



Bochner–Simons Formulas and the Rigidity of Biharmonic Submanifolds

Dorel Fetcu¹ · Eric Loubeau² · Cezar Oniciuc³

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Abstract

New integral formulas of Simons and Bochner type are found and then used to study biharmonic and biconservative submanifolds in space forms. This leads to new rigidity results and partial answers to conjectures on biharmonic submanifolds in spheres.

Keywords Stress-energy tensor · Constant mean curvature hypersurfaces · Biharmonic submanifolds · Biconservative submanifolds · Real space forms

Mathematics Subject Classification 53C42 · 53C24 · 53C21

1 Introduction

The rich history of using tensorial formulas to understand the geometry of hypersurfaces in Riemannian manifolds goes back to Simons' 1968 seminal paper [39], where, after finding the expression of the Laplacian of the squared norm of the second fundamental form of a minimal submanifold, which in the (simpler) case of minimal hypersurfaces in \mathbb{S}^{m+1} is

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✉ Dorel Fetcu
dorel.fetcu@etti.tuiasi.ro

Eric Loubeau
Eric.Loubeau@univ-brest.fr

Cezar Oniciuc
oniciucc@uaic.ro

- ¹ Department of Mathematics and Informatics, Gh. Asachi Technical University of Iasi, Bd. Carol I, 11, 700506 Iasi, Romania
- ² Univ. Brest, CNRS UMR 6205, LMBA, 29238 Brest, France
- ³ Faculty of Mathematics, Al. I. Cuza University of Iasi, Bd. Carol I, 11, 700506 Iasi, Romania

$$\frac{1}{2} \Delta |A|^2 = -|\nabla A|^2 - |A|^2(m - |A|^2),$$

A being the shape operator, he proves a very important rigidity result for compact minimal submanifolds of Euclidean spheres.

These results were generalized in 1969 to constant mean curvature (CMC) hypersurfaces in space forms by Nomizu and Smyth [33], and then by Erbacher [13] and Smyth [40] to the even more general case of submanifolds with parallel mean curvature vector field (PMC) in space forms.

In 1977, Cheng and Yau [10] proved a general Simons type formula for Codazzi tensors, i.e., symmetric $(1, 1)$ -tensors S on an m -dimensional Riemannian manifold M satisfying the classical Codazzi equation $(\nabla_X S)Y = (\nabla_Y S)X$:

$$\frac{1}{2} \Delta |S|^2 = -|\nabla S|^2 - \langle S, \text{Hess}(\text{trace } S) \rangle - \frac{1}{2} \sum_{i,j=1}^m R_{ijij}(\lambda_i - \lambda_j)^2, \tag{1.1}$$

where λ_i are the eigenvalues of S and R_{ijkl} are the components of the Riemannian curvature of M . Taking $S = A$, this equation recovers Nomizu and Smyth’s result as well as Simons’ after rewriting the last term.

However, when S fails to satisfy the Codazzi condition, Formula (1.1) ceases to work. For this case, a valuable tool is a non-linear Bochner type formula in a 1993 paper by Mok et al. [27]. More details on this formula can be found in Sect. 4 where this technique is applied to study the geometry of biharmonic and biconservative hypersurfaces in space forms, especially in the Euclidean sphere. For compact CMC hypersurfaces in space forms this formula again leads to the Nomizu–Smyth equation of [33], while, when working with biharmonic, or, more generally, biconservative surfaces in a Riemannian manifold, and a non-Codazzi tensor, one recovers Theorem 6 in Ref. [23].

A biharmonic map $\phi : M \rightarrow N$ between two Riemannian manifolds is a critical point of the bienergy functional

$$E_2 : C^\infty(M, N) \rightarrow \mathbb{R}, \quad E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, dv,$$

where M is compact and $\tau(\phi) = \text{trace } \nabla d\phi$ is the tension field of ϕ . The corresponding Euler–Lagrange equation, also known as the biharmonic equation, was obtained by Jiang [20] in 1986:

$$\tau_2(\phi) = -\Delta \tau(\phi) - \text{trace } R^N(d\phi, \tau(\phi))d\phi = 0, \tag{1.2}$$

where $\tau_2(\phi)$ is the bitension field of ϕ , $\Delta = -\text{trace}(\nabla\phi)^2 = -\text{trace}(\nabla\phi\nabla\phi - \nabla_{\nabla}^{\phi})$ is the rough Laplacian defined on sections of $\phi^{-1}(TN)$, and R^N is the curvature tensor of TN , given by $R^N(X, Y)Z = [\bar{\nabla}_X, \bar{\nabla}_Y]Z - \bar{\nabla}_{[X, Y]}Z$. Here, ∇^{ϕ} denotes the pull-back connection on $\phi^{-1}(TN)$, while ∇ and $\bar{\nabla}$ are the Levi-Civita connections on TM and TN , respectively. Henceforth, for the sake of simplicity, we will denote all connections on various fiber bundles by ∇ , the difference being clear from the context.

Since any harmonic map is biharmonic, the purpose is to study biharmonic non-harmonic maps, which are called proper biharmonic. A biharmonic submanifold of N is a biharmonic isometric immersion $\phi : M \rightarrow N$.

Biharmonic maps were introduced in 1964 by Eells and Sampson in Ref. [12] as a generalization of harmonic maps and nowadays this topic represents a well-established and dynamic research direction in differential geometry. In Euclidean spaces, Chen [8] proposed an alternative definition of biharmonic submanifolds. Chen's definition coincides with the previous one when the ambient space is \mathbb{E}^n and he conjectured that there are no proper biharmonic submanifolds in \mathbb{E}^n .

When the ambient space has (constant) non-positive sectional curvature all known results have suggested a similar conjecture called the generalized Chen conjecture (see [5,25,31,35]).

A special attention has been paid to biharmonic submanifolds in spheres and articles like [3,5,6,9] led to two conjectures.

Conjecture 1 [3] *Proper biharmonic submanifolds of S^n are CMC.*

Conjecture 2 [3] *The only proper biharmonic hypersurfaces of S^{m+1} are (open parts of) either hyperspheres $S^m(1/\sqrt{2})$ or standard products of spheres $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

The second conjecture remains difficult to prove even assuming that the hypersurface is also CMC and compact. This problem actually has a broader interest as any CMC hypersurface M^m in S^{m+1} is biharmonic if and only if the squared norm of its shape operator is constant and equal to m (see [3,34]). Therefore, CMC hypersurfaces with $|A|^2 = m$ are biharmonic and their classification is a natural goal after Chern et al.'s classification of minimal hypersurfaces with $|A|^2 = m$ in Ref. [11] (see also [1]).

The most recent results to support these two conjectures were obtained by Maeta and Luo in Ref. [24] and by Maeta and Ou in Ref. [26]. In this last article, the authors prove that any compact proper biharmonic hypersurface of the Euclidean sphere with constant scalar curvature has constant mean curvature. However, they cannot conclude that it is necessarily on the list of Conjecture 2.

Fix a map $\phi : M \rightarrow (N, h)$, where M is compact and h is a Riemannian metric on N , and think of E_2 as a functional on the set of all Riemannian metrics on M . Critical points of this new functional are characterized by the vanishing of the stress-energy tensor S_2 , and this tensor satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi \rangle.$$

A submanifold M in N with $\operatorname{div} S_2 = 0$ is called biconservative and it is characterized by the fact that the tangent part of its bitension field vanishes. It follows easily that any PMC submanifold in a space form is biconservative.

This paper deals mainly with Conjecture 2 under additional geometric hypotheses. For example, beside being biharmonic or biconservative, some of our hypersurfaces will have the same curvature properties as those studied by Cheng and Yau [10] in a different context. It is worth mentioning that there are currently no results concerning Conjecture 2 without pretty strong additional geometric hypotheses. We first present a

general collection of known (with one new) results on biharmonic and biconservative submanifolds and on the stress-energy tensor of the bienergy. In Sect. 3, we compute the Laplacian of the squared norm of the tensor S_2 for any hypersurface in a real space form and deduce a classification result for compact biconservative hypersurfaces with constant scalar curvature and non-negative sectional curvature (Theorem 3.9). It turns out however that this situation is less rigid than the biharmonic case as we find more examples than in Conjecture 2. Then, we give a positive answer to this conjecture, with additional assumptions on the scalar and sectional curvatures (Corollary 3.12). In the fourth section, we obtain a new general integral formula for tensors, apply it to S_2 , and show that compact biconservative submanifolds with parallel normalized mean curvature vector field (PNMC), dimension less than or equal to ten, and non-negative sectional curvature in space forms must be PMC (Theorem 4.6). As a consequence, for hypersurfaces with dimension less than or equal to ten, we obtain a similar result to Corollary 3.12 replacing the constant scalar curvature condition with nowhere vanishing mean curvature (Corollary 4.9).

Conventions We work in the smooth category and assume manifolds to be connected and without boundary. On compact Riemannian manifolds, we consider the canonical Riemannian measure.

2 Preliminaries

In this section, we briefly recall basic results on biharmonic and biconservative submanifolds and a general formula for the Laplacian of the biharmonic stress-energy tensor.

The stress-energy tensor associated to a variational problem, first described by Hilbert in Ref. [17], is a symmetric 2-covariant or $(1, 1)$ -tensor S conservative, i.e., divergence-free at critical points.

To study harmonic maps, Baird and Eells [2] (cf. also [38]) introduced the tensor

$$S = \frac{1}{2} |d\phi|^2 g - \phi^* h$$

for maps $\phi : (M, g) \rightarrow (N, h)$ and showed that S satisfies the equation

$$\operatorname{div} S = -\langle \tau(\phi), d\phi \rangle,$$

hence $\operatorname{div} S$ vanishes when ϕ is harmonic. For any isometric immersion, $\tau(\phi)$ is normal and therefore $\operatorname{div} S = 0$.

The stress-energy tensor S_2 of the bienergy, introduced in [20] and studied in Ref. [7, 14, 15, 22, 28, 29, 32], is

$$\begin{aligned} S_2(X, Y) = & \frac{1}{2} |\tau(\phi)|^2 \langle X, Y \rangle + \langle d\phi, \nabla \tau(\phi) \rangle \langle X, Y \rangle \\ & - \langle d\phi(X), \nabla_Y \tau(\phi) \rangle - \langle d\phi(Y), \nabla_X \tau(\phi) \rangle \end{aligned}$$

and duly satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi \rangle.$$

For isometric immersions, $(\operatorname{div} S_2)^\sharp = -\tau_2(\phi)^\top$ and, unlike the harmonic case, $\operatorname{div} S_2$ does not necessarily vanish.

Definition 2.1 A submanifold $\phi : M \rightarrow N$ of a Riemannian manifold N is called biconservative if $\operatorname{div} S_2 = 0$, i.e., $\tau_2(\phi)^\top = 0$.

For hypersurfaces of space forms, the biharmonic stress-energy tensor is parallel whenever the shape operator is so.

Proposition 2.2 Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a non-minimal hypersurface. Then $\nabla S_2 = 0$ if and only if $\nabla A = 0$.

Proof First assume that $\nabla A = 0$. It then easily follows that the mean curvature function $f = (1/m)\operatorname{trace} A$ is a non-zero constant. Let $H = f\eta = (1/m)\tau(\phi)$ be the mean curvature vector field of M , where η is the unit normal vector field. Since for a hypersurface $S_2 = -(m^2/2)f^2I + 2mfA$, one obtains $\nabla S_2 = 0$, where I denotes the identity operator on TM .

Assume now that $\nabla S_2 = 0$. Denote by W the set of all points of M where the number of distinct principal curvatures is locally constant. This subset is open and dense in M . On each connected component of W , which is also open in M , the principal curvatures are smooth functions and the shape operator A is (locally) diagonalizable.

We will work on such a connected component W_0 of W and prove that f is constant on W_0 . As W is open and dense this property will then hold throughout M , and combined with $\nabla S_2 = 0$ yields $\nabla A = 0$.

Assume that $\operatorname{grad} f$ does not vanish identically on W_0 . Take a connected and open subset U of W_0 where $\operatorname{grad} f \neq 0$ and $f \neq 0$ at each point in U . Consider an orthonormal frame field $\{E_i\}$ on U such that $AE_i = \lambda_i E_i$ and, from the symmetry of ∇S_2 and ∇A , we have

$$\begin{aligned} & -m^2 f(E_i f)E_j + 2m(E_i f)\lambda_j E_j \\ & = -m^2 f(E_j f)E_i + 2m(E_j f)\lambda_i E_i, \quad \forall i, j \in \{1, \dots, m\}. \end{aligned}$$

For $i \neq j$, it follows that

$$(2\lambda_j - mf)E_i f = 0,$$

so

$$(\lambda_i - \lambda_j)(2\lambda_j - mf)E_i f = 0, \quad \forall i, j \in \{1, \dots, m\}. \quad (2.1)$$

Since $\operatorname{grad} f \neq 0$, we can assume that there exists $i_0 \in \{1, \dots, m\}$ such that $E_{i_0} f \neq 0$ at any point in U . From (2.1), one obtains, on U ,

$$2\lambda_{i_0}\lambda_j - m\lambda_{i_0}f - 2\lambda_j^2 + m\lambda_j f = 0, \quad \forall j \in \{1, \dots, m\}$$

and, therefore,

$$(2 - m)mf\lambda_{i_0} - 2|A|^2 + m^2f^2 = 0. \quad (2.2)$$

The squared norms of A and S_2 are related by

$$16m^2f^2|A|^2 = 4|S_2|^2 - m^4f^4(m - 8),$$

and Eq. (2.2) shows that

$$(2 - m)mf\lambda_{i_0} = \frac{4|S_2|^2 - m^5f^4}{8m^2f^2}.$$

If $m > 2$, the above equation can be re-written as

$$2mf\lambda_{i_0} = \frac{4|S_2|^2 - m^5f^4}{4(2 - m)m^2f^2}. \quad (2.3)$$

Since $\nabla S_2 = 0$, we have that $|S_2|$ is constant on M and the eigenvalues of S_2 are also constant functions on M :

$$-\frac{m^2}{2}f^2 + 2mf\lambda_i = c_i = \text{constant}.$$

It follows, using (2.3), that on U , we have

$$-\frac{m^2}{2}f^2 + \frac{4|S_2|^2 - m^5f^4}{4(2 - m)m^2f^2} = c_{i_0},$$

which gives a polynomial equation in f^2 with constant coefficients forcing f to be constant on U and contradicting $E_{i_0}f \neq 0$ at any point of U .

If $m = 2$, Eq. (2.2) gives $|A|^2 = 2f^2$, which leads to $\lambda_1 = \lambda_2$ on U . Therefore, U is umbilical in N and f is constant on U . As we have already seen, this is a contradiction. \square

Remark 2.3 The case when $m \neq 4$ had already been proved, by a different method, in Ref. [22].

Remark 2.4 Hypersurfaces of space forms with $\nabla A = 0$ were studied in [21,36]. They only admit one or two distinct principal curvatures which must be constant. If they have two distinct principal curvatures they are intrinsically isometric to the product of two space forms and, using either the Moore Lemma in Ref. [30] or the Fundamental Theorem of hypersurfaces in space forms, one obtains a complete classification.

The basic characterization of hypersurfaces in space forms in terms of S_2 is given by the following proposition.

Proposition 2.5 [22] *Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a hypersurface in a space form N and S_2 its biharmonic stress-energy tensor.*

- (1) *If $m \neq 4$, then $S_2 = 0$ if and only if M is minimal;*
- (2) *If $m = 4$, then $S_2 = 0$ if and only if M is either minimal or umbilical;*
- (3) *$S_2 = a\langle \cdot, \cdot \rangle$, with $a \neq 0$, if and only if $m \neq 4$ and M is umbilical and non-minimal.*

Essential to further computations are the following properties of the shape operator A .

Lemma 2.6 *Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a hypersurface in a space form with the shape operator A . Then*

- (1) *A is symmetric;*
- (2) *∇A is symmetric;*
- (3) *$\langle (\nabla A)(\cdot, \cdot), \cdot \rangle$ is totally symmetric;*
- (4) *$\operatorname{div} A = \operatorname{trace} \nabla A = m \operatorname{grad} f$.*

The next result gives a general expression of the Laplacian of the biharmonic stress-energy tensor and will be used to derive a Simons type equation for hypersurfaces of space forms.

Theorem 2.7 [23] *Let $\phi : M \rightarrow N$ a smooth map between two Riemannian manifolds. Then the (rough) Laplacian of S_2 is the symmetric $(0, 2)$ tensor*

$$\begin{aligned}
 & (\Delta S_2)(X, Y) \\
 &= \left\{ \langle \Delta \tau(\phi), \tau(\phi) \rangle - 2|\nabla \tau(\phi)|^2 - 2 \sum \langle R(X_i, X_j) d\phi(X_i), \nabla_{X_j} \tau(\phi) \rangle \right. \\
 &\quad - 2 \langle d\phi(\operatorname{Ricci}(\cdot)), \nabla_{(\cdot)} \tau(\phi) \rangle - 2 \langle \nabla d\phi, \nabla^2 \tau(\phi) \rangle + \langle d\phi, \nabla(\Delta \tau(\phi)) \rangle \\
 &\quad - \langle \nabla(\operatorname{trace} R^N(d\phi(\cdot), \tau(\phi)) d\phi(\cdot)), d\phi \rangle \\
 &\quad \left. - \langle \operatorname{trace} R^N(d\phi(\cdot), \tau(\phi)) d\phi(\cdot), \tau(\phi) \rangle \right\} \langle X, Y \rangle \\
 &\quad + 2 \langle \nabla_X \tau(\phi), \nabla_Y \tau(\phi) \rangle + \sum \langle R(X_i, X) d\phi(X_i), \nabla_Y \tau(\phi) \rangle \\
 &\quad + \sum \langle R(X_i, Y) d\phi(X_i), \nabla_X \tau(\phi) \rangle \\
 &\quad + \langle d\phi(\operatorname{Ricci}(X)), \nabla_Y \tau(\phi) \rangle + \langle d\phi(\operatorname{Ricci}(Y)), \nabla_X \tau(\phi) \rangle \\
 &\quad + 2 \sum \langle \nabla d\phi(X_i, X), (\nabla^2 \tau(\phi))(X_i, Y) \rangle \\
 &\quad + 2 \sum \langle \nabla d\phi(X_i, Y), (\nabla^2 \tau(\phi))(X_i, X) \rangle \\
 &\quad - \langle d\phi(X), \nabla_Y(\Delta \tau(\phi)) \rangle - \langle d\phi(Y), \nabla_X(\Delta \tau(\phi)) \rangle \\
 &\quad + \sum \langle d\phi(X), R(X_i, Y) \nabla_{X_i} \tau(\phi) \rangle \\
 &\quad + \sum \langle d\phi(Y), R(X_i, X) \nabla_{X_i} \tau(\phi) \rangle \\
 &\quad + \sum \langle d\phi(X), (\nabla R)(X_i, X_i, Y, \tau(\phi)) + R(X_i, Y) \nabla_{X_i} \tau(\phi) \rangle
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum \langle d\phi(Y), (\nabla R)(X_i, X_i, X, \tau(\phi)) + R(X_i, X)\nabla_{X_i}\tau(\phi) \rangle \\
 &+ \langle d\phi(X), \nabla_{\text{Ricci}(Y)}\tau(\phi) \rangle + \langle d\phi(Y), \nabla_{\text{Ricci}(X)}\tau(\phi) \rangle,
 \end{aligned}
 \tag{2.4}$$

where $\{X_i\}$ is a local orthonormal frame field.

Remark 2.8 In Eq. (2.4), we have

$$(\nabla^2\tau(\phi))(X, Y) = \nabla_X\nabla_Y\tau(\phi) - \nabla_{\nabla_XY}\tau(\phi),$$

while R is the curvature tensor in $\phi^{-1}(TN)$ and

$$\begin{aligned}
 (\nabla R)(X, Y, Z, \sigma) &= (\nabla_X R)(Y, Z, \sigma) \\
 &= \nabla_X R(Y, Z)\sigma - R(\nabla_X Y, Z)\sigma - R(Y, \nabla_X Z)\sigma - R(Y, Z)\nabla_X\sigma.
 \end{aligned}$$

Here, while R^N denotes the curvature tensor on TN , for the curvature tensors on $\phi^{-1}(TN)$ and TM we use the same notations, the difference between them being made by the arguments.

Recall that the decomposition in normal and tangent parts of the biharmonic equation $\tau_2(\phi) = 0$ for a hypersurface M^m in N^{m+1} yields

$$\Delta f + f|A|^2 - f \text{Ricci}^N(\eta, \eta) = 0$$

and

$$2A(\text{grad } f) + mf \text{grad } f - 2f(\text{Ricci}^N(\eta))^\top = 0,$$

where $(\text{Ricci}^N(\eta))^\top$ is the tangent component of the Ricci curvature of N in the direction of η . It is easy to see that while any CMC hypersurface M in a space form $N^{m+1}(c)$ is biconservative, M is proper biharmonic if and only if $|A|^2 = cm$, hence c must be positive.

3 A Simons Type Formula for Hypersurfaces and Applications

In Ref. [26], assuming only compactness and constant scalar curvature and using the Weitzenböck formula for the differential df of the mean curvature function, proper biharmonic hypersurfaces are proved to be CMC. Using a different approach, we work with tensors to find the best tensorial formula possible to answer Conjecture 2.

The Laplacian of the squared norm of the biharmonic stress-energy tensor of an immersed hypersurface can be computed and put to use to prove some rigidity results.

Proposition 3.1 *Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a hypersurface in a space form. We have*

$$\begin{aligned} \frac{1}{2} \Delta |S_2|^2 &= -|\nabla S_2|^2 + 4cm^4 f^4 - 4m^3 f^3 (\text{trace } A^3) - 4m^2 f^2 |A|^2 (cm - |A|^2) \\ &\quad + m^4 (m - 16) f^2 |\text{grad } f|^2 + 4m^2 |A|^2 |\text{grad } f|^2 + 2m^2 |A|^2 \Delta f^2 \\ &\quad + 4m^2 f \langle \text{grad } s, \text{grad } f \rangle - 8m^2 \text{div}(f \text{Ricci}(\text{grad } f)) \\ &\quad + \frac{m^5}{8} \Delta f^4 - 4cm^2 (m - 1) \Delta f^2 - 10m^2 f \langle \tau_2^\top(\phi), \text{grad } f \rangle \\ &\quad - 4m^2 f^2 \text{div} \left(\tau_2^\top(\phi) \right) - 2 \left| \tau_2^\top(\phi) \right|^2 + 4mf \langle \nabla \tau_2^\top(\phi), A \rangle. \end{aligned}$$

Proof This is just an application of Formula (2.4) of ΔS_2 for an immersed hypersurface M^m in a space form $N(c)$. For the sake of simplicity, we consider a point $p \in M$ and a geodesic frame field around it, and compute all terms at p . First, since $\tau(\phi) = mH$, we have

$$\langle \Delta \tau(\phi), \tau(\phi) \rangle = m^2 \langle \Delta H, H \rangle$$

and

$$\begin{aligned} -2|\nabla \tau(\phi)|^2 &= -2m^2 |\nabla H|^2 = -2m^2 \sum |\nabla_{X_i} H|^2 \\ &= -2m^2 \sum | -fAX_i + (X_i f)\eta |^2 \\ &= -2m^2 f^2 |A|^2 - 2m^2 |\text{grad } f|^2. \end{aligned}$$

Next, using the expression of the curvature of a space form

$$R^N(X, Y)Z = c\{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \}, \tag{3.1}$$

one obtains

$$\begin{aligned} -2 \sum \langle R(X_i, X_j) d\phi(X_i), \nabla_{X_j} \tau(\phi) \rangle &= -2 \sum \langle R^N(d\phi(X_i), d\phi(X_j)) d\phi(X_i), \\ &\quad -mfAX_j + m\nabla_{X_j}^\perp H \rangle \\ &= 2cmf(1 - m)(\text{trace } A) = 2cm^2 f^2 - 2cm^3 f^2. \end{aligned}$$

In the same way, we get

$$-2 \langle d\phi(\text{Ricci}(\cdot)), \nabla_{(\cdot)} \tau(\phi) \rangle = 2mf \langle \text{Ricci}, A \rangle$$

and then, since $\text{Ricci} = c(m - 1)I + mfA - A^2$,

$$-2 \langle d\phi(\text{Ricci}(\cdot)), \nabla_{(\cdot)} \tau(\phi) \rangle = 2c(m - 1)m^2 f^2 + 2m^2 f^2 |A|^2 - 2mf(\text{trace } A^3).$$

Since in the case of immersions we have $(\nabla d\phi)(X_i, X_j) = B(X_i, X_j)$, a direct computation using the Weingarten equation shows that

$$-2\langle \nabla d\phi, \nabla^2 \tau(\phi) \rangle = 2mf(\text{trace } A^3) - 2m\langle A, \text{Hess } f \rangle.$$

Furthermore, for any hypersurface, we have

$$\begin{aligned} \langle A, \text{Hess } f \rangle &= \sum \langle AX_i, \nabla_{X_i} \text{grad } f \rangle = \sum \langle X_i, A(\nabla_{X_i} \text{grad } f) \rangle \\ &= \sum \langle X_i, \nabla_{X_i} A(\text{grad } f) - (\nabla_{X_i} A)(\text{grad } f) \rangle \\ &= \text{div}(A(\text{grad } f)) - m|\text{grad } f|^2 \end{aligned}$$

and, therefore,

$$-2\langle \nabla d\phi, \nabla^2 \tau(\phi) \rangle = 2mf(\text{trace } A^3) - 2m \text{div}(A(\text{grad } f)) + 2m^2|\text{grad } f|^2.$$

The next term in the formula of ΔS_2 is

$$\begin{aligned} \langle d\phi, \nabla(\Delta \tau(\phi)) \rangle &= m\langle d\phi, \nabla(\Delta H) \rangle = m \sum \langle d\phi(X_i), \nabla_{X_i}(\Delta H) \rangle \\ &= m \sum \{X_i \langle d\phi(X_i), \Delta H \rangle - \langle (\nabla d\phi)(X_i, X_i), \Delta H \rangle\} \\ &= -\text{div } \tau_2^\top(\phi) - m^2 \langle H, \Delta H \rangle. \end{aligned}$$

Again using Eq. (3.1), one obtains

$$\begin{aligned} &-\langle \nabla(\text{trace } R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot)), d\phi \rangle \\ &= cm^2 \langle \nabla H, d\phi(\cdot) \rangle = cm^2 \sum \langle -fAX_i, d\phi(X_i) \rangle \\ &= -cm^3 f^2 \end{aligned}$$

and

$$-\langle \text{trace } R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot), \tau(\phi) \rangle = cm^3 f^2.$$

The expressions of the following terms can be obtained by some direct computation and also using Lemma 2.6, in the same way as above,

$$\begin{aligned} 2\langle \nabla_X \tau(\phi), \nabla_Y \tau(\phi) \rangle &= 2m^2 f^2 \langle AX, AY \rangle + 2m^2 (Xf)(Yf), \\ \sum \langle R(X_i, X)d\phi(X_i), \nabla_Y \tau(\phi) \rangle &= \sum \langle R(X_i, Y)d\phi(X_i), \nabla_X \tau(\phi) \rangle \\ &= cm(m-1)f \langle AX, Y \rangle, \\ \langle d\phi(\text{Ricci}(X)), \nabla_Y \tau(\phi) \rangle &= -mf \langle \text{Ricci}(X), AY \rangle, \\ 2 \sum \langle \nabla d\phi(X_i, X), (\nabla^2 \tau(\phi))(X_i, Y) \rangle &= -2mf \langle A^2 Y, AX \rangle + 2m(\text{Hess } f)(AX, Y), \\ 2 \sum \langle \nabla d\phi(X_i, Y), (\nabla^2 \tau(\phi))(X_i, X) \rangle &= -2mf \langle A^2 X, AY \rangle + 2m(\text{Hess } f)(AY, X), \\ \sum \langle d\phi(X), R(X_i, Y)\nabla_{X_i} \tau(\phi) \rangle &= -cmf \langle AX, Y \rangle + cm^2 f^2 \langle X, Y \rangle, \\ \sum \langle d\phi(Y), R(X_i, X)\nabla_{X_i} \tau(\phi) \rangle &= -cmf \langle AX, Y \rangle + cm^2 f^2 \langle X, Y \rangle, \end{aligned}$$

$$\langle d\phi(X), (\nabla R)(X_i, X_i, Y, \tau(\phi)) + R(X_i, Y)\nabla_{X_i}\tau(\phi) \rangle = 0,$$

$$\langle d\phi(Y), (\nabla R)(X_i, X_i, X, \tau(\phi)) + R(X_i, X)\nabla_{X_i}\tau(\phi) \rangle = 0,$$

$$\langle d\phi(X), \nabla_{\text{Ricci}(Y)}\tau(\phi) \rangle = -mf\langle AX, \text{Ricci}(Y) \rangle,$$

$$\langle d\phi(Y), \nabla_{\text{Ricci}(X)}\tau(\phi) \rangle = -mf\langle AY, \text{Ricci}(X) \rangle.$$

Finally, for the remaining terms, we have

$$\begin{aligned} -\langle d\phi(X), \nabla_Y(\Delta\tau(\phi)) \rangle &= -m\langle d\phi(X), \nabla_Y(\Delta H) \rangle \\ &= -mY(\langle d\phi(X), \Delta H \rangle) + m\langle \nabla_Y d\phi(X), \Delta H \rangle \\ &= Y(\langle \tau_2^\top(\phi), X \rangle) + m\langle B(X, Y), \Delta H \rangle - \langle \nabla_X Y, \tau_2^\top(\phi) \rangle \end{aligned}$$

and

$$-\langle d\phi(Y), \nabla_X(\Delta\tau(\phi)) \rangle = X(\langle \tau_2^\top(\phi), Y \rangle) + m\langle B(X, Y), \Delta H \rangle - \langle \nabla_Y X, \tau_2^\top(\phi) \rangle.$$

Assembling all these terms and taking into account that

$$\tau_2^\top(\phi) = -2mA(\text{grad } f) - \frac{m^2}{2}\text{grad } f^2,$$

one obtains

$$\begin{aligned} (\Delta S_2)(X, Y) &= \left(2cm^2 f^2 - \frac{m^2}{2}\Delta f^2 \right) \langle X, Y \rangle \\ &\quad + 2m^2 f^2 \langle AX, AY \rangle + 2m^2 (Xf)(Yf) + 2cm(m-2)f \langle AX, Y \rangle \\ &\quad - 2mf \langle \text{Ricci}(X), AY \rangle - 2mf \langle \text{Ricci}(Y), AX \rangle - 4mf \langle A^2 X, AY \rangle \\ &\quad + 2m(\text{Hess } f)(AX, Y) + 2m(\text{Hess } f)(AY, X) \\ &\quad + \langle \nabla_Y \tau_2^\top(\phi), X \rangle + \langle \nabla_X \tau_2^\top(\phi), Y \rangle + 2m \langle B(X, Y), \Delta H \rangle. \end{aligned}$$

Now, using that, in the case of hypersurfaces, $S_2 = -(m^2 f^2/2)I + 2mfA$ and also

$$\begin{aligned} \langle \text{Hess } f, A^2 \rangle &= \langle \text{Hess } f, c(m-1)I + mfA - \text{Ricci} \rangle \\ &= -c(m-1)\Delta f - \frac{f}{2}\text{div}(\tau_2^\top(\phi)) + \frac{m^2 f}{4}\Delta f^2 \\ &\quad - m^2 f |\text{grad } f|^2 - \text{div}(\text{Ricci}(\text{grad } f)) + \frac{1}{2}\langle \text{grad } s, \text{grad } f \rangle \end{aligned}$$

and

$$\langle H, \Delta H \rangle = \frac{1}{2}\Delta f^2 + f^2|A|^2 + |\text{grad } f|^2,$$

a long but straightforward computation leads to the conclusion. \square

Remark 3.2 Let M^m be a hypersurface in a space form $N^{m+1}(c)$ and consider the operator T on M given by

$$T(X) = -\text{trace}(RA)(\cdot, X, \cdot),$$

where

$$RA(X, Y, Z) = R(X, Y)AZ - A(R(X, Y)Z), \quad \forall X, Y, Z \in C(TM).$$

At a point $p \in M$, consider an orthonormal basis $\{e_i\}$ of T_pM such that $Ae_i = \lambda_i e_i$. Using the operator T we can write (see [33])

$$\begin{aligned} 4cm^4 f^4 - 4m^3 f^3(\text{trace } A^3) - 4m^2 f^2 |A|^2 (cm - |A|^2) &= 4m^2 f^2 \langle T, A \rangle \\ &= -2m^2 f^2 \sum (\lambda_i - \lambda_j)^2 R_{ijij}. \end{aligned} \tag{3.2}$$

The next result, which is obtained by a straightforward computation, comes to further improve the above formula of the Laplacian of $|S_2|^2$.

Lemma 3.3 *Let M^m be a hypersurface in a space form $N^{m+1}(c)$ and A_H its shape operator in the direction of H , i.e., $A_H = fA$. Then*

$$|\nabla S_2|^2 = (m - 8)m^4 f^2 |\text{grad } f|^2 + 4m^2 |\nabla A_H|^2$$

and, furthermore,

$$\begin{aligned} |\nabla S_2|^2 &= (m - 8)m^4 f^2 |\text{grad } f|^2 + 4m^2 |A|^2 |\text{grad } f|^2 + 4m^2 f^2 |\nabla A|^2 \\ &\quad + 2m^2 \text{div}(|A|^2 \text{grad } f^2) + 2m^2 |A|^2 \Delta f^2. \end{aligned}$$

From Proposition 3.1 and the second equation in Lemma 3.3, we obtain a further formula for the Laplacian of $|S_2|^2$.

Theorem 3.4 *Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a hypersurface in a space form. Then*

$$\begin{aligned} \frac{1}{2} \Delta |S_2|^2 &= 4cm^4 f^4 - 4m^3 f^3(\text{trace } A^3) - 4m^2 f^2 |A|^2 (cm - |A|^2) \\ &\quad - 8m^4 f^2 |\text{grad } f|^2 - 4m^2 f^2 |\nabla A|^2 \\ &\quad + 4m^2 f \langle \text{grad } s, \text{grad } f \rangle \\ &\quad - 8m^2 \text{div}(f \text{Ricci}(\text{grad } f)) - 2m^2 \text{div}(|A|^2 \text{grad } f^2) \\ &\quad + \frac{m^5}{8} \Delta f^4 - 4cm^2(m - 1) \Delta f^2 - 10m^2 f \langle \tau_2^\top(\phi), \text{grad } f \rangle \\ &\quad - 4m^2 f^2 \text{div}(\tau_2^\top(\phi)) - 2 |\tau_2^\top(\phi)|^2 + 4mf \langle \nabla \tau_2^\top(\phi), A \rangle. \end{aligned} \tag{3.3}$$

Remark 3.5 Rewriting Eq. (3.3) in terms of the shape operator A yields a generalization of the well-known formula for CMC hypersurfaces in Ref. [33].

Theorem 3.4 leads to the next two results.

Theorem 3.6 Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a constant scalar curvature biconservative hypersurface in a space form. Then

$$\begin{aligned} \frac{3m^2}{2} \Delta f^4 &= 4f^2 \{cm^2 f^2 - mf(\text{trace } A^3) - |A|^2(cm - |A|^2) \\ &\quad - 2m^2 |\text{grad } f|^2 - |\nabla A|^2\}. \end{aligned} \quad (3.4)$$

Corollary 3.7 Let $\phi : M^m \rightarrow \mathbb{S}^{m+1}$ be a biharmonic hypersurface with constant scalar curvature. Then the following system holds

$$\begin{cases} \frac{3m^2}{2} \Delta f^4 = 4f^2 \{m^2 f^2 - mf(\text{trace } A^3) - |A|^2(m - |A|^2) \\ \quad - 2m^2 |\text{grad } f|^2 - |\nabla A|^2\} \\ \Delta f = f(m - |A|^2). \end{cases} \quad (3.5)$$

Remark 3.8 Since $\Delta f^4 = 4f^3 \Delta f - 12f^2 |\text{grad } f|^2$, a consequence of the last corollary is that a biharmonic hypersurface in the Euclidean sphere with constant scalar curvature satisfies

$$\begin{cases} \frac{3m^2}{2} \Delta f^4 = 4f^2 \{m^2 f^2 - mf(\text{trace } A^3) - |A|^2(m - |A|^2) \\ \quad - 2m^2 |\text{grad } f|^2 - |\nabla A|^2\} \\ \Delta f^4 = 4f^4(m - |A|^2) - 12f^2 |\text{grad } f|^2. \end{cases}$$

The next rigidity result is a direct application of the Simons type formula (3.4).

Theorem 3.9 Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a compact biconservative hypersurface in a space form $N^{m+1}(c)$, with $c \in \{-1, 0, 1\}$. If M is not minimal, has constant scalar curvature, and $\text{Riem}^M \geq 0$, then M is either

- (1) $\mathbb{S}^m(r)$, $r > 0$, if $c \in \{-1, 0\}$, i.e., N is either the hyperbolic space \mathbb{H}^{m+1} or the Euclidean space \mathbb{E}^{m+1} ; or
- (2) $\mathbb{S}^m(r)$, $r \in (0, 1)$, or the product $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$, where $r_1^2 + r_2^2 = 1$, $m_1 + m_2 = m$, and $r_1 \neq \sqrt{m_1/m}$, if $c = 1$, i.e., N is the Euclidean sphere \mathbb{S}^{m+1} .

Proof Integrating Eq. (3.4) over M , we have

$$\begin{aligned} \int_M \left\{ 4cm^2 f^4 - 4mf^3(\text{trace } A^3) - 4f^2 |A|^2 (cm - |A|^2) \right\} &= \int_M \left\{ 8m^2 f^2 |\text{grad } f|^2 \right. \\ &\quad \left. + 4f^2 |\nabla A|^2 \right\} \geq 0. \end{aligned} \quad (3.6)$$

Since $\text{Riem}^M \geq 0$, Eqs. (3.2) and (3.6) forces $f^2 |\text{grad } f|^2 = 0$ and $\nabla A = 0$, which implies $T = 0$. Therefore, M is a CMC hypersurface with $\nabla A = 0$ and we conclude using the classification of such hypersurfaces in [21,36,37]. \square

The following two results are partial answers to Conjecture 2.

Proposition 3.10 *Let $\phi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper biharmonic hypersurface in the Euclidean sphere. If the scalar curvature s of M is constant, and*

$$mf^2 \leq f(\text{trace } A^3),$$

then M is either $\mathbb{S}^m(1/\sqrt{2})$ or the product $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Proof Since M is a compact proper biharmonic hypersurface with constant scalar curvature, we have, using Eq. (3.6) and [26, Theorem 2.3],

$$m \int_M \left\{ mf^2 - f(\text{trace } A^3) \right\} = \int_M |\nabla A|^2.$$

It follows that $\nabla A = 0$ and we conclude using [5,19], where all proper biharmonic hypersurfaces satisfying $\nabla A = 0$ were determined. □

Remark 3.11 Consider the eigenvalue functions $\lambda_i, i \in \{1, \dots, m\}$, of the shape operator A . The hypotheses of Proposition 3.10 can be re-written as

$$\sum \lambda_i = \alpha, \quad \sum \lambda_i^2 = m, \quad \sum \lambda_i \leq \sum \lambda_i^3,$$

where $\alpha \in (0, m]$ is a real constant. At a fixed point $p \in M$ the above relations are numerical and it is easy to find real numbers satisfying them with strict inequality. However, Proposition 3.10 shows that such numbers cannot be the values at p of the eigenvalue functions.

It is easy to see that, for a CMC biharmonic hypersurface of the Euclidean sphere, $\text{Riem}^M \geq 0$ implies $mf^2 \leq f(\text{trace } A^3)$.

Corollary 3.12 *Let $\phi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper biharmonic hypersurface with constant scalar curvature and $\text{Riem}^M \geq 0$. Then M is either $\mathbb{S}^m(1/\sqrt{2})$ or the product $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Remark 3.13 Note that if M^m is a constant scalar curvature compact proper biharmonic hypersurface in \mathbb{S}^{m+1} , then we have the following constraint (see [34])

$$s \in (m(m - 2), 2m(m - 1)].$$

In Ref. [10], compact hypersurfaces M^m in \mathbb{S}^{m+1} with $\text{Riem}^M \geq 0$ and constant scalar curvature $s \geq m(m - 1)$ were classified. Observe that the hypotheses of Corollary 3.12 do not necessarily imply that $s \geq m(m - 1)$, but only $s > m(m - 2)$. Moreover, when M is only biconservative, as in Theorem 3.9, there is no restriction on the scalar curvature.

Remark 3.14 In the non-compact case, a constant scalar curvature proper biharmonic hypersurface of the Euclidean sphere with at most six distinct principal curvatures must be CMC [16].

4 A Bochner Type Formula and Applications

Results on Conjecture 2 obtained in the previous section rely heavily on the constant scalar curvature hypothesis. To circumvent this condition, we will prove a proposition inspired by a non-linear Bochner type formula in Ref. [27], involving the 4-tensor defined on a Riemannian manifold M :

$$Q(X, Y, Z, W) = \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle,$$

the map

$$\sigma_{24}(X, Y, Z, W) = (X, W, Z, Y),$$

which permutes the second and fourth variables, and, given a symmetric $(1, 1)$ -tensor S , the 1-form θ defined as the contraction $C((Q \circ \sigma_{24}) \otimes g^*, \nabla S \otimes S)$, where g denotes the metric tensor on M and g^* is its dual.

The next formula cannot be considered of Simons type as we do not compute a Laplacian and the shape operator is not involved. Moreover, this formula extends beyond Codazzi tensors as it involves the antisymmetric part of ∇S .

Proposition 4.1 *On a Riemannian manifold M with curvature tensor R we have*

$$\operatorname{div} \theta = \langle T, S \rangle + |\operatorname{div} S|^2 - |\nabla S|^2 + \frac{1}{2}|W|^2, \tag{4.1}$$

where $T(X) = -\operatorname{trace}(RS)(\cdot, X, \cdot)$ and $W(X, Y) = (\nabla_X S)Y - (\nabla_Y S)X$.

Proof Since we work with tensor products, it seems easier to use local coordinates. This way one can write

$$\begin{aligned} (Q \circ \sigma_{24}) \otimes g^* &= Q_{ilkj} g^{ab} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b}, \\ (\nabla S) \otimes S &= (\nabla_\alpha S_\beta^\sigma) S_\gamma^\delta dx^\alpha \otimes dx^\beta \otimes dx^\gamma \otimes \frac{\partial}{\partial x^\sigma} \otimes \frac{\partial}{\partial x^\delta}, \end{aligned}$$

and

$$(\nabla S) \otimes (\nabla S) = (\nabla_\alpha S_\beta^\sigma) (\nabla_\gamma S_\omega^\delta) dx^\alpha \otimes dx^\beta \otimes dx^\gamma \otimes dx^\omega \otimes \frac{\partial}{\partial x^\sigma} \otimes \frac{\partial}{\partial x^\delta}.$$

Therefore, we have

$$\theta_i = Q_i^{jkl} (\nabla_j S_l^a) S_k^b g_{ab}$$

and then

$$\theta^i = Q^{ijkl} (\nabla_j S_l^a) S_{ak}.$$

Using this expression and the commutation formula for $\nabla^2 S$, a straightforward computation leads to

$$\begin{aligned} \operatorname{div} \theta &= Q^{jkl} \{(\nabla_i \nabla_j S_l^a) S_{ak} + (\nabla_j S_l^a)(\nabla_i S_{ak})\} \\ &= Q^{jkl} \{(\nabla_i \nabla_j S_l^a) S_{ak} + (\nabla_j S_l^a)(\nabla_k S_{ia}) + (\nabla_j S_l^a)(\nabla_i S_{ka} - \nabla_k S_{ia})\} \\ &= \frac{1}{2} Q^{jkl} \{(S_l^b R_{bij}^a - S_b^a R_{lij}^b) S_{ak} + 2(\nabla_j S_l^a)(\nabla_k S_{ia}) \\ &\quad + 2(\nabla_j S_l^a)(\nabla_i S_{ka} - \nabla_k S_{ia})\} \\ &= \frac{1}{2} Q^{jkl} \{(RS(\partial_i, \partial_j, \partial_l))^a S_{ak} + 2(\nabla_j S_l^a)(\nabla_k S_{ia}) \\ &\quad + 2(\nabla_j S_l^a)((\nabla_i S_{ka} - \nabla_k S_{ia}))\}. \end{aligned}$$

Since $Q^{jkl} = g^{ik} g^{jl} - g^{jk} g^{il}$, we get that

$$\operatorname{div} \theta = \frac{1}{2} g^{ik} g^{jl} (RS(\partial_i, \partial_j, \partial_l))^a S_{ak} + \frac{1}{2} \langle T, S \rangle + |\operatorname{div} S|^2 - |\nabla S|^2 + \frac{1}{2} |W|^2.$$

Next, let us consider a point $p \in M$ and $\{e_1, \dots, e_m\}$ a basis at p such that $S e_i = \lambda_i e_i$. Then one obtains

$$\begin{aligned} g^{ik} g^{jl} (RS(\partial_i, \partial_j, \partial_l))^a S_{ak} &= (g^{jl} S_l^b R_{bij}^a - g^{jl} S_b^a R_{lij}^b) S_a^i \\ &= \sum_{i,k} \langle R(e_k, e_i) S e_i - S(R(e_k, e_i) e_i), S e_k \rangle \\ &= -\frac{1}{2} \sum_{i,k} (\lambda_i - \lambda_k)^2 R(e_k, e_i, e_k, e_i) \\ &= \langle T, S \rangle \end{aligned}$$

and replacing in the expression of $\operatorname{div} \theta$ we conclude. □

When M is a compact CMC hypersurface in a space form, taking A instead of S in Eq. (4.1), one obtains a classic formula from [33]. If M is a biconservative surface, taking S to be S_2 , we recover [23, Theorem 6] as well as [32, Proposition 5.1]. Still with S equals S_2 , but for biharmonic hypersurfaces in Euclidean spheres, we get the following result.

Proposition 4.2 *Let $\phi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper biharmonic hypersurface with $\operatorname{Riem}^M \geq 0$, such that*

$$f^2 |\nabla A|^2 - |A|^2 |\operatorname{grad} f|^2 + |A|^2 (m - |A|^2) f^2 \geq 0.$$

Then M is either $\mathbb{S}^m(1/\sqrt{2})$ or the product $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Proof Recall that the biharmonic stress-energy tensor S_2 of a hypersurface is given by

$$S_2 = -\frac{m^2 f^2}{2}I + 2mfA,$$

and a straightforward computation leads to

$$\begin{aligned} |W|^2 = \sum_{i,j} |W(X_i, X_j)|^2 &= 2m^5 f^2 |\text{grad } f|^2 + 8m^2 |A|^2 |\text{grad } f|^2 - 10m^4 f^2 |\text{grad } f|^2 \\ &\quad + 8m^3 f \langle \text{grad } f, A \text{ grad } f \rangle - 8m^2 |A \text{ grad } f|^2, \end{aligned}$$

where $\{X_i\}$ is a geodesic frame around a point $p \in M$.

From this formula and Lemma 3.3 it follows that

$$\begin{aligned} \frac{1}{2} |W|^2 - |\nabla S_2|^2 &= -4m^2 f^2 |\nabla A|^2 - 2m^2 \langle \text{grad } f^2, \text{grad } |A|^2 \rangle \\ &= -4m^2 f^2 |\nabla A|^2 - 2m^2 \text{div}(|A|^2 \text{ grad } f^2) - 2m^2 |A|^2 \Delta f^2 \\ &= -4m^2 \left\{ f^2 |\nabla A|^2 - |A|^2 |\text{grad } f|^2 + |A|^2 (m - |A|^2) f^2 \right\} \\ &\quad - 2m^2 \text{div}(|A|^2 \text{ grad } f^2). \end{aligned}$$

Next, by integrating (4.1) on M , from the hypotheses, it easily follows that

$$f^2 |\nabla A|^2 - |A|^2 |\text{grad } f|^2 + |A|^2 (m - |A|^2) f^2 = 0$$

and

$$\sum_{i,j} f^2 (\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j) = 0, \tag{4.2}$$

where λ_i are the principal curvatures of M .

Now, from (4.2) it follows that, on a connected component of $U = \{p \in M | f^2(p) > 0\}$, there are at most two distinct principal curvatures, not necessarily constant, and then, since M is biharmonic, we have that $\text{grad } f = 0$, and so $\Delta f = 0$, on that component and therefore on U (see [3]). Let $q \in M$ be a point such that $f(q) = 0$. From the normal part of the biharmonic equation (1.2), it can be easily seen that $(\Delta f)(q) = 0$, which means that $\Delta f = 0$ on M . Therefore, f is constant on M , i.e., M is a CMC hypersurface with at most two distinct principal curvatures, which implies $|A|^2 = m$ and $\nabla A = 0$. This concludes the proof. \square

In Ref. [9] it is proved that, for a biharmonic hypersurface M^m in S^{m+1} , we have

$$|\nabla A|^2 \geq \frac{m^2(m+26)}{4(m-1)} |\text{grad } f|^2. \tag{4.3}$$

Using this inequality, one obtains the following corollary of Proposition 4.2.

Corollary 4.3 *Let $\phi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper biharmonic hypersurface with $\text{Riem}^M \geq 0$, such that*

$$\left(\frac{m^2(m + 26)}{4(m - 1)} f^2 - |A|^2 \right) |\text{grad } f|^2 + |A|^2(m - |A|^2) f^2 \geq 0.$$

Then M is either $\mathbb{S}^m(1/\sqrt{2})$ or the product $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

In this last part, we will use Eq. (4.1) to study biconservative submanifolds with parallel normalized mean curvature vector field in space forms.

A non-minimal submanifold in a Riemannian manifold with the mean curvature vector field parallel in the normal bundle is called a PMC submanifold.

Let $\phi : M^m \rightarrow N^n$ be a submanifold with mean curvature vector field H such that $H \neq 0$ at any point in M . Henceforth, we will denote by $h = |H| > 0$ the mean curvature of M and by $\eta_0 = H/|H|$ a unit normal vector field with the same direction as H . If η_0 is parallel in the normal bundle, i.e., $\nabla^\perp \eta_0 = 0$, the submanifold M is said to have parallel normalized mean curvature vector field and it is then called a PNMC submanifold. It is easy to see that a PNMC submanifold is PMC if and only if it also is CMC.

Now, let us denote $A_0 = A_{\eta_0}$ the shape operator of M in the direction η_0 . We have the following straightforward properties of A_0 .

Lemma 4.4 *Let $\phi : M^m \rightarrow N^n(c)$ be a PNMC submanifold in a space form. Then, the following hold:*

- (1) A_0 is symmetric;
- (2) ∇A_0 is symmetric;
- (3) $\langle (\nabla A_0)(\cdot, \cdot), \cdot \rangle$ is totally symmetric;
- (4) $\text{trace } A_0 = mh$;
- (5) $\text{div } A_0 = \text{trace}(\nabla A_0) = m \text{ grad } h$.

We will need the following lemma, that provides an inequality similar to (4.3), for the last main result.

Lemma 4.5 *Let $\phi : M^m \rightarrow N^n(c)$ be a PNMC biconservative submanifold. Then*

$$|\nabla A_0|^2 \geq \frac{m^2(m + 26)}{4(m - 1)} |\text{grad } h|^2. \tag{4.4}$$

Proof Since M is biconservative, we have $\text{div } S_2 = 0$, which is equivalent to

$$\text{trace}(\nabla A_H) = \frac{m}{4} \text{grad } h^2.$$

We can rewrite this relation as follows. Consider a geodesic frame $\{X_i\}$ around a point $p \in M$. Then, at p , one obtains

$$\sum_i (\nabla(hA_0))(X_i, X_i) = \frac{m}{4} \operatorname{grad} h^2$$

and then

$$\sum_i ((X_i h)A_0 + h\nabla_{X_i} A_0)(X_i) = \frac{m}{4} \operatorname{grad} h^2,$$

that is

$$A_0 \operatorname{grad} h + h \operatorname{div} A_0 = \frac{m}{4} \operatorname{grad} h^2.$$

From the last property in Lemma 4.4, it follows that

$$A_0 \operatorname{grad} h = -\frac{m}{2} h \operatorname{grad} h. \quad (4.5)$$

Next, consider a point $p_0 \in M$. If $\operatorname{grad} h$ vanishes at p_0 , Inequality (4.4) obviously holds. Assume that $(\operatorname{grad} h)(p_0) \neq 0$ and then $\operatorname{grad} h$ does not vanish throughout an open neighborhood of p_0 . In this neighborhood, consider an orthonormal frame field $\{E_1 = \operatorname{grad} h / |\operatorname{grad} h|, E_2, \dots, E_m\}$. Then, from (4.5), we have

$$A_0 E_1 = -\frac{m}{2} h E_1. \quad (4.6)$$

Now, using Eq. (4.6) and the fact that A_0 is symmetric, one obtains

$$\begin{aligned} \langle (\nabla A_0)(E_1, E_1), E_1 \rangle &= \langle \nabla_{E_1} A_0 E_1 - A_0(\nabla_{E_1} E_1), E_1 \rangle \\ &= \left\langle -\frac{m}{2} \nabla_{E_1}(h E_1) - A_0(\nabla_{E_1} E_1), E_1 \right\rangle \\ &= \left\langle -\frac{m}{2} |\operatorname{grad} h| E_1 - \frac{m}{2} h \nabla_{E_1} E_1 - A_0(\nabla_{E_1} E_1), E_1 \right\rangle \\ &= -\frac{m}{2} |\operatorname{grad} h| \end{aligned} \quad (4.7)$$

and then, from the last property in Lemma 4.4, we have

$$\begin{aligned} \sum_{i=2}^m \langle (\nabla A_0)(E_i, E_i), E_1 \rangle &= \sum_{i=1}^m \langle (\nabla A_0)(E_i, E_i), E_1 \rangle - \langle (\nabla A_0)(E_1, E_1), E_1 \rangle \\ &= \langle \operatorname{div} A_0, E_1 \rangle + \frac{m}{2} |\operatorname{grad} h| \\ &= \frac{3m}{2} |\operatorname{grad} h|. \end{aligned} \quad (4.8)$$

Finally, using (4.7), (4.8), and the third property in Lemma 4.4, it follows that

$$\begin{aligned}
 |\nabla A_0|^2 &= \sum_{i,j=1}^m |(\nabla A_0)(E_i, E_j)|^2 = \sum_{i,j,k=1}^m \langle (\nabla A_0)(E_i, E_j), E_k \rangle^2 \\
 &\geq \langle (\nabla A_0)(E_1, E_1), E_1 \rangle^2 + \sum_{i=2}^m \langle (\nabla A_0)(E_1, E_i), E_i \rangle^2 \\
 &\quad + \sum_{i=2}^m \langle (\nabla A_0)(E_i, E_1), E_i \rangle^2 + \sum_{i=2}^m \langle (\nabla A_0)(E_i, E_i), E_1 \rangle^2 \\
 &= \frac{m^2}{4} |\text{grad } h|^2 + 3 \sum_{i=2}^m \langle (\nabla A_0)(E_i, E_i), E_1 \rangle^2 \\
 &\geq \frac{m^2}{4} |\text{grad } h|^2 + \frac{3}{m-1} \left(\sum_{i=2}^m \langle (\nabla A_0)(E_i, E_i), E_1 \rangle \right)^2 \\
 &= \frac{m^2(m+26)}{4(m-1)} |\text{grad } h|^2
 \end{aligned}$$

and we are finished. □

We are now ready to prove the main result of this section.

Theorem 4.6 *Let $\phi : M^m \rightarrow N^n(c)$ be a compact PNMC biconservative submanifold in a space form with $\text{Riem}^M \geq 0$ and $m \leq 10$. Then M is a PMC submanifold and $\nabla A_H = 0$.*

Proof First take $S = A_0$ in Proposition 4.1 and, since A_0 is a Codazzi tensor, by integrating over M and using Lemma 4.4, one obtains

$$\int_M \left\{ -\langle T, A_0 \rangle + |\nabla A_0|^2 \right\} = m^2 \int_M |\text{grad } h|^2. \tag{4.9}$$

Next, using Inequality (4.4), we can see that

$$\int_M \langle T, A_0 \rangle \geq \frac{3m^2(10-m)}{4(m-1)} \int_M |\text{grad } h|^2. \tag{4.10}$$

But $\langle T, A_0 \rangle = -(1/2) \sum_{i,j} (\lambda_i - \lambda_j)^2 R(e_i, e_j, e_i, e_j) \leq 0$ at any point $p \in M$, where $\{e_1, \dots, e_m\}$ is a basis at p such that $A_0 e_i = \lambda_i e_i$, and then, from (4.10), it follows that, if $m \leq 9$, then $\text{grad } h = 0$, i.e., h is constant and $\langle T, A_0 \rangle = 0$. Using again (4.9) we have that $\nabla A_0 = 0$ and therefore $\nabla A_H = 0$.

When $m = 10$, we can see from (4.10) that $\langle T, A_0 \rangle = 0$ and then, from (4.9), that

$$\int_M |\nabla A_0|^2 = 100 \int_M |\text{grad } h|^2,$$

which implies equality in (4.4).

Consider the open set $U = \{p \in M | (\text{grad } h)(p) \neq 0\}$ and an arbitrary point $p_0 \in U$. We will show that $\Delta h^2 = 0$ at p_0 , and therefore on U .

First, on an open neighborhood of p_0 , we consider an orthonormal frame field $\{E_1 = \text{grad } h / |\text{grad } h|, E_2, \dots, E_{10}\}$ and, since $A_0 E_1 = -5h E_1$, we have

$$\begin{cases} (\nabla A_0)(E_1, E_1) = -5|\text{grad } h|E_1 \\ (\nabla A_0)(E_i, E_j) = 0, \quad \forall i, j \in \{2, \dots, 10\}, \quad i \neq j \\ (\nabla A_0)(E_1, E_i) = \frac{5}{3}|\text{grad } h|E_i, \quad \forall i \in \{2, \dots, 10\} \\ (\nabla A_0)(E_i, E_i) = \frac{5}{3}|\text{grad } h|E_1, \quad \forall i \in \{2, \dots, 10\}. \end{cases} \tag{4.11}$$

From the commutation formula

$$(\nabla^2 A_0)(X, Y, Z) - (\nabla^2 A_0)(Y, X, Z) = R A_0(X, Y, Z),$$

one obtains

$$\sum_{i=1}^{10} \left\{ (\nabla^2 A_0)(E_i, Y, E_i) - (\nabla^2 A_0)(Y, E_i, E_i) \right\} = -T(Y).$$

Since $\langle T, A_0 \rangle = 0$, we have

$$\sum_{i,j=1}^{10} \left\{ \langle (\nabla^2 A_0)(E_i, E_j, E_i), A_0 E_j \rangle - \langle (\nabla^2 A_0)(E_j, E_i, E_i), A_0 E_j \rangle \right\} = 0. \tag{4.12}$$

After some long but otherwise simple computations, using Eqs. (4.11) and $A_0 E_1 = -5h E_1$, we get the expressions of $(\nabla^2 A_0)(E_1, E_1, E_1)$, $(\nabla^2 A_0)(E_i, E_1, E_i)$, $(\nabla^2 A_0)(E_1, E_j, E_1)$, $(\nabla^2 A_0)(E_j, E_j, E_j)$, and $(\nabla^2 A_0)(E_i, E_j, E_i)$, with $i, j \neq 1$ and $i \neq j$, and then

$$\begin{aligned} \sum_{i,j=1}^{10} \langle (\nabla^2 A_0)(E_i, E_j, E_i), A_0 E_j \rangle &= 50h(E_1 | \text{grad } h|) + \frac{200}{3}h(\text{div } E_1) | \text{grad } h| \\ &+ \frac{10}{3} | \text{grad } h| \sum_{i=2}^{10} \langle \nabla_{E_i} E_1, A_0 E_i \rangle \end{aligned}$$

and

$$\sum_{i,j=1}^{10} \langle (\nabla^2 A_0)(E_j, E_i, E_i), A_0 E_j \rangle = -50h(E_1 | \text{grad } h|) + 10 | \text{grad } h| \sum_{i=2}^{10} \langle \nabla_{E_i} E_1, A_0 E_i \rangle.$$

Replacing in Eq. (4.12), one obtains

$$15h(E_1|\text{grad } h) + 10h(\text{div } E_1)|\text{grad } h| - |\text{grad } h| \sum_{i=2}^{10} \langle \nabla_{E_i} E_1, A_0 E_i \rangle = 0. \tag{4.13}$$

We also have

$$\begin{aligned} \sum_{i=2}^{10} \langle \nabla_{E_i} E_1, A_0 E_i \rangle &= - \sum_{i=2}^{10} \langle E_1, (\nabla A_0)(E_i, E_i) + A_0(\nabla_{E_i} E_i) \rangle \\ &= - \langle E_1, 15 \text{grad } h \rangle - \sum_{i=2}^{10} \langle A_0 E_1, \nabla_{E_i} E_i \rangle \\ &= - 15|\text{grad } h| - 5h \sum_{i=2}^{10} \langle \nabla_{E_i} E_1, E_i \rangle \\ &= - 15|\text{grad } h| - 5h \text{div } E_1 \end{aligned}$$

and Eq. (4.13) becomes

$$h(E_1|\text{grad } h) + h(\text{div } E_1)|\text{grad } h| + |\text{grad } h|^2 = 0. \tag{4.14}$$

Now, we obtain $E_1|\text{grad } h| = (\text{Hess } h)(E_1, E_1)$ and

$$\text{div } E_1 = - \frac{(\text{Hess } h)(E_1, E_1) + \Delta h}{|\text{grad } h|}$$

and then, from (4.14), it follows that

$$-h\Delta h + |\text{grad } h|^2 = 0,$$

which is nothing but $\Delta h^2 = 0$.

Next, on $\text{int}(M \setminus U)$ we have $\text{grad } h = 0$ and therefore $\Delta h^2 = 0$. By continuity, it follows that $\Delta h^2 = 0$ throughout M , which means that h is constant, i.e., M is PMC. This also implies that $\nabla A_0 = 0$ and, therefore, that $\nabla A_H = 0$, which concludes the proof. \square

Remark 4.7 The (compact) PMC submanifolds in $N(c)$, $c \in \{0, 1\}$, with A_H parallel were classified in Refs. [40,41], and then such submanifolds which are also proper biharmonic were described in [4, Theorem 3.16].

Corollary 4.8 *Let $\phi : M^m \rightarrow N^{m+1}(c)$ be a compact biconservative hypersurface in a space form such that its mean curvature does not vanish at any point, $\text{Riem}^M \geq 0$, and $m \leq 10$. Then M is one of the hypersurfaces in Theorem 3.9.*

From the last corollary, we find another partial answer to Conjecture 2, which is a weaker result than that of Chen [9].

Corollary 4.9 *Let $\phi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper biharmonic hypersurface such that its mean curvature does not vanish at any point, $\text{Riem}^M \geq 0$, and $m \leq 10$. Then M is either $\mathbb{S}^m(1/\sqrt{2})$ or the product $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Open Problems

Our results concerning compact biconservative hypersurfaces in space forms satisfying certain additional geometric conditions raise the following natural question.

Is any compact biconservative hypersurface in a space form CMC?

Another open problem is the following (possible) partial answer to Conjecture 2

The only non-minimal solutions to Eqs. (3.5) are the hypersurfaces given by Conjecture 2.

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References

1. Alencar, H., do Carmo, M.: Hypersurfaces with constant mean curvature in spheres. Proc. Am. Math. Soc. **120**, 1223–1229 (1994)
2. Baird, P., Eells, J.: A Conservation Law for Harmonic Maps, Geometry Symposium Utrecht 1980. Lecture Notes in Mathematics, vol. 894, pp. 1–25. Springer, Berlin/New York (1981)
3. Balmuş, A., Montaldo, S., Oniciuc, C.: Classification results for biharmonic submanifolds in spheres. Israel J. Math. **168**, 201–220 (2008)
4. Balmuş, A., Oniciuc, C.: Biharmonic submanifolds with parallel mean curvature vector field in spheres. J. Math. Anal. Appl. **386**, 619–630 (2012)
5. Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds of \mathbb{S}^3 . Int. J. Math. **12**, 867–876 (2001)
6. Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds in spheres. Israel J. Math. **130**, 109–123 (2002)
7. Caddeo, R., Montaldo, S., Oniciuc, C., Piu, P.: Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor. Ann. Mat. Pura Appl. **193**, 529–550 (2014)
8. Chen, B.-Y.: Some open problems and conjectures on submanifolds of finite type. Soochow J. Math. **17**, 169–188 (1991)
9. Chen, J.H.: Compact 2-harmonic hypersurfaces in $\mathbb{S}^{n+1}(1)$. Acta Math. Sinica **36**, 49–56 (1993)
10. Cheng, S.-Y., Yau, S.-T.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)
11. Chern, S. S., do Carmo, M., Kobayashi, S.: Minimal submanifolds of a sphere with second fundamental form of constant length. In: Functional Analysis and Related Fields (Proceeding Conference for M. Stone, University of Chicago, Chicago), Springer, New York, pp. 59–75 (1970)
12. Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)
13. Erbacher, J.: Isometric immersions of constant mean curvature and triviality of the normal connection. Nagoya Math. J. **45**, 139–165 (1971)
14. Fetcu, D., Nistor, S., Oniciuc, C.: On biconservative surfaces in 3-dimensional space forms. Commun. Anal. Geom. **24**, 1027–1045 (2016)
15. Fu, Y.: Explicit classification of biconservative surfaces in Lorentz 3-space forms. Ann. Mat. Pura Appl. **194**, 805–822 (2015)

16. Fu, Y., Hong, M.-C.: Biharmonic hypersurfaces with constant scalar curvature in space forms. *Pacif. J. Math.* **294**, 329–350 (2018)
17. Hilbert, D.: Die grundlagen der physik. *Math. Ann.* **92**, 1–32 (1924)
18. Ichiyama, T., Inoguchi, J.I., Urakawa, H.: Clasification and isolation phenomena of bi-harmonic maps and bi-Yang-Mills fields. *Note Mat.* **30**, 15–48 (2010)
19. Jiang, G.Y.: 2-Harmonic isometric immersions between Riemannian manifolds. *Chin. Ann. Math. Ser. A* **7**, 130–144 (1986)
20. Jiang, G.Y.: 2-Harmonic maps and their first and second variational formulas. *Chin. Ann. Math. Ser. A* **7**(4), 389–402 (1986)
21. Lawson, H.B.: Local rigidity theorems for minimal hypersurfaces. *Ann. Math.* **89**, 187–197 (1969)
22. Loubeau, E., Montaldo, S., Oniciuc, C.: The stress-energy tensor for biharmonic maps. *Math. Z.* **259**, 503–524 (2008)
23. Loubeau, E., Oniciuc, C.: Biharmonic surfaces of constant mean curvature. *Pacif. J. Math.* **271**, 213–230 (2014)
24. Luo, Y., Maeta, S.: Biharmonic hypersurfaces in a sphere. *Proc. Am. Math. Soc.* **145**, 3109–3116 (2017)
25. Maeta, S.: Properly immersed submanifolds in complete Riemannian manifolds. *Adv. Math.* **253**, 139–151 (2014)
26. Maeta, S., Ou, Y.-L.: Some classifications of biharmonic hypersurfaces with constant scalar curvature, Preprint (2017) [arXiv:1708.08540](https://arxiv.org/abs/1708.08540)
27. Mok, N., Siu, Y.-T., Yeung, S.-K.: Geometric superrigidity. *Invent. Math.* **113**, 57–83 (1993)
28. Montaldo, S., Oniciuc, C., Ratto, A.: Proper biconservative immersions into the Euclidean space. *Ann. Mat. Pura Appl.* **195**, 403–422 (2016)
29. Montaldo, S., Oniciuc, C., Ratto, A.: Biconservative surfaces. *J. Geom. Anal.* **26**, 313–329 (2016)
30. Moore, J.D.: Isometric immersions of riemannian products. *J. Differ. Geom.* **5**, 159–168 (1971)
31. Nakauchi, N., Urakawa, H., Gudmundsson, S.: Biharmonic maps into a Riemannian manifold of non-positive curvature. *Geom. Dedic.* **169**, 263–272 (2014)
32. Nistor, S.: On biconservative surfaces. *Differ. Geom. Appl.* **54**, 490–502 (2017)
33. Nomizu, K., Smyth, B.: A formula of Simons' type and hypersurfaces with constant mean curvature. *J. Differ. Geom.* **3**, 367–377 (1969)
34. Oniciuc, C.: Biharmonic submanifolds in space forms, Habilitation Thesis, (2012) [www.researchgate.net](https://www.researchgate.net/publication/26149805605), <https://doi.org/10.13140/2.1.4980.5605>
35. Ou, Y.-L., Tang, L.: On the generalized Chen's conjecture on biharmonic submanifolds. *Mich. Math. J.* **61**, 531–542 (2012)
36. Ryan, P.J.: Homogeneity and some curvature conditions for hypersurfaces. *Tôhoku Math. J.* **21**, 363–388 (1969)
37. Ryan, P.J.: Hypersurfaces with parallel Ricci tensor. *Osaka Math. J.* **8**, 251–259 (1971)
38. Sanini, A.: Applicazioni tra varietà riemanniane con energia critica rispetto a deformazioni di metriche. *Rend. Mat.* **3**, 53–63 (1983)
39. Simons, J.: Minimal varieties in Riemannian manifolds. *Ann. Math.* **88**, 62–105 (1968)
40. Smyth, B.: Submanifolds of constant mean curvature. *Math. Ann.* **205**, 265–280 (1973)
41. Yau, S.T.: Submanifolds with constant mean curvature II. *Am. J. Math.* **97**, 76–100 (1975)