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C^k-Estimates for $\bar{\partial}$-Equation on Certain Convex Domains of Infinite Type in $\mathbb{C}^n$

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Abstract

In this paper, C^k-estimates are obtained for the Henkin solution operator of the Cauchy–Riemann system

$$\bar{\partial} u = \varphi$$

on a class of certain smoothly bounded, convex domains of infinite type in $\mathbb{C}^n$, where φ is a $\bar{\partial}$-closed $(0, q)$-differential form. It is proved that the Henkin solution of the $\bar{\partial}$-equation admits a suitable Hölder gain.

Keywords $\bar{\partial}$ · Henkin solution · Henkin operator · Hölder estimates for $\bar{\partial}$ · Infinite-type domains

Mathematics Subject Classification 32W05 · 32F32 · 32T25 · 32T99

1 Introduction

Let $z = (z_1, \ldots, z_n)$ be the standard coordinates in the complex Euclidean space $\mathbb{C}^n$, where $z_j = x_j + ix_{n+j}$, for $x_j \in \mathbb{R}$, $j = 1, \ldots, n$ and $i = \sqrt{-1}$. We define the following Wirtinger derivatives

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$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{n+j}}\right) \quad \text{for } j = 1, \ldots, n,$$

and their duals are $\mathrm{d}\bar{z}_j = \mathrm{d}x_j - i\mathrm{d}x_{n+j}$.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary $b\Omega$. Let $C(\Omega)$ denote the class of continuous functions on Ω which endowed with the compact-open topology. We set

$$D^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_{2n}^{\alpha_{2n}}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$ is a multi-index of length $2n$. We also define $C^k(\Omega)$ to be the set of those functions $u : \Omega \to \mathbb{C}$ such that for each $\alpha \in (\mathbb{N} \cup \{0\})^{2n}$ with $|\alpha| \le k$, the derivative $D^\alpha u$ exists and belongs to $C(\Omega)$.

Let φ be a $(0, q)$-differential form on Ω. Write φ as

$$\varphi(z) = \sum_{|J|=q} \varphi_J(z)\mathrm{d}\bar{z}^J,$$

where $J = (j_1, \ldots, j_q)$ is a multi-index of length $|J| = q$, and $\mathrm{d}\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}$. The Cauchy–Riemann complex $\bar{\partial}$ on $(0, q)$-differential forms is defined by

$$\bar{\partial}\varphi = \sum_{|J|=q} \bar{\partial}\varphi_J \wedge \mathrm{d}\bar{z}^J,$$

where $\varphi_J \in C^1(\Omega)$ for all $|J| = q$, and

$$\bar{\partial}\varphi_J = \sum_{j=1}^{n} \frac{\partial \varphi_J}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j.$$

Thus $\bar{\partial}\varphi$ is a $(0, q+1)$-differential form on Ω. For a given $(0, q)$-differential form φ with coefficients in $C(\Omega)$, the Cauchy–Riemann equation is the problem of looking for a $(0, q-1)$-differential form u with coefficients in $C^1(\Omega)$ so that

$$\bar{\partial}u = \varphi.$$

Researchers have been interested in the interaction between the functional-regularity properties of u and the geometric properties on $b\Omega$. In particular, the study of C^k-regularity has been an attractive topic in the area of partial differential equations in several complex variables. The interaction also provides many significant tools to studying of complex geometry.

In this purpose, there are two variously main methods to studying the $\bar{\partial}$-problem. The first one is called to be the Hilbert L^2-method introduced by Kohn. The Kohn methods are based on abstract L^2-technique in the theory of pseudo-differential operators developed by Hörmander. Ones of the main results are obtained from Kohn methods are sub-elliptic estimates. We refer the reader to Chapter 1 to Chapter 10 of the monograph [6] by Chen and Shaw. However, these methods do not allow us to estimate solutions for the $\bar{\partial}$-problem in other norms, such as supremum norm, Hölder norms, and C^k-norms. In 1968–1969, Henkin and Ramirez introduced another method to solving the $\bar{\partial}$-equations with supremum norm estimates on strongly pseudoconvex domains. All information about Henkin–Ramirez methods as well as their applications can be found in the bedside book in several complex variables [25] by Range. The main purpose of the present paper is to provide the C^k-regularity for the $\bar{\partial}$-equation by the Henkin–Ramirez method.

Definition 1.1 Let Ω be an open subset of $\mathbb{R}^{2n}$ with smooth boundary.

(1) For each $u \in C^k(\Omega)$, the C^k-norm of u is defined as

$$\|u\|_{C^k(\Omega)} = \sum_{\alpha:|\alpha|=k} \sup_{x\in\Omega} |D^\alpha u(x)|.$$

(2) For $q = 0, 1, \ldots, n$, $C^k_{(0,q)}(\Omega)$ is the class of $(0, q)$-differential forms with coefficients belonging to $(C^k(\Omega), \|.\|_{C^k(\Omega)})$. For each $\varphi = \sum_{|J|=q} \varphi_J \mathrm{d}\bar{z}_J \in C^k_{(0,q)}(\Omega)$, we define

$$\|\varphi\|_{C^k_{(0,q)}(\Omega)} = \sum_{|J|=q} \|\varphi_J\|_{C^k(\Omega)}.$$

(3) Let f be an increasing, positive function such that $\lim\limits_{t\to+\infty} f(t) = +\infty$. A function u is called to belong $\Lambda^f_k(\Omega)$ if $u \in C^k(\Omega)$ and

$$\|u\|_{\Lambda^f_k(\Omega)} = \|u\|_{C^k(\Omega)} + \sum_{\alpha:|\alpha|=k} \sup_{\substack{x,y\in\Omega\\ x\neq y}} f(|x-y|^{-1})|D^\alpha u(x) - D^\alpha u(y)| < +\infty.$$

(4) For $q = 0, 1, \ldots$, $\Lambda^{k,f}_{(0,q)}(\Omega)$ is the class of $(0, q)$-differential forms with coefficients belong to $(\Lambda^f_k(\Omega), \|.\|^f_{\Lambda^f_k(\Omega))}$. For each $\varphi = \sum_{|J|=q} \varphi_J \mathrm{d}\bar{z}_J \in \Lambda^{k,f}_{(0,q)}(\Omega)$, we define

$$\|\varphi\|_{\Lambda^{k,f}_{(0,q)}(\Omega)} = \sum_{|J|=q} \|\varphi_J\|_{\Lambda^f_k(\Omega)}.$$

It is clear that if $f(t) = t^\alpha$, for $0 < \alpha < 1$, the space $\Lambda^{t^\alpha}_k(\Omega)$, $\Lambda^{k,t^\alpha}_{(0,q)}(\Omega)$ coincide to the classical Hölder spaces of order $k + \alpha$ for functions and for $(0, q)$-differential forms, respectively.

In 1974, Siu [31] obtained uniform estimates for the derivatives in the $\bar{\partial}$-problem when Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^N-boundary $b\Omega$ for $N \geq 4$. In particular, let $\varphi \in C^k_{(0,1)}(\Omega)$ be a $\bar{\partial}$-closed differential form for $k \leq N - 4$, he proved that the Henkin–Ramirez solution $u = \mathcal{T}_q\varphi \in \Lambda^{t^{1/2}}_k(\Omega)$ and

$$\|u\|_{\Lambda_k^{t^{1/2}}(\Omega)} \leq C\|\varphi\|_{C^k_{(0,1)}(\Omega)}$$

uniformly.

In 1980, Saito [28], Lieb and Range [21] established sharper and better estimates which are improvements of Siu's result above. Furthermore, in the later paper, the authors used a continuation result of Seeley [29] to modify the Henkin–Ramirez solution.

In 1987, Ryczaj [27] considered the $\bar{\partial}$-equation for a certain class of non-strictly pseudoconvex domains in $\mathbb{C}^n$:

$$\Omega = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \rho(z) = \sum_{k=1}^{n} |z_k|^{2m_k} - 1 < 0 \right\}.$$

The result obtained is that: for $k \geq 1$ and $0 < \theta < \frac{1}{\max 2m_k}$, there is a constant $C_{k,\theta} > 0$ such that if $\varphi \in C^k_{(0,q)}(\Omega)$ is continuous on $b\Omega$ and $\bar{\partial}\varphi = 0$, the Henkin–Ramirez solution $u = \mathcal{T}_q\varphi \in \Lambda^{k,t^\theta}_{(0,q-1)}(\Omega)$ and

$$\|u\|_{\Lambda^{k,t^\theta}_{(0,q-1)}(\Omega)} \leq C_{k,\theta}\|\varphi\|_{C^k_{(0,q)}(\Omega)}.$$

More generally, let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type m in the sense of Catlin [3]. Assume that $\varphi \in C^k_{(0,q)}(\bar{\Omega})$ is a $\bar{\partial}$-closed differential form. In [1], Alexandre has constructed a special integral solution of Henkin–Ramirez type $u \in \Lambda^{k,t^{1/m}}_{(0,q-1)}(\bar{\Omega})$ such that

$$\|u\|_{\Lambda^{k,t^{1/m}}_{(0,q-1)}(\bar{\Omega})} \leq C_k\|\varphi\|_{C^k_{(0,q)}(\bar{\Omega})}.$$

In particular, the author obtains new estimates for the normal derivatives of the defining function and link them to the ε-extremal basis constructed by McNeal [20] on convex domains of finite type in the sense of Catlin.

Naturally, a question to ask is: when Ω no longer satisfies the finite-type condition, does the C^k-regularity hold? When $n = 2$ and $k = 0$, there are few papers relating to this question. In particular, in [32], the author obtained the sup-norm estimate of Henkin–Ramirez operators on smoothly bounded convex domains of the form

$$\Omega^\infty = \{z \in \mathbb{C}^2 : |z_1| + \psi(|z_2|^2) - 1 < 0\},$$

where ψ is a real function satisfying some conditions. Notice that Ω^∞ is infinite type in any sense. In [24], Range proved that there is no solution u of the $\bar{\partial}$ equation on Ω^∞ which belongs to $\Lambda_{t^\alpha}(\Omega^\infty)$, for any $0 < \alpha < 1$. One of interesting models is when $\psi(t) = \exp(-1/t^\alpha)$ for $0 < t \ll 1$ and $0 < \alpha < 1/2$, this is the case of infinite type at 0. Then, in 2013, Khanh [17] has proved new Hölder estimates for Henkin–Ramirez solutions on these domains. However, until now, we do not have any positive result for this problem when $n \geq 3$ or $k \geq 1$. The main result of this paper provides a suitable answer in a certain class of convex domains of infinite type in some sense.

Theorem 1.2 (Main theorem) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, admissibly decoupled, convex domain with $n \geq 2$. Assume that Ω admits the F-type at all boundary points for some function F (see Definition 2.2). For $k \geq 0$ and $1 \leq q \leq n-1$, let $\varphi \in C^k_{(0,q)}(\Omega)$ be a $\bar\partial$-closed differential form on Ω.*

(1) *If $n = 2$, there exists a function $u \in \Lambda^f_k(\Omega)$ so that $\bar\partial u = \varphi$ and*

$$\|u\|_{\Lambda^f_k(\Omega)} \leq C_k \|\varphi\|_{C^k_{(0,1)}(\Omega)},$$

where

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} \mathrm{d}t\right)^{-1}.$$

(2) *If $n \geq 3$, there exists a $(0, q-1)$-differential form $u \in \Lambda^{k,f}_{(0,q-1)}(\Omega)$ so that $\bar\partial u = \varphi$ and*

$$\|u\|_{\Lambda^{k,f}_{(0,q-1)}(\Omega)} \leq C_{k,s,n} \|\varphi\|_{C^k_{(0,q)}(\Omega)},$$

for every $0 \leq s \leq n-3$ and where

$$f(d^{-1}) = \left(\int_0^d \frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t} \mathrm{d}t\right)^{-1}.$$

Here F^* is the inverse function of F.

The structure of this paper is as follows. Section 2 deals with preliminary results relating to the notion of a domain admitting an F-type. Section 3 is concerned with some certain examples to illustrate the notion of F-type. In Sect. 4, the formula of higher derivatives of Henkin–Ramirez solution is provided. The proof of Main Theorem is given in Sect. 5.

2 Preliminaries

Let Ω be a bounded convex domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth boundary $b\Omega$. Let ρ be a defining function for Ω so that $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $b\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$, $\nabla\rho \neq 0$ on $b\Omega$. The convexity means

$$\sum_{i,j=1}^{2n} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(\zeta) a_i a_j \geq 0 \quad \text{on } b\Omega,$$

for every $a = (a_1, \ldots, a_{2n}) \in \mathbb{R}^{2n}$ with $\sum_{j=1}^{2n} a_j \frac{\partial \rho}{\partial x_j}(\zeta) = 0$ on $b\Omega$.

Let us define, for $\zeta \in b\Omega$ and $z \in \Omega$:

$$\Phi(\zeta, z) = \langle \partial\rho, \zeta - z\rangle = \sum\nolimits_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j). \tag{2.1}$$

The convexity of Ω implies

$$\mathrm{Re}\left(\sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j)\right) > 0$$

and so $\Phi(\zeta, z) \neq 0$ as well, for all $\zeta \in b\Omega$ and $z \in \Omega$.

Moreover, as a consequence of the definition, we also have:

Lemma 2.1 *There are positive constants δ, c such that for all boundary points $\zeta \in b\Omega \cap B(P, \delta)$, the followings are satisfied:*

(1) *$\Phi(\zeta, .)$ is holomorphic in $z \in B(\zeta, \delta)$;*
(2) *$\Phi(\zeta, \zeta) = 0$, and $d_z\Phi|_{z=\zeta} \neq 0$;*
(3) *$\rho(z) > 0$ for all z with $\Phi(\zeta, z) = 0$ and $0 < |z - \zeta| < c$.*
By multiplying ρ and Φ by suitable non-zero functions of ζ, one may assume more
(4) *$|\partial\rho(\zeta)| = 1$, and $\partial\rho(\zeta) = d_z\Phi|_{z=\zeta}$.*

It is well known that there are some pseudoconvex domains not admitting any holomorphic support function $\Phi(\cdot, \cdot)$, even of finite type. This phenomenon was established by Kohn and Nirenberg in [18]. Thus, as a first step, we should study any new ideas that would help to increase our understanding the $\bar{\partial}$-equations on convex domains of infinite type.

We recall a geometric condition which plays an important role in our analysis.

Definition 2.2 (see [12] or [10,11] for the case $n = 2$) The function $F\colon [0, \infty) \to [0, \infty)$ is called a *type in* $\mathbb{C}^n$ if the following conditions are satisfied:

(1) F is smooth and increasing;
(2) $F(0) = 0$;
(3) For all $k \in \{0, \ldots, n-2\}$,

$$\int_0^d (-\ln F(r^2))^{n-k-1}\mathrm{d}r < \infty,$$

for some small $d > 0$;
(4) $\frac{F(r)}{r}$ is non-decreasing.

The function F with the properties above is supposed to be given throughout this paper. Then, a (bounded convex, smooth) domain Ω in $\mathbb{C}^n$ is said to be *admitting an F-type* at the boundary point $P \in b\Omega$ if there are positive constants c, c' such that for all $\zeta \in b\Omega \cap B(P, c')$ we have

$$\rho(z) \gtrsim \sum_{k=1}^{n} \frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + F(|z - \zeta|^2),$$

for all $z \in B(\zeta, c)$ with $\Phi(\zeta, z) = 0$.

Here and in what follows, the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

For each $z \in \Omega$, let $P \in b\Omega$ so that $\text{dist}(z, b\Omega) = |z - P|$. We may assume that $\left|\frac{\partial\rho}{\partial\zeta_n}(\zeta)\right| \geq 1$ for $\zeta \in b\Omega \cap B(z, c)$. For $k = 1, \ldots, n-1$, we define

$$L_k = \frac{\partial\rho}{\partial\bar{\zeta}_k}\frac{\partial}{\partial\bar{\zeta}_n} - \frac{\partial\rho}{\partial\bar{\zeta}_n}\frac{\partial}{\partial\bar{\zeta}_k}$$

and choose L_n be the unit normal vector field of type (0, 1) on $b\Omega \cap B(z, c)$.

Definition 2.3 Ω is called to be an admissibly decoupled domain if

$$\left|L_i\left(\frac{\partial\rho}{\partial\zeta_j}\right)\right| \lesssim \delta_{ij}\frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j}(\zeta),$$

for all $\zeta \in b\Omega \cap B(z, c)$ and $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta.

It is mentioned that in case $n = 2$, or in case domains whose the Levi form of the boundary has at most one degenerate eigenvalue at all boundary points, the appearance of $\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2$ and the admissibly decoupled condition are not necessary.

In the proof of the Main Theorem, we shall apply Stoke's Theorem, so we need a continuation of $\Phi(\zeta, z)$ inside Ω, that is

$$\widetilde{\Phi}(\zeta, z) = \Phi(\zeta, z) - \rho(\zeta),$$

where $z, \zeta \in \bar{\Omega}$. The following is the main contribution in our analysis:

Lemma 2.4 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, convex domain admitting an F-type at $P \in b\Omega$. Then there is a positive constant c such that the support function $\Phi(\zeta, z)$ satisfies the following estimate:*

$$|\Phi(\zeta, z)| \gtrsim |\rho(z)| + |\operatorname{Im}\Phi(\zeta, z)| + \sum_{k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + F(|z - \zeta|^2), \quad (2.2)$$

for every $\zeta \in b\Omega \cap B(P, c)$, and $z \in \Omega$, $|z - \zeta| < c$.

Proof This Lemma was first proved by Range in [23,24] for $F(t) = t^m$ and by the present author for F-type functions in [12]. We recall the proof here for convenience. Let δ, c and $\rho(z)$, $\Phi(\zeta, z)$ be as above, and let $(w', w_n) = (w_1, \ldots, w_{n-1}, w_n)$. For any $\zeta \in b\Omega \cap B(P, \delta)$, we define the holomorphic map $\psi_\zeta : z \mapsto w = (w', w_n) = (z' - \zeta', \Phi(\zeta, z))$. The Jacobian matrix of ψ_ζ at ζ is unitary since (4) in Lemma 2.1. Hence, the inverse map ψ_ζ^{-1} exists and can be assumed that its Jacobian matrix is uniformly bounded. As an immediate result, $|\psi_\zeta(z) - \psi_\zeta(Z)| \approx |z - Z|$ for all

$z, Z \in B(\zeta, c)$. We define $\rho_\zeta(w', w_n) := \rho(\psi_\zeta^{-1}(w', w_n))$; then ρ_ζ is a defining function for $\psi_\zeta(\Omega \cap B(\zeta, c))$.

By the property (3) in Lemma 2.1 and the F-type condition, after shrinking c enough, for some d small, we obtain

$$\begin{aligned} &\rho_\zeta(w', 0) > 0 \quad \text{for } 0 < |w'| < d, \\ &\rho_\zeta(w', 0) \gtrsim \sum_{k=1}^{n-1} a_k |w_k|^2 + F(|w'|^2) \quad \text{for } 0 \le |w'| < d, \end{aligned} \tag{2.3}$$

where $a_k = \dfrac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta)$. Therefore, by Taylor's theorem, for any $|w| < d$, we have

$$\begin{aligned} \rho_\zeta(w', w_n) &= \rho_\zeta(w', 0) + 2\,\mathrm{Re}\left(\frac{\partial \rho_\zeta}{\partial w_n}(w', 0).w_n\right) + o(|w_n|) \\ &\ge 2\,\mathrm{Re}\, w_n + \sum_{k=1}^{n-1} a_k |w_k|^2 + F(|w'|^2) + A.F(|w'|^2) + o(1)|w_n|, \end{aligned} \tag{2.4}$$

where the last inequality follows from $\partial_w \rho_\zeta(0) = dw_n$ and $o(1) \to 0$ when $|w| \to 0$. Here, the convergence is uniform in ζ-variables, since the fact that $o(1)$ in our case only depends on the modulus of continuity of the first-order partial derivatives of $\rho_\zeta(w', w_n)$. So, let $0 < d^* < d$ be so small such that $o(1)|w_n| \le |\mathrm{Re}\, w_n| + |\mathrm{Im}\, w_n|$ for every $|w| \le d^*$. Hence, the above inequality implies that

$$-2\,\mathrm{Re}\, w_n + |\mathrm{Re}\, w_n| \ge \rho_\zeta(w', w_n) - |\mathrm{Im}\, w_n| + \sum_{k=1}^{n-1} a_k |w_k|^2 + F(|w'|^2) + A \cdot F(|w'|^2)$$

for $|w| < d^*$. This implies

$$|\mathrm{Re}\, w_n| \gtrsim \rho_\zeta(w', w_n) - |\mathrm{Im}\, w_n| + \sum_{k=1}^{n-1} a_k |w_k|^2 + A \cdot F(|w'|^2) \quad \text{for } |w| < d^*.$$

The last step is to convert $\rho_\zeta(w)$ to $\rho(z)$. To do this, we choose $c^* < c$ so small such that $\psi_\zeta(B(\zeta, c^*)) \subset B(0, d^*)$. Then, using the Taylor's formula and the fact that F is smooth, we have

$$F(|w'|^2) = F(|w|^2) + O(1) \cdot |w_n|^2,$$

so we obtain

$$|\mathrm{Re}\, \Phi(\zeta, z)| \gtrsim -\rho(z) - |\mathrm{Im}\, \Phi(\zeta, z)| + \sum_{k=1}^{n-1} \frac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta) |\zeta_k - z_k|^2 + A \cdot F(|\zeta - z|^2).$$

Replace the left-hand side by $C|\Phi|$, for $C > 0$ large enough, the following holds

$$|\Phi(\zeta, z)| \gtrsim |\rho(z)| + |\operatorname{Im} \Phi(\zeta, z)| + \sum_{k=1}^{n-1} \frac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta)|\zeta_k - z_k|^2 + F(|z - \zeta|^2).$$

This completes the proof. □

Remark 2.5 This proof also implies that

$$\rho(z) - \rho(\zeta) - 2\operatorname{Re} \sum_{k=1}^{n} \frac{\partial \rho}{\partial \zeta_k}(\zeta)(\zeta_k - z_k) \gtrsim \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_k}(\zeta)|\zeta_k - z_k|^2 + F(|z - \zeta|^2)$$

for $|\zeta - z| < c$. Then we have

$$|\widetilde{\Phi}(\zeta, z)| \gtrsim |\operatorname{Im} \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + F(|z - \zeta|^2)$$

for $|\zeta - z| < c$ and $\rho(z) \leq \rho(\zeta)$.

Moreover, it is not difficult to show that:

Corollary 2.6 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, convex domain admitting an F-type at all boundary points. The following inequality holds*

$$\left| \frac{\frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_j}(\zeta)}{\Phi(\zeta, z)} \right| \lesssim \frac{1}{\rho(\zeta) - \rho(z) + |\zeta_j - z_j|^2}$$

for $|z| + |\zeta| < 1$, $|\zeta - z| < c$ and $\rho(z) \leq \rho(\zeta)$.

3 Examples

In this section, we provide some examples to illustrate the notion of F-type. Firstly, we begin with some convex domains of finite type in the sense of Range (see [23,24]).

3.1 Domain of Finite Type

Example 3.1 ([25, p. 195]) Let $\Omega \subset \mathbb{C}^2$ be a bounded strictly convex domain with its smooth, strictly plurisubharmonic defining function ρ. For every $P \in b\Omega$, there exist positive constants c', c and C such that for all $\zeta \in \overline{\Omega} \cap B(P, c')$ we have

$$\operatorname{Re} \Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + C|\zeta - z|^2,$$

where $|\zeta - z| < c$.

Hence, when $\zeta \in b\Omega \cap B(P, c')$, $z \in \{|\zeta - z| < c\}$ and $\Phi(\zeta, z) = 0$, we have

$$\rho(z) \gtrsim F(|z - \zeta|^2),$$

with $F(t) = t$. So Ω is of F-type.

Example 3.2 ([24, Corollary 5.4]) Let us consider the following complex ellipsoid

$$\Omega = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^{n} |z_j|^{2m_j} - 1 < 0\} \quad (m_j \in \mathbb{N}).$$

Then there exist constants $c, C > 0$ such that

$$\mathrm{Re}\, \Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + C|\zeta - z|^{2m},$$

for $\zeta \in \bar{\Omega}$, $z \in \Omega$ with $|\zeta - z| < c$, and $m = \max\{m_j\}$. Thus Ω is a convex domain admitting an F-type, with $F(t) = t^m$.

Example 3.3 ([22, Proposition 1]) Let $\Omega \subset \mathbb{C}^2$ be a convex domain with real-analytic boundary, i.e., ρ is a real-analytic function. Then, there exist constants $c, C > 0$ and a positive integer m such that

$$\mathrm{Re}\, \Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + C|\zeta - z|^{2m},$$

for $\zeta \in \bar{\Omega}$, $z \in \Omega$ with $|\zeta - z| < c$. Therefore Ω is a domain admitting an F-type, with $F(t) = t^m$.

Example 3.4 Assume that Ω denote a bounded domain of the type

$$\Omega = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^{n} \rho_j(|z_j|^2) - 1 < 0, \right\}$$

where all functions ρ_j are assumed to be real-analytic in $[0, a_j]$ such that

(1) $\rho_j'(t) \geq 0$, $\rho_j'(t) + 2t\rho_j''(t) \geq 0$ for $0 \leq t \leq a_j$;
(2) $\rho_j'(0) = \rho_j(0) = 0$ and $\rho_j(a_j) > 1$.

In [2], Bruna and del Castillo obtained that there exists a positive integer m such that

$$\mathrm{Re}\, \Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_k \partial \bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + |\zeta - z|^{2m},$$

for $\zeta, z \in \bar{\Omega}$ (see [2, Formula (7)]), and

$$\left| L_i \left(\frac{\partial \rho}{\partial \zeta_j} \right) \right| \lesssim \delta_{ij} \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_j}(\zeta) \quad \text{(see [2, p. 534]).}$$

Therefore Ω is a smoothly bounded, admissibly decoupled, convex domain admitting an F-type, with $F(t) = t^m$.

3.2 Domains of Infinite Type

In this subsection, we consider a large class of certain convex domains of infinite type in $\mathbb{C}^n$.

Example 3.5

$$\Omega^\infty = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \rho(z) = \sum_{k=1}^n \exp\left(1 - \frac{1}{|z_k|^{2\alpha_k}}\right) - 1 < 0 \right\},$$

where $0 < \alpha_k < \dfrac{1}{2(n-1)}$. We are going to prove that Ω^∞ is a domain admitting F-type with $F(t) = \exp\left(1 - \dfrac{1}{t^s}\right)$, $0 < s < \dfrac{1}{2(n-1)}$.

When $n = 2$, in [32], Verdera proved that Ω^∞ is a convex domain admitting F-type with $F(t) = \exp\left(1 - \dfrac{1}{t^s}\right)$ for $0 < s < 1/2$.

Lemma 3.6 *Let* $0 < s < \dfrac{1}{2}$, *and* $g : [0, 1] \to \mathbb{R}$ *be defined by* $g(t) = \exp\left(1 - \dfrac{1}{t^s}\right)$. *There exists a constant* $\eta = \eta(s) > 0$, *and the following inequality holds:*

$$g(|\zeta + v|^2) - g(|\zeta|^2) - 2\,\mathrm{Re}\left(\frac{\partial g}{\partial \zeta}(|\zeta|^2) v\right) \gtrsim g(|v|^2), \tag{3.1}$$

where $\zeta, v \in \mathbb{C}$, $|\zeta| + |v| < \eta$.

Proof We may assume that $\zeta, v \neq 0$, and then write $\zeta = r\exp(i\theta)$, $v = R\exp(i\beta)$ and $v\zeta^{-1} = \mu\exp(i\gamma)$. Let us fix r, R and θ. Then, let β vary; the left-hand side of (3.1) equals to

$$F(\gamma) = g(r^2(1 + \mu^2 + 2\mu\cos\gamma)) - g(r^2) - 2g'(r^2)r^2\mu\cos\gamma.$$

Since r is fixed,

$$F'(\gamma) = 2r^2\mu\sin\gamma(g'(r^2) - g'(r^2(1 + \mu^2 + 2\mu\cos\gamma))).$$

Because $0 < t \leq 1$, $g'(t) > 0$ and there is $0 < \tau < 1$ such that $g''(t) > 0$ for all $0 < t < \tau$. Therefore, without loss of generality, it may be assumed that $r < \sqrt{\tau}$ which satisfies

$$g'(\tau) = \inf_{\tau \leq t \leq 1} g'(t).$$

Notice that $F'(\gamma) = 0$ if and only if $\gamma = 0, \pi$ or $\cos\gamma = -\mu/2$. We consider μ in two cases.

(1) If $\mu < 2$, then $\cos\gamma > -1$, and so F attains its absolute minimum value at γ with $\cos\gamma = -\mu/2$. At such γ,

$$F(\pi) = g'(r^2)R^2 > g'(R^2/4)R^2.$$

(2) If $\mu \geq 2$, F attains its absolute minimum value at $\gamma = \pi$, and

$$F(\pi) = g(r^2 + R(R - 2r)) - g(r^2) + 2rRg'(r^2).$$

Now let R vary, $F(\pi)$ is a function of R. Firstly, if $2r \leq R \leq 4r$, $F(\pi)$ attains its absolute minimum at $R = 2r$. Then at this R,

$$F(\pi) = 4g'(r^2)r^2 \geq g'(NR^2)NR^2 \quad (N = 1/16 \text{ for instance}).$$

Secondly, if $R > 4r$, then $R - 2r > R/2$. Hence, at such R, and for $R^2/16 < t < R^2/2$ we have

$$\begin{aligned} F(\pi) &\geq g\left(r^2 + \frac{R^2}{2}\right) - g(r^2) + 2rRg'(r^2) \\ &\geq g\left(r^2 + \frac{R^2}{2}\right) - g(r^2) \\ &\geq g\left(\frac{R^2}{2}\right) - g\left(\frac{R^2}{16}\right) \\ &= \frac{7}{16}R^2 g'(t). \end{aligned}$$

Finally, if $\tau > R^2/2$, $g'(t) \geq g'(R^2/16)$. Since F is minimized at $\gamma = \pi$, taking the infimum over $0 \leq \gamma \leq 2\pi$, we have

$$\inf_{0 \leq \gamma \leq 2\pi} F(\gamma) \geq g'(NR^2)NR^2, \quad \text{for } r, R < \sqrt{\tau}.$$

Note that $g''(t) > 0$ for all $0 < t < \tau$, so $g(t) \leq tg'(t)$ for $0 < t < \tau$. Applying this to the above inequality, the desired estimate is obtained.

In case $n = 2$, Lemma 3.6 is all that we need to prove the main theorem. However, in case $n \geq 3$, we need more sharp right-hand side.

Lemma 3.7 *Let α, g and η be defined as in Lemma 3.6. Then we have*

$$g(|\zeta+v|^2)-g(|\zeta|^2)-2\operatorname{Re}\left(\frac{\partial g}{\partial\zeta}(|\zeta|^2)v\right)\gtrsim\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}g(|\zeta|^2)|v|^2+g(|v|^2),\quad(3.2)$$

where $\zeta, v\in\mathbb{C}$, $|\zeta|+|v|<\eta$.

Proof By Taylor's series expansion, the left-hand side of (3.2) agrees with Taylor polynomial

$$T(\zeta,v)=\sum_{\alpha+\beta\geq 2}\frac{1}{\alpha!\beta!}\left(\frac{\partial}{\partial\zeta}\right)^{\alpha}\left(\frac{\partial}{\partial\bar{\eta}}\right)^{\beta}g(|\zeta|^2)v^{\alpha}\bar{v}^{\beta}.$$

Firstly, if $\alpha+\beta=2$, by an elementary calculus, we deduce that

$$\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}g(|\zeta|^2)|v|^2+\operatorname{Re}\left(\frac{\partial^2}{\partial\zeta^2}g(|\zeta|^2)v^2\right)\gtrsim\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}g(|\zeta|^2)|v|^2.\quad(3.3)$$

Secondly, when $|\alpha|+|\beta|\geq 3$, we consider $|v|\leq a|\zeta|$ for some $0<a<1$. Then, (3.3) implies that

$$T(\zeta,z)\gtrsim\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}g(|\zeta|^2)|v|^2$$

for $|v|\leq a|\zeta|$ and for a sufficiently small a.

Otherwise, if $|v|>a|z|$, then

$$g(|v|^2)\gtrsim\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}g(|\zeta|^2)|v|^2.$$

Thus, it follows from Lemma 3.6 that

$$T(\zeta,v)\gtrsim\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}g(|\zeta|^2)|v|^2.$$

Hence, gathering results, we obtain the desired inequality. □

Corollary 3.8 *For $\zeta\in b\Omega^{\infty}$, there are constants $c>0$ and $0<s<\dfrac{1}{2(n-1)}$ such that we have*

$$\rho(z)\gtrsim\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k-\zeta_k|^2+\exp\left(1-\frac{1}{|\zeta-z|^{2s}}\right),$$

for $|z-\zeta|<c$ and $z\in(T_{\zeta}(\Omega^{\infty}))^{\mathbb{C}}$—the complex tangent space to $b\Omega^{\infty}$ at ζ.

Proof Firstly, we have

$$\begin{aligned}\rho(z) &= \rho(\zeta) - 2\,\mathrm{Re}\,\Phi(\zeta,z) + (\rho(z) - \rho(\zeta) + 2\,\mathrm{Re}\,\Phi(\zeta,z))\\ &= \rho(\zeta) - 2\,\mathrm{Re}\,\Phi(\zeta,z) + \sum_{k=1}^{n}\left\{\exp\left(1-\frac{1}{|z_k|^{2s_k}}\right) - \exp\left(1-\frac{1}{|\zeta_k|^{2s_k}}\right)\right.\\ &\quad \left. - 2\,\mathrm{Re}\left[\frac{\partial}{\partial\zeta_k}\left(\exp\left(1-\frac{1}{|\zeta_k|^{2s_k}}\right)\right)(\zeta_k - z_k)\right]\right\}. \end{aligned} \tag{3.4}$$

Secondly, it follows from (3.2) that

$$\begin{aligned}\rho(z) &\gtrsim \rho(\zeta) - 2\,\mathrm{Re}\,\Phi(\zeta,z) + \sum_{k=1}^{n}\exp\left(1-\frac{1}{|\zeta_k - z_k|^{2s}}\right)\\ &\quad + \sum_{k=1}^{n}\frac{\partial^2}{\partial\zeta_k\partial\bar{\zeta}_k}\left\{\exp\left(1-\frac{1}{|\zeta_k|^{2s_k}}\right)\right\}|z_k-\zeta_k|^2\\ &\quad \text{(for } 0 < s := \max(s_k) < 1/2(n-1))\\ &\gtrsim \rho(\zeta) - 2\,\mathrm{Re}\,\Phi(\zeta,z) + \sum_{k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k-\zeta_k|^2 + \exp\left(1-\frac{1}{|\zeta - z|^{2s}}\right).\end{aligned}$$

Finally, for some small $c > 0$, if $\zeta \in b\Omega^\infty \cap B(P,c)$ we have

$$\rho(z) \gtrsim \sum_{k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k-\zeta_k|^2 + \exp\left(1-\frac{1}{|\zeta - z|^{2s}}\right),$$

for all $z \in B(\zeta, c)$ with $\Phi(\zeta,z) = 0$. That means Ω^∞ is a domain admitting F-type with $F(t) = \exp\left(1-\frac{1}{t^s}\right)$, $0 < s < \frac{1}{2(n-1)}$. □

Next we compute the integral

$$\int_0^d (-\ln t)^{n-s-2}\frac{\sqrt{F^*(t)}}{t}\mathrm{d}t$$

for $0 < d < 1$, where $F(t) = \exp\left(1-\frac{1}{t^\alpha}\right)$, $0 < \alpha < \frac{1}{2(n-1)}$. Since $F^*(t) = \frac{1}{t(1-\ln t)^{\frac{1}{\alpha}}}$, we have

$$\begin{aligned}
&\int_0^d (-\ln t)^{n-s-2} \frac{\sqrt{F^*(t)}}{t} \mathrm{d}t \\
&= \int_{1-\ln d}^{+\infty} \frac{(y-1)^{n-2-s}}{y^{\frac{1}{2\alpha}}} \mathrm{d}y \\
&= \int_{1-\ln d}^{+\infty} y^{-\frac{1}{2\alpha}} \left(\sum_{j=0}^{n-s-2} \binom{n-s-2}{j} y^{n-s-2-j} (-1)^j \right) \mathrm{d}y \\
&= \sum_{j=0}^{n-s-2} \frac{(-1)^j}{n-s-1-j-\frac{1}{2\alpha}} \binom{n-s-2}{j} y^{n-s-1-j-\frac{1}{2\alpha}} \Bigg|_{y=1-\ln d}^{y\to+\infty} .
\end{aligned}$$

Since $0 < \alpha < \dfrac{1}{2(n-1)}$,

$$\sum_{j=0}^{n-s-2} \frac{(-1)^j}{n-s-1-j-\frac{1}{2\alpha}} \binom{n-s-2}{j} y^{n-s-1-j-\frac{1}{2\alpha}} \Bigg|_{y\to+\infty} = 0.$$

Immediately we have

$$\begin{aligned}
&\int_0^d (-\ln t)^{n-s-2} \frac{\sqrt{F^*(t)}}{t} \mathrm{d}t \\
&= (1-\ln d)^{n-s-1-\frac{1}{2\alpha}} \sum_{j=0}^{n-s-2} \frac{(-1)^{j+1}}{n-s-1-j-\frac{1}{2\alpha}} \binom{n-s-2}{j} (1-\ln d)^{-j}.
\end{aligned}$$

This gives

$$f(d) = \frac{(1+|\ln d|)^{\frac{1}{2\alpha}+1+s-n}}{\sum_{j=0}^{n-s-2} \frac{(-1)^{j+1}}{n-s-1-j-\frac{1}{2\alpha}} \binom{n-s-2}{j} (1+|\ln d|)^{-j}},$$

where f is defined in Main Theorem. Therefore, as a consequence of Main Theorem, we have

Corollary 3.9 *Let $\varphi \in C^k_{(0,q)}(\Omega^\infty)$ be a $\bar{\partial}$-closed differential form on Ω^∞ with $k \geq 0$ and $1 \leq q \leq n-1$.*

(1) *If $n = 2$, there exists a function $u \in \Lambda^f_k(\Omega^\infty)$ so that $\bar{\partial} u = \varphi$ and*

$$\|u\|_{\Lambda^f_k(\Omega^\infty)} \leq C_k \|\varphi\|_{C^k_{(0,1)}(\Omega^\infty)},$$

where

$$f(d) = \left(\frac{1}{2\alpha} - 1\right) (1+|\ln d|)^{\frac{1}{2\alpha}-1}.$$

(2) *If* $n \geq 3$, *there exists a* $(0, q-1)$-*differential form* $u \in \Lambda^{k,f}_{(0,q-1)}(\Omega^\infty)$ *so that* $\bar{\partial} u = \varphi$ *and*

$$\|u\|_{\Lambda^{k,f}_{(0,q-1)}(\Omega^\infty)} \leq C_{k,s,n} \|\varphi\|_{C^k_{(0,q)}(\Omega^\infty)},$$

for every $0 \leq s \leq n-3$ *and where*

$$f(d) = \frac{(1 + |\ln d|)^{\frac{1}{2\alpha}+1+s-n}}{\sum_{j=0}^{n-s-2} \frac{(-1)^{j+1}}{n-s-1-j-\frac{1}{2\alpha}} \binom{n-s-2}{j} (1+|\ln d|)^{-j}}.$$

Example 3.10 By the same arguments above with some minor modifications, it is not difficult to show that

$$\Omega_0^\infty = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \rho(z) = 2 \sum_{k=1}^{n} \exp\left(-\frac{1}{|z_k| \cdot |\ln(|z_k|)|^{\alpha_k}} \right) - 1 < 0 \right\},$$

where $\alpha_k > 2$, is a convex domain admitting F-type with $F(t^2) = 2\exp\left(-\frac{1}{t|\ln t|^\alpha}\right)$, $\alpha = \max_k \alpha_k$.

4 Higher Derivatives of Henkin–Ramirez Solution on Convex Domains

As mentioned in Sect. 1, we briefly recall the construction of Henkin–Ramirez solution operators for the $\bar{\partial}$-equation in Ω.

For $1 \leq j \leq n$, and $\lambda \in [0, 1]$ we define

$$\begin{aligned}
\omega_j^1(\zeta, z) &= \frac{\frac{\partial \rho}{\partial \zeta_j}(\zeta)}{\widetilde{\Phi}(\zeta, z)} \quad \text{on } \{(\zeta, z) : \widetilde{\Phi}(\zeta, z) \neq 0\}, \\
P_j^0(\zeta, z) &= \bar{\zeta}_j - \bar{z}_j, \quad \Phi^0(\zeta, z) = |\zeta - z|^2, \\
\omega_j^0(\zeta, z) &= \frac{P_j^0(\zeta, z)}{\Phi^0(\zeta, z)} \quad \text{for } \zeta \neq z, \\
\omega_j(\zeta, z, \lambda) &= (1-\lambda)\omega_j^0(\zeta, z) + \lambda \omega_j^1(\zeta, z).
\end{aligned}$$

Assume that there exists a small $\delta_0 > 0$ such that for all $|\delta| \leq \delta_0$, domains $\{z : \rho(z) < \delta\}$ are convex. Let G be an open neighborhood of $\bar{\Omega}$ such that the closure $\bar{G} \subset \{z : \rho(z) < \delta_0\}$, and we set $K = \bar{G} \setminus \Omega$.

Since $\omega = (\omega_1, \dots, \omega_n)$ is well defined on $K \times \Omega \times [0, 1]$, for $q = 0, 1, \dots, n-1$, we set

$$\begin{aligned} &K_n(\zeta, z, \lambda) = K_{-1}(\zeta, z, \lambda) = 0, \text{ and} \\ &K_q(\zeta, z, \lambda) \\ &\quad = (-1)^q \binom{n-1}{q} \det(w, \underbrace{\bar\partial_z w, \dots, \bar\partial_z w}_{q-\text{times}}, \underbrace{\bar\partial_{\zeta,\lambda} w, \dots, \bar\partial_{\zeta,\lambda} w}_{(n-q-1)-\text{times}}) \wedge \mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n, \end{aligned}$$

where $\bar\partial_{\zeta,\lambda} = \bar\partial_\zeta + d_\lambda$. Notice that

$$\bar\partial_{\zeta,\lambda} K_q(\zeta, z, \lambda) = (-1)^q \bar\partial_z K_{q-1}(\zeta, z, \lambda).$$

The following lemma is a multi-dimensional version of the Cauchy–Pompeiu formula in one complex variable.

Lemma 4.1 (Bochner–Martinelli–Koppelman formula) ([26, Lemma 2.4]) *Let* $B_q(\zeta, z) = K_q(\zeta, z, 0)$. *For* $\varphi \in C^1_{(0,q)}(\bar\Omega)$ *and* $z \in \Omega$, *we have*

$$\begin{aligned} \varphi(z) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \Bigg(&\int_{b\Omega} \varphi(\zeta) \wedge B_q(\zeta, z) - \int_\Omega \bar\partial \varphi(\zeta) \wedge B_q(\zeta, z) \\ &- \bar\partial_z \int_\Omega \varphi(\zeta) \wedge B_{q-1}(\zeta, z) \Bigg). \end{aligned} \tag{4.1}$$

The Henkin–Ramirez solution operator for the $\bar\partial$-problem is given in the following lemma.

Lemma 4.2 [26, Sect. 2] *For* $\varphi \in C_{(0,q)}(\bar\Omega)$, $1 \le q \le n$, *and* $z \in \Omega$ *define*

$$\mathcal{T}_q \varphi(z) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \left(\int_{b\Omega \times [0,1]} \varphi(\zeta) \wedge K_{q-1}(\zeta, z, \lambda) - \int_\Omega \varphi(\zeta) \wedge B_{q-1}(\zeta, z) \right).$$

Then, if $\varphi \in C^1_{(0,q)}(\bar\Omega)$ *is* $\bar\partial$*-closed, then*

$$\bar\partial \mathcal{T}_q \varphi = \varphi$$

on Ω.

In order to apply Stoke's theorem to $\mathcal{T}_q\varphi$, we must modify $\mathcal{T}_q$ as I. Lieb and R.M. Range have done in [21]. To do so, we need a Seeley type operator (see [29] for details). We denote by $C^k_c(W)$ the space of $C^k(W)$-functions from $W \to \mathbb{C}$ with compact support.

Lemma 4.3 (Seeley extension) *If* $\Omega \subset \mathbb{R}^N$ *is an open set with smooth boundary and* G *is a neighborhood of* $\bar\Omega$. *Then there exists a linear operator* $E : C^0(\bar\Omega) \to C^0_c(G)$, *such that*

(1) $Eu|_{\bar\Omega} = u$;
(2) *for* $k = 0, 1, \ldots,$ *if* $u \in C^k(\bar\Omega)$, *then* $Eu \in C^k_c(G)$;
(3) *for* $k = 0, 1, \ldots$, *there exists a constant* C_k *so that*

$$\|Eu\|_{C^k(G)} \le C_k \|u\|_{C^k(\Omega)}.$$

Let $\varphi = \sum_I \varphi_I \mathrm{d}\bar z^I$ be a $(0, q)$-differential form on Ω. We also write $E\varphi = \sum_I E\varphi_I d\bar z^I$ as the extension of φ. Then the Henkin–Ramirez solution operator for the $\bar\partial$-equation is extended to G as follows.

Definition 4.4 For $1 \le q \le n$, $\varphi \in C_{(0,q)}(\bar\Omega)$, and $z \in \Omega$, we define

$$S_q\varphi(z) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\Bigg[\int_{b\Omega\times[0,1]} \varphi(\zeta)\wedge K_{q-1}(\zeta, z, \lambda) - \int_\Omega \varphi(\zeta)\wedge B_{q-1}(\zeta, z) - \int_{K\times\{1\}} E\varphi(\zeta)\wedge K_{q-1}(\zeta, z, \lambda) - \bar\partial_z \int_{K\times[0,1]} E\varphi(\zeta)\wedge K_{q-2}(\zeta, z, \lambda)\Bigg].$$

Lemma 4.5 [27, Lemma 3.2] *If* $\varphi \in C^1_{(0,q)}(\bar\Omega)$, $\bar\partial\varphi = 0$ *on* Ω, *then*

$$\bar\partial S_q\varphi(z) = \varphi(z)$$

for $z \in \Omega$. *In this case, we have*

$$S_q\varphi(z) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\left(\int_{K\times[0,1]} \bar\partial E\varphi(\zeta)\wedge K_{q-1}(\zeta, z, \lambda) - \int_G E\varphi(\zeta)\wedge B_{q-1}(\zeta, z)\right).$$

Proof Firstly, since

$$S_q\varphi(z) = \mathcal{T}_q\varphi(z) - \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\left(\int_{K\times[0,1]} E\varphi(\zeta)\wedge K_{q-1}(\zeta, z, \lambda) + \bar\partial_z \int_{K\times[0,1]} E\varphi(\zeta)\wedge K_{q-2}(\zeta, z, \lambda)\right)$$

and $\bar\partial_z \mathcal{T}_q\varphi = \varphi$, and $(\bar\partial)^2 = 0$, we obtain

$$\bar\partial_z S_q\varphi(z) = \varphi(z) - \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\underbrace{\bar\partial_z \int_{K\times\{1\}} E\varphi(\zeta)\wedge K_{q-1}(\zeta, z, \lambda)}_{=0} = \varphi(z).$$

Secondly, since $\bar\partial_z(E\varphi(\zeta)\wedge K_{q-2}(\zeta,z,\lambda)) = -E\varphi(\zeta)\wedge\bar\partial_{\zeta,\lambda}K_{q-1}(\zeta,z,\lambda)$ we have

$$\begin{aligned}
&\int_{K\times[0,1]}\bar\partial E\varphi(\zeta)\wedge K_{q-1}(\zeta,z,\lambda) - \bar\partial_z\int_{K\times[0,1]}E\varphi(\zeta)\wedge K_{q-2}(\zeta,z,\lambda)\\
&\quad=\int_{K\times[0,1]}\bar\partial E\varphi(\zeta)\wedge K_{q-1}(\zeta,z,\lambda)+\int_{K\times[0,1]}E\varphi(\zeta)\wedge\bar\partial_{\zeta,\lambda}K_{q-1}(\zeta,z,\lambda)\\
&\quad=\int_{K\times[0,1]}\bar\partial_{\zeta,\lambda}(E\varphi(\zeta)\wedge K_{q-1}(\zeta,z,\lambda))\\
&\quad=\int_{b(K\times[0,1])}E\varphi\wedge K_{q-1}(\zeta,z,\lambda)\quad\text{(by Stoke's Theorem).}
\end{aligned}$$

The fact $b(K\times[0,1]) = (bG\times[0,1])\cup(K\times\{1\})\setminus(b\Omega\times[0,1])\setminus(K\times\{0\})$ implies

$$\begin{aligned}
&\int_{K\times[0,1]}\bar\partial E\varphi(\zeta)\wedge K_{q-1}(\zeta,z,\lambda) - \bar\partial_z\int_{K\times[0,1]}E\varphi(\zeta)\wedge K_{q-2}(\zeta,z,\lambda)\\
&\quad=\underbrace{\int_{bG\times[0,1]}E\varphi\wedge K_{q-1}(\zeta,z,\lambda)}_{=0\text{ since }E\varphi=0\text{ on }bG}+\int_{K\times\{1\}}E\varphi\wedge K_{q-1}(\zeta,z,\lambda)\\
&\qquad-\int_{b\Omega\times[0,1]}E\varphi\wedge K_{q-1}(\zeta,z,\lambda)-\int_{K\times\{0\}}E\varphi\wedge K_{q-1}(\zeta,z,\lambda).
\end{aligned}$$

Hence the desired identity follows immediately. □

The second integral over $\int_G E\varphi(\zeta)\wedge B_{q-1}(\zeta,z)$ is not significant in our analysis since this operator is bounded from $C^k_{(0,q)}(\bar\Omega)$ into $\Lambda^{k,f}_{(0,q)}(\Omega)$ for all $0 < f(t) \lesssim t$. The problematic subject is the first integral

$$\mathcal{T}\varphi(z) = \int_{K\times I}\bar\partial E\varphi(\zeta)\wedge K_{q-1}(\zeta,z,\lambda),$$

since K contains the boundary $b\Omega$.

Next, we will write down the operator $\mathcal{T}\varphi$ into a linear combination of simple terms. The definition of K_q implies that K_{q-1} is a linear combination of the terms

$$\lambda^i(1-\lambda)^j\det(\omega^p,\underbrace{\bar\partial_z\omega^0,\ldots,\bar\partial_z\omega^1}_{(q-1)-\text{times}},(\omega^1-\omega^0)d\lambda,\underbrace{\bar\partial_\zeta\omega^0,\ldots,\bar\partial_\zeta\omega^1}_{(n-q-1)-\text{times}})\wedge d\zeta_1\wedge\cdots\wedge d\zeta_n$$

for $i+j=n-1$, $p=0,1$, and

$$\lambda^i(1-\lambda)^j\det(\omega^p,\underbrace{\bar\partial_z\omega^0,\ldots,\bar\partial_z\omega^1}_{(q-1)-\text{times}},\underbrace{\bar\partial_\zeta\omega^0,\ldots,\bar\partial_\zeta\omega^1}_{(n-q)-\text{times}})\wedge d\zeta_1\wedge\cdots\wedge d\zeta_n$$

for $i+j=n-1$, $p=0,1$.

Then, taking the wedge product by $\bar{\partial} E\varphi$ and integrating over $\lambda \in [0, 1]$, the second integrals involving terms of second terms equal to zero. Hence, $\mathcal{T}\varphi(z)$ is a linear combination of the terms

$$\int_K \bar{\partial} E\varphi \det(\omega^0, \omega^1, \bar{\partial}_z\omega^0 \ldots, \bar{\partial}_z\omega^1, \bar{\partial}_\zeta\omega^0 \ldots, \bar{\partial}_\zeta\omega^1) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

From the definition of Φ, we have

$$\frac{\partial \Phi}{\partial \zeta_i}(\zeta, z) = \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(\zeta_j - z_j) + \frac{\partial \rho}{\partial \zeta_j}(\zeta),$$

and so

$$\frac{\partial \Phi}{\partial \zeta_i}(\zeta, \zeta) = \frac{\partial \rho}{\partial \zeta_j}(\zeta), \quad \text{for all } \zeta \in K.$$

Since $\nabla\rho \neq 0$ on $b\Omega$, for each $\zeta_0 \in b\Omega$, there exists an index ν_{ζ_0} such that

$$\frac{\partial \Phi}{\partial \zeta_{\nu_{\zeta_0}}}(\zeta_0, \zeta_0) \neq 0.$$

The compactness of K implies that there are a finite covering $\{U_j\}_{j=1}^m$ of K and $a > 0$ such that for every j there is ν_j with

$$\left|\frac{\partial \Phi}{\partial \zeta_{\nu_j}}\right| \geq a \quad \text{on } U_j \times U_j.$$

Let $\{\chi_j\}_{j=1}^m \subset C^\infty(K)$ be a partition of unity of K with corresponding to the finite covering $\{U_j\}_{j=1}^m$, so

$$\sum_{j=1}^{m} \chi_j = 1 \quad \text{on } K.$$

Since

$$\det(\omega^0, \underbrace{\bar{\partial}_z\omega^0, \ldots, \bar{\partial}_\zeta\omega^0}_{s-\text{times}}, \omega^1, \underbrace{\bar{\partial}_z\omega^1, \ldots, \bar{\partial}_\zeta\omega^1}_{(n-s-2)-\text{times}})$$
$$= \frac{\det(P^0, \bar{\partial}_z P^0, \ldots, \bar{\partial}_\zeta P^0, P^1, \bar{\partial}_z P^1, \ldots, \bar{\partial}_\zeta P^1)}{(\Phi^0(\zeta, z))^{s+1}(\widetilde{\Phi}(\zeta, z))^{n-s-1}}$$

for all $0 \leq s \leq n-2$, it is sufficient to estimate for every j and J, $0 \leq s \leq n-2$,

$$\left| D_z^\alpha \int_{U_j} \chi_j(\zeta)\phi_j(\zeta) \frac{\det(P^0, \bar{\partial}_z P^0, \ldots, \bar{\partial}_\zeta P^0, P^1, \bar{\partial}_z P^1, \ldots, \bar{\partial}_\zeta P^1)}{(\Phi^0(\zeta,z))^{s+1}(\widetilde{\Phi}(\zeta,z))^{n-s-1}} \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n \right|, \tag{4.2}$$

where $\bar{\partial} E\varphi(\zeta) = \sum_{|J|=q+1} \phi_J(\zeta) d\bar{\zeta}^J$, and $2 \leq |\alpha| \leq k+1$ due to Lemma 5.1. Since $\phi_J = 0$ on Ω and $\phi_J \in C_c^{k-1}(G)$, $\chi_j \phi_J \in C_c^{k-1}(U_j)$ and $\chi_j \phi_J = 0$ on $U_j \cap \Omega$.

For convenience, we assume $\left|\frac{\partial \widetilde{\Phi}}{\partial \zeta_\nu}\right| \geq a$ on $U \times U$. We try to express the long integral (4.2) explicitly. Since Ω is admissibly decoupled, it is enough to consider the following integrals, for each $z \in U$ and $2 \leq |\alpha| \leq k+1$, $0 \leq s \leq n-2$,

$$D_z^\alpha \int_U \chi(\zeta)\phi_J(\zeta) \frac{(\bar{\zeta}_0 - \bar{z}_0)\frac{\partial\rho}{\partial\zeta_1}(\zeta)\frac{\partial^2\rho}{\partial\zeta_2\partial\bar{\zeta}_2}(\zeta)\ldots\frac{\partial^2\rho}{\partial\zeta_{n-s-1}\partial\bar{\zeta}_{n-s-1}}(\zeta)}{(\Phi^0(\zeta,z))^{s+1}(\widetilde{\Phi}(\zeta,z))^{n-s-1}} d\zeta.$$

Lemma 4.6 (Converting derivative lemma) *Assume*

$$\left|\frac{\partial \widetilde{\Phi}}{\partial \zeta_\nu}\right| \geq a \quad \text{on } U \times U, \quad \text{where } U \text{ is a member in the above } \{U_j\}_{j=1}^m.$$

Then for each $z \in U$ and $2 \leq |\alpha| \leq k+1$, the integral

$$D_z^\alpha \int_U \chi(\zeta)\phi_J(\zeta) \frac{(\bar{\zeta}_0 - \bar{z}_0)\frac{\partial\rho}{\partial\zeta_1}(\zeta)\frac{\partial^2\rho}{\partial\zeta_2\partial\bar{\zeta}_2}(\zeta)\ldots\frac{\partial^2\rho}{\partial\zeta_{n-s-1}\partial\bar{\zeta}_{n-s-1}}(\zeta)}{(\Phi^0(\zeta,z))^{s+1}(\widetilde{\Phi}(\zeta,z))^{n-s-1}} d\zeta$$

is a linear combination of the following terms

$$\int_U D_\zeta^\beta(\chi(\zeta)\phi_J(\zeta)) \frac{N_j(\zeta,z)\psi(\zeta)}{|\zeta - z|^p(\widetilde{\Phi}(\zeta,z))^{n-s-1+m}} \kappa(\zeta) d\zeta, \tag{4.3}$$

where

$$\psi(\zeta) = \begin{cases} \displaystyle\prod_{j=1}^{n-s-1} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j}(\zeta) & \text{if } \nu \notin \{2,\ldots,n-s-1\}, \\ \displaystyle\frac{\partial\rho}{\partial\zeta_1} \prod_{\substack{j=2 \\ j\neq\nu}}^{n-s-1} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j} & \text{if } \nu \in \{2,\ldots,n-s-1\}, \end{cases}$$

and $0 \le |\beta| \le |\alpha| - 2$*,* $\kappa(\zeta)$ *is a smooth, bounded function,* $m = 0, 1, 2$*, and* $N_j(\zeta, z)$*'s are products of* j*-factors of the form* $(\zeta_j - z_j)$ *or* $(\bar{z}_j - \bar{\zeta}_j)$ *with*

$$\frac{|N_j(\zeta, z)|}{|z - \zeta|^p} \le \frac{1}{|z - \zeta|^{2s+q+1}}, \quad \textit{for } (|\alpha| - |\beta|) \ge (m + q).$$

Proof This lemma is proved in [27, Proposition 3.1]. □

Since

$$\left|\frac{\partial \rho}{\partial z_j}(z)\right| \lesssim \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_j}(z)$$

for all $|z| < 1$,

$$|\psi(\zeta)| \lesssim \frac{\partial^2 \rho}{\partial \zeta_1 \partial \bar{\zeta}_1}(\zeta) \ldots \frac{\partial^2 \rho}{\partial \zeta_{n-s-2} \partial \bar{\zeta}_{n-s-2}}(\zeta). \tag{4.4}$$

5 Proof of the Main Theorem

In case $F(t) = t^m$, we use the Hardy–Littlewood Lemma to obtain standard Hölder estimates. However, since the F-type function F in this paper is more general, we must apply the following result which was proved in [17].

Lemma 5.1 (General Hardy–Littlewood lemma) *Let* Ω *be a smoothly bounded domain in* $\mathbb{R}^N$ *and let* ρ *be a defining function of* Ω*. Let* $G : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function such that* $\frac{G(t)}{t}$ *is decreasing and* $\int_0^d \frac{G(t)}{t} \mathrm{d}t < \infty$ *for* $d > 0$ *small enough. If* $u \in C^1(\Omega)$ *such that*

$$|\nabla u(x)| \lesssim \frac{G(|\rho(x)|)}{|\rho(x)|} \quad \textit{for every } x \in \Omega,$$

then

$$f(|x - y|^{-1})|u(x) - u(y)| < \infty$$

uniformly in $x, y \in \Omega$*,* $x \ne y$*, and where* $f(d^{-1}) := \left(\int_0^d \frac{G(t)}{t} \mathrm{d}t\right)^{-1}$.

Let $U \subset G$ be a member of $\{U_j\}$, where $\{U_j\}$ is defined in the previous section.

Lemma 5.2 *If* $v \in C_c^{k-1}(U)$*,* $v = 0$ *on* $\Omega \cap U$*, then*

$$\left|\int_U D^\beta v(\zeta) \frac{\psi(\zeta)}{|z - \zeta|^{2s+p+1} (\widetilde{\Phi}(\zeta, z))^{n-s-1+m}}\right| \lesssim \|v\|_{k-1,U} \frac{G(|\rho(z)|)}{|\rho(z)|},$$

where $m = 0, 1, 2$, $(k-1-|\beta|) - (p+m) \geq -2$, $|\beta| \leq k-1$, *and for small* $t > 0$

$$G(t) = \begin{cases} \sqrt{F^*(t)}, & \text{if } n = 2, \\ (-\ln t)^{n-s-2}\sqrt{F^*(t)}, & \text{if } n \geq 3, \quad \text{for } 0 \leq s \leq n-3. \end{cases}$$

Proof Since $v(\zeta) = 0$ for $\zeta \in U \cap \Omega$, by Mean Value Theorem, we have

$$|D^\beta v(\zeta)| \lesssim \|v\|_{k-1,U}|\zeta - z|^{k-1-|\beta|}, \quad z \in U \cap \Omega.$$

This implies

$$\begin{aligned} &\left| D^\beta v(\zeta) \frac{\psi(\zeta)}{|z-\zeta|^{2s+p+1}(\widetilde{\Phi}(\zeta,z))^{n-s-1+m}} \right| \\ &\lesssim \|v\|_{k-1,U} \left| \frac{\psi(\zeta)}{|\zeta - z|^{2s+p+1-(k-1-|\beta|)}(\widetilde{\Phi}(\zeta,z))^{n-s-1+m}} \right| \\ &\lesssim \|v\|_{k-1,U} \frac{|\psi(\zeta)|}{|\zeta - z|^{2s+M}|\widetilde{\Phi}(\zeta,z)|^{n-s+2-M}}, \\ &\quad \text{with } M = 3 - m = 1, 2, 3. \end{aligned}$$

Therefore, it is sufficient to show that

$$\begin{aligned} &\int_{K\cap U} \frac{|\psi(\zeta)|}{|\zeta - z|^{2s+M}|\widetilde{\Phi}(\zeta,z)|^{n-s+2-M}} d\zeta \\ &\lesssim \begin{cases} \dfrac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}, & \text{if } n = 2, \\ \dfrac{(-\ln|\rho(z)|)^{n-s-2}\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}, & \text{if } n \geq 3, \text{ for } 0 \leq s \leq n-3. \end{cases} \end{aligned}$$

Since the second estimate ($n = 2$) is proved similarly and simpler than the first one, we omit its proof here. This means we must show that

$$\int_{K\cap U} \frac{|\psi(\zeta)|}{|\zeta - z|^{2s+M}|\widetilde{\Phi}(\zeta,z)|^{n-s+2-M}} d\zeta \lesssim \frac{(-\ln|\rho(z)|)^{n-s-2}\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}, \quad (5.1)$$

if $n \geq 3$, for $0 \leq s \leq n-3$, and $M = 1, 2, 3$.

Recalling that (4.4) holds on $K \cap U$, by Corollary 2.6 we have

$$\left| \frac{\psi(\zeta)}{|\widetilde{\Phi}(\zeta,z)|^{n-s-2}} \right| \lesssim \prod_{j=1}^{n-s-2} \frac{1}{\rho(\zeta) - \rho(z) + |\zeta_j - z_j|^2} \lesssim \prod_{j=1}^{n-s-2} \frac{1}{|\rho(z)| + |\zeta_j - z_j|^2},$$

where $\rho(\zeta) > 0$ since $\zeta \in K \cap U$, $-\rho(z) = |\rho(z)|$ since $z \in \Omega \cap U$.

Hence, all of integrals of the form (5.1) are bounded from above by

$$I_1 = \int_{K\cap U} \frac{d\zeta}{|\widetilde{\Phi}(\zeta,z)|^3|\zeta - z|^{2s+1}\prod_{j=1}^{n-s-2}(|\rho(z)| + |\zeta_j - z_j|^2)},$$
$$I_2 = \int_{K\cap U} \frac{d\zeta}{|\widetilde{\Phi}(\zeta,z)|^2|\zeta - z|^{2s+2}\prod_{j=1}^{n-s-2}(|\rho(z)| + |\zeta_j - z_j|^2)},$$
$$I_3 = \int_{K\cap U} \frac{d\zeta}{|\widetilde{\Phi}(\zeta,z)||\zeta - z|^{2s+3}\prod_{j=1}^{n-s-2}(|\rho(z)| + |\zeta_j - z_j|^2)}. \quad (5.2)$$

Moreover, since Ω is bounded and ρ is smooth, $|\widetilde{\Phi}(\zeta,z)| \lesssim |\zeta - z|$. This gives that

$$I_2, I_3 \lesssim I_1.$$

Next, in order to estimate I_1, we localize the domain $\Omega\cap U$ by the so-called the Henkin coordinates see ([14, p. 608]).

Lemma 5.3 (Henkin coordinates) *There is a neighborhood W of $b\Omega$ such that if $z \in W$ and*

$$\frac{\partial\rho}{\partial\zeta_n}(z) \neq 0,$$

then the function $(x_1,\dots,x_{2n-2},y,t)$ form a set of real coordinates in some neighborhood of z, where

$$\begin{cases} y(\zeta) = \operatorname{Im}\widetilde{\Phi}(\zeta,z),\\ t(\zeta) = \rho(\zeta),\\ x_{2j} = \operatorname{Im}(\zeta_j - z_j), \quad x_{2j-1}(\zeta) = \operatorname{Re}(\zeta_j - z_j), \quad j = 1,\dots,n-1. \end{cases}$$

Applying the Henkin coordinates to I_1, with $(x,y,t) = (x_1,\dots,x_{2n-2},y,t)$, we obtain

$$I_1 \lesssim \int_{|(x,y,t)|\le c} \frac{dx\,dy\,dt}{(y + t + |\rho(z)| + F(|x'|^2))^3|x|^{2s+1}\prod_{j=1}^{n-s-2}(|\rho(z)| + x_{2j-1}^2 + x_{2j}^2)}$$
$$\lesssim \int_{|x|\le c} \frac{dx'}{(|\rho(z)| + F(|x|^2))|x|^{2s+1}\prod_{j=1}^{n-s-2}(|\rho(z)| + x_{2j-1}^2 + x_{2j}^2)}$$
$$\lesssim \int_{|(x'',x''')|\le c} \frac{dx'dx''}{(|\rho(z)| + F(|x''|^2))|x''|^{2s+1}\prod_{j=1}^{n-s-2}(|\rho(z)| + x_{2j-1}^2 + x_{2j}^2)}$$

(where $x' = (x_1,\dots,x_{2n-2s-4}) \in \mathbb{R}^{2n-2s-4}$, $x'' = (x_{2n-2s-3},\dots,x_{2n-2}) \in \mathbb{R}^{2s+2}$)

$$\lesssim \int_{|x'|\le c} \frac{\mathrm{d}x'}{\prod_{j=1}^{n-s-2}(|\rho(z)|+x_{2j-1}^2+x_{2j}^2)} \int_{|x''|\le c} \frac{\mathrm{d}x''}{(|\rho(z)|+F(|x''|^2))|x''|^{2s+1}}$$
$$\lesssim \prod_{j=1}^{n-s-2}\left[\iint_{[-c,c]^2} \frac{\mathrm{d}x_{2j-1}\mathrm{d}x_{2j}}{(|\rho(z)|+x_{2j-1}^2+x_{2j}^2)}\right] \int_{|x''|\le c} \frac{\mathrm{d}x''}{(|\rho(z)|+F(|x''|^2))|x''|^{2s+1}}.$$

Notice that

$$\int_0^c \frac{r\mathrm{d}r}{|\rho(z)|+r^2} \lesssim (-\ln|\rho(z)|).$$

Therefore, using polar coordinates for each (x_{2j-1}, x_{2j}), we get

$$I_1 \lesssim (-\ln|\rho(z)|)^{(n-s-2)} \int_{|x''|\le c} \frac{\mathrm{d}x''}{(|\rho(z)|+F(|x''|^2))|x''|^{2s+1}}.$$

Using the spherical coordinates $r=|x''|$ in $\mathbb{R}^{2s+2}$, we obtain immediately the following estimate

$$I_1 \lesssim (-\ln|\rho(z)|)^{(n-s-2)} \int_0^c \frac{\mathrm{d}r}{(|\rho(z)|+F(r^2))}.$$

Now, the last integral is estimated inspired by the techniques by Khanh in [17]. We split it into two parts

$$\int_0^c \frac{\mathrm{d}r}{|\rho(z)|+F(r^2)} = \underbrace{\int_0^{\sqrt{F^*(|\rho(z)|)}} \frac{\mathrm{d}r}{|\rho(z)|+F(r^2)}}_{\textbf{easy part}} + \underbrace{\int_{\sqrt{F^*(|\rho(z)|)}}^c \frac{\mathrm{d}r}{|\rho(z)|+F(r^2)}}_{\textbf{diff. part}}.$$

It is clear that the "easy part" is bounded from above by $\dfrac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}$.

For the "diff. part", if $r \ge \sqrt{F^*(|\rho(z)|)}$, then

$$\frac{F(r^2)}{r^2} \ge \frac{F(F^*(|\rho(z)|))}{F^*(|\rho(z)|)} = \frac{|\rho(z)|}{|F^*(|\rho(z)|)|}, \text{ (since } F \text{ is increasing).}$$

Then we have

$$\frac{F(r^2)}{|\rho(z)|} \ge \frac{r^2}{F^*(|\rho(z)|)}.$$

Therefore,

$$\int_{\sqrt{F^*(|\rho(z)|)}}^1 \frac{\mathrm{d}r}{|\rho(z)|+F(r^2)} \le \frac{1}{|\rho(z)|}\int_{\sqrt{F^*(|\rho(z)|)}}^1 \frac{\mathrm{d}r}{1+r^2/F^*(|\rho(z)|)}$$
$$\le \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}\int_1^\infty \frac{\mathrm{d}y}{1+y^2} = \frac{\pi}{4}\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

Combining all results, we have

$$I_1 \lesssim (-\ln|\rho(z)|)^{(n-s-2)}\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}. \tag{5.3}$$

Therefore the proof of the lemma is complete. □

Proposition 5.4 *Let $\varphi \in C^k_{(0,q)}(\Omega)$ be a $\bar\partial$-closed differential form with $k \ge 1$. Then,*

(1) *if $n = 2$, we have*

$$\|S_q\varphi\|_{\Lambda^{k,f}_{(0,q-1)}(\Omega)} \lesssim \|\varphi\|_{C^k_{(0,q)}(\Omega)},$$

where

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)}}{t}\mathrm{d}t\right)^{-1}.$$

(2) *if $n \ge 3$, we have*

$$\|S_q\varphi\|_{\Lambda^{k,f}_{(0,q-1)}(\Omega)} \lesssim \|\varphi\|_{C^k_{(0,q)}(\Omega)},$$

where

$$f(d^{-1}) = \left(\int_0^d \frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t}\mathrm{d}t\right)^{-1}.$$

Proof Let U be the open set in Lemma 5.2. We show that the assertions (1) and (2) hold on $\Omega\cap U$. Therefore, by the estimates in Lemma 5.2, it suffices to check that $\sqrt{F^*(t)}$ and $(-\ln t)^{n-s-2}\sqrt{F^*(t)}$ satisfy all conditions in General Hardy–Littlewood Lemma for $t > 0$ small. When $n = 2$, the proof is more easy than the case $n \ge 3$. The fact $(-\ln t)^{n-s-2}\sqrt{F^*(t)}$ is decreasing is trivial. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(-\ln F(t^2))^{n-s-2}t}{F(t^2)}\right)$$
$$= \frac{(-\ln F(t^2))^{n-s-2}}{F^2(t^2)}\left(\frac{2(n-s-2)t^2F'(t^2)}{(\ln F(t^2))} + F(t^2) - 2t^2F'(t^2)\right).$$

Since $\frac{F(t)}{t}$ is increasing, $tF'(t) \geq F(t)$. Therefore, for small $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(-\ln F(t^2))^{n-s-2}t}{F(t^2)}\right) < 0,$$

and so

$$\frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t}$$

is also decreasing. Finally, we show that

$$\int_0^c \frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t}\mathrm{d}t < \infty.$$

Let $t = F(y^2)$, we have

$$\begin{aligned}
&\int_0^c \frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t}\mathrm{d}t \\
&\quad = -\int_0^{F^*(c)} y\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{(-\ln F(y^2))^{n-s-1}}{n-s-1}\right)\mathrm{d}y \\
&\quad = -\, y\left(\frac{(-\ln F(y^2))^{n-s-1}}{n-s-1}\right)\Bigg|_{y=0}^{\sqrt{F^*(c)}} + \int_0^{F^*(c)}\left(-\ln F(y^2)\right)^{n-s-1}\mathrm{d}y
\end{aligned}$$

Since $(-\ln(F(t^2))^{n-s-1}$ is decreasing when $0 \leq t \leq \delta$, for $0 < \delta < c$ small enough, we have

$$(-\ln F(\eta^2))^{n-s-1}|\eta \leq \int_0^{\eta}(-\ln F(t^2))^{n-s-1}\mathrm{d}t \leq \int_0^{\delta}(-\ln F(t^2))^{n-s-1}\mathrm{d}t < \infty,$$

uniformly in $0 \leq \eta \leq \delta$. Hence, $\sqrt{F^*(t)}(-\ln t)^{n-s-1} < \infty$, for all $0 \leq t \leq \sqrt{F^*(\delta)}$, and $\lim_{t\to 0} t(-\ln F(t^2))^{n-s-1} = 0$. These imply

$$\int_0^c \frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t}\mathrm{d}t < \infty.$$

On $\Omega \setminus U$, the proof the these estimates are trivial. Indeed, for every small $\delta > 0$, since $\{z : \rho(z) < \delta\}$ is convex, $\widetilde{\Phi}(\zeta, z) \neq 0$ on $K \times (\Omega \setminus U)$. Hence $|\widetilde{\Phi}(\zeta, z)| \geq a > 0$ on the compact set $K \times (\Omega \setminus U)$. Shrinking $a > 0$ if necessary, we can assume that $|\zeta - z| > a > 0$ for all $(\zeta, z) \in K \times (\Omega \setminus U)$. That means $S_q\varphi(z)$ is not singular for $z \in \Omega \setminus U$. □

Remark 5.5 In the proof of Lemma 5.2, if we follow the same technique in [24] or [27], then we do certainly obtain $t^{-\alpha(n-s-2)}\dfrac{F^*(t)}{t}$ for $0<\alpha\ll 1$, $0\le s\le n-3$. Then we also need that $t^{-\alpha(n-s-2)}F^*(t)$ is increasing for some α since the general Hardy–Littlewood lemma requires. However, this requirement holds if and only if $F(t)\approx t^m$, for some $m\in\mathbb{Z}^+$. Then we obtain the results by R. M. Range in [24] or by J. Ryczaj in [27].

Proposition 5.6 (1) *If* $n=2$, *we have* $\|S_q\varphi\|_{\Lambda^{0,f}_{(0,q-1)}(\Omega)}\lesssim\|\varphi\|_{L^\infty_{(0,q)}(\Omega)}$, *for any* $\bar\partial$-*closed differential form* $\varphi\in C_{(0,q)}(\bar\Omega)$, *where*

$$f(d^{-1})=\left(\int_0^d\frac{\sqrt{F^*(t)}}{t}\mathrm{d}t\right)^{-1}.$$

(2) *If* $n\ge 3$, *for every* $0\le s\le n-3$, *we have* $\|S_q\varphi\|_{\Lambda^{0,f}_{(0,q-1)}(\Omega)}\lesssim\|\varphi\|_{L^\infty_{(0,q)}(\Omega)}$, *for any* $\bar\partial$-*closed differential form* $\varphi\in C_{(0,q)}(\bar\Omega)$, *where*

$$f(d^{-1})=\left(\int_0^d\frac{(-\ln t)^{n-s-2}\sqrt{F^*(t)}}{t}\mathrm{d}t\right)^{-1}.$$

Proof Since $\varphi\in C_{(0,q)}(\bar\Omega)$, we make use of them form of $S_q\varphi$ given by the definition 4.4. Moreover,

$$\begin{aligned}S_q\varphi(z)&=\mathcal{T}_q\varphi(z)\\&\quad-\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\left[\int_{K\times\{1\}}\bar\partial E\varphi(\zeta)\wedge K_{q-1}(\zeta,z,\lambda)-\int_G E\varphi(\zeta)\wedge B_{q-1}(\zeta,z)\right].\end{aligned}$$

In both of cases, we always have $\|\mathcal{T}_q\varphi\|_{\Lambda^{0,f}_{(0,q-1)}(\Omega)}\lesssim\|\varphi\|_{L^\infty_{(0,q)}(\Omega)}$, see [12, Main Theorem]. Hence it is enough to show that for $q>1$ we have

$$\left\|\bar\partial_z\int_{K\times[0,1]}E\varphi(\zeta)\wedge K_{q-2}(\zeta,z,\lambda)\right\|_{\Lambda^{0,f}_{(0,q-1)}(\Omega)}\lesssim\|\varphi\|_{L^\infty_{(0,q)}(\Omega)}.$$

Moreover, this estimate is a consequence of the fact: for $v\in C_0(U_j)$ and $z\in U_j$, the following estimate holds:

$$\begin{aligned}&\left|D_z^2\int_{U_j\cap K}v(\zeta)\frac{(\bar\zeta_0-\bar z_0)\dfrac{\partial\rho}{\partial\zeta_1}(\zeta)\dfrac{\partial^2\rho}{\partial\zeta_2\partial\bar\zeta_2}(\zeta)\dots\dfrac{\partial^2\rho}{\partial\zeta_{n-s-1}\partial\bar\zeta_{n-s-1}}(\zeta)}{(\Phi^0(\zeta,z))^{s+1}(\widetilde\Phi(\zeta,z))^{n-s-1}}\mathrm{d}\zeta\right|\\&\quad\lesssim\|v\|_{L^\infty(\Omega)}\frac{G(|\rho(z)|)}{|\rho(z)|},\end{aligned}\tag{5.4}$$

where

$$G(t) = \begin{cases} \sqrt{F^*(t)}, & \text{if } n = 2, \\ (-\ln t)^{n-s-2}\sqrt{F^*(t)}, & \text{if } n \geq 3, \text{ for } 0 \leq s \leq n-3. \end{cases}$$

It is clear that by applying the same technique in the proof of Lemma 5.2 we can easily obtain the estimate (5.4). Hence we have completed the proof of the proposition. □

Proof of Main Theorem

Let $|\alpha| = k$, and let v and V be the coefficients of $D^\alpha S_q\varphi$ and $D^\alpha S_q E\varphi$, respectively, where $E\varphi$ is the extension of φ on G by Lemma 4.3. For all $z, w \in \Omega$, from Proposition 5.6, we have

$$f(|z-w|^{-1})|v(z) - v(w)| \leq f(|z-w|^{-1})|V(z) - V(w)| \lesssim \|\varphi\|_{C^k_{(0,q)}(\Omega)}.$$

This completes the proof of Main theorem. □

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