The Journal of Geometric Analysis (2021) 31:9338–9365
https://doi.org/10.1007/s12220-020-00513-z

Atomic Decomposition for Mixed Morrey Spaces

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Received: 10 March 2020 / Accepted: 3 September 2020 / Published online: 30 September 2020

Abstract
In this paper, we consider some norm estimates for mixed Morrey spaces considered by the first author. Mixed Lebesgue spaces are realized as a special case of mixed Morrey spaces. What is new in this paper is a new norm estimate for mixed Morrey spaces that is applicable to mixed Lebesgue spaces as well. An example shows that the condition on parameters is optimal. As an application, the Olsen inequality adapted to mixed Morrey spaces can be obtained.

Keywords Mixed Morrey spaces · Fractional integral operators · Atoms

Mathematics Subject Classification 41A17 · 42B35

1 Introduction

We obtain some decomposition results for mixed Lebesgue spaces and mixed Morrey spaces. Let us first recall the definition of mixed Lebesgue spaces. Let $0 < q_1, q_2, \ldots, q_n \leq \infty$ be constants. Write $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. Then define the mixed Lebesgue norm $\|\cdot\|_{L^{\mathbf{q}}}$ by

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$$\|f\|_{L^{\mathbf{q}}} \equiv \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{q_1} \mathrm{d}x_1 \right)^{\frac{q_2}{q_1}} \mathrm{d}x_2 \right)^{\frac{q_3}{q_2}} \cdots \mathrm{d}x_n \right)^{\frac{1}{q_n}}.$$

A natural modification for x_i is made when $q_i = \infty$. We define the *mixed Lebesgue space* $L^{\mathbf{q}}(\mathbb{R}^n)$ to be the set of all measurable function f on $\mathbb{R}^n$ with $\|f\|_{L^{\mathbf{q}}} < \infty$. Here and below we use the notation

$$\mathbf{q} = (q_1, q_2, \ldots, q_n), \quad \mathbf{q}^* = (q_1^*, q_2^*, \ldots, q_n^*), \quad \mathbf{t} = (t_1, t_2, \ldots, t_n)$$

to denote the vectors in $\mathbb{R}^n$. The aim of this paper is to develop a theory of decompositions based on the following boundedness of the maximal operator.

Here and below, for $0 \le a \le b \le \infty$, $a \le \mathbf{q} \le b$ means that $a \le qi \le b$ for all $i = 1, 2, \ldots, n$.

Theorem 1 *Assume that*

$$1 \le t_k < \min\{q_1, \ldots, q_k\} \le \infty \quad (k = 1, \ldots, n).$$

Define

$$M^{(\mathbf{t})} f(x) = \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{\|\chi_Q\|_{L^{\mathbf{t}}}} \|f \chi_Q\|_{L^{\mathbf{t}}}$$

for a measurable function f. Then for all measurable functions f

$$\|M^{(\mathbf{t})} f\|_{L^{\mathbf{q}}} \lesssim \|f\|_{L^{\mathbf{q}}}.$$

For $0 < p < \infty$, and $0 < \mathbf{q} < \infty$ satisfying

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$$

recall that mixed Morrey spaces are defined by the norm given by

$$\|f\|_{\mathcal{M}^p_{\mathbf{q}}} \equiv \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p} - \frac{1}{n} \sum_{j=1}^n \frac{1}{q_j}} \|f \chi_Q\|_{L^{\mathbf{q}}}$$

for measurable functions $f : \mathbb{R}^n \to \mathbb{C}$, where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of all dyadic cubes. We denote by $\mathcal{Q}(\mathbb{R}^n)$ the set of all cubes whose edges are parallel to the coordinate axes. If there is no confusion, we substitute $\mathcal{D}$ and $\mathcal{Q}$ for $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{Q}(\mathbb{R}^n)$, respectively.

Using Theorem 1, we seek to prove the following decomposition result about the functions in mixed Morrey spaces.

This result extends [26, Chapter 8, Lemma 5]

Theorem 2 *Suppose that the parameters* $p, \mathbf{q}, s, \mathbf{t}$ *satisfy*

$$1 < p < s < \infty, \quad 1 < \max\{q_1, \dots, q_k\} < t_k < \infty \quad (k = 1, \dots, n),$$
$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}.$$

Assume that $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}_{\mathbf{t}}^{s}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, *and* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *fulfill*

$$\|a_j\|_{\mathcal{M}_{\mathbf{t}}^{s}} \le |Q_j|^{\frac{1}{s}}, \quad \text{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\mathbf{q}}^{p}} < \infty. \tag{1}$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n) \cap L_{\text{loc}}^{\mathbf{q}}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{M}_{\mathbf{q}}^{p}} \le C_{p,\mathbf{q},s,\mathbf{t}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\mathbf{q}}^{p}}. \tag{2}$$

The next assertion concerns the decomposition of functions in $\mathcal{M}_{\mathbf{q}}^{p}(\mathbb{R}^n)$. Hereafter, we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}_0$, denote by $\mathcal{P}_d(\mathbb{R}^n)$ the set of all polynomial functions with degree less than or equal to d, so that $\mathcal{P}(\mathbb{R}^n) \equiv \bigcup_{d=0}^{\infty} \mathcal{P}_d(\mathbb{R}^n)$. It is clear that $\mathcal{P}_{-1}(\mathbb{R}^n) = \{0\}$. Let $K \in \mathbb{N}_0$. The set $\mathcal{P}_K(\mathbb{R}^n)^{\perp}$ denotes the set of measurable function f for which $\langle \cdot \rangle^K f \in L^1(\mathbb{R}^n)$ and $\displaystyle\int_{\mathbb{R}^n} x^{\alpha} f(x) \mathrm{d}x = 0$ for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le K$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. Such a function f is said to satisfy the *moment condition of order* K. In this case, one also writes $f \perp \mathcal{P}_K(\mathbb{R}^n)$.

One writes $\mathbf{q} < \mathbf{t}$ if $q_j < t_j$ for each $j = 1, 2, \dots, n$.

The following theorem is a consequence of the paper [11].

Theorem 3 *Suppose that the real parameters* $p, \mathbf{q}, K$ *satisfy*

$$1 < p < \infty, \quad 1 < \mathbf{q} < \infty, \quad \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad K \in \mathbb{N}_0 \cap \left(\frac{n}{q_0} - n - 1, \infty \right),$$

where $q_0 = \min(q_1, \dots, q_n)$. *Let* $f \in \mathcal{M}_{\mathbf{q}}^{p}(\mathbb{R}^n)$. *Then there exists a triplet* $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, *and* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and that, for any* $v > 0$

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v} \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_{\mathbf{q}}^{p}} \le C_v \|f\|_{\mathcal{M}_{\mathbf{q}}^{p}}. \tag{3}$$

Here the constant $C_v > 0$ *is independent of* f.

We rephrase Theorems 2 and 3 in the case of mixed Lebesgue spaces.

Corollary 1 *Suppose that the parameters* $\mathbf{q}, \mathbf{t}$ *satisfy*

$$1 < \max\{q_1, \dots, q_k\} < t_k < \infty \quad (k = 1, \dots, n).$$

Assume that $\{a_j\}_{j=1}^{\infty} \subset L^{\mathbf{t}}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, *and* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *fulfill*

$$\|a_j\|_{L^{\mathbf{t}}} \le |Q_j|^{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{t_k}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{L^{\mathbf{q}}} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $L^{\mathbf{q}}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{L^{\mathbf{q}}} \le C_{p,q,s,t} \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{L^{\mathbf{q}}}.$$

Corollary 2 *Let* $1 < \mathbf{q} < \infty$ *and* $K \in \mathbb{N}_0 \cap \left(\frac{n}{q} - n - 1, \infty\right)$. *Let* $f \in L^{\mathbf{q}}(\mathbb{R}^n)$. *Then there exists a triplet* $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, *and* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $L^{\mathbf{q}}(\mathbb{R}^n)$ *and that, for any* $v > 0$

$$|a_j| \le \chi_{Q_j}, \quad \left\|\left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v\right)^{\frac{1}{v}}\right\|_{L^{\mathbf{q}}} \le C_v \|f\|_{L^{\mathbf{q}}}.$$

Here the constant $C_v > 0$ *is independent of* f.

Theorem 3 is a special case of Theorem 4 to follow, which concerns the decomposition of Hardy-mixed Morrey spaces. Based on [21], we define Hardy-mixed Morrey spaces. For $0 < \mathbf{q}, p < \infty$ satisfying $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$, the Hardy-mixed Morrey space $H\mathcal{M}_{\mathbf{q}}^{p}(\mathbb{R}^n)$ is defined as the set of any $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{H\mathcal{M}_{\mathbf{q}}^{p}} = \left\|\sup_{t>0} |e^{t\Delta} f|\right\|_{\mathcal{M}_{\mathbf{q}}^{p}}$ is finite, where $e^{t\Delta} f$ stands for the heat extension of f;

$$e^{t\Delta} f(x) = \left\langle \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x - \cdot|^2}{4t}\right), f\right\rangle \quad (t > 0, \ x \in \mathbb{R}^n).$$

See [28] for the equivalent norms of the Hardy–Morrey spaces. We rephrase Theorems 2 and 3 in full generality in terms of Hardy-mixed Morrey spaces. The following result is again a consequence of the paper [11].

Theorem 4 *Suppose that the real parameters p, $\mathbf{q}$, K satisfy*

$$0 < p < \infty, \quad 0 < \mathbf{q} < \infty, \quad \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad K \in \mathbb{N}_0 \cap \left(\frac{n}{q_0} - n - 1, \infty\right),$$

where $q_0 = \min(q_1, \dots, q_n)$. Let $f \in H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$. Then there exists a triplet $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$, and $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that, for any $v > 0$,

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\mathbf{q}}} \le C_v \|f\|_{H\mathcal{M}^p_{\mathbf{q}}}. \tag{4}$$

Here the constant C_v is a constant that is independent on v but not on f.

We remark that Theorems 2 and 4 are the special cases of the results in [11].

Theorem 2 has the following counterpart.

Theorem 5 *Suppose that the parameters p, $\mathbf{q}$, s, $\mathbf{t}$ satisfy*

$$1 < p < s < \infty, \quad 0 < \max\{1, q_1, \dots, q_k\} < t_k < \infty \quad (k = 1, \dots, n),$$
$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}.$$

Write $v(\mathbf{q}) \equiv \min\{1, q_1, \dots, q_n\}$ and $d_q = \left[n \left(\frac{1}{v(\mathbf{q})} - 1 \right) \right]$. Assume that a triple

$$(\{a_j\}_{j=1}^\infty, \{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) \in (\mathcal{M}^s_{\mathbf{t}}(\mathbb{R}^n) \cap \mathcal{P}_{d_q}^\perp(\mathbb{R}^n)) \times [0, \infty) \times \mathcal{Q}(\mathbb{R}^n)$$

fulfills

$$\|a_j\|_{\mathcal{M}^s_{\mathbf{t}}} \le |Q_j|^{\frac{1}{s}}, \quad \operatorname{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^{v(\mathbf{q})} \right)^{\frac{1}{v(\mathbf{q})}} \right\|_{\mathcal{M}^p_{\mathbf{q}}} < \infty.$$

Then $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and satisfies

$$\|f\|_{H\mathcal{M}^p_{\mathbf{q}}} \le C_{p,\mathbf{q},s,\mathbf{t}} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^{v(\mathbf{q})} \right)^{\frac{1}{v(\mathbf{q})}} \right\|_{\mathcal{M}^p_{\mathbf{q}}}.$$

Remark that in [14] Jia and Wang considered the case of $q_i = q \le 1$ for $i = 1, 2, \ldots, n$. We also remark that Theorems 4 and 5 with $q_i = q = p \le 1$ for $i = 1, 2, \ldots, n$ are included in [10, Theorems 2.1 and 2.2]. Theorem 2 is new and even in Theorem 3–5 we do not have to postulate $\mathbf{q} \le 1$. Concerning $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$ and $H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$ when $\mathbf{q} > 1$, we have the following assertion:

Proposition 1 *Let* $1 < p < \infty$ *and* $1 < \mathbf{q} < \infty$ *satisfy*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}.$$

(1) *If* $f \in \mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$, *then* $f \in H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$.
(2) *If* $f \in H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$, *then* f *can be represented by a locally integrable function and the representative belongs to* $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$.

We elaborate a detailed proof of Proposition 1 in Sect. 3.

As an application of Theorem 2, we can reprove the following Olsen inequality about the fractional integral operator I_α, where I_α $(0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, \mathrm{d}y.$$

The following result is known:

Proposition 2 [18, Theorem 1.11] *Suppose that the parameters* $\alpha, p, \mathbf{q}, s, \mathbf{t}$ *satisfy*

$$1 < p < s < \infty, \quad 1 < \mathbf{q} < \mathbf{t} < \infty, \quad \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}$$

and

$$\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s}, \quad \frac{q_j}{p} = \frac{t_j}{s} \quad (j = 1, 2, \ldots, n).$$

Then I_α *is bounded from* $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^s_{\mathbf{t}}(\mathbb{R}^n)$.

Based upon Proposition 2, we can prove the following result.

Theorem 6 *Suppose that the parameters* $\alpha, p, \mathbf{q}, p^*, \mathbf{q}^*, s, \mathbf{t}$ *satisfy*

$$\begin{aligned} &1 < p, p^*, s < \infty, \quad 1 < \mathbf{q}, \mathbf{q}^*, \mathbf{t} < \infty, \\ &\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{p^*} \le \sum_{j=1}^{n} \frac{1}{q^*_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}, \\ &\max\{t_1, \ldots, t_j\} < q^*_j, \quad \frac{1}{p} > \frac{\alpha}{n}, \quad \frac{1}{p^*} \le \frac{\alpha}{n}, \end{aligned} \tag{5}$$

for each $j = 1, 2, \ldots, n$, *and that*

$$\frac{1}{s} = \frac{1}{p^*} + \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t_j}{s} = \frac{q_j}{p} \quad (j = 1, 2, \ldots, n). \tag{6}$$

Then for all $f \in \mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$ *and* $g \in \mathcal{M}^{p^*}_{\mathbf{q}^*}(\mathbb{R}^n)$

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^s_{\mathbf{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}} \cdot \|f\|_{\mathcal{M}^p_{\mathbf{q}}},$$

where the constant C *is independent of* f *and* g.

This result recaptures [23, Proposition 1.8] as the special case of $q_i = q$ and $t_i = t$ for all $i = 1, 2, \ldots, m$. Note that a detailed calculation in [22, p. 6] shows that Theorem 6 is not just a combination of Proposition 2 and Lemma 1.

Lemma 1 *Suppose that the parameters* $p, \mathbf{q}, p^*, \mathbf{q}^*, s, \mathbf{t}$ *satisfy*

$$1 < p, p^*, s < \infty, \quad 1 < \mathbf{q}, \mathbf{q}^*, \mathbf{t} < \infty,$$
$$\frac{n}{p} \le \sum_{j=1}^n \frac{1}{q_j}, \quad \frac{n}{p^*} \le \sum_{j=1}^n \frac{1}{q_j^*}, \quad \frac{n}{s} \le \sum_{j=1}^n \frac{1}{t_j}.$$

Assume

$$\frac{1}{s} = \frac{1}{p^*} + \frac{1}{p}, \quad \frac{1}{t_j} = \frac{1}{q_j^*} + \frac{1}{q_j}.$$

Then

$$\|f \cdot g\|_{\mathcal{M}^s_{\mathbf{t}}} \le \|f\|_{\mathcal{M}^p_{\mathbf{q}}} \|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}} \quad (f \in \mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n), g \in \mathcal{M}^{p^*}_{\mathbf{q}^*}(\mathbb{R}^n)).$$

We can prove this lemma easily by using Hölder's inequality. So we omit the proof. We write $\infty' = 1$ and $s' = \frac{s}{s-1}$ for $1 < s < \infty$. We have the following proposition:

Proposition 3 *In addition to the assumption in Theorem 6, suppose that* $u \in (1, \infty]$ *satisfies* $u' < \min\{q_1, q_2, \ldots, q_n, p\}$. *Let* $\Omega \in L^u(\mathbb{S}^{n-1})$ *be homogeneous of degree zero, that is,* Ω *satisfies, for any* $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. *Then,*

$$\left\|g \cdot I_{\Omega,\alpha}(f)\right\|_{\mathcal{M}^s_{\mathbf{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}} \|\Omega\|_{L^u(\mathbb{S}^{n-1})} \|f\|_{\mathcal{M}^p_{\mathbf{q}}},$$

where

$$I_{\Omega,\alpha} f(x) \equiv \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) \mathrm{d}y.$$

Proposition 3 is a direct consequence of Theorem 6, the next lemma and the boundedness of the Hardy–Littlewood maximal operator M.

Lemma 2 [12] *If* $1 < u \le \infty$, *then we have*

$$|I_{\Omega,\alpha} f(x)| \le C\, \|\Omega\|_{L^u(\mathbb{S}^{n-1})}\, |I_\alpha F(x)|,$$

where $F(x) \equiv M\left(|f|^{u'}\right)(x)^{\frac{1}{u'}}$.

Hardy-mixed Morrey spaces admit a characterization by using the grand maximal operator. To formulate the result, we recall the following two fundamental notions [25].

(1) Topologize $\mathcal{S}(\mathbb{R}^n)$ by norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.

(2) Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| : t > 0,\ \psi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n), \tag{7}$$

where we choose and fix a large integer N.

The following proposition can be proved.

Proposition 4 *Let* $0 < \mathbf{q} < \infty$, $0 < p < \infty$, *and* $\frac{n}{p} \le \sum_{j=1}^n \frac{1}{q_j}$. *Then*

$$\|\mathcal{M}f\|_{\mathcal{M}^p_{\mathbf{q}}} \sim \|f\|_{H\mathcal{M}^p_{\mathbf{q}}}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

When $p \le 1$ and $q_1 = q_2 = \cdots = q_n$, this proposition is contained in [14]. Here for the sake of convenience, we give the proof of Proposition 4 in Sect. 3.

We plan to prove our results in the following manner. First of all, we elaborate the proof of Theorem 1 in Sect. 2. Next, we concentrate on Theorem 2 in Sect. 4.1. Subsequently, based on the argument of the proof of Theorem 2, we prove Theorem 5 in Sect. 4.2. Necessary lemmas for the proofs are stated in each subsection. Finally, Sect. 5 is devoted to the proof of Theorem 6.

2 Proof of Theorem 1

We invoke a result due to Bagby [2].

Lemma 3 *Let* $1 < q_1, \ldots, q_m < \infty$ *and* $1 < p < \infty$. *For* $i = 1, 2 \ldots, m$, *let* (Ω_i, μ_i) *be* σ*-finite measure spaces, and* $\Omega = \Omega_1 \times \cdots \times \Omega_m$. *For* $f \in L^0(\mathbb{R}^n \times \Omega)$,

$$\int_{\mathbb{R}^n} \|Mf(x,\cdot)\|^p_{L^{(q_1,\ldots,q_m)}}\, dx \lesssim \int_{\mathbb{R}^n} \|f(x,\cdot)\|^p_{L^{(q_1,\ldots,q_m)}}\, dx.$$

The following lemma is used in the induction step (see [18, (12)]).

Here and below, for $t > 0$ and $j = 1, 2, \ldots, n$, we denote by M_j the 1-dimensional maximal operator which acts on the j-th variable and write $M_j^{(t)} f = (M_j[|f|^t])^{\frac{1}{t}}$

Lemma 4 *Let* $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in (1, \infty)^n$ *and let*

$$t_n \in [1, \min\{q_1, q_2, \ldots, q_n\}).$$

Then

$$\left\| M_n^{(t_n)} f \right\|_{L^{\mathbf{q}}} \lesssim \|f\|_{L^{\mathbf{q}}}$$

for all $f \in L^{\mathbf{q}}(\mathbb{R}^n)$.

For the proof we use the following notation for $h \in L^0(\mathbb{R}^n)$:

$$\|h\|_{L^{(q_1,\ldots,q_m)}}(x_{m+1}, \ldots, x_n) \equiv \left\| \left[\|h\|_{L^{(q_1,\ldots,q_{m-1})}} \right] \right\|_{L^{(q_m)}}(x_{m+1}, \ldots, x_n)$$

and when $m = 1$, we define

$$\|h\|_{L^{(q_1)}}(x_2, \ldots, x_n) \equiv \left(\int_{\mathbb{R}} |h(x_1, \ldots, x_n)|^{q_1} \mathrm{d}x_1 \right)^{\frac{1}{q_1}}.$$

Proof Thanks to Lemma 3, we obtain

$$\begin{aligned}
\left\| M_n^{(t_n)} f \right\|_{L^{\mathbf{q}}}^{q_n} &= \int_{\mathbb{R}} \left\| M_n^{(t_n)} f(\cdot, x_n) \right\|_{L^{(q_1,\ldots,q_{n-1})}}^{q_n} \mathrm{d}x_n \\
&= \int_{\mathbb{R}} \left\| M_n[|f|^{t_n}](\cdot, x_n) \right\|_{L^{\left(\frac{q_1}{t_n},\ldots,\frac{q_{n-1}}{t_n}\right)}}^{\frac{q_n}{t_n}} \mathrm{d}x_n \\
&\lesssim \int_{\mathbb{R}} \left\| [|f(\cdot, x_n)|^{t_n}] \right\|_{L^{\left(\frac{q_1}{t_n},\ldots,\frac{q_{n-1}}{t_n}\right)}}^{\frac{q_n}{t_n}} \mathrm{d}x_n = \|f\|_{L^{\mathbf{q}}}^{q_n}.
\end{aligned}$$

Thus, we obtain the desired result. □

Proof of of Theorem 1 We start with a preliminary observation for maximal operators. Let $x \in \mathbb{R}^n$. Let $Q = I_1 \times \cdots \times I_n$ where each I_j is an interval in $\mathbb{R}$ with same length. Then,

$$\begin{aligned}
&\frac{\chi_Q(x)}{\|\chi_Q\|_{L^{\mathbf{t}}}} \|f\chi_Q\|_{L^{\mathbf{t}}} = \frac{\bigotimes_{j=1}^n \chi_{I_j}(x)}{\prod_{j=1}^n |I_j|^{\frac{1}{t_j}}} \left\| f\chi_{\prod_{j=1}^n I_j} \right\|_{L^{\mathbf{t}}} \\
&= \frac{\bigotimes_{j=2}^n \chi_{I_j}(x_2, \ldots, x_n)}{\prod_{j=2}^n |I_j|^{\frac{1}{t_j}}} \\
&\quad \times \left\| \left[\left(\frac{\chi_{I_1}(x_1)}{|I_1|} \int |f(y_1, \cdot)|^{t_1} \chi_{I_1}(y_1) \mathrm{d}y_1 \right)^{\frac{1}{t_1}} \right] \chi_{\prod_{j=2}^n I_j} \right\|_{L^{(t_2,\ldots,t_n)}}
\end{aligned}$$

$$\leq \frac{\bigotimes_{j=2}^{n} \chi_{I_j}(x_2, \ldots, x_n)}{\prod_{j=2}^{n} |I_j|^{\frac{1}{t_j}}} \left\| \left[M_1^{(t_1)} f \right] \chi_{\prod\limits_{j=2}^{n} I_j} \right\|_{L^{(t_2,\ldots,t_n)}}.$$

Continuing this procedure, we have

$$\frac{\chi_Q(x)}{\|\chi_Q\|_{L^{\mathbf{t}}}} \|f\chi_Q\|_{L^{\mathbf{t}}} \leq M_n^{(t_n)} \cdots M_1^{(t_1)}(f)(x).$$

Thus, it follows that

$$M^{(\mathbf{t})} f(x) \leq M_n^{(t_n)} \cdots M_1^{(t_1)}(f)(x).$$

Therefore, it suffices to show that

$$\left\| M_n^{(t_n)} \cdots M_1^{(t_1)}(f) \right\|_{L^{\mathbf{q}}} \lesssim \|f\|_{L^{\mathbf{q}}}. \tag{8}$$

We proceed by induction on n. For $n = 1$, the result follows by the classical case of the boundedness of the Hardy–Littlewood maximal operator.

Suppose that the result holds for $n = m - 1$ with $m > 1$ in $\mathbb{N}$: assume that

$$\|M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)} h\|_{L^{(q_1,\ldots,q_{m-1})}} \lesssim \|h\|_{L^{(q_1,\ldots,q_{m-1})}}$$

for $1 < t_k < \min\{q_1, \ldots, q_k\} < \infty$ for each $k = 1, \ldots, m-1$, and for $h \in L^0(\mathbb{R}^{m-1})$. Since $t_m < \min\{q_1, \ldots, q_m\}$, for $g \in L^0(\mathbb{R}^m)$, thanks to Lemma 4 we have

$$\begin{aligned} \left\| M_m^{(t_m)} g \right\|_{L^{(q_1,\ldots,q_m)}} &= \left\| \left[\left\| M_m^{(t_m)} g \right\|_{L^{(q_1,\ldots,q_{m-1})}} \right] \right\|_{L^{(q_m)}} \\ &= \left\| \left[\left\| M_m[|g|^{t_m}] \right\|_{L^{\left(\frac{q_1}{t_m},\ldots,\frac{q_{m-1}}{t_m}\right)}} \right] \right\|_{L^{\left(\frac{q_m}{t_m}\right)}}^{\frac{1}{t_m}} \\ &\lesssim \left\| \left[\|g\|_{L^{(q_1,\ldots,q_{m-1})}} \right] \right\|_{L^{(q_m)}} = \|g\|_{L^{(q_1,\ldots,q_m)}}. \end{aligned}$$

Thus, by the induction assumption, letting $g = M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f)$ in the above, we obtain

$$\begin{aligned} \left\| M_m^{(t_m)} \cdots M_1^{(t_1)}(f) \right\|_{L^{(q_1,\ldots,q_m)}} &= \left\| M_m^{(t_m)} \left[M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f) \right] \right\|_{L^{(q_1,\ldots,q_m)}} \\ &\lesssim \left\| M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f) \right\|_{L^{(q_1,\ldots,q_m)}} \\ &= \left\| \left\| M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f) \right\|_{L^{(q_1,\ldots,q_{m-1})}} \right\|_{L^{(q_m)}} \\ &\lesssim \left\| \|f\|_{L^{(q_1,\ldots,q_{m-1})}} \right\|_{L^{(q_m)}} \lesssim \|f\|_{L^{(q_1,\ldots,q_m)}}. \end{aligned}$$

Hence, inequality (8) holds for any dimension n. We obtain the desired result. □

One can show that the condition

$$t_k < \min\{q_1, q_2, \dots, q_k\}$$

is sharp.

Proposition 5 *In Theorem 1, for each $k = 1, 2, \dots, n$, the condition $t_k < \min\{q_1, q_2, \dots, q_k\}$ can not be removed.*

Proof We induct on n. The base case $n = 1$ is clear since the Hardy–Littlewood maximal operator is bounded on $L^p(\mathbb{R})$ if and only if $p > 1$. Assume that the conclusion of Proposition 5 is true for $n = m - 1$ and that $M^{(t_1,t_2,\dots,t_m)}$ is bounded on $L^{(q_1,q_2,\dots,q_m)}(\mathbb{R}^m)$. Let $h \in L^{(t_1,t_2,\dots,t_{m-1})}(\mathbb{R}^{m-1})$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}&\chi_{[-N,N]^m} \left(M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h \right] \otimes \chi_{[-N,N]} \right)\\ &\qquad \le M^{(t_1,t_2,\dots,t_m)} \left[(\chi_{[-N,N]^{m-1}} h) \otimes \chi_{[-N,N]} \right].\end{aligned}$$

Consequently,

$$\begin{aligned}&(2N)^{\frac{1}{q_m}} \left\| \chi_{[-N,N]^{m-1}} M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h \right] \right\|_{L^{(q_1,q_2,\dots,q_{m-1})}}\\ &= \left\| \chi_{[-N,N]^m} M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h \right] \otimes \chi_{[-N,N]} \right\|_{L^{(q_1,q_2,\dots,q_m)}}\\ &\le \left\| M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h \right] \otimes \chi_{[-N,N]} \right\|_{L^{(q_1,q_2,\dots,q_m)}}\\ &\le C \left\| \left(\chi_{[-N,N]^{m-1}} h \right) \otimes \chi_{[-N,N]} \right\|_{L^{(q_1,q_2,\dots,q_m)}}\\ &\le C (2N)^{\frac{1}{q_m}} \|h\|_{L^{(q_1,q_2,\dots,q_{m-1})}}.\end{aligned}$$

So, we are led to

$$\begin{aligned}&\left\| \chi_{[-N,N]^{m-1}} M^{(t_1,t_2,\dots,t_{m-1})} [\chi_{[-N,N]^{m-1}} h] \right\|_{L^{(q_1,q_2,\dots,q_{m-1})}}\\ &\quad \le C \|h\|_{L^{(q_1,q_2,\dots,q_{m-1})}}.\end{aligned}$$

Letting $N \to \infty$, we obtain

$$\left\| M^{(t_1,t_2,\dots,t_{m-1})} h \right\|_{L^{(q_1,q_2,\dots,q_{m-1})}} \le C \|h\|_{L^{(q_1,q_2,\dots,q_{m-1})}}.$$

By the induction assumption, we have $t_k < \min\{q_1, q_2, \dots, q_k\}$ for all $k = 1, 2, \dots, m - 1$. If we start from the inequality

$$\begin{aligned}&\chi_{[-N,N]^m} \left(\chi_{[-N,N]} \otimes M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h \right] \right)\\ &\quad \le M^{(t_1,t_2,\dots,t_m)} \left[\chi_{[-N,N]} \otimes \left(\chi_{[-N,N]^{m-1}} h \right) \right],\end{aligned}$$

and argue similarly, we obtain

$$\left\| M^{(t_2,t_3,\ldots,t_m)} h \right\|_{L^{(q_2,q_3,\ldots,q_m)}} \le C \|h\|_{L^{(q_2,q_3,\ldots,q_m)}}.$$

Thus $t_m < \min(q_2, q_3, \ldots, q_m)$ by the induction assumption. It remains to show that $t_m < q_1$. To this end, we consider the function of the form:

$$f(x_1, x_2, \ldots, x_m) = \sum_{j=-\infty}^{\infty} \chi_{([jN,(j+1)N]\times[-N,N]^{m-1})}(x_1, x_2, \ldots, x_m) h_j(x_m),$$

where $h_j \in L^{q_m}(\mathbb{R})$. Then for all $(x_1, x_2, \ldots, x_m)$

$$\begin{aligned} &\chi_{(\mathbb{R}\times[-N,N]^{m-1})}(x_1, x_2, \ldots, x_m) M^{(\mathbf{t})} f(x_1, x_2, \ldots, x_m) \\ &\quad \ge \sum_{j=-\infty}^{\infty} \chi_{([jN,(j+1)N]\times[-N,N]^{m-1})}(x_1, x_2, \ldots, x_m) M^{(t_m)} \left[\chi_{[-N,N]} h_j\right](x_m). \end{aligned}$$

We abbreviate

$$H_m(x) \equiv M^{(t_m)} \left[\chi_{[-N,N]} h_j\right](x_m).$$

Hence, we obtain

$$\begin{aligned} &\left\| \sum_{j=-\infty}^{\infty} \chi_{([jN,(j+1)N]\times[-N,N]^{m-1})} M^{(t_m)} \left[\chi_{[-N,N]} h_j\right] \right\|_{L^{\mathbf{q}}} \\ &= \left\| \left(\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} \chi_{[jN,(j+1)N]} \left(H_m(\cdot_m)\right)^{q_1} \mathrm{d}x_1 \right)^{\frac{1}{q_1}} \chi_{[-N,N]^{m-1}} \right\|_{L^{(q_2,\ldots,q_m)}} \\ &\sim (2N)^{\frac{1}{q_1}+\cdots+\frac{1}{q_{m-1}}} \left\| \left(\sum_{j=-\infty}^{\infty} \left(H_m(\cdot_m)\right)^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^{q_m}}. \end{aligned}$$

In the same way, we deduce

$$\begin{aligned} &\left\| \chi_{(\mathbb{R}\times[-N,N]^{m-1})} M^{(\mathbf{t})} f \right\|_{L^{\mathbf{q}}} \\ &\quad \lesssim (2N)^{\frac{1}{q_1}+\cdots+\frac{1}{q_{m-1}}} \left\| \left(\sum_{j=-\infty}^{\infty} \left(|\chi_{[-N,N]} h_j(\cdot_m)|\right)^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^{q_m}}, \end{aligned}$$

since $M^{(\mathbf{t})}$ is bounded. Thus, letting $N \to \infty$, we obtain

$$\left\| \{M^{(t_m)} h_j\}_{j=-\infty}^{\infty} \right\|_{L^{q_m}(\ell^{q_1})} \le \left\| \{h_j\}_{j=-\infty}^{\infty} \right\|_{L^{q_m}(\ell^{q_1})}.$$

This forces $q_1 > t_m$ (see [25, p. 75, § 5.1]). □

3 Proof of Propositions 1 and 4

3.1 Proof of Proposition 1

To prove Proposition 1, we need the description of the (pre) dual spaces of mixed Morrey spaces [19]. Recall that when $1 < p < \infty$ and $1 < \mathbf{q} < \infty$ satisfy

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j},$$

then the predual space $\mathcal{H}_{\mathbf{q}'}^{p'}(\mathbb{R}^n)$ of the mixed Morrey space $\mathcal{M}_{\mathbf{q}}^{p}(\mathbb{R}^n)$ is given by

$$\mathcal{H}_{\mathbf{q}'}^{p'}(\mathbb{R}^n) = \left\{ g = \sum_{j=1}^{\infty} \mu_j b_j \ : \ \{\mu_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}), \ \text{ each } b_j \text{ is a } (p', \mathbf{q}')\text{-block} \right\}.$$

Here by "a $(p', \mathbf{q}')$-block" we mean an $L^{\mathbf{q}'}(\mathbb{R}^n)$-function supported on a cube Q with $L^{\mathbf{q}'}(\mathbb{R}^n)$-norm lesser or equal to $|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q'_j} - \frac{1}{p'}}$. The norm of $\mathcal{H}_{\mathbf{q}'}^{p'}(\mathbb{R}^n)$ is defined by

$$\|g\|_{\mathcal{H}_{\mathbf{q}'}^{p'}} = \inf \sum_{j=1}^{\infty} |\mu_j|,$$

where inf is over all admissible expressions above. A fundamental fact about this space is that $\mathcal{H}_{\mathbf{q}'}^{p'}(\mathbb{R}^n)$ is separable, that the dual of $\mathcal{H}_{\mathbf{q}'}^{p'}(\mathbb{R}^n)$ is cannonically identified with $\mathcal{M}_{\mathbf{q}}^{p}(\mathbb{R}^n)$ and that

$$\|f\|_{\mathcal{M}_{\mathbf{q}}^{p}} = \sup \left\{ \|f \cdot g\|_{L^1} \ : \ \|g\|_{\mathcal{H}_{\mathbf{q}'}^{p'}} = 1 \right\}.$$

Proposition 1 was investigated by Long [16] and Zorko [29] when $q_j = q$ for all $j = 1, \ldots, n$; see [15] as well. We refer to [1], [9], and [19] for more recent characterizations of the predual spaces.

Example 1 Suppose that $1 \le t'_k < \min(q'_1, q'_2, \ldots, q'_k) < \infty$. If we let κ be the operator norm of the maximal operator $M^{(\mathbf{t}')}$ on $L^{\mathbf{q}'}(\mathbb{R}^n)$, whose finiteness is guaranteed by

Theorem 1, then we obtain $\kappa^{-1}\chi_Q M^{(\mathbf{t}')}g$ is a $(p', \mathbf{q}')$-block modulo a multiplicative constant for any $(p', \mathbf{q}')$-block g. Indeed, it is supported on a cube Q and it satisfies

$$\left\|\kappa^{-1}\chi_Q M^{(\mathbf{t}')}g\right\|_{L^{\mathbf{q}'}} \le \|\chi_Q g\|_{L^{\mathbf{q}'}} = \|g\|_{L^{\mathbf{q}'}} \le |Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q'_j}-\frac{1}{p'}}.$$

Proof of Proposition 1 (1) Denote by $B(R) = \{x \in \mathbb{R}^n : |x| < R\}$ for $R > 0$. Since

$$\|f\|_{L^1(B(R))} \le CR^{-\frac{n}{p}+n}\|f\|_{\mathcal{M}^p_{\mathbf{q}}},$$

we have $f \in \mathcal{S}'(\mathbb{R}^n)$. As is described in [6], we have a pointwise estimate $|e^{t\Delta}f| \le Mf$, where M denotes the Hardy–Littlewood maximal operator. Since M is shown to be bounded in [4], we have $f \in H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$.

(2) Let $f \in H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$. Then $\{e^{t\Delta}f\}_{t>0}$ is a bounded set of $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$, which admits a separable predual as we have seen. Therefore, there exists a sequence $\{t_j\}_{j=1}^{\infty}$ decreasing to 0 such that $\{e^{t_j\Delta}f\}_{j=1}^{\infty}$ converges to a function g in the weak-* topology of $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$. Meanwhile, it can be shown that $\lim_{t\downarrow 0} e^{t\Delta}f = f$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$ [21]. Since the weak-* topology of $\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$ is stronger than the topology of $\mathcal{S}'(\mathbb{R}^n)$, it follows that $f = g \in \mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$.

3.2 Proof of Proposition 4

The proof is similar to Hardy spaces with variable exponents [5,17]. We content ourselves with stating two fundamental estimates (13) and (14).

We define the (discrete) maximal function with respect to $e^{t\Delta}$ by

$$M_{\mathrm{heat}}f(x) \equiv \sup_{j\in\mathbb{Z}} |e^{2^j\Delta}f(x)| \quad (x \in \mathbb{R}^n). \tag{9}$$

Recall that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function is defined by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| : t > 0,\ \psi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n), \tag{10}$$

where $\mathcal{F}_N$ is given by

$$\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}. \tag{11}$$

Suppose that we are given an integer $K \gg 1$. We write

$$M^*_{\mathrm{heat}}f(x) \equiv \sup_{j\in\mathbb{Z}}\left(\sup_{y\in\mathbb{R}^n}\frac{|e^{2^j\Delta}f(y)|}{(1+4^j|x-y|^2)^K}\right) \quad (x \in \mathbb{R}^n). \tag{12}$$

The next lemma connects M^*_{heat} with M_{heat} in terms of the usual Hardy–Littlewood maximal function M.

Lemma 5 ([17, Lemma 3.2], [20, §4]) *For* $0 < \theta < 1$, *there exists* K_θ *so that for all* $K \geq K_\theta$, *we have*

$$M^*_{\rm heat} f(x) \leq C M^{(\theta)}[M_{\rm heat} f](x) = C M\left[\sup_{k\in\mathbb{Z}} |e^{2^k\Delta} f|^\theta\right](x)^{\frac{1}{\theta}} \quad (x \in \mathbb{R}^n) \quad (13)$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$, *where* $M^{(\theta)}$ *is the powered maximal operator given by*

$$M^{(\theta)} g(x) \equiv M[|g|^\theta](x)^{\frac{1}{\theta}} \quad (x \in \mathbb{R}^n)$$

for measurable functions g.

In the course of the proof of [17, Theorem 3.3], it can be shown that

$$\mathcal{M} f(x) \sim \sup_{\tau\in\mathcal{F}_N,\, j\in\mathbb{Z}} |\tau^j * f(x)| \lesssim M^*_{\rm heat} f(x) \quad (14)$$

once we fix an integer $K \gg 1$ and $N \gg 1$.

With the fundamental pointwise estimates (13) and (14), Proposition 4 can be proved with ease. We omit the details.

4 Proofs of Theorems 2–5

4.1 Proof of Theorem 2

By decomposing Q_j suitably, we may suppose each Q_j is dyadic.

To prove this, we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \geq N$. Let us assume in addition that a_j are non-negative. Fix a non-negative $(p', \mathbf{q}')$-block $g \in \mathcal{H}^{p'}_{\mathbf{q}'}(\mathbb{R}^n)$ with the associated cube Q.

Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that Q_j is the jth dyadic parent of Q for each $j \in \mathbb{N}$. Then by the Hölder inequality [3]

$$\int_{\mathbb{R}^n} f(x)g(x)\,{\rm d}x = \sum_{j=1}^\infty \lambda_j \int_Q a_j(x)g(x)\,{\rm d}x \leq \sum_{j=1}^\infty \lambda_j \|a_j\|_{L^{\mathbf{q}}(Q)} \|g\|_{L^{\mathbf{q}'}(Q)}$$

from $f = \sum_{j=1}^\infty \lambda_j a_j$. Due to the size condition of a_j and g, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,{\rm d}x &\leq \sum_{j=1}^\infty \lambda_j |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{1}{s}} |Q_j|^{\frac{1}{s}} |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q'_j} - \frac{1}{p'}} \\ &\leq \sum_{j=1}^\infty \lambda_j |Q|^{\frac{1}{p} - \frac{1}{s}} |Q_j|^{\frac{1}{s}}.\end{aligned}$$

Note that

$$\left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\mathbf{q}}^p} \geq \left\| \lambda_{j_0} \chi_{Q_{j_0}} \right\|_{\mathcal{M}_{\mathbf{q}}^p} = |Q_{j_0}|^{\frac{1}{p}} \lambda_{j_0}$$

for each j_0. Consequently, it follows from the condition $p < s$ that

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \sum_{j=1}^{\infty} |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_j|^{\frac{1}{s}-\frac{1}{p}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\mathbf{q}}^p} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\mathbf{q}}^p}.$$

Conversely assume that Q contains each Q_j. Then by the Hölder inequality

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x = \sum_{j=1}^{\infty} \lambda_j \int_{Q_j} a_j(x)g(x)\,\mathrm{d}x \leq \sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^{\mathbf{t}}(Q_j)} \|g\|_{L^{\mathbf{t}'}(Q_j)}.$$

Thanks to the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \sum_{j=1}^{\infty} \lambda_j |Q_j|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{t_j}-\frac{1}{s}} |Q_j|^{\frac{1}{s}} \|g\|_{L^{\mathbf{t}'}(Q_j)}.$$

Thus, in terms of the maximal operator $M^{(\mathbf{t}')}$ defined in Theorem 1, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x &\leq \sum_{j=1}^{\infty} \lambda_j |Q_j| \times \inf_{y \in Q_j} M^{(\mathbf{t}')} g(y) \\
&\lesssim \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(y) \right) M^{(\mathbf{t}')} g(y)\,\mathrm{d}y \\
&\leq \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(y) \right) \chi_Q(y) M^{(\mathbf{t}')} g(y)\,\mathrm{d}y.
\end{aligned}$$

Hence, by Example 1, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \kappa \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\mathbf{q}}^p}.$$

This is the desired result. Finally, we can remove the assumption that $\lambda_j = 0$ for large j by the monotone convergence theorem. Thus, the proof is complete.

4.2 Proof of Theorem 5

Recall again that the grand maximal operator $\mathcal{M}$ was given by

$$\mathcal{M}f(x) = \sup\{|\varphi_t * f(x)| : \varphi \in \mathcal{F}_N,\ t > 0\} \quad (x \in \mathbb{R}^n).$$

Then we know that

$$\mathcal{M}a_j(x) \le C\left(\chi_{3Q_j}(x)Ma_j(x) + (M\chi_{Q_j}(x))^{\frac{n+d_q+1}{n}}\right), \tag{15}$$

where $d_q = \left[n\left(\frac{1}{v(\mathbf{q})} - 1\right)\right]$ and $v(\mathbf{q}) = \min(1, q_1, \ldots, q_n)$. See [17, (5.2)] for more details. The first term can be controlled by an argument similar to Theorem 2. The second term can be handled by using the Fefferman–Stein maximal inequality for mixed Morrey spaces [18].

Proposition 6 *Let* $1 < \mathbf{q}, p < \infty$, $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$, *and* $1 < r \le \infty$. *Then*

$$\left\| \left(\sum_{j=1}^{\infty} (Mf_j)^r \right)^{1/r} \right\|_{\mathcal{M}^p_{\mathbf{q}}} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right\|_{\mathcal{M}^p_{\mathbf{q}}}$$

for all sequences of measurable functions $\{f_j\}_{j=1}^{\infty}$.

See [24, Theorem 2.2], [27, Lemma 2.5] for the case of classical Morrey spaces.

Let us show Theorem 5. Using Proposition 4 and (15), we have

$$\begin{aligned}
\|f\|_{H\mathcal{M}^p_{\mathbf{q}}} \sim \|\mathcal{M}f\|_{\mathcal{M}^p_{\mathbf{q}}} &\le \left\| \sum_{j=1}^{\infty} \lambda_j \mathcal{M}a_j \right\|_{\mathcal{M}^p_{\mathbf{q}}} \\
&\lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \left(\chi_{3Q_j} Ma_j + (M\chi_{Q_j})^{\frac{n+d_q+1}{n}} \right) \right\|_{\mathcal{M}^p_{\mathbf{q}}} \\
&\lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{3Q_j} Ma_j \right\|_{\mathcal{M}^p_{\mathbf{q}}} + \left\| \sum_{j=1}^{\infty} \lambda_j (M\chi_{Q_j})^{\frac{n+d_q+1}{n}} \right\|_{\mathcal{M}^p_{\mathbf{q}}} \equiv I_1 + I_2.
\end{aligned}$$

First, we consider I_1. The proof is similar to Theorem 2. For the sake of completeness, we supply the proof. Thanks to decomposing Q_j suitably, we may suppose each Q_j is dyadic. We will use duality again. We assume that there exists $N \in \mathbb{N}$ such that

$\lambda_j = 0$ whenever $j \ge N$. Let $r = \dfrac{p}{v(\mathbf{q})}$ and $\mathbf{w} = \dfrac{\mathbf{q}}{v(\mathbf{q})}$, so that $r, \mathbf{w} > 1$. Then,

$$\left\| \sum_{j=1}^{\infty} \lambda_j \chi_{3Q_j} M a_j \right\|_{\mathcal{M}^p_{\mathbf{q}}} \le \left\| \left(\sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j} M a_j \right]^{v(\mathbf{q})} \right)^{\frac{1}{v(\mathbf{q})}} \right\|_{\mathcal{M}^p_{\mathbf{q}}}$$

$$= \left\| \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j} M a_j \right]^{v(\mathbf{q})} \right\|_{\mathcal{M}^r_{\mathbf{w}}}^{\frac{1}{v(\mathbf{q})}}.$$

Fix a non-negative $(r', \mathbf{w}')$-block $g \in \mathcal{H}^{r'}_{\mathbf{w}'}(\mathbb{R}^n)$ with the associated cube Q. Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that Q_j is the jth dyadic parent of Q for each $j \in \mathbb{N}$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x) \right]^{v(\mathbf{q})} g(x) \mathrm{d}x = \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \int_Q \left[M a_j(x) \right]^{v(\mathbf{q})} g(x) \mathrm{d}x \\
&\le \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left\| \left[M a_j \right]^{v(\mathbf{q})} \right\|_{L^{\mathbf{w}}(Q)} \|g\|_{L^{\mathbf{w}'}(Q)} \\
&\le \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \| M a_j \|_{L^{\mathbf{q}}(Q)}^{v(\mathbf{q})} \|g\|_{L^{\mathbf{w}'}(Q)} \\
&\le \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left[\| M a_j \|_{L^{\mathbf{t}}(Q)} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{q_j} - \frac{1}{t_j} \right)} \right]^{v(\mathbf{q})} \|g\|_{L^{\mathbf{w}'}(Q)}.
\end{aligned}$$

Using the boundedness of the Hardy–Littlewood maximal operator on $\mathcal{M}^s_{\mathbf{t}}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x) \right]^{v(\mathbf{q})} g(x) \mathrm{d}x \\
&\le \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left[\| M a_j \|_{L^{\mathbf{t}}(Q)} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{q_j} - \frac{1}{t_j} \right)} \right]^{v(\mathbf{q})} \|g\|_{L^{\mathbf{w}'}(Q)} \\
&\le \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{w_j}} \left[|Q|^{-\frac{1}{s}} \| M a_j \|_{\mathcal{M}^s_{\mathbf{t}}} \right]^{v(\mathbf{q})} \|g\|_{L^{\mathbf{w}'}(Q)} \\
&\lesssim \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{w_j}} \left[|Q|^{-\frac{1}{s}} \| a_j \|_{\mathcal{M}^s_{\mathbf{t}}} \right]^{v(\mathbf{q})} \|g\|_{L^{\mathbf{w}'}(Q)}.
\end{aligned}$$

Thus, using the size condition of a_j and g, we obtain

$$\left(\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\left[\lambda_j\chi_{3Q_j}(x)Ma_j(x)\right]^{v(\mathbf{q})}g(x)\mathrm{d}x\right)^{\frac{1}{v(\mathbf{q})}}$$

$$\lesssim\left(\sum_{j=1}^{\infty}\lambda_j^{v(\mathbf{q})}|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{w_j}}\left[|Q|^{-\frac{1}{s}}|Q_j|^{\frac{1}{s}}\right]^{v(\mathbf{q})}|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{w_j'}-\frac{1}{r'}}\right)^{\frac{1}{v(\mathbf{q})}}$$

$$=|Q|^{\frac{1}{p}-\frac{1}{s}}\left(\sum_{j=1}^{\infty}\left[\lambda_j|Q_j|^{\frac{1}{s}}\right]^{v(\mathbf{q})}\right)^{\frac{1}{v(\mathbf{q})}}.$$

Note that

$$\left\|\left(\sum_{j=1}^{\infty}(\lambda_j\chi_{Q_j})^{v(\mathbf{q})}\right)^{\frac{1}{v(\mathbf{q})}}\right\|_{\mathcal{M}_{\mathbf{q}}^p}\geq\left\|\lambda_{j_0}\chi_{Q_{j_0}}\right\|_{\mathcal{M}_{\mathbf{q}}^p}=\lambda_{j_0}|Q_{j_0}|^{\frac{1}{p}}$$

for each $j_0\in\mathbb{N}$. Thus,

$$\left(\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\left[\lambda_j\chi_{3Q_j}(x)Ma_j(x)\right]^{v(\mathbf{q})}g(x)\mathrm{d}x\right)^{\frac{1}{v(\mathbf{q})}}$$

$$\lesssim\sum_{k=1}^{\infty}|Q|^{\frac{1}{p}-\frac{1}{s}}|Q_k|^{\frac{1}{s}-\frac{1}{p}}\left\|\left(\sum_{j=1}^{\infty}(\lambda_j\chi_{Q_j})^{v(\mathbf{q})}\right)^{\frac{1}{v(\mathbf{q})}}\right\|_{\mathcal{M}_{\mathbf{q}}^p}$$

$$\sim\left\|\left(\sum_{j=1}^{\infty}(\lambda_j\chi_{Q_j})^{v(\mathbf{q})}\right)^{\frac{1}{v(\mathbf{q})}}\right\|_{\mathcal{M}_{\mathbf{q}}^p}.$$

Conversely assume that Q contains each Q_j. Then by the Hölder inequality and the boundedness of the Hardy–Littlewood maximal operator on $L^{\mathbf{t}}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\left[\lambda_j\chi_{3Q_j}(x)Ma_j(x)\right]^{v(\mathbf{q})}g(x)\mathrm{d}x$$

$$=\sum_{j=1}^{\infty}\lambda_j^{v(\mathbf{q})}\int_{3Q_j}\left[Ma_j(x)\right]^{v(\mathbf{q})}g(x)\mathrm{d}x$$

$$\leq\sum_{j=1}^{\infty}\lambda_j^{v(\mathbf{q})}\left\|\left[Ma_j\right]^{v(\mathbf{q})}\right\|_{L^{\mathbf{ø}}(3Q_j)}\|g\|_{L^{\mathbf{ø}'}(3Q_j)}\quad\left(\mathbf{ø}=\frac{\mathbf{t}}{v(\mathbf{q})}\right)$$

$$\leq \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left\| M a_j \right\|_{L^{\mathbf{t}}(3Q_j)}^{v(\mathbf{q})} \|g\|_{L^{\mathbf{ø}'}(3Q_j)}$$

$$\leq \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left\| a_j \right\|_{L^{\mathbf{t}}}^{v(\mathbf{q})} \|g\|_{L^{\mathbf{ø}'}(3Q_j)}.$$

Considering the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\mathbf{q})} g(x) \mathrm{d}x$$

$$\lesssim \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left[|Q_j|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{t_j}-\frac{1}{s}} \left\| a_j \right\|_{\mathcal{M}_{\mathbf{t}}^{s}} \right]^{v(\mathbf{q})} \|g\|_{L^{\mathbf{ø}'}(3Q_j)}$$

$$\leq \sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} \left[|Q_j|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{t_j}} \right]^{v(\mathbf{q})} \|g\|_{L^{\mathbf{ø}'}(3Q_j)}.$$

Thus, in terms of the maximal operator $M^{(\mathbf{t}')}$ defined in Theorem 1, we obtain

$$\left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\mathbf{q})} g(x) \mathrm{d}x\right)^{\frac{1}{v(\mathbf{q})}}$$

$$\leq \left(\sum_{j=1}^{\infty} \lambda_j^{v(\mathbf{q})} |Q_j| \times \inf_{y \in Q_j} M^{(\mathbf{ø}')} g(y)\right)^{\frac{1}{v(\mathbf{q})}}$$

$$\leq \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j}(y)\right)^{v(\mathbf{q})}\right) M^{(\mathbf{ø}')} g(y)\, \mathrm{d}y\right)^{\frac{1}{v(\mathbf{q})}}$$

$$\leq \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j}(y)\right)^{v(\mathbf{q})}\right) \chi_Q(y) M^{(\mathbf{ø}')} g(y)\, \mathrm{d}y\right)^{\frac{1}{v(\mathbf{q})}}.$$

As in Example 1, $\kappa^{-1} \chi_Q M^{(\mathbf{ø}')} g$ is a $(r', \mathbf{w}')$-block as long as κ is the operator norm of $M^{(\mathbf{ø}')}$ on $L^{\mathbf{q}'}(\mathbb{R}^n)$. Hence, we obtain

$$\left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\mathbf{q})} g(x) \mathrm{d}x\right)^{\frac{1}{v(\mathbf{q})}}$$

$$\lesssim \kappa^{\frac{1}{v(\mathbf{q})}} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\mathbf{q})}\right) \right\|_{\mathcal{M}_{\mathbf{w}}^{r}}^{\frac{1}{v(\mathbf{q})}} = \kappa^{\frac{1}{v(\mathbf{q})}} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\mathbf{q})}\right)^{\frac{1}{v(\mathbf{q})}} \right\|_{\mathcal{M}_{\mathbf{q}}^{p}}.$$

Next, we consider I_2. Put

$$u = \frac{n + d_q + 1}{n} p, \quad \mathbf{v} = \frac{n + d_q + 1}{n} \mathbf{q}.$$

Then, by Proposition 6 and the embedding $\ell^{v(\mathbf{q})} \hookrightarrow \ell^1$, we have

$$I_2 = \left\| \left[\sum_{j=1}^{\infty} \lambda_j (M\chi_{Q_j})^{\frac{n+d_q+1}{n}} \right]^{\frac{n}{n+d_q+1}} \right\|_{\mathcal{M}^u_{\mathbf{v}}}^{\frac{n+d_q+1}{n}} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\mathbf{q}}}$$
$$\leq \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\mathbf{q})} \right)^{\frac{1}{v(\mathbf{q})}} \right\|_{\mathcal{M}^p_{\mathbf{q}}}.$$

Thus, we obtain the desired result.

4.3 Proof of Theorem 4

We outline the proof of Theorem 4 since this is similar to [13]. As in [21, Exercise 3.34], if $0 < r < 1$ and $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $f \in \mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$, then we can find $\{a_j\}_{j=1}^{\infty} \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_L^{\perp}(\mathbb{R}^n)$ and a sequence $\{Q_j\}_{j=1}^{\infty}$ of cubes:

(1) $\mathrm{supp}(a_j) \subset Q_j$,
(2) $f = \sum_{j=1}^{\infty} a_j$ in $\mathcal{S}'(\mathbb{R}^n)$,
(3) $\left\{ \sum_{j=1}^{\infty} (\|a_j\|_{L^\infty} \chi_{Q_j})^r \right\}^{\frac{1}{r}} \lesssim \mathcal{M} f$.

Using this inequality, we can prove Theorem 4.

Proof of Theorem 4 Let $f \in H\mathcal{M}^p_{\mathbf{q}}(\mathbb{R}^n)$. Then we consider the decomposition:

$$e^{t\Delta} f = \sum_{Q \in \mathcal{D}} \lambda^t_Q a^t_Q$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$, where $a^t_Q \in \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\lambda^t_Q \geq 0$ and

$$|a^t_Q| \leq \chi_{3Q}, \quad \left\| \sum_{Q \in \mathcal{D}} \lambda^t_Q \chi_{3Q} \right\|_{\mathcal{M}^p_{\mathbf{q}}} \lesssim \|\mathcal{M}[e^{t\Delta} f]\|_{\mathcal{M}^p_{\mathbf{q}}} \lesssim \|\mathcal{M} f\|_{\mathcal{M}^p_{\mathbf{q}}}.$$

Due to the weak-* compactness of the unit ball of $L^\infty(\mathbb{R}^n)$, there exists a sequence $\{t_l\}_{l=1}^{\infty}$ that converges to 0 such that

$$\lambda_Q = \lim_{l \to \infty} \lambda^{t_l}_Q, \quad a_Q = \lim_{l \to \infty} a^{t_l}_Q$$

exist for all $Q \in \mathcal{D}$ in the sense that

$$\lim_{l \to \infty} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^n} a_Q(x)\varphi(x)\mathrm{d}x$$

for all $\varphi \in L^1(\mathbb{R}^n)$. We claim

$$f = \sum_{Q \in \mathcal{D}} \lambda_Q a_Q$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a test function. Then we have

$$\langle f, \varphi \rangle = \lim_{l \to \infty} \langle e^{t_l \Delta} f, \varphi \rangle = \lim_{l \to \infty} \sum_{Q \in \mathcal{D}} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x$$

from the definition of the convergence in $\mathcal{S}'(\mathbb{R}^n)$. Once we fix m, we have

$$|\lambda_Q^{t_l}| \lesssim \frac{\|\mathcal{M} f\|_{\mathcal{M}_{\mathbf{q}}^p}}{\|\chi_{[0,2^{-m})^n}\|_{\mathcal{M}_{\mathbf{q}}^p}} \tag{16}$$

and

$$\left| \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x \right| \le \int_{3Q} |\varphi(x)|\mathrm{d}x.$$

Since

$$\sum_{Q \in \mathcal{D}_m} \frac{\|\mathcal{M} f\|_{\mathcal{M}_{\mathbf{q}}^p}}{\|\chi_{[0,2^{-m})^n}\|_{\mathcal{M}_{\mathbf{q}}^p}} \int_{3Q} |\varphi(x)|\mathrm{d}x = 3^n \frac{\|\mathcal{M} f\|_{\mathcal{M}_{\mathbf{q}}^p}}{\|\chi_{[0,2^{-m})^n}\|_{\mathcal{M}_{\mathbf{q}}^p}} \|\varphi\|_{L^1} < \infty,$$

we are in the position of using the Fubini theorem to have

$$\sum_{m \in \mathbb{Z}} \int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} a_Q^{t_l}(x) \right) \varphi(x)\mathrm{d}x = \sum_{m \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x.$$

With this in mind, let us set

$$a_{m,l} \equiv \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x$$

for each $m \in \mathbb{Z}$ and $l \in \mathbb{N}$. Then we have

$$|a_{m,l}| \le C 2^{\frac{nm}{p}} \|\mathcal{M} f\|_{\mathcal{M}_{\mathbf{q}}^p} \|\varphi\|_{L^1} \quad (m \in \mathbb{Z}) \tag{17}$$

thanks to (16).

Let $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} a_{m,l} &= \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{3Q} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x \\ &= \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{3Q} a_Q^{t_l}(x) \left(\varphi(x) - \sum_{|\beta| \le K} \frac{1}{\beta!} \partial^\beta \varphi(c(Q))(x - c(Q))^\beta \right) \mathrm{d}x \end{aligned}$$

since $a_Q^{t_l} \in \mathcal{P}_K^\perp(\mathbb{R}^n)$. Thus, by the mean-value theorem, we have

$$|a_{m,l}| \le C(\varphi) \sum_{Q \in \mathcal{D}_m} |\lambda_Q^{t_l}| \ell(Q)^{n+K+1} \sup_{y \in 3Q} \frac{1}{1+|y|^{n+1}}. \tag{18}$$

Here $C(\varphi)$ is a constant depending on φ.

Meanwhile, for each $\tilde{m} \in \mathbb{Z}^n$, we have

$$\left\| \sum_{Q \in \mathcal{D}_m, |c(Q)-\tilde{m}| \le n} \lambda_Q^{t_l} \chi_Q \right\|_{\mathcal{M}^p_{q_0}} \lesssim \|\mathcal{M} f\|_{\mathcal{M}^p_{\mathbf{q}}},$$

which implies

$$\left\| \sum_{Q \in \mathcal{D}_m, |c(Q)-\tilde{m}| \le n} \lambda_Q^{t_l} \chi_Q \right\|_{L^{q_0}} \lesssim \|\mathcal{M} f\|_{\mathcal{M}^p_{\mathbf{q}}}$$

or equivalently

$$\left(\sum_{Q \in \mathcal{D}_m, |c(Q)-\tilde{m}| \le n} 2^{-mn} |\lambda_Q^{t_l}|^{q_0} \right)^{\frac{1}{q_0}} \lesssim \|\mathcal{M} f\|_{\mathcal{M}^p_{\mathbf{q}}}.$$

Since $\ell^{q_0}(\mathbb{Z}^n) \hookrightarrow \ell^1(\mathbb{Z}^n)$,

$$\sum_{Q \in \mathcal{D}_m, |c(Q)-\tilde{m}| \le n} |\lambda_Q^{t_l}| \lesssim 2^{\frac{mn}{q_0}} \|\mathcal{M} f\|_{\mathcal{M}^p_{\mathbf{q}}}.$$

Combining this estimate with (18), we obtain

$$|a_{m,l}| \lesssim \sum_{\tilde{m} \in \mathbb{Z}^n} \sum_{Q \in \mathcal{D}_m, |c(Q)-\tilde{m}| \le n} |\lambda_Q^{t_l}| \ell(Q)^{n+K+1} \sup_{y \in 3Q} \frac{1}{1+|y|^{n+1}}$$

$$
\begin{aligned}
&\sim \sum_{\tilde{m}\in\mathbb{Z}^n} \sum_{Q\in\mathcal{D}_m, |c(Q)-\tilde{m}|\le n} \frac{|\lambda_Q^{t_l}|\ell(Q)^{n+K+1}}{1+|\tilde{m}|^{n+1}} \\
&\lesssim 2^{\frac{mn}{q_0}-(n+K+1)m} \|\mathcal{M}f\|_{\mathcal{M}^p_{\mathbf{q}}}.
\end{aligned} \tag{19}
$$

Since $K+1 > n\left(\frac{1}{q_0}-1\right)$, we obtain

$$
n+K+1 > \frac{n}{q_0}.
$$

Thus by (17) and (19), we obtain

$$
|a_{m,l}| \lesssim \min\left(2^{\frac{mn}{q_0}-(n+K+1)m}, 2^{\frac{mn}{p}}\right).
$$

Since

$$
\sum_{m=-\infty}^{\infty} \min\left(2^{\frac{mn}{q_0}-(n+K+1)m}, 2^{\frac{mn}{p}}\right) \lesssim 1,
$$

we are in the position of using the Lebesgue convergence theorem to have

$$
\lim_{l\to\infty} \sum_{m=-\infty}^{\infty} a_{m,l} = \sum_{m=-\infty}^{\infty} \left(\lim_{l\to\infty} a_{m,l}\right).
$$

That is,

$$
\langle f, \varphi\rangle = \lim_{l\to\infty} \langle e^{t_l\Delta} f, \varphi\rangle = \sum_{m=-\infty}^{\infty} \left(\lim_{l\to\infty} \sum_{Q\in\mathcal{D}_m} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x\right).
$$

Hence, using Fubini's theorem again, we obtain

$$
\begin{aligned}
\langle f, \varphi\rangle &= \sum_{m=-\infty}^{\infty} \left(\lim_{l\to\infty} \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}_m} \lambda_Q^{t_l} a_Q^{t_l}(x)\right) \varphi(x)\mathrm{d}x\right) \\
&= \sum_{m=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_m} \lim_{l\to\infty} \left(\int_{\mathbb{R}^n} \lambda_Q^{t_l} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x\right) \\
&= \sum_{m=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_m} \int_{\mathbb{R}^n} \lambda_Q a_Q(x)\varphi(x)\mathrm{d}x = \left\langle \sum_{Q\in\mathcal{D}} \lambda_Q a_Q, \varphi\right\rangle.
\end{aligned}
$$

Consequently, we obtain the desired result. □

5 Proof of Theorem 6

First, we prove two lemmas. We invoke an estimate from [7, Lemma 2.2] and [8, Lemma 2.1].

Lemma 6 *There exists a constant depending only on n and α such that, for every cube Q, we have $I_\alpha \chi_Q(x) \ge C\ell(Q)^\alpha \chi_Q(x)$ for all $x \in Q$.*

To prove the next estimate, we use Proposition 2. We invoke another estimate from [13, Lemma 4.2].

Lemma 7 *Let $K = 0, 1, 2, \dots$ Suppose that A is an $L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$-function supported on a cube Q. Then,*

$$|I_\alpha A(x)| \le C_{\alpha,K} \|A\|_{L^\infty} \ell(Q)^\alpha \sum_{k=1}^{\infty} \frac{1}{2^{k(n+K+1-\alpha)}} \chi_{2^k Q}(x) \quad (x \in \mathbb{R}^n). \tag{20}$$

Now we prove Theorem 6. We may assume that $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is a positive measurable function in view of the positivity of the integral kernel. We decompose f according to Theorem 3 with $K > \alpha - \frac{n}{p^*} - 1$; $f = \sum_{j=1}^\infty \lambda_j a_j$, where $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K^\perp(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ fulfill (3). Then by Lemma 7, we obtain

$$|g(x) I_\alpha f(x)| \le C \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j}{2^{k(n+K+1-\alpha)}} \left(\ell(Q_j)^\alpha |g(x)| \chi_{2^k Q_j}(x) \right).$$

Therefore, we conclude

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^s_{\mathbf{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}} \left\| \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}} \cdot \frac{\ell(2^k Q_j)^{\frac{n}{p^*}}}{\|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}}} |g| \chi_{2^k Q_j} \right\|_{\mathcal{M}^s_{\mathbf{t}}}.$$

For each $(j, k) \in \mathbb{N} \times \mathbb{N}$, write

$$\kappa_{jk} \equiv \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}}, \quad b_{jk} \equiv \frac{\ell(2^k Q_j)^{\frac{n}{p^*}}}{\|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}}} |g| \chi_{2^k Q_j}.$$

Then,

$$\sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}} \cdot \frac{\ell(2^k Q_j)^{\frac{n}{p^*}}}{\|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}}} |g| \chi_{2^k Q_j} = \sum_{j,k=1}^\infty \kappa_{jk} b_{jk},$$

each b_{jk} is supported on a cube $2^k Q_j$ and

$$\|b_{jk}\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}} \le \ell(2^k Q_j)^{\frac{n}{p^*}}.$$

Observe that $\chi_{2^k Q_j} \le 2^{kn} M\chi_{Q_j}$. Hence, if we choose $1 < \theta$ so that

$$K > \alpha - \frac{n}{p^*} - 1 + \theta n - n,$$

then we have

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \kappa_{jk} \chi_{2^k Q_j} \right\|_{\mathcal{M}^s_{\mathbf{t}}} \\
&= \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}} \chi_{2^k Q_j} \right\|_{\mathcal{M}^s_{\mathbf{t}}} \\
&= \left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha - \frac{n}{p^*}} (M\chi_{Q_j})^{\theta} \right\|_{\mathcal{M}^s_{\mathbf{t}}} \\
&\le C \left\| \sum_{j=1}^{\infty} \left(M\left[\lambda_j^{\frac{1}{\theta}} \ell(Q_j)^{\frac{1}{\theta}\left(\alpha - \frac{n}{p^*}\right)} \chi_{Q_j} \right] \right)^{\theta} \right\|_{\mathcal{M}^s_{\mathbf{t}}} \\
&\le C \left(\left\| \left\{ \sum_{j=1}^{\infty} \left(M\left[\lambda_j^{\frac{1}{\theta}} \ell(Q_j)^{\frac{1}{\theta}\left(\alpha - \frac{n}{p^*}\right)} \chi_{Q_j} \right] \right)^{\theta} \right\}^{\frac{1}{\theta}} \right\|_{\mathcal{M}^{\theta s}_{\theta \mathbf{q}}} \right)^{\theta}.
\end{aligned}$$

By virtue of Proposition 6, the Fefferman–Stein inequality for mixed Morrey spaces, with $f_j = \lambda_j^{\frac{1}{\theta}} \ell(Q_j)^{\frac{1}{\theta}\left(\alpha - \frac{n}{p^*}\right)} \chi_{Q_j}$, we can remove the maximal operator and we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^s_{\mathbf{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\mathbf{q}^*}} \left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha - \frac{n}{p^*}} \chi_{Q_j} \right\|_{\mathcal{M}^s_{\mathbf{t}}}.$$

We distinguish two cases here.

(1) If $\alpha = \frac{n}{p^*}$, then $p = s$ and $\mathbf{q} = \mathbf{t}$. Thus, we can use (3).
(2) If $\alpha > \frac{n}{p^*}$, then, by Proposition 2 and Lemma 6, we obtain

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha - \frac{n}{p^*}} \chi_{Q_j} \right\|_{\mathcal{M}^s_{\mathbf{t}}} &\le C \left\| I_{\alpha - \frac{n}{p^*}} \left[\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right] \right\|_{\mathcal{M}^s_{\mathbf{t}}} \\
&\le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\mathbf{q}}}.
\end{aligned}$$

Thus, we are still in the position of using (3).

Consequently, we obtain

$$\left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \kappa_{jk} \chi_{2^k Q_j} \right\|_{\mathcal{M}_{\mathbf{t}}^s} \lesssim \|f\|_{\mathcal{M}_{\mathbf{q}}^p} < \infty. \tag{21}$$

Observe also that $p^* > s$ and that $\mathbf{q}^* > \mathbf{t}$. Thus, by Theorem 2 and (21), it follows that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\mathbf{t}}^s} \leq C\|g\|_{\mathcal{M}_{\mathbf{q}^*}^{p^*}} \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \kappa_{jk} \chi_{2^k Q_j} \right\|_{\mathcal{M}_{\mathbf{t}}^s} \leq C\|g\|_{\mathcal{M}_{\mathbf{q}^*}^{p^*}} \|f\|_{\mathcal{M}_{\mathbf{q}}^p}.$$

Acknowledgements Toru Nogayama and Takahiro Ono were supported financially by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists (20J10403 and 20J20606). Daniel Salim was supported by P3MI-ITB Program 2017. Yoshihiro Sawano was supported by Grant-in-Aid for Scientific Research (C) (19K03546), the Japan Society for the Promotion of Science and People's Friendship University of Russia. The authors are thankful to Dr. Naoya Hatano for pointing out the mistake in the proof of Theorem 4.

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