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Cohomology Dimension Growth for Nakano q-Semipositive Line Bundles

Huan Wang[1]

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Abstract
We study the cohomology with high tensor powers of Nakano q-semipositive line bundles on complex manifolds. We obtain the asymptotic estimates for the dimension of cohomology with high tensor powers of semipositive line bundles over q-convex manifolds and various possibly non-compact complex manifolds, in which the order of estimates are optimal. Besides, estimates for the modified Dirac operator on Nakano q-positive line bundle on almost complex manifolds are given.

Keywords Positivity · Semipositivity · Cohomology · Fundamental estimates · Q-convex manifolds · Covering manifolds · Pseudoconvex domains · Weakly 1-complete manifolds · Complete manifolds · Bergman kernel · Dirac operator

1 Introduction

Let X be a complex manifold of dimension n and (E, h^E) be a holomorphic Hermitian vector bundle over X. Let ∇^E be the holomorphic Hermitian connection of (E, h^E) and $R^{(E,h^E)} = (\nabla^E)^2$ be the curvature of ∇^E. The Bochner–Kodaira–Nakano formula and its variation with boundary term, [2,14,18,26], play the central role in various vanishing theorems on complex manifolds. The latter have important applications in complex differential and algebraic geometry, such as the characterization of projective manifolds [22], Moishezon manifolds [13,36,37] and more recently the criterion for uniruledness and rationally connectedness and related results [6,10,43]. The key ingredient in these formulas is the curvature term $[\sqrt{-1}R^{(E,h^E)}, \Lambda]$, where Λ is the dual of Hermitian metric on manifolds. With appropriate assumptions on the positivity of R^E, one can achieve the curvature term is strictly positive, i.e., the pointwise

✉ Huan Wang
huanwang@gate.sinica.edu.tw; huanwang2016@hotmail.com

1 Institute of Mathematics, Academia Sinica, Taipei, Taiwan

Hermitian product $\left\langle [\sqrt{-1}R^{(E,h^E)}, \Lambda]s, s\right\rangle_h > 0$ for forms s with values in E, which is enough to prove vanishing theorems in various situations, see [16,21].

Instead of the strict positivity, we consider the q-semipositivity, which was introduced in [35] over Kähler manifolds. A holomorphic Hermitian line bundle on Kähler manifolds is called Nakano q-positive (resp. semipositive) which means that at every point the sum of any set of q eigenvalues of the curvature form is positive (resp. non-negative) when the eigenvalues are computed with respect to the Kähler metric. Another definition of the q-positivity is the Griffiths q-positive (resp. semipositive), which means that at every point the curvature form has at least $n-q+1$ positive (resp. semipositive) eigenvalues, see [32, Chapter 3, Sect. 1, Definition 1.1], [35] and [27]. More precisely, a holomorphic Hermitian line bundle (L, h^L) over a Hermitian manifold (X, ω) is Nakano q-semipositive with respect to the Hermitian metric ω of X, if for any (n, q)-forms s, $\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]s, s\right\rangle_h \geq 0$, see Definition 1.1, (2.9), (2.10) and [33]. In this setting, the vanishing of harmonic forms does not hold in general; however, the dimension of harmonic spaces with values in high tensor power of such line bundles still can be estimated, and moreover the estimate turns out to be optimal, see [4]. The solution of Grauert–Riemenschneider conjecture [13,36,37] shows that if $R^{(L,h^L)} \geq 0$ (i.e., Nakano 1-semipositive) on a compact complex manifold X then $\dim H^q(X, L^k) = o(k^n)$ as $k \to \infty$ for all $q \geq 1$. Demaily's solution involves holomorphic Morse inequalities [13]: $\dim H^q(X, L^k \otimes E) \leq \operatorname{rank}(E)\frac{k^n}{n!}\int_{X(q)}(-1)^q(\frac{\sqrt{-1}}{2\pi}R^{(L,h^L)})^n + o(k^n)$ as $k \to \infty$, where E is an arbitrary holomorphic vector bundle and $X(q)$ is the set where $\sqrt{-1}R^{(L,h^L)}$ has exactly q negative eigenvalues and $n-q$ positive eigenvalues. We refer to [26] for a comprehensive account of Demaily's holomorphic Morse inequalities and Bergman kernel asymptotics.

Let now E be an arbitrary holomorphic line bundle over X. Along the same lines, Berndtsson [4] showed that if $R^{(L,h^L)} \geq 0$ then $\dim H^q(X, L^k \otimes E) = O(k^{n-q})$ and it improves the estimate of Siu and Demailly, which gives only $\dim H^q(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ (since $X(q)$ is the empty set for a semipositive line bundle). The magnitude k^{n-q} is optimal. By adapting their methods to general (possibly non-compact) complex manifolds with L^2-cohomology [41], we obtain a local estimate of Bergman density function on compact subsets of the underling manifolds when $R^L \geq 0$. As applications, the estimates of the Berndtsson type still hold on covering manifolds, i.e., $\dim_\Gamma \overline{H}^{0,q}_{(2)}(X, L^k \otimes E) = O(k^{n-q})$ for all $q \geq 1$, and 1-convex manifolds, i.e., $\dim H^q(X, L^k \otimes E) = O(k^{n-q})$ for all $q \geq 1$, see [40,41]. With additional assumptions on the positivity of (L, h^L), the same estimates hold on pseudoconvex domains, weakly 1-complete manifolds and complete manifolds, see [40]. Note that, on projective manifolds, the estimate of $O(k^{n-q})$ type for nef line bundles can be found in [15], and the case of pseudo-effective line bundles was obtained in [29]. On an arbitrary compact manifolds, such estimates for semipositive line bundles equipped with Hermitian metric with analytic singularities were established by [39,40] (in the latter paper a vector bundle E of arbitrary rank is considered).

In this paper, in order to generalize such estimates to q-convex manifolds, we use the notion of Nakano q-semipositivity from [33,35], which includes the usual semipositivity $R^L \geq 0$ as a special case. We remark that, inspired by [4,26], this paper together with [40,41] give a unified approach to the optimal estimate of the dimension of cohomology of high tensor powers of line bundles with semipositivity on (compact and non-compact) complex manifolds.

Definition 1.1 [35] Let X be a complex manifold of $\dim X = n$ and ω a Hermitian metric on X. Let (L, h^L) be a holomorphic Hermitian line bundle over X. Let $1 \leq q \leq n$.

(A) (L, h^L) is called **Nakano** q**-positive** (resp. semipositive, negative, seminegative) with respect to ω at $x \in X$, if the sum of any set of q eigenvalues of the curvature form R^L_x is positive (resp. non-negative, negative, non-positive) when the eigenvalues are computed with respect to the Hermitian metric ω.

(B) (L, h^L) is called **Griffiths** q**-positive** (resp. semipositive, negative, seminegative) at $x \in X$, if the curvature form R^L_x has at least $n-q+1$ positive (resp. semipositive, negative, seminegative) eigenvalues.

For the relation of the notions of Griffiths and Nakano q-positivity, see Remark 2.4. The basic example of Nakano q-positivity is the dual of canonical bundle K^*_X on a compact Kähler manifold X of $\dim X = n$. With respect to a Kähler metric ω, the Ricci curvature of X is positive (resp. non-negative) if and only if K^*_X is Nakano 1-positive (resp. 1-semipositive); the scalar curvature of X is positive (resp. non-negative) if and only if K^*_X is Nakano n-positive (resp. n-semipositive). The basic example of Griffiths q-positivity is the dual of tautological line bundle $L(E^*)^*$, which is Griffiths $(n+1)$-positive on the projective bundle $P(E^*)$ of a holomorphic Hermitian vector bundle (E, h^E) over a compact complex manifold X of $\dim X = n$.

Firstly, we provide a refined local estimate on Bergman density functions for Nakano q-semipositive line bundles, which generalizes the main result in [4,41] and [40, Theorem 3.1]. The advantage is that it enables us to study the harmonic spaces of tensor powers of line bundles with weaker semipositivity on complex manifolds.

Theorem 1.2 *Let (X, ω) be a Hermitian manifold of dimension n and let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles over X. Let $1 \leq q \leq n$. Let $K \subset X$ be a compact subset and (L, h^L) be Nakano q-semipositive with respect to ω on a neighborhood of K. Then there exists $C > 0$ depending on K, ω, (L, h^L) and (E, h^E), such that*

$$B^j_k(x) \leq C k^{n-j} \quad \text{for all } x \in K, k \geq 1, q \leq j \leq n, \tag{1.1}$$

where $B^j_k(x)$ is the Bergman density function (3.1) of harmonic $(0, j)$-forms with values in $L^k \otimes E$. In particular, if (L, h^L) is semipositive on a neighborhood of K, the estimate holds on K for all $k \geq 1$ and $1 \leq j \leq n$.

As a direct application, it leads to the refinement of [4, Theorem 1.1] and [41, Theorem 1.2] as follows, refer to Definition 2.3 for Γ-covering manifolds.

Corollary 1.3 *Let (X, ω) be a Γ-covering manifold of dimension n, and let (L, h^L) and (E, h^E) be two Γ-invariant holomorphic Hermitian line bundles on X. Let $1 \leq q \leq n$*

and (L, h^L) be Nakano q-semipositive with respect to ω on X. Then there exists $C > 0$ such that for any $k \geq 1$, $q \leq j \leq n$, we have

$$\dim_\Gamma \overline{H}^{0,j}_{(2)}(X, L^k \otimes E) = \dim_\Gamma \mathscr{H}^{0,j}(X, L^k \otimes E) \leq Ck^{n-j}. \tag{1.2}$$

In particular, if (L, h^L) is semipositive on X, the estimate holds for all $k \geq 1$ and $1 \leq j \leq n$.

Note that holomorphic Morse inequalities on covering manifolds were obtained in [28,38].

Secondly, we obtain a refined estimate of L^2-cohomology on Hermitian manifolds from the local estimate of $B^j_k(x)$ as [40, Theorem 1.1]. It provides a uniform approach to study the cohomology of high tensor power of Nakano q-semipositive line bundles over various compact and non-compact manifolds.

Let (X, ω) be a Hermitian manifold of dimension n. Let $dv_X := \omega^n/n!$ be the volume form on X. Let (L, h^L) and (E, h^E) be holomorphic Hermitian vector bundles on X with $\operatorname{rank}(L) = 1$. We denote by $(L^2_{0,q}(X, L^k \otimes E), \|\cdot\|)$ the space of square integrable $(0, q)$-forms with values in $L^k \otimes E$ with respect to the L^2 inner product induced by the above data. We denote by $\overline{\partial}^E_k$ the maximal extension of the Dolbeault operator on $L^2_{0,\bullet}(X, L^k \otimes E)$ and by $\overline{\partial}^{E*}_k$ its Hilbert space adjoint. Let $\mathscr{H}^{0,q}(X, L^k \otimes E)$ be the space of harmonic $(0, q)$-forms with values in $L^k \otimes E$ on X. For a given $0 \leq q \leq n$, we say that **the concentration condition holds in bidegree** $(0, q)$ **for harmonic forms with values in** $L^k \otimes E$ **for large** k, if there exists a compact subset $K \subset X$ and $C_0 > 0$ such that for sufficiently large k, we have

$$\|s\|^2 \leq C_0 \int_K |s|^2 dv_X, \tag{1.3}$$

for $s \in \operatorname{Ker}(\overline{\partial}^E_k) \cap \operatorname{Ker}(\overline{\partial}^{E*}_k) \cap L^2_{0,q}(X, L^k \otimes E)$. The set K is called the exceptional compact set of the concentration. We say that **the fundamental estimate holds in bidegree** $(0, q)$ **for forms with values in** $L^k \otimes E$ **for large** k, if there exists a compact subset $K \subset X$ and $C_0 > 0$ such that for sufficiently large k, we have

$$\|s\|^2 \leq C_0 \left(\|\overline{\partial}^E_k s\|^2 + \|\overline{\partial}^{E,*}_k s\|^2 + \int_K |s|^2 dv_X \right), \tag{1.4}$$

for $s \in \operatorname{Dom}(\overline{\partial}^E_k) \cap \operatorname{Dom}(\overline{\partial}^{E*}_k) \cap L^2_{0,q}(X, L^k \otimes E)$. The set K is called the exceptional compact set of the estimate.

Theorem 1.4 *Let (X, ω) be a Hermitian manifold of dimension n and let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let $1 \leq q \leq n$. Let the concentration condition holds in bidegree $(0, q)$ for harmonic forms with values in $L^k \otimes E$ for large k. Let (L, h^L) be Nakano q-semipositive with respect to ω on a neighborhood of the exceptional set K. Then there exists $C > 0$ such that for sufficiently large k, we have*

$$\dim \mathscr{H}^{0,q}(X, L^k \otimes E) \le Ck^{n-q}. \tag{1.5}$$

The same estimate also holds for reduced L^2-Dolbeault cohomology groups,

$$\dim \overline{H}^{0,q}_{(2)}(X, L^k \otimes E) \le Ck^{n-q}. \tag{1.6}$$

In particular, if the fundamental estimate holds in bidegree $(0, q)$ for forms with values in $L^k \otimes E$ for large k, the same estimate holds for L^2-Dolbeault cohomology groups,

$$\dim H^{0,q}_{(2)}(X, L^k \otimes E) \le Ck^{n-q}. \tag{1.7}$$

Finally, by Theorem 1.4, we can study the dimension of cohomology on q-convex manifolds with semipositive line bundles. Holomorphic Morse inequalities for q-convex manifolds were obtained by Bouche [5] and [26, Sect. 3.5].

Theorem 1.5 *Let X be a q-convex manifold of dimension n and $1 \le q \le n$, and let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Assume $(L, h^L) \ge 0$ on a neighborhood of the exceptional subset K. Then, there exists $C > 0$ such that for every $j \ge q$ and $k \ge 1$,*

$$\dim H^j(X, L^k \otimes E) \le Ck^{n-j}. \tag{1.8}$$

The extremal case is also interesting when the ω-trace of $R^{(L,h^L)}$ is non-negative (i.e., n-semipositive), see Sect. 2. We obtain the finiteness of dimension of cohomology of high tensor power of such line bundles in Sect. 3 and 4. Related to the Nakano n-semipositive and the ω-trace of curvature tensor, a direct consequence from [44], [21, Ch.III.(1.34)] and [10, Corollary 5.1], which strengthens [42, Theorem B (A)], is as follows: If a compact Kähler manifold X possesses a quasi-positive $(1, 1)$-form representing the first Chern class $c_1(X)$, then X is projective and rationally connected. And a compact, simply connected, Kähler manifolds with non-negative bisectional curvature is projective and rationally connected, see Proposition 4.19 and 4.20.

For the Nakano q-positive cases, inspired by [27], [25, Theorem 1.1, 2.5] and [26, Sect. 1.5], we generalize the estimates of modified Dirac operator $D^{c,A}_k$ (see [26, Definition 1.3.6, Sect. 1.5]) of high tensor powers of positive line bundles to the Nakano q-positive case for all $1 \le q \le n$ as follows.

Theorem 1.6 *Let (X, J) be a compact smooth manifold with almost complex structure J and $\dim_{\mathbb{R}} X = 2n$. Let g^{TX} be a Riemannian metric compatible with J and $\omega := g^{TX}(J\cdot, \cdot)$ be the real $(1, 1)$-forms on X induced by g^{TX} and J. Let (E, h^E) and (L, h^L) be Hermitian vector bundles on X with $\operatorname{rank}(L) = 1$. Let ∇^E and ∇^L be Hermitian connections on (E, h^E) and (L, h^L) and let $R^E := (\nabla^E)^2$ and $R^L := (\nabla^L)^2$ be the curvatures. Let $\frac{\sqrt{-1}}{2\pi} R^L$ be compatible with J. Assume $1 \le q \le n$ and (L, h^L) is Nakano q-positive with respect to ω on X (see also (2.10)). Then there exists $C_L > 0$ such that for any $k \in \mathbb{N}$ and any $s \in \Omega^{0,\ge q}(X, L^k \otimes E)$,*

$$\|D^{c,A}_k s\|^2 \ge (2\mu_q k - C_L)\|s\|^2, \tag{1.9}$$

where the constant $\mu_q > 0$ *defined in (5.3). Especially, for k large enough,*

$$\mathrm{Ker}\left(D_k^{c,A}|_{\Omega^{0,\geq q}(X,L^k\otimes E)}\right) = 0. \tag{1.10}$$

This paper is organized as follows. In Sect. 2 we introduce the notions and basic facts on Definition 1.1. In Sect. 3 we provide the local estimate of the Bergam density function associated with Nakano q-semipositive line bundles, Theorem 1.2, and its applications, Corollary 1.3 and Theorem 1.4. In Sect. 4 we prove Theorem 1.5 and related results. In Sect. 5, estimates for the modified Dirac operator on Nakano q-positive line bundle on almost complex manifolds, Theorem 1.6, are given. From [4], we see Theorem 1.2, Corollary 1.3, Theorems 1.4 and 1.5 give the optimal order $O(k^{n-j})$ of dimension of the cohomology. And Theorem 1.6 provides a precise bound μ_q for q-positive line bundles along the lines of [25,26]. For techniques and formulations of this paper, we refer the reader to [4,26,37,40,41].

2 Preliminaries

2.1 L^2-cohomology

Let (X,ω) be a Hermitian manifold of dimension n and (F,h^F) a holomorphic Hermitian vector bundle over X. Let $\Omega^{p,q}(X,F)$ be the space of smooth (p,q)-forms on X with values in F for $p,q\in\mathbb{N}$. The volume form is $dv_X := \frac{\omega^n}{n!}$.

The L^2-scalar product is given by $\langle s_1,s_2\rangle = \int_X\langle s_1(x),s_2(x)\rangle_h \mathrm{d}v_X(x)$ on $\Omega^{p,q}(X,F)$, where $\langle\cdot,\cdot\rangle_h := \langle\cdot,\cdot\rangle_{h^F,\omega}$ is the pointwise Hermitian inner product induced by ω and h^F. We denote by $L^2_{p,q}(X,F)$, the L^2 completion of $\Omega_0^{p,q}(X,F)$, which is the subspace of $\Omega^{p,q}(X,F)$ consisting of elements with compact support.

Let $\overline{\partial}^F : \Omega_0^{p,q}(X,F)\to L^2_{p,q+1}(X,F)$ be the Dolbeault operator and let $\overline{\partial}^F_{\max}$ be its maximal extension (see [26, Lemma 3.1.2]). From now on we still denote the maximal extension by $\overline{\partial}^F := \overline{\partial}^F_{\max}$ and the associated Hilbert space adjoint by $\overline{\partial}^{F*} := \overline{\partial}^{F*}_H := (\overline{\partial}^F_{\max})^*_H$. Consider the complex of closed, densely defined operators $L^2_{p,q-1}(X,F)\xrightarrow{\overline{\partial}^F} L^2_{p,q}(X,F)\xrightarrow{\overline{\partial}^F} L^2_{p,q+1}(X,F)$. Note that $(\overline{\partial}^F)^2 = 0$. By [26, Proposition 3.1.2], the operator defined by

$$\begin{aligned}\mathrm{Dom}(\square^F) &= \left\{s\in\mathrm{Dom}(\overline{\partial}^F)\cap\mathrm{Dom}(\overline{\partial}^{F*}) : \overline{\partial}^F s\in\mathrm{Dom}(\overline{\partial}^{F*}),\ \overline{\partial}^{F*}s\in\mathrm{Dom}(\overline{\partial}^F)\right\},\\ \square^F s &= \overline{\partial}^F\overline{\partial}^{F*}s+\overline{\partial}^{F*}\overline{\partial}^F s \quad \text{for } s\in\mathrm{Dom}(\square^F),\end{aligned} \tag{2.1}$$

is a positive, self-adjoint extension of Kodaira Laplacian, called the Gaffney extension.

Definition 2.1 [26] The space of harmonic forms $\mathscr{H}^{p,q}(X,F)$ is defined by

$$\mathscr{H}^{p,q}(X,F) := \mathrm{Ker}(\square^F) = \{s\in\mathrm{Dom}(\square^F)\cap L^2_{p,q}(X,F) : \square^F s = 0\}. \tag{2.2}$$

The qth reduced L^2-Dolbeault cohomology is defined by

$$\overline{H}^{0,q}_{(2)}(X, F) := \frac{\mathrm{Ker}(\overline{\partial}^F) \cap L^2_{0,q}(X, F)}{[\mathrm{Im}(\overline{\partial}^F) \cap L^2_{0,q}(X, F)]}, \tag{2.3}$$

where $[V]$ denotes the closure of the space V. The qth (non-reduced) L^2-Dolbeault cohomology is defined by

$$H^{0,q}_{(2)}(X, F) := \frac{\mathrm{Ker}(\overline{\partial}^F) \cap L^2_{0,q}(X, F)}{\mathrm{Im}(\overline{\partial}^F) \cap L^2_{0,q}(X, F)}. \tag{2.4}$$

According to the general regularity theorem of elliptic operators, $s \in \mathscr{H}^{p,q}(X, F)$ implies $s \in \Omega^{p,q}(X, F)$. By weak Hodge decomposition (cf. [26, (3.1.21) (3.1.22)]),

$$\overline{H}^{0,q}_{(2)}(X, F) \cong \mathscr{H}^{0,q}(X, F) \tag{2.5}$$

for any $q \in \mathbb{N}$. The qth cohomology of the sheaf of holomorphic sections of F is isomorphic to the qth Dolbeault cohomology, $H^q(X, F) \cong H^{0,q}(X, F)$.

For a given $0 \leq q \leq n$, we say **the fundamental estimate holds in bidegree** $(0, q)$ **for forms with values in** F, if there exists a compact subset $K \subset X$ and $C > 0$ such that

$$\|s\|^2 \leq C \left(\|\overline{\partial}^F s\|^2 + \|\overline{\partial}^{F*}\|^2 + \int_K |s|^2 \mathrm{d}v_X \right), \tag{2.6}$$

for $s \in \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F,*}) \cap L^2_{0,q}(X, F)$. K is called the exceptional compact set of the estimate. If the fundamental estimate holds, the reduced and non-reduced L^2-Dolbeault cohomology coincide, see [26, Theorem 3.1.8]. For a given $0 \leq q \leq n$, we say that **the concentration condition holds in bidegree** $(0, q)$ **for harmonic forms with values in** F, if there exists a compact subset $K \subset X$ and $C > 0$ such that

$$\|s\|^2 \leq C \int_K |s|^2 \mathrm{d}v_X, \tag{2.7}$$

for $s \in \mathrm{Ker}(\overline{\partial}^F) \cap \mathrm{Ker}(\overline{\partial}^{F*}) \cap L^2_{0,q}(X, F)$. We call K the exceptional compact set of the concentration. Note if the fundamental estimate holds, the concentration condition also.

2.1.1 The Convexity of Complex Manifolds and Γ-Coverings

Definition 2.2 A complex manifold X of dimension n is called q-convex if there exists a smooth function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ such that the sublevel set $X_c = \{\varrho < c\} \Subset X$ for all $c \in \mathbb{R}$ and the complex Hessian $\partial\overline{\partial}\varrho$ has $n - q + 1$ positive eigenvalues outside a compact subset $K \subset X$. Here $X_c \Subset X$ means that the closure $\overline{X}_c$ is compact in X. We call ϱ an exhaustion function and K exceptional set. X is q-complete if $K = \emptyset$ in additional.

Every compact complex manifold is q-convex for all $1 \le q \le n$. By definition, a compact complex manifold is exactly a 0-convex manifold. For non-compact manifolds, Greene-Wu [16, Ch. IX. (3.5) Theorem] showed that: Every connected non-compact complex manifold of dimension n is n-complete. Moreover, every connected complex manifold of dimension n is n-convex. Thus, if X is a connected non-compact complex manifold of dimension n and E a holomorphic vector bundle over X, $H^n(X, E) = 0$, see [1]. We denote the jth Dolbeault cohomology with compact supports by $[H^{0,j}(X, E)]_0$, see [19, (20.8) (20.17)]. Note that if X is compact, $[H^{0,j}(X, E)]_0$ is equal to the usual cohomology. The duality between it and the usual Dolbeault cohomology on q-convex manifold of dimension n with $1 \le q \le n$ is given by

$$\dim[H^{0,j}(X, E)]_0 = \dim H^{0,n-j}(X, E^* \otimes K_X) \le \infty \quad \text{for all } 0 \le j \le n-q. \tag{2.8}$$

If $q = 1$, then, moreover, $\dim[H^{0,n}(X, E)]_0 = \dim H^0(X, E^* \otimes K_X)$, where $K_X = \wedge^n T^{1,0*}X$.

Let M be a relatively compact domain with smooth boundary bM in a complex manifold X. Let $\rho \in \mathscr{C}^\infty(X, \mathbb{R})$ such that $M = \{x \in X : \rho(x) < 0\}$ and $d\rho \neq 0$ on $bM = \{x \in X : \rho(x) = 0\}$. We denote the closure of M by $\overline{M} = M \cup bM$. We say that ρ is a defining function of M. Let $T^{(1,0)}bM := \{v \in T_x^{(1,0)}X : \partial\varrho(v) = 0\}$ be the analytic tangent bundle to bM at $x \in bM$. The Levi form of ρ is the 2-form $\mathscr{L}_\rho := \partial\overline{\partial}\rho \in \mathscr{C}^\infty(bM, T^{(1,0)*}bM \otimes T^{(0,1)*}bM)$. M is called strongly (resp. (weakly)) pseudoconvex if the Levi form $\mathscr{L}_\rho$ is positive definite (resp. semidefinite). Note any strongly pseudoconvex domain is 1-convex.

A complex manifold X is called weakly 1-complete if there exists a smooth plurisubharmonic function $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$ such that $\{x \in X : \varphi(x) < c\} \Subset X$ for any $c \in \mathbb{R}$. φ is called an exhaustion function. Note any 1-convex manifold is weakly 1-complete.

A Hermitian manifold (X, ω) is called complete, if all geodesics are defined for all time for the underlying Riemannian manifold.

Definition 2.3 Let (X, ω) be a Hermitian manifold of dimension n on which a discrete group Γ acts holomorphically, freely and properly such that ω is a Γ-invariant Hermitian metric and the quotient X/Γ is compact. We say X is a Γ-covering manifold, see also [3,26,41].

2.1.2 Kodaira Laplacian with $\overline{\partial}$-Neumann Boundary Conditions

Let (X, ω) be a Hermitian manifold of dimension n and (F, h^F) be a holomorphic Hermitian vector bundles over X. Let M be a relatively compact domain in X. Let ρ be a defining function of M satisfying $M = \{x \in X : \rho(x) < 0\}$ and $|d\rho| = 1$ on bM, where the pointwise norm $|\cdot|$ is given by g^{TX} associated to ω.

Let $e_{\boldsymbol{n}} \in TX$ be the inward pointing unit normal at bM and $e_{\boldsymbol{n}}^{(0,1)}$ its projection on $T^{(0,1)}X$. In a local orthonormal frame $\{w_1, \ldots, \omega_n\}$ of $T^{(1,0)}X$, we have $e_{\boldsymbol{n}}^{(0,1)} = -\sum_{j=1}^n w_j(\rho)\overline{w}_j$. Let $B^{0,q}(X, F) := \{s \in \Omega^{0,q}(\overline{M}, F) : i_{e_{\boldsymbol{n}}^{(0,1)}} s = 0 \text{ on } bM\}$. Then $B^{0,q}(M, F) = \mathrm{Dom}(\overline{\partial}_H^{F*}) \cap \Omega^{0,q}(\overline{M}, F)$ and the Hilbert space adjoint $\overline{\partial}_H^{F*}$ of $\overline{\partial}^F$ coincides with the formal adjoint $\overline{\partial}^{F*}$ of $\overline{\partial}^F$ on $B^{0,q}(M, F)$, see [26, Proposition 1.4.19]. The operator $\Box_N s := \overline{\partial}^F \overline{\partial}^{F*} s + \overline{\partial}^{F*} \overline{\partial}^F s$ for $s \in \mathrm{Dom}(\Box_N) := \{s \in B^{0,q}(M, F) : \overline{\partial}^F s \in B^{0,q+1}(M, F)\}$. The Friedrichs extension of $\Box_N$ is a self-adjoint operator and is called the Kodaira Laplacian with $\overline{\partial}$-Neumann boundary conditions, which coincides with the Gaffney extension of the Kodaira Laplacian, see [26, Proposition 3.5.2]. $\Omega^{0,\bullet}(\overline{M}, F)$ is dense in $\mathrm{Dom}(\overline{\partial}^F)$ in the graph norms of $\overline{\partial}^F$, and $B^{0,\bullet}(M, F)$ is dense in $\mathrm{Dom}(\overline{\partial}_H^{F*})$ and in $\mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}_H^{F*})$ in the graph norms of $\overline{\partial}_H^{F*}$ and $\overline{\partial}^E + \overline{\partial}_H^{E*}$, respectively, see [26, Lemma 3.5.1]. Here the graph norm is defined by $\|s\| + \|Rs\|$ for $s \in \mathrm{Dom}(R)$.

2.2 Nakano *q*-Semipositive Line Bundles and the *ω*-Trace

Let (X, ω) be a Hermitian manifold of dimension n and (E, h^E) a holomorphic Hermitian vector bundle over X. Let ∇^E be the holomorphic Hermitian connection of (E, h^E) and $R^{(E,h^E)} = (\nabla^E)^2$ be the curvature. Let $\bigwedge^{p,q} T^*X := \bigwedge^p T^{1,0*}X \otimes \bigwedge^q T^{0,1*}X$ and let $\bigwedge^{p,q} T_x^*X$ be the fiber of the bundle $\bigwedge^{p,q} T^*X$ for $x \in X$, and $\Omega^{p,q}(X, E) := \mathscr{C}^\infty(X, \bigwedge^{p,q} T^*X \otimes E)$ the space of smooth (p, q)-forms with values in E. We set $\langle , \rangle_h$ the induced pointwise Hermitian metric in the context.

Let (L, h^L) be a holomorphic Hermitian line bundle over X. Then $R^L = \overline{\partial}\partial \log |s|_{h^L}^2$ for any local holomorphic frame s, and the Chern–Weil form of the first Chern class of L is $c_1(L, h^L) = \frac{\sqrt{-1}}{2\pi} R^L$, which is a real $(1, 1)$-form on X. We use the notion of positive (p, p)-form, see [16, Chapter III, §1, (1.1) (1.2) (1.5) (1.7)]. If a (p, p)-form T is positive, we write $T \geq 0$. Let Λ be the dual of the operator $\mathcal{L} := \omega \wedge \cdot$ on $\Omega^{p,q}(X)$ with respect to the Hermitian inner product $\langle , \rangle_h$ on X. In a local orthonormal frame $\{w_j\}_{j=1}^n$ of $T^{1,0}X$ and its dual $\{w^j\}$ of $T^{1,0*}X$, $R^{(L,h^L)} = R^{(L,h^L)}(w_i, \overline{w}_j) w^i \wedge \overline{w}^j$, $\mathcal{L} = \sqrt{-1} \sum_{j=1}^n w^j \wedge \overline{w}^j \wedge \cdot$ and $\Lambda = -\sqrt{-1} \sum_{j=1}^n i_{\overline{w}^j} i_{w^j}$. For $s \in \Omega^{p,q}(X)$, $\left\langle [\sqrt{-1} R^{(L,h^L)}, \Lambda] s, s \right\rangle_h \in \mathscr{C}^\infty(X, \mathbb{R})$.

Recall the notion of q-semipositivity of line bundle in Definition 1.1, see [33, (1)] and [35, Sect. 4]. By the definition, for $1 \leq q \leq n$, (L, h^L) is **Nakano q-semipositive with respect to ω at** $x \in X$, which means that

$$\left\langle [\sqrt{-1} R^{(L,h^L)}, \Lambda] \alpha, \alpha \right\rangle_h \geq 0 \quad \text{for all } \alpha \in \wedge^{n,q} T_x^* X. \tag{2.9}$$

We denoted it by $(\star_q) \geq 0$ at x. For $1 \leq q \leq n$, (L, h^L) is **Nakano q-positive with respect to ω at** x, which means that

$$\left\langle [\sqrt{-1} R^{(L,h^L)}, \Lambda] \alpha, \alpha \right\rangle_h > 0 \quad \text{for all } \alpha \in \wedge^{n,q} T_x^* X \backslash \{0\}. \tag{2.10}$$

We denoted it by $(\star_q) > 0$ at x. For a subset $Y \subset Y$, (L, h^L) is **Nakano q-semipositive (resp. positive) with respect to ω on Y**, if $(\star_q) \geq 0$ (resp. > 0) at every point of Y. In a local orthonormal frame $\{\omega_j\}$ of $T^{1,0}X$ around x, (2.9) is equivalent to

$$\left\langle R^{(L,h^L)}(\omega_i, \overline{\omega}_j)\overline{\omega}^j \wedge i_{\overline{\omega}_i}\alpha, \alpha\right\rangle_h \geq 0 \quad \text{for all } \alpha \in \wedge^{0,q} T_x^* X. \tag{2.11}$$

And (2.10) can be represented by replacing $\bigwedge^{n,q} T_x^* X$ and $\geq$ by $\bigwedge^{n,q} T_x^* X \backslash \{0\}$ and $>$ in (2.11), respectively.

Remark 2.4 (notions of the q-positivity) Note that the notion of Nakano q-positive depends on the choice of Hermitian metric ω. On the other hand, the notion of Griffiths q-positive is independent of the choice of ω. If L is Nakano q-positive at x, then L is also Griffiths q-positive at x. If L is Griffiths q-positive at x, then there exists a metric ω such that L is Nakano q-positive at x with respect to ω. Actually, if L is Griffiths q-positive on X, then for any compact set K there exists a Hermitian metric ω on X such that L is Nakano q-positive on K with respect to ω, see [26, (3.5.7)] or [27, (9)] for the construction of ω. The Nakano 1-semipositivity (resp. positivity) coincides with the Griffiths 1-semipositivity (resp. positivity), i.e., the usual semipositivity (resp. positivity). By definition, q-positivity implies the q-semipositivity. For vector bundles, we refer to [32,35] for the definitions of the q-positive in the sense of Nakano and Griffiths.

2.2.1 The Special Case $(\star_1) \geq 0$

An important special case is $(\star_1) \geq 0$, which is equivalent to (L, h^L) is semipositive as follows.

Definition 2.5 A holomorphic Hermitian line bundle (L, h^L) is semipositive at $x \in X$, if $R^{(L,h^L)}(U, \overline{U}) \geq 0$ for $U \in T_x^{1,0}X$, denoted by $(L, h^L)_x \geq 0$. For a subset $Y \subset X$, (L, h^L) is semipositive on Y if $(L, h^L)_x \geq 0$ at all $x \in Y$. The definition of $(L, h^L)_x > 0$ is analogue.

From the definition, $(L, h^L)_x \geq 0$ implies $(\star_q) \geq 0$ at x for all $1 \leq q \leq n$. Conversely, $(\star_1) \geq 0$ at x implies $(L, h^L)_x \geq 0$. Thus, $(\star_q) \geq 0$ is a refinement of $(L, h^L) \geq 0$.

Proposition 2.6 *Let $(L, h^L) \geq 0$ (resp. > 0) at $x \in X$. Then, for any Hermitian metric ω on X, $1 \leq q \leq n$ and $\alpha \in \bigwedge^{n,q} T_x^* X$ $\big($resp. $\bigwedge^{n,q} T_x^* X \backslash \{0\}\big)$,*

$$\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]\alpha, \alpha\right\rangle_h \geq 0 \; (\textit{resp.} \; > 0). \tag{2.12}$$

Proof Let $\{\omega_j\}$ be an orthonormal frame around x such that $\sqrt{-1}R_x^{(L,h^L)} = \sqrt{-1}c_j(x)\omega^j \wedge \overline{\omega}^j$. Let $C_J(x) := \sum_{j \in J} c_j(x)$ for each ordered $J = (j_1, \ldots, j_q)$

with $|J| = q$. Let $\alpha \in \bigwedge^{n,q} T_x^* X$ and $\alpha = \sum_J f_{NJ}(x) w^N \wedge \overline{w}^J$, $N = (1, \ldots, n)$ and $|J| = q$,

$$\left\langle [\sqrt{-1} R^{(L,h^L)}, \Lambda]\alpha, \alpha \right\rangle_h (x) = \sum_J C_J(x) |f_{NJ}(x)|^2. \tag{2.13}$$

Since $(L, h^L)_x \geq 0$, $c_j(x) \geq 0$ for all $1 \leq j \leq n$ and thus $C_J(x) \geq 0$ for all $1 \leq |J| \leq n$. And the positive case follows similarly. □

Proposition 2.7 $(L, h^L)_x \geq 0$ *if and only if* $(\star_1) \geq 0$ *at* x.

Proof Suppose $(\star_1) \geq 0$ at x. Let $U \in T_x^{1,0} X$ with $U = \sum_{k=1}^n u_k \omega_k$ in a local orthonormal frame $\{\omega_j\}_{j=1}^n$ of $T^{1,0}X$ around x. We set $\alpha = u_k \overline{\omega}^k \in T_x^{0,1*} X$, and then $R^{(L,h^L)}(U, \overline{U}) = \overline{u}_j R^L(\omega_i, \overline{\omega}_j) u_i = \langle R^L(\omega_i, \overline{\omega}_j) \overline{\omega}^j \wedge i_{\overline{\omega}_i} \alpha, \alpha \rangle_h \geq 0$. □

The general relation among $(\star_q) \geq 0$, $1 \leq q \leq n$, is as follows.

Proposition 2.8 *If* $(\star_q) \geq 0$ *at* x, *then* $(\star_{q+1}) \geq 0$ *at* x.

Proof From $(\star_q) \geq 0$ at x and (2.13), $C_J(x) \geq 0$ for each ordered $|J| = q$. Let $C_K(x) := \sum_{k \in K} c_k(x)$ for each ordered $|K| = q + 1$. Then $C_K(x) = \frac{1}{q} \sum_{|J|=q, J \subset K} C_J(x) \geq 0$, and thus $(\star_{q+1}) \geq 0$ at x by (2.13). □

Remark 2.9 Clearly, the positive case $(>)$ of Proposition 2.7 and 2.8 also holds.

2.2.2 The Special Case $(\star_n) \geq 0$

Another interesting case is $(\star_n) \geq 0$, which is equivalent to the ω-trace of Chern curvature tensor $R^{(L,h^L)}$ is non-negative as follows.

Definition 2.10 The ω-trace of Chern curvature tensor $R^{(L,h^L)}$, $\tau(L, h^L, \omega) \in \mathscr{C}^\infty(X, \mathbb{R})$, is defined by $\sqrt{-1} R^{(L,h^L)} \wedge \omega_{n-1} = \tau(L, h^L, \omega) \omega_n$.

Equivalently, let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$ with respect to ω and $\{w^j\}$ the dual frame of $T^{(1,0)*}X$,

$$\begin{aligned} \tau(L, h^L, \omega) := \mathrm{Tr}_\omega R^{(L,h^L)} &= \sum_{j=1}^n R^{(L,h^L)}(w_j, \overline{w}_j) \\ &= \sum_{i,k} R^{(L,h^L)} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_k} \right) \langle dz_i, d\overline{z}_k \rangle_{g^{T^*X}}. \end{aligned}$$

We say the ω-trace of Chern curvature tensor $R^{(L,h^L)}$ is semipositive (resp. positive) if $\tau(L, h^L, \omega) \geq 0$ (resp. > 0). From (2.13), it follows immediately:

Proposition 2.11 *We have the following:*

(1) $\tau(L, h^L, \omega)|\alpha|_h^2 = \left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]\alpha, \alpha \right\rangle_h$ *for all* $\alpha \in \wedge^{n,n} T_x^* X$, $x \in X$.
(2) $\tau(L, h^L, \omega)_x \geq 0$ *if and only if* $(\star_n) \geq 0$ *at* x.
(3) $\tau(L, h^L, \omega)_x > 0$ *if and only if* $(\star_n) > 0$ *at* x.
(4) $\tau(L, h^L, \omega) = -\tau(L^*, h^{L^*}, \omega)$.

Example 2.12 Let (X, ω) be a Kähler manifold of dimension n, let (K_X^*, h_ω) be the dual of canonical line bundle $K_X := \wedge^n T^{(1,0)*} X$ associated with the Hermitian metric h_ω induced from ω. The ω-trace of $R^{(K_X^*, h_\omega)}$ coincides with the scalar curvature r_ω^X of (X, ω) up to the multiplication of 2, i.e., for $r_\omega^X := 2 \sum_j Ric(\omega_j, \overline{\omega}_j)$,

$$2\tau(K_X^*, h_\omega, \omega) = 2\mathrm{Tr}_\omega R^{(K_X^*, h_\omega)} = r_\omega^X. \tag{2.14}$$

2.2.3 The ω-Trace of Chern Curvature Tensor of Vector Bundles

Let (E, h^E) be a holomorphic Hermitian vector bundle over a complex manifold (X, ω). The ω-trace of Chern curvature tensor $R^{(E,h^E)}$, $\tau(E, h^E, \omega) := \mathrm{Tr}_\omega R^{(E,h^E)} \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ is defined by

$$\sqrt{-1}R^{(E,h^E)} \wedge \omega_{n-1} = \tau(E, h^E, \omega)\omega_n, \tag{2.15}$$

see [10, Sect. 1.5.]. Note in [26, (4.15)] $\Lambda_\omega(R^E)$ is the contraction of R^E with respect to ω and thus $\sqrt{-1}\Lambda_\omega(R^E) = \tau(E, h^E, \omega)$ in our notations. We define $\tau(E, h^E, \omega) \geq 0$ (resp. > 0) at $x \in X$ by

$$\langle \tau(E, h^E, \omega)s, s \rangle_{h^E} \geq 0 \tag{2.16}$$

(resp. > 0) for $s \in E_x$ (resp. $s \in E_x \backslash \{0\}$). Similarly, we can define $\tau(E, h^E, \omega) \leq 0$ (resp. < 0). Let (E^*, h^{E^*}) be the dual bundle with the induced metric given by (E, h^E),

$$\tau(E, h^E, \omega) = -\tau(E^*, h^{E^*}, \omega) \tag{2.17}$$

coincide as Hermitian matrices, see [21]. For the projection $\pi : P(E^*) \to X$ and the dual of tautological line bundle $O_{E^*}(1) := (L(E^*))^*$ over $P(E^*)$ with Hermitian metrics $\omega_{P(E^*)}$ and $h^{O_{E^*}(1)}$ induced from ω and h^E, see [21, Ch. III, Sect. 5], we set

$$\tau(O_{E^*}(1)) := \tau\left(O_{E^*}(1), h^{O_{E^*}(1)}, \omega_{P(E^*)}\right). \tag{2.18}$$

3 Bergman Density Function and Applications

3.1 Local Estimates for Bergman Density Functions

Let (X, ω) be a Hermitian (paracompact) manifold of dimension n and (L, h^L) and (E, h^E) be Hermitian holomorphic line bundles over X. For $k \in \mathbb{N}$ we form the

Hermitian line bundles $L^k := L^{\otimes k}$ and $L^k \otimes E$, the latter endowed with the metric $h_k = (h^L)^{\otimes k} \otimes h^E$. Let ∇^L be the holomorphic Hermitian connection of (L, h^L). The curvature of (L, h^L) is defined by $R^L = (\nabla^L)^2$, then the Chern–Weil form of the first Chern class of L is $c_1(L, h^L) = \frac{\sqrt{-1}}{2\pi} R^L$, which is a real $(1, 1)$-form on X.

Let $\mathrm{d}v_X := \frac{\omega^n}{n!}$ be the volume form on X. We denote that the maximal extension of the Dolbeault operator $\overline{\partial}_k^E := (\overline{\partial}^{L^k \otimes E})_{\max}$, its Hilbert space adjoint $\overline{\partial}_k^{E*} := (\overline{\partial}^{L^k \otimes E})_H^*$, and the Gaffney extension of Kodaira Laplacian $\Box_k^E := \Box^{L^k \otimes E}$. Let $\mathscr{H}^{o,q}(X, L^k \otimes E) := \mathrm{Ker}(\Box_k^E) \cap L^2_{0,q}(X, L^k \otimes E)$ be the space of harmonic $(0, q)$-forms with values in $L^k \otimes E$ on X. For simplifying the notations, sometimes we will denote $\overline{\partial}$, $\overline{\partial}^*$ and $\Box$. For forms with values in $L^k \otimes E$, we denote the Hermitian norm $|\cdot| := |\cdot|_{h_k,\omega}$ induced by ω, h^L, h^E and the L^2-inner product $\|\cdot\| := \|\cdot\|_{L^2_{0,q}(X, L^k \otimes E)}$ for each $q \in \mathbb{N}$. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$ with respect to ω with dual frame $\{w^j\}$ of $T^{(1,0)*}X$.

Let $\mathscr{H}^{0,q}(X, L^k \otimes E)$ be the space of harmonic $(0, q)$-forms with values in $L^k \otimes E$. Let $\{s_j^k\}_{j\geq 1}$ be an orthonormal basis and denote by B_k^q the Bergman density function defined by

$$B_k^q(x) = \sum_{j\geq 1} |s_j^k(x)|^2_{h_k,\omega}, \quad x \in X, \tag{3.1}$$

where $|\cdot|_{h_k,\omega}$ is the pointwise norm of a form. The function (3.1) is well defined by an adaptation of [11, Lemma 3.1] to form case.

We follow the notations in [41, Sect. 3.2] and show the sub-meanvalue formulas of harmonic forms in $\mathscr{H}^{n,q}(X, L^k \otimes E)$. Let (L, h^L) and (E, h^E) be Hermitian holomorphic line bundles over X. For any compact subset K in X, the interior of K is denoted by $\mathring{K}$. Let K_1, K_2 be compact subsets in X, such that $K_1 \subset \mathring{K}_2$. Then there exists a constant $c_0 = c_0(\omega, K_1, K_2) > 0$ such that for any $x_0 \in K_1$, the holomorphic normal coordinate around x_0 is $V \cong W \subset \mathbb{C}^n$, where

$$W := B(c_0) := \{z \in \mathbb{C}^n : |z| < c_0\}, \quad V := B(x_0, c_0) \subset \mathring{K}_2 \subset K_2,$$

$z(x_0) = 0$, and $\omega(z) = \sqrt{-1}\sum_{i,j} h_{ij}(z) dz_i \wedge d\overline{z}_j$ with $h_{ij}(0) = \frac{1}{2}\delta_{ij}$.

Lemma 3.1 *Let (X, ω) be a Hermitian manifold of dimension n and (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles over X. Let K_1 and K_2 be compact subsets in X such that $K_1 \subset \mathring{K}_2$. Let $1 \leq q \leq n$. Assume (L, h^L) satisfies (2.9) for $x \in \mathring{K}_2$. Then,*

(1) *there exists a constant $C > 0$ such that*

$$\int_{|z|<r} |\alpha|^2_{h_k,\omega} \mathrm{d}v_X \leq C r^{2q} \int_X |\alpha|^2_{h_k,\omega} \mathrm{d}v_X \tag{3.2}$$

for any $\alpha \in \mathscr{H}^{n,q}(X, L^k \otimes E)$ and $0 < r < \frac{C_0}{2^n}$;

(2) *there exists a constant* $C > 0$ *such that*

$$|\alpha(x_0)|^2_{h_k,\omega} \leq Ck^n \int_{|z|<\frac{2}{\sqrt{k}}} |\alpha|^2_{h_k,\omega} \mathrm{d}v_X \tag{3.3}$$

for any $x_0 \in K_1$, $\alpha \in \mathscr{H}^{n,q}(X, L^k \otimes E)$ *and* k *sufficiently large,*

where $|\cdot|^2_{h_k,\omega}$ *is the pointwise Hermitian norm induced by* ω, h^L *and* h^E.

Proof In [41, Lemma 3.4, 3.5] the assertion was proved for all $1 \leq q \leq n$ for a semipositive line bundle on $\mathring{K}_2$. However, in order to prove the assertion for a fixed q, it is enough to assume (L, h^L) is Nakano q-semipositive, i.e., it satisfies (2.9) for $x \in \mathring{K}_2$. Indeed, if (L, h^L) satisfies (2.9) for $x \in \mathring{K}_2$, we have

$$c_1(L, h^L) \wedge T_\alpha \wedge \omega_{q-1} = (2\pi)^{-1}\Big\langle[\sqrt{-1}R^L, \Lambda]\alpha, \alpha\Big\rangle_h \omega_n \geq 0 \tag{3.4}$$

on $\mathring{K}_2$. Thus, the inequality in [41, (3.11)], $i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) \geq -C_4|\alpha|^2_h\omega_n$, still holds for $\alpha \in \mathscr{H}^{n,q}(X, L^k \otimes E)$ and the rest part of the proof is unchanged. Thus this sub-meanvalue proposition analogue to [41, Lemma 3.4, Lemma 3.5] follows. □

Analogue to [4,40,41], we obtain a local estimates for the Bergman density functions as follows.

Theorem 3.2 *Let* (X, ω) *be a Hermitian manifold of dimension* n *and let* (L, h^L) *and* (E, h^E) *be holomorphic Hermitian line bundles over* X, *and* $1 \leq q \leq n$. *Let* $K \subset X$ *be a compact subset and* (L, h^L) *is Nakano* q*-semipositive with respect to* ω *on a neighborhood of* K. *Then there exists* $C > 0$ *depending on* K, ω, (L, h^L) *and* (E, h^E), *such that*

$$B^q_k(x) \leq Ck^{n-q} \quad \text{for all } x \in K, k \geq 1, \tag{3.5}$$

where $B^q_k(x)$ *is defined by* (3.1) *for harmonic* $(0, q)$*-forms with values in* $L^k \otimes E$.

Proof We repeat the procedure in the proof of [41, Theorem 1.1] by using Lemma 3.1 instead of [41, Lemma 3.4, 3.5]. By combine (3.3) and the case $r = \frac{2}{\sqrt{k}}$ of (3.2), we have there exists $C > 0$ such that

$$S^q_k(x) := \sup\left\{\frac{|\alpha(x)|^2_{h_k,\omega}}{\|\alpha\|^2_{L^2}} : \alpha \in \mathscr{H}^{n,q}(X, L^k \otimes E)\right\} \leq Ck^{n-q} \tag{3.6}$$

for any $x \in K_1$ and $k \geq 1$. Finally, it follows from the fact $S^q_k(x) \leq B^q_k(x) \leq CS^q_k(x)$ and replacing $E \bigotimes \Lambda^n(T^{(1,0)}X)$ for E in $\mathscr{H}^{n,q}(X, L^k \otimes E)$. □

Proof of Theorem 1.2 Combining Theorem 3.2 and Proposition 2.8. □

3.2 The Growth of Cohomology on Coverings

Proof of Corollary 1.3 For a fundamental domain $U \Subset X$ with respect to Γ, by Theorem 1.2 and Proposition 2.8, $\dim_\Gamma \mathscr{H}^{0,j}(X, L^k \otimes E) = \int_U B_k^j(x) dv_X \le Ck^{n-j}$ for all $j \ge q$. □

Corollary 3.3 *Let (X, ω) be a Γ-covering Hermitian manifold of dimension n. Let (L, h^L) and (E, h^E) be two Γ-invariant holomorphic Hermitian line bundles on X.*

(1) *If $\tau(L, h^L, \omega) \ge 0$ on X, then there exists $C > 0$ such that for any $k \ge 1$,*

$$\dim_\Gamma \overline{H}^{0,n}_{(2)}(X, L^k \otimes E) \le C. \tag{3.7}$$

(2) *If $\tau(L, h^L, \omega) \le 0$ on X, then there exists $C > 0$ such that for any $k \ge 1$,*

$$\dim_\Gamma \overline{H}^{0,0}_{(2)}(X, L^k \otimes E) \le C. \tag{3.8}$$

Proof Apply Proposition 2.11(2)(4), Corollary 1.3 and Serre duality [9, 6.3.15]. □

Since connected complex manifolds are either compact or n-complete, see [16, IX.(3.5)], we can rephrase Corollary 3.3 for the trivial Γ by (2.8) as follows. Let (X, ω) be a connected Hermitian manifold of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. If $\tau(L, h^L, \omega) \ge 0$ on X, then $\dim H^n(X, L^k \otimes E) \le C$ for any $k \ge 1$; if $\tau(L, h^L, \omega) \le 0$ on X, then $\dim[H^{0,0}(X, L^k \otimes E)]_0 \le C$ for any $k \ge 1$.

3.3 The Growth of Cohomology on General Hermitian Manifolds

As another application, we can refine the main result in [40].

Proof of Theorem 1.4 By Theorem 3.2 and the concentration condition, we have

$$\begin{aligned} \dim \overline{H}^{0,q}_{(2)}(X, L^k \otimes E) &= \dim \mathscr{H}^{0,q}(X, L^k \otimes E) \\ &= \sum_{j \ge 1} \|s_j^k\|^2 \le C_0 \int_K B_k^q(x) \mathrm{d}v_X \\ &\le C_0 C k^{n-q} \operatorname{vol}(K) \end{aligned} \tag{3.9}$$

for sufficiently large k. Note that $H^{0,q}_{(2)}(X, F) = \overline{H}^{0,q}_{(2)}(X, F)$ and the dimension is finite, when the fundamental estimate holds in bidegree $(0, q)$ for forms with values in a holomorphic Hermitian vector bundle (F, h^F). □

4 Refined Estimates on Complex Manifolds with Convexity

4.1 Proof of the Result on q-convex Manifolds

Let X be a q-convex manifold of dimension n. Let ϱ be a exhaustion function of X and K a compact exceptional set in X. By definition, $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies $X_c := \{\varrho < c\} \Subset X$ for all $c \in \mathbb{R}$, $\sqrt{-1}\partial\overline{\partial}\varrho$ has $n - q + 1$ positive eigenvalues on $X \backslash K$. In this section, we fix real numbers u_0, u and v satisfying $u_0 < u < c < v$ and $K \subset X_{u_0}$.

We outline the idea of our proof of Theorem 1.5. Let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. The fundamental estimate holds in bidegree $(0, j)$ for forms with values in $L^k \otimes E$ for large k and each $q \leq j \leq n$ on X_c when X is a q-convex manifold, see Proposition 4.2, which was obtained in [26, Theorem 3.5.8] for the proof of Morse inequalities on q-convex manifolds. We observe that the Nakano q-semipositive is preserved by the modification of h^L, see Proposition 4.3. By Theorem 1.4, Proposition 2.6 and related cohomology isomorphism, we obtain the desired results for $j \geq q$.

Firstly, we choose now a Hermitian metric ω on X from [26, Lemma 3.5.3].

Lemma 4.1 *For any $C_1 > 0$ there exists a metric g^{TX} (with Hermitian form ω) on X such that for any $j \geq q$ and any holomorphic Hermitian vector bundle (F, h^F) on X,*

$$\left\langle (\partial\overline{\partial}\varrho)(w_l, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_l} s, s \right\rangle_h \geq C_1 |s|^2, \quad s \in \Omega_0^{0,j}(X_v \backslash \overline{X}_{u_0}, F), \tag{4.1}$$

where $\{w_l\}_{l=1}^n$ is a local orthonormal frame of $T^{(1,0)}X$ with dual frame $\{w^l\}_{l=1}^n$ of $T^{(1,0)*}X$.

Now we consider the q-convex manifold X associated with the metric ω obtained above as a Hermitian manifold (X, ω). Note for arbitrary holomorphic vector bundle F on a relatively compact domain M in X, the Hilbert space adjoint $\overline{\partial}^{F*}_H$ of $\overline{\partial}^F$ coincides with the formal adjoint $\overline{\partial}^{F*}$ of $\overline{\partial}^F$ on $B^{0,j}(M, F) = \mathrm{Dom}(\overline{\partial}^{F*}_H) \cap \Omega^{0,j}(\overline{M}, F)$, $1 \leq j \leq n$. So we simply use the notion $\overline{\partial}^{F*}$ on $B^{0,j}(M, F)$, $1 \leq j \leq n$.

Secondly, we will modify Hermitian metric h^L_χ on L and show the fundamental estimate fulfilled. Let $\chi(t) \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi'(t) \geq 0$, $\chi''(t) \geq 0$, which will be determined later. We define a Hermitian metric $h^L_\chi := h^L e^{-\chi(\varrho)}$ on L, and thus the modified curvature is

$$R^{L_\chi} = R^L + \chi'(\varrho)\partial\overline{\partial}\varrho + \chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho. \tag{4.2}$$

Proposition 4.2 *Let X be a q-convex manifold of dimension n with the exceptional set $K \subset X_c$. Then there exists a compact subset $K' \subset X_c$ and $C_0, C_3 > 0$ such that for sufficiently large k, we have*

$$\|s\|^2 \leq \frac{C_0}{k}\left(\|\overline{\partial}^E_k s\|^2 + \|\overline{\partial}^{E*}_{k,H} s\|^2 \right) + C_0 \int_{K'} |s|^2 dv_X \tag{4.3}$$

for any $s \in \mathrm{Dom}(\overline{\partial}^E_k) \cap \mathrm{Dom}(\overline{\partial}^{E*}_{k,H}) \cap L^2_{0,j}(X_c, L^k \otimes E)$ *and* $q \leq j \leq n$*, where* $\chi'(\varrho) \geq C_3$ *on* $X_v \backslash \overline{X}_u$ *and the* L^2*-norm is given by* ω*,* $h^{L^k}_\chi$ *and* h^E *on* X_c*.*

Proof See [40, Proposition 3.8] or [26, Theorem 3.5.8]. □

Thirdly, we will show that (L_χ, h^{L_χ}) preserves the certain semipositivity of (L, h^L) by choosing a appropriate χ as follows. Let $C_3 > 0$ be in Lemma 4.2. We choose $\chi \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi''(t) \geq 0$, $\chi'(t) \geq C_3$ on (u, v) and $\chi(t) = 0$ on $(-\infty, u_0)$. Therefore, $\chi'(\varrho(x)) \geq C_3 > 0$ on $X_v \backslash \overline{X}_u$ and $\chi(\varrho(x)) = \chi'(\varrho(x)) = 0$ on X_{u_0}. Note $K \subset X_{u_0}$ and $u_0 < u < c < v$. Now we have a fixed χ which leads to the following proposition.

Proposition 4.3 *X is a q-convex manifold with Hermitian metric* ω *given by Lemma 4.1. Let* $j \geq q$*. Suppose* (L, h^L) *satisfies*

$$\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]\alpha, \alpha \right\rangle_h \geq 0 \ \ \text{for all } \alpha \in \wedge^{n,j} T^*_x X, x \in X_c. \tag{4.4}$$

Then, (L_χ, h^{L_χ}) *satisfies*

$$\left\langle [\sqrt{-1}R^{(L_\chi,h^{L_\chi})}, \Lambda]\alpha, \alpha \right\rangle_h \geq 0 \ \ \text{for all } \alpha \in \wedge^{n,j} T^*_x X, x \in X_c. \tag{4.5}$$

In particular, if $(L, h^L) \geq 0$ *on* X_c*,* (L_χ, h^{L_χ}) *satisfies*

$$\left\langle [\sqrt{-1}R^{(L_\chi,h^{L_\chi})}, \Lambda]\alpha, \alpha \right\rangle_h \geq 0 \ \ \text{for all } \alpha \in \wedge^{n,j} T^*_x X, x \in X_c, j \geq q. \tag{4.6}$$

Proof $\sqrt{-1}R^{L_\chi} = \sqrt{-1}R^L + \sqrt{-1}\chi'(\varrho)\partial\overline{\partial}\varrho + \sqrt{-1}\chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho$ on X_c. From the above definition of χ, we have $\chi'(\varrho) \geq 0$ on X, $\chi'(\varrho) = 0$ on $\overline{X}_{u_0}$, and $\chi''(\varrho) \geq 0$ on X. Since $\sqrt{-1}\partial\varrho \wedge \overline{\partial}\varrho \geq 0$ on X_c, we have $\sqrt{-1}\chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho \geq 0$ on X_c. Therefore, we only need to show that, for all $\alpha \in \wedge^{n,j} T^*_x X$, $x \in X_c \backslash \overline{X}_{u_0}$,

$$\left\langle [\sqrt{-1}\partial\overline{\partial}\varrho, \Lambda]\alpha, \alpha \right\rangle_h \geq 0. \tag{4.7}$$

In fact, from Lemma 4.1, for $s \in \Omega^{n,j}_0(X_v \backslash \overline{X}_{u_0}) = \Omega^{0,j}_0(X_v \backslash \overline{X}_{u_0}, K_X)$ with $s(x) = \alpha \in \wedge^{n,j} T^*_x X$, $x \in X_c \backslash \overline{X}_{u_0}$,

$$\begin{aligned} \left\langle [\sqrt{-1}\partial\overline{\partial}\varrho, \Lambda]\alpha, \alpha \right\rangle_h &= \left\langle [\sqrt{-1}\partial\overline{\partial}\varrho, \Lambda]s, s \right\rangle_h (x) = \left\langle \sqrt{-1}\partial\overline{\partial}\varrho \wedge \Lambda s, s \right\rangle_h (x) \\ &= \left\langle (\partial\overline{\partial}\varrho)(w_l, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_l} s, s \right\rangle_h (x) \\ &\geq C_1 |s|^2_h(x) = C_1 |\alpha|^2_h \geq 0. \end{aligned} \tag{4.8}$$

Thus the proof is complete. □

Now we combine the above components and obtain:

Theorem 4.4 *Let X be a q-convex manifold of dimension n with a Hermitian metric ω given by Lemma 4.1. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let the exceptional set $K \subset X_c$. Let $j \geq q$ and (L, h^L) satisfies, with respect to ω,*

$$\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]\alpha, \alpha \right\rangle_h \geq 0 \quad \textit{for all } \alpha \in \wedge^{n,j} T^*_x X, x \in X_c. \tag{4.9}$$

Then, for all $k \geq 1$, $\dim H^j(X, L^k \otimes E) \leq Ck^{n-j}$.

Proof Proposition 4.2 entails the fundamental estimate holds in bidegree $(0, j)$ for forms with values in $L^k \otimes E$ for large k on X_c with respect to ω, h^L_χ and h^E and $j \geq q$. Thus, by Proposition 4.3 and Theorem 1.4, there exists $C > 0$ such that for sufficiently large k,

$$\dim H^{0,j}_{(2)}(X_c, L^k \otimes E) = \dim \mathscr{H}^{0,j}(X_c, L^k \otimes E) \leq Ck^{n-j} \tag{4.10}$$

holds with respect to h^E and the chosen metrics ω and h^L_χ on X_c (as in [40]). By results of Hörmander [26, Theorem 3.5.6], Andreotti–Grauert [26, Theorem 3.5.7] and the Dolbeault isomorphism [26, Theorem B.4.4], we have, for $j \geq q$,

$$\begin{aligned} H^j(X, L^k \otimes E) &\cong H^j(X_v, L^k \otimes E) \cong H^{0,j}(X_v, L^k \otimes E) \\ &\cong H^{0,j}_{(2)}(X_c, L^k \otimes E). \end{aligned} \tag{4.11}$$

Thus the conclusion holds for sufficiently large k. Note that for any holomorphic vector bundle F, $\dim H^j(X, F) < \infty$ for $j \geq q$ by the result of Andreotti–Grauert [26, Theorem B.4.8]. So the conclusion holds for all $k \geq 1$. □

Proof of Theorem 1.5 Let X_c be a sublevel set including K such that $(L, h^L) \geq 0$ on X_c. From Proposition 2.6, $(L, h^L) \geq 0$ on X_c implies for any Hermitian metric ω,

$$\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]\alpha, \alpha \right\rangle_h \geq 0 \quad \text{for all } \alpha \in \wedge^{n,j} T^*_x X, x \in X_c, j \geq 1. \tag{4.12}$$

Then the conclusion follows by Theorem 4.4. □

By adapting the duality formula [19, 20.7 Theorem] to Theorem 1.5, we have the analogue result to [41, Remark 4.4] for seminegative line bundles.

Corollary 4.5 *Let X be a q-convex manifold of dimension n and let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. Let (L, h^L) be seminegative on a neighborhood of the exceptional subset K. Then there exists $C > 0$ such that for any $0 \leq j \leq n - q$ and $k \geq 1$, the jth cohomology with compact supports*

$$\dim[H^{0,j}(X, L^k \otimes E)]_0 \leq Ck^j. \tag{4.13}$$

Proof For any $q \leq s \leq n$, $\dim[H^{0,n-s}(X, L^k \otimes E)]_0 = \dim H^{0,s}(X, L^{k*} \otimes E^* \otimes K_X) \leq Ck^{n-s}$ by Theorem 1.5 and (2.8), see [1] and [19, 20.7 Theorem]. □

Remark 4.6 (Vanishing theorems on q-convex manifolds) Let (E, h^E) be a holomorphic vector bundle on X. If $(L, h^L) > 0$ on X_c with $K \subset X_c$ instead of the hypothesis $(L, h^L) \geq 0$ on X_c in Theorem 1.5, then for $j \geq q$ and sufficiently large k, $\dim H^j(X, L^k \otimes E) = 0$, see [26, Theorem 3.5.9]. And it can be generalized to Nakano q-positive as follows.

Theorem 4.7 *Let (X, ω) be a q-convex manifold of dimension n with the Hermitian metric ω given by Lemma 4.1 and $1 \leq q \leq n$. Let E, L be holomorphic vector bundle with* $\operatorname{rank}(L) = 1$. *Let $K \subset X$ be the exceptional set. If (L, h^L) is Nakano p-positive with respect to ω on X_c with $K \subset X_c$, then for $j \geq \max\{p, q\}$ and k sufficiently large,*

$$H^j(X, L^k \otimes E) = 0. \tag{4.14}$$

Proof We can shrink X_c with $K \subset X_c$ such that (L, h^L) is p-positive with respect to ω on the closure $\overline{X}_c$. By (2.11), there exists $C_L > 0$ such that

$$\langle R^L(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i} s, s\rangle_h \geq C_L |s|_h^2 \tag{4.15}$$

for any $s \in B^{0,j}(X_c, F)$ with arbitrary holomorphic line bundle F and $j \geq p$. Thus there exists $C_2 > 0$, for each $s \in B^{0,j}(X_c, L^k \otimes E)$ with $j \geq \max\{p, q\}$ and k sufficiently large,

$$\|s\|^2 \leq \frac{C_2}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_k^{E*} s\|^2) \tag{4.16}$$

holds with respect to h^L and ω as in [26, Lemma 3.5.4], and thus it holds for $s \in \mathscr{H}^{0,j}(X_c, L^k \otimes E)$ with $j \geq \max\{p, q\}$. Then, for k sufficiently large, $H^j(X, L^k \otimes E) \cong \mathscr{H}^{0,j}(X_c, L^k \otimes E) = 0$ with $j \geq \max\{p, q\}$. □

Remark 4.8 (Complex spaces) Let X be a j-convex Kähler manifold with $\dim X = n$ and $1 \leq j \leq n$. Let (L, h^L) be a holomorphic Hermitian line bundle and $(L, h^L) \geq 0$ on X. Let S be a complex space and $f : X \to S$ a proper surjective holomorphic map. Then, by Theorem 1.5 and [30], $\dim H^p(S, R^q f_*(K_X \otimes L^k)) = O(k^{n-p-q})$ for all (p, q) with $p + q \geq j$, where $R^q f_*(\cdot)$ is the qth higher direct image sheaf.

4.2 Pseudoconvex, Weakly 1-Complete, and Complete Manifolds

Analogue to the case of q-convex manifolds, we can generalize other results in [40] as follows. Holomorphic Morse inequalities for weakly 1-complete manifolds and pseudoconvex domain were obtained in [27] and [26, Theorem 3.5.10, 3.5.12].

Theorem 4.9 *Let $M \Subset X$ be a smooth (weakly) pseudoconvex domain in a complex manifold X of dimension n. Let ω be a Hermitian metric on X. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let $1 \leq q \leq n$. Assume (L, h^L) is Nakano q-semipositive with respect to ω on M, and (L, h^L) is Nakano q-positive* *with respect to ω in a neighborhood of bM. Then there exists $C > 0$ such that for sufficiently large k, we have*

$$\dim H^{0,j}_{(2)}(M, L^k \otimes E) \leq Ck^{n-j} \quad \text{for all } q \leq j \leq n. \tag{4.17}$$

Proof We follow [40, Theorem 1.5, (3.29)] and [26, Theorem 3.5.10]. Let $\rho \in \mathscr{C}^\infty(X, \mathbb{R})$ be a defining function of M such that $M = \{x \in X : \rho(x) < 0\}$ with $|d\rho| = 1$ on the boundary bM. Let $x \in bM$. For $s \in \Omega^{0,\bullet}(\overline{M}, L^k \otimes E)$, the Levi form defined by $\mathscr{L}_\rho(s, s)(x) := \sum_{j,k=2}^n (\partial\overline{\partial}\rho)(w_k, \overline{w}_j)\langle \overline{w}^j \wedge i_{\overline{w}_k} s(x), s(x)\rangle_h$. Since M is pseudoconvex, it follows that, for $s \in B^{0,q}(M, L^k \otimes E)$,

$$\int_{bM} \mathscr{L}_\rho(s, s)\mathrm{d}v_{bM} \geq 0. \tag{4.18}$$

Let $X_c := \{x \in X : \rho(x) < c\}$ for $c \in \mathbb{R}$. We fix $u < 0 < v$ such that L is Nakano q-positive with respect to ω on a open neighborhood of $X_v \backslash \overline{X}_u$, then there exists $C_L > 0$ such that for any holomorphic Hermitian vector bundle (F, h^F) on X,

$$\langle R^L(w_l, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_l} s, s\rangle_h \geq C_L |s|^2, \quad s \in \Omega^{0,q}_0(X_v \backslash \overline{X}_u, F). \tag{4.19}$$

By the Bochner–Kodaira–Nakano formula with boundary term [26, Corollary 1.4.22], there exist $C_4 \geq 0$ and $C_5 \geq 0$ such that for any $s \in B^{0,q}(M, L^k \otimes E)$ with $\mathrm{supp}(s) \in X_v \backslash \overline{X}_u$,

$$\begin{aligned}\frac{3}{2}(\|\partial^E_k s\|^2 + \|\partial^{E*}_k s\|^2) &\geq \langle R^{L^k \otimes E \otimes K^*_X}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s\rangle \\ &\quad + \int_{bM} \mathscr{L}_\rho(s, s)\mathrm{d}v_{bM} - C_4\|s\|^2 \\ &\geq \int_M (kC_L - C_5 - C_4)|s|^2 \mathrm{d}v_X. \end{aligned} \tag{4.20}$$

For any $k \geq k_0 := [2\frac{C_4+C_5}{C_L}] + 1$, we have $C_L - \frac{C_4+C_5}{k} \geq \frac{1}{2}C_L$. Let $C_2 := \frac{3}{C_L}$. For any $s \in B^{0,q}(M, L^k \otimes E)$ with $\mathrm{supp}(s) \subset X_v \backslash \overline{X}_u$ and $k \geq k_0 > 0$, we have

$$\|s\|^2 \leq \frac{C_2}{k}(\|\overline{\partial}^E_k s\|^2 + \|\overline{\partial}^{E*}_k s\|^2) \tag{4.21}$$

where the L^2-norm $\|\cdot\|$ is given by ω, h^{L^k} and h^E on M.

Note the fact that $B^{0,q}(M, L^k \otimes E)$ is dense in $\mathrm{Dom}(\overline{\partial}^E_k) \cap \mathrm{Dom}(\overline{\partial}^{E*}_{k,H}) \cap L^2_{0,q}(M, L^k \otimes E)$ with respect to the graph norm of $\overline{\partial}^E_k + \overline{\partial}^{E*}_{k,H}$. Following the same argument in Lemma 4.2 (without the modification of h^L by χ), we conclude that there exists a compact subset $K' \subset M$ and $C_0 > 0$ such that for sufficiently large k, we have

$$\|s\|^2 \leq \frac{C_0}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_{k,H}^{E*} s\|^2) + C_0 \int_{K'} |s|^2 dv_X \tag{4.22}$$

for any $s \in \mathrm{Dom}(\overline{\partial}_k^E) \cap \mathrm{Dom}(\overline{\partial}_{k,H}^{E*}) \cap L^2_{0,q}(M, L^k \otimes E)$, where the L^2-norm is given by ω, h^{L^k} and h^E on M. That is, the fundamental estimate holds in bidegree $(0, q)$ for forms with values in $L^k \otimes E$ for large k. Finally, we apply Theorem 1.4 and Proposition 2.8. □

The polynomial growth of dimension of cohomology of Griffiths q-positive line bundles on weakly 1-complete manifolds via holomorphic Morse inequalities, we refer to [27]. For the Nakano q-positive cases, by applying Theorem 4.9 as in [40, Proof of Theorem 1.6], we obtain:

Corollary 4.10 *Let X be a weakly 1-complete manifold of dimension n with a smooth plurisubharmonic exhaustion function ρ and ω be a Hermitian metric on X. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let $1 \leq q \leq n$ and (L, h^L) is Nakano q-semipositive with respect to ω on X.*

(1) *Assume (L, h^L) is Nakano q-positive with respect to ω on $X \backslash K$ for a compact subset K. Then, for any sublevel set $X_c := \{\rho < c\}$ with smooth boundary and $K \subset X_c$, there exists $C > 0$ such that for k sufficiently large,*

$$\dim H^{0,j}_{(2)}(X_c, L^k \otimes E) \leq Ck^{n-j} \quad \text{for all } q \leq j \leq n. \tag{4.23}$$

(2) *Assume (L, h^L) is positive on $X \backslash K$ for a compact subset K. Then there exists $C > 0$ such that for k sufficiently large,*

$$\dim H^j(X, L^k \otimes E) \leq Ck^{n-j} \quad \text{for all } q \leq j \leq n. \tag{4.24}$$

Proof (1) is from X_c is a smooth pseudoconvex domain and Theorem 4.9; (2) follows from (1) and $H^j(X, L^k \otimes E) \cong H^{0,j}_{(2)}(X_c, L^k \otimes E)$ for all $j \geq q$ and sufficiently large k. □

Similarly, we also can refine [40, Theorem 1.2] on complete manifolds.

Theorem 4.11 *Let (X, ω) be a complete Hermitian manifold of dimension n. Let (L, h^L) be a holomorphic Hermitian line bundle on X. Assume there exists a compact subset $K \subset X$ such that $\sqrt{-1}R^{(L,h^L)} = \omega$ on $X \backslash K$. Let $1 \leq q \leq n$ and (L, h^L) is Nakano q-semipositive with respect to ω on K. Then there exists $C > 0$ such that for sufficiently large k, we have*

$$\dim H^{0,j}_{(2)}(X, L^k \otimes K_X) \leq Ck^{n-j} \quad \text{for all } q \leq j \leq n. \tag{4.25}$$

Proof Since (X, ω) is complete, $\overline{\partial}_{k,H}^{E*} = \overline{\partial}_k^{E*}$ for arbitrary holomorphic Hermitian vector bundle (E, h^E). In a local orthonormal frame $\{\omega_j\}_{j=1}^n$ of $T^{(1,0)}X$ with dual frame $\{w^j\}_{j=1}^n$ of $T^{(1,0)*}X$, $\omega = \sqrt{-1}\sum_{j=1}^n \omega^j \wedge \overline{\omega}^j$ and $\Lambda = -\sqrt{-1} i_{\overline{w}_j} i_{w_j}$. Thus

$\sqrt{-1}R^{(L,h^L)} = \sqrt{-1}\sum_{j=1}^n \omega^j \wedge \overline{\omega}^j$ outside K. Let $\{e_k\}$ be a local frame of L^k. For $s \in \Omega_0^{n,q}(X\backslash K, L^k)$, we can write $s = \sum_{|J|=q} s_J \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^J \otimes e_k$ locally, thus

$$[\sqrt{-1}R^L, \Lambda]s = \sum_{|J|=q} (q s_J \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^J) \otimes e_k = qs. \tag{4.26}$$

Since $(X\backslash K, \sqrt{-1}R^{(L,h^L)})$ is Kähler, we apply Nakano's inequality [26, (1.4.52)],

$$\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2 \geq k\langle[\sqrt{-1}R^L, \Lambda]s, s\rangle \geq qk\|s\|^2 \geq k\|s\|^2. \tag{4.27}$$

Therefore, we have $\|s\|^2 \leq \frac{1}{k}(\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2)$ for $s \in \Omega_0^{n,q}(X\backslash K, L^k)$ with $1 \leq q \leq n$ with respect to h^L and ω.

Next we follow the analogue argument in [40, Proposition 3.8] to obtain the fundamental estimates as follows. Let V and U be open subsets of X such that $K \subset V \Subset U \Subset X$. We choose a function $\xi \in \mathscr{C}_0^\infty(U, \mathbb{R})$ such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $\overline{V}$. We set $\phi := 1 - \xi$, thus $\phi \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfying $0 \leq \phi \leq 1$ and $\phi \equiv 0$ on $\overline{V}$.

Now let $s \in \Omega_0^{n,q}(X, L^k)$, thus $\phi s \in \Omega_0^{n,q}(X\backslash K, L^k)$. We set $K' := \overline{U}$, then

$$\|\phi s\|^2 \geq \|s\|^2 - \int_{K'} |s|^2 dv_X, \tag{4.28}$$

and similarly there exists a constant $C_1 > 0$ such that

$$\frac{1}{k}(\|\overline{\partial}_k(\phi s)\|^2 + \|\overline{\partial}_k^*(\phi s)\|^2) \leq \frac{5}{k}(\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2) + \frac{12C_1}{k}\|s\|^2. \tag{4.29}$$

By combining the above three inequalities, there exists $C_0 > 0$ such that for any $s \in \Omega_0^{n,q}(X, L^k) = \Omega_0^{0,q}(X, L^k \otimes K_X)$ and k large enough

$$\|s\|^2 \leq \frac{C_0}{k}(\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2) + C_0 \int_{K'} |s|^2 dv_X. \tag{4.30}$$

Finally, since $\Omega_0^{0,\bullet}(X, L^k \otimes K_X)$ is dense in $\mathrm{Dom}(\overline{\partial}_k^{K_X}) \cap \mathrm{Dom}(\overline{\partial}_k^{K_X *})$ in the graph norm, the fundamental estimate holds in bidegree $(0, q)$ for forms with values in $L^k \otimes K_X$ for k large. So the conclusion follows from Theorem 1.4 and Proposition 2.8. □

4.3 Vanishing Theorems and the Estimate $O(k^{n-q})$

In this section, we restrict to Kähler manifolds X and $E = K_X$. Firstly, inspired by [17,31], we see the injectivity for Nakano q-semipositive line bundles.

Lemma 4.12 *Let (X, ω) be a compact Kähler manifold of dimension n and let (L, h^L) be holomorphic Hermitian line bundle on X. Let $1 \leq q \leq n$ and (L, h^L) be Nakano q-semipositive with respect to ω on X. Let $s \in H^0(X, L^k)\backslash\{0\}$ for some $k > 0$. Then, for every $j \geq q$ and $m \geq 1$, the multiplication map $\cdot \otimes s$:*

$$H^j(X, K_X \otimes L^m) \to H^j(X, K_X \otimes L^{m+k}) \tag{4.31}$$

is injective. In particular, if (L, h^L) is semipositive, it holds for all $j \geq 1$.

Proof We follow [17, 1.5 Enoki's proof]. By Proposition 2.8 and Hodge theorem, we only need to show the multiplication map $\cdot \otimes s$ between the harmonic spaces

$$\mathscr{H}^{n,q}(X, L^m) \to \mathscr{H}^{n,q}(X, L^{m+k}) \tag{4.32}$$

is injective for $m \geq 1$. Let $u \in \mathscr{H}^{n,q}(X, L^m)$. Since $s \in H^0(X, L^k)$, $\overline{\partial}^{L^{m+k}}(s \otimes u) = 0$. From the q-semipositive and Nakano's inequality [26, (1.4.51)], $\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]u, u \right\rangle_h = 0$ on X. From [26, (1.4.44),(1.4.38c)], $(\nabla^{L^m})^{1,0*}(s \otimes u) = s \otimes ((\nabla^{L^m})^{1,0*}u) = 0$. Also we have $\left\langle [\sqrt{-1}R^{L^{m+k}}, \Lambda](s \otimes u), (s \otimes u) \right\rangle_h = 0$. Thus

$$\|\overline{\partial}^{L^{m+k}*}(s \otimes u)\|^2 = \|(\nabla^{L^{m+k}})^{1,0*}(s \otimes u)\|^2 = 0.$$

We obtain $s \otimes u \in \mathscr{H}^{n,q}(X, L^{m+k})$. Suppose $s \otimes u = 0$ on X. Since $s \neq 0$ and [16, Ch.VII.3. (2.4) Lemma], $u = 0$ on X. □

Let $\kappa(L)$ be the Kodaira dimension of L on a compact complex manifold X given by

$$\kappa(L) := -\infty, \text{ when } H^0(X, L^k) = 0 \text{ for all } k > 0; \text{ otherwise}, \tag{4.33}$$

$$\kappa(L) := \max\{m \in \mathbb{N} : \limsup_{k\to\infty} \frac{\dim H^0(X, L^k)}{k^m} > 0\} \in [0, \dim X]. \tag{4.34}$$

By the above lemma and Corollary 1.3 with the trivial Γ, we obtain:

Theorem 4.13 *Let (X, ω) be a compact Kähler manifold of dimension n and let (L, h^L) be holomorphic Hermitian line bundle on X. Let $1 \leq q \leq n$ and (L, h^L) be Nakano q-semipositive with respect to ω on X. Then, for all $j > \max\{n - \kappa(L),\ q - 1\}$ and $m > 0$,*

$$H^j(X, K_X \otimes L^m) = 0. \tag{4.35}$$

Proof We follow [31, Theorem 4.5]. Suppose there exist $m > 0$ and $j > n - \kappa(L)$ with $j \geq q$ such that $H^j(X, K_X \otimes L^m) \neq 0$. Let $u \in H^j(X, K_X \otimes L^m)\backslash\{0\}$ and let $\{s_j\}_{i=1}^N \subset H^0(X, L^k)$ be linearly independent. By the injectivity Lemma 4.12,

$\{s_i \otimes u\}_{i=1}^N \subset H^j(X, K_X \otimes L^{m+k})$ are linearly independent. By Corollary 1.3 for compact Kähler manifolds, we see

$$\frac{\dim H^0(X, L^k)}{k^{\kappa(L)}} \le \frac{\dim H^j(X, K_X \otimes L^{k+m})}{k^{n-j+1}} \le \frac{C(k+m)^{n-j}}{k^{n-j+1}} \le \frac{C}{k}. \quad (4.36)$$

By applying $\limsup_{k\to+\infty}$, there is a contradiction. □

Corollary 4.14 *Let (X, ω) be a compact Kähler manifold and let (L, h^L) be a holomorphic Hermitian line bundle. If $Tr_\omega R^{(L^*, h^{L^*})} \ge 0$ on X and $\kappa(L^*) > 0$, then $\kappa(L) = -\infty$. In particular, if the scalar curvature $r_\omega = 0$ on X and $\kappa(K_X^*) > 0$, then $\kappa(K_X) = -\infty$.*

Proof $H^0(X, L^m) \cong H^n(X, K_X \otimes L^{m*}) = 0$ and $r_\omega = 2\sum_j Ric(\omega_j, \overline{\omega}_j) = 2\mathrm{Tr}_\omega R^{K_X^*}$. □

Secondly, as applications of Bochner–Kodaira–Nakano formulas, certain Kodaira type vanishing theorems of Nakano q-semipositive line bundles hold as follows.

Proposition 4.15 *Let (X, ω) be a complete Kähler manifold of dimension n and $1 \le q \le n$. Let (L, h^L) be a Nakano q-semipositive line bundle with respect to ω on X. Assume there exists $C_0 > 0$ and a compact subset $K \subsetneq X$ such that $\sqrt{-1}R^{(L,h^L)} \ge C_0\omega$ on $X\backslash K$. Then,*

$$H^{0,j}_{(2)}(X, K_X \otimes L) = 0 \ \ \text{for all } j \ge q. \quad (4.37)$$

Proof Since (X, ω) is complete, $\overline{\partial}^{L*}_H = \overline{\partial}^{L*}$. For $s \in \Omega_0^{n,j}(X, L)$ for $j \ge q$, from Bochner–Kodaira–Nakano formula, we have

$$\|\overline{\partial}^L s\|^2 + \|\overline{\partial}^{L*} s\|^2 \ge \langle[\sqrt{-1}R^L, \Lambda]s, s\rangle \ge C_0\|s\|^2_{X\backslash K} = C_0\|s\|^2 - C_0\|s\|^2_K. \quad (4.38)$$

Since $\Omega_0^{n,j}(X, L)$ is dense in $\mathrm{Dom}(\overline{\partial}^L) \cap \mathrm{Dom}(\overline{\partial}^{L*})$, (4.38) holds for $s \in \mathrm{Dom}(\overline{\partial}^L) \cap \mathrm{Dom}(\overline{\partial}^{L*})$. Since $K \subsetneq X$, $s|_{X\backslash K} = 0$ for $s \in \mathscr{H}^{n,j}(X, L)$, and then $\mathscr{H}^{n,j}(X, L) = 0$. From (4.38), the fundamental estimate holds for $(0, j)$-form with values in $K_X \otimes L$, and thus $H^{0,j}_{(2)}(X, K_X \otimes L) \cong \mathscr{H}^{0,j}(X, K_X \otimes L) = 0$. □

Proposition 4.16 *Let (X, ω) be a weakly 1-complete Kähler manifold of dimension n and $1 \le q \le n$. Let (L, h^L) be a Nakano q-semipositive line bundle with respect to ω on X. Assume there exists a compact subset $K \subsetneq X$ and $\sqrt{-1}R^{(L,h^L)} = \omega$ on $X\backslash K$. Then,*

$$H^j(X, K_X \otimes L) = 0 \ \ \text{for all } j \ge q. \quad (4.39)$$

Proof Let $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$ be an exhaustion function of X such that $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ on X and $X_c := \{\varphi < c\} \Subset X$ for all $c \in \mathbb{R}$. We choose a regular value $c \in \mathbb{R}$ of φ such that $K \subsetneqq X_c$ by Sard's theorem. Thus X_c is a smooth pseudoconvex domain and $\sqrt{-1}R^L = \omega > 0$ on a neighborhood of bX_c, in particular on $X_c \backslash K$. It follows that for $s \in \Omega^{n,j}(X_c, L)$, $j \geq q$,

$$\langle[\sqrt{-1}R^L, \Lambda]s, s\rangle = \langle[\sqrt{-1}R^L, \Lambda]s, s\rangle_K + \langle[\omega, \Lambda]s, s\rangle_{X_c\backslash K} \geq \|s\|^2_{X_c\backslash K}. \tag{4.40}$$

If $s \in B^{n,j}(X_c, L)$, $\|\overline{\partial}^L s\|^2 + \|\overline{\partial}^{L*} s\|^2 \geq \langle[\sqrt{-1}R^L, \Lambda]s, s\rangle + \int_{bM_c} \mathscr{L}_\rho(s, s)dv_{X_c}$ by [26]. Since X_c is pseudoconvex, $\int_{bM_c} \mathscr{L}_\rho(s, s)dv_{X_c} \geq 0$. Since $\overline{\partial}^{L*}_H = \overline{\partial}^{L*}$ on $B^{0,j}(X_c, K_X \otimes L)$,

$$\|\overline{\partial}^L s\|^2 + \|\overline{\partial}^{L*}_H s\|^2 \geq \|s\|^2 - \|s\|^2_K \tag{4.41}$$

holds for $s \in B^{0,j}(X_c, K_X \otimes L)$, thus for $s \in \mathrm{Dom}(\overline{\partial}^L) \cap \mathrm{Dom}(\overline{\partial}^{L*}) \cap L^2_{n,j}(X_c, L)$. In particular, if $s \in \mathscr{H}^{n,q}(X_c, L)$, $s|_{X_c\backslash K} = 0$ and so $\mathscr{H}^{n,j}(X, L) = 0$ for $j \geq q$. Since the fundamental estimate holds for $(0, j)$-form with values in $K_X \otimes L$ on X_c, $H^{0,j}_{(2)}(X_c, K_X \otimes L) = \mathscr{H}^{0,j}(X_c, K_X \otimes L) = 0$ for $j \geq q$. Moreover, by [34, Theorem 1.2] and $\omega = \sqrt{-1}R^L$ on $X\backslash X_c$, it follows $H^j(X, K_X \otimes L) \cong H^{n,j}(X, L) \cong H^{n,j}_{(2)}(X_c, L) = 0$. □

For a pseudoconvex domain M, we follow the above argument for X_c and obtain:

Proposition 4.17 *Let M be a smooth pseudoconvex domain in a Kähler manifold (X, ω) of dimension n and $1 \leq q \leq n$. Let (L, h^L) be a Nakano q-semipositive line bundle with respect to ω on M. Assume (L, h^L) is Nakano q-positive with respect to ω on a neighborhood of bM. Then for every $j \geq q$,*

$$H^{0,j}_{(2)}(M, K_X \otimes L) = 0. \tag{4.42}$$

Proof Let (L, h^L) be Nakano q-positive with respect to ω on a neighborhood U of bM such that $\overline{U}$ is compact. Let $V \Subset U$ and V be a smaller neighborhood of bM. By the Bochner–Kodaira–Nakano formula with boundary term [26, Corollary 1.4.22], for any $s \in B^{0,q}(M, L \otimes K_X)$, $k \geq 0$,

$$\begin{aligned}\frac{3}{2}\|\overline{\partial}^{K_X} s\|^2 + \|\overline{\partial}^{K_X,*} s\|^2 &\geq \langle R^L(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s\rangle \\ &\geq \langle R^L(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s\rangle_{M\cap V} \\ &\geq C\|s\|^2_{M\cap V} = C(\|s\|^2 - \|s\|^2_{M\backslash V}),\end{aligned} \tag{4.43}$$

where $C > 0$, given by the Nakano q-positive line bundle L with respect to ω on U and the compactness of $\overline{M \cap V} \subset U$ is independent of the choice of s. Thus, we follow the argument for $X_c = M$ in Proposition 4.16 and obtain $\mathscr{H}^{0,q}(M, L \otimes K_X) = 0$. Since the fundamental estimate holds, $H^{0,q}_{(2)}(M, L \otimes K_X) = 0$. And the assertion holds for all $j \geq q$ by Proposition 2.8 and Remark 2.9. □

4.4 Remarks on ω-Trace and Kodaira Type Vanishing Theorems

Let (E, h^E) be a holomorphic Hermitian vector bundle on a Hermitian manifold (X, ω). The ω-trace of $R^{(E,h^E)}$ can be represented by

$$\tau(E, h^E, \omega) := \mathrm{Tr}_\omega R^{(E,h^E)} := \sum_j R^{(E,h^E)}(\omega_j, \overline{\omega}_j)$$
$$= \sum_{i,k} R^{(E,h^E)}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_k}\right) \langle dz_i, d\overline{z}_k \rangle_{g^{T^*X}}.$$

Comparing to the usual trace $\mathrm{Tr}[R^{(E,h^E)}] \in \Omega^{1,1}(X)$ depending only on h^E, $\tau(E, h^E, \omega) := \mathrm{Tr}_\omega R^{(E,h^E)} \in \mathrm{End}(E)$ depends on h^E and ω. By Bochner–Kodaira–Nakano formulas, Serre duality and Le Potier's Theorem [21, 3.5.1, (3.5.8)], it follows that:

Proposition 4.18 *Let (E, h^E) be a holomorphic Hermitian vector bundle over a compact Kähler manifold (X, ω). (1) If $\tau(O_{E^*}(1)) \leq 0$ and < 0 at one point on $P(E^*)$, then $H^0(X, S^m(E)) = 0$ for all $m \geq 1$. (2) If $\tau(E) \leq 0$ and < 0 at one point on X, then $H^0(X, E^m) = 0$ for all $m \geq 1$.*

Proof The case $m = 1$ of (2) follows from Bochner–Kodaira–Nakano formulas (or using the Lichnerowicz formula [26, (1.4.31)])). From the fact $\tau(E^{\otimes m}) = \tau(E)^{\otimes m}$, refer to [21, III.(1.12)] or [43, (3.7)], we have (2) holds for all $m \geq 1$. And (1) is from Le Potier's Theorem [21, 3.5.1, (3.5.8)] and (2) for $E = O_{E^*}(1)$. □

Recall that a compact complex manifold X is said to be rationally connected if any two points of X can be joined by a chain of rational curves, see [10]. We say a real $(1, 1)$-form $\alpha \in \Omega^{1,1}(X)$ is quasi-positive on X, if $\alpha \geq 0$ on X and > 0 at one point.

Proposition 4.19 *Let X be a compact Kähler manifold with a quasi-positive $(1, 1)$-form representing the first Chern class $c_1(X)$. Then X is projective and rationally connected.*

Proof Calabi–Yau theorem [44] provides a Kähler metric ω on X such that the Ricci form $\sqrt{-1}R^{K_X^*} = \mathrm{Ric}_\omega$ is quasi-positive, so K_X^* is big and X is projective. Since (X, ω) is Kähler, $\mathrm{Ric}_\omega = \sqrt{-1}\mathrm{Tr}[R^{T^{1,0}X}]$ and it coincides with $\tau(T^{1,0}X, h_\omega, \omega) = \mathrm{Tr}_\omega R^{T^{1,0}X}$ as Hermitian matrices. Thus, $\tau(T^{1,0}X, h_\omega, \omega) \geq 0$ and > 0 at one point. By $\tau(T^{1,0}X) = -\tau(T^{1,0*}X)$ and Proposition 4.18 (2), we have $H^0(X, (T^{1,0*}X)^m) = 0$ for all $m \geq 1$, and the rationally connected follows from [10, 5.1. Corollary]. □

Equivalently, it follows from [21, Ch.III. (1.34)] and [10, 5.1 Corollary] that: A compact Käher manifold with quasi-positive Ricci curvature is projective and rationally connected. It strengthens [42, Theorem B (A)] which asserted such a manifold is simply connected and has no nonzero holomorphic q-forms for $q > 0$, since any rationally connected projective manifold has these properties, see [12, Corollary 4.18]. And it also leads to the fact [8,24] that every smooth Fano manifold X is rationally connected (See [12,23,45]).

Proposition 4.20 *Let X be a compact Kähler manifold of non-negative bisectional curvature. The following conditions are equivalent: (A) X is simply connected; (B) The first Betti number is zero; (C) X has quasi-positive Ricci curvature; (D) X is projective and rationally connected.*

Proof From [20, Corollary 1] and Proposition 4.19, we see (A), (B) and (C) are equivalent and (C) implies (D). And [12, Corollary 4.18] entails (D) implies (A). □

5 Dirac Operator on Nakano q-positive Line Bundles

Inspired by [27] and [25, Theorem 1.1, 2.5], we consider q-positive line bundles and the Dirac operators. We give some estimates of modified Dirac operators on high tensor powers of q-positive line bundles based on [26, Sect. 1.5].

5.1 Nakano q-positive Line Bundles with Respect to ω

In this section, we work on the following setting. Let (X, J) be a smooth manifold with almost complex structure J and $\dim_{\mathbb{R}} X = 2n$. Let g^{TX} be a Riemannian metric compatible with J and $\omega := g^{TX}(J\cdot, \cdot)$ be the real $(1, 1)$-forms on X induced by g^{TX} and J. Let (E, h^E) and (L, h^L) be Hermitian vector bundles on X with $\operatorname{rank}(L) = 1$. Let ∇^E and ∇^L be Hermitian connections on (E, h^E) and (L, h^L) and let $R^E := (\nabla^E)^2$ and $R^L := (\nabla^L)^2$ be the curvatures. Assume that $\frac{\sqrt{-1}}{2\pi} R^L$ is compatible with J. Thus, the Chern–Weil form $c_1(L, h^L) := \frac{\sqrt{-1}}{2\pi} R^L$ representing the first Chern class $c_1(L)$ of L is a real $(1, 1)$-forms on X. (For example, X is a compact complex manifold and (E, h^E, ∇^E), (L, h^L, ∇^L) are holomorphic Hermitian).

The almost complex structure J induced a splitting of the complexification of the tangent bundle, i.e., $TX \otimes \mathbb{C} = T^{1,0}X \bigoplus T^{0,1}X$, and the cotangent bundle. Let $0 \le p, q \le n$, and let $\bigwedge^{p,q} T_x^* X$ be the fiber of the bundle $\bigwedge^{p,q} T^*X := \wedge^p T^{1,0*}X \otimes \wedge^q T^{0,1*}X$ for $x \in X$. For $k \in \mathbb{N}$, we denote by $\Omega^{p,q}(X, L^k \otimes E)$ the space of (p, q)-forms with values in $L^k \otimes E$ on X and set $\Omega^{0,\ge q}(X, L^k \otimes E) := \bigoplus_{j\ge q}^n \Omega^{0,j}(X, L^k \otimes E)$. As defined in Sect. 2, we denote by $\langle\cdot,\cdot\rangle_h$ and $|\cdot|_h$ the pointwise Hermitian inner product and Hermitian norm, and by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the L^2 inner product and L^2-norm. Let Λ be the dual of the operator $\mathcal{L} := \omega \wedge \cdot$ on $\Omega^{p,q}(X)$ with respect to the Hermitian inner product $\langle\cdot,\cdot\rangle_h$ on X. In a local orthonormal frame $\{w_j\}_{j=1}^n$ of $T^{1,0}X$ with respect to g^{TX} and its dual $\{w^j\}$ of $T^{1,0*}X$, $R^{(L,h^L)} = R^{(L,h^L)}(w_i, \overline{w}_j) w^i \wedge \overline{w}^j$, $\mathcal{L} = \sqrt{-1}\sum_{j=1}^n w^j \wedge \overline{w}^j$ and $\Lambda = -\sqrt{-1}\sum_{j=1}^n i_{\overline{w}^j} i_{w^j}$. For any $s \in \Omega^{p,q}(X, L^k \otimes E)$, $\left\langle [\sqrt{-1}R^{(L,h^L)}, \Lambda]s, s\right\rangle_h \in \mathscr{C}^\infty(X, \mathbb{R})$. We set

$$w_d = -\sum_{i,j} R^L(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i} \in \operatorname{End}(\Lambda(T^{*0,1}X)). \tag{5.1}$$

For a real 3-form A on X, one can define modified Dirac operator $D_k^{c,A}$ acting on $\Omega^{0,\bullet}(X, L^k \otimes E) = \bigoplus_{j\geq 0} \Omega^{0,j}(X, L^k \otimes E)$, see [26, Definition 1.3.6, (1.5.27)]. Ma-Marinescu obtained the precise lower bound of $D_k^{c,A}$ as follows. The proof is based on a application of Lichnerowicz formula, see [26, (1.5.34), (1.5.30)] and [25].

Theorem 5.1 *[26] There exists $C > 0$ such that for any $k \in \mathbb{N}$, $s \in \Omega^{0,\bullet}(X, L^k \otimes E)$,*

$$\|D_k^{c,A} s\|^2 \geq 2k\langle -w_d s, s\rangle - C\|s\|^2. \tag{5.2}$$

As a consequence, they obtained the spectral gap property [26, Theorem 1.5.7, 1.5.8], which play the essential role in their approach to the Bergman kernel. In this section, we generalize [26, Theorem 1.5.7] to the case of Nakano q-positive line bundles.

Definition 5.2 For each $1 \leq q \leq n$, the number $\mu_q \in \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\mu_q(x) := \inf_{u \in \wedge^{n,q} T_x^* X} \frac{\langle [\sqrt{-1} R^L, \Lambda] u, u\rangle_h}{|u|_h^2}, \quad \mu_q := \inf_{x \in X} \mu_q(x). \tag{5.3}$$

In terms of local orthonormal frame $\{\omega_j\}$ of $T^{1,0}X$, it follows that

$$\mu_q = \inf_{\alpha \in \wedge^{0,q} T_x^* X, x \in X} \frac{\left\langle R^L(\omega_i, \overline{\omega}_j)\overline{\omega}^j \wedge i_{\overline{\omega}_i}\alpha, \alpha\right\rangle_h}{|\alpha|_h^2} = \inf_{\alpha \in \wedge^{0,q} T_x^* X, x \in X} \frac{\langle -w_d \alpha, \alpha\rangle_h}{|\alpha|_h^2}. \tag{5.4}$$

In other words, if $\lambda_1(x) \leq \lambda_1(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of R_x^L with respect to ω at $x \in X$, then $\mu_q(x) = \sum_{j=1}^q \lambda_j(x)$ and $\mu_q = \inf_{x\in X} \mu_q(x)$.

Theorem 5.3 *Let X be compact. Let $1 \leq q \leq n$ and (L, h^L) is Nakano q-positive line bundle with respect to ω on X. Then there exists $C_L > 0$ such that for any $k \in \mathbb{N}$ and any $s \in \Omega^{0,\geq q}(X, L^k \otimes E)$,*

$$\|D_k^{c,A} s\|^2 \geq (2\mu_q k - C_L)\|s\|^2, \tag{5.5}$$

where the constant $\mu_q > 0$ defined in (5.3). Especially, for k large enough,

$$\mathrm{Ker}\left(D_k^{c,A}|_{\Omega^{0,\geq q}(X, L^k\otimes E)}\right) = 0. \tag{5.6}$$

Proof As in (2.13), we choose a local orthonormal frame around $x \in X$ such that $R_x^L(\omega_i, \omega_j) = \delta_{ij} c_i(x)$ for $1 \leq i, j \leq n$. Then

$$w_d = -\sum_{j\geq 1} c_j(x) \overline{w}^j \wedge i_{\overline{w}_j} \in \mathrm{End}(\Lambda(T_x^{*0,1}X)). \tag{5.7}$$

Let $C_J(x) := \sum_{j\in J} c_j(x)$ for each ordered $J = (j_1, \dots, j_q)$ with $|J| = q$. For $\alpha \in \bigwedge_x^{0,q} X\backslash\{0\}$, we represent it by $\alpha = \sum_J \alpha_J \overline{w}^J$ with $|J| = q$. From (5.4) and (5.7), we have

$$\mu_q = \inf_{x\in X} \inf_{\alpha_J\in\mathbb{C}} \frac{\sum_J C_J(x)|\alpha_J|^2}{\sum_J |\alpha_J|^2} = \inf_{x\in X} \inf_{|J|=q} C_J(x). \tag{5.8}$$

For $s(x) \in \bigwedge_x^{0,q} X \otimes L_x^k \otimes E_x$, we can represent it by $s(x) = \sum_{J,i} s_{J,i}(x)\overline{w}^J \otimes e_i^k$ for a local orthonormal frame $\{e_i^k\}$ of $L^k \otimes E$. Thus $|s(x)|_h^2 = \sum_{J,i} |s_{J,i}(x)|^2$. By (5.7) and (5.8), Theorem 5.1 entails that, for any $s \in \Omega^{0,q}(X, L^k \otimes E)$,

$$\|D_k^{c,A}s\|^2 \geq 2k\langle -w_d s, s\rangle - C\|s\|^2 = 2k\int_X \sum_{J,i} C_J(x)|s_{J,i}|^2 \mathrm{d}v_X - C\|s\|^2 \tag{5.9}$$

By (2.9), (2.11) and (5.3), we have $\mu_q > 0$. By (5.8), it follows that

$$\|D_k^{c,A}s\|^2 \geq 2k\mu_q\|s\|^2 - C\|s\|^2 \tag{5.10}$$

holds for $s \in \Omega^{0,q}(X, L^k \otimes E)$. By Proposition 2.8 and Remark 2.9, we see $\mu_{j+1} > \mu_j > 0$ for each $j \geq q$. Thus the assertion holds for $s \in \Omega^{0,\geq q}(X, L^k \otimes E)$. □

Remark 5.4 From Remark 2.9, the positive assumption [26, (1.5.21)] is equivalent to Nakano 1-positive line bundle with respect to ω. By (5.8), $\mu_1 = \inf_{x\in X, 1\leq j\leq n} c_j(x)$. Thus [26, Theorem 1.5.7] follows from Theorem 5.3 by choosing $q = 1$.

In general, for a real 3-form A on X, $(D_k^{c,A})^2$ may not preserve the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^k \otimes E)$. As a special case, we can consider Kodaira Laplacian $\square^{L^k\otimes E}$, which preserves the $\mathbb{Z}$-grading. On a complex manifold X, Hodge–Dolbeault operator satisfies $D_k := \sqrt{2}(\overline{\partial}^{L^k\otimes E} + \overline{\partial}^{L^k\otimes E,*}) = D_k^{c,A}$, for $A = -\frac{1}{4}T_{as}$, see [26, (1.4.17)], and the Kodaira Laplacian satisfies $\square^{L^k\otimes E} = \frac{1}{2}D_k^2$. Then, from Hodge theorem, Serre duality and the equivalent definition of the q-positive line bundle (see Remark 2.4), Theorem 5.3 leads to Andreotti–Grauert vanishing theorem [1, Proposition 27] (see also [16, (5.1) Theorem]):

Corollary 5.5 *([1]) Let X be a compact complex manifold of dimension n and (E, h^E) and (L, h^L) be holomorphic Hermitian vector bundles on X with* $\mathrm{rank}(L) = 1$. *If R^L has at least p positive eigenvalues and at least q negative eigenvalues at every $x \in X$, then, for $j \in \{j \in \mathbb{N} : j \leq q-1$ or $j \geq n-p+1\}$ and sufficiently large k, $H^j(X, L^k \otimes E) = 0$.*

By the same argument in [25, Theorem 4.4, Corollary 4.5-4.6] and [26, (6.1.15)], Theorem 5.3 still holds on Γ-covering manifolds as follows. Let $\widetilde{X}$ be a Γ-covering manifold of dimension n. Let $\widetilde{J}$ be Γ-invariant almost complex structure on $\widetilde{X}$. Let $g^{T\widetilde{X}}$ be a Γ-invariant Riemannian metric compatible with $\widetilde{J}$ and $\omega := g^{T\widetilde{X}}(\widetilde{J}\cdot, \cdot)$ be the real $(1,1)$-forms on $\widetilde{X}$ induced by $g^{T\widetilde{X}}$ and $\widetilde{J}$. Let $(\widetilde{E}, h^{\widetilde{E}})$ and $(\widetilde{L}, h^{\widetilde{L}})$ be Γ-invariant holomorphic Hermitian vector bundles on $\widetilde{X}$ with $\operatorname{rank}(\widetilde{L}) = 1$. Let $\nabla^{\widetilde{E}}$ and $\nabla^{\widetilde{L}}$ be Chern connections on $(\widetilde{E}, h^{\widetilde{E}})$ and $(\widetilde{L}, h^{\widetilde{L}})$ and let $R^{\widetilde{E}} := (\nabla^{\widetilde{E}})^2$ and $R^{\widetilde{L}} := (\nabla^{\widetilde{L}})^2$ be the curvatures. Let $\widetilde{D}_k := \sqrt{2}(\overline{\partial}^{\widetilde{L}^k\otimes\widetilde{E}} + \overline{\partial}^{\widetilde{L}^k\otimes\widetilde{E},*})$ be the Hodge–Dolbeault operator defined on $\operatorname{Dom}(\widetilde{D}_k) = \operatorname{Dom}(\overline{\partial}^{\widetilde{L}^k\otimes\widetilde{E}}) \cap \operatorname{Dom}(\overline{\partial}^{\widetilde{L}^k\otimes\widetilde{E},*})$ and $\square^{\widetilde{L}^k\otimes\widetilde{E}} := \frac{1}{2}\widetilde{D}_k^2$ the self-adjoint extension of Kodaira Laplacian.

Theorem 5.6 *Assume $1 \le q \le n$ and $(\widetilde{L}, h^{\widetilde{L}})$ is Nakano q-positive with respect to ω on $\widetilde{X}$. Then there exists $C_{\widetilde{L}} > 0$ such that for any $k \in \mathbb{N}$ and any $\widetilde{s} \in \operatorname{Dom}(\widetilde{D}_k) \cap L^2_{0,\ge q}(\widetilde{X}, \widetilde{L}^k \otimes \widetilde{E})$,*

$$\|\widetilde{D}_k\widetilde{s}\|^2 \ge (2\mu_q k - C_{\widetilde{L}})\|\widetilde{s}\|^2, \tag{5.11}$$

where the constant $\mu_q > 0$ defined in (5.3).

For the L^2 Andreotti–Grauert theorem on covering manifolds, see [7, Theorem 3.5] and [25, Sect. 4].

5.2 Semipositive Line Bundles of Type q

Let X be a complex manifold of dimension n and (L, h^L) be a holomorphic Hermitian line bundle. For $1 \le q \le n$, we have the notion of semipositive line bundles of type q as follows, refer to [32, Chapter 3, Sect. 1, Definition 1.1]. We say (L, h^L) is semipositive of type q if $(L, h^L) \ge 0$ everywhere and $\sqrt{-1}R_x^{(L,h^L)}$ is positive on a $(n-q+1)$-dimensional subspace of $T_x^{(1,0)}X$ at every $x \in X$,

We remark that, by [32, Chapter 3, Sect. 2, Proposition 2.1 (1),(2)] and Definition 1.1, if (L, h^L) is semipositive of type q on a complex manifold X, then (L, h^L) is Nakano q-positive at every point $x \in X$ with respect to arbitrary Hermitian metric ω on X. As a consequence, by replacing the hypothesis Nakano q-positive with respect to ω by semipositive of type q in Theorem 5.3 and 5.6, the conclusion therein still holds.

Besides, by adapting the notion of semipositive of type q to Theorem 4.7, we obtain another generalization of [26, Theorem 3.5.9] as follows.

Corollary 5.7 *Let (X, ω) be a q-convex manifold of dimension n. Let E, L be holomorphic vector bundle with $\operatorname{rank}(L) = 1$. Let $K \subset X$ be the exceptional set and $1 \le p \le n$. If (L, h^L) is semipositive of type p on X_c with $K \subset X_c$, then for $j \ge \max\{p, q\}$ and k sufficiently large,*

$$H^j(X, L^k \otimes E) = 0. \tag{5.12}$$

Proof Since (L, h^L) is semipositive of type p on X_c, (L, h^L) is Nakano q-positive with respect to ω given by Lemma 4.1 on X_c. Finally, we use Theorem 4.7. □

Corollary 5.8 *Let M be a smooth pseudoconvex domain in a Kähler manifold (X, ω) of dimension n and $1 \leq q \leq n$. Let (L, h^L) be a semipositive line bundle on M. Assume (L, h^L) is semipositive of type q on a neighborhood of bM. Then for every $j \geq q$,*

$$H^{0,j}_{(2)}(M, K_X \otimes L) = 0. \tag{5.13}$$

Proof Propositions 2.6 and 4.17 and the fact that (L, h^L) is Nakano q-positive with respect to any Hermitian metric ω on a neighborhood of bM. □

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