by zero to all of Ω. Since we already showed f is approximated by fh_j, we just have to estimate $\|f\widetilde{h}_j - fh_j\|_{\mathrm{BMO}}$. This means estimating the mean oscillation over cubes for which $g = \widetilde{h}_j - h_j$ does not vanish. In particular this implies not both $\widetilde{h}_j, h_j$ are equal to zero or both equal to 1. Since $\Omega_j \subset \Omega$, for $x \in \Omega_j$, $\operatorname{dist}(x, \partial\Omega_j) \leq \operatorname{dist}(x, \partial\Omega)$ so if $\operatorname{dist}(x, \partial\Omega_j) \geq 1$ then $\operatorname{dist}(x, \partial\Omega) \geq 1$, which implies $\widetilde{h}_j(x) = 1 = h_j(x)$. Thus for $x \in \Omega_j$,

$$
\begin{aligned}
\widetilde{h}_j(x) \neq h_j(x) \qquad &\Longrightarrow \operatorname{dist}(x, \partial\Omega_j) < \min(1, \operatorname{dist}(x, \partial\Omega)) \\
&\Longrightarrow \operatorname{dist}(x, \partial B(0, j))) < 1 \implies |x| > j - 1.
\end{aligned}
$$

Note that for $x \in \Omega \setminus \Omega_j$, we always have $|x| \geq j > j - 1$.

Thus, we only need to estimate the oscillation of fg on a cube Q with $Q \cap B(0, j-1)^c \neq \emptyset$. As we saw in the proof of Lemma 2 above, $\lim_{\beta\to\infty}\gamma_\Omega(f,\beta) = 0$ implies that given $\epsilon > 0$, the oscillation of f is bounded by $\epsilon/3$ on every Q which is not contained in $B(0, \beta)$ for some $\beta > 0$; take j sufficiently large and set $\beta = j - 1$. Furthermore, as in [4], the mean oscillation of φ_Ω on any cube is bounded independently of Ω, meaning that $\|g\|_{\mathrm{BMO}} \leq \|\widetilde{h}_j\|_{\mathrm{BMO}} + \|h_j\|_{\mathrm{BMO}} \lesssim \frac{1}{j} \to 0$. Also by definition $\|g\|_\infty \le 1$. Thus we have, as in [4], that

$$\begin{aligned}
\fint_Q |f(x)g(x) - (fg)_Q| &\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x)g(x) - f(y)f(y)|\mathrm{d}x\mathrm{d}y \\
&\le \frac{\|g\|_\infty}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{\|f\|_\infty}{|Q|^2}\int_Q\int_Q |g(x) - g(y)|\mathrm{d}x\mathrm{d}y \\
&< \frac{2\epsilon}{3} + \frac{C\|f\|_\infty}{j} < \epsilon
\end{aligned}$$

for j sufficiently large. □

Proof of Proposition 3 Once we have Lemmas 2 and 3, to prove the Proposition we only need to approximate $f \in L^\infty(\Omega) \cap \mathrm{VMO}_1(\Omega)$ with compact support in Ω by continuous functions with compact support. This can be done as in [4], by convolution with a test function with sufficiently small support, and the test function can be as smooth as we like. Alternatively, we can apply the procedure of Proposition 1 with a grid of sufficiently small sidelength (say $\delta < \mathrm{dist}(\mathrm{supp}(f), \partial\Omega)/2\sqrt{n}$) to guarantee that the Lipschitz function produced by the averaging is supported inside Ω. □

2.3 Uniform Domains

We will follow Jones in [20], introducing them via Whitney cubes.

Let $\Omega \subset \mathbb{R}^n$ be a domain and Ω' be the interior of its complement in $\mathbb{R}^n$. Let E and E' be the Whitney decompositions of Ω and Ω', respectively, (see e.g. [29] for the definition and properties of Whitney decompositions).

For Q_1, Q_2 in E, we define two distance functions d_1 and d_2 as follows. By $d_1(Q_1, Q_2)$ we denote the length of (i.e. number of cubes in) a shortest chain of adjacent cubes in E connecting Q_1 to Q_2. Assuming the cubes in the Whitney decomposition are closed, here and below, *adjacent* cubes means those with nonempty intersections, so they are either neighbors or coinciding (the word *touching* is used in [20]). A chain of adjacent Whitney cubes is called a *Whitney chain*.

The other function is defined by

$$d_2(Q_1, Q_2) := \left|\log\left(\frac{\ell(Q_1)}{\ell(Q_2)}\right)\right| + \log\left(2 + \frac{\mathrm{dist}(Q_1, Q_2)}{\ell(Q_1) + \ell(Q_2)}\right), \tag{14}$$

where here, as above, $\mathrm{dist}(Q_1, Q_2)$ is the usual Euclidean distance between Q_1 and Q_2.

Definition 3 We say that Ω is a uniform domain if there exists constant κ such that for all cubes $S_1, S_2 \in E$

$$d_1(S_1, S_2) \le \kappa d_2(S_1, S_2).$$

Remark 1 It was shown in [15] that Ω is uniform in the above sense if and only if there are constants $C, D > 0$ such that any two points $x, y \in \Omega$ are connected by a rectifiable curve $\gamma \subset \Omega$ with

1. $\text{length}(\gamma) \leq C|x-y|$
2. $s(\gamma(x,z)), s(\gamma(y,z)) \leq D \operatorname{dist}(z, \partial\Omega), \quad \forall z \in \gamma.$

Here $s(\gamma(z_1, z_2))$ is the arclength of the part of γ connecting z_1 to z_2. Condition 1 shows that uniform domains are necessarily quasi-convex, while the existence of a curve γ satisfying condition 2 means that Ω is a John domain.

We want to use d_1 and d_2 to control the difference in average values of functions. Note that by the properties of the Whitney decomposition, if S_1, S_2 are adjacent Whitney cubes then

$$\frac{1}{4} \leq \frac{\ell(S_1)}{\ell(S_2)} \leq 4. \tag{15}$$

This means that along a shortest Whitney chain between two cubes S_1 and S_2, the side-length cannot grow by more than $4^{d_1(S_1,S_2)}$. The following lemma follows immediately from this fact and the proof of Lemma 2.2 in [20].

Lemma 4 *Let $f \in L^1_{\text{loc}}(\Omega)$ and S_1, S_2 be two Whitney cubes in E of sidelenghts $\ell(S_1) \leq \ell(S_2)$. Then the average values of f over S_1, S_2 satisfy*

$$|f_{S_1} - f_{S_2}| \leq C d_1(S_1, S_2)\omega_\Omega(f, 4^{d_1(S_1,S_2)}\ell(S_1)),$$

where $\omega_\Omega(f,t)$ is the modulus of oscillation of f.

Combined with Definition 3, the lemma gives the following corollary.

Corollary 2 *Let Ω be a uniform domain, $f \in L^1_{\text{loc}}(\Omega)$, and S_1, S_2 be Whitney cubes in E. If $d_2(S_1, S_2) \leq K$ then there are constants $C, C' > 0$ depending on K and κ such that*

$$|f_{S_1} - f_{S_2}| \leq C\omega_\Omega(f, C'\ell(S_1)).$$

3 Proof of Theorem 3: Extension from a Uniform Domain

We first define the extension T in Theorem 3 for an unbounded uniform domain Ω. Given a function ϕ in $L^1_{\text{loc}}(\Omega)$, we want to extend ϕ to a function on $\mathbb{R}^n$. By Corollary 2.9 in [20], $\partial\Omega$ has measure zero. Therefore, to define the extension almost everywhere on $\mathbb{R}^n$, we only need to define it on $\Omega' := \overline{\Omega}^c$.

As above, we decompose Ω and Ω' into collections of Whitney cubes E and E', respectively, assuming that Ω contains Whitney cubes of arbitrary large size, which, as shown in [20] for a uniform domain, is the case if and only if Ω is unbounded. For every $S' \in E'$, there exists $S \in E$ with $\ell(S) \geq \ell(S')$. We say such a cube S is a *matching cube* to S' if it is nearest to S' (in Euclidean distance). There may be several choices for S. As pointed out in [20], if S is a matching cubes of S', then

$$\ell(S') \leq \ell(S) \leq 2\ell(S'), \tag{16}$$

(otherwise $\ell(S) \geq 4\ell(S')$ so by (15) its neighbors will have sidelength at least $\ell(S)$ and one of them will be closer to S'). Moreover, for a uniform domain, Lemma 2.10 in [20] shows

$$\operatorname{dist}(S, S') \leq 65\kappa^2 \ell(S'), \tag{17}$$

where κ is the constant in Definition 3.

Following the definition of the extension in [20], for each Whitney cube S_i' in E', we fix a matching cube $S_i \in E$. Denoting by ϕ_{S_i} the average of ϕ on S_i, we set

$$\phi(x) = \phi_{S_i}, \quad x \in S_i'. \tag{18}$$

Note that the function ϕ, now defined almost everywhere on $\mathbb{R}^n$, is the original extension of Jones [20]. While we use ϕ for both the function in Ω and its extension in Ω', Jones denotes this function by $\tilde{\phi}$ and the extension mapping by Λ. It is clearly linear.

Since ϕ is a step function on $\mathcal{O} = \Omega'$, we are back in the setup of Sect. 2 with the Whitney decomposition $\{S_i'\}$ forming the collection of cubes. For $x \in \Omega'$, noting that $\partial\Omega' = \partial\Omega$, we define

$$R(x) = c_n \operatorname{dist}(x, \partial\Omega), \tag{19}$$

for some constant $c_n \in (0, 1)$ to be determined. By the properties of the Whitney decomposition, each $S_i' \in E'$ satisfies

$$\ell(S_i') \leq \operatorname{dist}(S_i', \partial\Omega) \leq 4\sqrt{n}\,\ell(S_i'). \tag{20}$$

Combined with (15), this shows we can choose c_n sufficiently small (say $(16\sqrt{n})^{-1}$) so that if $x \in S_i'$ then $R(x) \leq \ell(S_j')$ for every Whitney cube S_j' adjacent to S_i', which means $\mathcal{N}(x)$, defined by (4), consists exactly of S_i' and the Whitney cubes adjacent to it.

Definition 4 Given ϕ in $L^1_{\mathrm{loc}}(\Omega)$ and extended by the Jones extension (18) to $\Omega' := \overline{\Omega}^c$, we set

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x \in \Omega, \\ A(\phi)(x) & x \in \Omega', \end{cases}$$

where $\tilde{\phi} = A(\phi)$ is defined from ϕ on Ω' as in Definition 1, using the function R given by (19) for an appropriate choice of c_n. Denote by T the operator $\phi \to \tilde{\phi}$, so that T is a composition of the Jones' extension map Λ and the map which is the identity on Ω and the averaging operator A on Ω', hence it is a linear map.

For the rest of the section, we will assume the functions $\phi, \tilde{\phi}$ are as in Definition 4.

The following is an immediate consequence of Lemma 1 and the fact that by the properties of the distance function, R is Lipschitz with Lipschitz constant c_n.

Corollary 3 *For $x_1, x_2 \in \Omega'$,*

$$|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| \leq C \sup_{S_i, S_j} \left|\phi_{S_i} - \phi_{S_j}\right| \min\left(\frac{|x_1 - x_2|}{\min_{i=1,2} \operatorname{dist}(x_i, \partial\Omega)}, 1\right),$$

where the supremum is taken over all cubes S_i and S_j in the Whitney decomposition of Ω which are matching cubes to the cubes in $\mathcal{N}(x_1) \cup \mathcal{N}(x_2)$.

Furthermore, we obtain the following refinement for points in the adjacent cubes

Corollary 4 *For $x_1, x_2 \in \Omega'$, if $x \in S_i'$, $i = 1, 2$, with S_1' and S_2' adjacent, then there exist C and C' so that*

$$|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| \leq C\omega_\Omega(\phi, C'\ell(S_1'))\frac{|x_1 - x_2|}{\ell(S_1')}.$$

Proof Due to Corollary 3 , (15) and (20), it is enough to show that if S_1 and S_2 are matching cubes to adjacent cubes S_1' and S_2', respectively, then

$$|\phi_{S_1} - \phi_{S_2}| \leq C\omega_\Omega(\phi, C'\ell(S_1)).$$

This follows immediately from Corollary 2 once one shows that $d_2(S_1, S_2) \leq K$ for some $K > 0$. In order to see the latter, note that from (17) (Lemma 2.10 in [20]) we know that

$$\begin{aligned}\operatorname{dist}(S_1, S_2') &\leq \operatorname{dist}(S_1, S_1') + \operatorname{diam}(S_1') + \operatorname{dist}(S_1', S_2')\\ &\leq 65\kappa^2\ell(S_1') + \operatorname{diam}(S_1') = (65\kappa^2 + \sqrt{n})\ell(S_1').\end{aligned}$$

Similarly

$$\operatorname{dist}(S_1, S_2) \leq \operatorname{dist}(S_1, S_1') + \operatorname{diam}(S_1') + \operatorname{dist}(S_1', S_2) \leq (130\kappa^2 + \sqrt{n})\ell(S_2').$$

By the condition on the sidelength of matching cubes, we have that

$$\frac{\operatorname{dist}(S_1, S_2)}{\ell(S_1) + \ell(S_2)} \leq \frac{\operatorname{dist}(S_1, S_2)}{\ell(S_2')} \leq 130\kappa^2 + \sqrt{n}.$$

Then, by (16) we have

$$d_2(S_1, S_2) = \left|\log\left(\frac{\ell(S_1)}{\ell(S_1')}\frac{\ell(S_1')}{\ell(S_2')}\frac{\ell(S_2')}{\ell(S_2)}\right)\right| + \log\left(2 + \frac{\operatorname{dist}(S_1, S_2)}{\ell(S_1) + \ell(S_2)}\right) \leq K.$$

□

Finally, this gives an estimate on the oscillation of $\tilde{\phi}$ on cubes in Ω' terms of the modulus of oscillation of ϕ in Ω.

Corollary 5 *There exists constants C, C' and c such that if a cube $Q \subset \Omega'$ has sidelength $\ell(Q) \leq c\delta(Q)$, where $\delta(Q) = \mathrm{dist}(Q, \partial\Omega)$, then*

$$\fint_Q |\tilde{\phi}(x) - \tilde{\phi}_Q| \mathrm{d}x \leq C\omega_\Omega(\phi, C'\delta(Q)) \frac{\ell(Q)}{\delta(Q)}.$$

In $\omega_\Omega(\phi, t)$ is assumed to be concave, we have furthermore that

$$\fint_Q |\tilde{\phi}(x) - \tilde{\phi}_Q| \mathrm{d}x \leq C\omega_\Omega(\phi, C'\ell(Q)).$$

Proof Since $\tilde{\phi}$ is continuous, we have $\tilde{\phi}_Q = \tilde{\phi}(x_0)$ for some $x_0 \in Q$. Take a Whitney cube S_1' containing x_0. Then by (20),

$$\begin{aligned}\mathrm{diam}(Q) &\leq c\sqrt{n}\, \delta(Q) \leq c\sqrt{n}\, \mathrm{dist}(x_0, \partial\Omega) \leq c\sqrt{n}\, (\mathrm{dist}(S_1', \partial\Omega) + \sqrt{n}\, \ell(S_1')) \\ &\leq c5n\, \ell(S_1') < \frac{\ell(S_1')}{4},\end{aligned}$$

for $c < \frac{1}{20n}$. By (15), this means every $x \in Q$ lies in S_1' or one of its adjacent cubes, so we can apply Corollary 4 to get

$$\fint |\tilde{\phi}(x) - \tilde{\phi}_Q| \mathrm{d}x = \fint |\tilde{\phi}(x) - \tilde{\phi}(x_0)| \mathrm{d}x \leq C\omega_\Omega(\phi, C'\ell(S_1')) \frac{\ell(Q)}{\delta(Q)}.$$

Applying (20) again, we also get that $\ell(S_1') \leq \mathrm{dist}(S_1', \partial\Omega) \leq \mathrm{dist}(x_0, \partial\Omega) \leq \delta(Q) + \mathrm{diam}(Q) \leq 2\delta(Q)$. Since the modulus of oscillation is increasing, the proof of the first inequality is complete.

For the second inequality, we just need to note that a concave function $f(x)$ which vanishes at the origin satisfies $tf(x) \leq f(tx)$ for all $t \in [0, 1]$. □

3.1 The BMO Extension

Here, we assume $\phi \in \mathrm{BMO}(\Omega)$, so Jones' result gives us that the extended function $\Lambda\phi$, which we still call ϕ, is in $\mathrm{BMO}(\mathbb{R}^n)$, with $\|\phi\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C\|\phi\|_{\mathrm{BMO}}(\Omega)$. We will show

$$\|\tilde{\phi} - \phi\|_\infty \leq c\|\phi\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

and therefore $\|\tilde{\phi}\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C(c+1)\|\phi\|_{\mathrm{BMO}(\Omega)}$, proving that our extension is bounded.

Since $\tilde{\phi} = \phi$ on Ω, we only need to show the L^∞ estimate on Ω'. Recall that by (5) we have, if $x \in S_i'$ for some Whitney cube of S_i' of Ω',

$$|\phi(x) - \tilde{\phi}(x)| \leq \sup_{S_j' \in \mathcal{N}(x)} |\phi_{S_i'} - \phi_{S_j'}|. \tag{21}$$

Since in the definition of R in (19) we chose c_n sufficiently small to guarantee that $\mathcal{N}(x)$ consists only of cubes S_j' adjacent with S_i', hence satisfying $d_1(S_i', S_j') \leq 1$, we can apply Lemma 4 (actually it enough to apply Lemma 2.2 in [20]) to get that the supremum on the right-hand-side of (21) is bounded by $c\|\phi\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

3.2 The $\mathbf{VMO_1(\Omega)}$ Extension

In this section, we show that the extension defined above maps $\mathrm{VMO}_1(\Omega)$ to $\mathrm{VMO}(\mathbb{R}^n)$, so we assume in what follows that $\phi \in \mathrm{BMO}(\Omega)$ with $\lim_{t\to 0^+} \omega_\Omega(f, t) = 0$. From the results in the preceding section we have that $\|\tilde{\phi}\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|\phi\|_{\mathrm{BMO}(\Omega)}$.

The next step is to show that $\tilde{\phi}$ on Ω' inherits the vanishing mean oscillation condition from ϕ on Ω.

Lemma 5 *For $\phi, \tilde{\phi}$ as above,*

$$\lim_{t\to 0^+} \omega_{\Omega'}(\tilde{\phi}, t) = 0.$$

Furthermore, if $\omega_\Omega(\phi, t)$ is concave in t, we get that there exist constants C, C' with

$$\omega_{\Omega'}(\tilde{\phi}, t) \leq C\omega_\Omega(\phi, C't). \tag{22}$$

Proof As discussed above (see (13)), it is shown in Lemma A1.1 and Theorem A1.1 of [4] that the BMO norm can be controlled by looking only at cubes with $\ell(Q) \leq c\mathrm{dist}(Q, \partial\Omega)$ for some fixed c, and the proofs do not depend on the fact that the domain is bounded. In fact, the proof of Theorem A1.1 in [4] reduces to estimating the oscillation in a single ball by balls that are contained inside it, giving the following result when applied to $\tilde{\phi}$ on Ω':

$$\omega_{\Omega'}(\tilde{\phi}, t) \approx \sup_{\substack{Q\subset\Omega' \\ \ell(Q)\leq \min(t, c\delta(Q))}} \fint_Q |\tilde{\phi}(x) - \tilde{\phi}_Q| \mathrm{d}x, \tag{23}$$

where we have used the notation $\delta(Q) := \mathrm{dist}(Q, \partial\Omega)$ as in Corollary 5.

Fix C, C' and c as in Corollary 5. Let $\epsilon > 0$ be given and take $\delta > 0$ sufficiently small so that $\omega_\Omega(\phi, C'\delta) < \epsilon/C$. Then if $\ell(Q) \leq c\delta(Q)$ with $\delta(Q) \leq \delta$, that corollary implies

$$\fint_Q |\tilde{\phi}(x) - \tilde{\phi}_Q| \mathrm{d}x \leq C\omega_\Omega(\phi, C'\delta) < \epsilon.$$

For Q with $\delta(Q) > \delta$, Corollary 5 gives

$$\fint_Q |\tilde{\phi}(x) - \tilde{\phi}_Q| \mathrm{d}x \leq C\|\phi\|_{\mathrm{BMO}(\Omega)} \frac{\ell(Q)}{\delta},$$

which can be made smaller than ϵ for $\ell(Q) < \delta(C\|\phi\|_{\mathrm{BMO}(\Omega)})^{-1}\epsilon$.

The case where $\omega_\Omega(\phi, t)$ is concave in t is simpler as the bound follows immediately from (23) and Corollary 5. □

Note that in case $\omega_\Omega(\phi, t)$ is not concave, we can always replace it by its *least concave majorant* (see for example Sect. 2.6 in [10]) in order to obtain (22). As the proof above shows, however, the estimate that Corollary 5 gives us without this assumption is better since away from the boundary the extension $\tilde{\phi}$ is Lipschitz.

To show that ϕ on Ω and $\tilde{\phi}$ on Ω' combine into one function in $\mathrm{VMO}(\mathbb{R}^n)$, we use the following lemma, which is a version of Proposition 3 in [6]. The proof given there is a quantified version of the proof of Lemma 2.11 in [20], keeping track of the modulus of continuity instead of the BMO norm, and applies on any uniform domain, not necessarily bounded.

Lemma 6 *Let Ω be a uniform domain and ϕ_1, ϕ_2 be* BMO *functions on Ω and Ω', respectively, satisfying*

$$\lim_{t\to 0^+} \omega_\Omega(\phi_1, t) = 0 = \lim_{t\to 0^+} \omega_{\Omega'}(\phi_2, t).$$

If there exists a bounded, nondecreasing function $\eta : [0, \infty) \to [0, \infty)$ which is continuous at 0, *with $\eta(0) = 0$, such that for each $S' \in E'$ with $S' \subset \Omega'$, and for some $S \in E$ which is a matching cube of S',*

$$|(\phi_1)_S - (\phi_2)_{S'}| \leq \eta(\ell(S')), \tag{24}$$

then the function Φ defined by

$$\Phi(x) = \begin{cases} \phi_1(x), & x \in \Omega \\ \phi_2(x), & x \in \Omega' \end{cases}$$

is in $\mathrm{VMO}(\mathbb{R}^n)$ *with*

$$\|\Phi\|_{\mathrm{BMO}} = \|\omega_{\mathbb{R}^n}(\Phi, \cdot)\|_\infty \lesssim \|\omega_\Omega(\phi_1, \cdot)\|_\infty + \|\omega_{\Omega'}(\phi_2, \cdot)\|_\infty + \|\eta\|_\infty.$$

We want to apply the lemma with $\phi_1 = \phi$ on Ω and $\phi_2 = \tilde{\phi}$ on Ω'. Note that an estimate on the BMO norm is already given to us by the BMO extension in the previous section, so we just need to use the lemma to conclude that the extension $\tilde{\phi}$ is in $\mathrm{VMO}(\mathbb{R}^n)$. It therefore remains to verify (24). We only need to check this for for some $S \in E$ which is a matching cube of S', meaning we can take the matching cube that we had chosen in the definition of the Jones extension of ϕ to Ω' in (18). That means that

$$|(\phi_1)_S - (\phi_2)_{S'}| = |\phi_{S'} - \tilde{\phi}_{S'}| \leq \sup_{x \in S', S'_j \in \mathcal{N}(x)} |\phi_{S'} - \phi_{S'_j}|,$$

where we have used (21). As in the previous section, we can use Lemma 4 with $f = \tilde{\phi}$ on Ω', and recalling that $R(x)$ was chosen so that $\mathcal{N}(x)$ consists of only adjacent cubes to $S' \ni x$, to bound the right-hand-side by $\omega_{\Omega'}(\tilde{\phi}, 4\ell(S'))$. By Lemma 5 and the comments following it above, this is controlled by $\eta(\ell(S'))$, where we take $\eta(t)$ to be the least concave majorant of $C\omega_\Omega(\phi, 4C't)$. Thus, the hypothesis (24) has been verified and we get the conclusion that our extension $\Phi = \tilde{\phi} \in \mathrm{VMO}(\mathbb{R}^n)$. For the bound on the norm, since the least concave majorant preserves the L^∞ norm, we have

$$\|\tilde{\phi}\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|\omega_\Omega(\phi, \cdot)\|_\infty + \|\omega_{\Omega'}(\tilde{\phi}, \cdot)\|_\infty + \|\eta\|_\infty \lesssim \|\phi\|_{\mathrm{BMO}(\Omega)},$$

as we already knew from the previous section.

3.3 The Lipschitz Extension

Assume ϕ is Lipschitz in Ω with Lipschitz constant L and apply the extension operator T in Definition 4. We claim that $\tilde{\phi} = T\phi$ is Lipschitz on $\mathbb{R}^n$ with Lipschitz constant bounded by a multiple of L.

Let us first show that $\tilde{\phi}$ is Lipschitz on Ω'. Suppose that x_1 and x_2 belong to Whitney cubes S'_1 and S'_2 in Ω', respectively, and assume without loss of generality that $\ell(S_1) \leq \ell(S_2)$.

If S'_1 and S'_2 are adjacent, we can apply Corollary 4 and the Lipschitz continuity of ϕ on Ω to conclude that

$$|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| \lesssim \frac{\omega_\Omega(\phi, C'\ell(S_1))}{\ell(S_1)}|x_1 - x_2| \lesssim L|x_1 - x_2|. \tag{25}$$

Now suppose S'_1 and S'_2 are not adjacent, which means $|x_1 - x_2| \gtrsim \ell(S'_1) + \ell(S'_2) \gtrsim \min_{i=1,2} \mathrm{dist}(x_i, \partial\Omega)$. Let x'_1 and x'_2 be the centers of cubes S'_1 and S'_2 respectively. Then by construction of $\tilde{\phi}$

$$|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| \leq |\tilde{\phi}(x_1) - \tilde{\phi}(x'_1)| + |\tilde{\phi}(x'_2) - \tilde{\phi}(x_2)| + \left|\phi_{S_1} - \phi_{S_2}\right|.$$

Applying (25) to the first two terms we get

$$|\tilde{\phi}(x_1) - \tilde{\phi}(x_2)| \lesssim \left|\phi_{S_1} - \phi_{S_2}\right| + L|x_1 - x_2|.$$

Furthermore,

$$\left|\phi_{S_1} - \phi_{S_2}\right| \leq \fint_{S_1}\fint_{S_2} |\phi(x) - \phi(y)|\mathrm{d}x\mathrm{d}y \leq L(\mathrm{dist}(S_1, S_2) + \ell(S_1) + \ell(S_2)).$$

By (17) (Lemma 2.10 in [20]), we know that $\operatorname{dist}(S_i, S_i') \lesssim \ell(S_i') \approx \ell(S_i)$ so by the triangle inequality we get

$$\left|\phi_{S_1} - \phi_{S_2}\right| \lesssim L(\operatorname{dist}(S_1', S_2') + \ell(S_1') + \ell(S_2')) \lesssim L|x_1 - x_2|.$$

This shows that $\tilde{\phi}$ is Lipschitz on Ω'.

Once we know that $\tilde{\phi}$ is Lipschitz on Ω and Ω' separately, by uniform continuity we can extend it to $\partial\Omega$ from either side. To see that this is well-defined, we need to show that if $x \in \partial\Omega$ and $x_i \to x$, $x_i' \to x$ with $x_i \in \Omega$, $x_i' \in \Omega'$ then $\lim \tilde{\phi}(x_i) = \lim \tilde{\phi}(x_i')$. Note that by the uniform continuity of ϕ and the properties of Whitney cubes, there is a unique limit of ϕ_{S_i} for any sequence of Whitney cubes S_i in Ω with $\operatorname{dist}(S_i, x) \to 0$. Thus, all that remains to show is that if $x_i' \in \Omega'$, $x_i \to x$ and $x_i \in S_i'$, then $\operatorname{dist}(S_i, x) \to 0$ where S_i are matching cubes for the S_i'. But this again follows from the Jones estimate that $\operatorname{dist}(S_i, S_i') \lesssim \ell(S_i')$ which must go to zero.

So $\tilde{\phi}$ is now defined pointwise everywhere on $\mathbb{R}^n$ and is Lipschitz continuous on $\overline{\Omega}$ and $\overline{\Omega'}$, with a constant bounded by a multiple of L.

Finally, if $x \in \Omega$ and $x' \in \Omega'$, there must be a point in $y \in \partial\Omega$ which lies on the straight line between them. Applying the Lipschitz continuity on $\overline{\Omega}$ to the pair x, y, and the Lipschitz continuity on $\overline{\Omega'}$ to the pair y, x', we get the desired Lipschitz estimate.

3.4 The VMO$_2$ Extension

It follows from the definition of VMO$_2$ that in order to prove part (iii) of Theorem 3, it suffices to prove parts (i) and (v), namely that the same extension is bounded on BMO and on Lipschitz functions. To see this, suppose $\phi \in \mathrm{VMO}(\Omega)$ and let $\{\phi_j\}$ be a sequence of Lipschitz functions on Ω such that $\|\phi_j - \phi\|_{\mathrm{BMO}(\Omega)} \to 0$. Then, by (v), each $T\phi_j$ is Lipschitz on $\mathbb{R}^n$, and by (i), $\|T\phi_j - T\phi\|_{\mathrm{BMO}(\mathbb{R}^n)} \to 0$. Thus, $T\phi_j$ is the limit in $\mathrm{BMO}(\mathbb{R}^n)$ of Lipschitz functions, hence is in the closure of the uniformly continuous function in $\mathrm{BMO}(\mathbb{R}^n)$, which is $\mathrm{VMO}(\mathbb{R}^n)$. Note that the boundedness in the norm is just part (i). Also note that it would have sufficed to prove that the extension maps uniformly continuous functions on Ω to uniformly continuous function on $\mathbb{R}^n$.

3.5 The CMO Extension

By Proposition 2, compactly supported Lipschitz functions are dense in CMO, and since we have shown the extension maps Lipschitz to Lipschitz and is bounded in the BMO norm, it suffices to show it preserves compact support.

Lemma 7 *Fix a Whitney cube S_0' in Ω' and a matching Whitney cube S_0 in Ω. If there is a constant $C \in \mathbb{R}$ such that*

$$\phi \equiv C, \text{ on all Whitney cubes } S \subset \Omega \text{ with } d_2(S, S_0) > M,$$

and M is sufficiently large then

$$\tilde{\phi} \equiv C, \text{ on all Whitney cubes } S' \subset \Omega' \text{ with } d_2(S', S_0') > 4M.$$

From the definition on d_2 in (14), and the relationship between the size of the Whitney cubes and the distance to the boundary, (20), we see that the condition $d_2(S, S_0) > M$ for large M is equivalent to S being close to boundary $\partial\Omega$ or S being far away from S_0 in the Euclidean distance, hence the hypothesis is the same as saying that ϕ has compact support in Ω. Similarly, the conclusion is equivalent to $\tilde{\phi}$ having compact support in Ω'. This shows the Lemma implies our desired conclusion.

Corollary 6 *If ϕ is compactly supported in Ω, then so is $\tilde{\phi}$.*

We now prove the Lemma.

Proof Let S' be a Whitney cube in Ω' with $d_2(S', S_0') > M$. We denote by $S \subset \Omega$ a matching cube of S'. We will show that for sufficiently large M, $d_2(S', S_0') > M$ implies $d_2(S, S_0) > M/4$.

We consider two possibilities

Case 1: Suppose

$$\left|\log \frac{\ell(S')}{\ell(S_0')}\right| \geq M/2,$$

Then for a matching cube $S \subset \Omega$, we have, by (16), for M sufficiently large,

$$\left|\log \frac{\ell(S)}{\ell(S_0)}\right| \geq \left|\log \frac{\ell(S')}{\ell(S_0')}\right| - \log 4 \geq M/2 - \log 4 > M/4.$$

Case 2:

$$\left|\log \frac{\ell(S')}{\ell(S_0')}\right| < M/2, \qquad \log\left(2 + \frac{\operatorname{dist}(S', S_0')}{\ell(S_0') + \ell(S')}\right) \geq M/2,$$

In this case, writing

$$\operatorname{dist}(S', S_0') \leq \operatorname{dist}(S', S) + \operatorname{diam}(S) + \operatorname{dist}(S, S_0) + \operatorname{diam}(S_0) + \operatorname{dist}(S_0, S_0'),$$

and using both (16) and (17) (Lemma 2.10 in [20]), we get

$$\begin{aligned}
\frac{\operatorname{dist}(S, S_0)}{\ell(S) + \ell(S_0)} &\geq \frac{\operatorname{dist}(S', S_0')}{\ell(S) + \ell(S_0)} - \frac{\operatorname{dist}(S', S)}{\ell(S) + \ell(S_0)} - \frac{\operatorname{dist}(S_0', S_0)}{\ell(S) + \ell(S_0)} - \sqrt{n} \\
&\geq \frac{\operatorname{dist}(S', S_0')}{2(\ell(S') + \ell(S_0'))} - \frac{\operatorname{dist}(S', S)}{\ell(S)} - \frac{\operatorname{dist}(S_0', S_0)}{\ell(S_0)} - \sqrt{n} \\
&\geq \frac{\operatorname{dist}(S', S_0')}{2(\ell(S') + \ell(S_0'))} - 130\kappa^2 - \sqrt{n}.
\end{aligned}$$

Therefore, taking M sufficiently large so that the hypotheses of this case imply $\frac{1}{4}\left(2+\frac{\operatorname{dist}(S',S'_0)}{\ell(S'_0)+\ell(S')}\right) \geq 130\kappa^2 - \sqrt{n}$, we have

$$\begin{aligned}\log\left(2+\frac{\operatorname{dist}(S,S_0)}{\ell(S_0)+\ell(S)}\right) &\geq \log\left(\frac{1}{4}\left(2+\frac{\operatorname{dist}(S',S'_0)}{\ell(S'_0)+\ell(S')}\right)\right.\\ &\qquad\left.+\frac{1}{4}\left(2+\frac{\operatorname{dist}(S',S'_0)}{\ell(S'_0)+\ell(S')}\right)-130\kappa^2-\sqrt{n}\right)\\ &\geq \log\left(\frac{1}{4}\left(2+\frac{\operatorname{dist}(S',S'_0)}{\ell(S'_0)+\ell(S')}\right)\right) \geq M/2-\log 4,\end{aligned}$$

and this again can be made greater than $M/4$. □

3.6 The Bounded Case

Now suppose Ω is a bounded uniform domain. Recall from the discussion in Sect. 2.2 that for a bounded domain, we have $\mathrm{VMO}_1(\Omega) = \mathrm{VMO}_2(\Omega) = \mathrm{CMO}(\Omega)$, so as above it is enough to show that the extension maps $\mathrm{BMO}(\Omega)$ to $\mathrm{BMO}(\mathbb{R}^n)$ and Lipschitz functions to Lipschitz functions. In [20], pp. 57–58, Jones adjusts his extension operator Λ to the case of a bounded domain by setting $\Lambda\phi$ on S'_i to be ϕ_{S_0} whenever S'_i is a Whitney cube in Ω' with $\ell(S'_i) > L$, where L is the maximum sidelength of the Whitney cubes in Ω, and S_0 is a fixed Whitney cube in Ω with sidelength equal to L. The definition for the case $\ell(S'_i) \leq L$ is the same as in the unbounded case above. The effect of this is to make the extension $\Lambda\phi$ constant on the region of Ω' lying sufficiently far away from $\partial\Omega$, thus creating an extension of compact support. The averaging operator A preserves this property. Thus the BMO and Lipschitz boundedness go through in the same way as for the unbounded case.

Alternatively, the bounded case can be proved via a different extension, also of compact support, which the authors previously constructed in [6]. That extension was continuous on Ω' and had the property that the values of $\tilde{\phi}$ on a Whitney cube S' only depended on the values of ϕ on a matching cube S in Ω, and not on those in neighboring cubes. This required the use of a bump function introduced in the proof of Theorem 5, which is Theorem 1 in [4].

4 Proof of Theorem 4

Let Ω be any domain in $\mathbb{R}^n$. In the proof of the necessity in Theorem 1 [20], Jones fixes a Whitney cube S_0 in Ω and defines the function ϕ_{S_0} on Ω by $d_1(S_0, S_x)$, where S_x is a Whitney cube containing x (this is well-defined up to a set of measure zero). He then shows that $\phi_{S_0} \in \mathrm{BMO}(\Omega)$ with the norm bounded independently of the choice of S_0.

In our case, we want to create a "test function" in $\mathrm{CMO}(\Omega)$, or more precisely a Lipschitz function with compact support. We begin by truncating the Jones function:

for $M > 0$, define

$$\phi_{S_0}^M(x) = \min(d_1(S_0, S_x), M).$$

By (12) and Lemma 2.4 in [20],

$$\|\phi_{S_0}^M\|_{\mathrm{BMO}(\Omega)} \le \|\phi_{S_0}\|_{\mathrm{BMO}(\Omega)} \le C.$$

Then we take $\tilde{\phi}$ to be the averaged version of $\phi_{S_0}^M$, applying the averaging operator A defined in Sect. 2 to the step function $\phi_{S_0}^M$ on Ω, with the function $R(x)$ defined as in (19), namely $R(x) = c_n \mathrm{dist}(x, \partial\Omega)$ with c_n sufficiently small.

We claim that for any $M > 0$

1. $\tilde{\phi} \in \mathrm{BMO}(\Omega)$ with BMO norm uniformly bounded in M and S_0.
2. $\tilde{\phi}$ has a compact support that depends on M and S_0
3. $\tilde{\phi} \in \mathrm{Lip}(\Omega)$ with Lip constant depending on M and S_0.

We already know that $\phi_{S_0}^M$ had BMO norm bounded by a constant independent of M and S_0, and this is not changed by the averaging process since, as we saw in Sect. 3.1, $\|\tilde{\phi} - \phi_{S_0}^M\|_\infty \le C\|\phi_{S_0}^M\|_{\mathrm{BMO}}$.

Note that $\tilde{\phi}$ is constantly equal to M on Whitney cubes S which are sufficiently far (depending on M) from S_0 in the d_1 distance. This means that the support of $\tilde{\phi}$, i.e. the set on which it is not constant, is contained in some d_1-"ball" (i.e. a ball in the quasi-hyperbolic distance in Ω) centred at S_0. Properties (15) and (20) of the Whitney cubes imply that the Euclidean distance $\mathrm{dist}(x, S_0)$ is bounded above and $\mathrm{dist}(x, \partial\Omega)$ is bounded below on the support of $\tilde{\phi}$, showing that it has compact support in Ω.

Furthermore, $R(x)$ is bounded below on this support by some constant $\delta > 0$ depending on S_0 and M. Now Lemma 1 can be applied to conclude that $\tilde{\phi}$ is Lipschitz with a Lipschitz constant bounded by a multiple of δ^{-1}.

Suppose any $f \in \mathrm{CMO}(\Omega)$ can be linearly extended to a function $Tf \in \mathrm{BMO}(\mathbb{R}^n)$ and there exists $K > 0$ independent of f such that

$$\|Tf\|_{\mathrm{BMO}(\mathbb{R}^n)} \le K\|f\|_{\mathrm{BMO}(\Omega)},$$

Thus, by the Fefferman-Stein lemma (see e.g. Lemma 2.1 in [20]), for any two cubes $Q_0, Q_1 \in \Omega$

$$|(f)_{Q_0} - (f)_{Q_1}| \le Cd_2(Q_0, Q_1) \cdot \|Tf\|_{\mathrm{BMO}(\mathbb{R}^n)} \le CKd_2(Q_0, Q_1)\ \|f\|_{\mathrm{BMO}(\Omega)}. \tag{26}$$

Then for any two Whitney cubes $S_0, S_1 \in \Omega$, we can choose $f = \tilde{\phi} = A\phi_{S_0}^M$ as above, with $M = d_1(S_0, S_1) + 1$. Furthermore we choose Q_0, Q_1 as subcubes of S_0 and S_1 respectively such that $\tilde{\phi} = 0$ and $\tilde{\phi}_{Q_1} = d_1(S_0, S_1)$. Such a choice is possible for c_n sufficiently small, with $\ell(Q_0), \ell(Q_1)$ comparable in sidelength to $\ell(S_0), \ell(S_1)$, respectively, by (3). The latter guarantees that $d_2(Q_0, Q_1) \lesssim d_2(S_0, S_1)$, which turns

(26) into

$$d_1(S_0, S_1) \leq CK d_2(S_0, S_1)\ \|\tilde{\phi}\|_{\mathrm{BMO}(\Omega)}.$$

As $\|\tilde{\phi}\|_{\mathrm{BMO}(\Omega)}$ is uniformly bounded and S_0, S_1 were arbitrary chosen, we see that Ω is a uniform domain with $\kappa \lesssim CK\|\tilde{\phi}^M_{S_0}\|_{\mathrm{BMO}(\Omega)}$.

As a corollary of the proof, we see that the hypothesis in Theorem 4 can be weakened to assuming the extension maps Lipschitz functions with compact support in Ω (modulo constants) to functions in $\mathrm{BMO}(\mathbb{R}^n)$, boundedly in the BMO norm.

5 Examples

In this section, we consider two examples complementing Theorem 3 and Corollary 1. The first example shows that unlike CMO and VMO, there are subspaces of BMO for which no extension can be simultaneously an extension from $\mathrm{BMO}(\Omega) \to \mathrm{BMO}(\mathbb{R}^n)$, even when Ω is Lipschitz. The second example shows that for certain non-uniform domains $\mathrm{VMO}_1(\Omega)$ is strictly larger than $\mathrm{VMO}_2(\Omega)$.

Example 7 Let $\mathrm{BMO}_{odd}(\mathbb{R})$ be the set of odd BMO functions (modulo constants) on the real line. Namely, $f \in \mathrm{BMO}_{odd}$ if and only if $f \in \mathrm{BMO}(\mathbb{R})$ and for almost all $x, y \in \mathbb{R}$

$$f(x) + f(-x) = f(y) + f(-y).$$

Note that BMO_{odd} is a subspace of $\mathrm{BMO}(\mathbb{R})$.

We claim that there is no simultaneous extension operator E from $\mathrm{BMO}(\mathbb{R}_+)$ to $\mathrm{BMO}(\mathbb{R})$ and from $[\mathrm{BMO}_{odd}(\mathbb{R})]|_{\mathbb{R}_+}$ to $\mathrm{BMO}_{odd}(\mathbb{R})$.

Let $f(x) = \sqrt{\log^+ \frac{1}{|x|}}$. Then $f \in \mathrm{VMO}(\mathbb{R})$ and as it follows from [4], there are UC functions $\{f_n\}$ compactly supported on $(0, 1)$ such that $f_n \to f$ in $\mathrm{BMO}(\mathbb{R}_+)$. If E is a continuous extension from $\mathrm{BMO}(\mathbb{R}_+)$ to $\mathrm{BMO}(\mathbb{R})$, then

$$Ef = \lim_n Ef_n.$$

in $\mathrm{BMO}(\mathbb{R})$. Moreover, f_n are in $[\mathrm{BMO}_{odd}(\mathbb{R})]|_{\mathbb{R}_+}$ and if E simultaneously extends the latter class to $\mathrm{BMO}_{odd}(\mathbb{R})$, then $Ef \in \mathrm{BMO}_{odd}(\mathbb{R})$. This however contradicts to Proposition 12 in [2] which says every $f \in \mathrm{BMO}_{odd}$ must satisfy

$$\sup_{a>0} \frac{1}{a} \int_0^a |f(t)| dt < \infty.$$

We can construct similar examples in higher dimensions e.g. by defining

$$B(\mathbb{R}^n) = \{f(x_1, \ldots, x_n) = g(x_n),\ g \in \mathrm{BMO}_{odd}(\mathbb{R})\};$$

this class of functions is a subspace of $\mathrm{BMO}(\mathbb{R}^n)$. The same reasoning as above shows that there is no linear extension T bounded from $B(\mathbb{R}^n)|_{\mathbb{R}^n{}_+}$ to $B(\mathbb{R}^n)$ and from $\mathrm{BMO}(\mathbb{R}^n{}_+)$ to $\mathrm{BMO}(\mathbb{R}^n)$ simultaneously.

Example 8 Let $\Omega = \mathbb{R}^2 \setminus \{(x, 0), x \geq 0\}$. Consider a smooth function

$$g(x) = \begin{cases} \log x, & \text{if } x \geq 2 \\ 0, & \text{if } x \leq 1, \end{cases}$$

and $f(x, y)$ defined on Ω as

$$f(x, y) = \begin{cases} g(x), & \text{if } y \geq 0 \\ -g(x), & \text{if } y \leq 0, \end{cases}$$

It is clear that $f(x, y)$ is smooth on Ω and therefore a $\mathrm{VMO}_1(\Omega)$. We claim, however, that $f \notin \mathrm{VMO}_2(\Omega)$. More specifically, for some ϵ_0 there is no $g \in \mathrm{UC}(\Omega)$ such that

$$\|f - g\|_{\mathrm{BMO}} \leq \epsilon_0. \tag{27}$$

Indeed, consider cubes Q_k^+ and Q_k^- that have sidelengths 2^{-k-2} and centred at points $(2^k, 2^{-k})$ and $(2^k, -2^{-k})$ respectively.

Then

$$|f_{Q_k^+} - f_{Q_k^-}| \approx 2k \tag{28}$$

Note that

$$d_1(Q_k^+, Q_k^-) \approx k,$$

and by Lemma 2.2 in [20]

$$||f_{Q_k^+} - f_{Q_k^-}| - |g_{Q_k^+} - g_{Q_k^-}|| \lesssim \|f - g\|_{\mathrm{BMO}(\Omega)} \cdot d_1(Q_k^+, Q_k^-),$$

so

$$||f_{Q_k^+} - f_{Q_k^-}| - |g_{Q_k^+} - g_{Q_k^-}|| \lesssim k\|f - g\|_{\mathrm{BMO}(\Omega)} \tag{29}$$

If (27) held for any ϵ_0, then (28) and (29) would imply

$$|g_{Q_k^+} - g_{Q_k^-}| \gtrsim k,$$

but then

$$k \lesssim |g_{Q_k^+} - g_{Q_k^-}| \leq |Q_k^+|^{-1} \left| \int_{Q_k^+} g(x, y) - g(x, -y) \mathrm{d}x \mathrm{d}y \right| \leq \omega_g(2^{-k}).$$

This means $g \notin \mathrm{UC}(\Omega)$.

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