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The Heisenberg Group and Its Relatives in the Work of Elias M. Stein

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Abstract
We survey the work of Elias M. Stein in the field of analysis on the Heisenberg group and other nilpotent Lie groups, together with its applications to complex analysis in several variables and partial differential equations.

Keywords Harmonic analysis · Heisenberg group · Homogeneous group · Nilpotent Lie group

Mathematics Subject Classification Primary 43A80 · Secondary 32V20 · 35B65 · 42B20 · 42B37

1 Introduction

A substantial part of Elias M. Stein's research, beginning around 1970 and continuing through the rest of his life, had to do with analysis on the Heisenberg group and more general noncommutative nilpotent Lie groups, as well as analysis on other manifolds for which such groups provide model cases and analytic tools. The purpose of this article is to offer a brief survey of this work.

In order to keep the scope within reasonable bounds, I adopted two general principles. First, the central focus is on the Heisenberg group. The level of detail provided for results in more general settings is, so to speak, a decreasing function of the distance from the Heisenberg group, dropping to zero when the connection with nilpotent groups becomes negligible or nonexistent. Second, although many other people have contributed to the research in this area, I do not discuss any papers of which Stein is not the author or co-author. The only exceptions to this rule are a few citations of papers that provide some essential background material for Stein's work.

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Stein and his collaborators frequently announced their results via notes in the *Bulletin of the American Mathematical Society* or the *Proceedings of the National Academy of Sciences* before publishing a more complete account elsewhere. In this survey I have generally cited only the fully detailed papers, passing over the research announcements except in a few cases where there is a reason for mentioning them specifically.

2 Background

In this section we review some concepts, terminology, and notation that will be needed for the rest of the paper.

Let $\mathfrak{g}$ be a real Lie algebra. Its *lower central series* is the descending sequence $\{\mathfrak{g}^{(k)}\}$ of ideals defined by $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}]$ for $k > 1$. $\mathfrak{g}$ is *nilpotent of step k* if $\mathfrak{g}^{(k)} \neq \{0\} = \mathfrak{g}^{(k+1)}$. We shall be interested in the following nested family of subclasses within the class of nilpotent Lie algebras:

$$\text{homogeneous} \supset \text{graded} \supset \text{stratified} \supset \text{H-type} \supset \text{Heisenberg}.$$

Here are the definitions: A *homogeneous* Lie algebra is a nilpotent Lie algebra $\mathfrak{g}$ equipped with a one-parameter family $\{\delta_r : r > 0\}$ of automorphisms of the form $\delta_r = r^A$ $(= \exp(A \log r))$ where A is a diagonalizable linear transformation of $\mathfrak{g}$ with positive eigenvalues; we shall call the automorphisms in such a family *dilations*. A nilpotent Lie algebra $\mathfrak{g}$ equipped with a vector space decomposition $\mathfrak{g} = \bigoplus_{k\geq 1} V_k$ (with all but finitely many V_k equal to $\{0\}$) such that $[V_j, V_k] \subset V_{j+k}$ is called *graded*. The canonical family of dilations on a graded Lie algebra is given by $\delta_r\big(\sum v_k\big) = \sum r^k v_k$ for $v_k \in V_k$. A graded Lie algebra $\mathfrak{g}$ is *stratified* if V_1 generates $\mathfrak{g}$ as a Lie algebra. An *H-type* (or *Heisenberg-type*) Lie algebra is a 2-step graded Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ such that (i) V_2 is the center of $\mathfrak{g}$ and (ii) for each nonzero linear functional λ on V_2 the bilinear form $(X, Y) \mapsto \lambda([X, Y])$ on V_1 is nondegenerate. Finally, for $n \geq 1$ the *Heisenberg algebra* $\mathfrak{h}_n$ is the $(2n + 1)$-dimensional Lie algebra whose underlying vector space is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and whose Lie bracket is given by

$$\big[(p, q, s), \big(p', q', s'\big)\big] = \big(0, 0, p \cdot q' - q \cdot p'\big). \tag{1}$$

This is clearly H-type, with $V_1 = \mathbb{R}^n \times \mathbb{R}^n \times \{0\}$ and $V_2 = \{0\} \times \{0\} \times \mathbb{R}$, and its dilations are given by $\delta_r(p, q, s) = (rp, rq, r^2 s)$.

If $\mathfrak{g}$ is any nilpotent Lie algebra, the exponential map is a diffeomorphism from $\mathfrak{g}$ onto the corresponding simply connected group G, so we may identify G with $\mathfrak{g}$ as sets with the group law determined from the Lie algebra law by the Campbell–Hausdorff formula, and the terminology introduced above for $\mathfrak{g}$ will be taken to apply also to G.[1] In particular, when $\mathfrak{g}$ is homogeneous, the dilations δ_r are also group automorphisms of G. Lebesgue measure dx on $\mathfrak{g}$ is a bi-invariant Haar measure on G, and the number

[1] In the literature of sub-Riemannian geometry, stratified groups are commonly called *Carnot groups*.

Q such that $\mathrm{d}(\delta_r x) = r^Q\,\mathrm{d}x$ (namely, the sum of the eigenvalues of A when $\delta_r = r^A$) is called the *homogeneous dimension* of $\mathfrak{g}$ or G.

Note: All nilpotent Lie groups in the sequel will be assumed to be simply connected without explicit specification.

The *Heisenberg group* H_n is the simply connected group corresponding to the Heisenberg algebra $\mathfrak{h}_n$. Since for 2-step nilpotent algebras the Campbell–Hausdorff formula is simply $(\exp X)(\exp Y) = \exp\big(X + Y + \frac{1}{2}[X, Y]\big)$, the group law on H_n is

$$(p,\ q,\ s)\big(p',\ q',\ s'\big) = \big(p + p',\ q + q',\ s + s' + \tfrac{1}{2}(p \cdot q' - q \cdot p')\big). \tag{2}$$

For our purposes, however, it will be convenient to describe H_n in a slightly different way. Namely, if we identify $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with $\mathbb{C}^n \times \mathbb{R}$ via the map

$$(p, q, s) \mapsto (z, t) \equiv (q + ip, 4s),$$

one can easily check that the group law (2) becomes

$$(z,\ t)(z',\ t') = \big(z + z',\ t + t' + 2\mathrm{Im} z \cdot \overline{z'}\big). \tag{3}$$

(Here and below, $z \cdot w = \sum z_j w_j$.) Henceforth we think of H_n as $\mathbb{C}^n \times \mathbb{R}$ endowed with the group law (3).

The group H_n acts as a group of biholomorphic transformations of $\mathbb{C}^{n+1}$, as follows. For $(z, t) \in H_n$, we define $T_{(z,t)} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by

$$T_{(z,t)}(\omega, \zeta_1, \ldots, \zeta_n) = \big(\omega + t + i|z|^2 + 2i\overline{z} \cdot \zeta,\ \zeta_1 + z_1, \ldots, \zeta_n + z_n\big). \tag{4}$$

The reader may easily verify that (i) the correspondence $(z, t) \mapsto T_{(z,t)}$ is a group homomorphism; (ii) $T_{(z,t)}$ maps the domain

$$D^{n+1} = \big\{(\omega, \zeta) = (\omega, \zeta_1, \ldots, \zeta_n) \in \mathbb{C}^{n+1} : \mathrm{Im}\omega > |\zeta|^2\big\} \tag{5}$$

and its boundary[2]

$$bD^{n+1} = \big\{(\omega, \zeta) : \mathrm{Im}\omega = |\zeta|^2\big\}$$

onto themselves; (iii) the map

$$(z, t) \mapsto T_{(z,t)}(0, 0) = (t + i|z|^2, z) \tag{6}$$

is a bijection from H_n to bD^{n+1}.

[2] In complex geometry, boundaries of domains are denoted by b rather than ∂ to avoid confusion with the holomorphic exterior derivative.

The domain D^{n+1} is biholomorphically equivalent to the unit ball B^{n+1} in $\mathbb{C}^{n+1}$ via the "Cayley transform" $C : B^{n+1} \to D^{n+1}$ defined by

$$C(\omega_0, \ldots, \omega_n) = \frac{1}{i\omega_0 + 1}(\omega_0 + i,\ \omega_1, \ldots, \omega_n). \tag{7}$$

Thus B^{n+1} and D^{n+1} are the higher dimensional analogues of the unit disc B^1 and the upper half plane D^1 in $\mathbb{C}^1$, where the group "H_0" is just $\mathbb{R}$, acting by horizontal translations. But the geometry for $n > 0$ is more complicated. On any smooth real hypersurface $M \subset \mathbb{C}^{n+1}$, the complexified tangent space at any point p splits into the "holomorphic" and "antiholomorphic" spaces $T_{1,0}$ and $T_{0,1}$—that is, the spaces of linear combinations of the $\partial/\partial z_j$'s and the $\partial/\partial \bar{z}_j$'s, respectively, that are tangent to M at p—together with an additional one-dimensional subspace. (In what follows we shall call vectors in $T_{1,0} \oplus T_{0,1}$ *complex-tangential*.) When we identify bD^{n+1} with H_n with coordinates (z, t) via (6), these subspaces become the spans of Z_j $(1 \le j \le n)$, $\overline{Z}_j$ $(1 \le j \le n)$, and T, where

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad \overline{Z}_j = \frac{\partial}{\partial \bar{z}_j} - i z_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}. \tag{8}$$

These are left-invariant vector fields on H_n, and their only nonzero commutators are $[Z_j, \overline{Z}_j] = -2iT$ $(1 \le j \le n)$.

This noncommutativity is a reflection of a fundamental curvature property of D^{n+1} known as *strong pseudoconvexity*, which is defined as the positive definiteness of a certain Hermitian form known as the Levi form. Among several equivalent definitions of the Levi form, the following one will be most relevant for our purposes. Suppose Ω is an open set in $\mathbb{C}^{n+1}$ with smooth boundary $b\Omega$; let $\iota : b\Omega \to \mathbb{C}^{n+1}$ be the inclusion map and let r be a defining function for Ω. (That is, r is a smooth real-valued function on $\mathbb{C}^{n+1}$ such that $\mathrm{d}r \neq 0$ on $b\Omega$, and Ω is the set where $r < 0$.) Then $\tau = \iota^*[(\partial - \bar{\partial})r]$ is a one-form on $b\Omega$ that annihilates the complex-tangential space at each point. The *Levi form* is the Hermitian form on $T_{1,0}$-valued vector fields on $b\Omega$ (defined up to a positive scalar-valued factor depending on the choice of r) given by $L(V, W) = \langle \tau, [V, \overline{W}] \rangle$; it measures how the complex-tangential space $T_{1,0} \oplus T_{0,1}$ fails to be closed under Lie brackets. (When $\Omega = D^{n+1}$, with notation as in (5) we take $r = |\zeta|^2 - \operatorname{Im} \omega$; then it turns out that if $V = \sum a_j Z_j$ and $W = \sum b_j Z_j$ as in (8), $L(V, W) = 2 \sum a_j \bar{b}_j$.)

3 Complex Analysis and Differential Equations

Stein's first contribution to analysis on nilpotent groups came in the late 1960s in his joint work with Anthony Knapp, announced in [11] and presented in detail in [12]. Suppose G is a homogeneous group of homogeneous dimension Q, and K is a smooth function on $G \setminus \{0\}$ ($0 =$ the origin in $\mathfrak{g} =$ the identity element of G) that is homogeneous of degree $-Q$ and hence just on the edge of integrability at both 0 and infinity. Suppose also that K has "mean value zero," or equivalently, that K

defines a distribution that is homogeneous of degree $-Q$ as a distribution.[3] The first main theorem of [11] and [12] is that *the operator* $Sf = f * K$, *initially defined for* $f \in C_c^\infty(G)$, *is bounded on* $L^2(G)$. In the Abelian case ($G = \mathbb{R}^Q$) this is easy, as the Fourier transform of the distribution K is a bounded function, but non-commutative convolutions necessitate an entirely different proof.

That proof represents the debut of the technique of *almost orthogonal decompositions*, which has become a standard tool in harmonic analysis. In more detail, one writes K as a sum of pieces K_j that are supported in compact subsets of $G\backslash\{0\}$ with disjoint interiors and satisfy $\|K_j\|_1 \leq C < \infty$, in such a way that the corresponding operators S_j are "almost orthogonal" in a suitable sense and are uniformly bounded on $L^2(G)$. One then invokes a theorem to the effect that whenever a sequence S_j of operators on a Hilbert space has these properties, the partial sums of the series $\sum S_j$ are uniformly bounded, and the series converges in the weak (and usually strong) operator topology to a bounded operator. This theorem was proved independently by Mischa Cotlar, but the elegant proof in [12] is the one now universally known.

The motivation for these results in [12] was the study of intertwining operators for principal series representations of semisimple matrix groups. Indeed, if G is such a group with Iwasawa decomposition KAN and $\dim(A) = 1$, the action of A by conjugation on N defines a family of dilations, and the intertwining operators can be realized as convolution operators on N of the sort just described. But these noncommutative singular integrals $f \mapsto f * K$ immediately started to take on a life of their own. In particular, soon after the publication of [11], several people independently realized that the Calderón–Zygmund theory—for which the standard reference is now Stein's classic book [39]—could be generalized to cover this situation and even more general ones, so that these operators are bounded not only on $L^2(G)$ but on $L^p(G)$ for $1 < p < \infty$; moreover, the somewhat more elementary arguments that prove that classical singular integrals preserve Hölder continuity also generalize (see Coifman–Weiss [2] and Korányi–Vági [14]). Hence, the whole machinery of singular integral operators on Euclidean space, with its manifold applications, was ripe to expand into new areas where noncommutativity plays an essential role.

At around the same time, the study of boundary values of holomorphic functions on domains in $\Omega \subset \mathbb{C}^{n+1}$ (or, more generally, in complex manifolds) was an active area that attracted Stein's attention. In [40] he showed that in this setting, the appropriate analogue of the non-tangential convergence to boundary values that applies to holomorphic functions on domains in $\mathbb{C}$ (and for harmonic functions in higher dimensions) is what he called "admissible convergence," in which the complex-tangential directions along the boundary are weighted differently than the remaining real direction. He continued this train of thought in the note [41], where he investigated some Lipschitz classes on Ω with similar nonisotropic behavior on $b\Omega$ and showed that some recently obtained integral formulas for solving $\bar\partial u = f$ on Ω (which become nonisotropic singular integrals on $b\Omega$) yield solutions in these classes.

These two lines of research joined together in a very fruitful way in the big paper [5] that Stein and I wrote in 1972–1973 on the $\bar\partial_b$ complex and the Heisenberg group.

[3] More precisely, the mean-zero condition means that K acts on test functions by a principal-value integral, $\langle K, \phi\rangle = \text{p.v.} \int K\phi$, which satisfies $\langle K, \phi \circ \delta_r\rangle = \langle K, \phi\rangle$ for all $r > 0$.

Let us recall the setting: On any real manifold one has the exterior derivative d acting on differential forms of various degrees to form the de Rham complex. On a complex manifold one obtains the $\overline{\partial}$ complex or Dolbeault complex by replacing $\partial/\partial x_j$ by $\partial/\partial \overline{z}_j$ and $\mathrm{d}x_j$ by $\mathrm{d}\overline{z}_j$. Then, on the boundary $b\Omega$ of a domain Ω in a complex manifold, one obtains the $\overline{\partial}_b$ complex by throwing away the part of $\overline{\partial}$ that is not tangent to $b\Omega$. Just as the Laplacian $\Delta = d^*d + dd^*$ is used in the study of the de Rham complex, the *Kohn Laplacian*

$$\Box_b = \overline{\partial}_b^*\overline{\partial}_b + \overline{\partial}_b\overline{\partial}_b^*, \tag{9}$$

first studied by Joseph Kohn [13], is an essential tool in the study of the $\overline{\partial}_b$ complex. Unlike Δ, $\Box_b$ is *not elliptic*, as it is genuinely second-order only in the complex-tangential directions. However, Kohn showed that if Ω is strongly pseudoconvex and $0 < q < n$ where $\dim_{\mathbb{C}}(\Omega) = n+1$, $\Box_b$ acting on q-forms satisfies L^2 estimates that are "half as strong" as elliptic estimates, and in particular it is *hypoelliptic*: that is, a q-form u must be C^∞ on any open set where $\Box_b u$ is C^∞.

On the Heisenberg group H_n, identified with the boundary of the domain D^{n+1} as in (4) and (6), $\overline{\partial}_b$ is simply given by

$$\overline{\partial}_b\Big(\sum_J f_J \mathrm{d}\overline{z}^J\Big) = \sum_J \sum_j (\overline{Z}_j f_j)\mathrm{d}\overline{z}_j \wedge \mathrm{d}\overline{z}^J$$

($\mathrm{d}\overline{z}^J$ denotes a wedge product of $\mathrm{d}\overline{z}_j$'s), and $\Box_b$ is given on q-forms by

$$\Box_b\Big(\sum_{|J|=q} f_J \mathrm{d}\overline{z}^J\Big) = \sum_{|J|=q} (\mathcal{L}_{n-2q} f_J)\mathrm{d}\overline{z}^J,$$

where, for any $\alpha \in \mathbb{C}$,

$$\mathcal{L}_\alpha = -\frac{1}{2}\sum (Z_j\overline{Z}_j + \overline{Z}_j Z_j) + i\alpha T. \tag{10}$$

We therefore focus our attention on the operators $\mathcal{L}_\alpha$, which, like $\Box_b$, are not elliptic.

The calculation that enables the analysis in [5] to proceed is that *the function*

$$\phi_\alpha(z,t) = (|z|^2 - it)^{-(n+\alpha)/2}(|z|^2 + it)^{-(n-\alpha)/2}$$

satisfies $\mathcal{L}_\alpha\phi_\alpha = c_\alpha\delta$, *where* $c_\alpha = 2^{2-2n}\pi^{n+1}/\Gamma(\frac{1}{2}(n+\alpha))\Gamma(\frac{1}{2}(n-\alpha))$ *and* δ *is the point mass at the origin*. Thus, except in the "inadmissible" cases $\pm\alpha = n, n+2, n+4, \ldots$ where $c_\alpha = 0$, $c_\alpha^{-1}\phi_\alpha$ is a fundamental solution for $\mathcal{L}_\alpha$, so for any reasonable f—say, continuous with compact support—the function $u = f * c_a^{-1}\phi_a$ satisfies $\mathcal{L}_\alpha u = f$; it follows easily that $\mathcal{L}_\alpha$ is hypoelliptic.

But we can say more. Since ϕ_α is homogeneous of degree $-2n$ with respect to the dilations $\delta_r(z,t) = (rz, r^2t)$, whereas the operators Z_j, $\overline{Z}_j$, and T decrease degrees of homogeneity by 1, 1, and 2, respectively, the distributions $Z_jZ_k\phi_\alpha$, $Z_j\overline{Z}_k\phi_\alpha$, $\overline{Z}_j\overline{Z}_k\phi_\alpha$, and $T\phi_\alpha$ are all homogeneous of degree $-2n-2$, the homogeneous dimension of H_n. They are therefore singular integral kernels in the sense discussed above, so the convolution operators they define are bounded on L^p for $1 < p < \infty$. We therefore obtain a sharp L^p regularity theorem for the operators $\mathcal{L}_\alpha$ that looks just like the elliptic regularity theorem provided we replace "derivatives of order k" by "derivatives of homogeneous degree k," where the homogeneous degrees of Z_j, $\overline{Z}_j$ and T are 1, 1, and 2, respectively. To wit, *if α is admissible and $\mathcal{L}_\alpha f$ has derivatives of homogeneous degree $\leq k$ in L^p on an open set U, then f has derivatives of homogeneous degrees $\leq k+2$ in L^p on U*. This improves the original results of Kohn [13] for $\Box_b$ in the case $p=2$ and is entirely new for other p. We also obtain sharp Lipschitz or Hölder regularity for $\mathcal{L}_\alpha$ by specifying moduli of continuity in terms of the homogeneous "norm" $|(z,t)| = (|z|^4+t^2)^{1/4}$ instead of the Euclidean norm $(|z|^2+t^2)^{1/2}$.

These results give sharp regularity theorems for $\Box_b$ on q-forms on bD^{n+1} with $0 < q < n$. Moreover, those theorems can be transferred to the boundary $b\Omega$ of any smoothly bounded strongly pseudoconvex domain Ω by locally approximating $b\Omega$ by the Heisenberg group in a sufficiently careful way. This procedure yields an integral kernel $K(x,y)$ on $b\Omega \times b\Omega$ that gives an approximate inverse or *parametrix* for $\Box_b$ on q-forms on $b\Omega$ and qualitatively resembles the kernels $c_\alpha^{-1}\phi_\alpha(y^{-1}x)$ that give the exact inverse of $\mathcal{L}_\alpha$ on H_n. The sharp L^p and Lipschitz estimates follow since appropriate derivatives of K give singular integral kernels as before.

On q-forms with $q=0$ or $q=\dim_{\mathbb{C}}(\Omega)-1$ (which correspond to the inadmissible values $\alpha = \pm n$ for the Heisenberg group), $\Box_b$ is *not* hypoelliptic. For 0-forms (i.e., functions) this is clear from the fact that $\Box_b$ annihilates functions u on $b\Omega$ that are boundary values of holomorphic functions on Ω and hence satisfy $\overline{\partial}_b u = 0$[4]; such functions need not be smooth. Nonetheless, Peter Greiner, Kohn, and Stein [8] showed that on the Heisenberg group the methods described above yield interesting results here too.

First, by differentiating the equation $\mathcal{L}_\alpha\phi_\alpha = c_\alpha\delta$ with respect to α at $\alpha = n$, one finds that the function

$$\psi(z,t) = \frac{2^{n-2}\Gamma(n)}{\pi^{n+1}}(|z|^2-it)^{-n}\log\left(\frac{|z|^2-it}{|z|^2+it}\right)$$

is a "relative fundamental solution" for $\mathcal{L}_n = \Box_b$ in the following sense: denoting by R the operator $Rf = f * \psi$ and by P the orthogonal projection of $L^2(H_n)$ onto $\mathcal{N} = \{f \in L^2(H_n) : \overline{\partial}_b f = 0\}$ (the space of boundary values of functions in $H^2(D^{n+1})$), one has

$$R\Box_b = \Box_b R = I - P. \tag{11}$$

Since P is given by convolution with the so-called Cauchy–Szegő kernel, which is another singular integral kernel of the sort discussed above, this yields regularity for the solution of $\Box_b u = f$ in $\mathcal{N}^\perp$ when f itself is in $\mathcal{N}^\perp$. It also yields a striking local solvability result: *if $f \in L^2(H_n)$, the equation $\Box_b u = f$ has a solution on an open*

[4] The second term in (9) is absent when $q=0$.

set U if and only if Pf is real-analytic on U. As a corollary, in the case $n = 1$ where $\Box_b = Z\overline{Z}$ with $Z = \partial/\partial z + i\bar{z}\partial/\partial t$, the equation $Zu = f$ (which, up to a simple change of variable, is Hans Lewy's original example of a non-locally-solvable linear differential equation [15]) is solvable on U if and only if Pf is real-analytic on U.

There is one more chapter to this story. It follows rather easily from (11) that for any $\lambda \neq 0$ the operator $\Box_b + \lambda I$ on functions on H_n is locally solvable and hypoelliptic. It takes more delicate and precise estimates, however, to show that $\Box_b + \lambda I$ is also analytic–hypoelliptic (i.e., u must be real-analytic on any open set where $\Box_b u + \lambda u$ is). Stein performed that analysis in [42].

The results and techniques of [5] immediately suggested several lines of further research in which Stein played a central role. We shall discuss these in roughly chronological order, but first we need to mention one more piece of background. In 1967 Lars Hörmander [10] studied differential operators of the form $\mathcal{L} = \sum X_j^2$ where $X_1, \ldots, X_m$ are real vector fields on a manifold M. (Note that the operator $-\mathcal{L}_0$ defined by (10) is of this form, with $X_j = (Z_j + \overline{Z}_j)/2$ and $X_{j+n} = (Z_j - \overline{Z}_j)/2i$ for $1 \leq j \leq n$.) He showed that $\mathcal{L}$ *is hypoelliptic provided that at each point $p \in M$, the vector fields X_j and their iterated commutators $[X_i, X_j]$, $[X_h, [X_i, X_j]]$, ..., up to some order k span the tangent space T_pM*. (This hypothesis on the X_j's quickly became known as the *Hörmander condition of step k*.) Here, as with $\Box_b$, the operator $\mathcal{L}$ is not elliptic (unless the X_j's already span the tangent space at each point), and the regularity depends on the fact that the "missing directions" can be controlled by the X_j's through their commutators.

Just as the essential model cases for elliptic operators are the constant-coefficient operators, the essential model cases for vector fields satisfying the Hörmander condition are provided by left-invariant vector fields on nilpotent Lie groups. More precisely, we take G to be a stratified group, with Lie algebra $\mathfrak{g} = \bigoplus_1^k V_j$, and take $X_1, \ldots, X_n$ to be a basis for V_1; the operator $\mathcal{L} = -\sum X_j^2$ is then called a *sub-Laplacian* for the group G.[5] In [4] I showed that $\mathcal{L}$ has a fundamental solution with qualitative properties like those of the operators $\mathcal{L}_\alpha$ (in particular, the sub-Laplacian $\mathcal{L}_0$) on the Heisenberg group and, by arguments similar to those in [5], derived sharp regularity properties of $\mathcal{L}$ in terms of non-isotropic Sobolev and Lipschitz norms.

In a major paper, Linda Rothschild and Stein [37] showed how to generalize these results to sums of squares of general vector fields X_j on a manifold M satisfying the Hörmander condition of step k. This requires a more complicated technique than the approximation procedure in [5], because the minimal Lie algebra whose structure can reflect the essential commutators of the X_j's may vary from point to point and will in general be of higher dimension than M. For example, on $\mathbb{R}^2$ let $X_1 = \partial/\partial x$ and $X_2 = x\partial/\partial y$. Away from the y-axis the operator $X_1^2 + X_2^2$ is already elliptic, but on the y-axis the commutator $[X_1, X_2] = \partial/\partial y$ is needed to span the tangent space, and the Lie algebra spanned by X_1, X_2, and $[X_1, X_2]$ (an isomorph of $\mathfrak{h}_1$) is 3-dimensional although the X_j's act on $\mathbb{R}^2$. Rothschild and Stein's solution to this problem was to employ a stratified group G, the "free nilpotent group of step k," with a sufficiently rich commutator structure to reflect the essential commutation relations of the X_j's

[5] The inclusion of the minus sign is a matter of taste. It has the advantage of making $\mathcal{L}$ a positive operator in the sense of Hilbert space theory.

at every point, and to lift the X_j's to vector fields $\widetilde{X}_j$ on a manifold $\widetilde{M}$ of the same dimension as G in such a way that $-\sum \widetilde{X}_j^2$ can be approximated near each point by the sub-Laplacian on G. This gives sharp regularity results for $\sum \widetilde{X}_j^2$ on $\widetilde{M}$, which can then be projected back down to M to yield the regularity results for $\sum X_j^2$. Rothschild and Stein also obtain similar results for other related classes of second-order operators, including the Kohn Laplacians for domains satisfying more general pseudoconvexity conditions.

The parametrices for differential operators and the associated singular integrals in [5] and [37] are all described in terms of integral kernels. In the classical theory of elliptic differential equations, it has been very useful to have an alternative description of these operators as Fourier integrals, which yields a "symbolic calculus": the calculus of pseudodifferential operators. The standard symbol classes $S^m_{1,0}$ for pseudodifferential operators have been extended in several ways to encompass the operators needed for various purposes. In [28], Alexander Nagel and Stein introduced a new class of symbols that is adapted to the problems we have just discussed and studied the resulting class of pseudodifferential operators in relation to those problems.

The detailed analysis of the phenomena studied in [37] requires a study of the geometry associated to a family of vector fields X_j on a manifold M that satisfy the Hörmander condition. More precisely, such a family gives rise to a nonisotropic notion of "distance," or rather a collection of equivalent notions of distance. Here is one: if $p, q \in M$, we say that $\rho(p,q) < \delta$ if there is a curve $\phi : [0,1] \to M$ with $\phi(0) = p$ and $\phi(1) = q$, whose tangent vector $\phi'(t)$ at each point has the form $\sum a_j^{(1)} X_j + \sum a_{jk}^{(2)} [X_j, X_k] + \cdots$, where the coefficients $a_*^{(l)}$ at level l are bounded by δ^l. In the announcement [31] and the paper [32] Nagel, Stein, and Stephen Wainger made a detailed study of the balls associated to these distance functions and derived estimates for the parametrices and related kernels of Rothschild–Stein. (Incidentally, the promised Part II of [32], dealing with generalizations of results in [40] to certain weakly pseudoconvex domains, never appeared; the authors have let the statements and sketches of proofs in the announcement [31] stand on their own.)

To return to complex analysis: the methods and results of [5] have a bearing on the study not only of the operator $\Box_b$ but on several related operators arising from complex analysis on a strongly pseudoconvex domain Ω. The first of these is the solution of the $\overline{\partial}$-*Neumann problem*, a boundary value problem of central importance for the Laplacian $\Box = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ for the Dolbeault complex on Ω. (In brief, the problem is to solve $\Box u = f$ on q-forms, subject to boundary conditions that guarantee that u and $\overline{\partial} u$ are in the domain of the Hilbert space adjoint of $\overline{\partial}$ on the space of L^2 forms on M. These conditions are "non-coercive," which implies that one cannot expect regularity properties of the solution as strong as the estimates for more classical boundary value problems such as the Dirichlet problem.) The original solution of the $\overline{\partial}$-Neumann problem by Kohn was obtained by Hilbert space methods and L^2 estimates, without any explicit formulas. In [9], Greiner and Stein applied the technology of [5] to construct an explicit parametrix that yields sharp L^p and Lipschitz estimates; it involves a mixture of operators that possess non-isotropic regularity properties as in [5] and operators of a more classical sort. Many years later, Stein and his collaborators embedded such "mixed" operators into a more general theory that we shall describe in Sect. 5.

Two other operators of primary interest in complex analysis on a strongly pseudoconvex domain Ω are the Bergman projection P_B (the orthogonal projection from $L^2(\Omega)$ to the subspace of L^2 holomorphic functions) and the Szegő projection P_S (the orthogonal projection from $L^2(b\Omega)$ to the subspace of boundary values of H^2 holomorphic functions). Using the ideas of [5] together with the known asymptotic formulas for the integral kernels of P_B and P_S, Duong Hong Phong and Stein [33] showed that P_B and P_S are not only bounded on the appropriate Sobolev and Lipschitz spaces but actually bounded from these spaces to the corresponding nonisotropic spaces with greater regularity in the complex-tangential directions.

With all of these results in hand, a natural next step was to extend them to appropriate classes of smoothly bounded weakly pseudoconvex domains—in particular, domains of "finite type" where the extra real direction on the boundary is obtained from *higher-order* commutators of complex-tangential vector fields. These matters are rather far removed from our principal focus, so we shall just give a brief list of the relevant papers and their topics. For domains of finite type in $\mathbb{C}^2$, the regularity properties of the Bergman and Szegő kernels, the $\overline{\partial}$-Neumann problem, and the heat equation for $\Box_b$ have been investigated by Nagel, Jean-Pierre Rosay, Stein, and Wainger [27], Der-Chen Chang, Nagel, and Stein [1], and Nagel and Stein [29], respectively. In higher dimensions (where geometric complications are an obstacle to the complete extension of the results in $\mathbb{C}^2$ to arbitrary domains of finite type), Jeffrey McNeal and Stein [16,17] have investigated the properties of the Bergman and Szegő kernels for convex domains of finite type, and Nagel and Stein [29] have studied the $\overline{\partial}_b$ complex on a class of model domains which they call "decoupled." Nagel and Stein [30] have also studied the L^p regularity of $\Box_b$ and the Szegő projection on a class of model CR submanifolds of $\mathbb{C}^n$ of higher codimension. These last two works involve the use of singular integrals of a more general type whose kernels have singularities on subspaces rather than single points, which we shall discuss in Sect. 5.

4 More Neo-classical Analysis on Groups

The research described in the preceding section has roots in classical harmonic analysis on $\mathbb{R}^n$ and its applications to differential equations. In this section we discuss four more works of Stein and his collaborators that fall into this general category.

(1) The theory of Hardy spaces H^p ($0 < p < \infty$) was originally part of complex analysis in one variable, but in the 1960s and 1970s it spawned a real-variable theory of Hardy spaces H^p on $\mathbb{R}^n$: these are spaces of distributions that possess suitable maximal functions in L^p, which can also be characterized for $p \leq 1$ in terms of "atomic decompositions." (For $p > 1$, H^p is simply L^p.) Stein played a big part in the growth of this theory, first through joint work first with Guido Weiss [43] and then with Charles Fefferman [3]. In [6] Stein and I developed the theory of real-variable H^p spaces on homogeneous groups: their characterization in terms of various maximal functions, square functions, and atomic decompositions; their dual spaces; boundedness of singular and fractional integrals; and connections with Poisson integrals on symmetric spaces. To some extent the Euclidean theory

could be easily generalized, but in many places there were technical obstacles to be overcome, and occasionally some novel techniques were needed.

(2) Nagel, Fulvio Ricci, and Stein [23] (see also [22] for a summary of the results) investigated the properties of fundamental solutions for a class of hypoelliptic left-invariant differential operators $\mathcal{L}$ on arbitrary (not necessarily homogeneous) nilpotent groups G. (The following may be taken as a model case: $\mathcal{L} = -\sum X_j^2$ where $X_1, \ldots, X_n$ are any left-invariant vector fields that generate $\mathfrak{g}$ as a Lie algebra; however, the authors' class of operators also includes higher-order ones.) They show that there are two *homogeneous* groups G_0 and G_∞, built on the same underlying vector space as G, such that the corresponding operators L_0 and L_∞ on G_0 and G_∞ are homogeneous with respect to the dilations on these groups and possess homogeneous fundamental solutions K_0 and K_∞. Their main result is that the original operator $\mathcal{L}$ has a fundamental solution that resembles K_0 near the origin and resembles K_∞ near infinity, from which they derive some L^p estimates.

(3) Let G be a stratified group with sub-Laplacian $\mathcal{L} = -\sum X_j^2$. If m is any bounded Borel function on $(0, \infty)$, one can define the bounded operator $m(\mathcal{L})$ on $L^2(G)$ by the spectral functional calculus, and it is of interest to know when $m(\mathcal{L})$ is also bounded on other L^p spaces. (When $G = \mathbb{R}^n$, this is subsumed in the more general problem of the boundedness of Fourier multipliers, the principal results on which are associated with the names of Marcinkiewicz and Hörmander. See Stein's book [39] and the references given there.) Through the work of several people, it was established that a sufficient condition for $m(\mathcal{L})$ to be bounded on L^p $(1 < p < \infty)$ and weak type (1.1) is that $\|m\|_{(\alpha)} < \infty$ for some $\alpha > Q/2$, where Q is the homogeneous dimension of G and $\|\cdot\|_{(\alpha)}$ is a localized, scale-invariant version of the L^2 Sobolev norm of order α on $(0, \infty)$. But in [20], Detlef Müller and Stein showed that for groups related to the Heisenberg groups the hypothesis $\alpha > Q/2$ could be improved to $\alpha > d/2$, where d is the *Euclidean* dimension of G; this condition is sharp when G is a Heisenberg group or a Euclidean group. (Their precise hypotheses are that $G = \prod_1^K G_k$ where each G_k is either a Heisenberg group or a Euclidean group, and $\mathcal{L} = \sum_1^K \mathcal{L}_k$ where $\mathcal{L}_k$ is a sub-Laplacian on G_k.) The crux of the proof is a weighted L^2 estimate for the distribution kernel of $m(\mathcal{L})$ on the Heisenberg group that is actually *stronger* than the corresponding Euclidean estimate.

(4) For the standard wave equation $\partial_t^2 u - \Delta u = 0$ on $\mathbb{R}^n$, it is easy to see that there can be a loss of about $n/2$ continuous derivatives in passing from the initial data $u|_{t=0}$ and $\partial_t u|_{t=0}$ to the solution $u|t = t_0$ for any $t_0 > 0$, but that there is no loss of L^2 derivatives. This suggests that for other values of p there should be an L^p estimate involving a loss of an intermediate number of derivatives, and this is indeed the case; the basic result is that the operator $e^{it\sqrt{-\Delta}}(1-\Delta)^{-\alpha}$ is bounded on L^p if $\alpha \geq (n-1)|1/p - 1/2|$. Müller and Stein [21] established an analogous but slightly weaker result for the wave equation $\partial_t^2 u + \mathcal{L}_0 u = 0$ on the Heisenberg group H_n [where $\mathcal{L}_0$ is defined by (10)]: the operator $e^{it\sqrt{\mathcal{L}_0}}(1+\mathcal{L}_0)^{-a}$ is bounded on L^p if $\alpha > (2n)|1/p - 1/2|$. The proof is substantially more difficult than in the Euclidean case because of the nonellipticity of $\mathcal{L}_0$.

5 Strongly Singular Integrals on Groups

Much of Stein's work in the latter part of his career has to do with integral operators that are "more singular" than Calderón–Zygmund operators: convolution operators whose kernels have singularities on submanifolds rather than single points, or whose kernels are supported on submanifolds, or which are "twisted" by oscillatory factors, as well as more general operators for which such convolutions are model cases. In particular, he and his collaborators produced an important series of papers on such operators in the context of the Heisenberg group and other nilpotent groups.

The greater generality in some of these situations necessitates additional technicalities in the statements of the results, with which it would be tedious to burden the readers of this survey paper. We shall therefore let the following brief indications suffice. The setting is a homogeneous group G, and the operators to be studied are convolutions with distributions K that are C^∞ away from a lower-dimensional singular set S; we may be imprecise in distinguishing between the distribution K and the function $\widetilde{K} = K|(G \setminus S)$. In the classical case, $S = \{0\}$ and K is obtained from $\widetilde{K}$ a principal-value integral, which exists since $\widetilde{K}$ is required to satisfy a suitable mean-zero condition. As we indicated briefly in Sect. 2, this condition is equivalent to the condition that $\langle K, \phi \circ \delta_r \rangle$ is independent of $r > 0$ for any test function ϕ. In the general setting we do not specify exactly how K is determined from $\widetilde{K}$, and it is most convenient to replace the mean-zero condition by suitable hypotheses on the uniform boundedness of the action of K on test functions composed with dilations. Without trying to state these hypotheses precisely, we shall refer to them as "appropriate cancellation conditions."

Stein's work on strongly singular integrals on groups began with a paper with Daryl Geller [7] on convolution operators on the Heisenberg group H_n with kernels of the form $K(z, t) = L(z)\delta(t)$ where L is a Calderón–Zygmund kernel on $\mathbb{C}^n$ and δ is the Dirac distribution on $\mathbb{R}$: their main result is that such operators are bounded on L^p, $1 < p < \infty$. For ordinary (Euclidean) convolution on $\mathbb{R}^{2n+1}$ this would immediately reduce to the classical theory for convolution with L on $\mathbb{C}^n$, but the noncommutative convolution on H_n displays unexpected new features. For example, although convolution on $\mathbb{R}^{2n+1}$ with the distribution $D(z, t) = (\partial_t^n \delta)(t)$ is far from being a bounded operator on any common function space, it is not hard to see that Heisenberg convolution with D is unitary on L^2 up to a scalar factor. With this in mind, Geller and Stein showed that there is an analytic family of distributions K_γ with $K_0 = K$, becoming more strongly singular in t as $\operatorname{Re}\gamma$ decreases, such that Heisenberg convolution with K_γ is bounded on L^2 for $\operatorname{Re}\gamma = -n$ and bounded on all L^p for $\operatorname{Re}\gamma > 0$. The boundedness for $\gamma = 0$ then follows from Stein's interpolation theorem [38] for analytic families — one of his earliest contributions to analysis and one of his favorite tools ever since.

A couple of years later, Ricci and Stein [34,35] embedded this result into a more general L^p theory of singular integral operators on a homogeneous group G, of the following two types: (1) $Tf(x) = \int_G e^{iP(x,y)} K(xy^{-1}) f(y)\,\mathrm{d}y$, where K is a kernel of Calderón–Zygmund type and P is a real polynomial; (2) $Sf(x) = \int_V f(xy^{-1}) K(y) \mathrm{d}\sigma(y)$, where V is a smooth submanifold of G that is invariant under dilations, σ is surface measure on V, and K is a kernel such that the distribution

$K\,d\sigma$ has the appropriate critical homogeneity and satisfies the appropriate cancellation conditions. (These two types of operators are closely related, via Euclidean Fourier analysis on the center of G.) Moreover, they extended these results to the situation where the homogeneity of the kernels is with respect to a one-parameter family of maps that satisfy the properties of dilations except that they need not be automorphisms of G (so G need not even be homogeneous). The idea is that in this case, G can be realized as a quotient of a larger homogeneous group $\widetilde{G}$ in such a way that its "dilations" come from the automorphic dilations of $\widetilde{G}$; the results on $\widetilde{G}$ can then be transferred to G. Finally, in a third paper [36], Ricci and Stein extended this study to "fractional integrals": operators of the above two types where the kernel is homogeneous of some degree higher than the critical one.

At this point another line of investigation comes into the picture: singular integrals and spectral multipliers associated to a *multi-parameter* dilation structure, which may or may not consist of automorphisms of the group in question. The most straightforward situation is a product $G = G_1 \times G_2$ where G_1 and G_2 are stratified groups, each carrying its own family of dilations δ^1_r, δ^2_r and its own sub-Laplacian $\mathcal{L}_1 = -\sum X_j^2$, $\mathcal{L}_2 = -\sum Y_j^2$. G is then endowed with the two-parameter family of dilations $\delta_{(r,s)}(x, y) = \left(\delta^1_r x, \delta^2_s y\right)$, and one can consider (1) convolution with "product-type" singular integral kernels $K(x, y)$ that satisfy estimates of the form

$$\left|X^I Y^J K(x, y)\right| \leq C_{IJ}|x|^{-Q_1-|I|}|y|^{-Q_2-|J|} \tag{12}$$

(where Q_j is the homogeneous dimension of G_j) as well as appropriate cancellation conditions; and (2) multiplier operators $m(\mathcal{L}_1, \mathcal{L}_2)$ defined by the spectral functional calculus (which applies since $\mathcal{L}_1$ and $\mathcal{L}_2$ commute), where m is a function on $(0, \infty) \times (0, \infty)$ that satisfies the Marcinkiewicz-type conditions

$$\left|(\xi\partial_\xi)^\alpha(\eta\partial_\eta)^\beta m(\xi, \eta)\right| \leq C_{\alpha,\beta} \tag{13}$$

for sufficiently many α, β. It is not too hard to show, based on older results, that such convolution and multiplier operators are bounded on L^p, $1 < p < \infty$.

These results appear in the 1995 paper [18] of Müller, Ricci, and Stein but they are preliminary to the main purpose of the paper: the study of multiplier operators $m(\mathcal{L}_0, iT)$ on the Heisenberg group where $\mathcal{L}_0$ is the sub-Laplacian (10) and T is the derivative in the central variable, and m satisfies the conditions (13). (Such operators arise, for example, in the solution of the $\overline{\partial}$-Neumann problem on the domain (5), as worked out in Greiner–Stein [9].) The essential point is that these conditions are invariant under *independent* rescalings of $\mathcal{L}_0$ and T, as opposed to the joint rescaling that comes from the natural dilations on H_n. Nonetheless, the authors show that if (13) is satisfied for sufficiently many α and β, $m(\mathcal{L}_0, T)$ is bounded on L^p. The method is to relate $m(\mathcal{L}_0, T)$ on H_n to $m(\mathcal{L}_0, iU)$ on $H_n \times \mathbb{R}$, where U is the derivative along the $\mathbb{R}$ factor, to which the product theory described in the preceding paragraph applies. Moreover, they show that the integral kernels for these operators satisfy estimates of the form

$$\left|\partial_z^I \partial_t^j K(z, t)\right| \leq C_{Ij}|z|^{-2n-|I|}\left(|z|^2 + |t|\right)^{-1-j} \tag{14}$$

together with appropriate cancellation conditions.

The argument employed in [18], however, does not give the optimal smoothness hypotheses on m. In [19], Müller, Ricci, and Stein give a different, more complicated argument to sharpen and generalize the boundedness theorem of [18]. Instead of working on H_n, they work on an arbitrary H-type group G and consider multiplier operators of the form $M = m(\mathcal{L}, iT_1, \ldots, iT_n)$ where $\mathcal{L}$ is the sub-Laplacian on G and $T_1, \ldots, T_n$ are a basis for the center of $\mathfrak{g}$. The precise conditions on m needed to ensure L^p boundedness of M are a bit too technical to state here, but they essentially require control of derivatives up to order $d/2+\epsilon$ where (as in [20]) d is the Euclidean dimension of G.

The techniques used in [18], as well as other situations where the analysis involves quotients of products of stratified groups, led Nagel, Ricci, and Stein [24] to develop a theory of singular integrals with "flag kernels." These are integral operators whose kernels are of product type and so satisfy estimates like (12), but also satisfy stronger estimates that generalize (14) (where the product-type estimate would have $|t|$ in place of $|z|^2 + |t|$).

We briefly describe their general setup. A *flag* in $\mathbb{R}^N$ is an increasing family of linear subspaces, $\{0\} = V_0 \subset V_1 \subset \cdots V_n = \mathbb{R}^N$. We assume that $\mathbb{R}^N$ is endowed with a family of dilations under which each V_j is invariant, and that W_j is a complementary subspace to V_{j-1} in V_j that is also invariant under the dilations. Then $\mathbb{R}^N = \bigoplus_1^n W_j$, and if $x \in \mathbb{R}^N$ we write $x = (x_1, \ldots, x_n)$ with $x_j \in W_j$. A *flag kernel* relative to the flag $\{V_j\}$ is a distribution K on $\mathbb{R}^N$ that is C^∞ away from the subspace $x_n = 0$ and satisfies estimates of the form

$$\left|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} K(x)\right| \le C_\alpha \prod_{k=1}^{n} (|x_k| + \cdots + |x_n|)^{-Q_k - |\alpha_k|}, \tag{15}$$

where each α_j is a multi-index of length $\dim W_j$, and $|x_j|$, $|\alpha_j|$ and Q_j denote (respectively) a homogeneous norm on W_j, the homogeneous degree of α_j, and the homogeneous dimension of W_j *with respect to the given family of dilations*. The distribution K must also satisfy appropriate cancellation conditions defined by induction on the length n of the flag.

Nagel, Ricci, and Stein showed that if K is a flag kernel, Euclidean convolution with K is bounded on L^p for $1 < p < \infty$; moreover, the set of all these convolution operators, for a fixed flag, forms an algebra. They showed, moreover, that these results still hold if $\mathbb{R}^N$, with the given dilations, is the underlying space for a homogeneous group G and Euclidean convolution is replaced by convolution on G—but only under some rather strong restrictions on the group structure and its relation to the dilations. This is sufficient for the applications to the study of the $\overline{\partial}_b$ complex that is the second objective of [24] (as we mentioned in Sect. 3), but it remained desirable to develop the theory in greater generality. This was accomplished by Nagel, Ricci, Stein, and Wainger [25], where the main theorems above—L^p boundedness and closure under composition—are proved (by considerably more involved arguments) for convolution with flag kernels on a homogeneous group G, assuming merely that the spaces $W_j \subset \mathfrak{g}$ are eigenspaces for the dilations.

Since the set of singularities of a flag kernel is, in general, a linear subspace of positive dimension rather than just the origin, the corresponding convolution operator is not pseudolocal (i.e., does not preserve local smoothness). However, flag kernels provide a framework for studying collections of convolution operators that are pseudolocal but whose kernels have a structure related to more complicated families of dilations than the one-parameter groups that define the homogeneity of Calderón–Zygmund kernels. For example, one would like to study operators that are compositions of two or more Calderón–Zygmund operators, each associated to a *different* one-parameter group of dilations. This situation arises, for example, in the solution of the $\overline{\partial}$-Neumann problem in [9], which involves compositions of Heisenberg-type singular integrals with standard pseudodifferential operators. In [25], a sequel was promised that would deal with such matters.

The promised sequel, although not entitled as such, is Nagel, Ricci, Stein, and Wainger [26]. In it the authors study kernels on $\mathbb{R}^N$ that satisfy differential inequalities adapted to an n-parameter family of dilations ($n \leq N$) specified by an $n \times n$ matrix $\mathbf{E}$ of positive numbers (essentially, the exponents for the dilations), as well as appropriate cancellation conditions; the collection of such kernels is denoted $\mathcal{P}(\mathbf{E})$. Here $\mathbb{R}^N$ may be taken as an Abelian group or as the underlying space for a non-Abelian homogeneous group G; in the latter case the n-parameter family of dilations must be compatible with the homogenous structure on G in a suitable sense. This general framework includes kernels arising from composition of Calderón–Zygmund operators with different homogeneity as described above, as well as "two-flag kernels": kernels that are adapted to a flag V_j as well as its dual flag $\widetilde{V}_j = \bigoplus_j^n W_j$. The theorems at the heart of the paper, again, are that the convolution operators with kernels in $\mathcal{P}(\mathbf{E})$ are bounded on L^p and form an algebra. The authors also extend the results to suitable "variable-coefficient" operators.

6 Conclusion

Our story ends here, on a rather high level of generality and abstraction, but with direct connections back to the problems in complex analysis with which it all began. Much more could be said, as research encapsulated in this article has direct connections to work by many other people as well as many resonances with results in other areas of analysis.

I wish to add only a few final comments about a striking feature of the thirty-seven papers and monographs of Stein cited here: all but five of them are joint work with sixteen collaborators, individually and in various combinations. Stein's enthusiasm for working with other people is a major factor in the central role he played in the development of harmonic analysis over his 65-year career; it enlarged the scope of his own work by incorporating the expertise of others, brought forth some of the best work of his collaborators, and fostered the development of the sort of mathematical community to which it has been a pleasure to belong. All mathematicians in the field of harmonic analysis, and especially those of us who have had the privilege of working with and learning from Eli Stein, have many reasons to be grateful.

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