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Hankel Measures for Hardy Spaces

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Abstract
In this paper, we study the so-called Hankel measures on the open unit disk. We obtain several new characterizations for such measures and answer a question raised by J. Xiao in 2000.

Keywords Hankel measure · Carleson measure · Hardy space · Hankel matrix

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. Recall that for $0 < p < \infty$ the Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.$$

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Every function $f \in H^p$ has non-tangential limits $f(\zeta)$ for almost every ζ on the unit circle $\mathbb{T}$. This enables us to view H^p as a closed subspace of $L^p(\mathbb{T}, \mathrm{d}\theta)$. See [8,11] for the theory of Hardy spaces.

A positive Borel measure μ on $\mathbb{D}$ is called a Carleson measure if there exists a constant $C = C_p > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}\mu(z) \le C\|f\|_{H^p}^p$$

for all $f \in H^p$. Carleson measures play a fundamental role in the theory of Hardy spaces. According to a celebrated theorem of L. Carleson, the definition of Carleson measures is actually independent of the exponent p used above. More specifically, a positive Borel measure μ on $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $C > 0$ such that $\mu(S_I) \le C|I|$ for all arcs $I \subset \mathbb{T}$, where $|I|$ is the length of I and

$$S_I = \left\{ z = r\mathrm{e}^{it} : 0 < 1 - r < |I|, \mathrm{e}^{it} \in I \right\}$$

is the so-called Carleson box based on I.

Motivated by the study of Hankel matrices and Hankel operators on the Hardy space and in parallel to the notion of Carleson measures, Xiao introduced the notion of Hankel measures on the unit disk in [20], namely, a complex Borel measure μ on $\mathbb{D}$ is called a Hankel measure if there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{D}} f^2(z) \mathrm{d}\mu(z) \right| \le C\|f\|_{H^2}^2 \tag{1}$$

for $f \in H^2$ (or more precisely, for f in a dense subspace of H^2). It is clear that every Carleson measure is a Hankel measure and every Hankel measure must be finite. Several characterizations of Hankel measures are obtained in [20] in terms of $BMOA$ defined in Sect. 2 and certain integral transforms of μ.

In this paper, we further explore the notion of Hankel measures. We obtain several new characterizations of Hankel measures and settle a question that was left open by Xiao in [20].

2 Some Integral Transforms of Measures

It is well known that a positive Borel measure μ on $\mathbb{D}$ is a Carleson measure if and only if the following Poisson-type transform of μ,

$$P(\mu)(w) = \int_{\mathbb{D}} \frac{1 - |w|^2}{|1 - w\bar{z}|^2} \, \mathrm{d}\mu(z),$$

is a bounded function of w on $\mathbb{D}$. See [8,11,26] for example.

We consider two similar integral transforms for complex Borel measures μ on the open unit disk, that is,

$$P_1(\mu)(w) = \int_{\mathbb{D}} \frac{1}{1 - w\overline{z}}\, \mathrm{d}\mu(z), \qquad w \in \overline{\mathbb{D}},$$

and

$$P_2(\mu)(w) = \int_{\mathbb{D}} \frac{1 - |w|^2}{(1 - w\overline{z})^2}\, \mathrm{d}\mu(z), \qquad w \in \overline{\mathbb{D}}.$$

It is clear that every Hankel measure μ has the property that

$$\sup_{w\in\mathbb{D}} |P_2(\overline{\mu})(w)| = \sup_{w\in\mathbb{D}} \left| \int_{\mathbb{D}} \frac{1 - |w|^2}{(1 - \overline{w}z)^2}\, \mathrm{d}\mu(z) \right| < \infty. \tag{2}$$

It was further asked in [20] whether the above condition is sufficient for μ to be a Hankel measure. We show in this section that the answer to this question is negative.

We say that an $L^1(\mathbb{T})$-function f belongs to BMO, the space of functions having bounded mean oscillation on $\mathbb{T}$, if

$$\sup_{I\subseteq\mathbb{T}} \frac{1}{|I|} \int_I \left| f(\mathrm{e}^{i\theta}) - \frac{1}{|I|} \int_I f(\mathrm{e}^{it})\, \mathrm{d}t \right| \mathrm{d}\theta < \infty.$$

Let $BMOA$ be the space of functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\mathbb{T}$. It is well known (cf. [3,12]) that the space $BMOA$ consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^2_{BMOA} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)\, \mathrm{d}A(z) < \infty,$$

where

$$dA(z) = \frac{1}{\pi}\, \mathrm{d}x\mathrm{d}y = \frac{1}{\pi} r\, \mathrm{d}r\, \mathrm{d}\theta, \qquad z = x + iy = r\mathrm{e}^{i\theta},$$

and

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}$$

is the Möbius transformation of $\mathbb{D}$ interchanging a and 0.

We begin with the following characterization of Hankel measures from [20].

Theorem 1 *Let μ be a complex Borel measure on $\mathbb{D}$. Then the following conditions are equivalent.*

(a) *μ is a Hankel measure.*

(b) *There exists a positive constant C such that*

$$\left|\int_{\mathbb{D}} f(z)\mathrm{d}\mu(z)\right| \leq C\|f\|_{H^1}$$

for all $f \in H^1$.
(c) *$P_1(\overline{\mu})$ is in $BMOA$.*

Recall that the Bloch space $\mathcal{B}$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

It is well known that $BMOA$ is a proper subspace of $\mathcal{B}$. In particular, every function in $BMOA$ has boundary values almost everywhere, while there are examples of functions in $\mathcal{B}$ that have no radial limit at every point on $\mathbb{T}$. See [26].

Lemma 2 *Let μ be a complex Borel measure on $\mathbb{D}$. Then the following two statements are equivalent.*

(a) *The measure $\overline{\mu}$ satisfies condition (2), that is, $P_2(\overline{\mu}) \in L^\infty(\mathbb{D})$.*
(b) *The function $P_1(\overline{\mu})$ belongs to the Bloch space $\mathcal{B}$.*

Proof For $w \in \mathbb{D}$ we have

$$wP_1(\overline{\mu})(w) = \int_{\mathbb{D}} \frac{w}{1-w\overline{z}}\,\mathrm{d}\overline{\mu}(z) = \sum_{n=0}^{\infty}\left(\int_{\mathbb{D}} \overline{z}^n\,\mathrm{d}\overline{\mu}(z)\right)w^{n+1}.$$

It follows that

$$\begin{aligned}
&\sup_{w\in\mathbb{D}}(1-|w|^2)|\,(wP_1(\overline{\mu})(w))'\,| \\
&\quad= \sup_{w\in\mathbb{D}}(1-|w|^2)\left|\sum_{n=0}^{\infty}(n+1)\left(\int_{\mathbb{D}} \overline{z}^n\,\mathrm{d}\overline{\mu}(z)\right)w^n\right| \\
&\quad= \sup_{w\in\mathbb{D}}\left|\int_{\mathbb{D}} \frac{1-|w|^2}{(1-w\overline{z})^2}\,\mathrm{d}\overline{\mu}(z)\right| \\
&\quad= \sup_{w\in\mathbb{D}}\left|\int_{\mathbb{D}} \frac{1-|w|^2}{(1-\overline{w}z)^2}\,\mathrm{d}\mu(z)\right|.
\end{aligned}$$

Consequently, the function $w \mapsto wP_1(\overline{\mu})(w)$ belongs to $\mathcal{B}$ if and only if condition (2) holds for $\overline{\mu}$.

It is easy to see that $P_1(\overline{\mu}) \in \mathcal{B}$ if and only if the function $w \mapsto wP_1(\overline{\mu})(w)$ belongs to $\mathcal{B}$. This proves the desired result. □

Recall that the Dirichlet-type space $\mathcal{D}_0^1$ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{\mathcal{D}_0^1} = |f(0)| + \int_{\mathbb{D}} |f'(z)|\,\mathrm{d}A(z) < \infty.$$

Equivalently, $\mathcal{D}_0^1$ is the space of functions $f \in H(\mathbb{D})$ satisfying

$$\int_{\mathbb{D}} |f''(z)|(1-|z|^2)\,\mathrm{d}A(z) < \infty.$$

It is well known that $\mathcal{D}_0^1 \subsetneq H^1$; see [9] for example.

Denote by X the space of functions $f \in H(\mathbb{D})$ admitting an integral representation

$$f(w) = \int_{\mathbb{D}} \frac{\mathrm{d}\overline{\mu}(z)}{1-w\overline{z}}, \quad w \in \mathbb{D},$$

where μ is a complex Borel measure with $|\mu|(\mathbb{D}) < \infty$. For any non-negative integer n, it is easy to check that

$$w^n = \int_{\mathbb{D}} \frac{(n+1)z^n\,\mathrm{d}A(z)}{1-w\overline{z}}, \quad w \in \mathbb{D}.$$

Consequently, every polynomial belongs to X.

Lemma 3 *Let $f \in \mathcal{D}_0^1$ and $g(z) = z^4 f(z)$, $z \in \mathbb{D}$. Then $g \in X$.*

Proof Note that

$$g'(z) = 4z^3 f(z) + z^4 f'(z).$$

Since $f \in \mathcal{D}_0^1$ and $\mathcal{D}_0^1 \subseteq H^1$ (and H^1 is contained in the Bergman space A^1 of area integrable analytic functions on $\mathbb{D}$), we see that $g \in \mathcal{D}_0^1$ as well.

For any $h \in \mathcal{D}_0^1$ with $h(0) = h'(0) = 0$, it is easy to check that

$$h(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)h''(w)}{\overline{w}^2(1-z\overline{w})}\,\mathrm{d}A(w), \quad z \in \mathbb{D}. \tag{3}$$

See [26]. Let $h = g$. We obtain

$$g(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)[12w^2 f(w) + 8w^3 f'(w) + w^4 f''(w)]}{\overline{w}^2(1-z\overline{w})}\,\mathrm{d}A(w).$$

Define a complex Borel measure μ by

$$d\overline{\mu}(w) = \frac{(1-|w|^2)[12w^2 f(w) + 8w^3 f'(w) + w^4 f''(w)]}{\overline{w}^2}\,\mathrm{d}A(w).$$

Then $|\mu|(\mathbb{D}) < \infty$ and

$$g(z) = \int_{\mathbb{D}} \frac{\mathrm{d}\overline{\mu}(w)}{1-z\overline{w}}, \quad z \in \mathbb{D}.$$

This shows that $g \in X$. □

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. The function φ induces a composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$, namely, $C_\varphi f = f \circ \varphi$. It is well known that C_φ maps the Bloch space back into itself.

We can now prove the main result of this section, which settles a question raised by Xiao in [20].

Theorem 4 *There exists a complex Borel measure μ on $\mathbb{D}$ such that condition (2) holds but μ is not a Hankel measure.*

Proof Consider the function $\varphi(z) = (1+z)/2$, which is an analytic self-map of $\mathbb{D}$ and satisfies

$$|\varphi(z) - \varphi(w)| \le |z - w|$$

for all z and w in $\mathbb{D}$. By [6, p. 2988], φ fails to have the so-called Bloch-to-$BMOA$ pullback property, namely, there exists a function $f \in \mathcal{B}$ such that $g = f \circ \varphi$ is not in $BMOA$. Since C_φ is bounded on $\mathcal{B}$, we have $g \in \mathcal{B} \setminus BMOA$.

By [24, Theorem 1], if ϕ is an analytic self-map of $\mathbb{D}$, then the composition operator C_ϕ is bounded from $\mathcal{B}$ to $\mathcal{D}_0^1$ if and only if

$$\int_{\mathbb{D}} \frac{|\phi'(z)|}{1 - |\phi(z)|^2} \, \mathrm{d}A(z) < \infty. \tag{4}$$

We proceed to show that (4) holds for the linear function φ above.

Let

$$I = \int_{\mathbb{D}} \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2} \, \mathrm{d}A(z).$$

Then

$$\begin{aligned} I &= \frac{2}{\pi} \int_0^1 r \, \mathrm{d}r \int_0^{2\pi} \frac{1}{3 - r^2 - 2r\cos\theta} \, \mathrm{d}\theta \\ &= \frac{4}{\pi} \int_0^1 r \mathrm{d}r \int_0^{\pi} \frac{1}{3 - r^2 - 2r\cos\theta} \mathrm{d}\theta \\ &= \frac{2}{\pi} \int_0^1 \mathrm{d}r \int_0^{\pi} \frac{1}{C_r - \cos\theta} \mathrm{d}\theta, \end{aligned}$$

where $C_r = (3 - r^2)/(2r) > 1$. We make the change of variables $x = \tan(\theta/2)$, so that

$$\mathrm{d}\theta = \frac{2}{1+x^2} \mathrm{d}x, \qquad \cos\theta = \frac{1-x^2}{1+x^2}.$$

It is clear that

$$\int_0^{\pi} \frac{1}{C_r - \cos\theta} \, \mathrm{d}\theta = \frac{2}{C_r - 1} \int_0^{+\infty} \frac{\mathrm{d}x}{1 + \frac{C_r+1}{C_r-1} x^2} = \frac{\pi}{\sqrt{C_r^2 - 1}}.$$

It follows that

$$I = 2\int_0^1 \frac{\mathrm{d}r}{\sqrt{C_r^2 - 1}} \leq 2\sqrt{2}\int_0^1 \frac{\mathrm{d}r}{\sqrt{(1-r)(r+3)}} < \infty.$$

Thus we have $g \in \mathcal{D}_0^1$. Consequently, $g \in (\mathcal{D}_0^1 \cap \mathcal{B}) \setminus BMOA$. Let $h(z) = z^4 g(z)$. It follows from Lemma 3 that $h \in X$. Thus there exists a complex Borel measure μ with $|\mu|(\mathbb{D}) < \infty$ such that $h = P_1(\overline{\mu})$. It is easy to see that $h \in \mathcal{B} \setminus BMOA$. Combining this with Theorem 1 and Lemma 2, we obtain the desired result. □

Note that the essential part of the proof above is that there exists a function in $(\mathcal{D}_0^1 \cap \mathcal{B}) \setminus BMOA$. It is natural to try to look for a lacunary series for this purpose. But it can be shown that there are no lacunary series in the set $(\mathcal{D}_0^1 \cap \mathcal{B}) \setminus BMOA$. This explains why we had to use such a detour in the proof of Theorem 4.

3 New Characterizations of Hankel Measures

In this section, we obtain several new characterizations for Hankel measures and indicate how to apply them to the study of Hankel operators on various function spaces.

Note that for $f \in H(\mathbb{D})$, f^n is well defined for all positive integers. If p is not a positive integer, then f^p as a function in $H(\mathbb{D})$ is defined only when f is non-vanishing on $\mathbb{D}$.

Lemma 5 *Suppose $p > 0$ and μ is a complex Borel measure on $\mathbb{D}$. If μ is a Hankel measure, then there exists a positive constant C such that*

$$\left|\int_{\mathbb{D}} f^p(z)\mathrm{d}\mu(z)\right| \leq C\|f\|_{H^p}^p$$

for all non-vanishing functions f in H^p.

Proof If f belongs to H^p and vanishes nowhere in $\mathbb{D}$, then $f^p \in H^1$ (cf. [8, p. 47]). Since μ is a Hankel measure, it follows from part (b) of Theorem 1 that there is a positive constant C satisfying

$$\left|\int_{\mathbb{D}} f^p(z)\mathrm{d}\mu(z)\right| \leq C\|f^p\|_{H^1} = C\|f\|_{H^p}^p,$$

which finishes the proof. □

We can now generalize the equivalence of (a) and (b) in Theorem 1 to the case of some other Hardy spaces.

Theorem 6 *Let μ be a complex Borel measure on $\mathbb{D}$. If n is a positive integer, then the following two conditions are equivalent.*

(a) *μ is a Hankel measure.*
(b) *There exists a positive constant C such that*

$$\left|\int_{\mathbb{D}} f^n(z)\,\mathrm{d}\mu(z)\right| \leq C\|f\|_{H^n}^n$$

for all f in the Hardy space H^n.

Proof That (a) implies (b) follows from the proof of Lemma 5.

To show that (b) implies (a), we use the method of Xiao [20, p. 137]. Suppose $f \in H^1$ and f is not identically zero. It is well known (cf. [8]) that there exists an inner function B and an outer function O such that $f(z) = B(z)O(z)$ for all $z \in \mathbb{D}$. Set

$$f_1(z) = \frac{[B(z) - 1]O(z)}{2}, \qquad z \in \mathbb{D},$$

and

$$f_2(z) = \frac{[B(z) + 1]O(z)}{2}, \qquad z \in \mathbb{D}.$$

Because $|B(z)| < 1$ for $z \in \mathbb{D}$ and $|B(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$, we must have $\|f_k\|_{H^1} \leq \|f\|_{H^1}$ for $k = 1, 2$.

Since f_1 and f_2 are both non-vanishing on $\mathbb{D}$, there exist analytic functions g_1 and g_2 such that $g_1^n = f_1$ and $g_2^n = f_2$. Clearly, $g_1, g_2 \in H^n$, Consequently, for $k = 1, 2$,

$$\left|\int_{\mathbb{D}} f_k(z)\mathrm{d}\mu(z)\right| = \left|\int_{\mathbb{D}} g_k^n(z)\mathrm{d}\mu(z)\right| \leq C\|g_k\|_{H^n}^n = C\|f_k\|_{H^1}.$$

Combining this with $f = f_1 + f_2$, we deduce that

$$\begin{aligned}\left|\int_{\mathbb{D}} f(z)\mathrm{d}\mu(z)\right| &\leq \left|\int_{\mathbb{D}} f_1(z)\mathrm{d}\mu(z)\right| + \left|\int_{\mathbb{D}} f_2(z)\mathrm{d}\mu(z)\right| \\ &\leq C\|f_1\|_{H^1} + C\|f_2\|_{H^1} \\ &\leq 2C\|f\|_{H^1}.\end{aligned}$$

By Theorem 1, μ is a Hankel measure. This completes the proof. □

Next we are going to show that if μ is a positive Borel measure supported on the real interval $[0, 1)$ then μ is a Hankel measure if and only if condition (2) holds for $\overline{\mu}$. Thus Xiao's original conjecture actually holds in this situation. As an application, we will obtain some characterizations of bounded Hankel operators on Hardy spaces, weighted Bergman spaces, and Dirichlet-type spaces induced by such measures.

For a positive Borel measure μ on $[0, 1)$ and a non-negative integer n, we denote by $\mu[n]$ the n-th moment of μ, that is,

$$\mu[n] = \int_0^1 t^n\,\mathrm{d}\mu(t).$$

Let $\mathcal{H}_\mu$ be the Hankel matrix $\big(\mu[n+k]\big)_{n,k\geq 0}$.

The Hankel matrix $\mathcal{H}_\mu$ formally induces a linear operator on $H(\mathbb{D})$, sometimes called the Hankel operator induced by μ and still denoted by $\mathcal{H}_\mu$, as follows. For any function

$$z \mapsto f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in $H(\mathbb{D})$, we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu[n+k] a_k \right) z^n.$$

If we identify a function in $H(\mathbb{D})$ with its sequence of Taylor coefficients (which can be thought of as an infinite-dimensional column vector), then the action of $\mathcal{H}_\mu$ is exactly matrix multiplication (the matrix on the left and the infinite column vector on the right).

A classical example is when μ is the Lebesgue measure on [0, 1). In this case, the Hankel matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix

$$\mathcal{H} = \left(\frac{1}{n+k+1} \right)_{n,k\geq 0}.$$

See [4,5,13–15,19,20] for some recent work on the Hilbert matrix and other Hankel matrices.

The action of the Hankel operator $\mathcal{H}_\mu$ on Hardy spaces has been studied for a long time. For example, the Hankel operator $\mathcal{H}_\mu$ acting on H^2 was studied in [18,19], the action of $\mathcal{H}_\mu$ on H^1 was studied in [10,15], and the operator $\mathcal{H}_\mu$ on other Hardy spaces H^p was investigated in [5,13].

The operators $\mathcal{H}_\mu$ have also been studied on various other analytic function spaces in recent years. We mention two of them here. First, for $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z) < \infty.$$

Second, for $0 < p < \infty$ and $\alpha > -1$, the Dirichlet-type space $\mathcal{D}^p_\alpha$ consists of $f \in H(\mathbb{D})$ with $f' \in A^p_\alpha$. D. Girela and N. Merchán [15] characterized the boundedness of the operator $\mathcal{H}_\mu$ acting on A^p_α and $\mathcal{D}^p_\alpha$ spaces. See [4,7,14,16,17] for more results about $\mathcal{H}_\mu$ acting on other spaces of analytic functions.

For the sake of comparison, we mention the following results obtained in [5,13,15].

Theorem 7 *Let μ be a positive Borel measure on [0, 1). Suppose $1 < s < \infty$, $-1 < \alpha < p-2$, $1 < q < \infty$, and $q-2 < \beta \leq q-1$. Then the following conditions are equivalent.*

(a) *$\mathcal{H}_\mu$ is bounded on H^s.*
(b) *$\mathcal{H}_\mu$ is bounded on A^p_α.*
(c) *$\mathcal{H}_\mu$ is bounded on $\mathcal{D}^q_\beta$.*
(d) *μ is a Carleson measure.*
(e)

$$\sup_{w\in\mathbb{D}}\int_0^1 \frac{1-|w|^2}{|1-\overline{w}t|^2}\mathrm{d}\mu(t) < \infty.$$

(f) $\mu[n] = O\left(\frac{1}{n}\right)$.

Our next result shows that Xiao's conjecture about the characterization of Hankel measures by condition (2) is true for positive Borel measures supported on [0, 1). Furthermore, in this case, condition (2) holds if and only if μ is a Carleson measure.

Recall from [2,22,23] that for any $0 < p < \infty$ the space $\mathcal{Q}_p$ consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{Q}_p} = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2(1-|\sigma_a(z)|^2)^p \,\mathrm{d}A(z) < \infty.$$

It is well known that $\mathcal{Q}_p$ coincides with $BMOA$ when $p = 1$, and $\mathcal{Q}_p$ is the Bloch space $\mathcal{B}$ for all $p > 1$. So the most interesting cases of $\mathcal{Q}_p$ are when $0 < p < 1$.

Lemma 8 *Suppose $0 < p < \infty$ and*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $\{a_n\}$ is a decreasing sequence of non-negative numbers. Then $f \in \mathcal{Q}_p$ if and only if $a_n = O(n^{-1})$.

This result can be found in [22, p. 29] and [13, pp. 589-590]. We now state and prove our next main result.

Theorem 9 *Let μ be a positive Borel measure on [0, 1). Then the following conditions are equivalent.*

(a) *μ is a Hankel measure.*
(b) *μ satisfies condition (2), that is,*

$$\sup_{w\in\mathbb{D}}\left|\int_0^1 \frac{1-|w|^2}{(1-\overline{w}t)^2}\,\mathrm{d}\mu(t)\right| < \infty.$$

(c) *$P_1(\mu) \in \mathcal{Q}_p$ for some $0 < p < \infty$.*
(d) *$P_1(\mu) \in \mathcal{Q}_p$ for all $0 < p < \infty$.*
(e) $\mu[n] = O\left(\frac{1}{n}\right)$.

Proof If μ is a positive Borel measure on [0, 1), then for any $w \in \mathbb{D}$ we have

$$P_1(\overline{\mu})(w) = P_1(\mu)(w) = \sum_{n=0}^{\infty} \left(\int_0^1 t^n \, d\mu(t) \right) w^n = \sum_{n=0}^{\infty} \mu[n] w^n.$$

Clearly, $\{\mu[n]\}$ is a decreasing sequence of non-negative numbers. From Lemma 8 we see that conditions (c), (d), and (e) are equivalent.

On the other hand, by Theorem 1, $P_1(\overline{\mu}) \in BMOA$ if and only if μ is a Hankel measure. By Lemma 2, $P_1(\overline{\mu}) \in \mathcal{B}$ if and only if $\overline{\mu}$ satisfies condition (2). Since $\mathcal{B} = \mathcal{Q}_2$ and $BMOA = \mathcal{Q}_1$, both (a) and (b) are equivalent to (e). The proof is complete. □

Combining Theorem 9 with Theorem 7, we obtain new characterizations for the Hankel operator $\mathcal{H}_\mu$ to be bounded on Hardy spaces, weighted Bergman spaces, and Dirichlet-type spaces in terms of Hankel measures and $\mathcal{Q}_p$ functions.

4 Balayage of Carleson Measures

In this section, we re-examine the condition

$$\left| \int_{\mathbb{D}} f(z) \, d\mu(z) \right| \leq C \|f\|_{H^1} \tag{5}$$

for all $f \in H^1$ and explore its relationship to the classical notion of balayage (see [11] for definition).

It is clear that condition (5) simply says that the linear functional

$$L(f) = \int_{\mathbb{D}} f(z) \, d\mu(z)$$

is bounded on H^1. Since the dual space of H^1 can be identified with $BMOA$ under the usual integral pairing over the unit circle, condition (5) holds if and only if there exists a function $g \in BMOA$ such that

$$\int_{\mathbb{D}} f(z) \, d\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta$$

for all $f \in H^1$. Furthermore, by Fubini's theorem, we have

$$\begin{aligned} L(f) &= \int_{\mathbb{D}} f(z) \, d\mu(z) \\ &= \int_{\mathbb{D}} \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta}) \, d\theta}{1 - z e^{-i\theta}} \right] d\mu(z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta, \end{aligned}$$

where

$$g(w)=\int_{\mathbb{D}}\frac{\mathrm{d}\overline{\mu}(z)}{1-w\overline{z}}=P_1(\overline{\mu})(w),\qquad w\in\overline{\mathbb{D}}.$$

This was the proof in [20] for the equivalence of conditions (b) and (c) in Theorem 1.

On the other hand, if we reproduce f using the Poisson transform instead of the Cauchy transform in the arguments above, we obtain

$$\begin{aligned}L(f)&=\int_{\mathbb{D}}f(z)\,\mathrm{d}\mu(z)\\&=\int_{\mathbb{D}}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|1-z\mathrm{e}^{-i\theta}|^2}f(\mathrm{e}^{i\theta})\,\mathrm{d}\theta\right]\mathrm{d}\mu(z)\\&=\frac{1}{2\pi}\int_0^{2\pi}f(\mathrm{e}^{i\theta})\overline{g(\mathrm{e}^{i\theta})}\,\mathrm{d}\theta,\end{aligned}$$

where

$$g(\mathrm{e}^{i\theta})=\int_{\mathbb{D}}\frac{1-|z|^2}{|1-z\mathrm{e}^{-i\theta}|^2}\,\mathrm{d}\overline{\mu}(z)$$

is the balayage of $\overline{\mu}$. Note that the balayage has only been studied for positive Borel measures before, but it is clear that it can also be defined for complex Borel measures on $\mathbb{D}$.

It is well known that if μ is a (positive) Carleson measure on $\mathbb{D}$, then its balayage belongs to BMO; see [11] for example. Since a BMO function is the sum of a function in $BMOA$ and the conjugate of another function in $BMOA$, each BMO function induces a bounded linear functional on H^1. Thus every Carleson measure is automatically a Hankel measure, which of course is also clear from the definitions.

The main result of this section is the following.

Theorem 10 *Suppose μ is a finite real-valued Borel measure on $\mathbb{D}$. Then μ is a Hankel measure if and only if the balayage of μ is in BMO.*

Proof Suppose that μ is a real-valued Borel measure on $\mathbb{D}$ and its balayage is g. By an earlier argument, we have

$$\int_{\mathbb{D}}f(z)\,\mathrm{d}\mu(z)=\frac{1}{2\pi}\int_0^{2\pi}f(\mathrm{e}^{i\theta})\overline{g(\mathrm{e}^{i\theta})}\,\mathrm{d}\theta. \tag{6}$$

If g is in BMO, then it can be written as $g=h+\overline{h}$, where h belongs to $BMOA$, because g is real. It follows that

$$\int_{\mathbb{D}}f(z)\,\mathrm{d}\mu(z)=f(0)h(0)+\frac{1}{2\pi}\int_0^{2\pi}f(\mathrm{e}^{i\theta})\overline{h(\mathrm{e}^{i\theta})}\,\mathrm{d}\mathrm{d}\theta.$$

By the well-known duality relation $(H^1)^* = BMOA$, condition (5) is valid. This along with Theorem 1 shows that μ is a Hankel measure.

Conversely, if μ is a Hankel measure, then by Theorem 1 again, we have (5) for all $f \in H^1$. Combining this with (6), we can find a positive constant C such that

$$\frac{1}{2\pi}\left|\int_0^{2\pi} f(\mathrm{e}^{i\theta})\overline{g(\mathrm{e}^{i\theta})}\,\mathrm{d}\theta\right| \le C\|f\|_{H^1}$$

for all $f \in H^1$. Since μ is real, we can write $g = h + \overline{h}$, where h is analytic. It follows that there is another positive constant C such that

$$\frac{1}{2\pi}\left|\int_0^{2\pi} f(\mathrm{e}^{i\theta})\overline{h(\mathrm{e}^{i\theta})}\,\mathrm{d}\theta\right| \le C\|f\|_{H^1}$$

for all $f \in H^1$. Since the dual space of H^1 is isomorphic to $BMOA$, we conclude that $h \in BMOA$, which gives $g \in BMO$. This completes the proof of the theorem. □

5 Further Remarks

Hankel measures have also been studied in [21] in the context of weighted Bergman spaces. More specifically, a complex Borel measure μ on $\mathbb{D}$ is called a Hankel measure for the weighted Bergman space A^2_α if there exists a positive constant C such that

$$\left|\int_{\mathbb{D}} f^2\,\mathrm{d}\mu\right| \le C\|f\|^2_{A^2_\alpha}, \qquad f \in A^2_\alpha.$$

Note that the term "pseudo-Carleson measure" was used in [21] instead of Hankel measures. Since these measures are closely related to Hankel operators and a systematic study of them was first carried out by Xiao in [20,21], it is probably more appropriate to stick to the term "Hankel measures" or "Hankel–Xiao measures."

It is well known that the Hardy space H^p can be thought of as the limiting case of A^p_α when $\alpha \to -1^+$; see [25] for example. Sometimes, it is possible to derive results about Hardy spaces from the corresponding results about weighted Bergman spaces, and vice versa, as was demonstrated in [25] by several classical examples in complex analysis. But this is not always the case, as we will see below.

First, our Theorem 6 can be considered as the limiting case (when $\alpha \to -1^+$) of the second inequality on page 452 of [21].

Second, the equivalence of conditions (a) and (b) in our Theorem 9 is the limiting case (when $\alpha \to -1^+$) of the equivalence of conditions (i) and (iii) in [21]. In particular, a positive measure μ supported on [0, 1) is a Hankel measure for the Hardy space H^2 if and only if it is a Hankel measure for A^2_α. Although we provided a proof here for completeness, experts in the field would recognize the equivalence of Theorems 7 and 9 .

Third, our Theorem 4 shows that there are cases when it is NOT possible (and actually wrong) to obtain a result for the Hardy space H^2 by taking the limit (when $\alpha \to -1^+$) of the corresponding result for weighted Bergman spaces A^2_α. It was shown in [21] that, for any given $\alpha > -1$, a complex Borel measure μ on $\mathbb{D}$ is a Hankel measure for A^2_α if and only if

$$\sup_{w\in\mathbb{D}} \left| \int_{\mathbb{D}} \left[\frac{1-|w|^2}{(1-\overline{w}z)^2} \right]^{\alpha+2} \mathrm{d}\mu(z) \right| < \infty.$$

If we formally take the limit $\alpha \to -1^+$, the condition above becomes our condition (2), which, according to our Theorem 4, does NOT characterize Hankel measures for H^2.

Finally, we mention that Hankel measures (or Hankel–Xiao measures) have also been studied in 2011 by [1, Theorem 5] in the context of the classical Dirichlet space on the unit disk.

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