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Stability of the Spacetime Positive Mass Theorem in Spherical Symmetry

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Abstract

The rigidity statement of the positive mass theorem asserts that an asymptotically flat initial data set for the Einstein equations with zero ADM mass, and satisfying the dominant energy condition, must arise from an embedding into Minkowski space. In this paper, we address the question of what happens when the mass is merely small. In particular, we formulate a conjecture for the stability statement associated with the spacetime version of the positive mass theorem, and give examples to show how it is basically sharp if true. This conjecture is then established under the assumption of spherical symmetry in all dimensions. More precisely, it is shown that a sequence of asymptotically flat initial data satisfying the dominant energy condition, without horizons except possibly at an inner boundary, and with ADM masses tending to zero must arise from isometric embeddings into a sequence of static spacetimes converging to Minkowski space in the pointed volume preserving intrinsic flat sense. The difference of second fundamental forms coming from the embeddings and initial data must converge to zero in L^p, $1 \leq p < 2$. In addition some minor tangential results are also given, including the spacetime version of the Penrose inequality with rigidity statement in all dimensions for spherically symmetric initial data, as well as symmetry inheritance properties for outermost apparent horizons.

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1 Introduction

Let (M^n, g, k) be an initial data set for the Einstein equations. This means that (M^n, g) is a complete Riemannian manifold, possibly with boundary, and k is a symmetric 2-tensor representing the second fundamental form of an embedding into spacetime. These satisfy the constraint equations

$$16\pi\mu = R_g + (\mathrm{Tr}_g k)^2 - |k|_g^2, \qquad 8\pi J = \mathrm{div}_g\left(k - (\mathrm{Tr}_g k)g\right), \tag{1.1}$$

where μ and J are the energy and momentum density of the matter fields, and R_g denotes scalar curvature. The dominant energy condition is satisfied if

$$\mu \geq |J|_g. \tag{1.2}$$

We will say that the initial data are *asymptotically flat* if there is an asymptotic end in the manifold M^n that is diffeomorphic to the complement of a ball $\mathbb{R}^n \setminus B_0(\rho_0)$, and there exists a constant C such that in the coordinates x provided by this asymptotic diffeomorphism

$$\left|\partial^{\beta_1}(g_{ij} - \delta_{ij})\right| \leq \frac{C}{|x|^{n-2+|\beta_1|}}, \qquad \left|\partial^{\beta_2} k_{ij}\right| \leq \frac{C}{|x|^{n-1+|\beta_2|}}, \tag{1.3}$$

for multi-indices $\beta_1 \leq 2$, $\beta_2 \leq 1$ and

$$\left|R_g\right| \leq \frac{C}{|x|^{n+1}}, \qquad \left|\mathrm{Tr}_g k\right| \leq \frac{C}{|x|^n}. \tag{1.4}$$

These fall-off conditions are modeled on those of the original Schoen-Yau proof of the positive mass theorem [34]. We believe our results should follow assuming the weaker asymptotic decay as in the work of Eichmair et al. [12,13]; however, for the sake of simplicity of exposition this will not be done.

With the above setting, the ADM energy and linear momentum of the asymptotic end are finite, well-defined, and given by

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu^j, \tag{1.5}$$

$$P_i = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (k_{ij} - (\mathrm{Tr}_g k)g_{ij})\nu^j, \tag{1.6}$$

where S_r are coordinate spheres with unit outer normal ν and ω_{n-1} is the volume of the standard sphere $\mathbb{S}^{n-1}$. The ADM mass is then the Lorentz length of the energy-momentum 4-vector

$$m = \sqrt{E^2 - |P|^2}. \tag{1.7}$$

In this paper the main results will be concerned with spherically symmetric initial data. It turns out that in spherical symmetry, under the definition of asymptotic flatness in (1.3) and (1.4), the linear momentum vanishes $|P| = 0$ and hence $m = E$ as is shown in Proposition 3.6.

The positive mass inequality asserts that an asymptotically flat complete initial data set satisfying the dominant energy condition has

$$E \geq |P|. \tag{1.8}$$

This was established by Eichmair et al. in [13] for dimensions $3 \leq n \leq 7$ by using stable marginally outer trapped surfaces (MOTS) in analogy with the minimal hypersurface technique deployed in the time-symmetric case, and in all dimensions $n \geq 3$ for spin manifolds by Bartnik [5] and Witten [40] (see also work of Parker and Taubes [30]). Earlier, the weaker inequality $E \geq 0$ was initially proven by Schoen and Yau [34] when $n = 3$ with the help of Jang's equation, and this reduction argument was later extended by Eichmair [12] to include dimensions $3 \leq n \leq 7$.

The rigidity of the positive mass theorem may be broken into two statements. The first asserts:

$$E = |P| \quad \Longrightarrow \quad E = |P| = 0. \tag{1.9}$$

This was proven by Huang and Lee [17] for $3 \leq n \leq 7$. Their approach only uses the positive mass inequality as input but not its proof, and thus can be extended to higher dimensions for spin manifolds. The spin case was previously treated by Beig and Chrusciel [6] for $n = 3$ and Chrusciel and Maerten [10] for higher dimensions. The second statement is that

$$E = 0 \Longrightarrow (M^n, g, k) \text{ embeds as initial data in Minkowski space.} \tag{1.10}$$

As with the inequality, this was originally established by Schoen and Yau in [34] for three dimensions and extended by Eichmair in [12] to dimensions less than eight. Finally in the spin case this was treated for all dimensions in work of Beig et al. [6,10]. Here we state the positive mass rigidity theorem in a particular way that allows us to propose a natural almost rigidity (or stability) conjecture.

Theorem 1.1 (Positive mass rigidity theorem [12,17,34]) *Let (M^n, g, k) be a complete asymptotically flat initial data set, with $3 \leq n \leq 7$, and satisfying the dominant energy condition. If the ADM mass vanishes $m = 0$, then M^n is diffeomorphic to $\mathbb{R}^n$ and (M^n, g) can be isometrically embedded as a graph in Minkowski space. That is*

$$F : (M^n, g) \to (\mathbb{R}^{1,n}, -\mathrm{d}t^2 + g_{\mathbb{E}}), \qquad F(x) = (f(x), x), \tag{1.11}$$

where $g_{\mathbb{E}}$ is the Euclidean metric and

$$g = F^*(-\mathrm{d}t^2 + g_{\mathbb{E}}) = -\mathrm{d}f^2 + g_{\mathbb{E}}, \tag{1.12}$$

and the second fundamental form, h, of the embedding agrees with that of the initial data

$$h = k. \tag{1.13}$$

The purpose of this paper is to establish an almost rigidity or stability version of this theorem in the spherically symmetric setting. We will say that the initial data are *spherically symmetric* if M^n is diffeomorphic to $\mathbb{R}^n \setminus B_0(r_0)$ or $\mathbb{R}^n$ and the metric and second fundamental form may be expressed by

$$g = g_{11}(r)\mathrm{d}r^2 + \rho(r)^2 g_{S^{n-1}}, \qquad k_{ij} = n_i n_j k_n(r) + (g_{ij} - n_i n_j)k_t(r), \tag{1.14}$$

for some radial functions g_{11}, ρ, k_n, and k_t, where $n = \sqrt{g^{11}}\partial_r$ is the unit normal to coordinate spheres. This decomposition for k exhibits its normal and tangential components with respect to the coordinate spheres, and is motivated by the implicit assumption that the initial data come from a spherically symmetric spacetime in which k is the 'time derivative' of g which already has this structure.

The boundary, if nonempty, of the initial data will consist of apparent horizons. Recall that the strength of the gravitational field around a hypersurface $\Sigma \subset M^n$ may be measured by the null expansions (null mean curvatures) given by

$$\theta_{\pm} := H_{\Sigma} \pm Tr_{\Sigma}k, \tag{1.15}$$

where H_{Σ} is the mean curvature with respect to the unit normal pointing towards spatial infinity. These quantities can be interpreted as the rate at which the area of a shell of light changes as it moves away from the surface in the outward future/past direction $(+/-)$. Future or past trapped surfaces are defined by the inequalities $\theta_+ < 0$ or $\theta_- < 0$, respectively, and may be thought of as lying in a region of strong gravity. If $\theta_+ = 0$ or $\theta_- = 0$, then Σ is called a future or past apparent horizon; these naturally arise as boundaries of future or past trapped regions. Furthermore, such surfaces will be referred to as an *outermost apparent horizon* if it is not enclosed by any other apparent horizon. In Lemma 3.1 it is shown that the outermost apparent horizon inherits the symmetry of its ambient space. In this text the abbreviated term *horizon* will often be used for these objects.

We will consider asymptotically flat (M^n, g, k) that have either no horizons or only a horizon on an inner boundary, in which case the boundary is an outermost apparent horizon. Under these conditions for spherically symmetric initial data, it is shown in Lemma 3.4 that the areas (or $n-1$ dimensional volumes) of the level sets of ρ are increasing. Thus we may define the level set

$$\Sigma_A = \rho^{-1}(\rho_A) \text{ such that } \mathrm{Vol}_g(\Sigma_A) = A = \omega_{n-1}\rho_A^{n-1}. \tag{1.16}$$

We will study regions within and between these level sets

$$\Omega_A = \rho^{-1}[0, \rho_A], \qquad \Omega_{A_1,A_2} = \rho^{-1}[\rho_{A_1}, \rho_{A_2}], \tag{1.17}$$

as well as their tubular neighborhoods

$$T_D(\Sigma_A) = \{p \in M^n \mid \exists q \in \Sigma_A \ with \ d_g(p,q) < D\}, \tag{1.18}$$

where $d_g(p,q)$ denotes the distance between p and q.

In order to state the main theorem we need the notion of *uniform asymptotic flatness*. A sequence of initial data (M_j^n, g_j, k_j) will be referred to as *uniformly asymptotically flat* if each member of the sequence is asymptotically flat according to (1.3) and (1.4), and the constants ρ_0 and C in the definition are independent of j.

Theorem 1.2 *Fix $A > 0$ and $D > \rho_A$, and consider a sequence of uniformly asymptotically flat spherically symmetric initial data sets (M_j^n, g_j, k_j) satisfying the dominant energy condition and with no closed horizons except possibly the inner boundary. If their ADM masses converge to zero $m_j \to 0$, then there exist Riemannian manifolds $(\bar{M}_j^n, \bar{g}_j)$ diffeomorphic to (M_j^n, g_j) with graphical isometric embeddings*

$$F_j : \left(M_j^n, g_j\right) \to \left(\mathbb{R} \times \bar{M}_j^n, -\mathrm{d}t^2 + \bar{g}_j\right), \qquad F_j(x) = (f_j(x), x), \tag{1.19}$$

$$g_j = F_j^*(-\mathrm{d}t^2 + \bar{g}_j) = -\mathrm{d}f_j^2 + \bar{g}_j, \tag{1.20}$$

such that the static spacetimes

$$\left(\mathbb{R} \times \bar{M}_j^n, -\mathrm{d}t^2 + \bar{g}_j\right) \ \textit{converge to Minkowski space} \ (\mathbb{R}^{1,n}, -\mathrm{d}t^2 + g_{\mathbb{E}}) \tag{1.21}$$

in that the base manifolds converge in the pointed volume preserving intrinsic flat sense to Euclidean space. More precisely, regions within Σ_A in $(\bar{M}_j^n, \bar{g}_j)$ converge to balls in Euclidean space

$$\left(\Omega_A^j \cap T_D\left(\Sigma_A^j\right), \bar{g}_j\right) \xrightarrow{\mathcal{VF}} \left(B_0(\rho_A), g_{\mathbb{E}}\right). \tag{1.22}$$

Furthermore if there is a uniform constant C such that $\| k_j \|_{L^2(M_j^n)} \leq C$, then for any $1 \leq p < 2$ the second fundamental forms h_j of the graphs satisfy

$$\| h_j - k_j \|_{L^p\left(\Omega_A^j \cap T_D(\Sigma_A^j), \bar{g}_j\right)} \to 0. \tag{1.23}$$

The intrinsic flat distance $d_{\mathcal{F}}(\Omega_j, \Omega_j')$ between pairs of compact oriented Riemannian manifolds with boundary was first introduced by the third author with Wenger in [38]. Intuitively it measures the filling volume between the given manifolds. It is 0 if and only if there is an orientation preserving isometry between the manifolds Ω_j

and Ω_j' [38]. The volume preserving intrinsic flat distance was introduced in [36] and includes an extra term involving the global difference of volumes

$$d_{\mathcal{VF}}(\Omega_j, \Omega_j') = d_{\mathcal{F}}(\Omega_j, \Omega_j') + |\operatorname{Vol}_j(\Omega_j) - \operatorname{Vol}_\infty(\Omega_j')|. \tag{1.24}$$

This has been studied by Portegies in [32] and by Jauregui-Lee in [20]. In particular they have shown that

$$d_{\mathcal{VF}}(\Omega_j, \Omega_\infty) \to 0 \quad \Longrightarrow \quad \operatorname{Vol}(B_{p_j}(r)) \to \operatorname{Vol}(B_{p_\infty}(r)), \tag{1.25}$$

for sequences of points $p_j \in \Omega_j$ converging to $p_\infty \in \Omega_\infty$.

Theorem 1.2 has been proven for time-symmetric initial data sets by Lee and the third author in [23]. In that setting $k_j = 0$ and f_j can be taken to be constant so that $h_j = 0$ and (1.23) follows trivially. The intrinsic flat convergence is proven in [23] by constructing an explicit filling manifold. In fact, LeFloch and the third author have proven in [24] that the metric tensors converge in the H^1_{loc} sense. Note that examples in [23] demonstrate that even in the spherically symmetric time-symmetric setting one can have sequences with ADM mass converging to 0 which do not converge to regions in Euclidean space in the smooth or Gromov–Hausdorff sense. Applying techniques from [23] in our Example 2.4, it is shown why one needs a tubular neighborhood in (1.22). Furthermore, in Examples 2.1 and 2.2 we demonstrate the need to assume that there are no interior horizons.

Without time symmetry, when $k_j \neq 0$, Theorem 1.2 makes no claim as to the convergence of the original sequence of Riemannian manifolds (M_j^n, g_j). Example 2.7 illustrates why the initial sequence (M_j^n, g_j) need not converge in any reasonable sense. There we construct sequences of initial data sets of zero mass lying in Minkowski space which become increasingly null on large regions, so that volumes disappear instead of converging.

Conjecture 1.3 Theorem 1.2 holds without requiring spherical symmetry when suitable definitions are made for the regions Ω_A. To achieve the conclusion exactly as stated we expect that $E \to 0$ should replace $m \to 0$ in the hypotheses for the general case. It is possible that a similar statement holds for $m \to 0$, but the approach would have to be different from the one used here in light of examples with boost. In the outline below, we clarify which steps strongly use spherical symmetry and which hold more generally.

The corresponding almost rigidity or stability conjecture in the time-symmetric case was stated and proven in the spherically symmetric setting by Lee et al. [23]. It has been confirmed in the graph setting by Huang et al. in [18] and for geometrostatic manifolds by the third author with Stavrov in [37]. Initial controls on the metric tensor towards proving the time-symmetric conjecture have been found by Allen for regions covered by smooth inverse mean curvature flow in [1], and by the first author for axisymmetric manifolds [8].

With the definition of asymptotic flatness used here the ADM mass m and ADM energy E agree in spherical symmetry since the linear momentum P vanishes (Proposition 3.6). In general when mass and energy differ, Conjecture 1.3 could be quite

subtle in the case of large linear momentum, as the construction of the base manifolds $(\bar{M}^n, \bar{g})$ presented here does not behave well in this setting. The methods used here and based on the Jang equation are tailored to the situation when E is small, which will not be the case if $|P|$ stays uniformly away from zero.

We now give an outline of the proof of Theorem 1.2, which is modeled on the Schoen-Yau approach to the positive mass theorem [34]. It should be noted that some of the arguments do not require spherical symmetry. The first step is to solve the 2nd order quasi-linear elliptic Jang equation for each (M_j^n, g_j, k_j) to obtain solutions f_j with asymptotically cylindrical blow-up at the outermost apparent horizon. See Sect. 4 for details. The original study of this equation in [34] observed that the solution only blows-up at apparent horizons. Prescribed blow-up at the outermost horizon was obtained in work of Eichmair et al. for low dimensions [12,14,27]. In the spherically symmetric case, the equation can be reduced to a 1st order ODE and the desired solutions can be produced in any dimension [Theorem 4.1]. From the solutions a sequence of Riemannian manifolds,

$$\text{the Jang deformations: } \left(\bar{M}_j^n, \bar{g}_j = g_j + df_j^2\right), \tag{1.26}$$

can be constructed which serve as the base for the ambient static spacetimes of Theorem 1.2. Schoen-Yau showed that the scalar curvature of the Jang metric is nonnegative modulo a divergence term as stated in (4.8). The Jang manifolds are also uniformly asymptotically flat [Lemma 4.3], and have the same ADM masses as the original initial data [Corollary 4.2]. A primary difference is that they have a cylindrical end where previously there was a boundary. In Example 2.5 we explicitly solve the Jang equation for a constant time slice of the Schwarzschild spacetime so that one can see precisely how this step may be implemented constructively.

The nonnegativity property of the scalar curvature of $(\bar{M}_j^n, \bar{g}_j)$ allows one to further conformally transform the Jang deformations to Riemannian manifolds of zero scalar curvature,

$$\text{the conformal transformations: } \left(\tilde{M}_j^n, \tilde{g}_j = u_j^{4/(n-2)}\bar{g}_j\right). \tag{1.27}$$

Along the cylindrical ends u_j decays exponentially fast to zero and hence conformally closes this end (see [34] and Proposition 5.1). In general the masses of the conformal deformations converge to zero. Thus if no horizons are present or one restricts attention to domains outside the outermost minimal surface, it is expected (by the time-symmetric almost rigidity conjecture) that regions in $(\tilde{M}_j^n, \tilde{g}_j)$ converge to balls in Euclidean space in the intrinsic flat sense. The hope is then to prove that the conformal factors u_j are sufficiently close to 1 in order to establish that regions in $(\bar{M}_j^n, \bar{g}_j)$ converge to balls in Euclidean space as well. See Remark 5.2.

In the spherically symmetric setting the conformal deformation is Euclidean space $\tilde{g}_j = g_{\mathbb{E}}$, so the mass is 0 and there are no horizons (cf. Lemma 5.3). Therefore $\bar{g}_j$ is related to the Euclidean metric via the conformal factor u_j, and establishing (1.22) is reduced to controlling u_j. A global L^2 gradient bound in terms of the mass is obtained from the stability property associated with the Jang surface in Lemma 5.6, and this is

parlayed into $C^{0,\frac{1}{2}}$ control away from the center of the manifold in Proposition 5.10 by using the uniform asymptotically flat assumption and Lemma 5.9. We then have that $u_j \to 1$ uniformly on appropriate subdomains avoiding the center. This work is completed in Sect. 5.

The volume preserving intrinsic flat convergence (1.22) is proven in Sect. 6 in two main steps. The overall approach is to apply a result of Lakzian and the third author [22], as stated in Proposition 6.1, to achieve intrinsic flat convergence. First, control on u_j is used to show that regions avoiding the center are smoothly close to annuli in Euclidean space (see Lemma 6.2). Secondly, we prove that the volumes of the regions closer to the center are small using area monotonicity and the coarea formula in Lemma 6.5. The remaining required terms are estimated in various lemmas throughout the section. Moreover, volume convergence follows from the above and is stated in Lemma 6.8. Without spherical symmetry one might imagine doing something similar, cutting out many wells rather than just the center as in joint work of the third author with Stavrov in [37], or using a completely different approach as in joint work of the third author with Huang and Lee in [18].

In Sect. 7 convergence of second fundamental forms (1.23) is established, where the proof relies on nonnegativity of the spacetime Hawking mass. Control on $|h_j - k_j|_{\bar{g}_j}$ away from the center is given in Proposition 7.1 using estimates for the conformal factors u_j, as well as the stability property associated with the Jang surface. While $|h_j - k_j|_{\bar{g}_j}$ might be large near the center, with the additional hypothesis on k_j and the small volume inside in Proposition 7.2, convergence in the desired tubular neighborhood is achieved in Theorem 7.3. Finally, in Sect. 8 we prove Theorem 1.2 using all the above.

2 Examples

In this section we provide some examples which illustrate the importance of various hypotheses in Theorem 1.2, and some intuition as to what is happening in the proof. In the time-symmetric setting, where the objects of study are manifolds with nonnegative scalar curvature, examples are given with closed interior horizons (Examples 2.1 and 2.2) that fail to have volume preserving intrinsic flat convergence to Euclidean space.

The additional assumption of no closed interior horizons and no boundary is also considered. In this setting, a time-symmetric initial data set is a graph over itself and the solution to Jang's equation is constant. An example within this context is provided that contains a deep well, demonstrating why tubular neighborhoods are introduced in order to obtain volume preserving intrinsic flat convergence (Example 2.4).

The proof of Theorem 1.2 will then be applied to slices of the Schwarzschild spacetime, so one can see what happens when there is a boundary horizon. Difficulties arise even in this time-symmetric example because solutions to Jang's equation blow-up near the horizon so that the base spaces $(\bar{M}^n, \bar{g})$ possess an asymptotically cylindrical end. In Example 2.5 we see exactly how Schwarzschild slices arise as graphs over base spaces which are close to Euclidean space in the volume preserving intrinsic flat sense. This clarifies why the proof of Theorem 1.2 is delicate when the manifolds have boundary.

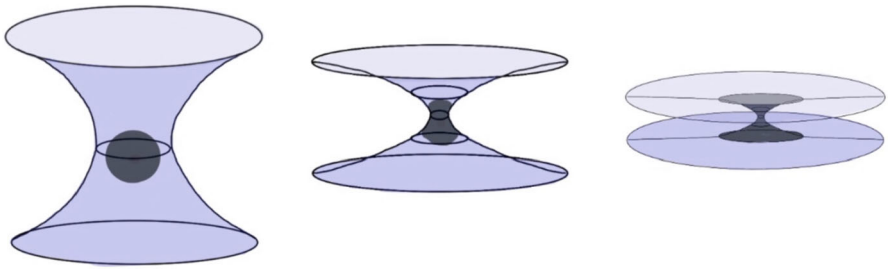

Fig. 1 In Example 2.1 we see that a sequence of balls centered on the horizons of a sequence of Schwarzschild manifolds with $m_j \to 0$ has volume converging to twice the volume of a Euclidean ball

Finally we consider examples which are not time-symmetric. In Example 2.7 it is demonstrated how even the restriction to sequences of spacelike graphs (M_j^n, g_j, k_j) in Minkowski space does not allow for proper control over the original sequence of Riemannian manifolds (M_j^n, g_j). This justifies why Theorem 1.2 only deals with convergence of the base manifolds, and does not address convergence of the given sequence of initial data.

2.1 Horizons in Time-Symmetric Examples

The assumption of no closed interior horizons is necessary to avoid the formation of bubbles and other phenomena which may occur behind a horizon. Since the inner boundary is allowed to be a horizon, these hypotheses mean that the main theorem applies within the domain of outer communication. This is consistent with the basic intuition that the ADM mass cannot effectively 'see' within a black hole. The following well-known examples explain why one cannot hope to obtain volume preserving intrinsic flat convergence without these assumptions.

Example 2.1 Riemannian Schwarzschild space is a constant time slice of the Schwarzschild spacetime, with a metric that can be written as

$$g = \left(1 + z'(r)^2\right) \mathrm{d}r^2 + r^2 g_{S^{n-1}} \tag{2.1}$$

where $z'(r) = \sqrt{2m/(r-2m)}$. It has a horizon (minimal surface) at $r = 2m$, and can be extended smoothly past the horizon by writing r as a function of z. In fact the graph is that of a parabola. If we take a sequence of Riemannian Schwarzschild spaces of smaller and smaller mass m, this parabola becomes more vertical and the vertex decreases to the origin. See Fig. 1. This sequence of Riemannian Schwarzschild spaces converges smoothly to Euclidean space on compact sets that avoid the increasingly thin necks, and by any weak notion of convergence is seen to converge to a double sheeted Euclidean space. The volumes of balls centered around points on the horizon converge to twice the volume of a Euclidean ball. It is only by removing the part behind the horizon that one may consider the limit to be a single Euclidean space, and obtain volume preserving convergence to Euclidean space.

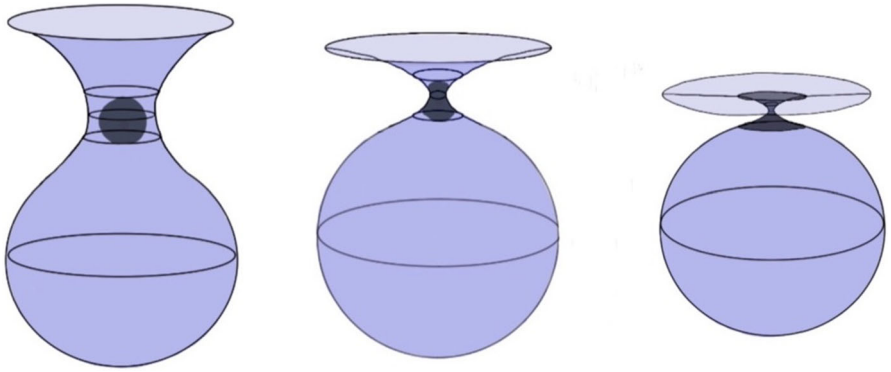

Fig. 2 In Example 2.2 we see that a sequence of balls centered on the horizons of a sequence of manifolds with $m_j \to 0$ has volume converging to the sum of the volumes of a Euclidean ball and a ball in a sphere

Example 2.2 Start with the Riemannian Schwarzschild initial data viewed as a parabola using the function $r(z)$ described in the previous example. Keep the region outside the horizon exactly isometric to Riemannian Schwarzschild, but behind the black hole attach a round sphere in a C^1 way. This is achieved by ensuring that the induced metrics and mean curvatures of the interface surfaces agree from both sides. This guarantees that the scalar curvature is distributionally nonnegative across the interface surface. Furthermore since Riemannian Schwarzschild is scalar flat and the sphere has positive scalar curvature, applying Ricci flow for a very short time, one obtains a smooth metric which is C^1 close to the original and has positive scalar curvature everywhere. The almost spherical region of the resulting manifold is called a bubble. See Fig. 2. Now perform this construction with a sequence of Schwarzschild spaces having masses converging to zero, while keeping the bubble the same size throughout the sequence. By any notion of weak convergence this sequence converges to a Euclidean space with a sphere attached to it. In analogy with the previous example, balls of a fixed radius centered at points on the horizon have volumes converging to the sum of the volume of a ball of the same radius in Euclidean space plus a ball of the same radius in the sphere. Again we do not obtain volume preserving intrinsic flat convergence to Euclidean space, unless the part inside the horizon is cut out.

2.2 Deep Wells in Time-Symmetric Examples

Let us now consider Theorem 1.2 in the time-symmetric case where there are no horizons and no boundary. In such a setting the solution to Jang's equation is constant, so the theorem states that volume preserving intrinsic flat convergence occurs within the initial data themselves, as opposed to convergence of ambient spacetimes in which the data embed. This was established by Lee et al. in [23]. In this subsection we first recall in Lemma 2.3 an example construction technique from [23]. We then review an example, Example 2.4, with deep wells demonstrating the need to use tubular neighborhoods to obtain volume preserving intrinsic flat convergence.

Recall the definition of Hawking mass for a surface Σ in a Riemannian 3-manifold

$$m_H(\Sigma) = \frac{1}{2}\sqrt{\frac{A}{\omega_2}}\left(1 - \frac{1}{4\pi}\int_\Sigma \left(\frac{H}{2}\right)^2\right), \tag{2.2}$$

where A and H are the area and mean curvature of Σ. As described in Sect. 3.3 this may be generalized in spherical symmetry to higher dimensions n by

$$m(s) = \frac{\rho^{n-2}(s)}{2}(1 - \rho'(s)^2), \tag{2.3}$$

where the metric is expressed in a radial arclength coordinate s and with area radius function $\rho(s)$. The first variation of Hawking mass becomes

$$m'(s) = \frac{\rho^{n-1}(s)\rho'(s)}{2(n-1)}R, \tag{2.4}$$

which is nonnegative for nondecreasing area radius functions and nonnegative scalar curvature.

Lemma 2.3 *Let $\mathcal{M}$ denote the collection of asymptotically flat spherically symmetric manifolds*

$$\left(r^{-1}[r_{min}, \infty) \subset \mathbb{R}^n,\ g = (1 + [z'(r)]^2)\mathrm{d}r^2 + r^2 g_{S^{n-1}}\right) \tag{2.5}$$

with nonnegative scalar curvature that have no closed interior minimal surfaces and either no boundary, or minimal surface boundary $r^{-1}(r_{min})$. Let $\mathcal{H}$ be the collection of admissible Hawking mass functions, that is increasing functions $m : [r_{min}, \infty) \to \mathbb{R}$ such that

$$m(r_{min}) = \frac{1}{2}r_{min}^{n-2}, \tag{2.6}$$

and

$$m(r) < \frac{1}{2}r^{n-2}, \tag{2.7}$$

for $r > r_{min} \geq 0$. There is a constructive bijection between $\mathcal{M}$ and $\mathcal{H}$ such that

$$m(r) = \frac{r^{n-2}}{2}\left(\frac{z'(r)^2}{1 + z'(r)^2}\right) < \frac{1}{2}r^{n-2}. \tag{2.8}$$

In [23] this result was used to construct an example with an arbitrarily deep well. Here we also describe the volumes in this example, justifying the necessity of cutting off the region using tubular neighborhoods of fixed size D to obtain volume preserving intrinsic flat convergence in Theorem 1.2.

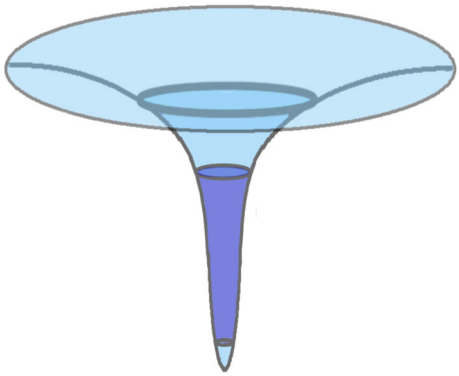

Fig. 3 This manifold (M^n_j, g_j) of Example 2.4 has a deep well with large volume in the shaded region $\Omega^j_{A_{\delta_j},A} \subset M^n_j$

Example 2.4 Given $A > 0$, $L > 0$, and $\delta > 0$ there exists $(M^n, g) \in \mathcal{M}$ with ADM mass $m < \delta$ such that the distance $d(\Sigma_{min}, \Sigma_A) > L$, where Σ_A is a symmetry sphere with area A and Σ_{min} is either the boundary ∂M^n or the pole. See Fig. 3. In fact the example is constructed by first choosing a radius r_ϵ depending on δ, and then constructing the admissible Hawking mass function so that the distance between the levels r_ϵ and $r_\epsilon/2$ is an arbitrary value L. Thus the volume between these level sets, computed with the coarea formula, provides a lower bound

$$\mathrm{Vol}_g(\Omega_A) \geq A_\delta L, \tag{2.9}$$

where A_δ is the area of the level set $r^{-1}(r_\epsilon)$ which depends on δ but not L. Taking a sequence with

$$\delta_j \to 0 \quad \text{and} \quad L_j = j/A_{\delta_j} \to \infty, \tag{2.10}$$

we obtain a sequence of spherically symmetric examples which are increasingly deep and have

$$\mathrm{Vol}_{g_j}(\Omega^j_A) \geq \mathrm{Vol}_{g_j}(\Omega^j_{A_{\delta_j},A}) \to \infty. \tag{2.11}$$

By (1.22) of Theorem 1.2 it holds that for any fixed $D > 0$,

$$\mathrm{Vol}_{g_j}\left(\Omega^j_A \cap T_D(\Sigma^j_A)\right) \to \mathrm{Vol}_{g_{\mathbb{E}}}\left(B_0(\rho_A)\right) < \infty. \tag{2.12}$$

This fixed distance $D > 0$ of the tubular neighborhood is needed to cut off the arbitrarily large volumes in the arbitrarily deep wells.

2.3 Riemannian Schwarzschild Space

Let us now consider Theorem 1.2 in the time-symmetric case where there is a boundary horizon; for simplicity we restrict the discussion in this subsection to dimension $n = 3$.

Even though this setting is time-symmetric the solutions of Jang's equation will not be constant, rather they will blow-up at the horizon boundary. These solutions are then used to embed the initial data $(M^3, g, 0)$ into $(\mathbb{R} \times \bar{M}^3, -\mathrm{d}t^2 + \bar{g})$ as a graph over the base space $(\bar{M}^3, \bar{g})$. This base will have different properties than the original data, in particular it will have an asymptotically cylindrical end. Next an appropriate conformal factor u is found so that the new metric $\tilde{g} = u^4 \bar{g}$ is scalar flat. These are some of the main steps in the proof of Theorem 1.2, and will in this subsection be computed explicitly for Riemannian Schwarzschild initial data.

Example 2.5 Recall that the induced metric on a time slice $M^3 = r^{-1}[2m, \infty)$ of the Schwarzschild spacetime of mass m can be written in the form

$$g = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 g_{S^2}. \tag{2.13}$$

The Jang equation may be solved explicitly in this case for a blow-up solution. To see this observe that from [7] and the discussion in Sect. 4.2, Jang's equation may be reduced to a first order ODE by setting

$$v = \frac{\sqrt{g^{11}} f'}{\sqrt{1 + g^{11} f'^2}}, \tag{2.14}$$

where $g^{11} = 1 - 2m/r$. Namely, the Jang equation in this case becomes simply

$$v' + \frac{2}{r} v = 0, \tag{2.15}$$

and the blow-up solution is $v = \left(\frac{2m}{r}\right)^2$. It follows that

$$\begin{aligned} g^{11} f'^2 = \frac{v^2}{1 - v^2} &= \frac{1}{(r/2m)^4 - 1} \\ &= \frac{1}{(1 - 2m/r)(1 + 2m/r)(r/2m)^2[1 + (r/2m)^2]}, \end{aligned} \tag{2.16}$$

so that

$$f' = \frac{(1 - 2m/r)^{-1}}{(r/2m)\sqrt{(1 + 2m/r)[1 + (r/2m)^2]}}. \tag{2.17}$$

Therefore the Jang metric is

$$\bar{g} = g + \mathrm{d}f^2 = \bar{g}_{11} \mathrm{d}r^2 + r^2 g_{S^2} = (g_{11} + f'(r)^2) \mathrm{d}r^2 + r^2 g_{S^2} \tag{2.18}$$

with

$$\bar{g}_{11} = g_{11} + f'(r)^2$$

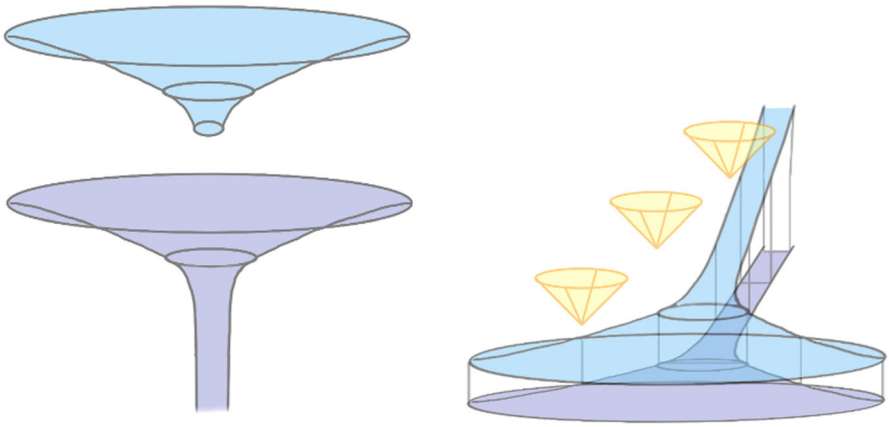

Fig. 4 On the left, the Riemannian Schwarzschild geometry (light blue) and its Jang perturbation $(\bar{M}^3, \bar{g})$ in purple are shown as embedded into $\mathbb{E}^4$, with the graphical height z coordinate directed upwards. On the right, the Riemannian Schwarzschild space is illustrated as a graph over the base Jang perturbation. This depiction takes place in 5-dimensional Minkowski space $\mathbb{R}^{1,4} \supset (\mathbb{R} \times \bar{M}^3, -\mathrm{d}t^2 + \bar{g})$, where the spatial z coordinate is directed inwards. Light cones are shown in yellow (Color figure online)

$$= (1 - 2m/r)^{-2} \left[(1 - 2m/r) + \frac{1}{(r/2m)^2 (1 + 2m/r)[1 + (r/2m)^2]} \right]. \tag{2.19}$$

This clearly has an asymptotically cylindrical end as $r \to 2m$. Figure 4 illustrates how the Riemannian Schwarzschild geometry embeds as a graph over this Jang deformation. Note that it becomes increasingly null upon approach to the horizon.

The conformal deformation to zero scalar curvature can also be given explicitly for this Schwarzschild example. To do this let $r = r(\tilde{r})$ be such that

$$\bar{g}_{11} \left(\frac{\mathrm{d}r}{\mathrm{d}\tilde{r}} \right)^2 = \left(\frac{r}{\tilde{r}} \right)^2, \tag{2.20}$$

then

$$\bar{g} = \bar{g}_{11} \left(\frac{\mathrm{d}r}{\mathrm{d}\tilde{r}} \right)^2 \mathrm{d}\tilde{r}^2 + \left(\frac{r}{\tilde{r}} \right)^2 \tilde{r}^2 g_{S^2} = \left(\frac{r}{\tilde{r}} \right)^2 \left[\mathrm{d}\tilde{r}^2 + \tilde{r}^2 g_{S^2} \right] = \left(\frac{r}{\tilde{r}} \right)^2 g_{\mathbb{E}}. \tag{2.21}$$

We may solve for $\tilde{r}$ in terms of r by

$$\log \tilde{r} = \int_{4m}^{r} \sqrt{\bar{g}_{11}} r^{-1} \mathrm{d}r. \tag{2.22}$$

Now set $u^{-4} = (r/\tilde{r})^2$ so that $g_{\mathbb{E}} = u^4 \bar{g}$. Thus, $u = \sqrt{\tilde{r}/r}$ serves as the desired conformal factor yielding a scalar flat deformation. The fact that this conformal change resulted in a Euclidean metric is not special to the Schwarzschild example, as will be seen in Sect. 5.

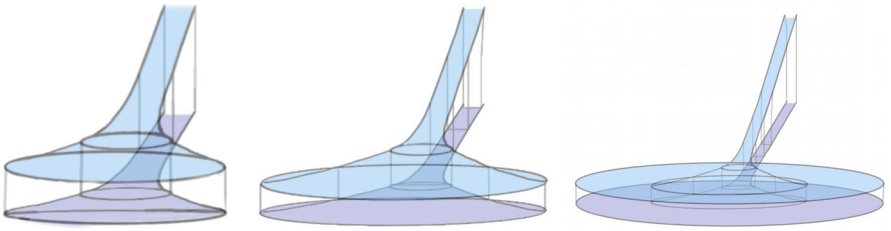

Fig. 5 A sequence of Riemannian Schwarzschild manifolds as graphs over their Jang deformations with masses tending to zero. See also Fig. 4

Figure 5 depicts a sequence of Riemannian Schwarzschild manifolds (M_j^3, g_j) with masses $m_j \to 0$, embedded as graphs over a sequence of base Jang deformations $(\bar{M}_j^3, \bar{g}_j)$ with asymptotically cylindrical ends that converge in the pointed volume preserving intrinsic flat sense to Euclidean space.

2.4 Graphs in Minkowski Space

If (M_j^n, g_j, k_j) arise as spacelike graphs $t = f_j(x)$ in Minkowski space, then the Jang metric $\bar{g}_j$ obtained by solving Jang's equation is exactly the Euclidean metric and k_j is the second fundamental form of the graph. In the notation of Theorem 1.2 we have

$$\bar{g}_j = g_{\mathbb{E}}, \qquad g_j = -\mathrm{d}f_j^2 + g_{\mathbb{E}}, \qquad h_j = k_j. \tag{2.23}$$

Theorem 1.2 is trivially true and there is nothing to prove. On the other hand, such examples can exhibit pathological behavior from the point of view of establishing volume preserving intrinsic flat convergence of (M_j^n, g_j). In this subsection an example is presented to demonstrate why we say nothing about the limiting behavior of (M_j^n, g_j) in Theorem 1.2. In particular, even for sequences of spacelike graphs in Minkowski space one cannot hope for more than subsequential convergence, and the limiting space need not be well-behaved.

Remark 2.6 If (M_j^n, g_j, k_j) arise as spacelike graphs $t = f_j(x)$ in Minkowski space, then $g_j = -\mathrm{d}f_j^2 + g_{\mathbb{E}}$ is positive definite, so the Lipschitz norm satisfies $Lip_{g_{\mathbb{E}}}(f_j) < 1$. Thus by Arzela–Ascoli a subsequence of the f_j converge to a Lipschitz function f_∞ with $Lip_{g_{\mathbb{E}}}(f_j) \leq 1$. However, the graph of f_∞ need not be spacelike!

Example 2.7 Consider a sequence of graphs $t = f_j(x)$ in Minkowski space which are spherically symmetric, with $Lip_{g_{\mathbb{E}}}(f_j) < 1$, and converging to f_∞ that satisfies

$$f_\infty'(r) = 1 \quad \text{for} \quad r \in [\rho_{A_1}, \rho_{A_2}] \tag{2.24}$$

as in Fig. 6, where r is the radial distance function for $g_{\mathbb{E}}$. It can be arranged that g_j converge in the C^0 sense. The limit is a semidefinite metric g_∞ with

$$g_\infty(\partial_r, \partial_r) = -f_\infty'(r)^2 + 1 = 0 \quad \text{for} \quad r \in [\rho_{A_1}, \rho_{A_2}]. \tag{2.25}$$

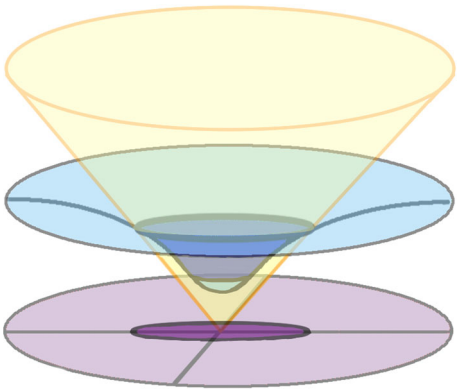

Fig. 6 In Example 2.7 we see that even if our given manifold is a graph in Minkowski space the manifold (in blue) can have regions (in dark blue) of very small volume because it is almost null (Color figure online)

By the coarea formula

$$\mathrm{Vol}_{g_j}(\Omega_{A_1,A_2}) \leq A_2 d_{g_j}(\Sigma_{A_1}, \Sigma_{A_2}) \to 0, \tag{2.26}$$

since

$$d_{g_j}(\Sigma_{A_1}, \Sigma_{A_2}) = \int_{\rho_{A_1}}^{\rho_{A_2}} \sqrt{g_j(\partial_r, \partial_r)} \mathrm{d}r \to 0. \tag{2.27}$$

In contrast

$$\mathrm{Vol}_{\bar{g}_j}(\Omega_{A_1,A_2}) \to \mathrm{Vol}_{g_{\mathbb{E}}}(B_0(\rho_{A_2}) \setminus B_0(\rho_{A_1})) \neq 0. \tag{2.28}$$

Thus we find nice behavior of the base spaces $(\bar{M}_j^n, \bar{g}_j)$ as described in Theorem 1.2, with pathological limiting behavior for the original sequence (M_j^n, g_j).

3 Manifolds with Spherical Symmetry

In this section we prove that outermost apparent horizons inherit the symmetries of the asymptotically flat initial data sets in which they lie (Lemma 3.1), that the areas of symmetry spheres are monotonic in spherically symmetric initial data sets without horizons or with outermost apparent horizon boundary (Lemma 3.4), and establish the spacetime Penrose Inequality in all dimensions under the assumption of spherically symmetry (Theorem 3.5). These results are of use to us when proving Theorem 1.2. Prior work in these directions is reviewed within.

3.1 Horizons in Initial Data with Symmetry

Lemma 3.1 *Let (M^n, g, k) be an asymptotically flat initial data set which admits a continuous symmetry with generator η. This means that η is a Killing field which*

leaves k invariant, and thus the following Lie derivatives vanish $\mathfrak{L}_\eta g = \mathfrak{L}_\eta k = 0$. If the outermost apparent horizon is smooth, then η must be tangential to it.

In particular, if (M^n, g, k) is spherically symmetric with smooth outermost apparent horizon then this surface is also spherically symmetric.

Remark 3.2 The existence of outermost apparent horizons due to appropriate trapping is proven in [2,11]. Like minimal surfaces, they are shown to have a singular set which is no larger than codimension 7. Thus for $2 \leq n \leq 7$ the outermost apparent horizon is smooth.

Remark 3.3 The symmetry inheritance property for stable MOTS was already observed in [3, Theorem 8.1] (see also [29]) when $n = 3$. There a spacetime perspective was taken, as opposed to the initial data point of view used here.

Proof The following argument is a generalization of that in [9] for outermost minimal surfaces in axisymmetry. Suppose that the outermost apparent horizon Σ does not admit the stated symmetry. Then the Killing field η is not tangential to Σ at all points. Thus, if φ_t denotes the flow of this Killing field so that $\partial_t \varphi_t = \eta \circ \varphi_t$, then there is a nonzero t_0 near zero such that a domain within $\varphi_{t_0}(\Sigma)$ lies outside of Σ. Furthermore, observe that since (M^n, g, k) is invariant under the action of φ_t the surface $\varphi_{t_0}(\Sigma)$ is an apparent horizon of the same type.

Consider now the compact set $\mathcal{U}$ which is the union of all smooth compact embedded apparent horizons within M^n, and define the trapped region $\mathcal{T}$ to be the union of $\mathcal{U}$ with all the bounded components of $M^n \setminus \mathcal{U}$. As described in [4, Theorem 3.3] the outermost apparent horizon arises as the boundary $\partial\mathcal{T}$, moreover it is embedded and smooth away from a singular set of Hausdorff codimension at most 7; in fact it will be smooth by the assumptions of this lemma. Because $\partial\mathcal{T}$ must enclose both Σ and $\varphi_{t_0}(\Sigma)$, it cannot agree with Σ at all points. This, however, contradicts the outermost assumption for Σ. □

3.2 Monotonicity of Area

In this and the following subsection, the discussion is relevant to the spacetime Penrose inequality in spherical symmetry. The next result may be derived from the arguments in [26, Sect. 4].

Lemma 3.4 *Let (M^n, g, k) be a spherically symmetric asymptotically flat initial data set as in (1.14). Outside of the outermost apparent horizon, the area of symmetry spheres in (M^n, g) is a strictly increasing function of the radial coordinate. In particular, the warping function ρ defining g is also an increasing function.*

Proof Let S_r denote the level sets of r. Since

$$k = k_n g_{11} \mathrm{d}r^2 + k_t \rho^2 g_{S^{n-1}} \tag{3.1}$$

the null expansions (null mean curvatures) are given by

$$\theta_\pm = H \pm \mathrm{Tr}_{S_r} k = (n-1)\left(\sqrt{g^{11}}\frac{\partial_r \rho}{\rho} \pm k_t\right). \tag{3.2}$$

Since the null expansions are both positive near infinity, as the mean curvature H dominates $\mathrm{Tr}_{S_r} k$ according to decay rates, when moving inwards from infinity they must remain positive outside of the outermost apparent horizon where $\theta_+ = 0$ or $\theta_- = 0$. Note that here we are using Lemma 3.1 which asserts that the outermost apparent horizon (if present) is one of the spheres S_{r_0}. Therefore

$$\theta_\pm > 0 \quad \text{for} \quad r > r_0. \tag{3.3}$$

To finish the proof simply add the two null expansions, and observe that since both are positive we obtain

$$0 < \theta_+ + \theta_- = 4\sqrt{g^{11}}\frac{\partial_r \rho}{\rho} \quad \text{for} \quad r > r_0. \tag{3.4}$$

Hence, the warping function ρ is an increasing function. □

3.3 The Spacetime Penrose Inequality in All Dimensions

The spherically symmetric Penrose inequality without the maximal assumption was established in dimension $n = 3$ in [15], although the case of equality was not treated. A similar result is stated in [16] for all dimensions, but the hypotheses are too strong for our purposes and they also do not address the case of equality. The full result including the case of equality was given in [7] for $n = 3$, and here we easily extend it to all dimensions.

Theorem 3.5 *Let (M^n, g, k), $n \geq 3$ be an asymptotically flat spherically symmetric initial data set satisfying the dominant energy condition, and let $\mathcal{A}_0$ denote the area of the outermost apparent horizon. Then*

$$m \geq \frac{1}{2}\left(\frac{\mathcal{A}_0}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \tag{3.5}$$

and equality holds if and only if the initial data outside the outermost apparent horizon arise from an embedding into the Schwarzschild spacetime. In particular, for a sequence of initial data with $m_j \to 0$ we have $\mathcal{A}_j \to 0$.

Proof We will follow and generalize the arguments of [7] to higher dimensions. The proof is based on the generalized Jang equation introduced in [7]. The corresponding Jang deformation is similar to that of the original with the addition of an extra function ϕ that plays the role of warping factor for embeddings into a static spacetime, namely $\bar{g} = g + \phi^2 df^2$ where f satisfies Eq. (3) of [7] for a canonical choice of ϕ. The scalar curvature of the spherically symmetric generalized Jang metric

$$\bar{g} = \mathrm{d}\bar{s}^2 + \rho^2(\bar{s}) g_{S^{n-1}} \tag{3.6}$$

is given by

$$\overline{R} = (n-1)\rho^{-2}[(n-2)\left(1-\rho_{\bar{s}}^2\right) - 2\rho\rho_{\bar{s}\bar{s}}]. \tag{3.7}$$

Here $\bar{s}$ denotes radial distance from the boundary so that $\bar{s} = 0$ corresponds to the outermost apparent horizon. The fact that the outermost apparent horizon is a level set of $\bar{s}$ is a consequence of Lemma 3.1.

By comparing arbitrary spherically symmetric metrics to that of Schwarzschild we may derive and generalize the Hawking mass (Misner–Sharp mass in spherical symmetry [28]) to higher dimensions

$$\bar{m}(\bar{s}) := \frac{1}{2}\rho^{n-2}\left(1-\rho_{\bar{s}}^2\right) = \frac{1}{2}\left(\frac{A(\bar{s})}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}\left[1 - \frac{1}{(n-1)^2\omega_{n-1}^{\frac{2}{n-1}}A(\bar{s})^{\frac{n-3}{n-1}}}\int_{S_{\bar{s}}}\bar{H}^2\right], \tag{3.8}$$

where

$$A(\bar{s}) = \omega_{n-1}\rho^{n-1}(\bar{s}), \qquad \bar{H} = (n-1)\frac{\rho_{\bar{s}}}{\rho}. \tag{3.9}$$

A direct computation yields

$$2\bar{m}_{\bar{s}} = \frac{1}{n-1}\rho_{\bar{s}}\rho^{n-1}\bar{R}. \tag{3.10}$$

Therefore integrating produces

$$\bar{m}(\infty) - \bar{m}(0) = \int_0^\infty \frac{\rho_{\bar{s}}\rho^{n-1}}{2(n-1)}\bar{R}\mathrm{d}\bar{s} = \frac{1}{2(n-1)\omega_{n-1}}\int_{\bar{M}^n}\rho_{\bar{s}}\bar{R}\mathrm{d}V_{\bar{g}}. \tag{3.11}$$

We may now choose $\phi = \rho_{\bar{s}}$ and follow the arguments in [7, p. 750]. This allows one to integrate away the divergence term appearing in $\bar{R}$, leaving only nonnegative terms on the right-hand side of (3.11). It follows that $\bar{m}(\infty) \geq \bar{m}(0)$, and this gives the desired inequality

$$m \geq \frac{\rho^{n-2}(0)}{2} = \frac{1}{2}\left(\frac{\mathcal{A}_0}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \tag{3.12}$$

since the ADM mass of the Jang metric agrees with that of the given initial data in addition to the fact that the area of the apparent horizon agrees in both metrics as well. The case of equality follows directly from the arguments of [7]. □

3.4 Decay of the Second Fundamental Form

We now prove that under the definition of asymptotic flatness for spherical symmetry given in Sect. 1, the ADM linear momentum vanishes $|P| = 0$, and hence the ADM mass coincides with the ADM energy $m = E$.

Proposition 3.6 *Under the asymptotic decay conditions* (1.3) *and* (1.4)*, a spherically symmetric initial data set* (M^n, g, k) *satisfies the stronger decay*

$$|k|_g = O\left(\frac{1}{|x|^n}\right). \tag{3.13}$$

In particular, the ADM linear momentum vanishes $|P| = 0$ *and the ADM mass agrees with the ADM energy* $m = E$.

Proof Recall that the spherically symmetric initial data (M^n, g, k) may be expressed by

$$g = \mathrm{d}s^2 + r(s)^2 g_{S^{n-1}}, \qquad k_{ab} = n_a n_b k_n + (g_{ab} - n_a n_b)k_t, \tag{3.14}$$

where $n = \partial_s$. Consider the divergence constraint

$$J = \mathrm{div}_g\left(k - (\mathrm{Tr}_g k)g\right). \tag{3.15}$$

Observe that

$$\nabla_a n_b = \langle \nabla_a n, \partial_b \rangle = 2rr'\left(g_{S^{n-1}}\right)_{ab}, \tag{3.16}$$

and therefore

$$\nabla_a k_{ab} = 2rr'\left[\left(g_{S^{n-1}}\right)_{aa}(k_n - k_t)n_b + \left(g_{S^{n-1}}\right)_{ab}(k_n - k_t)n_a\right] + n_a n_b(\partial_a k_n - \partial_a k_t). \tag{3.17}$$

It follows that

$$\left(\mathrm{div}_g\, k\right)(\partial_b) = (k_n' - k_t')n_b + \frac{2(n-1)r'}{r}(k_n - k_t)n_b. \tag{3.18}$$

Furthermore

$$\mathrm{div}_g\left((\mathrm{Tr}_g k)g\right)(\partial_b) = \partial_b \mathrm{Tr}_g k = \partial_b k_n + (n-1)\partial_b k_t, \tag{3.19}$$

and hence

$$J(\partial_b) = (k_n' - k_t')n_b + \frac{2(n-1)r'}{r}(k_n - k_t)n_b - \partial_b k_n - (n-1)\partial_b k_t. \tag{3.20}$$

The only nonzero component is in the ∂_s direction, and from this we find that

$$k_t' + \frac{2(n-1)r'}{nr}k_t = \frac{1}{n}J(\partial_s) + \frac{2(n-1)r'}{nr}k_n. \tag{3.21}$$

Since

$$\mathrm{Tr}_g k = k_n + (n-1)k_t \tag{3.22}$$

this may be rewritten as

$$k_t' + \frac{2(n-1)r'}{r}k_t = \frac{1}{n}J(\partial_s) + \frac{2(n-1)r'}{nr}\mathrm{Tr}_g k =: \mathcal{K}. \tag{3.23}$$

A priori the assumed decay for k leads to $k_t = O(s^{1-n})$. However the ODE (3.23) shows that the fall-off is stronger. Indeed, from the dominant energy condition and the assumed decay

$$\begin{aligned} \mathrm{Tr}_g k &= O\left(\frac{1}{s^n}\right), \\ |J(\partial_s)|_g \leq |J|_g \leq \mu &= \frac{1}{16\pi}\left(R_g + (\mathrm{Tr}_g k)^2 - |k|_g^2\right) = O\left(\frac{1}{s^{n+1}}\right), \end{aligned} \tag{3.24}$$

and thus

$$k_t = \frac{1}{r^{2(n-1)}}\left[\int_{s_0}^{s} r^{2(n-1)}\mathcal{K}\mathrm{d}s + C\right] = O\left(\frac{1}{s^n}\right). \tag{3.25}$$

It then follows from the trace decay in (3.24) that k_n also satisfies the fall-off in (3.25), and hence

$$|k|_g = O\left(\frac{1}{s^n}\right), \tag{3.26}$$

which implies that the ADM linear momentum vanishes $|P| = 0$. □

4 Solving Jang's Equation to Obtain the Base Manifolds

4.1 Review of Jang's Equation Without Symmetry

Given an initial data set (M^n, g, k), the following quantities may be used to measure how far away it is from being realized as a graph $t = f(x)$ in Minkowski space

$$\bar{g} = g + \mathrm{d}f^2, \qquad \bar{k} = k - \frac{\nabla_g^2 f}{1 + |\nabla f|_g^2}. \tag{4.1}$$

In particular, such an embedding exists if and only if $\bar{g} = g_{\mathbb{E}}$ and $\bar{k} = 0$. A necessary condition for this to occur is the Jang equation [4]

$$\mathrm{Tr}_{\bar{g}}\bar{k} = 0 \quad \Leftrightarrow \quad \left(g^{ab} - \frac{f^a f^b}{1+|\nabla f|_g^2}\right)\left(\frac{\nabla_{ab} f}{\sqrt{1+|\nabla f|_g^2}} - k_{ab}\right) = 0, \tag{4.2}$$

where $f^a = g^{ab}\partial_b f$. Thus, one may think of Jang's equation as an attempt to find a candidate graph for an embedding into Minkowski space.

Even though such an embedding may not exist for the given initial data, we may use these ideas to construct an isometric embedding into a relevant static spacetime. Namely consider the map

$$F : (M^n, g) \to (\mathbb{R} \times M^n, -\mathrm{d}t^2 + \bar{g}) \tag{4.3}$$

defined by

$$F(x) = (f(x), x). \tag{4.4}$$

Then

$$F^*(-\mathrm{d}t^2 + \bar{g}) = -\mathrm{d}f^2 + \bar{g} = g, \tag{4.5}$$

and the second fundamental form is

$$h = \frac{\nabla_{\bar{g}}^2 f}{\sqrt{1-|\nabla f|_{\bar{g}}^2}} = \frac{\nabla_g^2 f}{\sqrt{1+|\nabla f|_g^2}}. \tag{4.6}$$

Another motivation for the Jang equation which is pertinent to the positive mass theorem, is to consider it as a method for deforming initial data to obtain weakly non-negative scalar curvature. In this setting one may view the Jang graph as a submanifold of the $(n+1)$-dimensional dual Riemannian manifold $(\mathbb{R} \times M^n, \mathrm{d}t^2 + g)$. The Jang metric $\bar{g}$ is then the induced metric on the graph and h is again the second fundamental form. If k is extended trivially off the $t = 0$ slice to the whole $(n+1)$-dimensional ambient space, then the Jang equation simply states that the Jang surface $\bar{M}^n$ satisfies the apparent horizon equation

$$H_{\bar{M}^n} - \mathrm{Tr}_{\bar{M}^n} k = 0, \tag{4.7}$$

where $H_{\bar{M}^n}$ is the mean curvature of the Jang surface. A computation [34] then shows that the scalar curvature of the Jang metric is nonnegative modulo a divergence term whenever the dominant energy condition is satisfied, that is

$$R_{\bar{g}} = 16\pi(\mu - J(w)) + |h-k|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2 - 2\mathrm{div}_{\bar{g}} q, \tag{4.8}$$

where

$$w^a = \frac{f^a}{\sqrt{1+|\nabla f|_g^2}}, \qquad q_a = \frac{f^b}{\sqrt{1+|\nabla f|_g^2}}(h_{ab} - k_{ab}). \tag{4.9}$$

This positivity property for $R_{\bar{g}}$ allows one to conformally transform $\bar{g}$ to zero scalar curvature. Hence through the Jang deformation combined with a conformal transformation, the initial data is taken into the time-symmetric setting.

4.2 Existence of Solutions to Jang's Equation

The Jang deformation preserves uniform asymptotic flatness as will be shown below, and preserves the mass so that $\bar{m} = m$. An interesting feature of the Jang equation's existence theory is its ability to detect apparent horizons. That is, it can only blow-up at apparent horizons in which case it approximates a cylinder over these surfaces [34]. In dimension $n = 3$ it has been shown that this cylindrical blow-up behavior can in fact be prescribed at the outermost apparent horizon [12,14,27]. Suppose that the boundary is decomposed into a disjoint union of future (+) and past (−) apparent horizon components $\partial M^3 = \partial_+ M^3 \cup \partial_- M^3$, and that there are no other apparent horizons present. If in a neighborhood of $\partial_\pm M^3$ there are constants $l \geq 1$ and $c > 0$ such that

$$c^{-1}\tau^l \leq \theta_\pm(S_\tau) \leq c\tau^l, \tag{4.10}$$

where $\tau(x) = \text{dist}(x, \partial M^3)$ and S_τ are surfaces of constant distance to the boundary, then there exists a smooth solution f of Jang's equation with the property that $f(x) \to \pm\infty$ as $x \to \partial_\pm M^3$. Furthermore, the asymptotics for this blow-up are given by

$$\begin{aligned} c_1^{-1}\tau^{-\frac{l-1}{2}} + c_2^{-1} \leq \pm f \leq c_1\tau^{-\frac{l-1}{2}} + c_2 \qquad &\text{if} \qquad l > 1, \\ -c_1^{-1}\log\tau + c_2^{-1} \leq \pm f \leq -c_1\log\tau + c_2 \qquad &\text{if} \qquad l = 1, \end{aligned} \tag{4.11}$$

for some positive constants c_1, c_2. In the spherically symmetric case, this type of existence result may be established in all dimensions (Fig. 7).

Recall the form of the spherically symmetric initial data

$$g = g_{11}(r)\mathrm{d}r^2 + \rho^2(r)g_{S^{n-1}}, \quad k_{ab} = n_a n_b k_n + (g_{ab} - n_a n_b)k_t, \tag{4.12}$$

defined on the compliment of a ball $M^n = \mathbb{R}^n \setminus B_0(r_0)$. It is assumed that $\partial M^n = S_{r_0}$ is the only apparent horizon, which means that the null expansions satisfy

$$\theta_\pm(r) = (n-1)\left(\sqrt{g^{11}}\frac{\rho_r}{\rho} \pm k_t\right) > 0, \quad r > r_0, \tag{4.13}$$

and that either $\theta_+(r_0) = 0$, $\theta_-(r_0) = 0$, or $\theta_+(r_0) = \theta_-(r_0) = 0$ depending on whether S_{r_0} is a future horizon, past horizon, or both, respectively. As observed in

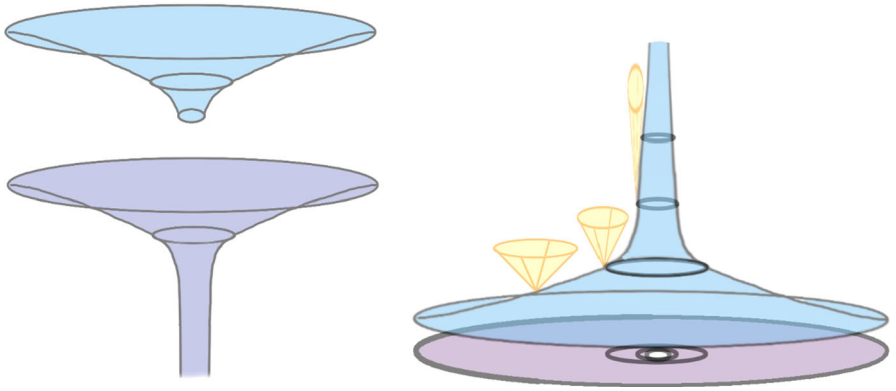

Fig. 7 On the left we see the Riemannian Schwarzschild manifold (in blue), and its Jang perturbation (in purple). On the right we see the Jang perturbation viewed as the base of a static spacetime. Due to the asymptotically cylindrical end, the light cones of this spacetime are increasingly narrow as we move towards the central cylinder. Above the base we see the embedding of the Riemannian Schwarzschild space in blue. The graph of the Jang map has an asymptote at the central cylinder, but it is becoming increasingly null so that the Schwarzschild manifold in blue does not have a cylindrical inner end (Color figure online)

[25] the Jang equation in spherical symmetry may be reduced to a first order ODE by setting

$$v = \frac{\sqrt{g^{11}} f_r}{\sqrt{1 + g^{11} f_r^2}}. \tag{4.14}$$

The equation (4.2) then becomes

$$\sqrt{g^{11}} v_r + (n-1)\left(\sqrt{g^{11}} \frac{\rho_r}{\rho} v - k_t\right) + (v^2 - 1) k_n = 0. \tag{4.15}$$

Observe that $|v| \leq 1$ and blow-up occurs precisely when $v = \pm 1$. A maximum principle type argument shows that the outermost horizon condition (4.13) ensures that $|v| < 1$ away from S_{r_0}. Building upon this estimate, existence and uniqueness for the spherically symmetric Jang equation may be established following the arguments of [7, Theorem 2]; this prior results was stated for dimension three but the proof carries over to higher dimensions. The result may be stated as follows, under the hypothesis that the initial data satisfy the following fall-off conditions in the asymptotic end

$$\begin{aligned} |k|_g &= O_1(r^{1-n}), & \mathrm{Tr}_g k &= O_1(r^{-n}), \\ g_{11} - 1 &= O_1(r^{2-n}), & \rho - r &= O_2(1). \end{aligned} \tag{4.16}$$

Theorem 4.1 *Assume that the initial data set is spherically symmetric, smooth, either complete or with outermost apparent horizon boundary, and satisfies the asymptotics* (4.16). *Then there exists a unique solution* $v \in C^\infty((r_0, \infty)) \cap C^1([r_0, \infty))$ *of* (4.15) *(the spherically symmetric Jang equation) such that* $-1 < v(r) < 1, r > r_0$, *with* $v(r_0)$

taking the value 0 *or* ± 1 *depending on whether* $r_0 = 0$ *and the manifold is complete, or* S_{r_0} *is a past (future) horizon, respectively. Furthermore, in the asymptotic end the decay is of the form*

$$v = O_2(r^{1-n}), \qquad as \qquad r \to \infty. \tag{4.17}$$

In all cases this gives rise to a spherically symmetric solution $f \in C^\infty(M^n)$ *of the Jang equation* (4.2) *satisfying*

$$f = O_3(r^{2-n}), \qquad as \qquad r \to \infty. \tag{4.18}$$

Corollary 4.2 *Under the hypotheses of Theorem* 4.1 *the Jang deformed initial data* $(\bar{M}^n, \bar{g})$ *is smooth, spherically symmetric, and asymptotically flat with the same mass* $\bar{m} = m$. *If* (M^n, g, k) *has a boundary then the Jang deformation, in addition, has an asymptotically cylindrical end satisfying the asymptotics* (4.11).

4.3 Uniform Asymptotics for the Solution of Jang's Equation

The asymptotic fall-off for solutions of Jang's equation given in the previous theorem depend on spherical symmetry and are not necessarily uniform. By allowing for a slightly weaker fall-off we may obtain uniform fall-off in the general case independent of any symmetry.

Lemma 4.3 *Let* (M_j^n, g_j, k_j) *be a sequence of uniformly asymptotically flat initial data, and let* $f_j \in C^\infty(M^n)$ *be corresponding solutions of Jang's equation with* $f_j(x) \to 0$ *as* $|x| \to \infty$ *where* $x \in \mathbb{R}^n \setminus B_{\rho_A}$ *are coordinates given by the asymptotic diffeomorphisms. Then for any small* $\varsigma > 0$ *there exist uniform constants* $\mathcal{C}$ *and* $\bar{r}$, *depending only on* ς, *such that*

$$|\partial^\beta f(x)| \leq \frac{\mathcal{C}}{|x|^{n-2-\varsigma+|\beta|}} \qquad for \qquad |x| \geq \bar{r}, \tag{4.19}$$

where $\bar{r} > \rho_A$. *In particular, the sequence of Jang deformations* $(\bar{M}_j^n, \bar{g}_j)$ *is uniformly asymptotically flat.*

Proof We shall adapt to our purposes an argument of Schoen-Yau which can be found in [34, pp. 248–249] (see also [12, Proposition 4]). Let $r(x) = |x|$ and for $0 < p < n-2$, $\lambda > 0$, $r > \lambda^{\frac{1}{p+1}}$ define the radial function

$$\bar{f}(r) = \lambda \int_r^\infty \left(s^{2p+2} - \lambda^2\right)^{-\frac{1}{2}} \mathrm{d}s. \tag{4.20}$$

Observe that there is a constant $c_1 = c_1(p)$ such that

$$0 < \bar{f}(r) \leq c_1 \lambda r^{-p}, \tag{4.21}$$

and

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}r} \to -\infty \qquad \text{as} \qquad r \to \lambda^{\frac{1}{p+1}}. \tag{4.22}$$

A computation shows that the Jang operator evaluated at this radial function yields

$$\begin{aligned}\left(g^{ab} - \frac{\bar{f}^a \bar{f}^b}{1+|\nabla \bar{f}|_g^2}\right)\left(\frac{\nabla_{ab}\bar{f}}{\sqrt{1+|\nabla \bar{f}|_g^2}} - k_{ab}\right)\\ \leq -\lambda(n-2-p)r^{-p-2} + c_2\left(r^{-n} + \lambda r^{-p-n}\right),\end{aligned} \tag{4.23}$$

where c_2 is a uniform constant arising from the uniformly asymptotically flat condition. We may then choose a uniform λ large enough to ensure that the right-hand side of (4.23) is nonpositive for $r > \lambda^{\frac{1}{p+1}}$, and thus $\bar{f}$ is a super-solution. Similarly, $-\bar{f}$ is a sub-solution on this domain. Since f and $\bar{f}$ both vanish at spatial infinity, and the derivative (4.22) is infinity, a maximum principle argument guarantees that $-\bar{f} \leq f \leq \bar{f}$. Therefore

$$|f(x)| \leq c_1 \lambda |x|^{-p} \qquad \text{for} \qquad |x| \geq \lambda^{\frac{1}{1+p}}. \tag{4.24}$$

From this, higher order fall-off follows by rescaling combined with the Schauder estimates as in Proposition 3 of [34]. Lastly, we may set $\bar{r} = \lambda^{\frac{1}{p+1}}$ to obtain the statement of this lemma. □

5 The Conformal Transformations

In this section we construct the conformal transformations and control the conformal factor as described in the introduction.

5.1 Review of Conformal Change Without Symmetry

In the previous section we have obtained, from the given initial data (M^n, g, k), a Jang deformation $(\bar{M}^n, \bar{g})$ which is complete, asymptotically flat, and with an additional asymptotically cylindrical end if the original data possessed a boundary. The positivity property (4.8) for the scalar curvature of the Jang metric leads to a stability-type inequality via integration by parts combined with Cauchy–Schwarz

$$\begin{aligned}&\int_{\bar{M}^n} \left(c_n^{-1}|\nabla\phi|_{\bar{g}}^2 + R_{\bar{g}}\phi^2\right)\mathrm{d}V_{\bar{g}}\\ &\quad\geq \int_{\bar{M}^n}\left(16\pi(\mu - J(w)) + |h-k|_{\bar{g}}^2 + |q|_{\bar{g}}^2\right)\phi^2 \mathrm{d}V_{\bar{g}}\end{aligned} \tag{5.1}$$

for all $\phi \in C_c^\infty(\bar{M}^n)$, where $c_n = \frac{n-2}{4(n-1)}$. The left-hand side arises from the basic quadratic form associated with the conformal Laplacian $L_{\bar{g}} = \Delta_{\bar{g}} - c_n R_{\bar{g}}$, and asserts that on compact subsets this operator has nonnegative spectrum (for the Dirichlet problem). In fact the spectrum is strictly positive, since if the principal eigenvalue is zero each term on the right-hand side of (5.1) would vanish, implying that $R_{\bar{g}} = 0$ and the principal eigenfunction is harmonic, which is impossible. Thus, a standard exhaustion argument together with asymptotic analysis [12,34] shows that there is a positive solution $u > 0$ of the zero scalar curvature equation

$$\begin{aligned} &L_{\bar{g}} u = 0 \qquad \text{on} \qquad \bar{M}^n, \\ &u(x) = 1 + \frac{\alpha}{|x|^{n-2}} + O_2\left(\frac{1}{|x|^{n-1}}\right) \qquad \text{as} \qquad |x| \to \infty, \end{aligned} \tag{5.2}$$

for some constant α. This allows a conformal transformation $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$ to zero scalar curvature in which the relation between the masses is given by $\tilde{m} = \bar{m} + 2\alpha$ [34, p. 259].

Moreover in the case that $(\bar{M}^n, \bar{g})$ possesses an additional cylindrical end, the solution u tends to zero in the limit along that end. In fact the decay along the cylindrical end is exponentially fast $u \sim e^{-\gamma s}$, where s is an arclength parameter along the cylindrical end and γ is the principal eigenvalue of $\Delta_{\partial M^n} - c_n R_{\partial M^n}$. In spherical symmetry additional assumptions are not required to obtain $\gamma > 0$ since the scalar curvature of the outermost apparent horizon $R_{\partial M^n}$ is positive, although in the general case a sufficient condition is for the dominant energy condition to be strict near the horizon as was used in [34]. The next proposition records these observations.

Proposition 5.1 *Given a smooth Jang deformation $(\bar{M}^n, \bar{g})$ there exists a positive solution $u \in C^\infty(\bar{M}^n)$ of (5.2), so that the conformal metric $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$ has zero scalar curvature $R_{\tilde{g}} = 0$ and $(\tilde{M}^n, \tilde{g})$ is asymptotically flat with mass*

$$\tilde{m} = \bar{m} + 2\alpha. \tag{5.3}$$

If an asymptotically cylindrical end is present in the Jang deformation, then the conformal factor is asymptotic to $e^{-\gamma s}$ along this end with an arclength parameter s. Furthermore, if the initial data are spherically symmetric then the function u and hence metric $\tilde{g}$ are also spherically symmetric.

Remark 5.2 Without any symmetry in dimension $n = 3$, it follows from a slightly generalized positive mass inequality [34] that $\tilde{m} \geq 0$, and in addition that $\alpha \leq 0$ (see (5.8) below). Thus if $m = \bar{m} \to 0$ then $\tilde{m} \to 0$. The almost rigidity conjecture in the time-symmetric setting then suggests, modulo horizon issues, that $(\tilde{M}^3, \tilde{g})$ is close in the intrinsic flat sense to Euclidean space.

5.2 Spherically Symmetry Gives $\tilde{g} = g_E$

We point out that in the spherically symmetric case the existence of a conformal transformation to zero scalar curvature may be obtained from the alternate observation

that all spherically symmetric metrics are conformally flat. This is related to a rigidity phenomena associated to zero scalar curvature in spherical symmetry. The following result, which is similar to Birkhoff's Theorem [39] in general relativity, is well-known although the authors do not know of a proper reference in the literature and thus include it here. It should be noted that a proof may be obtained from the arguments of [23], although here we give another approach.

Lemma 5.3 *Let (M^n, g), $n \geq 3$ be spherically symmetric, complete, scalar flat, and asymptotically flat. Then either it is isometric to flat Euclidean space $(\mathbb{R}^n, \delta)$ or the constant time slice of a Schwarzschild spacetime, that is $M^n \cong \mathbb{R}^n \setminus \{0\}$ and*

$$g = \left(1 + \tfrac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta. \tag{5.4}$$

Proof This conclusion may be derived from the observation that the Hawking mass of radial spheres is constant under the assumption of zero scalar curvature. The inverse mean curvature flow proof of the Penrose inequality [19] then guarantees the desired result where the parameter m is the value of the constant Hawking mass.

An alternative proof is to directly compute the scalar curvature of the spherically symmetric metric in polar form as in (3.7), and analyze the ODE as is done in [31, p. 70]. It is found that $\rho' = 1 + c\rho^{2-n}$ for some constant $c < 0$, the area radius function $\rho(r) > 0$ for all $r \in (-\infty, \infty)$, and it has a unique minimum (corresponding to a minimal surface) at $\rho = |c|^{\frac{1}{n-2}}$. It follows that ρ may be treated as a radial coordinate and so

$$g = \mathrm{d}r^2 + \rho^2(r) g_{S^{n-1}} = \left(1 + \tfrac{c}{\rho^{n-2}}\right)^{-1} \mathrm{d}\rho^2 + \rho^2 g_{S^{n-1}} \qquad \text{for} \qquad \rho \geq |c|^{\frac{1}{n-2}}. \tag{5.5}$$

Since $\rho(r)$ has a reflection symmetry across the minimal surface, this may be doubled to obtain the stated conclusion. □

This rigidity result suggests that the conformal transformation obtained in Proposition 5.1, in the case of spherical symmetry, gives rise to Euclidean space as we will now see.

Corollary 5.4 *Let (M^n, g, k) be a spherically symmetric, asymptotically flat initial data set satisfying the dominant energy condition which is either complete or has an outermost apparent horizon boundary. Then the conformally transformed Jang deformation of Proposition 5.1 is isometric to Euclidean space $(\tilde{M}^n, \tilde{g}) \cong (\mathbb{R}^n, g_{\mathbb{E}})$.*

Proof We only treat the case with boundary, as the case without boundary is similar. Let τ be the radial distance function from the boundary for (M^n, g). With the help of (4.11) the Jang metric takes the form

$$\bar{g} = (1 + f_\tau^2)\mathrm{d}\tau^2 + \rho^2 g_{S^{n-1}} = \left(1 + c\tau^{-l-1} + O(\tau^{-l})\right) \mathrm{d}\tau^2 + \rho^2 g_{S^{n-1}}. \tag{5.6}$$

Set $s = \frac{2\sqrt{c}}{l}\tau^{-l/2}$ and use the expansion $\rho = \rho(0) + O(\tau)$ to obtain

$$\bar{g} = \left(1 + O(s^{-2-\frac{2}{l}})\right) \mathrm{d}s^2 + \left(\rho(0)^2 + O(s^{-\frac{2}{l}})\right) g_{S^{n-1}}, \tag{5.7}$$

which illustrates the cylindrical asymptotics. Since $u \sim e^{-\gamma s}$, a straightforward computation shows that the Hawking mass (3.8) of the radial spheres $m(s) \to 0$ as $s \to \infty$. Moreover, since $R_{\bar{g}} = 0$, as in the proof of Lemma 5.3 the Hawking mass of these spheres must be constant, and hence zero. The desired conclusion now follows. □

Remark 5.5 The results of this subsection rely heavily on the spherically symmetric assumption. Without this hypothesis, the conformally changed manifold is only known to be scalar flat rather than isometric to Euclidean space. Any progress in showing that this manifold is close to Euclidean space will most likely be obtained in conjunction with advances toward the Riemannian version of the stability conjecture.

5.3 Controlling the Conformal Factor *u*

For the remainder of this section we will examine how the mass may be used to control the conformal factor u and the Jang deformation. Since $\tilde{g}$ is flat its mass vanishes $\tilde{m} = 0$, and therefore the formula of Proposition 5.1 relating the masses of each deformation yields $-2\alpha = m$; where we have also used that the Jang transformation preserves mass. Furthermore, multiplying equation (5.2) through by u and integrating by parts, and using the divergence structure present in $R_{\bar{g}}$ yields

$$\begin{aligned} &-c_n^{-1}(n-2)\omega_{n-1}\underbrace{\alpha}_{-\frac{m}{2}} \\ &\geq \int_{\tilde{M}^n}\left[\frac{4}{n-2}|\nabla u|_{\bar{g}}^2 + \left(16\pi(\mu - J(w)) + |h-k|_{\bar{g}}^2 + |q|_{\bar{g}}^2\right)u^2\right]\mathrm{d}V_{\bar{g}}. \end{aligned} \tag{5.8}$$

It follows that L^2 gradient bounds for the conformal factor are given in terms of the mass.

Lemma 5.6 *Assume that the given spherically symmetric initial data (M^n, g, k) satisfies the dominant energy condition. If u is the solution of (5.2) given by Proposition 5.1, then*

$$\| \nabla u \|_{L^2(\tilde{M}^n, \bar{g})}^2 \leq \frac{(n-2)^2\omega_{n-1}}{8c_n} m. \tag{5.9}$$

Remark 5.7 We remark that a version of Lemma 5.6 is likely to hold without the assumption of spherical symmetry, when a smooth Jang deformation exists. Namely, if the positive mass theorem is valid for $(\tilde{M}^n, \tilde{g})$ then $-2\alpha \leq m$, and (5.9) again follows from (5.8). The issue is that the n-dimensional Riemannian positive mass theorem [33,35] is not immediately applicable, as $(\tilde{M}^n, \tilde{g})$ may not be a smooth manifold,

geometrically or topologically. The later pathology arises from the fact that in higher dimensions, although the outermost apparent horizon is of positive Yamabe invariant, it may not be of spherical topology. Nonetheless, Eichmair [12] has found an effective way to deal with these concerns.

Next observe that the L^2 gradient bound for u on the Jang surface may be translated into a similar bound for $\log u$ on Euclidean space. This follows from Corollary 5.4 since $g_{\mathbb{E}} = u^{-\frac{4}{n-2}}\bar{g}$, and in particular

$$|\nabla u|^2_{\bar{g}} = u^{\frac{4}{n-2}}|\nabla u|^2_{g_{\mathbb{E}}}, \qquad \mathrm{d}V_{\bar{g}} = u^{-\frac{2n}{n-2}}\mathrm{d}V_{g_{\mathbb{E}}}. \tag{5.10}$$

Corollary 5.8 *Under the hypotheses of Lemma 5.6*

$$\| \nabla \log u \|^2_{L^2(\mathbb{R}^n, g_{\mathbb{E}})} \leq \frac{(n-2)^2\omega_{n-1}}{8c_n} m. \tag{5.11}$$

The global Sobolev bounds for the conformal factor obtained from the stability inequality can be parlayed into C^0 and even Hölder estimates away from the central fixed point of the spherical symmetry. To accomplish we will first need to obtain uniform control for u in the asymptotically flat end.

Lemma 5.9 *Let (M^n_j, g_j, k_j) be a sequence of spherically symmetric, uniformly asymptotically flat initial data satisfying the dominant energy condition and with either outermost apparent horizon boundary or no boundary. Let $(\bar{M}^n_j, \bar{g}_j)$ be the corresponding Jang deformations conformally related to $(\tilde{M}^n_j, \tilde{g}_j)$ via conformal factors u_j solving (5.2). Then there exist uniform constants c and $\bar{r}$ such that*

$$u_j(r) \leq \exp cr^{-2(n-2)} \qquad \text{for} \qquad r \geq \bar{r}, \tag{5.12}$$

where $r = |x|$ is the radial coordinate from (1.3) and $\bar{r}$ is as in Proposition 4.3.

Proof For convenience, within the proof the subscript j will be suppressed. Coordinate spheres in the asymptotic region will be denoted by S_r; note that they are distinct from the surfaces Σ_A used in other parts of the manuscript. Let $\bar{M}^n_r$ denote the component of $\bar{M}^n$ lying inside S_r. Multiply Eq. (5.2) through by u and integrate by parts up to a coordinate sphere S_r, $r \geq \bar{r}$, and use the divergence structure present in $R_{\bar{g}}$ as in (5.8) to find

$$\begin{aligned}
0 &= -\int_{\bar{M}^n_r} u\left(c_n^{-1}\Delta_{\bar{g}}u - R_{\bar{g}}u\right) dV_{\bar{g}} \\
&= \int_{\bar{M}^n_r}\left(c_n^{-1}|\nabla u|^2_{\bar{g}} + R_{\bar{g}}u^2\right)\mathrm{d}V_{\bar{g}} - \int_{S_r} c_n^{-1}u\partial_{\bar{\nu}}u\mathrm{d}A_{\bar{g}} \\
&\geq \int_{\bar{M}^n_r}\left(c_n^{-1}|\nabla u|^2_{\bar{g}} + 2u^2|q|^2_{\bar{g}} - 2u^2\mathrm{div}_{\bar{g}}q\right)\mathrm{d}V_{\bar{g}} - \int_{S_r} c_n^{-1}u\partial_{\bar{\nu}}u\mathrm{d}A_{\bar{g}} \\
&\geq -\int_{S_r}\left(\tfrac{1}{2c_n}\partial_{\bar{\nu}}u^2 + 2q(\bar{\nu})u^2\right)\mathrm{d}A_{\bar{g}},
\end{aligned} \tag{5.13}$$

where $\bar{\nu}$ is the unit outer normal with respect to $\bar{g}$. Since all quantities are spherically symmetric and r is arbitrary, we obtain the differential inequality

$$\partial_r \log u \geq -2c_n q(\partial_r) \qquad \text{for} \qquad r \geq \bar{r}. \tag{5.14}$$

According to the uniform fall-off (4.16) and (4.19), $q(\partial_r)$ may be estimated to yield

$$\partial_r \log u \geq -\bar{c} r^{-2n+1+2\varsigma} \tag{5.15}$$

where $\bar{c} > 0$ is a uniform constant. Integrating on the interval $[r, r_1]$ produces

$$\log u(r_1) - \log u(r) \geq \frac{\bar{c}}{2n-2-2\varsigma} \left(r_1^{-2n+2+2\varsigma} - r^{-2n+2+2\sigma} \right). \tag{5.16}$$

Now let $r_1 \to \infty$ and use that $u(r_1) \to 1$ to obtain

$$\log u(r) \leq c r^{-2(n-2)}, \tag{5.17}$$

with $c = \bar{c}(2n-2-2\varsigma)^{-1}$. □

Let $\tilde{r}$ denote the radial distance function for $g_{\mathbb{E}} = \tilde{g}$, so that

$$g_{\mathbb{E}} = \mathrm{d}\tilde{r}^2 + \tilde{r}^2 g_{S^{n-1}}. \tag{5.18}$$

Consider annular domains $\Omega_{\tilde{A}_1, \tilde{A}_2} \subset \mathbb{E}^n$ whose boundary consists of two coordinate spheres having areas $\tilde{A}_1 < \tilde{A}_2$. The next result shows that the conformal factors defining $\tilde{g}$ are uniformly close to 1 in Hölder space away from the center of the spherical symmetry. Note that in light of Example 2.4 we see that it is not possible to have such C^0 control near the center.

Proposition 5.10 *Let (M_j^n, g_j, k_j) be a sequence of spherically symmetric, uniformly asymptotically flat initial data satisfying the dominant energy condition and with either outermost apparent horizon boundary or no boundary. Let $(\bar{M}_j^n, \bar{g}_j)$ be the corresponding Jang deformations conformally related to $(\tilde{M}_j^n, \tilde{g}_j)$ via conformal factors u_j solving (5.2). Let $\tilde{A}_0 > 0$ be fixed, assume $m_j < \tilde{A}_0^{-2}$ is uniformly small, and set $\tilde{A}_j = \frac{1}{\sqrt{m_j}}$. Then there exists a uniform constant C such that*

$$\| \log u_j \|_{C^{0,\frac{1}{2}}\left(\Omega_{\tilde{A}_0, \tilde{A}_j}, g_{\mathbb{E}}\right)} \leq \frac{C m_j^{\frac{1}{4}}}{\sqrt{\tilde{A}_0}}. \tag{5.19}$$

Proof For convenience the subscript j will be suppressed in the proof. Estimate (5.11) shows that for any $\tilde{r}_1 < \tilde{r}_2$ we have

$$\tilde{r}_1^{n-1} \int_{\tilde{r}_1}^{\tilde{r}_2} (\partial_{\tilde{r}} \log u)^2 \, \mathrm{d}\tilde{r} \leq \tilde{c}_n m, \tag{5.20}$$

where $\tilde{c}_n = (n-2)^2 c_n^{-1}/8$. With the help of Hölder's inequality it follows that

$$\begin{aligned} |\log u(\tilde{r}_2) - \log u(\tilde{r}_1)| &= \left| \int_{\tilde{r}_1}^{\tilde{r}_2} \partial_{\tilde{r}} \log u \, \mathrm{d}\tilde{r} \right| \\ &\leq |\tilde{r}_2 - \tilde{r}_1|^{\frac{1}{2}} \left(\int_{\tilde{r}_1}^{\tilde{r}_2} (\partial_{\tilde{r}} \log u)^2 \, \mathrm{d}\tilde{r} \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\tilde{c}_n m}{\tilde{r}_1^{n-1}} |\tilde{r}_2 - \tilde{r}_1| \right)^{\frac{1}{2}}. \end{aligned} \tag{5.21}$$

If

$$\tilde{r}_0 = \left(\frac{\tilde{A}_0}{\omega_{n-1}} \right)^{\frac{1}{n-1}} \tag{5.22}$$

denotes the area radius for the inner boundary of $\Omega_{\tilde{A}_0, \tilde{A}_j}$ then

$$\frac{|\log u(\tilde{r}_2) - \log u(\tilde{r}_1)|}{|\tilde{r}_2 - \tilde{r}_1|^{\frac{1}{2}}} \leq \sqrt{\frac{\tilde{c}_n m}{\tilde{r}_0^{n-1}}} = \sqrt{\frac{\tilde{c}_n \omega_{n-1} m}{\tilde{A}_0}} \tag{5.23}$$

for $\tilde{r}_1 \geq \tilde{r}_0$, which yields one half of the desired Hölder estimate.

The next goal is to obtain C^0 bounds. Note that (5.21) implies

$$|\log u(\tilde{r}_1)| \leq |\log u(\tilde{r}_2)| + \sqrt{\frac{\tilde{c}_n m \tilde{r}_2}{\tilde{r}_0^{n-1}}}. \tag{5.24}$$

In order to control $u(\tilde{r}_2)$ uniformly we will utilize Lemma 5.9. The estimate there, however, is given in terms of the radial coordinate r associated with the uniform asymptotic coordinates of $\bar{g}$. The two coordinates r and $\tilde{r}$ may be compared in the asymptotic end by relating the volumes of coordinate spheres. Let S_r denote a coordinate sphere of radius r, then uniform asymptotic flatness shows that its volume with respect to $\bar{g}$ satisfies

$$|S_r|_{\bar{g}} \leq \omega_{n-1} r^{n-1} + c_1 r^{n-2} \qquad \text{for} \qquad r \geq \bar{r}, \tag{5.25}$$

for some uniform constant c_1 (uniform constants will be denoted by c_i, $i = 1, 2, \ldots$). With the help of (5.12), the volume of this same sphere computed with respect to

$g_{\mathbb{E}} = u^{\frac{4}{n-2}}\bar{g}$ may be estimated by

$$\begin{aligned} |S_r|_{g_{\mathbb{E}}} &= u^{\frac{2(n-1)}{n-2}}|S_r|_{\bar{g}} \\ &\leq e^{\frac{2c(n-1)}{n-2}r^{-2(n-2)}}\left(\omega_{n-1}r^{n-1} + c_1 r^{n-2}\right) \\ &\leq \omega_{n-1}r^{n-1} + c_2 r^{n-2}. \end{aligned} \tag{5.26}$$

On the other hand

$$|S_r|_{g_{\mathbb{E}}} = \omega_{n-1}\tilde{r}^{n-1}, \tag{5.27}$$

and hence

$$\tilde{r} \leq r + c_3 \qquad \Rightarrow \qquad \frac{1}{r} \leq \frac{1}{\tilde{r}} + \frac{c_4}{\tilde{r}^2}. \tag{5.28}$$

We may now combine (5.12), (5.24), and (5.28) to find for large $\tilde{r}_2$ that

$$\begin{aligned} |\log u(\tilde{r}_1)| &\leq \frac{c}{r(\tilde{r}_2)^{2(n-2)}} + \sqrt{\frac{\tilde{c}_n m \tilde{r}_2}{\tilde{r}_0^{n-1}}} \\ &\leq \frac{c_5}{\tilde{r}_2^{2(n-2)}} + \sqrt{\frac{\tilde{c}_n m \tilde{r}_2}{\tilde{r}_0^{n-1}}}. \end{aligned} \tag{5.29}$$

By choosing $\tilde{r}_2 = m^{-\frac{1}{2}}$ the desired result follows. □

6 Intrinsic Flat Convergence of the Base Manifolds

In this section we prove the following proposition which implies (1.22) of Theorem 1.2. Recall that

$$\Omega_A^j = \left\{x \in M_j^n \mid \rho_j(x) \leq \rho_A\right\} \tag{6.1}$$

is the region within the level set Σ_A^j of area A with respect to g_j and $\bar{g}_j$. A priori we do not know the area of the level set Σ_A^j with respect to the metric $\tilde{g}_j = u_j^{\frac{4}{n-2}}\bar{g}_j = g_{\mathbb{E}}$. This section strongly uses spherical symmetry and the fact that the metrics $\bar{g}$ and g are monotone as was proven in Lemma 3.4.

Proposition 6.1 *Given any $A > 0$, $D > 0$, and $\epsilon > 0$ there exists $\delta = \delta(A, D, \epsilon) > 0$ such that if mass $m_j < \delta$ then*

$$d_{\mathcal{VF}}\left(\left(\Omega_j, \bar{g}_j\right), \left(\Omega'_j, g_{\mathbb{E}}\right)\right) < \epsilon, \tag{6.2}$$

where

$$\Omega_j = \Omega_A^j \cap T_D(\Sigma_A^j) \subset \bar{M}_j^n \text{ with metric tensor } \bar{g}_j \tag{6.3}$$

and

$$\Omega_j' = \Omega_A^j \subset \mathbb{E}^n \text{ with metric tensor } \tilde{g}_j = u_j^{\frac{4}{n-2}} \bar{g}_j = g_{\mathbb{E}}. \tag{6.4}$$

Furthermore for fixed $A > 0$ and $D > 0$, if $m_j \to 0$ then

$$(\Omega_j', g_{\mathbb{E}}) \to (B_0(\rho_A), g_{\mathbb{E}}) \text{ smoothly as } j \to \infty. \tag{6.5}$$

To prove the volume preserving intrinsic flat convergence statement (6.2), we will show they have diffeomorphic subregions W_j and W_j' where the metric tensors are C^0 close and the volumes not covered by these subregions are small. This will be stated more precisely below. Based on the examples of Sect. 2, we cannot expect the region near the asymptotically cylindrical end of $(\Omega_j, \bar{g}_j)$ to be C^0 close to $(\Omega_j', g_{\mathbb{E}})$. We will therefore choose an

$$A_\epsilon = A_\epsilon(A, D) > 0, \tag{6.6}$$

and then cut out the annular domain $\Omega^j_{A_\epsilon, A}$ from inside Ω_j and Ω_j' to create regions where we will have the appropriate control. Set

$$W_j = \Omega^j_{A_\epsilon, A} \subset \Omega_A^j \cap T_D\left(\Sigma_A^j\right) \subset \bar{M}_j^n, \tag{6.7}$$

and

$$W_j' = \Omega^j_{A_\epsilon, A} \subset \mathbb{E}^n. \tag{6.8}$$

The precise choice of A_ϵ will be made later in Sect. 6.5.

6.1 Estimates on W_j

To begin the proof of Proposition 6.1 we apply the results of the previous section to establish uniform closeness between $\bar{g}_j$ and $g_{\mathbb{E}}$ on diffeomorphic domains. This is a primary step towards proving the intrinsic flat distance estimate (6.2). Secondly we show the smooth convergence to a ball in $\mathbb{E}^n$ (6.5).

Lemma 6.2 *Given any $\varepsilon > 0$ and fixed $A > A_\epsilon > 0$ there exists a $\delta = \delta(\varepsilon, A_\epsilon) > 0$ such that*

$$\bar{g}_j \leq (1+\varepsilon)^2 g_{\mathbb{E}} \quad \text{and} \quad g_{\mathbb{E}} \leq (1+\varepsilon)^2 \bar{g}_j \quad \text{on} \quad \Omega^j_{A_\epsilon, A}, \tag{6.9}$$

whenever the mass $m_j < \delta$.

Proof Choose $\tilde{A}_0 < A_\epsilon$ and $\tilde{A} > A$. Then, according to Proposition 5.10, there is a uniform constant C such that

$$\| u_j - 1 \|_{C^0\left(\Omega_{\tilde{A}_0,\tilde{A}}\right)} \leq \frac{C m_j^{\frac{1}{4}}}{\sqrt{\tilde{A}_0}}, \tag{6.10}$$

where $\Omega_{\tilde{A}_0,\tilde{A}} \subset \mathbb{E}^n$ is the annular domain whose boundary consists of two coordinate spheres having areas $\tilde{A}_0 < \tilde{A}$. Thus, for all m_j sufficiently small we find that $\Omega^j_{A_\epsilon,A} \subset \Omega_{\tilde{A}_0,\tilde{A}}$ and

$$(1+\varepsilon)^{-2} \leq u_j^{\frac{4}{n-2}} \leq (1+\varepsilon)^2, \tag{6.11}$$

from which the desired result follows. □

Lemma 6.3 *Fix $A > 0$ and $D > 0$, and let the symmetry spheres $\Sigma^j_A \subset \bar{M}^n_j$ be defined by*

$$\text{Area}_{g_j}\left(\Sigma^j_A\right) = \text{Area}_{\bar{g}_j}\left(\Sigma^j_A\right) = A. \tag{6.12}$$

If $m_j \to 0$ then

$$\text{Area}_{g_{\mathbb{E}}}\left(\Sigma^j_A\right) \to A, \tag{6.13}$$

and so the spheres

$$\left(\Sigma^j_A, g_{\mathbb{E}}\right) \to (\partial B_0(\rho_A), g_{\mathbb{E}}) \text{ smoothly} \tag{6.14}$$

and the balls

$$\left(\Omega'_j, g_{\mathbb{E}}\right) \to (B_0(\rho_A), g_{\mathbb{E}}) \text{ smoothly}. \tag{6.15}$$

Proof Observe that (6.13) follows from the C^0 convergence of the metric tensors in Lemma 6.2. Moreover (6.14) and (6.15) follow immediately from (6.13). □

6.2 Applying the Lakzian–Sormani Theorem

In work of Lakzian et al. [22], the following theorem was proven providing a concrete means to estimate the intrinsic flat distance. Intuitively this theorem observes that two manifolds Ω_j and Ω'_j, as in (6.3) and (6.4), are close in the intrinsic flat sense if they have diffeomorphic subregions W_j and W'_j, as in (6.7) and (6.8) where the metric tensors are C^0 close and the volumes not covered by these subregions are small. See Fig. 8. When using this result for the current problem note that a dictionary between the notation is $\Omega = \Omega_j$, $\Omega' = \Omega'_j$, $W = W_j$, $W' = W'_j$, $g = \bar{g}_j$, and $g' = g_{\mathbb{E}}$.

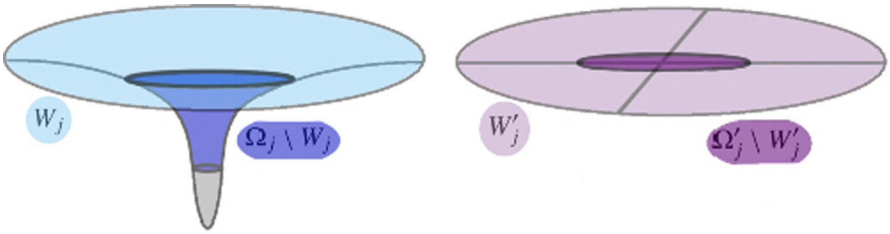

Fig. 8 To prove that two regions Ω_j and Ω'_j are close in the intrinsic flat sense using the Lakzian-Sormani theorem, one first identifies subregions $W_j \subset \Omega_j$ and $W'_j \subset \Omega'_j$ that are C^0 close and then shows that the volumes of the excesses $\Omega_j \setminus W_j$ and $\Omega'_j \setminus W'_j$ are small. It must also be ensured that the distance distortions are small. See Theorem 6.4

Theorem 6.4 [22] *Suppose that* (Ω, g) *and* (Ω', g') *are oriented precompact Riemannian manifolds with diffeomorphic subregions* $W \subset \Omega$ *and* $W' \subset \Omega'$. *Identifying* $W = W'$ *assume that*

$$g \leq (1+\varepsilon)^2 g' \quad \text{and} \quad g' \leq (1+\varepsilon)^2 g \quad \text{on} \quad W. \tag{6.16}$$

Taking the extrinsic diameters

$$\max\{\text{diam}_g(\Omega), \text{diam}_{g'}(\Omega')\} \leq D_0 \tag{6.17}$$

define a hemispherical width

$$\omega > \frac{\arccos(1+\varepsilon)^{-1}}{\pi} D_0 \quad \text{which goes to } 0 \text{ as } \varepsilon \to 0. \tag{6.18}$$

Taking the difference in distances with respect to the outside manifolds, set

$$\lambda = \sup_{x,y \in W} |d_{\Omega,g}(x,y) - d_{\Omega',g'}(x,y)| \leq 2D_0 \tag{6.19}$$

and define the height

$$\Lambda = \max\left\{\sqrt{2\lambda D_0},\ D_0\sqrt{\varepsilon^2+2\varepsilon}\right\}. \tag{6.20}$$

Then

$$\begin{aligned} d_{\mathcal{F}}(\Omega,\Omega') \leq (2\Lambda+\omega)\Big(&\text{Vol}_g(W) + \text{Vol}_{g'}(W') + \text{Area}_g(\partial W) + \text{Area}_{g'}(\partial W')\Big) \\ &+ \text{Vol}_g(\Omega \setminus W) + \text{Vol}_{g'}(\Omega' \setminus W'). \end{aligned} \tag{6.21}$$

6.3 Estimating the Volumes

In this subsection all volumes and areas appearing in the first line of (6.21) will be shown to be uniformly bounded, and those in the second line will be shown to be arbitrarily small. In addition, it will be established that the volumes $\mathrm{Vol}_{\bar{g}_j}(\Omega_j)$ converges to that of a ball in Euclidean space.

Lemma 6.5 *Volumes outside the diffeomorphic subregions may be estimated by*

$$\mathrm{Vol}_{\bar{g}_j}(\Omega_j \setminus W_j) \leq DA_\epsilon, \tag{6.22}$$

and

$$\mathrm{Vol}_{g_{\mathbb{E}}}(\Omega'_j \setminus W'_j) \leq (1+\varepsilon)^{n-1} DA_\epsilon. \tag{6.23}$$

In particular, both are less than $2^{n-1}DA_\epsilon$.

Proof First observe that

$$\Omega_j \setminus W_j = \Omega^j_A \cap T_D(\Sigma^j_A) \setminus \Omega^j_{A_\epsilon, A} = \Omega^j_{A_\epsilon} \cap T_D(\Sigma^j_A). \tag{6.24}$$

By Lemma 3.4, the largest area of the radial levels in this set is A_ϵ. Applying the coarea formula to these levels and the fact that the depth of the set is D, we have

$$\mathrm{Vol}_{\bar{g}_j}(\Omega_j \setminus W_j) \leq DA_\epsilon. \tag{6.25}$$

Applying the coarea formula again and using (6.16) yields

$$\mathrm{Vol}_{g_{\mathbb{E}}}(\Omega'_j \setminus W'_j) \leq D \,\mathrm{Vol}_{g_{\mathbb{E}}}(\Sigma^j_{A_\epsilon}) \leq D(1+\varepsilon)^{n-1} \,\mathrm{Vol}_{\bar{g}}(\Sigma^j_{A_\epsilon}) = D(1+\varepsilon)^{n-1} A_\epsilon. \tag{6.26}$$

□

Lemma 6.6 *Volumes of the diffeomorphic subregions may be estimated by*

$$\mathrm{Vol}_{\bar{g}_j}(W_j) \leq DA, \tag{6.27}$$

and

$$\mathrm{Vol}_{g_{\mathbb{E}}}(W'_j) \leq (1+\varepsilon)^{n-1} DA. \tag{6.28}$$

In particular, both are less than $2^{n-1}DA$.

Proof This follows from Lemma 3.4 and the coarea formula

$$\mathrm{Vol}_{\bar{g}_j}(W_j) \leq \mathrm{Vol}_{\bar{g}_j}(\Omega_j) \leq D \,\mathrm{Vol}_{\bar{g}}\left(\Sigma^j_A\right) = DA. \tag{6.29}$$

Furthermore

$$\mathrm{Vol}_{g_{\mathbb{E}}}\left(W_j'\right) \leq D\,\mathrm{Vol}_{g_{\mathbb{E}}}\left(\Sigma_A^j\right) = D(1+\varepsilon)^{n-1}A. \tag{6.30}$$

□

Lemma 6.7 *Boundary areas of the diffeomorphic subregions may be estimated by*

$$\mathrm{Area}_{\bar{g}_j}(\partial W_j) \leq A_\epsilon + A \leq 2A, \tag{6.31}$$

and

$$\mathrm{Area}_{g_{\mathbb{E}}}\left(\partial W_j'\right) \leq (1+\varepsilon)^{n-1}(A_\epsilon + A) \leq 2(1+\varepsilon)^{n-1}A. \tag{6.32}$$

In particular, both are less than $2^n A$.

Proof We know that ∂W_j has at most two components, both of which are radial levels of area less than A by monotonicity. The same holds for $\partial W_j'$, except that as above the upper bound on the outer area is $(1+\varepsilon)^{n-1}A$. □

The next lemma will be used to obtain *volume preserving* intrinsic flat convergence. It follows from the last few lemmas.

Lemma 6.8 *The difference of total volumes may be estimated by*

$$\left|\mathrm{Vol}_{\bar{g}_j}(\Omega_j) - \mathrm{Vol}_{g_{\mathbb{E}}}(\Omega_j')\right| < \left((1+\varepsilon)^n - 1\right)DA + 2D(1+\varepsilon)^{n-1}A_\epsilon. \tag{6.33}$$

Proof From Lemma 6.5 we have

$$\left|\mathrm{Vol}_{g_{\mathbb{E}}}(\Omega_j') - \mathrm{Vol}_{g_{\mathbb{E}}}(W_j')\right| \leq D(1+\varepsilon)^{n-1}A_\epsilon, \tag{6.34}$$

and

$$\left|\mathrm{Vol}_{\bar{g}_j}(\Omega_j) - \mathrm{Vol}_{\bar{g}_j}(W_j)\right| \leq DA_\epsilon < D(1+\varepsilon)^{n-1}A_\epsilon. \tag{6.35}$$

Moreover by (6.16)

$$\begin{aligned} |\,\mathrm{Vol}_{g_{\mathbb{E}}}(W_j') - \mathrm{Vol}_{\bar{g}_j}(W_j)| &\leq \left((1+\varepsilon)^n - 1\right) \\ \mathrm{Vol}_{\bar{g}_j}(W_j) &\leq \left((1+\varepsilon)^n - 1\right)DA. \end{aligned} \tag{6.36}$$

□

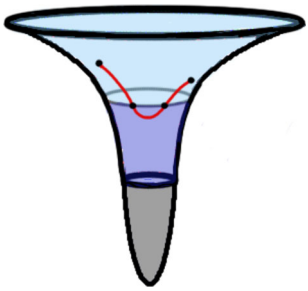

Fig. 9 Fig. 9 Line segment trajectory

6.4 Estimating Distances and Diameters

Lemma 6.9 *Let $D \geq \rho_A$, then*

$$\max\{\text{diam}_{\tilde{g}_j}(\Omega_j), \text{diam}_{g_{\mathbb{E}}}\left(\Omega'_j\right)\} \leq D_0 \tag{6.37}$$

where $D_0 \leq 4\pi D$.

Proof This follows because the depth of the tubular neighborhood is D, and the largest symmetry sphere satisfies

$$\text{diam}_{\tilde{g}_j}\left(\Sigma_A^j\right) = \pi\rho_A \leq \pi D. \tag{6.38}$$

So by the triangle inequality the diameter of Ω_j is no larger than $2D + \pi D \leq 4\pi D$. In addition

$$\text{diam}_{g_{\mathbb{E}}}\left(\Sigma_A^j\right) \leq (1+\varepsilon)\,\text{diam}_{\tilde{g}_j}\left(\Sigma_A^j\right) = (1+\varepsilon)\pi D. \tag{6.39}$$

So by the triangle inequality the diameter of Ω'_j is no larger than $2D + (1+\varepsilon)\pi D \leq 4\pi D$. □

Lemma 6.10 *The difference of distances satisfies*

$$\lambda_j = \sup_{x,y\in W_j} |d_{\tilde{g}_j}(x,y) - d_{g_{\mathbb{E}}}(x,y)| \leq (1+\varepsilon)\pi\rho_{A_\epsilon} + 4\pi\varepsilon D, \tag{6.40}$$

where W_j is identified with W'_j.

Proof Let $\sigma = \sigma_{x,y}$ be a line segment from $\sigma(0) = x$ to $\sigma(1) = y$ in Ω'_j so that

$$l_{g_{\mathbb{E}}}(\sigma) = |x-y| = d_{g_{\mathbb{E}}}(x,y). \tag{6.41}$$

Then there is a first time that $\sigma(t_1) \in \Sigma^j_{A_\epsilon}$ and a last time that $\sigma(t_2) \in \Sigma^j_{A_\epsilon}$. See Fig. 9.

By triangle inequality we have

$$\begin{aligned} d_{\bar{g}_j}(x,y) &\leq d_{\bar{g}_j}(\sigma(0),\sigma(t_1)) + d_{\bar{g}_j}(\sigma(t_1),\sigma(t_2)) + d_{\bar{g}}(\sigma(t_2),\sigma(1)) \\ &\leq l_{\bar{g}_j}(\sigma([0,t_1])) + \pi\rho_{A_\epsilon} + l_{\bar{g}_j}(\sigma([t_2,1])) \\ &\leq (1+\varepsilon)\, l_{g_{\mathbb{E}}}(\sigma([0,t_1])) + \pi\rho_{A_\epsilon} + (1+\varepsilon)\, l_{g_{\mathbb{E}}}(\sigma([t_2,1])) \\ &\leq \pi\rho_{A_\epsilon} + (1+\varepsilon) l_{g_{\mathbb{E}}}(\sigma([0,1])) \\ &= \pi\rho_{A_\epsilon} + (1+\varepsilon) d_{g_{\mathbb{E}}}(x,y) \\ &\leq \pi\rho_{A_\epsilon} + \varepsilon D_0 + d_{g_{\mathbb{E}}}(x,y). \end{aligned} \tag{6.42}$$

Here we have used Lemma 6.9 and the fact that $\Sigma^j_{A_\epsilon}$ is connected.

Let $\gamma = \gamma_{x,y}$ be a $\bar{g}_j$-length minimizing geodesic from $\gamma(0) = x$ to $\gamma(1) = y$ in Ω_j so that

$$l_{\bar{g}_j}(\gamma) = d_{\bar{g}_j}(x,y). \tag{6.43}$$

Then there is a first time that $\gamma(t_1) \in \Sigma^j_{A_\epsilon}$ and a last time that $\gamma(t_2) \in \Sigma^j_{A_\epsilon}$. By the triangle inequality we have

$$\begin{aligned} d_{g_{\mathbb{E}}}(x,y) &\leq d_{g_{\mathbb{E}}}(\gamma(0),\gamma(t_1)) + d_{g_{\mathbb{E}}}(\gamma(t_1),\gamma(t_2)) + d_{g_{\mathbb{E}}}(\gamma(t_2),\gamma(1)) \\ &\leq l_{g_{\mathbb{E}}}(\gamma([0,t_1])) + \operatorname{diam}_{g_{\mathbb{E}}}(\Sigma^j_{A_\epsilon}) + l_{g_{\mathbb{E}}}(\gamma([t_2,1])) \\ &\leq (1+\varepsilon) l_{\bar{g}_j}(\gamma([0,t_1])) + (1+\varepsilon) \\ &\quad \operatorname{diam}_{\bar{g}_j}(\Sigma^j_{A_\epsilon}) + (1+\varepsilon) l_{\bar{g}_j}(\gamma([t_2,1])) \\ &\leq (1+\varepsilon)\pi\rho_{A_\epsilon} + (1+\varepsilon) l_{\bar{g}_j}(\gamma([0,1])) \\ &= (1+\varepsilon)\pi\rho_{A_\epsilon} + (1+\varepsilon) d_{\bar{g}_j}(x,y) \\ &\leq (1+\varepsilon)\pi\rho_{A_\epsilon} + \varepsilon D_0 + d_{\bar{g}_j}(x,y). \end{aligned} \tag{6.44}$$

Combining these with $D_0 \leq 4\pi D$ from Lemma 6.9 we obtain the desired result. □

6.5 Proof of Proposition 6.1

First note that by Lemmas 6.5–6.9 applied to Theorem 6.4 we have

$$d_{\mathcal{F}}(\Omega_j, \Omega'_j) \leq \left(2\Lambda_j + \omega\right)\left(2D2^{n-1}A + 2A2^n\right) + 2D2^{n-1}A_\epsilon, \tag{6.45}$$

where

$$\Lambda_j = \max\left\{\sqrt{2\lambda_j D_0},\ D_0\sqrt{\varepsilon^2+2\varepsilon}\right\}, \tag{6.46}$$

Furthermore by Lemma 6.8

$$|\operatorname{Vol}_{\bar{g}_j}(\Omega_j) - \operatorname{Vol}_{g_{\mathbb{E}}}(\Omega_j)| < \left((1+\varepsilon)^n - 1\right) DA + D2^n A_\epsilon. \tag{6.47}$$

Given $A > 0$, $D > 0$, and $\epsilon > 0$ we must choose $\delta = \delta(A, D, \epsilon) > 0$ sufficiently small so that

$$d_{\mathcal{VF}}(\Omega_j, \Omega_j') \;=\; d_{\mathcal{F}}(\Omega_j, \Omega_j') + |\operatorname{Vol}_{\bar{g}_j}(\Omega_j) - \operatorname{Vol}_{g_{\mathbb{E}}}(\Omega_j)| < \epsilon \tag{6.48}$$

whenever $\delta < m_j$. Therefore choose ε and A_ϵ appropriately in order to satisfy

$$D2^n A_\epsilon < \epsilon/4, \tag{6.49}$$
$$\left((1+\varepsilon)^n - 1\right) DA < \epsilon/4, \tag{6.50}$$

and

$$\left(2\Lambda_j + \omega\right) C_{A,D} < \epsilon/4, \tag{6.51}$$

where

$$C_{A,D} = (2D2^{n-1}A + 2A2^n). \tag{6.52}$$

Observe that these together imply (6.48). It remains only to show that (6.51) may be obtained from choosing ε and A_ϵ small.

According to the definition of Λ_j and ω, (6.51) will be valid if

$$2D_0\sqrt{\varepsilon^2 + 2\varepsilon}C_{A,D} < \epsilon/8, \tag{6.53}$$
$$2\sqrt{2\lambda_j D_0}C_{A,D} < \epsilon/8, \tag{6.54}$$

and

$$\omega C_{A,D} = 2\frac{\arccos(1+\varepsilon)^{-1}}{\pi} D_0 C_{A,D} < \epsilon/8. \tag{6.55}$$

Moreover, from Lemma 6.10 it follows that (6.54) holds if

$$2\sqrt{8\pi\varepsilon D D_0}C_{A,D} < \epsilon/16, \tag{6.56}$$

and

$$2\sqrt{4\pi\rho_{A_\epsilon} D_0}C_{A,D} < \epsilon/16. \tag{6.57}$$

Thus we choose the area of the inner level set $A_\epsilon = A_\epsilon(A, D) > 0$ small enough so that (6.49), and (6.57) hold. Recall that ρ_{A_ϵ} is the radius of the $n-1$-sphere whose area is A_ϵ. In addition, we choose $\varepsilon = \varepsilon(A, D, \epsilon)$ small enough so that (6.50), (6.53), (6.55), and (6.56) hold. It follows that there exists $\delta = \delta(\varepsilon, A_\epsilon) > 0$ such that $m_j < \delta$ implies $d_{\mathcal{VF}}(\Omega_j, \Omega_j') < \epsilon$. Finally (6.5) was established in Lemma 6.3.

7 Convergence of the Second Fundamental Form Differences

In Theorem 1.2 a statement (1.23) is given concerning the convergence of the difference of the second fundamental forms of the initial data and Jang graphs. In this section we give a proof of that statement.

7.1 Outside Control

The pointwise control on the conformal factor gives rise to L^2 control of the difference of second fundamental forms. Recall that Ω_A is the domain inside the symmetry sphere of volume A in $(\bar{M}^n, \bar{g})$ and (M^n, g). By combining Proposition 5.10 and (5.8) we obtain the following.

Proposition 7.1 *Let (M_j^n, g_j, k_j) be a sequence of spherically symmetric, uniformly asymptotically flat initial data satisfying the dominant energy condition and with either outermost apparent horizon boundary or no boundary. For every $A_0 > 0$ there exists a uniform constant C such that*

$$\| h_j - k_j \|^2_{L^2(\bar{M}_j^n \setminus \Omega^j_{A_0}, \bar{g}_j)} \leq C m_j. \tag{7.1}$$

Proof According to Proposition 5.10 and the asymptotics of u_j, we find by choosing $0 < \tilde{A}_0 < A_0$ that

$$|u_j - 1| \leq c_1 m_j^{\frac{1}{4}} \qquad \text{on} \qquad \bar{M}_j^n \setminus \Omega^j_{A_0}, \tag{7.2}$$

for some uniform constant c_1. Thus for large j, the inequality (5.8) implies

$$\begin{aligned} c_n^{-1}(n-2)\omega_{n-1} m_j \;&\geq \int_{\bar{M}_j^n} |h_j - k_j|^2_{\bar{g}_j} u_j^2 \mathrm{d}V_{\bar{g}_j} \\ &\geq \tfrac{1}{2} \int_{\bar{M}_j^n \setminus \Omega^j_{A_0}} |h_j - k_j|^2_{\bar{g}_j} \mathrm{d}V_{\bar{g}_j}. \end{aligned} \tag{7.3}$$

□

7.2 Inside Control

The conformal factors u_j decay exponentially fast along the cylindrical ends of $(\bar{M}_j^n, \bar{g}_j)$, and thus the techniques used above to obtain estimates outside a fixed level surface do not extend down the cylindrical end. Nevertheless, with an extra hypothesis of uniform L^2 control for the initial data second fundamental forms k_j, we are able to obtain uniform L^2 bounds for the difference of second fundamental forms on tubular neighborhoods of the anchor surface. It may be possible to weaken this hypothesis, however we expect that some condition on k_j is necessary in order to control the difference of second fundamental forms.

Proposition 7.2 *Let (M_j^n, g_j, k_j) be a sequence of spherically symmetric, uniformly asymptotically flat initial data satisfying the dominant energy condition and with either outermost apparent horizon boundary or no boundary. Further, assume the uniform bound $\| k_j \|_{L^2(M_j^n)} \leq B$. Given $A > 0$ and $D > 0$ there exists C depending only on A, B, and D such that*

$$\| h_j - k_j \|_{L^2(\Omega_A^j \cap T_D(\Sigma_A^j), \bar{g}_j)} \leq C. \tag{7.4}$$

Proof For convenience in this proof we will drop the subscript j. Recall from (5.1) that

$$\int_{\bar{M}^n} |h-k|_{\bar{g}}^2 \phi^2 \mathrm{d}V_{\bar{g}} \leq \int_{\bar{M}^n} \left(c_n^{-1} |\nabla \phi|_{\bar{g}}^2 + R_{\bar{g}} \phi^2 \right) \mathrm{d}V_{\bar{g}} \tag{7.5}$$

for all $\phi \in C_c^\infty(\bar{M}^n)$. Choose a particular test function such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $T_{D+1}(\Sigma_A)$, and $|\nabla \phi|_{\bar{g}} \leq 2$ then

$$\begin{aligned} \int_{T_D(\Sigma_A)} |h-k|_{\bar{g}}^2 \mathrm{d}V_{\bar{g}} &\leq 2c_n^{-1} \mathrm{Vol}_{\bar{g}}(T_{D+1}(\Sigma_A)) \\ &+ \int_{T_{D+1}(\Sigma_A)} R_{\bar{g}} \phi^2 \mathrm{d}V_{\bar{g}}. \end{aligned} \tag{7.6}$$

Write

$$\bar{g} = \mathrm{d}\bar{s}^2 + \rho(\bar{s})^2 g_{S^{n-1}} \tag{7.7}$$

such that the interval $0 < \bar{s} < D+2$ covers $T_{D+1}(\Sigma_A)$, and notice that

$$R_{\bar{g}} = -2(n-1)\frac{\ddot{\rho}}{\rho} + (n-1)(n-2)\frac{1-\dot{\rho}^2}{\rho^2}. \tag{7.8}$$

Integrating by parts produces

$$\begin{aligned} &\int_{T_{D+1}(\Sigma_A)} R_{\bar{g}} \phi^2 \mathrm{d}V_{\bar{g}} \\ &= \omega_{n-1} \int_0^{D+2} \left[-2(n-1)\frac{\ddot{\rho}}{\rho} + (n-1)(n-2)\frac{1-\dot{\rho}^2}{\rho^2} \right] \phi^2 \rho^{n-1} \mathrm{d}\bar{s} \\ &= \omega_{n-1} \int_0^{D+2} \Big[4(n-1)\phi\dot{\phi}\rho^{n-2}\dot{\rho} + (n-1)(n-2)\phi^2\rho^{n-3}\dot{\rho}^2 \\ &\quad + (n-1)(n-2)\phi^2\rho^{n-3} \Big] \mathrm{d}\bar{s}. \end{aligned} \tag{7.9}$$

Since the mean curvature of coordinate spheres is $\bar{H} = (n-1)\rho^{-1}\dot{\rho}$, this may be rewritten as

$$\int_{T_{D+1}(\Sigma_A)} R_{\bar{g}}\phi^2 \mathrm{d}V_{\bar{g}} = \int_{T_{D+1}(\Sigma_A)} \left(4\langle\phi\nabla\phi, \partial_{\bar{s}}\rangle\bar{H} + \frac{n-2}{n-1}\phi^2\bar{H}^2\right)\mathrm{d}V_{\bar{g}} + \int_0^{D+2} \omega_{n-1}(n-1)(n-2)\phi^2\rho^{n-3}\mathrm{d}\bar{s}. \tag{7.10}$$

We now estimate each term of (7.10) separately. By uniform asymptotic flatness of $\bar{g}$ in Lemma 4.3 and asymptotic control (7.2) of u, we have that

$$\rho(D+2) = \left(\frac{A(D+2)}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \tag{7.11}$$

is uniformly bounded and thus

$$\int_0^{D+2} \phi^2\rho^{n-3}d\bar{s} \leq (D+2)\left(\frac{A(D+2)}{\omega_{n-1}}\right)^{\frac{n-3}{n-1}} \leq c_1. \tag{7.12}$$

Next according to Lemma 6.8 it holds that $\mathrm{Vol}_{\bar{g}}(T_{D+1}(\Sigma))$ is uniformly bounded, and hence by Hölder's inequality

$$\int_{T_{D+1}(\Sigma_A)} \langle\phi\nabla\phi, \partial_{\bar{s}}\rangle\bar{H}\mathrm{d}V_{\bar{g}} \leq 2\mathrm{Vol}_{\bar{g}}(T_{D+1}(\Sigma))^{\frac{1}{2}}\left(\int_{T_{D+1}(\Sigma_A)} \bar{H}^2\mathrm{d}V_{\bar{g}}\right)^{\frac{1}{2}} \leq c_2 \parallel \bar{H} \parallel_{L^2(T_{D+1}(\Sigma_A),\bar{g})}. \tag{7.13}$$

The remaining term of (7.10) may also be estimated by the L^2 norm of $\bar{H}$.

To complete the proof we will use the extra hypothesis concerning k to bound $\bar{H}$ in L^2. First observe that

$$\bar{g} = g + \mathrm{d}f^2 = (1+f_s^2)\mathrm{d}s^2 + \rho(s)^2 g_{S^{n-1}} \quad \Rightarrow \quad \frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \sqrt{1+f_s^2}, \tag{7.14}$$

so that

$$\bar{H} = (n-1)\rho^{-1}\frac{\mathrm{d}\rho}{\mathrm{d}\bar{s}} = (n-1)\rho^{-1}\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\bar{s}} = \frac{1}{\sqrt{1+f_s^2}}H \tag{7.15}$$

where H is the mean curvature with respect to g. Furthermore, let $\psi : T_{D+1}(\Sigma_A) \to M^n$ be the diffeomorphism onto its image associated with the coordinate change $\bar{s} \to s$. Then

$$\begin{aligned}\int_{T_{D+1}(\Sigma_A)} \bar{H}^2 \mathrm{d}V_{\bar{g}} &= \int_{T_{D+1}(\Sigma_A)} \left[\bar{H}^2 - \frac{(\mathrm{Tr}_{S_{\bar{s}}} k)^2}{1+f_s^2}\right] \mathrm{d}V_{\bar{g}} \\ &\quad + \int_{T_{D+1}(\Sigma_A)} \frac{(\mathrm{Tr}_{S_{\bar{s}}} k)^2}{1+f_s^2} \mathrm{d}V_{\bar{g}} \\ &= \int_0^{D+2} (1+f_s^2)^{-1} \int_{S_{\bar{s}}} \left[H^2 - (\mathrm{Tr}_{S_{\bar{s}}} k)^2\right] dA_{\bar{g}} \mathrm{d}\bar{s} \\ &\quad + \int_{\psi[T_{D+1}(\Sigma_A)]} \frac{(\mathrm{Tr}_{S_{\bar{s}}} k)^2}{\sqrt{1+f_s^2}} \mathrm{d}V_g \\ &\leq \int_0^{D+2} (n-2)^2 \omega_{n-1}^{\frac{n-3}{n-2}} A(\bar{s})^{\frac{n-3}{n-2}} \mathrm{d}\bar{s} \\ &\quad + \int_{M^n} (n-1)^2 |k|_g^2 \mathrm{d}V_g,\end{aligned} \tag{7.16}$$

where we have used that the spacetime Hawking mass

$$m(s) := \frac{1}{2}\left(\frac{A(s)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \left[1 - \frac{1}{(n-1)^2 \omega_{n-1}^{\frac{2}{n-1}} A(s)^{\frac{n-3}{n-1}}} \int_{S_s} \left(H^2 - (\mathrm{Tr}_{S_s} k)^2\right) d\mathrm{d}A_g\right] \tag{7.17}$$

is nonnegative [21], and the areas of coordinate spheres agree with respect to both metrics g and $\bar{g}$, that is $A(s) = A(\bar{s})$. Again using that $\rho(D+2)$ is uniformly bounded produces

$$\int_{T_{D+1}(\Sigma_A)} \bar{H}^2 \mathrm{d}V_{\bar{g}} \leq (D+2)(n-2)^2 \omega_{n-1}^{\frac{n-3}{n-2}} c_3 + (n-1)^2 \parallel k \parallel_{L^2(M^n)}^2, \tag{7.18}$$

from which the desired result follows. □

7.3 Global Control

Here we show that the difference of second fundamental forms tends to zero in L^p, $1 \leq p < 2$ on the tubular neighborhoods $\Omega_A \cap T_D(\Sigma_A) \subset \bar{M}^n$, when the masses go to zero. This result, and in fact even a uniform bound, cannot be extended to the entire domain Ω_A. The reason is due to the cylindrical asymptotics contained within this domain, and the observation that in the limit along the cylinder $h - k$ converges to $II - i^*k$, where II is the second fundamental form of ∂M^n in M^n and $i : \partial M^n \hookrightarrow M^n$ is inclusion. Since $II - i^*k$ does not necessarily vanish, it follows that $h - k$ is not necessarily in $L^p(\Omega_A)$.

Theorem 7.3 *Let (M_j^n, g_j, k_j) be a sequence of spherically symmetric, uniformly asymptotically flat initial data satisfying the dominant energy condition and with either outermost apparent horizon boundary or no boundary. Further, assume the*

uniform bound $\| k_j \|_{L^2(M_j^n)} \leq B$ *and that* $m_j \to 0$. *Then for any* $1 \leq p < 2$, $A > 0$, *and* $D > 0$

$$\| h_j - k_j \|_{L^p(\Omega_A^j \cap T_D(\Sigma_A^j), \bar{g}_j)} \to 0. \tag{7.19}$$

Proof Let $A_\varepsilon > 0$ be fixed. Then by Hölder's inequality

$$\begin{aligned}
&\int_{\Omega_A^j \cap T_D(\Sigma_A^j)} |h_j - k_j|_{\bar{g}_j}^p \mathrm{d}V_{\bar{g}_j} \\
&= \int_{\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \cap \Omega_{A_\varepsilon}^j} |h_j - k_j|_{\bar{g}_j}^p \mathrm{d}V_{\bar{g}_j} + \int_{\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \setminus \Omega_{A_\varepsilon}^j} |h_j - k_j|_{\bar{g}_j}^p \mathrm{d}V_{\bar{g}_j} \\
&\leq \mathrm{Vol}_{\bar{g}_j}\left(\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \cap \Omega_{A_\varepsilon}^j\right)^{\frac{2-p}{2}} \left(\int_{\Omega_A^j \cap T_D(\Sigma_A^j)} |h_j - k_j|_{\bar{g}_j}^2 \mathrm{d}V_{\bar{g}_j}\right)^{\frac{p}{2}} \\
&\quad + \mathrm{Vol}_{\bar{g}_j}\left(\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \setminus \Omega_{A_\varepsilon}^j\right)^{\frac{2-p}{2}} \left(\int_{\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \setminus \Omega_{A_\varepsilon}^j} |h_j - k_j|_{\bar{g}_j}^2 \mathrm{d}V_{\bar{g}_j}\right)^{\frac{p}{2}}.
\end{aligned} \tag{7.20}$$

Using the notation and results of Lemma 6.5, observe that

$$\mathrm{Vol}_{\bar{g}_j}\left(\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \cap \Omega_{A_\varepsilon}^j\right) = \mathrm{Vol}_{\bar{g}_j}(\Omega_j \setminus W_j) \leq DA_\varepsilon. \tag{7.21}$$

Furthermore by Lemma 6.8

$$\mathrm{Vol}_{\bar{g}_j}\left(\left[\Omega_A^j \cap T_D(\Sigma_A^j)\right] \setminus \Omega_{A_\varepsilon}^j\right) \leq \mathrm{Vol}_{\bar{g}_j}(\Omega_j) \leq C_1, \tag{7.22}$$

by Proposition 7.1

$$\| h_j - k_j \|_{L^2(\bar{M}_j^n \setminus \Omega_{A_\varepsilon}^j, \bar{g}_j)} \leq \sqrt{C_3(A_\varepsilon) m_j}, \tag{7.23}$$

and by Proposition 7.2

$$\| h_j - k_j \|_{L^2(\Omega_A^j \cap T_D(\Sigma_A^j), \bar{g}_j)} \leq C_2. \tag{7.24}$$

It follows that

$$\| h_j - k_j \|_{L^p(\Omega_A^j \cap T_D(\Sigma_A^j), \bar{g}_j)}^p \leq (DA_\varepsilon)^{\frac{2-p}{2}} C_2^p + C_1^{\frac{2-p}{2}} (C_3(A_\varepsilon) m_j)^{\frac{p}{2}}. \tag{7.25}$$

This can be made arbitrarily small by choosing A_ε small and j appropriately large. □

8 Proof of Theorem 1.2

The proof of the main theorem follows quickly from Proposition 6.1 and Theorem 7.3. Recall that

$$\Omega_j \;=\; \Omega_A^j \cap T_D(\Sigma_A^j) \;\subset\; \bar{M}_j^n \text{ with metric tensor } \bar{g}_j, \tag{8.1}$$

and

$$\Omega_j' \;=\; \Omega_A^j \;\subset\; \mathbb{E}^n \text{ with metric tensor } \tilde{g}_j = u_j^{\frac{4}{n-2}}\bar{g}_j = g_{\mathbb{E}}. \tag{8.2}$$

By the triangle inequality

$$\begin{aligned} d_{\mathcal{VF}}\left((\Omega_j, \bar{g}_j), (B_0(\rho_A), g_{\mathbb{E}})\right) \le\; & d_{\mathcal{VF}}\left((\Omega_j, \bar{g}_j), (\Omega_j', g_{\mathbb{E}})\right) \\ & + d_{\mathcal{VF}}\left((\Omega_j', g_{\mathbb{E}}), (B_0(\rho_A), g_{\mathbb{E}})\right). \end{aligned} \tag{8.3}$$

As $m_j \to 0$ we have that (6.2) produces

$$d_{\mathcal{VF}}\left((\Omega_j, \bar{g}_j), (\Omega_j', g_{\mathbb{E}})\right) \to 0, \tag{8.4}$$

and (6.5) yields

$$d_{\mathcal{VF}}\left((\Omega_j', g_{\mathbb{E}}), (B_0(\rho_A), g_{\mathbb{E}})\right) \to 0, \tag{8.5}$$

since smooth convergence implies convergence in the volume preserving intrinsic flat sense. Finally, Theorem 7.3 gives (1.23).

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