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Estimates on Partial Derivatives and Logarithmic Partial Derivatives of Holomorphic Functions on Polydiscs and Beyond

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Abstract
In this paper, we give estimates on partial derivatives and logarithmic partial derivatives for holomorphic functions on polydiscs. Estimates will also be utilized to characterize entire solutions of a class of partial differential equations, which gives a new form of Picard's theorem and its higher dimensional version.

Keywords Holomorphic function · Partial derivative · Logarithmic derivative · Entire solution · Partial differential equation

Mathematics Subject Classification 32A10 · 35G20 · 32A22 · 30D35

1 Introduction

Estimates on the norm of holomorphic functions play an important role in complex analysis and applications to differential equations, number theory, etc. Estimating derivatives and logarithmic derivatives traces back to the well-known Borel–Carathòdory theorem and Nevanlinna logarithmic derivative lemma (see e.g., [18,27]), and there are various works in this direction, see e.g., [2,3,6,10,14,17,21,24,28–30], to list a few. In the recent paper [15], an estimate on the average $\frac{1}{2\pi}\int_0^{2\pi}|f^{(n)}(re^{i\theta})|d\theta$ was given by the real part of a holomorphic function f in an elementary way:

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Theorem A *Let f be a holomorphic function in $|z| \leq R$ in the complex plane. Then for $0 < r < R$ and $n \geq 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|\mathrm{d}\theta \leq \frac{2n!}{(R-r)^n}(A(R, f) - \Re\{f(0)\}),$$

where $A(r, f) = \max_{|z|\leq r} \Re\{f(z)\}$.

An application of Theorem A yields the following estimate on logarithmic derivatives of holomorphic functions:

Theorem B *Let f be a holomorphic function without zeros in $|z| \leq R$. Then for $0 < r < R$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|\mathrm{d}\theta$$
$$\leq \log^+ \frac{1}{R-r} + \log^+ \log^+ M(R, f) + \log^+ \log^+ \frac{1}{|f(0)|} + 3\log 2,$$

where $M(r, f) = \max\limits_{|z|\leq r}\{|f(z)|\}$.

On the above, $\log^+ M(R, f)$ can be controlled by and thus be replaced by the proximity function $m(R, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\theta})|\mathrm{d}\theta$ in a standard way (see [15]*). But the use of the maximum modulus $M(R, f)$ has its advantage (see Sect. 3), which makes it possible for us to extend the approach to more general settings in an elementary and efficient way.*

There are various estimates on logarithmic derivatives in various settings (see the references cited above). The so-called Logarithmic Derivative Lemma is a central tool in Nevanlinna theory and its applications to other problems such as complex differential equations. The estimate in Theorem B takes a particular form with an elementary proof (via essentially Cauchy's formula only) suitable for a general audience and for situations where the full force of Nevanlinna theory is not necessary (see [15]).

The present paper has two objectives: firstly to generalize Theorems A and B to holomorphic functions of several complex variables and further extend Theorem B so that it allows f to have zeros and also allows higher order partial derivatives in the estimate, with still an elementary approach; secondly to connect the above questions to characterizing entire solutions of certain partial differential equations and to the well-known Picard theorem.

In $\mathbb{C}^n$, there are two natural notions of "neighborhood": the ball and the polydiscs. It is a well-known fact, due to H. Poincaré, that for $n > 1$ there does not exist a biholomorphic mapping between the ball and the polydisc in $\mathbb{C}^n$ (see e.g., [11]). Most of the known $\mathbb{C}^n$-versions of the Logarithmic Derivative Lemma are for holomorphic(meromorphic) functions on the balls (see e.g., [2,14,28,30]) and few results on polydiscs are known (see [26] and Sect. 3). But the easiest approach to some fundamental facts about holomorphic functions of several complex variables is based on polydiscs rather than balls (see [23], p. 2 and [22]). We will give estimates on partial derivatives and logarithmic partial derivatives parallel to Theorems A and B for holomorphic functions on polydiscs with still an elementary approach that essentially only invoke Cauchy's formula. The estimates obtained will be utilized in the last section to characterize entire solutions of a class of partial differential equations, which, as a consequence, gives a new form of the well-known Picard theorem and its higher dimensional version.

2 Estimates on Partial Derivatives

We first introduce some standard notations. Let

$$\mathbb{C}^n = \{z = (z_1, z_2, \ldots, z_n) : \ z_j \in \mathbb{C}, \ j = 1, 2, \ldots, n\}$$

denote the n-dimensional complex vector space normed by

$$\|z\| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}.$$

For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}_+^n$, define the polydiscs

$$D_{\mathbf{r}}(a) = \{z \in \mathbb{C}^n : \ |z_j - a_j| < r_j, \ j = 1, 2, \ldots, n\}$$

and

$$\bar{D}_{\mathbf{r}}(a) = \{z \in \mathbb{C}^n : \ |z_j - a_j| \le r_j, \ j = 1, 2, \ldots, n\}.$$

For brevity we shall denote by $\bar{D}_{\mathbf{r}} = \bar{D}_{\mathbf{r}}(0)$ the polydisc with center at the origin.

The Bergman–Shilov boundary is given by

$$\partial_o D_{\mathbf{r}}(a) = \{z \in \mathbb{C}^n : \ |z_j - a_j| = r_j, \ j = 1, 2, \ldots, n\},$$

which is also called the skeleton of the polydisc and is the product of n circles. Note that this skeleton of the polydisc contains only part of the points on the boundary $\partial D_{\mathbf{r}}(a)$, i.e., $\partial_o D_{\mathbf{r}}(a) \subset \partial D_{\mathbf{r}}(a)$.

Recall Cauchy's formula for polydiscs (see e.g., [11], p 31): Let f be a function holomorphic in an open set containing $D_{\mathbf{r}}(a)$. Then

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial_o D_{\mathbf{r}}(a)} \frac{f(z) \mathrm{d}z_1 \cdots \mathrm{d}z_n}{\prod_{j=1}^n (z_j - a_j)}, \tag{2.1}$$

or equivalently, with $z_j - a_j = r_j e^{i\theta_j}$ $(j = 1, 2, \ldots, n)$,

$$f(a) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(a_1 + r_1 e^{i\theta_1}, \ldots, a_n + r_n e^{i\theta_n}) \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \tag{2.2}$$

Let $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index with $\alpha_j \in \mathbb{Z}_+$, $1 \le j \le n$. We denote the length of I by $|I| = \sum_{j=1}^n \alpha_j$, and also denote

$$D^I f = \frac{\partial^{|I|} f}{\partial z^I} = \frac{\partial^{|I|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$$

for any holomorphic function f in $(z_1, z_2, \ldots, z_n)$.

Theorem 2.1 *Let f be a function holomorphic in $\bar{D}_{\mathbf{R}}$. Then for $0 < r_j < R_j$ $(j = 1, 2, \ldots, n)$ and $I = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ with $|I| \ge 1$,*

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| D^I f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ \le \frac{2\alpha_1! \ldots \alpha_n!}{\prod_{j=1}^n (R_j - r_j)^{\alpha_j}} \big(A(\mathbf{R}, f) - \Re\{f(0)\} \big), \end{aligned}$$

where $A(\mathbf{R}, f) = \max_{z \in \bar{D}_{\mathbf{R}}} \Re\{f(z)\}$.

Proof For any $w = (w_1, w_2, \ldots, w_n) \in \bar{D}_{\mathbf{r}}$ and $\text{æ} = (\rho_1, \ldots, \rho_n)$ with $\rho_j \le R_j - r_j$ $(j = 1, \ldots, n)$, we have by Cauchy's formula for polydiscs (after taking partial derivatives) that

$$\begin{aligned} D^I f(w) =& \frac{\alpha_1! \ldots \alpha_n!}{(2\pi i)^n} \int_{\partial_o D_{\text{æ}}(w)} \frac{f(z) \mathrm{d}z_1 \cdots \mathrm{d}z_n}{\prod_{j=1}^n (z_j - w_j)^{\alpha_j + 1}} \\ =& \frac{\alpha_1! \ldots \alpha_n!}{(2\pi)^n \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n}} \int_0^{2\pi} \cdots \int_0^{2\pi} \\ & f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n}) e^{-i(\sum_{j=1}^n \alpha_j \theta_j)} \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \end{aligned} \tag{2.3}$$

On the other hand, it follows from Cauchy's formula for polydiscs that

$$\begin{aligned} &\int_{\partial_o D_{\text{æ}}(w)} f(z) \prod_{j=1}^n (z_j - w_j)^{\alpha_j - 1} \, \mathrm{d}z_1 \cdots \mathrm{d}z_n \\ &= \int_{\partial_o D_{\text{æ}}(w)} \frac{f(z) \prod_{j=1}^n (z_j - w_j)^{\alpha_j}}{\prod_{j=1}^n (z_j - w_j)} \, \mathrm{d}z_1 \cdots \mathrm{d}z_n = 0. \end{aligned}$$

Since $z_j - w_j = \rho_j e^{i\theta_j}$ $(j = 1, 2, \ldots, n)$,

$$\int_0^{2\pi} \cdots \int_0^{2\pi} f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n}) e^{i(\sum_{j=1}^n \alpha_j \theta_j)} \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n = 0.$$

After taking conjugate of the integral above, we have

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \overline{f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n})} e^{-i(\sum_{j=1}^n \alpha_j \theta_j)} \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n = 0. \tag{2.4}$$

Then by (2.3) and (2.4) we have

$$D^I f(w) = \frac{2\alpha_1! \ldots \alpha_n!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Re\{f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n})\} e^{-i(\sum_{j=1}^n \alpha_j \theta_j)} \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Therefore, we have

$$|D^I f(w)| \le \frac{2\alpha_1! \ldots \alpha_n!}{(2\pi)^n \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n}} \int_0^{2\pi} \cdots \int_0^{2\pi} |\Re\{f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n})\}| \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \tag{2.5}$$

Applying (2.5) to the holomorphic function $f - A(\mathbf{R}, f)$ with $\rho_j = R_j - r_j$ $(j = 1, 2, \ldots, n)$ and noting that

$$A(\mathbf{R}, f) - \Re\{f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n})\} \ge 0,$$

we have

$$|D^I f(w)| \le \frac{2\alpha_1! \ldots \alpha_n!}{(2\pi)^n (R_1 - r_1)^{\alpha_1} \cdots (R_n - r_n)^{\alpha_n}} \int_0^{2\pi} \cdots \int_0^{2\pi} (A(\mathbf{R}, f) - \Re\{f(w_1 + \rho_1 e^{i\theta_1}, \ldots, w_n + \rho_n e^{i\theta_n})\}) \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Thus

$$|D^I f(w)| \le \frac{2\alpha_1! \ldots \alpha_n!}{\prod_{j=1}^n (R_j - r_j)^{\alpha_j}} \big(A(\mathbf{R}, f) - \Re\{f(w)\}\big)$$

in view of (2.2). Integrating the both sides for $w \in \partial_0 D_{\mathbf{r}}$, we obtain

$$\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left|D^I f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})\right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \le \frac{2\alpha_1! \ldots \alpha_n!}{\prod_{j=1}^n (R_j - r_j)^{\alpha_j}} \big(A(\mathbf{R}, f) - \Re\{f(0)\}\big)$$

by (2.2) again. This completes the proof. □

3 Estimates on Logarithmic Derivatives

Theorem 2.1 implies easily the following estimate (3.1) on logarithmic derivatives of holomorphic functions without zeros on polydiscs, allowing higher order derivatives. The method given below for Theorem 3.1 however does not go through for general holomorphic functions with zeros, which will be treated in a different way, see Theorem 3.2. We should note that the estimate in Theorem 3.1 is different from the one in Theorem 3.2 since the latter contains extra terms arising from possible zeros of the function.

As in one variable, the proximity function $m(\mathbf{r}, f)$ of f for $\mathbf{r} \in \mathbb{R}_+^n$ is defined as

$$m(\mathbf{r}, f) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \log^+ \left| f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n,$$

where $\log^+ |x| = \max\{\log |x|, 0\}$. We have the following

Theorem 3.1 *Let f be a holomorphic function without zeros in $\bar{D}_{\mathbf{R}} \subset \mathbb{C}^n$. Then for $0 < r_j < R_j$ $(j = 1, 2, \ldots, n)$ and $1 \leq j \leq n$,*

$$m\left(\mathbf{r}, \frac{\frac{\partial^k f}{\partial z_j^k}}{f}\right) \leq c_k \left(\log^+ \log \frac{M(\mathbf{R}, f)}{|f(0)|} + \log^+ \frac{1}{R_j - r_j} + 1\right), \tag{3.1}$$

where $M(\mathbf{R}, f) = \max_{z \in \bar{D}_{\mathbf{R}}}\{|f(z)|\}$ and c_k is a positive constant depending only on k.

Throughout the paper, c_k denotes a positive constant depending on k, the actual value of which may vary at each occurrence.

Proof We prove the theorem by induction on the number k. Without loss of generality, we take $j = 1$. First, we assume that $k = 1$. Since f is a holomorphic function without zeros in $\bar{D}_{\mathbf{R}}$, we can apply Theorem 2.1 with $I = (1, 0, \ldots, 0)$ to $\log f(z)$ for a holomorphic branch of the logarithm to obtain, noting that $A(\mathbf{R}, \log f) = \log M(\mathbf{R}, f)$, that

$$\begin{aligned} &\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \frac{\frac{\partial f}{\partial z_1}}{f} \left(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}\right) \right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &\leq \frac{2}{R_1 - r_1} \big(\log M(\mathbf{R}, f) - \log |f(0)|\big). \end{aligned}$$

By this inequality and the convex property of $\log x$, we deduce that

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\left|\frac{\frac{\partial f}{\partial z_1}}{f}\left(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n}\right)\right|\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\\
&\le \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\left(\left|\frac{\frac{\partial f}{\partial z_1}}{f}(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\right|+1\right)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\\
&\le \log\left(\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left(\left|\frac{\frac{\partial f}{\partial z_1}}{f}(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\right|+1\right)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\right)\\
&\le \log^+\left(\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left(\left|\frac{\frac{\partial f}{\partial z_1}}{f}(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\right|\right)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\right)+\log 2\\
&\le \log^+\frac{1}{R_1-r_1}+\log^+\log\frac{M(\mathbf{R},f)}{|f(0)|}+3\log 2.
\end{aligned}$$

Therefore, (3.1) is valid when $k=1$.

Suppose that (3.1) is true for $k=1,2,\ldots,l-1$. Let us prove that it is also valid for $k=l$. It is easy to check that

$$\frac{\partial^l}{\partial z_1^l}(\log f)=\frac{\partial^{l-1}}{\partial z_1^{l-1}}\left(\frac{f_{z_1}}{f}\right)=\frac{f_{z_1^l}}{f}+P_l\left(\frac{f_{z_1}}{f},\frac{f_{z_1^2}}{f},\ldots,\frac{f_{z_1^{l-1}}}{f}\right), \tag{3.2}$$

where P_l is a polynomial of degree l and $f_{z_1^j}=\frac{\partial^j f}{\partial z_1^j}$.

By (3.2) and the property of $\log^+ x$ we have

$$\log^+\left|\frac{f_{z_1^l}}{f}\right|\le c_l\left\{\sum_{j=1}^{l-1}\log^+\left|\frac{f_{z_1^j}}{f}\right|+1\right\}+\log^+\left|\frac{\partial^l}{\partial z_1^l}(\log f)\right|. \tag{3.3}$$

Applying Theorem 2.1 with $I=(l,0,\ldots,0)$ to $\log f(z)$ we have

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\frac{\partial^l}{\partial z_1^l}(\log f)(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\right|\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\\
&\quad\le\frac{2l!}{(R_1-r_1)^l}\big(\log M(\mathbf{R},f)-\log|f(0)|\big).
\end{aligned}$$

Similarly, by this inequality and the convex property of $\log x$, we deduce that

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\left|\frac{\partial^l}{\partial z_1^l}(\log f)(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\right|\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\\
&\le l\log^+\frac{1}{R_1-r_1}+\log^+\log\frac{M(\mathbf{R},f)}{|f(0)|}+\log(4l!).
\end{aligned}$$

Here we omit the details in order to avoid unnecessary repetition. This together with (3.3) and the hypothesis of induction yield that

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\left|\frac{\frac{\partial^l f}{\partial z_j^l}}{f}\left(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n}\right)\right|\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n$$
$$\le c_l\left(\log^+\log\frac{M(\mathbf{R},f)}{|f(0)|}+\log^+\frac{1}{R_j-r_j}+1\right),$$

where c_l denotes a positive constant depending on l. Thus (3.1) is also true for $k=l$. This completes the proof. □

We next derive a more general estimate on logarithmic derivatives for holomorphic functions in polydiscs, allowing zeros and higher order partial derivatives.

Theorem 3.2 *Let f be a holomorphic function in $\bar{D}_{\mathbf{R}}\subset\mathbb{C}^n$ with $f(0)\neq 0$ being a finite number. Then for $0<r_j<R_j$ and $1\le j\le n$,*

$$m\left(\mathbf{r},\frac{\frac{\partial^k f}{\partial z_j^k}}{f}\right)$$
$$\le c_k\left(\log^+\log\frac{M(\mathbf{R},f)}{|f(0)|}+\log^+\frac{1}{R_j-r_j}+\log^+R_j+\log^+\frac{1}{r_j}+1\right) \tag{3.4}$$

where c_k is a positive constant depending only on k.

As mentioned in Sect. 1, the use of the maximum modulus on the estimate (3.4) enables us to easily extend the bound from one dimension to higher dimensions (see the proof below); moreover, the maximum modulus can also be replaced by the proximity function $m(\mathbf{R},f)$ for a more familiar form. We refer to [26] for related but different results with very involved proofs.

Corollary 3.3 *Under the same conditions of Theorem 3.2, we have for $0<r_j<R_j$ $(j=1,2,\ldots,n)$ and $1\le j\le n$ that*

$$m\left(\mathbf{r},\frac{\frac{\partial^k f}{\partial z_j^k}}{f}\right)\le c_k\Big(\log^+m(\mathbf{R},f)+\log^+\log^+\frac{1}{|f(0)|}$$
$$+\sum_{j=1}^{n}\left(\log^+\frac{1}{R_j-r_j}+\log^+R_j+\log^+\frac{1}{r_j}+1\right), \tag{3.5}$$

where c_k is a positive constant depending only on k.

Proof of Theorem 3.2 Assume for the moment that the theorem holds when $n = 1$. Assume also, without loss of generality, that $j = 1$. Rewrite $\frac{\partial^k f}{\partial z_1^k}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, \ldots, r_n e^{i\theta_n})$ as $g^{(k)}(z)$ in the single variable z evaluated at $r_1 e^{i\theta_1}$, where $g(z) = f(z, r_2 e^{i\theta_2}, \ldots, r_n e^{i\theta_n})$. Then applying the theorem when $n = 1$ to g in $|z| \le r_1$ we immediately obtain, thanks to the maximum modulus with $M(r_1, g) \le M(\mathbf{R}, f)$, that

$$\begin{aligned} &\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\left|\frac{\frac{\partial^k f}{\partial z_1^k}}{f}\left(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n}\right)\right|\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \\ &\le c_k\Bigg(\log^+\frac{1}{R_1-r_1}+\log^+\frac{1}{r_1}+\log^+ R_1+1 \\ &\quad+\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\log\frac{M(\mathbf{R},f)}{|g(0)|}\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n\Bigg). \end{aligned} \tag{3.6}$$

To estimate the last term in (3.6) with $g(0) = f(0, r_2 e^{i\theta_2}, \ldots, r_n e^{i\theta_n})$, we use the convexity of the logarithmic function to deduce that

$$\begin{aligned} &\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\log\frac{M(\mathbf{R},f)}{|g(0)|}\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n \\ &\le \frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\left\{\log\frac{M(\mathbf{R},f)}{|g(0)|}+1\right\}\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n \\ &\le \log\left\{\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\frac{M(\mathbf{R},f)}{|g(0)|}\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n+1\right\} \\ &\le \log^+\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\frac{M(\mathbf{R},f)}{|g(0)|}\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n+\log 2 \\ &= \log^+\{\log M(\mathbf{R},f) \\ &-\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|f(0,r_2e^{i\theta_2},\ldots,r_ne^{i\theta_n})|\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n\}+\log 2. \end{aligned} \tag{3.7}$$

We claim that for any holomorphic function G in $\bar{D}_{\mathbf{r}} \subset \mathbb{C}^n$ with $G(0) \neq 0$,

$$\log|G(0)| \le \frac{1}{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\left|G(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\right|\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n. \tag{3.8}$$

When $n = 1$, this holds clearly by applying Cauchy's formula (2.2) (taking the real parts after applying the formula) to the function $G(z)\prod_{j=1}^m \frac{r^2-\bar{a}_j z}{r(z-a_j)}$ if G has zeros $a_1, \ldots, a_m$ in $|z| < r$, noting that $|\frac{\bar{z}z-\bar{a}_j z}{r(z-a_j)}| = 1$ when $|z| = r$. When $n > 1$, repeatedly using this one variable result we deduce that

$$
\begin{aligned}
&\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|G(r_1e^{i\theta_1}, r_2e^{i\theta_2}, \ldots, r_ne^{i\theta_n})|\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n\\
&\quad\geq \frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|G(0, r_2e^{i\theta_2}, r_3e^{i\theta_3}, \ldots, r_ne^{i\theta_n})\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n\cdots\\
&\quad\cdots\\
&\quad\geq \log|G(0,\ldots,0)|.
\end{aligned}
$$

This proves the claim. Applying (3.8) to the integral in the last inequality of (3.7) we obtain that

$$
\begin{aligned}
&\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+\log\frac{M(\mathbf{R}, f)}{|g(0)|}\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n\\
&\quad\leq \log^+\log\frac{M(\mathbf{R}, f)}{f(0)} + \log 2.
\end{aligned}
$$

This and (3.6) yield the conclusion (3.4) of the theorem.

It remains to prove (3.4) when $n = 1$, which was assumed to be true above. We include a proof using Theorem 3.1 and invoking Cauchy's formula only. To this end, take $\rho = \frac{R+r}{2}$, and assume $a_1, \ldots, a_s$ are the zeros of f in $|z| \leq \rho$. Let $g(z) = f(z)h(z)$, where

$$
h(z) = \prod_{j=1}^{s}\frac{\rho^2 - \bar{a}_j z}{\rho(z - a_j)}.
$$

Then g has no zeros in $|z| \leq \rho$ and $|h(z)| = 1$ on $|z| = \rho$. It is easy to check (or by the Leibniz rule),

$$
\frac{(fh)^{(k)}}{fh} = \frac{f^{(k)}}{f} + c_1\frac{f^{(k-1)}}{f}\frac{h'}{h} + \cdots + c_{k-1}\frac{f'}{f}\frac{h^{(k-1)}}{f} + \frac{h^{(k)}}{h}, \tag{3.9}
$$

where c_j are positive constants depending only on k (actually $c_j = \binom{j}{k}$), and also for $k \geq 1$,

$$
\left(\frac{h'}{h}\right)^{(k-1)} = \frac{h^{(k)}}{h} + P_k\left(\frac{h'}{h}, \ldots, \frac{h^{(k-1)}}{h}\right), \tag{3.10}
$$

where P_l is a polynomial of degree k. It is clear that

$$
\frac{h'}{h} = \sum_{j=1}^{s}\left(\frac{-1}{z - a_j} - \frac{\bar{a}_j}{\rho^2 - \bar{a}_j z}\right)
$$

and thus for $k \geq 1$ and $|z| = r$,

$$\begin{aligned}\left|\left(\frac{h'}{h}\right)^{(k-1)}\right| &\leq (k-1)! \sum_{j=1}^{s} \left(\frac{1}{|z-a_j|^k} + \frac{\rho^k}{|\rho^2 - \bar{a}_j z|^k}\right) \\ &\leq (k-1)! \left(\frac{s}{(d(z))^k} + \frac{1}{(\rho - r)^k}\right),\end{aligned}$$

where $d(z)$ is the least of the distances $|z - a_j|$, $j = 1, 2, \ldots, s$. To estimate the term with $d(z)$ we follow the known method using the following elementary lemma (see e.g., [8], p. 35): Let $z_1, \ldots, z_l$ be l points in the complex plane and let $d(z)$ be the least of the distances $|z - z_j|$, $j = 1, \ldots, l$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{r}{d(re^{i\theta})} \mathrm{d}\theta \leq 2 \log l + \frac{1}{2}.$$

It follows from this lemma that for $k \geq 1$,

$$m\left(r, \left(\frac{h'}{h}\right)^{(k-1)}\right) \leq c_k \left(\log s + \log^+ \frac{1}{r} + \log^+ \frac{1}{\rho - r} + 1\right). \tag{3.11}$$

Applying Cauchy's formula (2.2) to the function

$$\log\left(f(z) \prod_{j=1}^{q} \frac{R^2 - \bar{a}_j z}{R(z - a_j)}\right),$$

where $a_1, \ldots, a_s, a_{s+1}, \ldots, a_q$ are the zeros of f in $|z| \leq R$, we have that

$$\begin{aligned}\log\left(|f(0)| \left(\frac{R}{\rho}\right)^s\right) &\leq \log\left(|f(0)| \prod_{|a_j| \leq \rho} \frac{R}{|a_j|}\right) \\ &\leq \log\left(|f(0)| \prod_{j=1}^{q} \frac{R}{|a_j|}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| \mathrm{d}\theta.\end{aligned}$$

Thus, $s \log \frac{R}{\rho} \leq \log \frac{M(R,f)}{|f(0)|}$. Noting that $\log \frac{R}{\rho} = \int_\rho^R \frac{1}{t} \mathrm{d}t \geq \frac{R-\rho}{R}$, we obtain that

$$\log s \leq \log \log \frac{M(R, f)}{|f(0)|} + \log \frac{R}{R - \rho}.$$

This together with (3.11) yields that for $k \geq 1$,

$$m\left(r, \left(\frac{h'}{h}\right)^{(k-1)}\right)$$
$$\leq c_k \left(\log\log\frac{M(R, f)}{|f(0)|} + \log R + \log^+\frac{1}{r} + \log^+\frac{1}{R-r} + 1\right) \tag{3.12}$$

in view of the fact that $\rho = \frac{R+r}{2}$. We conclude by (3.10) and a simple induction on k that

$$m\left(r, \frac{h^{(k)}}{h}\right)$$
$$\leq c_k \left\{\log\log\frac{M(R, f)}{|f(0)|} + \log^+ R + \log^+\frac{1}{r} + \log^+\frac{1}{R-r} + 1\right\}. \tag{3.13}$$

In (3.9), the function fh is holomorphic without any zeros and thus the estimate (3.4) already holds for fh by Theorem 3.1. That is, for $k \geq 1$ we have that

$$m\left(r, \frac{(fh)^{(k)}}{fh}\right)$$
$$\leq c_k \left\{\log^+\log\frac{M(\rho, hf)}{(fh)(0)} + \log^+\frac{1}{\rho - r} + \log^+\rho + 1\right\}$$
$$\leq c_k \left\{\log^+\log\frac{M(\rho, f)}{|f(0)|} + \log^+\frac{1}{\rho - r} + \log^+\rho + \log^+\frac{1}{r} + 1\right\}$$

in view of the fact that $|h(0)| \geq 1$ and $h(z) = 1$ when $|z| = \rho$. This estimate when $k = 1$ and the estimate (3.13) when $k = 1$ yield the estimate (3.4) when $k = n = 1$. Using (3.9) with an induction on k, we obtain the conclusion (3.4) when $n = 1$. This completes the proof. □

Proof of Corollary 3.3 The conclusion clearly follows from the following fact, which will be proved using Cauchy's formula: *Suppose that f is a holomorphic function in $\bar{D}_{\mathbf{R}} \subset \mathbb{C}^n$*. Then,

$$\log M(\mathbf{r}, f)$$
$$\leq \left(\prod_{j=1}^{n}\frac{R_j + r_j}{R_j - r_j}\right)\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+|f(R_1 e^{i\theta_1}, \ldots, R_n e^{i\theta_n})|\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n, \tag{3.14}$$

where $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathbf{R} = (R_1, \ldots, R_n)$ with $r_j < R_j$ $(1 \leq j \leq n)$. This follows from an induction on n. When $n = 1$, this is a well-known result (see e.g., [8]). We however include a direct proof for completeness. Letting $h(z) = \frac{R^2 - \bar{a}_j z}{R(z - a_j)}$, where a_j's are the zeros of f in $|z| \leq R$, and applying Cauchy's formula to $\frac{R^2 - |w|^2}{R^2 - z\bar{w}}\log(f(z)h(z))$, where $|f(w)| = M(\mathbf{r}, f)$ for some w with $|w| = r < R$, we deduce that

$$\log(f(w)h(w)) = \frac{1}{2\pi i}\int_{|z|=R} \frac{\log(f(z)h(z))}{z-w}\frac{R^2-|w|^2}{z\bar{z}-z\bar{w}}\mathrm{d}z$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \log(f(Re^{i\theta})h(Re^{i\theta}))\frac{R^2-|w|^2}{|Re^{i\theta}-w|^2}\mathrm{d}\theta,$$

in view of the fact that $|h(z)| = 1$ on $|z| = R$. Taking the real parts and noting that $|h(w)| \geq 1$, we obtain that

$$\log M(\mathbf{r}, f) \leq \frac{R+r}{R-r}\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta})|\mathrm{d}\theta.$$

We then use an induction on n. It is true already when $n = 1$. Suppose that $f(\zeta_1, \ldots, \zeta_n) = M(\mathbf{r}, f)$ with $\mathbf{r} = (r_1, \ldots, r_n)$ and $|\zeta_j| \leq r_j, 1 \leq j \leq n$. Then by the induction hypothesis we have that

$$\begin{aligned}
&\log|f(\zeta_1, \ldots, \zeta_n)| \\
&\quad\leq \prod_{j=2}^{n}\frac{R_j+r_j}{R_j-r_j}\frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+ f(\zeta_1, R_2e^{i\theta_1}, \ldots, R_ne^{i\theta_n})\,\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n \\
&\quad\leq \prod_{j=1}^{n}\frac{R_j+r_j}{R_j-r_j}\frac{1}{(2\pi)^{n}}\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+ f(R_1e^{i\theta_1}, \ldots, R_ne^{i\theta_n})\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n.
\end{aligned}$$

4 Entire Solutions of PDEs and Picard's Theorem

Characterizing complex analytic solutions of differential equations is a topic of a long history (see e.g., the monographs [9,12]). It is well known that there is no systematic way to solve nonlinear PDEs. Characterization of entire solutions of a class of PDEs (see (4.2) and (4.6)) will be given by utilizing the estimates from the previous sections; this is motivated by its connection to seemingly unrelated Picard's theorem on entire functions.

Recall that the famous Picard theorem ([19]) for entire functions asserts that a nonconstant entire function in $\mathbb{C}$ must assume every complex number with at most one exception, which also immediately implies, by a simple transform, that a nonconstant meromorphic function in $\mathbb{C}$ assumes every complex values with at most two exceptions. Later in a separate paper ([20]), Picard further quantified his theorem and proved the following

Theorem C (Picard's Theorem) *An entire function that is not a polynomial must assume every complex number with at most one exception infinitely often.*

Picard's theorem is among the most striking results in complex analysis and plays a decisive role in the development of the theory of entire and meromorphic functions, with various proofs and extensions, see e.g., [1,4,5,13,31] and see also [25] for an exposition (the history, methods and references) of the theorem. In [16], a connection between Picard's theorem and entire solutions of a differential equation was given, which suggests the following heuristic: A Picard-type theorem is apt to imply a characterization of entire solutions of a differential equation, and vice versa. Inspired by this, we establish the following

Theorem 4.1 *Consider the ordinary differential equation*

$$p(z)f'(z) + a(z)g(f(z)) = 0 \tag{4.1}$$

where p is a nonzero polynomial, $a(z)$ a nonzero entire function in $\mathbb{C}$, and g a nonzero meromorphic function in $\mathbb{C}$ with at least two distinct zeros. Then an entire solution f to (4.1) *must be a polynomial in $\mathbb{C}$.*

While Theorem 4.1 and Theorem C seem unrelated, they turn out to be equivalent in the sense that one implies the other. This equivalence enables us to discover more general results on differential equations and also new Picard-type theorems (see below). As a matter of fact, we are able to prove the equivalence between Theorems 4.2 and 4.3, which are natural higher dimensional versions of Theorem C and Theorem 4.1, respectively.

We say that the zero set $Z(f)$ (counted with multiplicity) of a function f in $\mathbb{C}^n$ is transcendental if $Z(f)$ is not contained in the zero set of a nonzero polynomial. In the one variable case, that $Z(f)$ is transcendental coincides with the statement that $Z(f)$ contains infinitely many points.

Theorem 4.2 (A $\mathbb{C}^n$-Version of Picard's Theorem) *Suppose that f is an entire function in $\mathbb{C}^n$ that is not a polynomial. Then for every complex number a with at most one exception, the zero set $Z(f-a)$ must be transcendental. In particular, when $n=1$, f must assume every complex number with at most one exception infinitely often.*

Theorem 4.3 *Consider the partial differential equation*

$$p(z)\frac{\partial f(z)}{\partial z_j} + a(z)g(f(z)) = 0 \tag{4.2}$$

in $\mathbb{C}^n$, where $1 \leq j \leq n$, $p(z)$ is a nonzero polynomial and $a(z)$ an entire function in $\mathbb{C}^n$, and g a nonzero meromorphic function in $\mathbb{C}$ with at least two distinct zeros. Then an entire solution f of (4.2) *must be a polynomial in $\mathbb{C}^n$.*

We now give the proof of the equivalence between Theorems 4.2 and 4.3, which, as a consequence, gives the equivalence between Theorems C and 4.1.

Theorem 4.3 $\Longrightarrow$ **Theorem** 4.2. Assume that f is an entire function that is not a polynomial such that $Z(f-c)$ and $Z(f-d)$ are not transcendental, where c, d are two distinct complex numbers. Then it is clear by the definition that there exists a nonzero polynomial $p(z)$ in $\mathbb{C}^n$ such that $Z(f-c) \cup Z(f-d) \subset Z(p)$. Thus, $a(z) := \frac{p(z)\frac{\partial f(z)}{\partial z_j}}{(f-c)(f-d)}$ is entire. Clearly,

$$p(z)\frac{\partial f(z)}{\partial z_1} - a(z)(f-c)(f-d) = 0,$$

which is of the form in (4.2). Thus, f must be polynomial by Theorem 4.3, a contradiction. □

Theorem 4.2 $\Longrightarrow$ Theorem 4.3. Without loss of generality, we assume that $j = 1$ and then the given Eq. (4.2) is of the form

$$p(z)\frac{\partial f(z)}{\partial z_1} + a(z)g(f(z)) = 0. \tag{4.3}$$

Since the meromorphic function g has at least two distinct zeros, say c, d, we can write $g(z) = (z-c)^m(z-d)^n h(z)$, where m, n are two positive integers and h is a nonzero meromorphic function in $\mathbb{C}$ and holomorphic at c and d with $h(c)h(d) \neq 0$. We can then write (4.3) as

$$p(z)\frac{\partial f(z)}{\partial z_1} = -a(z)(f(z)-c)^m(f(z)-d)^n h(f(z)). \tag{4.4}$$

Suppose that f is an entire solution of this equation and f is not a polynomial. We will derive a contradiction below.

We may assume that $p(z)\frac{\partial f(z)}{\partial z_1} \not\equiv 0$, since otherwise $f \equiv c$ or $f \equiv d$, a contradiction. Thus, there is a point $a = (a_1, a_2, \ldots, a_n)$ such that $(p\frac{\partial f}{\partial z_1})(a_1, a_2, \ldots, a_n) \neq 0$, which implies that both p and $\frac{\partial f(z)}{\partial z_1}$ do not vanish at point a. In particular, the one variable function $F(z_1) := f(z_1, a_2, \ldots, a_n)$, the restriction of f to the first variable when $a' = (a_2, \ldots, a_n)$ is fixed, is a nonconstant entire function in z_1 and $p(z_1, a')$ is a nonzero polynomial in z_1. We have from (4.4) that

$$p(z_1, a')F'(z_1) = -a(z_1, a')(F(z_1)-c)^m(F(z_1)-d)^n h(F(z_1)). \tag{4.5}$$

By Picard's theorem, at least one of the nonconstant entire functions $F(z_1) - a$ and $F(z_1) - b$ has infinitely many zeros. We can obviously take such a zero ζ so that it is not a zero of $p(z_1, a')$. It is clear that both sides of (4.5) vanish at ζ with however different multiplicities due to the derivative $F'(z_1)$, which is absurd. This completes the proof. □

We next give a proof of Theorem 4.3 independent of Picard's theorem. Due to its equivalence to Theorem 4.2 we thus furnish another proof of Theorem C and thus Picard's theorem. The method turns out to be better than what we desired and applies to more general higher order partial differential equations, which clearly includes the one in Theorem 4.3 as a special case:

Theorem 4.4 *Consider the partial differential equation*

$$\sum_{|\alpha|=1}^{m} a_\alpha(z)\frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1}\cdots\partial^{\alpha_n} z_n} + a(z)g(f(z)) = 0 \tag{4.6}$$

in a polydisc $\bar{D}_{\mathbf{R}} \subseteq \mathbb{C}^n$, where a_α's are polynomials (not all zeros) and $a(z)$ a nonzero holomorphic function in $\bar{D}_{\mathbf{R}}$, and g a nonzero meromorphic function in $\mathbb{C}$ with at least two distinct zeros. Then a holomorphic solution f in $\bar{D}_{\mathbf{R}}$ of (4.6) satisfies that

$$m(\mathbf{r}, f) = O\left\{\log^+ m(\mathbf{R}, f) + \sum_{j=1}^{n} \log \frac{1}{R_j - r_j}\right\}, \tag{4.7}$$

as $r_j \to R_j$, where $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathbf{R} = (R_1, \ldots, R_n)$ with $r_j < R_j < +\infty$, $1 \leq j \leq n$. In particular, if $R_j = +\infty$, $1 \leq j \leq n$ and f is an entire solution of (4.6) in $\mathbb{C}^n$, then f must be a polynomial in $\mathbb{C}^n$.

Proof Suppose that f is a nonconstant holomorphic solution of the PDE (4.6) (If f is constant, the conclusion already holds). Since the nonzero meromorphic function g in $\mathbb{C}$ has at least two distinct zeros, say c, d, we can write $g(z) = (z-c)(z-d)h(z)$, where h is a nonzero meromorphic function in $\mathbb{C}$ with $h(c) \neq \infty$ and $h(d) \neq \infty$. Thus, the given equation can be written as

$$\sum_{|\alpha|=1}^{m} a_\alpha(z) \frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} = b(z)(f(z)-c)(f(z)-d), \tag{4.8}$$

where $b(z) = -a(z)h(f(z))$. We claim that the function $b(z)$ must be an entire function. In fact, if $b(z)$ is not holomorphic at a point w in $\mathbb{C}^n$, then h is not holomorphic at $f(w)$ since $a(z)$ is entire. Thus, $f(w) \neq c, d$ since h is holomorphic at c, d. Then the right-hand side of (4.8) and thus the left-hand side of (4.8) is not holomorphic at w, which is impossible since the left-hand side of (4.8) is an entire function. We now write (4.6) as

$$\frac{\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}}{(f(z)-c)(f(z)-d)} = b(z). \tag{4.9}$$

Corollary 3.3, by a standard argument using induction on the order of derivatives, implies that for any constant A,

$$m\left(\mathbf{r}, \frac{\frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}}{f - A}\right) = O\left(\log^+ m(\mathbf{R}, f) + \sum_{j=1}^{n} \log \frac{1}{R_j - r_j}\right) \tag{4.10}$$

as $r_j \to R_j$.

Note that $\log b = \log^+ b - \log^+ \frac{1}{b}$. Since b is entire, we have, in view of (3.8), that

$$m\left(\mathbf{r}, \frac{1}{b}\right) = m(\mathbf{r}, b) - \frac{1}{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left|b(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})\right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n$$
$$\leq m(\mathbf{r}, b) + O(1).$$

(When using (3.8), if $b(0) = 0$ we can easily take a ζ such that $b(\zeta) \neq 0$ and then replace z by $z + \zeta$ throughout the proof.) We thus deduce from (4.10) that

$$\begin{aligned} m\left(\mathbf{r}, \frac{1}{b}\right) &\leq m(\mathbf{r}, b) + O(1) \\ &\leq \sum_{|\alpha|=1}^{m} m\left(\mathbf{r}, \frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \left(\frac{1}{f-c} - \frac{1}{f-d}\right) + O\left(\sum_{j=1}^{n} \log r_j\right)\right) \\ &= O\left\{\log^+ m(\mathbf{R}, f) + \sum_{j=1}^{n} \log \frac{1}{R_j - r_j}\right\} \end{aligned}$$

as $r_j \to R_j$. Write (4.9) as

$$f - c = \frac{\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}}{b(z)(f-d)}.$$

We obtain using (4.10) again that

$$\begin{aligned} m(\mathbf{r}, f) &\leq m(\mathbf{r}, f - c) + O(1) \\ &\leq m\left(\mathbf{r}, \frac{1}{b}\right) + \sum_{|\alpha|=1}^{m} m\left(\mathbf{r}, \frac{\frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}}{(f-d)}\right) + O(1) \\ &= O\left\{\log^+ m(\mathbf{R}, f) + \sum_{j=1}^{n} \log \frac{1}{R_j - r_j}\right\}, \end{aligned}$$

as $r_j \to R_j$. This proves (4.7).

When the polydisc $\bar{D}_{\mathbf{R}}$ is the entire space $\mathbb{C}^n$ and f is an entire solution, the term $\frac{1}{R_j - r_j}$ in the above proof can clearly be replaced by $\log R_j$. In fact, parallel to (4.10), it follows from Corollary 3.3 that

$$m\left(\mathbf{r}, \frac{\frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}}{f - A}\right) = O\left(\log^+ m(\mathbf{R}, f) + \sum_{j=1}^{n} \log R_j\right) \tag{4.11}$$

as $R_j \to \infty$. We then have, by the above arguments, that

$$m(\mathbf{r}, f) = O\left\{\log^+ m(\mathbf{R}, f) + \sum_{j=1}^{n} \log R_j\right\}, \tag{4.12}$$

as $R_j \to \infty$. We next prove that f must be a polynomial in this case. We will utilize the following version of the Borel Lemma (see [16], p. 449; cf. [8], p. 38): Let $\phi(r) > 0$ be a function of $r \geq r_0$ such that ϕ is bounded on $|z| \leq r$ for large r and $A > 1$ is a constant. Then there is a sequence $\{r_l\}$ with $r_0 < r_l \to \infty$ as $l \to \infty$ such that

$$\phi(r_l + \frac{1}{\phi(r_l)}) \leq A\phi(r_l).$$

Taking $\mathbf{r} = (r, r, \ldots, r)$ and $\mathbf{R} = (R, R, \ldots, R)$ with $R = r + \frac{1}{\phi(r)}$, where $\phi(r) = m(\mathbf{r}, f)$, we then obtain a sequence $\{r_l\}$ with $r_l \to \infty$ as $l \to \infty$ such that $m(\mathbf{R}, f) \leq Am(\mathbf{r}, f)$ for some $A > 1$ when $r = r_l$. It then follows from (4.12) that

$$m(\mathbf{r}, f) = O(\log^+ m(\mathbf{r}, f) + \log r)$$

and thus $m(\mathbf{r}, f) = O(\log r)$ when $r = r_l$ and $l \to \infty$. Write $r_l = 2t_l, \mathbf{t}_l = (t_l, \ldots, t_l)$, and $\mathbf{r}_l = (r_l, \ldots, r_l)$. Then we have, in view of (3.14), that

$$\begin{aligned}\log M(\mathbf{t}_l, f) &\leq \left(\frac{2t_l + t_l}{2t_l - t_l}\right)^n m(\mathbf{r}_l, f)\\ &= O(\log r_l) = O(\log t_l).\end{aligned}$$

Thus, $M(\mathbf{t}_l, f) \leq At_l^N$ for some constants $C, N > 0$, and all large l. For any $w \in \mathbb{C}^n$, we then deduce by Cauchy's formula (2.1) that for any multi-index $I = (\alpha_1, \ldots, \alpha_n)$ with $|I| > N$,

$$\begin{aligned}|D^I f(w)| &= \left|\frac{\alpha_1! \ldots \alpha_n!}{(2\pi i)^n} \int_{\partial_o D_{\mathbf{t}_l}(w)} \frac{f(z)\mathrm{d}z_1 \cdots \mathrm{d}z_n}{\prod_{j=1}^n (z_j - w_j)^{\alpha_j+1}}\right|\\ &\leq \frac{\alpha_1! \ldots \alpha_n!}{(2\pi)^n} \frac{C(t_l + ||w||)^N}{t_l^{|I|}} \to 0\end{aligned}$$

as $l \to \infty$. Thus, $D^I f(w) = 0$ for all I with $|I| > N$. This is true for all $w \in \mathbb{C}^n$. Hence, f is a polynomial by integration. This completes the proof. □

To conclude the paper, we note that the functions in our interests of the present paper are holomorphic or entire functions. While meromorphic functions may be considered, some results obtained above however do not hold any more; e.g., the meromorphic function $f(z) = \frac{1}{e^{z_1+\cdots+z_n}-1}$ is a solution of the equation $\frac{\partial f(z)}{\partial z_1} + a(z)g(f(z)) = 0$ in $\mathbb{C}^n$, which is of the form (4.2) in Theorem 4.3 with $a(z) = p(z) = 1$ and $g(w) = w(w + 1)$. But, f is transcendental.

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