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Hilbert Matrix and Its Norm on Weighted Bergman Spaces

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Abstract

It is well known that the Hilbert matrix H is bounded on weighted Bergman spaces A^p_α if and only if $1 < \alpha + 2 < p$ with the conjectured norm $\pi / \sin \frac{(\alpha+2)\pi}{p}$. The conjecture was confirmed in the case when $\alpha = 0$ and also in the case when $\alpha > 0$ and $p \geq 2(\alpha+2)$, which reduces the conjecture in the case when $\alpha > 0$ to the interval $\alpha+2 < p < 2(\alpha+2)$. In the remaining case when $-1 < \alpha < 0$ and $p > \alpha+2$ there has been no progress so far in proving the conjecture, moreover, there is no even an explicit upper bound for the norm of the Hilbert matrix H on weighted Bergman spaces A^p_α. In this paper we obtain results which are better than known related to the validity of the mentioned conjecture in the case when $\alpha > 0$ and $\alpha+2 < p < 2(\alpha+2)$. On the other hand, we also provide for the first time an explicit upper bound for the norm of the Hilbert matrix H on weighted Bergman spaces A^p_α in the case when $-1 < \alpha < 0$ and $p > \alpha + 2$.

Keywords Hilbert matrix · Norm · Weighted Bergman spaces

Mathematics Subject Classification Primary 47B35 · Secondary 30H20

1 Introduction

The Hilbert matrix H and its action on the space ℓ^2 consisting of square summable sequences was first studied in [11], where Magnus described the spectrum of the Hilbert matrix. Thereafter Diamantopoulos and Siskakis in [3,4] begin to study the action of the Hilbert matrix on Hardy and Bergman spaces, which can be seen as the beginning of studying of the Hilbert matrix as an operator on spaces of holomorphic functions. They obtained some partial results concerning the questions of boundedness

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and exact norm of the Hilbert matrix on Hardy and Bergman spaces, which have been improved in [5] by Dostanić, Jevtić and Vukotić. We note also that Aleman, Montes-Rodríguez and Sarafoleanu provide a closed formula for the eigenvalues of the Hilbert matrix in a more general context (see [1]). Following the above results, it was known that Hilbert matrix H is bounded on Bergman space A^p if and only if $2 < p < \infty$ and

$$\|\mathrm{H}\|_{A^p \to A^p} = \frac{\pi}{\sin \frac{2\pi}{p}},$$

when $4 \leq p < \infty$. It was also conjectured that previous equality remains valid in the remaining case when $2 < p < 4$. This conjecture was actually proven in [2], where the new method based on the new way to use monotonicity of the integral means was introduced (see also [9]).

The starting point for studying the boundedness of the Hilbert matrix H on weighted Bergman spaces A^p_α was paper [6] by Galanopoulos, Girela, Peláez and Siskakis, where the corresponding partial results were obtained. A complete characterization of the boundedness of the Hilbert matrix H on the spaces A^p_α is given in [7], where it is proved

$$\mathrm{H} \text{ is bounded on } A^p_\alpha \text{ if and only if } 1 < \alpha + 2 < p.$$

On the other hand, the preceding result opened the way to the question of the exact norm of the Hilbert matrix acting on the weighted Bergman spaces. In [8] it was proved that

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} \geq \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}} \quad \text{for } 1 < \alpha + 2 < p, \tag{1.1}$$

and it was conjectured that this lower bound is the exact norm of the Hilbert matrix. This implies that it is necessary to have the following upper bound

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} \leq \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}} \quad \text{for } 1 < \alpha + 2 < p,$$

to prove mentioned conjecture. The conjecture was confirmed [8] in the case when $\alpha \geq 0$ and $p \geq 2(\alpha+2)$, which reduces the conjecture in the case $\alpha \geq 0$ to the interval $\alpha + 2 < p < 2(\alpha + 2)$. When $\alpha = 0$ this was completely solved in [2] (see also [9]). Very recently, Lindström, Miihkinen and Wikman in [10] confirmed the conjecture in the case $\alpha > 0$ when

$$\alpha + 2 + \sqrt{\alpha^2 + \frac{7}{2}\alpha + 3} = \alpha + 2 + \sqrt{(\alpha+2)^2 - \frac{1}{2}(\alpha+2)} \leq p < 2(\alpha + 2).$$

Among other things in this paper, we improved the previous result by confirming the conjecture in the case $\alpha > 0$ when

$$\alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)} \le p < 2(\alpha+2).$$

On the other hand, we note that in the case $-1 < \alpha < 0$ there has been no progress so far in proving the conjecture, moreover, there is no even an explicit upper bound for the norm of the Hilbert matrix H on weighted Bergman spaces A^p_α. Finally, in this paper we also provide an explicit upper bound for the norm of the Hilbert matrix H on weighted Bergman spaces A^p_α in the case $-1 < \alpha < 0$ when $p > \alpha + 2$.

1.1 Basic Notation

Let $\mathrm{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ be the open Euclidean disc of radius $r > 0$ centered at the point z_0 in the complex plane $\mathbb{C}$. Let also $\mathcal{H}(\mathbb{D})$ be the space of all holomorphic functions in the open unit disc $\mathbb{D} = \mathrm{D}(0, 1)$. An annulus centered at the point z_0 in the complex plane is defined as follows $\mathrm{A}(z_0, r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ where $r < R$. The Euclidean area measure on the complex plane will be denoted by dm, that is

$$\mathrm{dm}(z) = \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta, \quad \text{where } z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}.$$

Given a function f holomorphic in the unit disc $\mathbb{D}$, then for $0 < p < \infty$ and $0 < r < 1$, we consider its integral means of order p defined in the following way

$$\mathrm{M}_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}}.$$

It is well known that $r \mapsto \mathrm{M}_p(r, f)$ is an nondecreasing function. This is a simple consequence of the subharmonicity of $|f|^p$. The Beta function is defined by

$$\mathrm{B}(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t,$$

where a and b are real numbers such that $a > 0$ and $b > 0$. If $0 < a < 1$ then we will use the following well known formula

$$\mathrm{B}(a, 1-a) = \frac{\pi}{\sin \pi a}.$$

1.2 Hilbert Matrix and Weighted Bergman Spaces

The Hilbert matrix is an infinite matrix

$$\mathrm{H} = \left[\frac{1}{n+k+1}\right]_{n,k=0}^{\infty}.$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a holomorphic function in the unit disc $\mathbb{D}$, that is $f \in \mathcal{H}(\mathbb{D})$, then the Hilbert matrix can be viewed as an operator on spaces of holomorphic functions in the following way

$$\mathrm{H}f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n.$$

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space is defined as follows

$$A_{\alpha}^{p} = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{A_{\alpha}^{p}} = \left(\frac{\alpha+1}{\pi}\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^{\alpha} \mathrm{dm}(z)\right)^{1/p} < \infty \right\}.$$

We note that if $\alpha = 0$ then $A^p = A_0^p$ are standard unweighted Bergman spaces. It is well known (see [3,4,8]) that if a function f belongs to weighted Bergman space A_{α}^{p} then we have

$$\mathrm{H}f(z) = \int_0^1 \mathrm{T}_t f(z)\,\mathrm{d}t,$$

where

$$\mathrm{T}_t f(z) = \omega_t(z) f\left(\phi_t(z)\right), \quad \omega_t(z) = \frac{1}{1-(1-t)z} \quad \text{and} \quad \phi_t(z) = \frac{t}{1-(1-t)z}.$$

Recall that the Hilbert matrix H is bounded on weighted Bergman space A_{α}^{p} if and only if $1 < \alpha + 2 < p$. In that case, by following [8], from the continuous version of Minkowski inequality we have estimate

$$\|\mathrm{H}f\|_{A_{\alpha}^{p}} \le \int_0^1 \|\mathrm{T}_t f\|_{A_{\alpha}^{p}}\,\mathrm{d}t,$$

and

$$\|\mathrm{T}_t f\|_{A_{\alpha}^{p}} = t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}}\left(\frac{\alpha+1}{\pi}\int_{\mathrm{D}_t} |w|^{p-4}|f(w)|^p g_t(w)^{\alpha}\,\mathrm{dm}(w)\right)^{\frac{1}{p}}, \tag{1.2}$$

where

$$g_t(w) = \frac{2\,\mathrm{Re}\, w - t - (2-t)|w|^2}{(1-t)|w|^2}, \quad \mathrm{D}_t = \mathrm{D}\,(c_t, \rho_t)\,, \quad c_t = \frac{1}{2-t} \;\text{ and }\; \rho_t = \frac{1-t}{2-t}.$$

In the rest of the paper we will use the following function

$$\psi_{p,\alpha}(t) = t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}},$$

where $0 < t < 1$. It is easy to check that

$$g_t(w) = \frac{1}{|w|^2} \cdot \frac{\rho_t^2 - |w-c_t|^2}{\rho_t} \;\text{ for }\; w \in \mathrm{D}\,(c_t, \rho_t)\,.$$

Therefore we conclude

$$\|\mathrm{T}_t f\|_{A^p_\alpha} = \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi} \int_{\mathrm{D}_t} |w|^{p-2(\alpha+2)} |f(w)|^p \left(\frac{\rho_t^2 - |w-c_t|^2}{\rho_t} \right)^\alpha \mathrm{dm}(w) \right)^{\frac{1}{p}}.$$

The previous formula is valid for all $1 < \alpha + 2 < p$ and it will be used in the last section of this paper. On the other hand, following [8] in the special case when $\alpha > 0$ we obtain

$$g_t(w)^\alpha \leq \left(\frac{1 + |w|^2 - t - (2-t)|w|^2}{(1-t)|w|^2} \right)^\alpha = \left(\frac{1-|w|^2}{|w|^2} \right)^\alpha. \tag{1.3}$$

By combining (1.2) and (1.3) we get

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi} \int_{\mathrm{D}_t} |w|^{p-2(\alpha+2)} |f(w)|^p \left(1-|w|^2\right)^\alpha \mathrm{dm}(w) \right)^{\frac{1}{p}}.$$

1.3 The Functions Ψ_α and Φ_α

Let $\alpha > 0$. Then we define the functions Ψ_α and Φ_α as follows

$$\Psi_\alpha(x) = 2x^2 - (4(\alpha+2)+1)\,x - 2\sqrt{\alpha+2}\sqrt{x} + \alpha + 2,$$

and

$$\Phi_\alpha(x) = 2x^2 - (4(\alpha+2)+1)\,x + 2\sqrt{\alpha+2}\sqrt{x} + \alpha + 2,$$

where $x \in (\alpha+2, 2(\alpha+2))$. We note that these functions will play a crucial role in our paper. Next we obtain

$$\Psi'_\alpha(x) = 4x - 4(\alpha+2) - 1 - \frac{\sqrt{\alpha+2}}{\sqrt{x}} \;\text{ and }\; \Psi''_\alpha(x) = 4 + \frac{\sqrt{\alpha+2}}{2x\sqrt{x}},$$

for $x \in (\alpha+2, 2(\alpha+2))$. Therefore

$$\Psi''_\alpha(x) > 0,$$

for every $x \in (\alpha+2, 2(\alpha+2))$. This leads that function Ψ'_α is increasing on interval $(\alpha+2, 2(\alpha+2))$. On the other hand, we find

$$\Psi'_\alpha(\alpha+2) = -2 < 0 \ \text{ and } \ \Psi'_\alpha(2(\alpha+2)) = 4(\alpha+2) - 1 - \frac{1}{\sqrt{2}} > 0.$$

Based on the above considerations we can conclude that it is valid

$$\Psi_\alpha(x) \le \max\left\{\Psi_\alpha(\alpha+2), \Psi_\alpha(2(\alpha+2))\right\},$$

for every $x \in (\alpha+2, 2(\alpha+2))$. Since

$$\Psi_\alpha(\alpha+2) = -2(\alpha+2)(\alpha+3) < 0 \ \text{ and } \ \Psi_\alpha(2(\alpha+2)) = -\left(2\sqrt{2}+1\right)(\alpha+2) < 0,$$

we have

$$\Psi_\alpha(x) < 0, \tag{1.4}$$

for every $x \in (\alpha+2, 2(\alpha+2))$. By straightforward calculations we also derive

$$\Phi'_\alpha(x) = 4x - 4(\alpha+2) - 1 + \frac{\sqrt{\alpha+2}}{\sqrt{x}} \ \text{ and } \ \Phi''_\alpha(x) = 4 - \frac{\sqrt{\alpha+2}}{2x\sqrt{x}},$$

for $x \in (\alpha+2, 2(\alpha+2))$. Function Φ''_α is increasing on interval $(\alpha+2, 2(\alpha+2))$ which implies

$$\Phi''_\alpha(x) > \Phi''_\alpha(\alpha+2) = 4 - \frac{1}{2(\alpha+2)} > 0,$$

for every $x \in (\alpha+2, 2(\alpha+2))$. Hence function Φ'_α is also increasing on interval $(\alpha+2, 2(\alpha+2))$. This leads to

$$\Phi'_\alpha(x) > \Phi'_\alpha(\alpha+2) = 4(\alpha+2) - 4(\alpha+2) - 1 + 1 = 0,$$

where $x \in (\alpha+2, 2(\alpha+2))$, whence it follows that Φ_α is an increasing function on interval $(\alpha+2, 2(\alpha+2))$. Then

$$\Phi_\alpha(\alpha+2) = -2(\alpha+1)(\alpha+2) < 0 \ \text{ and } \ \Phi_\alpha(2(\alpha+2)) = \left(2\sqrt{2}-1\right)(\alpha+2) > 0.$$

This means that function Φ_α has a unique zero α_0 on the interval $(\alpha+2, 2(\alpha+2))$. Moreover, we get $\Phi_\alpha < 0$ on $(\alpha+2, \alpha_0)$ and $\Phi_\alpha > 0$ on $(\alpha_0, 2(\alpha+2))$. The previous notation will be used in the rest of the paper.

1.4 The Main Results

Let $\alpha > 0$ and let α_0 be a unique zero of the function Φ_α on the interval $(\alpha+2, 2(\alpha+2))$. We are now ready to state the main results of the paper.

Theorem 1.1 *Let $\alpha > 0$ and $\alpha_0 \le p < 2(\alpha+2)$. Then* $\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} = \dfrac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}$.

An immediate consequence we obtain the following result.

Corollary 1.1 *Let $\alpha > 0$ and*

$$\alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)} \le p < 2(\alpha+2).$$

Then

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} = \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}.$$

Proof It is enough to prove that

$$\Phi_\alpha\left(\alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)}\right) > 0.$$

Namely, the previous inequality implies

$$\alpha_0 < \alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)},$$

whence by Theorem 1.1 it follows the required conclusion. We can split the function Φ_α into two parts

$$\Phi_\alpha(x) = \Upsilon_\alpha(x) + \Lambda_\alpha(x),$$

where we denoted

$$\Upsilon_\alpha(x) = 2\left[(x - (\alpha+2))^2 - \left((\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)\right)\right],$$

and

$$\Lambda_\alpha(x) = 2\sqrt{\alpha+2}\sqrt{x} - x - 2\left(\sqrt{2} - 1\right)(\alpha+2).$$

Note that

$$\Upsilon_\alpha\left(\alpha+2+\sqrt{(\alpha+2)^2-\left(\sqrt{2}-\frac{1}{2}\right)(\alpha+2)}\right)=0.$$

So it is enough to prove

$$\Lambda_\alpha\left(\alpha+2+\sqrt{(\alpha+2)^2-\left(\sqrt{2}-\frac{1}{2}\right)(\alpha+2)}\right)>0,$$

or equivalently

$$\Lambda_\alpha\left(a+\sqrt{a^2-\left(\sqrt{2}-\frac{1}{2}\right)a}\right)>0,$$

where $a=\alpha+2$. This leads to

$$2\sqrt{a}\sqrt{a+\sqrt{a^2-\left(\sqrt{2}-\frac{1}{2}\right)a}}>\left(2\sqrt{2}-1\right)a+\sqrt{a^2-\left(\sqrt{2}-\frac{1}{2}\right)a},$$

whence after squaring we get the following equivalent form

$$\left(3-2\sqrt{2}\right)\sqrt{a^2-\left(\sqrt{2}-\frac{1}{2}\right)a}>\left(3-2\sqrt{2}\right)a-\left(\frac{\sqrt{2}}{2}-\frac{1}{4}\right).$$

Since

$$\left(3-2\sqrt{2}\right)\sqrt{a^2-\left(\sqrt{2}-\frac{1}{2}\right)a}>0,$$

it is enough to prove

$$\left(\left(3-2\sqrt{2}\right)\sqrt{a^2-\left(\sqrt{2}-\frac{1}{2}\right)a}\right)^2>\left(\left(3-2\sqrt{2}\right)a-\left(\frac{\sqrt{2}}{2}-\frac{1}{4}\right)\right)^2,$$

which after some calculations reduces to

$$a>\frac{\sqrt{2}-\frac{1}{2}}{8\left(5\sqrt{2}-7\right)}=\frac{13+9\sqrt{2}}{16}.$$

The last inequality is true, since

$$a = \alpha + 2 > 2 > \frac{13 + 9\sqrt{2}}{16}.$$

This completes the proof. □

Remark 1.1 Note that Corollary 1.1 improves the last best known result recently obtained by Lindström, Miihkinen and Wikman in [10], where they get the same conclusion under the assumptions $\alpha > 0$ and

$$\alpha + 2 + \sqrt{(\alpha+2)^2 - \frac{1}{2}(\alpha+2)} \le p < 2(\alpha+2),$$

which was discussed earlier. □

On the other hand, let

$$\beta = \alpha + 2 + \sqrt{(\alpha+2)^2 - (\alpha+2)}.$$

Then by straightforward calculations we obtain

$$\Phi_\alpha(\beta) = 2\underbrace{\left(\beta^2 - 2(\alpha+2)\beta + \alpha + 2\right)}_{=0} - \left(\sqrt{\beta} - \sqrt{\alpha+2}\right)^2 < 0,$$

which implies that $\beta < \alpha_0$. In the case when $\alpha > 0$ and $\alpha + 2 < p \le \beta$ we obtain the following partial result.

Theorem 1.2 *Let* $\alpha > 0$, $\alpha + 2 < p \le \alpha + 2 + \sqrt{(\alpha+2)^2 - (\alpha+2)}$ *and suppose that the following condition holds*

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t \le \frac{1}{\alpha+1}\int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t = \frac{1}{\alpha+1}\mathrm{B}\left(\frac{\alpha+2}{p}, 1 - \frac{\alpha+2}{p}\right), \tag{1.5}$$

where we denoted

$$\psi_{p,\alpha}(t) = t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}} \quad \textit{and} \quad \xi_{p,\alpha}(t) = \int_{\left(\frac{t}{2-t}\right)^2}^1 \rho^{\frac{p}{2}-(\alpha+2)}(1-\rho)^\alpha\,\mathrm{d}\rho,$$

for $0 < t < 1$. *Then*

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

Remark 1.2 We note that condition (1.5) is not always satisfied under the given conditions even when $p/2 - (\alpha + 2) + 1 > 0$, that is $p > 2\alpha + 2$, which actually allows the convergence of the integral $\xi_{p,\alpha}(0)$. Namely a calculation involving *Mathematica* shows that when $\alpha = 1$ then $\beta \approx 5.449$ and $\alpha_0 \approx 5.487$ and also for

$$\alpha + 2 < 2\alpha + 2 < p = 4.4 < \beta < \alpha_0,$$

we have

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t - \frac{1}{\alpha+1}\int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t \approx 0.962 > 0.$$

On the other hand, if $\alpha = 1$ and $\alpha + 2 < 2\alpha + 2 < p = 5.2 < \beta < \alpha_0$ then

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t - \frac{1}{\alpha+1}\int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t \approx -0.103 < 0,$$

which allows the application of Theorem 1.2 in some cases where it is not possible to apply Theorem 1.1. We also have

$$\beta = \alpha + 2 + \sqrt{(\alpha+2)^2 - (\alpha+2)} < \alpha_0 < \alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)},$$

and since $\sqrt{2} - 1/2 \approx 0.914$ (actually $\sqrt{2} - 1/2 > 0.914$) we can write

$$\alpha + 2 + \sqrt{(\alpha+2)^2 - (\alpha+2)} < \alpha_0 < \alpha + 2 + \sqrt{(\alpha+2)^2 - 0.914(\alpha+2)}. \quad (1.6)$$

From (1.6) we can conclude that there remains a small gap between β and α_0 to which we cannot apply the above Theorem 1.1 and Theorem 1.2. ◇

Finally, in the case when

$$-1 < \alpha < 0 \text{ and } p > \alpha + 2,$$

for the first time we obtain an explicit upper bound for the norm of the Hilbert matrix H on weighted Bergman spaces A^p_α. Namely, we have the following result.

Theorem 1.3 *Let* $-1 < \alpha < 0$ *and* $p > \alpha + 2$.

(i) *If* $p \geq 2(\alpha + 2)$ *then*

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} \leq 2^{\frac{\alpha+2}{p}} \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

(ii) *If* $\alpha + 2 < p < 2(\alpha + 2)$ *then*

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} \le 2^{\frac{\alpha+2}{p}} \left(1 + 2^{\frac{2(\alpha+2)}{p}-1}\right) \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}.$$

In the rest of the paper, we present the proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3.

2 Preliminaries

Let $\alpha > 0$ and $\alpha + 2 < p < 2(\alpha + 2)$. We consider functions

$$\psi_{p,\alpha}(t) = t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}} \quad \text{and} \quad \xi_{p,\alpha}(t) = \int_{\left(\frac{t}{2-t}\right)^2}^1 \rho^{\frac{p}{2}-(\alpha+2)}(1-\rho)^\alpha \, \mathrm{d}\rho,$$

where $0 < t < 1$ as in the statement of Theorem 1.2. For $s \in [0, 1]$ we denote

$$F_{p,\alpha}(s) = \xi_{p,\alpha}(s) \int_0^s \psi_{p,\alpha}(t)\, \mathrm{d}t + \int_s^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\, \mathrm{d}t \\ -\frac{1}{\alpha+1}\left(1-\left(\frac{s}{2-s}\right)^2\right)^{\alpha+1} \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t.$$

As we will see later it is of interest to examine under what conditions the function $F_{p,\alpha}$ is nonpositive on the segment $[0, 1]$. Let also

$$B_{p,\alpha} = \mathrm{B}\left(\frac{\alpha+2}{p}, 1 - \frac{\alpha+2}{p}\right) = \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t.$$

Then

$$\begin{aligned} F'_{p,\alpha}(s) &= \xi'_{p,\alpha}(s) \int_0^s \psi_{p,\alpha}(t)\, \mathrm{d}t + \xi_{p,\alpha}(s)\psi_{p,\alpha}(s) - \psi_{p,\alpha}(s)\xi_{p,\alpha}(s) \\ &\quad -\frac{1}{\alpha+1} \cdot (\alpha+1) \cdot \left(1-\left(\frac{s}{2-s}\right)^2\right)^{\alpha} \cdot (-1) \cdot \frac{4s}{(2-s)^3} \cdot \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t \\ &= -\left(\left(\frac{s}{2-s}\right)^2\right)^{\frac{p}{2}-(\alpha+2)} \left(1-\left(\frac{s}{2-s}\right)^2\right)^{\alpha} \frac{4s}{(2-s)^3} \int_0^s \psi_{p,\alpha}(t)\, \mathrm{d}t \\ &\quad + \frac{4s}{(2-s)^3}\left(1-\left(\frac{s}{2-s}\right)^2\right)^{\alpha} \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t \\ &= \frac{4s\left(1-\left(\frac{s}{2-s}\right)^2\right)^{\alpha}}{(2-s)^3} \left(B_{p,\alpha} - \left(\frac{s}{2-s}\right)^{p-2(\alpha+2)} \int_0^s \psi_{p,\alpha}(t)\, \mathrm{d}t\right). \end{aligned}$$

Therefore

$$F'_{p,\alpha}(s) = \frac{4s}{(2-s)^3}\left(1-\left(\frac{s}{2-s}\right)^2\right)^{\alpha}\left(\frac{s}{2-s}\right)^{p-2(\alpha+2)} G_{p,\alpha}(s), \tag{2.1}$$

where we denote

$$G_{p,\alpha}(s) = \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p} B_{p,\alpha} - \int_0^s \psi_{p,\alpha}(t)\,\mathrm{d}t. \tag{2.2}$$

We find

$$\begin{aligned} G'_{p,\alpha}(s) &= (2(\alpha+2)-p)\left(\frac{s}{2-s}\right)^{2(\alpha+2)-p-1} \frac{2}{(2-s)^2} B_{p,\alpha} - \psi_{p,\alpha}(s) \\ &= (4(\alpha+2)-2p)\, B_{p,\alpha}\left(\frac{s}{2-s}\right)^{2(\alpha+2)-p-1} \frac{1}{(2-s)^2} - \psi_{p,\alpha}(s). \end{aligned}$$

Hence

$$G'_{p,\alpha}(s) = (4(\alpha+2)-2p)\, B_{p,\alpha}\psi_{p,\alpha}(s) E_{p,\alpha}(s), \tag{2.3}$$

where

$$E_{p,\alpha}(s) = \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p-1} \frac{s^{-\frac{\alpha+2}{p}+1}(1-s)^{\frac{\alpha+2}{p}}}{(2-s)^2} - \frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}},$$

or equivalently

$$E_{p,\alpha}(s) = \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p+1} s^{-\frac{\alpha+2}{p}-1}(1-s)^{\frac{\alpha+2}{p}} - \frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}}. \tag{2.4}$$

By differentiation we obtain

$$\begin{aligned} E'_{p,\alpha}(s) &= (2(\alpha+2)-p+1)\left(\frac{s}{2-s}\right)^{2(\alpha+2)-p} \frac{2}{(2-s)^2} s^{-\frac{\alpha+2}{p}-1}(1-s)^{\frac{\alpha+2}{p}} \\ &\quad + \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p+1}\left(-\frac{\alpha+2}{p}-1\right) s^{-\frac{\alpha+2}{p}-2}(1-s)^{\frac{\alpha+2}{p}} \\ &\quad + \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p+1} s^{-\frac{\alpha+2}{p}-1}\frac{\alpha+2}{p}(1-s)^{\frac{\alpha+2}{p}-1}(-1) \\ &= \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p+1} s^{-\frac{\alpha+2}{p}-2}(1-s)^{\frac{\alpha+2}{p}} K_{p,\alpha}(s), \end{aligned}$$

where

$$K_{p,\alpha}(s) = (4(\alpha+2)-2p+2)\frac{1}{2-s} - \frac{\alpha+2}{p} - 1 - \frac{\alpha+2}{p}\frac{s}{1-s},$$

or

$$K_{p,\alpha}(s) = (4(\alpha+2) - 2p + 2)\,\frac{1}{2-s} - 1 - \frac{\alpha+2}{p}\frac{1}{1-s}.$$

Next we can write

$$K_{p,\alpha}(s) = \frac{1}{(2-s)(1-s)} L_{p,\alpha}(s),$$

where we denote

$$L_{p,\alpha}(s) = (4(\alpha+2) - 2p + 2)\,(1-s) - (2-s)(1-s) - \frac{\alpha+2}{p}(2-s).$$

Finally we obtain

$$L_{p,\alpha}(s) = -s^2 - \left(4(\alpha+2) - 2p - 1 - \frac{\alpha+2}{p}\right) s + 4(\alpha+2) - 2p - \frac{2(\alpha+2)}{p},$$

and

$$E'_{p,\alpha}(s) = \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p+2} s^{-\frac{\alpha+2}{p}-3}(1-s)^{\frac{\alpha+2}{p}-1} L_{p,\alpha}(s). \tag{2.5}$$

Let

$$A_{p,\alpha} = 4(\alpha+2) - 2p - \frac{2(\alpha+2)}{p}.$$

Note that

$$L_{p,\alpha}(s) = -s^2 - \left(A_{p,\alpha} + \frac{\alpha+2}{p} - 1\right) s + A_{p,\alpha}. \tag{2.6}$$

We denote the discriminant of the quadratic equation $L_{p,\alpha}(s) = 0$ by $D_{p,\alpha}$. Then

$$D_{p,\alpha} = \left(A_{p,\alpha} + \frac{\alpha+2}{p} - 1\right)^2 + 4A_{p,\alpha},$$

or

$$D_{p,\alpha} = A_{p,\alpha}^2 + 2\left(1 + \frac{\alpha+2}{p}\right) A_{p,\alpha} + \left(1 - \frac{\alpha+2}{p}\right)^2.$$

Equivalently, we get

$$D_{p,\alpha} = \left(A_{p,\alpha} + \left(1 + \sqrt{\frac{\alpha+2}{p}}\right)^2\right)\left(A_{p,\alpha} + \left(1 - \sqrt{\frac{\alpha+2}{p}}\right)^2\right).$$

In view of Sect. 1.3 we have

$$A_{p,\alpha} + \left(1 + \sqrt{\frac{\alpha+2}{p}}\right)^2 = -\frac{1}{p} \cdot \Psi_\alpha(p) > 0,$$

where we used the fact that $\alpha + 2 < p < 2(\alpha + 2)$ and inequality (1.4). This means that discriminant $D_{p,\alpha}$ has the same sign as the factor

$$A_{p,\alpha} + \left(1 - \sqrt{\frac{\alpha+2}{p}}\right)^2 = -\frac{1}{p} \cdot \Phi_\alpha(p).$$

We actually have

$$D_{p,\alpha} = \frac{1}{p^2} \cdot \Psi_\alpha(p) \cdot \Phi_\alpha(p).$$

Now we are ready to state our first preliminary result.

Lemma 2.1 *Let* $\alpha > 0$ *and* $\alpha_0 \le p < 2(\alpha + 2)$. *Then* $F_{p,\alpha}(s) \le 0$ *for all* $s \in [0, 1]$.

Proof Since $\alpha_0 \le p < 2(\alpha + 2)$ we have (see Sect. 1.3) that

$$\Psi_\alpha(p) < 0 \text{ and } \Phi_\alpha(p) \ge 0,$$

which implies

$$D_{p,\alpha} = \frac{1}{p^2} \cdot \Psi_\alpha(p) \cdot \Phi_\alpha(p) \le 0.$$

On the other hand, since $D_{p,\alpha}$ is a discriminant of a quadratic function $L_{p,\alpha}$ given by (2.6) we can conclude that

$$L_{p,\alpha} \le 0 \text{ on } [0, 1].$$

By using Eq. (2.5) we get

$$E'_{p,\alpha} \le 0 \text{ on } [0, 1],$$

which in turn implies that $E_{p,\alpha}$ is nonincreasing function on [0, 1]. By Eq. (2.4) we obtain

$$\begin{aligned} E_{p,\alpha}(s) &= \left(\frac{s}{2-s}\right)^{2(\alpha+2)-p+1} s^{-\frac{\alpha+2}{p}-1}(1-s)^{\frac{\alpha+2}{p}} - \frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}} \\ &= \left(\frac{1}{2-s}\right)^{2(\alpha+2)-p+1} s^{2(\alpha+2)-p-\frac{\alpha+2}{p}}(1-s)^{\frac{\alpha+2}{p}} - \frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}}. \end{aligned}$$

Hence

$$E_{p,\alpha}(s) = \left(\frac{1}{2-s}\right)^{2(\alpha+2)-p+1} s^{\frac{A_{p,\alpha}}{2}}(1-s)^{\frac{\alpha+2}{p}} - \frac{1}{(4(\alpha+2)-2p)\,B_{p,\alpha}}. \tag{2.7}$$

Also

$$A_{p,\alpha} + \underbrace{\left(1-\sqrt{\frac{\alpha+2}{p}}\right)^2}_{>0} = -\frac{1}{p}\cdot\Phi_\alpha(p) \le 0,$$

which leads to

$$A_{p,\alpha} < 0. \tag{2.8}$$

Combining (2.7) and (2.8) we obtain

$$\lim_{s\to 0^+} E_{p,\alpha}(s) = +\infty \ \text{ and } \ E_{p,\alpha}(1) = -\frac{1}{(4(\alpha+2)-2p)\,B_{p,\alpha}} < 0.$$

As we have already concluded that the function $E_{p,\alpha}(s)$ is nonincreasing on [0, 1] we derive that there exists $s_0 \in (0, 1)$ such that

$$E_{p,\alpha} \ge 0 \ \text{ on } \ [0, s_0] \ \text{ and } \ E_{p,\alpha} \le 0 \ \text{ on } \ [s_0, 1]\,.$$

From (2.3) we find that

$$G'_{p,\alpha} \ge 0 \ \text{ on } \ [0, s_0] \ \text{ and } \ G'_{p,\alpha} \le 0 \ \text{ on } \ [s_0, 1]\,.$$

Therefore the function $G_{p,\alpha}$ is nondecreasing on $[0, s_0]$ and nonincreasing on $[s_0, 1]$. So we have

$$G_{p,\alpha}(s) \ge \min\left\{G_{p,\alpha}(0), G_{p,\alpha}(1)\right\},$$

for all $s \in [0, 1]$. Since (see (2.2))

$$G_{p,\alpha}(0) = 0 \ \text{ and } \ G_{p,\alpha}(1) = B_{p,\alpha} - \int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t = 0,$$

we get $G_{p,\alpha} \ge 0$ on [0, 1]. From (2.1) we find $F'_{p,\alpha} \ge 0$ on [0, 1] which implies that $F_{p,\alpha}$ is nondecreasing function on [0, 1]. Finally

$$F_{p,\alpha}(s) \le F_{p,\alpha}(1),$$

for all $s \in [0, 1]$. Since

$$F_{p,\alpha}(1) = \underbrace{\xi_{p,\alpha}(1)}_{=0} \int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t = 0 \cdot B_{p,\alpha} = 0,$$

we conclude that

$$F_{p,\alpha}(s) \leq 0,$$

for all $s \in [0, 1]$. This finishes the proof. □

We need also the following preliminary result which will be used later.

Lemma 2.2 *Let $\alpha > 0$, $\alpha + 2 < p \leq \beta = \alpha + 2 + \sqrt{(\alpha+2)^2 - (\alpha+2)}$ and suppose that the following condition holds*

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t \leq \frac{1}{\alpha+1}\int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t.$$

Then $F_{p,\alpha}(s) \leq 0$ for all $s \in [0, 1]$.

Proof Note that given condition

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t \leq \frac{1}{\alpha+1}\int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t,$$

is actually equivalent to

$$F_{p,\alpha}(0) \leq 0. \tag{2.9}$$

In view of Sect. 1.3 we have

$$\Psi_\alpha(p) < 0 \ \text{ and } \ \Phi_\alpha(p) < 0,$$

which implies

$$D_{p,\alpha} = \frac{1}{p^2} \cdot \Psi_\alpha(p) \cdot \Phi_\alpha(p) > 0.$$

We also notice that

$$L_{p,\alpha}(s) = -s^2 - C_{p,\alpha}s + A_{p,\alpha} \ \text{ for } \ s \in [0, 1],$$

where we denote

$$C_{p,\alpha} = A_{p,\alpha} + \frac{\alpha+2}{p} - 1.$$

The roots of the quadratic equation $L_{p,\alpha}(s) = 0$ are given by

$$x_{p,\alpha} = \frac{-C_{p,\alpha} - \sqrt{D_{p,\alpha}}}{2} \quad \text{and} \quad X_{p,\alpha} = \frac{-C_{p,\alpha} + \sqrt{D_{p,\alpha}}}{2}.$$

Then

$$\left(C_{p,\alpha} + 2\right)^2 = C_{p,\alpha}^2 + 4\left(C_{p,\alpha} + 1\right) > C_{p,\alpha}^2 + 4A_{p,\alpha} = D_{p,\alpha}. \tag{2.10}$$

We also have

$$\begin{aligned} C_{p,\alpha} + 2 &= 4(\alpha+2) - 2p - \frac{2(\alpha+2)}{p} + \frac{\alpha+2}{p} - 1 + 2 \\ &= 4(\alpha+2) - 2p - \frac{\alpha+2}{p} + 1 \\ &= -\frac{1}{p} \cdot \left(2p^2 - (4(\alpha+2)+1)\,p + 2\sqrt{\alpha+2}\sqrt{p} + \alpha + 2 - 2\sqrt{\alpha+2}\sqrt{p}\right), \end{aligned}$$

that is

$$C_{p,\alpha} + 2 = -\frac{1}{p} \cdot \left(\underbrace{\Phi_\alpha(p)}_{<0} - 2\sqrt{\alpha+2}\sqrt{p} \right) > 0. \tag{2.11}$$

From (2.10) and (2.11) we find

$$C_{p,\alpha} + 2 > \sqrt{D_{p,\alpha}},$$

which leads to

$$X_{p,\alpha} = \frac{-C_{p,\alpha} + \sqrt{D_{p,\alpha}}}{2} < 1.$$

Therefore

$$x_{p,\alpha} < X_{p,\alpha} < 1, \tag{2.12}$$

and

$$L_{p,\alpha}(s) = -\left(s - x_{p,\alpha}\right)\left(s - X_{p,\alpha}\right), \tag{2.13}$$

for $s \in [0, 1]$. It is also easy to see that

$$A_{p,\alpha} = 4(\alpha+2) - 2p - \frac{2(\alpha+2)}{p} = -\frac{2}{p}\left(p^2 - 2(\alpha+2)p + \alpha + 2\right),$$

and

$$A_{p,\alpha} = -\frac{2}{p} \underbrace{\left(p - \left(\alpha + 2 - \sqrt{(\alpha+2)^2 - (\alpha+2)}\right)\right)}_{>0} (p - \beta), \tag{2.14}$$

where we used the fact that $p > \alpha + 2 > \alpha + 2 - \sqrt{(\alpha+2)^2 - (\alpha+2)}$. Then we can consider the following two cases.
Case $\alpha + 2 < p < \beta$ In this case from (2.14) we obtain that $A_{p,\alpha} > 0$. Hence

$$D_{p,\alpha} = C_{p,\alpha}^2 + 4A_{p,\alpha} > C_{p,\alpha}^2,$$

which implies $\sqrt{D_{p,\alpha}} > -C_{p,\alpha}$ and $\sqrt{D_{p,\alpha}} > C_{p,\alpha}$. Therefore

$$x_{p,\alpha} = \frac{-C_{p,\alpha} - \sqrt{D_{p,\alpha}}}{2} < 0 \quad \text{and} \quad X_{p,\alpha} = \frac{-C_{p,\alpha} + \sqrt{D_{p,\alpha}}}{2} > 0,$$

and by using (2.12) we have

$$x_{p,\alpha} < 0 < X_{p,\alpha} < 1.$$

Recall that $L_{p,\alpha}(s) = -\left(s - x_{p,\alpha}\right)\left(s - X_{p,\alpha}\right)$ for $s \in [0, 1]$ (see (2.13)). So we get

$$L_{p,\alpha} \geq 0 \quad \text{on} \quad \left[0, X_{p,\alpha}\right] \quad \text{and} \quad L_{p,\alpha} \leq 0 \quad \text{on} \quad \left[X_{p,\alpha}, 1\right].$$

By using (2.5) we have

$$E'_{p,\alpha} \geq 0 \quad \text{on} \quad \left[0, X_{p,\alpha}\right] \quad \text{and} \quad E'_{p,\alpha} \leq 0 \quad \text{on} \quad \left[X_{p,\alpha}, 1\right].$$

We can conclude that function $E_{p,\alpha}$ is nondecreasing on $\left[0, X_{p,\alpha}\right]$ and nonincreasing on $\left[X_{p,\alpha}, 1\right]$. Therefore

$$E_{p,\alpha}\left(X_{p,\alpha}\right) = \max_{s\in[0,1]} E_{p,\alpha}(s).$$

We recall that (see (2.7))

$$E_{p,\alpha}(s) = \left(\frac{1}{2-s}\right)^{2(\alpha+2)-p+1} s^{\frac{A_{p,\alpha}}{2}} (1-s)^{\frac{\alpha+2}{p}} - \frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}},$$

which leads to

$$E_{p,\alpha}(0) = E_{p,\alpha}(1) = -\frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}} < 0,$$

where we used the fact that $A_{p,\alpha} > 0$ in this case. Next we claim $E_{p,\alpha}\left(X_{p,\alpha}\right) > 0$. Assume to the contrary that $E_{p,\alpha}\left(X_{p,\alpha}\right) \leq 0$. This implies that $E_{p,\alpha}(s) \leq 0$ for all $s \in [0, 1]$. From (2.3) we get $G'_{p,\alpha} \leq 0$ on $[0, 1]$. Hence function $G_{p,\alpha}$ is nonincreasing on $[0, 1]$. Since

$$G_{p,\alpha}(0) = G_{p,\alpha}(1) = 0,$$

we find that $G_{p,\alpha} \equiv 0$ on $[0, 1]$ which in turn implies $G'_{p,\alpha} \equiv 0$ on $[0, 1]$. This leads to $E_{p,\alpha} \equiv 0$ on $[0, 1]$ (see again (2.3)). This is in a contradiction with (2.7), because formula (2.7) implies that function $E_{p,\alpha}$ cannot be identically equal to zero on $[0, 1]$. In this way we have proved that

$$E_{p,\alpha}\left(X_{p,\alpha}\right) > 0.$$

We have already proved that $E_{p,\alpha}(0) = E_{p,\alpha}(1) < 0$ and that $E_{p,\alpha}$ is nondecreasing on $\left[0, X_{p,\alpha}\right]$ and nonincreasing on $\left[X_{p,\alpha}, 1\right]$. Thus there exists $s_1 \in \left(0, X_{p,\alpha}\right)$ such that

$$E_{p,\alpha} \leq 0 \ \text{ on } \ [0, s_1] \ \text{ and } \ E_{p,\alpha} \geq 0 \ \text{ on } \ \left[s_1, X_{p,\alpha}\right],$$

and there exists $s_2 \in \left(X_{p,\alpha}, 1\right)$ such that

$$E_{p,\alpha} \geq 0 \ \text{ on } \ \left[X_{p,\alpha}, s_2\right] \ \text{ and } \ E_{p,\alpha} \leq 0 \ \text{ on } \ [s_2, 1].$$

So we obtain

$$E_{p,\alpha} \leq 0 \ \text{ on } \ [0, s_1], \ \ E_{p,\alpha} \geq 0 \ \text{ on } \ [s_1, s_2] \ \text{ and } \ E_{p,\alpha} \leq 0 \ \text{ on } \ [s_2, 1].$$

By using (2.3) we find

$$G'_{p,\alpha} \leq 0 \ \text{ on } \ [0, s_1], \ \ G'_{p,\alpha} \geq 0 \ \text{ on } \ [s_1, s_2] \ \text{ and } \ G'_{p,\alpha} \leq 0 \ \text{ on } \ [s_2, 1].$$

Therefore function $G_{p,\alpha}$ is nonincreasing on $[0, s_1]$, nondecreasing on $[s_1, s_2]$ and nonincreasing on $[s_2, 1]$. Since $G_{p,\alpha}(0) = G_{p,\alpha}(1) = 0$ there exists $s_3 \in [s_1, s_2]$ such that

$$G_{p,\alpha} \leq 0 \ \text{ on } \ [0, s_3] \ \text{ and } \ G_{p,\alpha} \geq 0 \ \text{ on } \ [s_3, 1].$$

From (2.1) we can conclude that

$$F'_{p,\alpha} \leq 0 \ \text{ on } \ [0, s_3] \ \text{ and } \ F'_{p,\alpha} \geq 0 \ \text{ on } \ [s_3, 1].$$

Hence function $F_{p,\alpha}$ is nonincreasing on $[0, s_3]$ and nondecreasing on $[s_3, 1]$. This implies that

$$F_{p,\alpha}(s) \leq \max\left\{F_{p,\alpha}(0), F_{p,\alpha}(1)\right\},$$

for all $s \in [0, 1]$. By (2.9) we have $F_{p,\alpha}(0) \leq 0$ and since $F_{p,\alpha}(1) = 0$ we finally conclude that $F_{p,\alpha}(s) \leq 0$ for all $s \in [0, 1]$.

Case $p = \beta$ It remains for us to consider what is happening in this case. From (2.14) we find that $A_{p,\alpha} = 0$. Hence we have $D_{p,\alpha} = C^2_{p,\alpha} + 4A_{p,\alpha} = C^2_{p,\alpha}$ and

$$C_{p,\alpha} = A_{p,\alpha} + \frac{\alpha + 2}{p} - 1 = \frac{\alpha + 2}{p} - 1 < 0,$$

which implies $\sqrt{D_{p,\alpha}} = -C_{p,\alpha}$. Therefore

$$x_{p,\alpha} = \frac{-C_{p,\alpha} - \sqrt{D_{p,\alpha}}}{2} = 0 \text{ and } X_{p,\alpha} = \frac{-C_{p,\alpha} + \sqrt{D_{p,\alpha}}}{2} = -C_{p,\alpha} > 0,$$

and by using (2.12) again we get

$$x_{p,\alpha} = 0 < X_{p,\alpha} < 1.$$

Since $L_{p,\alpha}(s) = -\left(s - x_{p,\alpha}\right)\left(s - X_{p,\alpha}\right)$ for $s \in [0, 1]$ (see (2.13)) we find that

$$L_{p,\alpha} \geq 0 \text{ on } \left[0, X_{p,\alpha}\right] \text{ and } L_{p,\alpha} \leq 0 \text{ on } \left[X_{p,\alpha}, 1\right],$$

and by using (2.5) we have

$$E'_{p,\alpha} \geq 0 \text{ on } \left[0, X_{p,\alpha}\right] \text{ and } E'_{p,\alpha} \leq 0 \text{ on } \left[X_{p,\alpha}, 1\right].$$

Therefore we can conclude that function $E_{p,\alpha}$ is nondecreasing on $\left[0, X_{p,\alpha}\right]$ and nonincreasing on $\left[X_{p,\alpha}, 1\right]$. Recall that (see (2.7))

$$E_{p,\alpha}(s) = \left(\frac{1}{2-s}\right)^{2(\alpha+2)-p+1} s^{\frac{A_{p,\alpha}}{2}} (1-s)^{\frac{\alpha+2}{p}} - \frac{1}{(4(\alpha+2)-2p)\, B_{p,\alpha}},$$

and since

$$A_{p,\alpha} = 4(\alpha+2) - 2p - \frac{2(\alpha+2)}{p} = 0,$$

we obtain

$$E_{p,\alpha}(s) = \left(\frac{1}{2-s}\right)^{\frac{\alpha+2}{p}+1} (1-s)^{\frac{\alpha+2}{p}} - \frac{1}{\frac{2(\alpha+2)}{p} B_{p,\alpha}}.$$

Let

$$c = \frac{\alpha+2}{p} = \frac{\alpha+2}{\beta} = \frac{\alpha+2}{\alpha+2+\sqrt{(\alpha+2)^2-(\alpha+2)}}.$$

It is easy to check that

$$\frac{1}{2} < c = \frac{\alpha+2}{\alpha+2+\sqrt{(\alpha+2)^2-(\alpha+2)}} < 2-\sqrt{2},$$

because of $\alpha + 2 > 2$. Then

$$E_{p,\alpha}(s) = \left(\frac{1}{2-s}\right)^{c+1} (1-s)^c - \frac{\sin \pi c}{2\pi c},$$

which implies that

$$E_{p,\alpha}(0) = \frac{\pi c - 2^c \sin \pi c}{2^{c+1}\pi c} \text{ and } E_{p,\alpha}(1) = -\frac{\sin \pi c}{2\pi c} < 0.$$

Since $1/2 < c < 2 - \sqrt{2}$ we have

$$2^c \sin \pi c \le 2^c < 2^{2-\sqrt{2}} < \frac{\pi}{2} < \pi c.$$

Therefore

$$E_{p,\alpha}(0) = \frac{\pi c - 2^c \sin \pi c}{2^{c+1}\pi c} > 0 \text{ and } E_{p,\alpha}(1) = -\frac{\sin \pi c}{2\pi c} < 0.$$

We have also already proved that function $E_{p,\alpha}$ is nondecreasing on $\left[0, X_{p,\alpha}\right]$ and nonincreasing on $\left[X_{p,\alpha}, 1\right]$. Thus there exists $S \in \left(X_{p,\alpha}, 1\right)$ such that

$$E_{p,\alpha} \ge 0 \text{ on } [0, S] \text{ and } E_{p,\alpha} \le 0 \text{ on } [S, 1].$$

Then from (2.3) we obtain that

$$G'_{p,\alpha} \ge 0 \text{ on } [0, S] \text{ and } G'_{p,\alpha} \le 0 \text{ on } [S, 1].$$

Hence the function $G_{p,\alpha}$ is nondecreasing on $[0, S]$ and nonincreasing on $[S, 1]$. So we conclude that

$$G_{p,\alpha}(s) \ge \min\left\{G_{p,\alpha}(0), G_{p,\alpha}(1)\right\},$$

for all $s \in [0, 1]$. On the other hand, since (see (2.2))

$$G_{p,\alpha}(0) = 0 \text{ and } G_{p,\alpha}(1) = B_{p,\alpha} - \int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t = 0,$$

we obtain $G_{p,\alpha} \ge 0$ on $[0, 1]$. Then from (2.1) we find $F'_{p,\alpha} \ge 0$ on $[0, 1]$ which implies that $F_{p,\alpha}$ is nondecreasing function on $[0, 1]$. Therefore $F_{p,\alpha}(s) \le F_{p,\alpha}(1)$ for all $s \in [0, 1]$. Since

$$F_{p,\alpha}(1) = \xi_{p,\alpha}(1) \int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t = 0 \cdot B_{p,\alpha} = 0,$$

we have that $F_{p,\alpha}(s) \le 0$ for all $s \in [0, 1]$. On the other hand, note that in this case it was not necessary to further assume at the beginning that inequality (2.9) is valid. This completes the proof. □

Remark 2.1 It will be interesting to see what happens in the case when $\beta < p < \alpha_0$. Then we can apply the same procedure as in the proof of Lemma 2.2. Namely, in this case from (2.14) we conclude that $A_{p,\alpha} < 0$. Therefore from (2.7) we obtain

$$\lim_{s\to 0^+} E_{p,\alpha}(s) = +\infty \quad \text{and} \quad E_{p,\alpha}(1) = -\frac{1}{(4(\alpha+2)-2p)\,B_{p,\alpha}} < 0.$$

On the other hand, we have

$$C_{p,\alpha} = \underbrace{A_{p,\alpha}}_{<0} - \underbrace{\left(1-\frac{\alpha+2}{p}\right)}_{>0} < 0,$$

and $D_{p,\alpha} = C_{p,\alpha}^2 + 4A_{p,\alpha} < C_{p,\alpha}^2$ which implies $\sqrt{D_{p,\alpha}} < -C_{p,\alpha}$. Hence

$$x_{p,\alpha} = \frac{-C_{p,\alpha} - \sqrt{D_{p,\alpha}}}{2} > 0,$$

and by using (2.12) we get

$$0 < x_{p,\alpha} < X_{p,\alpha} < 1.$$

This already complicates the determination of the sign of the quadratic function $L_{p,\alpha}(s)$ on the interval [0, 1] and proceeding further similarly as in the proof of Lemma 2.2, it turns out that it is not easy to conclude under which conditions the function $F_{p,\alpha}(s)$ is nonpositive on the interval [0, 1] in all possible cases. ◇

3 Proofs of Theorem 1.1 and Theorem 1.2

Let us first consider the case when $\alpha > 0$ and $\alpha + 2 < p < 2(\alpha + 2)$. Later we will focus on the special cases when $\alpha_0 \leq p < 2(\alpha + 2)$ or $\alpha + 2 < p \leq \beta$ as in the statements of Theorem 1.1 or Theorem 1.2, respectively. Let $f \in A_\alpha^p$. In view of (1.1) we have the corresponding lower bound, so we need to prove

$$\|\mathrm{H}f\|_{A_\alpha^p} \leq \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}\|f\|_{A_\alpha^p}. \tag{3.1}$$

We will use the technique developed in the paper [2]. Denote

$$\varphi(r) = 2\mathrm{M}_p^p(r, f) \quad \text{and} \quad \chi(r) = \varphi(r) - \varphi(0),$$

for $0 \leq r < 1$. Of course, functions φ and χ depend on the choice of the initial function f which we assume to be fixed in the considerations that follow. Then φ is nondecreasing and differentiable function on the interval (0, 1), which implies that function χ is also nondecreasing and differentiable on (0, 1). Therefore

$$\chi' \geq 0 \ \text{ on } \ (0,1) \ \text{ and } \ \chi(r) = \int_0^r \chi'(s)\,\mathrm{d}s, \tag{3.2}$$

for $0 \leq r < 1$. As shown in Sect. 1.2 we know that

$$\|\mathrm{H}f\|_{A^p_\alpha} \leq \int_0^1 \|\mathrm{T}_t f\|_{A^p_\alpha}\,\mathrm{d}t, \tag{3.3}$$

and

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \psi_{p,\alpha}(t)\left(\frac{\alpha+1}{\pi}\int_{\mathrm{D}_t} |w|^{p-2(\alpha+2)}|f(w)|^p\left(1-|w|^2\right)^\alpha \mathrm{dm}(w)\right)^{\frac{1}{p}},$$

where

$$\mathrm{D}_t = \mathrm{D}\left(c_t,\rho_t\right), \quad c_t = \frac{1}{2-t} \ \text{ and } \ \rho_t = \frac{1-t}{2-t}.$$

Note that it is valid

$$\mathrm{D}_t \subset \mathrm{A}\left(0, c_t-\rho_t, c_t+\rho_t\right) = \mathrm{A}\left(0, \frac{t}{2-t}, 1\right).$$

We denote

$$\mathrm{A}_t = \mathrm{A}\left(0, \frac{t}{2-t}, 1\right).$$

Consequently

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \psi_{p,\alpha}(t)\left(\frac{\alpha+1}{\pi}\int_{\mathrm{A}_t} |w|^{p-2(\alpha+2)}|f(w)|^p\left(1-|w|^2\right)^\alpha \mathrm{dm}(w)\right)^{\frac{1}{p}},$$

or equivalently

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \psi_{p,\alpha}(t)\left((\alpha+1)\int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1}\left(1-r^2\right)^\alpha \varphi(r)\,\mathrm{d}r\right)^{\frac{1}{p}}. \tag{3.4}$$

On the other hand

$$\frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}\|f\|_{A^p_\alpha} = \int_0^1 \psi_{p,\alpha}(t)\left((\alpha+1)\int_0^1 r\left(1-r^2\right)^\alpha \varphi(r)\,\mathrm{d}r\right)^{\frac{1}{p}}\mathrm{d}t. \tag{3.5}$$

Because of (3.3), (3.4) and (3.5), we can conclude that (3.1) holds if the following inequality is true

$$\int_0^1 \psi_{p,\alpha}(t)\left(I_{p,\alpha}(t)^{1/p} - J_\alpha^{1/p}\right)\mathrm{d}t \le 0, \tag{3.6}$$

where

$$I_{p,\alpha}(t) = \int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1}\left(1-r^2\right)^\alpha \varphi(r)\,\mathrm{d}r,$$

and

$$J_\alpha = \int_0^1 r\left(1-r^2\right)^\alpha \varphi(r)\,\mathrm{d}r.$$

Note that $I_{p,\alpha}(t)$ and J_α also depend on the function φ, that is on the function f that was initially selected. We obtain

$$I_{p,\alpha}(t)^{1/p} - J_\alpha^{1/p} \le \frac{1}{p} J_\alpha^{\frac{1}{p}-1}\left(I_{p,\alpha}(t) - J_\alpha\right),$$

where we used the well known fact that $x^\gamma - y^\gamma \le \gamma y^{\gamma-1}(x-y)$ for $x \ge 0$, $y \ge 0$ and $\gamma \in (0, 1)$ as well as the fact that it is valid $1/p \in (0, 1)$ because of $p > \alpha + 2 > 2$. Thus we have that (3.6) holds if the following inequality is true

$$\int_0^1 \psi_{p,\alpha}(t)\left(I_{p,\alpha}(t) - J_\alpha\right)\mathrm{d}t \le 0,$$

or

$$\int_0^1 \psi_{p,\alpha}(t)\left(\int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1}\left(1-r^2\right)^\alpha \varphi(r)\,\mathrm{d}r - \int_0^1 r\left(1-r^2\right)^\alpha \varphi(r)\,\mathrm{d}r\right)\mathrm{d}t \le 0,$$

or equivalently

$$V_{p,\alpha} + \varphi(0) W_{p,\alpha} \le U_{p,\alpha}, \tag{3.7}$$

where we denoted

$$V_{p,\alpha} = \int_0^1 \psi_{p,\alpha}(t)\int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1}\left(1-r^2\right)^\alpha \chi(r)\,\mathrm{d}r\mathrm{d}t,$$

and

$$W_{p,\alpha} = \int_0^1 \psi_{p,\alpha}(t)\int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1}\left(1-r^2\right)^\alpha \mathrm{d}r\mathrm{d}t$$
$$- \int_0^1 \psi_{p,\alpha}(t)\int_0^1 r\left(1-r^2\right)^\alpha \mathrm{d}r\mathrm{d}t,$$

and

$$U_{p,\alpha} = \int_0^1 \psi_{p,\alpha}(t) \int_0^1 r \left(1 - r^2\right)^{\alpha} \chi(r)\, \mathrm{d}r \mathrm{d}t.$$

Then by using a change of variable $\rho = r^2$ we get

$$\begin{aligned}
W_{p,\alpha} &= \frac{1}{2} \int_0^1 \psi_{p,\alpha}(t) \int_{\frac{t}{2-t}}^1 \left(r^2\right)^{\frac{p}{2}-(\alpha+2)} \left(1 - r^2\right)^{\alpha} \mathrm{d}\left(r^2\right) \mathrm{d}t \\
&\quad -\frac{1}{2} \int_0^1 \left(1 - r^2\right)^{\alpha} \mathrm{d}\left(r^2\right) \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t \\
&= \frac{1}{2} \int_0^1 \psi_{p,\alpha}(t) \int_{\left(\frac{t}{2-t}\right)^2}^1 \rho^{\frac{p}{2}-(\alpha+2)} (1-\rho)^{\alpha}\, \mathrm{d}\rho \mathrm{d}t \\
&\quad -\frac{1}{2} \int_0^1 (1-\rho)^{\alpha}\, \mathrm{d}\rho \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t \\
&= \frac{1}{2} \left(\int_0^1 \psi_{p,\alpha}(t) \xi_{p,\alpha}(t)\, \mathrm{d}t - \frac{1}{\alpha+1} \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t \right) \\
&= \frac{1}{2} F_{p,\alpha}(0).
\end{aligned}$$

On the other hand by using Fubini theorem we obtain

$$\begin{aligned}
V_{p,\alpha} &= \int_0^1 \psi_{p,\alpha}(t) \left(\int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1} \left(1-r^2\right)^{\alpha} \chi(r)\, \mathrm{d}r \right) \mathrm{d}t \\
&= \int_0^1 \psi_{p,\alpha}(t) \left(\int_{\frac{t}{2-t}}^1 r^{p-2(\alpha+2)+1} \left(1-r^2\right)^{\alpha} \int_0^r \chi'(s)\, \mathrm{d}s \mathrm{d}r \right) \mathrm{d}t \\
&= \int_0^1 \psi_{p,\alpha}(t) \left(\int_{\frac{t}{2-t}}^1 \int_0^r r^{p-2(\alpha+2)+1} \left(1-r^2\right)^{\alpha} \chi'(s)\, \mathrm{d}s \mathrm{d}r \right) \mathrm{d}t \\
&= \int_0^1 \psi_{p,\alpha}(t) \left(\int_0^1 \int_{\max\left\{s, \frac{t}{2-t}\right\}}^1 r^{p-2(\alpha+2)+1} \left(1-r^2\right)^{\alpha} \chi'(s)\, \mathrm{d}r \mathrm{d}s \right) \mathrm{d}t \\
&= \int_0^1 \psi_{p,\alpha}(t) \left(\int_0^1 \chi'(s) \int_{\max\left\{s, \frac{t}{2-t}\right\}}^1 r^{p-2(\alpha+2)+1} \left(1-r^2\right)^{\alpha}\, \mathrm{d}r \mathrm{d}s \right) \mathrm{d}t \\
&= \int_0^1 \psi_{p,\alpha}(t) \left(\int_0^1 \frac{\chi'(s)}{2} \int_{\max^2\left\{s, \frac{t}{2-t}\right\}}^1 \rho^{\frac{p}{2}-(\alpha+2)} (1-\rho)^{\alpha}\, \mathrm{d}\rho \mathrm{d}s \right) \mathrm{d}t,
\end{aligned}$$

and by using change of variable $s = u/(2-u)$ we get

$$V_{p,\alpha} = \int_0^1 \psi_{p,\alpha}(t) \left(\int_0^1 \frac{\chi'\left(\frac{u}{2-u}\right)}{(2-u)^2} \int_{\max^2\left\{\frac{u}{2-u}, \frac{t}{2-t}\right\}}^1 \rho^{\frac{p}{2}-(\alpha+2)} (1-\rho)^{\alpha} \, \mathrm{d}\rho \mathrm{d}u \right) \mathrm{d}t,$$

or equivalently

$$V_{p,\alpha} = \int_0^1 \psi_{p,\alpha}(t) \left(\int_0^1 \frac{1}{(2-u)^2} \chi'\left(\frac{u}{2-u}\right) \xi_{p,\alpha}\left(\max\{u,t\}\right) \mathrm{d}u \right) \mathrm{d}t.$$

This implies that

$$V_{p,\alpha} = \int_0^1 \frac{1}{(2-u)^2} \chi'\left(\frac{u}{2-u}\right) \left(\int_0^1 \psi_{p,\alpha}(t) \xi_{p,\alpha}\left(\max\{u,t\}\right) \mathrm{d}t \right) \mathrm{d}u,$$

or

$$V_{p,\alpha} = \int_0^1 \frac{1}{(2-u)^2} \chi'\left(\frac{u}{2-u}\right) \left(\xi_{p,\alpha}(u) \int_0^u \psi_{p,\alpha}(t) \mathrm{d}t + \int_u^1 \psi_{p,\alpha}(t) \xi_{p,\alpha}(t) \, \mathrm{d}t \right) \mathrm{d}u.$$

Similarly we have

$$\begin{aligned} U_{p,\alpha} &= \int_0^1 r\left(1-r^2\right)^{\alpha} \chi(r) \, \mathrm{d}r \int_0^1 \psi_{p,\alpha}(t) \, \mathrm{d}t \\ &= \int_0^1 r\left(1-r^2\right)^{\alpha} \int_0^r \chi'(s) \, \mathrm{d}s \mathrm{d}r \int_0^1 \psi_{p,\alpha}(t) \, \mathrm{d}t \\ &= \int_0^1 \chi'(s) \int_s^1 r\left(1-r^2\right)^{\alpha} \mathrm{d}r \mathrm{d}s \int_0^1 \psi_{p,\alpha}(t) \, \mathrm{d}t \\ &= \int_0^1 \frac{\chi'(s)}{2} \frac{1}{\alpha+1} \left(1-s^2\right)^{\alpha+1} \mathrm{d}s \int_0^1 \psi_{p,\alpha}(t) \, \mathrm{d}t, \end{aligned}$$

which leads to

$$U_{p,\alpha} = \int_0^1 \frac{1}{(2-u)^2} \chi'\left(\frac{u}{2-u}\right) \frac{1}{\alpha+1} \left(1-\left(\frac{u}{2-u}\right)^2\right)^{\alpha+1} \mathrm{d}u \int_0^1 \psi_{p,\alpha}(t) \, \mathrm{d}t,$$

or equivalently

$$U_{p,\alpha} = \int_0^1 \frac{1}{(2-u)^2} \chi'\left(\frac{u}{2-u}\right) \left(\frac{1}{\alpha+1} \left(1-\left(\frac{u}{2-u}\right)^2\right)^{\alpha+1} \int_0^1 \psi_{p,\alpha}(t) \, \mathrm{d}t \right) \mathrm{d}u.$$

Based on the foregoing considerations we can conclude

$$V_{p,\alpha} - U_{p,\alpha} = \int_0^1 \frac{1}{(2-u)^2} \chi' \left(\frac{u}{2-u} \right) F_{p,\alpha}(u)\, \mathrm{d}u.$$

Finally to obtain (3.7) it is enough to prove that

$$\int_0^1 \frac{1}{(2-u)^2} \chi' \left(\frac{u}{2-u} \right) F_{p,\alpha}(u)\, \mathrm{d}u + \frac{\varphi(0)}{2} F_{p,\alpha}(0) \le 0.$$

Note also that $\varphi(0) = 2|f(0)|^p$. So we can conclude the following. In the case when $\alpha > 0$ and $\alpha + 2 < p < 2(\alpha + 2)$ if

$$\int_0^1 \frac{1}{(2-u)^2} \chi' \left(\frac{u}{2-u} \right) F_{p,\alpha}(u)\, \mathrm{d}u + |f(0)|^p F_{p,\alpha}(0) \le 0, \tag{3.8}$$

then it is valid inequality (3.1). Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $f \in A^p_\alpha$ where $\alpha > 0$ and $\alpha_0 \le p < 2(\alpha + 2)$. In view of (1.1) it is enough to prove that inequality (3.1) is valid. By using Lemma 2.1 we have that

$$F_{p,\alpha}(u) \le 0, \tag{3.9}$$

for all $u \in [0, 1]$. From (3.2) we conclude that

$$\frac{1}{(2-u)^2} \chi' \left(\frac{u}{2-u} \right) \ge 0. \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$\int_0^1 \frac{1}{(2-u)^2} \chi' \left(\frac{u}{2-u} \right) F_{p,\alpha}(u)\, \mathrm{d}u + |f(0)|^p F_{p,\alpha}(0) \le 0,$$

which leads that inequality (3.8) is valid. This implies that inequality (3.1) is also valid, which completes the proof. □

Next we will use the previously presented results as well as Lemma 2.2 from Sect. 2. Thus we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Let $f \in A^p_\alpha$ where $\alpha > 0$ and $\alpha + 2 < p \le \beta$. We can use Lemma 2.2, because we know that under the assumptions of Theorem 1.2 the following condition is satisfied

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\, \mathrm{d}t \le \frac{1}{\alpha+1} \int_0^1 \psi_{p,\alpha}(t)\, \mathrm{d}t = \frac{1}{\alpha+1} \mathrm{B} \left(\frac{\alpha+2}{p}, 1 - \frac{\alpha+2}{p} \right).$$

Therefore

$$F_{p,\alpha}(u) \leq 0,$$

for all $u \in [0, 1]$. Since $\chi' \geq 0$ we derive that

$$\int_0^1 \frac{1}{(2-u)^2} \chi' \left(\frac{u}{2-u} \right) F_{p,\alpha}(u) \, \mathrm{d}u + |f(0)|^p F_{p,\alpha}(0) \leq 0,$$

which implies validity of the inequality (3.1). This finishes the proof. □

4 Proof of Theorem 1.3

In this section we consider the case when $-1 < \alpha < 0$ and $p > \alpha + 2$. Let also $f \in A^p_\alpha$. As we showed in Sect. 1.2 we have

$$\|\mathrm{H} f\|_{A^p_\alpha} \leq \int_0^1 \|\mathrm{T}_t f\|_{A^p_\alpha} \, \mathrm{d}t,$$

and

$$\|\mathrm{T}_t f\|_{A^p_\alpha} = \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi} \int_{\mathrm{D}_t} |w|^{p-2(\alpha+2)} |f(w)|^p \left(\frac{\rho_t^2 - |w - c_t|^2}{\rho_t} \right)^\alpha \mathrm{dm}(w) \right)^{\frac{1}{p}},$$

where

$$\mathrm{D}_t = \mathrm{D}\left(c_t, \rho_t\right), \quad c_t = \frac{1}{2-t} \quad \text{and} \quad \rho_t = \frac{1-t}{2-t}.$$

Let

$$\eta_t(z) = \rho_t z + c_t \quad \text{for} \quad z \in \mathbb{D} \quad \text{and} \quad 0 < t < 1.$$

Note that $\eta_t(\mathbb{D}) = \mathrm{D}_t \subset \mathbb{D}$. We will also use the following well known result, which is a consequence of Littlewood Subordination Principle (see Chapter 11 in [12]).

Lemma 4.1 (see Theorem 11.6 in [12]) *If $\eta : \mathbb{D} \to \mathbb{D}$ is holomorphic function, $p > 0$ and $\alpha > -1$ then*

$$\int_{\mathbb{D}} |(f \circ \eta)(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z) \leq \left(\frac{1 + |\eta(0)|}{1 - |\eta(0)|} \right)^{\alpha+2} \int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z),$$

for all holomorphic functions f on $\mathbb{D}$.

After the preliminary results mentioned above, we are now ready to prove Theorem 1.3 from the Sect. 1.4.

Proof of Theorem 1.3 Let $f \in A^p_\alpha$. Then we consider the following two cases as in the statement of Theorem 1.3.

Case (i) $p \geq 2(\alpha + 2)$ In this case we actually have that $|w|^{p-2(\alpha+2)} \leq 1$ for all $w \in \mathrm{D_t} = \eta_t(\mathbb{D})$. Therefore

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi}\right)^{\frac{1}{p}} \left(\int_{\eta_t(\mathbb{D})} |f(w)|^p \left(\frac{\rho_t^2 - |w - c_t|^2}{\rho_t}\right)^\alpha \mathrm{dm}(w)\right)^{\frac{1}{p}}.$$

After change of variable $w = \eta_t(z)$ we obtain

$$\begin{aligned}\|\mathrm{T}_t f\|_{A^p_\alpha} &\leq \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi}\right)^{\frac{1}{p}} \left(\rho_t^\alpha \int_{\mathbb{D}} |(f \circ \eta_t)(z)|^p \left(1 - |z|^2\right)^\alpha \left|\eta_t'(z)\right|^2 \mathrm{dm}(z)\right)^{\frac{1}{p}} \\ &= \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi}\right)^{\frac{1}{p}} \left(\rho_t^{\alpha+2} \int_{\mathbb{D}} |(f \circ \eta_t)(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z)\right)^{\frac{1}{p}}.\end{aligned}$$

Since

$$\rho_t^{\alpha+2} \left(\frac{1 + |\eta_t(0)|}{1 - |\eta_t(0)|}\right)^{\alpha+2} = \left(\rho_t \cdot \frac{1 + c_t}{1 - c_t}\right)^{\alpha+2} = \left(\frac{1-t}{2-t} \cdot \frac{3-t}{1-t}\right)^{\alpha+2} = \left(\frac{3-t}{2-t}\right)^{\alpha+2},$$

by using Lemma 4.1 we obtain

$$\rho_t^{\alpha+2} \int_{\mathbb{D}} |(f \circ \eta_t)(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z) \leq \left(\frac{3-t}{2-t}\right)^{\alpha+2} \int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z).$$

Hence we conclude

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \left(\frac{3-t}{2-t}\right)^{\frac{\alpha+2}{p}} \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi} \int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z)\right)^{\frac{1}{p}},$$

or

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq \left(\frac{3-t}{2-t}\right)^{\frac{\alpha+2}{p}} \psi_{p,\alpha}(t) \|f\|_{A^p_\alpha}.$$

Note that

$$\left(\frac{3-t}{2-t}\right)^{\frac{\alpha+2}{p}} \leq 2^{\frac{\alpha+2}{p}},$$

for all $0 < t < 1$. This leads to

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \leq 2^{\frac{\alpha+2}{p}} \psi_{p,\alpha}(t) \|f\|_{A^p_\alpha}.$$

Finally we obtain

$$\|\mathrm{H}f\|_{A^p_\alpha} \le \int_0^1 \|\mathrm{T}_t f\|_{A^p_\alpha}\,\mathrm{d}t \le 2^{\frac{\alpha+2}{p}} \cdot \int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t \cdot \|f\|_{A^p_\alpha} = 2^{\frac{\alpha+2}{p}} \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}} \|f\|_{A^p_\alpha}.$$

Therefore

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} \le 2^{\frac{\alpha+2}{p}} \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

Case (ii) $\boldsymbol{\alpha + 2 < p < 2(\alpha + 2)}$ In this case we have

$$|w| \ge c_t - |w - c_t| > c_t - \rho_t = \frac{t}{2-t} \quad \text{for all} \;\; w \in \mathrm{D}_t.$$

Therefore

$$|w|^{p-2(\alpha+2)} \le \left(\frac{t}{2-t}\right)^{p-2(\alpha+2)} = \left(\frac{2-t}{t}\right)^{2(\alpha+2)-p} \quad \text{for all} \;\; w \in \mathrm{D}_t.$$

This implies that $\|\mathrm{T}_t f\|_{A^p_\alpha}$ is not greater than

$$\left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)}{p}-1} \psi_{p,\alpha}(t) \left(\frac{\alpha+1}{\pi}\int_{\eta_t(\mathbb{D})} |f(w)|^p \left(\frac{\rho_t^2 - |w-c_t|^2}{\rho_t}\right)^{\alpha} \mathrm{dm}(w)\right)^{\frac{1}{p}}.$$

Similar to the Case (i) we get that it is valid

$$\left(\frac{\alpha+1}{\pi}\int_{\eta_t(\mathbb{D})} |f(w)|^p \left(\frac{\rho_t^2 - |w-c_t|^2}{\rho_t}\right)^{\alpha} \mathrm{dm}(w)\right)^{\frac{1}{p}} \le \left(\frac{3-t}{2-t}\right)^{\frac{\alpha+2}{p}} \|f\|_{A^p_\alpha},$$

which leads to

$$\left(\frac{\alpha+1}{\pi}\int_{\eta_t(\mathbb{D})} |f(w)|^p \left(\frac{\rho_t^2 - |w-c_t|^2}{\rho_t}\right)^{\alpha} \mathrm{dm}(w)\right)^{\frac{1}{p}} \le 2^{\frac{\alpha+2}{p}} \|f\|_{A^p_\alpha}.$$

Hence we obtain

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \le 2^{\frac{\alpha+2}{p}} \left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)}{p}-1} \psi_{p,\alpha}(t) \|f\|_{A^p_\alpha}. \tag{4.1}$$

On the other hand, we have

$$\begin{aligned}\left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)}{p}-1}\psi_{p,\alpha}(t) &= \left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)}{p}-1} t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}}\\ &= (t+2(1-t))^{\frac{2(\alpha+2)}{p}-1} t^{-\frac{\alpha+2}{p}}(1-t)^{-\frac{\alpha+2}{p}}\\ &\le \frac{t^{\frac{2(\alpha+2)}{p}-1}+2^{\frac{2(\alpha+2)}{p}-1}(1-t)^{\frac{2(\alpha+2)}{p}-1}}{t^{\frac{\alpha+2}{p}}(1-t)^{\frac{\alpha+2}{p}}},\end{aligned}$$

where we actually used the known fact which states that the following inequality is valid

$$(x+y)^{\gamma} \le x^{\gamma}+y^{\gamma},$$

for all $x \ge 0$, $y \ge 0$ and $\gamma \in (0,1)$ as well as the fact that

$$\frac{2(\alpha+2)}{p}-1 \in (0,1),$$

because of $\alpha+2 < p < 2(\alpha+2)$. So we get

$$\left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)}{p}-1}\psi_{p,\alpha}(t) \le t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}} + 2^{\frac{2(\alpha+2)}{p}-1}(1-t)^{\frac{\alpha+2}{p}-1}t^{-\frac{\alpha+2}{p}},$$

or

$$\left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)}{p}-1}\psi_{p,\alpha}(t) \le \psi_{p,\alpha}(t)+2^{\frac{2(\alpha+2)}{p}-1}\psi_{p,\alpha}(1-t). \tag{4.2}$$

From (4.1) and (4.2) we obtain

$$\|\mathrm{T}_t f\|_{A^p_\alpha} \le 2^{\frac{\alpha+2}{p}}\left(\psi_{p,\alpha}(t)+2^{\frac{2(\alpha+2)}{p}-1}\psi_{p,\alpha}(1-t)\right)\|f\|_{A^p_\alpha}.$$

Since

$$\int_0^1 \psi_{p,\alpha}(1-t)\,\mathrm{d}t = \int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t = \mathrm{B}\left(\frac{\alpha+2}{p}, 1-\frac{\alpha+2}{p}\right) = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}},$$

we find

$$\begin{aligned}\|\mathrm{H}f\|_{A^p_\alpha} &\le \int_0^1 \|\mathrm{T}_t f\|_{A^p_\alpha}\,\mathrm{d}t\\ &\le 2^{\frac{\alpha+2}{p}}\left(\int_0^1 \psi_{p,\alpha}(t)\,\mathrm{d}t + 2^{\frac{2(\alpha+2)}{p}-1}\int_0^1 \psi_{p,\alpha}(1-t)\,\mathrm{d}t\right)\|f\|_{A^p_\alpha}\end{aligned}$$

$$= 2^{\frac{\alpha+2}{p}} \left(\frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}} + 2^{\frac{2(\alpha+2)}{p}-1} \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}} \right) \|f\|_{A^p_\alpha}$$
$$= 2^{\frac{\alpha+2}{p}} \left(1 + 2^{\frac{2(\alpha+2)}{p}-1}\right) \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}} \|f\|_{A^p_\alpha}.$$

Finally we obtain

$$\|\mathrm{H}\|_{A^p_\alpha \to A^p_\alpha} \leq 2^{\frac{\alpha+2}{p}} \left(1 + 2^{\frac{2(\alpha+2)}{p}-1}\right) \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}.$$

This completes the proof. □

References

1. Aleman, A., Montes-Rodríguez, A., Sarafoleanu, A.: The eigenfunctions of the Hilbert matrix. Constr. Approx. **36**, 353–374 (2012)
2. Božin, V., Karapetrović, B.: Norm of the Hilbert matrix on Bergman spaces. J. Funct. Anal. **274**, 525–543 (2018)
3. Diamantopoulos, E.: Hilbert matrix on Bergman spaces. Ill. J. Math. **48**, 1067–1078 (2004)
4. Diamantopoulos, E., Siskakis, A.G.: Composition operators and the Hilbert matrix. Stud. Math. **140**, 191–198 (2000)
5. Dostanić, M., Jevtić, M., Vukotić, D.: Norm of the Hilbert matrix on Bergman and Hardy spaces and theorem of Nehari type. J. Funct. Anal. **254**, 2800–2815 (2008)
6. Galanopoulos, P., Girela, D., Peláez, J.A., Siskakis, A.G.: Generalized Hilbert operators. Ann. Acad. Sci. Fenn. Math. **39**, 231–258 (2014)
7. Jevtić, M., Karapetrović, B.: Hilbert matrix on spaces of Bergman-type. J. Math. Anal. Appl. **453**, 241–254 (2017)
8. Karapetrović, B.: Norm of the Hilbert matrix operator on the weighted Bergman spaces. Glasgow Math. J. **60**, 513–525 (2018)
9. Lindström, M., Miihkinen, S., Wikman, N.: Norm estimates of weighted composition operators pertaining to the Hilbert matrix. Proc. Am. Math. Soc. **147**, 2425–2435 (2019)
10. Lindström, M., Miihkinen, S., Wikman, N.: On the exact value of the norm of the Hilbert matrix operator on the weighted Bergman spaces, to appear in Ann. Acad. Sci. Fenn. Math. (https://arxiv.org/abs/2001.10476)
11. Magnus, W.: On the spectrum of Hilbert's matrix. Am. J. Math. **72**, 699–704 (1950)
12. Zhu, K.: Operator Theory in Function Spaces, Second Edition, Mathematical Surveys and Monographs 138. American Mathematical Society, Providence, RI (2007)