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Parabolicity, Brownian Exit Time and Properness of Solitons of the Direct and Inverse Mean Curvature Flow

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Abstract

We study some potential theoretic properties of homothetic solitons Σ^n of the MCF and the IMCF. Using the analysis of the extrinsic distance function defined on these submanifolds in $\mathbb{R}^{n+m}$, we observe similarities and differences in the geometry of solitons in both flows. In particular, we show that parabolic MCF-solitons Σ^n with $n > 2$ are self-shrinkers and that parabolic IMCF-solitons of any dimension are self-expanders. We have studied too the geometric behavior of parabolic MCF and IMCF-solitons confined in a ball, the behavior of the mean exit time function for the Brownian motion defined on Σ as well as a classification of properly immersed MCF-self-shrinkers with bounded second fundamental form, following the lines of Cao and Li (Calc Var 46:879–889, 2013).

Keywords Extrinsic distance · Parabolicity · Soliton · Self-shrinker · Self-expander · Mean exit time function · Laplace operator · Brownian motion · Mean curvature flow · Inverse mean curvature flow

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1 Introduction

The potential theory on a complete manifold is mainly devoted to the study of harmonic (or subharmonic) functions defined on it, and more generally, to the study of the relation among the geometry of the manifold and the properties of the solutions of some distinguished PDEs raised using the Laplace–Beltrami operator such us Laplace and Poisson equations. The interplay between geometric information (encoded in the form of bounds for the curvature, for example) and functional theoretic properties (such as the existence of bounded harmonic or subharmonic functions) constitutes a rich arena at the crossroads of functional analysis, differential geometry and PDEs theory where the problems we are going to study are placed. To address these problems, we will add in this paper the point of view of submanifold theory, in relation with some distinguished submanifolds in the Euclidean space. In particular, we are going to focus in the study of the parabolicity of homothetic solitons for the mean curvature flow and for the inverse mean curvature flow and the relation of this concept with the geometry of these submanifolds. We are going to apply the same technique, namely, the analysis of the extrinsic distance defined on the submanifold on MCF and IMCF solitons, in order to highlight similarities and differences among them.

We recall that a non-compact, complete n-dimensional manifold M^n is *parabolic* if and only if every subharmonic, and bounded from above function defined on it is constant. If such non-constant function exists, then M is *non-parabolic*. This functional property holds in compact manifolds as a direct application of the strong Maximum Principle, so parabolicity can be viewed as generalization of compactness.

On the other hand, parabolicity implies the following *weak Maximum Principle*: given M a (not necessarily complete) Riemannian manifold, it satisfies the weak Maximum Principle if and only if for any bounded function $u \in C^2(M)$ with $\sup_M u = u^* < \infty$, there exists a sequence of points $\{x_k\}_{k\in N} \subseteq M$ such that $u(x_k) > u^* - \frac{1}{k}$ and $\Delta u(x_k) < 0$ (see [34], see also [1]).

Let us consider now an isometric immersion $X : \Sigma \to \mathbb{R}^{n+m}$ of the manifold Σ^n in $\mathbb{R}^{n+m}$. A question that arises naturally when studying the parabolicity of Σ consists in to obtain a geometric description of this potential theoretic property, relating it, for example, with the behavior of its mean curvature. In this sense, when the dimension of the submanifold is $n = 2$, minimality does not imply parabolicity nor non-parabolicity: some minimal surfaces in $\mathbb{R}^3$ are parabolic (e.g. Costa's surface, Helicoid, Catenoid), while some others (like P-Schwartz surface or Scherk doubly periodic surface) are non-parabolic.

However, something can be said in this context. In particular, we have, by one hand, that *complete and minimal isometric immersions* $\varphi : \Sigma^2 \to \mathbb{R}^n$ *included in a ball* $\varphi(\Sigma) \subseteq B^n_R$ *are non-parabolic*. The proof of this theorem follows from the fact that coordinate functions $x_i : \Sigma \to \mathbb{R}$ are harmonic, bounded in $\varphi(\Sigma) \subseteq B^n_R$ and non-constant. Recall that in the paper [29], Nadirashvili constructed a complete (non-proper) immersion of a minimal disk into the unit ball in $\mathbb{R}^3$.

On the other hand, when the dimension of the submanifold is bigger or equal than 3, we have that *complete and minimal proper isometric immersions* $\varphi : \Sigma^n \to \mathbb{R}^{n+m}$ *with* $n \geq 3$ *are non-parabolic* (see [26]). The proof in this case is based on obtaining bounds for the *capacity* at infinity of a suitable precompact set in the submanifold.

Since solitons for MCF and IMCF satisfy a geometric condition on its mean curvature, namely, Eqs. (2.8) and (2.9) in Definitions 2.7 and 2.9 respectively, and inspired by the results above mentioned, it could be interesting to establish a geometric description of parabolicity of a complete and non-compact soliton for the MCF and IMCF, and to study the behavior of parabolic solitons confined in a ball. To do that, we have used the analysis of the Laplacian of radial functions depending on the extrinsic distance, and Theorem 2.3 (see [1,34]), where it is proved that parabolicity implies the weak Maximum Principle alluded above.

In what follows, we are going to give an account of our main results concerning these and other related questions.

In Theorem 3.1, we prove that (not necessarily proper) parabolic solitons for the MCF with dimension $n \geq 3$ are self-shrinkers and in Corollary 3.3, we prove that (again not necessarily proper) self-expanders for the MCF are non-parabolic. Concerning solitons for the IMCF (not necessarily proper as before) and using the techniques above mentioned, we have proved in Theorem 4.1 that parabolic solitons for the IMCF are self-expanders, and that self-shrinkers for the IMCF with $n \geq 2$, and self-expanders for the IMCF with $n \geq 3$ and velocity $C > \frac{1}{n-2}$ are non-parabolic (Corollary 4.2).

Another line of research that we mentioned above is the study of the behavior of solitons included in a ball or in a half-space containing the origin. We can find in the literature several works dealing with this question, for example the paper [35], where it is presented a half-space theorem for self-shrinkers of the MCF and the paper [5], where the results in [35] are extended to shrinking cylinders.

Our results in this line of research, where we consider again not necessarily proper solitons for MCF and IMCF, are Theorem 5.1, where it is proved that complete and parabolic self-shrinkers for the MCF confined in the ball $B^{n+m}\left(\sqrt{\frac{n}{\lambda}}\right)$ centered at $\vec{0} \in \mathbb{R}^{n+m}$ must be minimal submanifolds of the sphere $S^{n+m-1}\left(\sqrt{\frac{n}{\lambda}}\right)$ and, as a corollary, that the only complete and connected parabolic self-shrinkers for the MCF with codimension 1 confined in the ball $B^{n+m}\left(\sqrt{\frac{n}{\lambda}}\right)$ are the spheres of radius $\sqrt{\frac{n}{\lambda}}$. Moreover, we have proved that there are not complete and non-compact parabolic self-expanders for MCF confined in a ball of any radius (Theorem 5.4). Concerning solitons for the IMCF we have proved in Theorem 5.5 that complete and non-compact parabolic solitons confined in an R-ball are compact minimal submanifolds of a sphere of radius less or equal than R.

In regard to classification results using bounds for the norm of the second fundamental form, in the paper [2], the authors obtained a classification theorem for complete self-shrinkers of MCF without boundary and with polynomial volume growth satisfying that the squared norm of its second fundamental form is less or equal than 1 (λ in the case we consider λ-self-shrinkers). Using the mean exit time function (whose behavior is closely related with the notion of parabolicity) defined on the extrinsic balls of the solitons, we have obtained some classification results for them. In particular, in first place (Theorem 6.2), we have established an isoperimetric inequality satisfied by properly immersed MCF-self-shrinkers $X : \Sigma^n \to \mathbb{R}^{n+m}$ and, from this result we have shown: first, that the properly immersed self-shrinkers confined in the $\sqrt{\frac{n}{\lambda}}$-ball

$B^{n+m}(\sqrt{\frac{n}{\lambda}})$ or included in the complementary set $\overline{\mathbb{R}^{n+m} \setminus B^{n+m}(\sqrt{\frac{n}{\lambda}})}$ must be compact minimal submanifolds of the sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ (Theorem 6.3), and secondly, (Theorem 6.9), that the properly immersed λ-self-shrinkers with the squared norm of its second fundamental form bounded from above by the quantity $\frac{5}{3}\lambda$, must be the sphere $S^{n+m-1}\left(\sqrt{\frac{n}{\lambda}}\right)$, or alternatively, this sphere separates the soliton into two parts. We present finally a characterization of IMCF-solitons in terms of the Mean Exit Time function defined on its extrinsic balls (Theorem 7.3).

1.1 Outline of the Paper

The structure of the paper is as follows: in the preliminaries, Sects. 2, 2.1, we recall the preliminary concepts and properties of extrinsic distance function. In Sect. 2.2 it is presented and studied the notion of *parabolicity*, together a result which relates parabolicity with the weak Maximum Principle alluded in the Introduction and that shall be widely used along the paper. We finish the preliminaries defining the solitons for the MCF and IMCF (Sect. 2.3) and relating them with the minimal spherical immersions (Sect. 2.4). Sections 3 and 4 are devoted to the geometric description of the parabolicity of solitons for the MCF and IMCF. In Sect. 5 we shall study the behavior of parabolic solitons for the MCF and IMCF confined in a ball. In Sect. 6 we study the mean exit time function and the volume of solitons for the MCF and we shall present a classification Theorem for complete and proper self-shrinkers with bounded norm of its second fundmental form. Finally, in Sect. 7 it is studied the mean exit and the volume of solitons for the IMCF.

2 Preliminaries

2.1 The Extrinsic Distance Function

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete isometric immersion into the Euclidean space $\mathbb{R}^{n+m}$. The *extrinsic distance function of* X to the origin $\vec{0} \in \mathbb{R}^{n+m}$ is given by

$$r : \Sigma \to \mathbb{R}, \quad r(p) = \operatorname{dist}_{\mathbb{R}^{n+m}}\left(\vec{0},\ X(p)\right) = \|X(p)\|.$$

In the above equality, $\| \, , \|$ denotes the norm of vectors in $\mathbb{R}^{n+m}$ induced by the usual metric $g_{\mathbb{R}^{n+m}}$. The gradients of $r(x) = \operatorname{dist}_{\mathbb{R}^{n+m}}(\vec{0},\ x)$ in $\mathbb{R}^{n+m}$ and in Σ are denoted by $\nabla^{\mathbb{R}^{n+m}} r$ and $\nabla^{\Sigma} r$, respectively. Then we have the following basic relation:

$$\nabla^{\mathbb{R}^{n+m}} r = \nabla^{\Sigma} r + (\nabla^{\mathbb{R}^{n+m}} r)^{\perp} \ \text{ on } \Sigma, \tag{2.1}$$

where $(\nabla^{\mathbb{R}^{n+m}} r)^{\perp}(X(x)) = \nabla^{\perp} r(X(x))$ is perpendicular to $T_x\Sigma$ for all $x \in \Sigma$.

Definition 2.1 Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete isometric immersion into the Euclidean space $\mathbb{R}^{n+m}$. We denote the *extrinsic metric balls* of radius $R > 0$ and center $\vec{0} \in \mathbb{R}^{n+m}$ by D_R. They are defined as the subset of Σ:

$$D_R = \{x \in \Sigma : r(x) < R\} = \{x \in \Sigma : X(x) \in B_R^{n+m}(\vec{0})\} = X^{-1}\left(B_R^{n+m}\left(\vec{0}\right)\right),$$

where $B_R^{n+m}(\vec{0})$ denotes the open geodesic ball of radius R centered at the pole $\vec{0} \in \mathbb{R}^{n+m}$. Note that the set $X^{-1}(\vec{0})$ can be the empty set.

Remark a When the immersion X is proper, the extrinsic domains D_R are precompact sets, with smooth boundaries ∂D_R for a.e. $R > 0$. The assumption on the smoothness of ∂D_R makes no restriction. Indeed, the distance function r is smooth in $\mathbb{R}^{n+m} - \{\vec{0}\}$ since $\vec{0}$ is a pole of $\mathbb{R}^{n+m}$. Hence the composition $r|_\Sigma$ is smooth in Σ and consequently the radii R that produce non-smooth boundaries ∂D_R have 0-Lebesgue measure in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

Remark b Along the paper, we shall denote as $S^{n+m-1}(R)$ and as $B^{n+m}(R)$ or $B_R^{n+m}(\vec{0})$ the spheres and the balls centered at $\vec{0}$ in $\mathbb{R}^{n+m}$. In the classification results, (as Corollaries 5.3 and 5.7, or Theorem 6.9), we are also using this notation to denote the n-dimensional R-spheres $\mathbb{S}^n(R)$ considered as Riemannian manifolds, where the center it is not relevant. Another place where the center of the balls and spheres is not relevant is in the Poisson problem (6.2). In all the cases we are using the same notation, and the relevance or not of the center and if we are considering the spheres immersed or not will be clear from the context.

A technical result which we will use is the following:

Lemma 2.2 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete isometric immersion into the Euclidean space $\mathbb{R}^{n+m}$. Let $r : \Sigma \to \mathbb{R}$, $r(p) = dist_{\mathbb{R}^{n+m}}(X(p), \vec{0}) = \|X(p)\|$ the extrinsic distance of the points in Σ to the origin $\vec{0} \in \mathbb{R}^{n+m}$. Given any function $F : \mathbb{R} \to \mathbb{R}$, we have that*

$$\begin{aligned}\Delta^\Sigma F(r(x)) &= \left(\frac{F''(r(x))}{r^2(x)} - \frac{F'(r(x))}{r^3(x)}\right)\|X^T\|^2 \\ &\quad + \frac{F'(r(x))}{r(x)}\left(n + \langle X, \vec{H}\rangle\right),\end{aligned} \tag{2.2}$$

where X^T denotes here the tangential component of X with respect to $X(\Sigma)$ and $\vec{H}$ denotes the mean curvature vector field of Σ.

2.2 Parabolicity and Capacity Estimates

Parabolicity extends the maximum principle to complete and non-compact parabolic manifolds in the following way (see [34], p.79):

Theorem 2.3 *Let M be a complete noncompact and parabolic Riemannian manifold. Then for each $u \in C^2(M)$, $\sup u < \infty$, u nonconstant on M, there exists a sequence $\{x_k\} \subset M$ such that $u(x_k) > \sup u - \frac{1}{k}$, $\Delta u(x_k) < 0$, $\forall k \in \mathbb{N}$.*

Remark c Along this subsection, the symbols Δ and ∇ denote the *intrinsic* Laplacian and the gradient in the Riemannian manifold M.

To relate this functional property with the geometry of the underlying manifold, we shall establish bounds for the *capacity* of M. When $\Omega \subset M$ is precompact, it can be proved (see [17]) that the *capacity* of the compact K in Ω is given as the following integral:

$$\operatorname{cap}(K, \Omega) = \int_{\Omega} \|\nabla \phi\|^2 \, \mathrm{d}V_g = \int_{\partial K} \|\nabla \phi\| \mathrm{d}\mu,$$

where ϕ is the solution of the Laplace equation on $\Omega - K$ with Dirichlet boundary values:

$$\begin{cases} \Delta u = 0, \\ u \mid_{\partial K} = 1, \\ u \mid_{\partial \Omega} = 0. \end{cases} \tag{2.3}$$

Moreover, for any compact $K \subset M$ and any open set $G \subset M$ containing K, we have

$$\operatorname{cap}(K, M) \leq \operatorname{cap}(K, G). \tag{2.4}$$

The relation among capacity and parabolicity is given by the following result (see [17]):

Theorem 2.4 *Let (M, g) be a Riemannian manifold. M is parabolic iff M has* zero capacity*, i.e., there exists a non-empty precompact open set $D \subseteq M$ such that* $\operatorname{cap}(D, M) = 0$.

On the other hand, it can be proved that given $K \subset M$ a (pre)compact subset of M, if we consider $\{\Omega_i\}_{i=1}^{\infty}$ an exhaustion of M by nested and precompact sets, such that $K \subseteq \Omega_i$ for some i, then the capacity of K in all the manifold (the *capacity at infinity* $\operatorname{cap}(K, M) = \operatorname{cap}(K)$) is given as the following limit:

$$\operatorname{cap}(K, M) = \lim_{i \to \infty} \operatorname{cap}(K, \Omega_i).$$

This definition is independent of the exhaustion. Another result concerning bounds for the capacity of a manifold is following:

Theorem 2.5 ([17]). *Let M be a complete and non-compact Riemannian manifold. Let $G \subset M$ be a precompact open set and $K \subset G$ be compact. Suppose that a Lipschitz function u is defined in $\overline{G \setminus K}$ such that $u = a$ on ∂K and $u = b$ on ∂G where $a < b$ are real constants. Then*

$$\operatorname{cap}(K, G) \leq \left(\int_a^b \frac{\mathrm{d}t}{\int\limits_{\{x \,:\, u(x) = t\}} \|\nabla u(x)\| \mathrm{d}A(x)} \right)^{-1}. \tag{2.5}$$

To obtain sufficient conditions for parabolicity, we shall apply the following criterion of Has'minskii.

Theorem 2.6 ([18]). *Let M be a Riemannian manifold. If there exists $v : M \to \mathbb{R}$ superharmonic outside a compact set, and $v(x) \to \infty$ when $x \to \infty$, then M is parabolic.*

2.3 Solitons

Let $X_0 : \Sigma^n \to \mathbb{R}^{n+m}$ be an isometric immersion of an n-dimensional manifold Σ into the Euclidean space $\mathbb{R}^{n+m}$. The evolution of X_0 by mean curvature flow (MCF) is a smooth one-parameter family of immersions satisfying

$$\begin{cases} \frac{\partial}{\partial t} X(p,t) = \vec{H}(p,t) & \forall p \in \Sigma, \quad \forall t \geq 0, \\ X(p,0) \quad = X_0(p), & \forall p \in \Sigma. \end{cases} \tag{2.6}$$

Here, $\vec{H}_t = \vec{H}(\ ,t)$ is the mean curvature vector of the immersion $X_t = X(\ ,t)$ i.e., the trace of the second fundamental form α_t ($\vec{H}_t = \mathrm{tr}_{g_t}\,\alpha_t = \Delta_{g_t} X_t$). Likewise, the evolution of the initial immersion X_0 by the inverse of the mean curvature flow (IMCF) is a one-parameter family of immersions satisfying

$$\begin{cases} \frac{\partial}{\partial t} X(p,t) = -\frac{\vec{H}(p,t)}{\|\vec{H}(p,t)\|^2} & \forall p \in \Sigma, \quad \forall t \geq 0, \\ X(p,0) \quad = X_0(p), & \forall p \in \Sigma. \end{cases} \tag{2.7}$$

We are going to fix the notions we shall use along the paper (see [11,23] for the definition of soliton).

Definition 2.7 A complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a λ-soliton for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ ($\lambda \in \mathbb{R}$) if and only if

$$\vec{H} = -\lambda X^{\perp}, \tag{2.8}$$

where $X^{\perp}$ stands for the normal component of X and $\vec{H}$ is the mean curvature vector of the immersion X.

Remark d Note that, if we have a complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ satisfying the geometric condition (2.8), and we consider the family of homothetic immersions $X_t = \sqrt{1 - 2\lambda t}\, X$, it is straightforward to check that $\{X_t\}_{t=0}^{\infty}$ satisfies Eq. (2.6), so X becomes the 0-slice of the family $\{X_t\}_{t=0}^{\infty}$ of solutions of Eq. (2.6).

Definition 2.8 A λ-soliton for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ is called a self-shrinker if and only if $\lambda > 0$. It is called a self-expander if and only if $\lambda < 0$.

Remark e Note that a complete and minimal immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ can be considered as a "limit case" of λ-soliton for the MCF when $\lambda = 0$, because as $\vec{H}_\Sigma = \vec{0}$, then it satisfies Eq. (2.8).

For the inverse mean curvature flow we have the following definition:

Definition 2.9 The complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a C-soliton for the IMCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ ($C \in \mathbb{R}$) if and only if

$$\frac{\vec{H}(p)}{\|\vec{H}(p)\|^2} = -CX^{\perp}, \tag{2.9}$$

where $X^{\perp}$ stands for the normal component of X and $\vec{H}$ is the mean curvature vector of the immersion X.

Remark f Note that if we have a complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ satisfying the geometric condition (2.9) and we consider the family of homothetic immersions $X_t = e^{Ct}X$, it is straightforward to check that $\{X_t\}_{t=0}^{\infty}$ satisfies Eq. (2.7), so X becomes the 0-slice of the family $\{X_t\}_{t=0}^{\infty}$ of solutions of Eq. (2.7).

Definition 2.10 A C-soliton for the IMCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ is called a self-shrinker if and only if $C < 0$. It is called a self-expander if and only if $C > 0$.

Remark g A complete and minimal immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ cannot be considered as a C-soliton for the IMCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ for any constant C because X cannot satisfy Eq. (2.9).

2.4 Solitons and Spherical Immersions

Let us consider now a *spherical* immersion, namely, an isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ such that $X(\Sigma) \subseteq S^{n+m-1}(R)$ for some radius $R > 0$. Then, we have the following well-known characterization of self-shrinkers of MCF and self-expanders of IMCF. Assertion (3) concerning solitons for the MCF was proved by Smoczyk in [39] and concerning solitons for the IMCF was proved in [14]. It was proved in [4] that *closed* C-solitons for the IMCF are minimal spherical immersions with velocity $C = \frac{1}{n}$. For completeness, we present here a proof based in the following Takahashi's Theorem (see [40]):

Theorem 2.11 *If an isometric immersion $\varphi : M^n \to \mathbb{R}^{n+m}$ of a Riemannian manifold satisfies $\Delta^M\varphi + \lambda\varphi = 0$ for some constant $\lambda \neq 0$, then $\lambda > 0$ and φ realizes a minimal immersion in a sphere $S^{n+m-1}(R)$ with $R = \sqrt{\frac{n}{\lambda}}$.*

Now, the mentioned characterization:

Proposition 2.12 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete spherical immersion. We have that:*

(1) *If X is a λ-soliton for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$, then $\lambda = \frac{n}{R^2}$ and $X : \Sigma^n \to S^{n+m-1}(R)$ is a minimal immersion.*
(2) *If X is a C-soliton for the IMCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$, then $C = \frac{1}{n}$ and $X : \Sigma^n \to S^{n+m-1}(R)$ is a minimal immersion.*

(3) *Conversely, if* $X : \Sigma^n \to S^{n+m-1}(R)$ *is a minimal immersion, then* X *is, simultaneously, a* $\frac{n}{R^2}$*-soliton for the MCF with respect to* $\vec{0} \in \mathbb{R}^{n+m}$ *and a* $\frac{1}{n}$*-soliton for the IMCF with respect to* $\vec{0} \in \mathbb{R}^{n+m}$.

Proof First of all, a remark about notation: along this proof, we shall denote as $\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}$ the mean curvature vector of Σ, considered as a submanifold of $\mathbb{R}^{n+m}$, and as $\vec{H}_{\Sigma\subseteq S^{n+m-1}(R)}$, the mean curvature vector of Σ, considered as a submanifold of the sphere $S^{n+m-1}(R)$.

Starting with the proof, let us note that, as $\|X\| = R$ on Σ, then $X(q) \perp T_q\Sigma$ for all $q \in \Sigma$. Hence

$$X^\perp = X \text{ and } X^T = 0.$$

To see (1), we have, as Σ is a λ-soliton for the MCF, that

$$\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}} = -\lambda X^\perp = -\lambda X.$$

On the other hand, $\lambda \neq 0$ because as $r = R$ on Σ, then, applying Lemma 2.2,

$$0 = \Delta^\Sigma r^2 = 2n - 2\lambda R^2,$$

and hence $\lambda = \frac{n}{R^2} \neq 0$. Therefore, $\Delta^\Sigma X = \vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}} = -\frac{n}{R^2}X$. We apply now Takahashi's Theorem to conclude that $X : \Sigma^n \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.

To see assertion (2), we have that, as Σ is a C-soliton for the IMCF, that

$$\frac{\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}}{\|\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}\|^2} = -CX^\perp = -CX.$$

On the other hand, $C \neq 0$ because as $r = R$ on Σ, then, applying Lemma 2.2,

$$0 = \Delta^\Sigma r^2 = 2\left(n - \frac{1}{C}\right)$$

and hence $C = \frac{1}{n} \neq 0$. Moreover,

$$\left\| \frac{\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}}{\|\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}\|^2} \right\| = \frac{R}{n}$$

so $\|\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}\| = \frac{n}{R}$, and therefore,

$$\Delta^\Sigma X = \vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}} = -C\|\vec{H}_{\Sigma\subseteq\mathbb{R}^{n+m}}\|^2 X = -\frac{n}{R^2}X.$$

Again we use Takahashi's Theorem to conclude that $X : \Sigma^n \to S^{n+m-1}(R)$ is a minimal immersion.

To prove assertion (3), let us suppose that $X : \Sigma^n \to S^{n+m-1}(R)$ is a minimal immersion. Then use the equation (see [6]),

$$\vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}} = \vec{H}_{\Sigma \subseteq S^{n+m-1}(R)} - \frac{n}{R^2} X = -\frac{n}{R^2} X = -\frac{n}{R^2} X^{\perp}$$

and we have that Σ is a λ-soliton for the MCF with $\lambda = \frac{n}{R^2}$.

On the other hand, $\|\vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}}\| = \frac{n}{R^2}\|X\| = \frac{n}{R}$, and hence

$$\frac{\vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}}}{\|\vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}}\|^2} = -\frac{1}{n} X^{\perp}$$

and we have that Σ is a C-soliton for the IMCF, independently of the radius R. □

3 A Geometric Description of Parabolicity of MCF-Solitons

3.1 Geometric Necessary Conditions for Parabolicity

We start proving that parabolic solitons for MCF with dimension strictly greater than 2 are self-shrinkers.

Theorem 3.1 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and parabolic λ-soliton for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$, with $n > 2$. Then X is a self-shrinker ($\lambda > 0$) for the MCF.*

Proof To prove the theorem we are going to apply Theorem 2.3 with a family of bounded functions depending on $\epsilon > 0$ and constructed using the distance function. For any $\epsilon > 0$, let us consider the function $f_1^\epsilon : \mathbb{R}_+^* \to (-\infty, \frac{1}{\epsilon})$ defined as

$$f_1^\epsilon(s) = \frac{1}{\epsilon}\left(1 - \frac{1}{s^\epsilon}\right).$$

The function f_1^ϵ is smooth in $\mathbb{R}_+^*$ and strictly increasing in $\mathbb{R}_+^*$, so it is a bijection among $\mathbb{R}_+^*$ and its image $\operatorname{Im} f_1^\epsilon$. Moreover, as $\lim_{r\to 0^+} f_1^\epsilon(r) = -\infty$ and $\lim_{r\to\infty} f_1^\epsilon(r) = \frac{1}{\epsilon}$, then $\sup_{\mathbb{R}_+} f_1^\epsilon \leq \frac{1}{\epsilon} < \infty$.

We are going to divide the rest of the proof in two cases. First, we shall consider a soliton Σ such that $\vec{0} \notin X(\Sigma)$. In this case, $r^{-1}(0) = \emptyset$, and we define the functions

$$u_1^\epsilon : \Sigma \to \mathbb{R}, \quad x \to u_1^\epsilon(x) := f_1^\epsilon(r(x)). \tag{3.1}$$

We have that $\sup_\Sigma u_1^\epsilon = u_1^{\epsilon^*} \leq \frac{1}{\epsilon} < \infty$, and as $\vec{0} \notin \Sigma$, then $r^{-1}(0) = \emptyset$, and these functions are smooth in Σ. Then we can apply to them directly Theorem 2.3 in the following way:

If, for some $\epsilon > 0$, the function u_1^ϵ is constant, then it is straightforward to check that *all* functions u_1^ϵ are constant and moreover, $r|_\Sigma = R$, so $X(\Sigma) \subseteq S^{n+m-1}(R)$, namely, X is a spherical immersion and hence, we apply Proposition 2.12 to get the conclusion (1) (for all $n \geq 1$).

Alternatively, let us suppose that the test functions u_1^ϵ are nonconstant on Σ^n. Given $\epsilon > 0$, since $\sup_\Sigma u < \infty$ and Σ is parabolic, we know by using Theorem 2.3 that there exists a sequence $\{x_k\} \subset \Sigma$ (depending on ϵ) such that

$$\Delta^\Sigma u_1^\epsilon(x_k) < 0.$$

Moreover, by Eq. (2.2)

$$\begin{aligned} 0 > \Delta^\Sigma u_1^\epsilon(x_k) &= -\frac{2+\epsilon}{r^{4+\epsilon}(x_k)}\|X^T(x_k)\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}(n + \langle H, X\rangle) \\ &\geq -\frac{2+\epsilon}{r^{4+\epsilon}(x_k)}\|X(x_k)\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}(n + \langle H, X\rangle) \\ &= \frac{-2-\epsilon+n-\lambda\|X^\perp(x_k)\|^2}{r^{2+\epsilon}(x_k)}, \end{aligned}$$

where we have used that $\langle H, X\rangle = -\lambda\|X^\perp\|^2$ because $X : \Sigma \to \mathbb{R}^{n+m}$ is a λ-soliton for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$.Therefore, for any $\epsilon > 0$, and for its associated sequence $\{x_k\} \subset \Sigma$,

$$\lambda\|X^\perp(x_k)\|^2 > n - 2 - \epsilon.$$

Then, if $n > 2$ there exists ϵ_0 such that $n - 2 - \epsilon_0 > 0$ and we have that

$$\lambda\|X^\perp(x_k^{\epsilon_0})\|^2 > n - 2 - \epsilon_0 > 0.$$

so we conclude that $\lambda > 0$ and we have proved the theorem.

In the second case to consider, we assume that $\vec{0} \in \Sigma$, namely that $X^{-1}(\vec{0}) \neq \emptyset$. Then $r^{-1}(0) \neq \emptyset$, so u_1^ϵ is not smooth in $r^{-1}(0) \subseteq \Sigma$. We are going to modify u_1^ϵ to get $u_2^\epsilon \in \mathcal{C}^\infty(\Sigma)$ and we shall use the same argument as before on these modified functions with some care. This modification is given by the following

Lemma 3.2 *Let $X : \Sigma \to \mathbb{R}^{n+m}$ be an isometric immersion. Suppose that $X^{-1}(\vec{0}) \neq \emptyset$. Then given $\epsilon > 0$ and the function u_1^ϵ defined in Eq. (3.1), there exist a smooth function $u_2^\epsilon : \Sigma \to \mathbb{R}$ and a positive real number $x_0 > 0$ such that*

(1) *The function u_2^ϵ satisfies that*

$$u_2^\epsilon = \begin{cases} u_1^\epsilon & \text{on } \Sigma \setminus D_{\frac{x_0}{2}}, \\ f_1^\epsilon(\frac{x_0}{4}) & \text{on } \quad D_{\frac{x_0}{4}}. \end{cases}$$

(2) *The function u_2 is not constant on Σ, and* $\sup_\Sigma u_2^\epsilon > \sup_{D_{\frac{x_0}{2}}} u_2^\epsilon$. *Therefore,*

$$\sup_\Sigma u_2^\epsilon \leq u_1^{\epsilon^*} < \infty.$$

Proof To prove the lemma, and given the function u_1^ϵ defined in Eq. (3.1), let us consider an extrinsic ball $D_\rho(\vec{0}) \subseteq \Sigma$ such that $\Sigma \setminus D_\rho(\vec{0}) \neq \emptyset$. We have that $u_1^\epsilon \in C^\infty(\Sigma \setminus D_\rho(\vec{0}))$. Let us fix $0 < x_0 < \rho$ and $0 < \delta_0 < f_1^\epsilon(x_0) - f_1^\epsilon(\frac{x_0}{2})$, and let us define the function $g^\epsilon : (-\infty, \frac{x_0}{4}] \cup [\frac{x_0}{3}, \frac{x_0}{2}) \to (f_1^\epsilon(\frac{x_0}{4}) - \delta_0, f_1^\epsilon(\frac{x_0}{2}) + \delta_0)$ as

$$g^\epsilon(s) := \begin{cases} f_1^\epsilon\left(\frac{x_0}{4}\right) & \text{for } \ s \leq \frac{x_0}{4}, \\ f_1^\epsilon(s) & \text{for } \ s \geq \frac{x_0}{3}. \end{cases} \tag{3.2}$$

The set $A := (-\infty, \frac{x_0}{4}] \cup [\frac{x_0}{3}, \frac{x_0}{2})$ is closed in $N := (-\infty, \frac{x_0}{2})$, and if we denote as $M := (f_1^\epsilon(\frac{x_0}{4}) - \delta_0, f_1^\epsilon(\frac{x_0}{2}) + \delta_0)$, then $g^\epsilon \in \mathcal{C}^\infty(A, M)$. Moreover, there is a continuous extension of g^ϵ to N, given by

$$h^\epsilon(s) := \begin{cases} f_1^\epsilon\left(\frac{x_0}{4}\right) & \text{for } \quad s \leq \frac{x_0}{4}, \\ \left(s - \frac{x_0}{4}\right) \frac{f_1^\epsilon(\frac{x_0}{3}) - f_1^\epsilon(\frac{x_0}{4})}{\frac{x_0}{3} - \frac{x_0}{4}} + f_1^\epsilon\left(\frac{x_0}{4}\right) & \text{for } \quad \frac{x_0}{4} \leq s \leq \frac{x_0}{3}, \\ f_1^\epsilon(s) & \text{for } \quad \frac{x_0}{3} \leq s \leq \frac{x_0}{2}. \end{cases} \tag{3.3}$$

Then applying the Extension Lemma for smooth maps (see [21]), there exists an smooth extension $\bar{h^\epsilon} : N \to M$ of g^ϵ, i.e. $\bar{h^\epsilon}|_A = g^\epsilon$. This function $\bar{h^\epsilon}$ can be trivially extended smoothly to all the real line defining $f_2^\epsilon : (-\infty, \infty) \to (f_1^\epsilon(\frac{x_0}{4}) - \delta_0, \frac{1}{\epsilon})$ as

$$f_2^\epsilon(s) := \begin{cases} \bar{h^\epsilon}(s) & \text{for } s < \frac{x_0}{2}, \\ f_1^\epsilon(s) & \text{for } s \geq \frac{x_0}{2}, \end{cases} \tag{3.4}$$

because $\bar{h^\epsilon}(s) = g^\epsilon(s) = f_1^\epsilon(s)$ for any $s > \frac{x_0}{3}$, and hence, $f_2^\epsilon = f_1^\epsilon = \bar{h^\epsilon}$ in the open set $(\frac{x_0}{3}, \frac{x_0}{2})$.

Now, let us define, for each $\epsilon > 0$, the function $u_2^\epsilon : \Sigma \to \mathbb{R}$ as $u_2^\epsilon(p) := f_2^\epsilon(r(p))$. Then $u_2^\epsilon \in \mathcal{C}^\infty(\Sigma)$. Observe that this u_2^ϵ satisfies the statement (1) of the lemma.

To prove statement (2) of the lemma note that since $X^{-1}(\vec{0}) \neq \emptyset$ there exists at least one point $p \in \Sigma$ such that $p \in D_{\frac{x_0}{4}}$ (on the contrary, $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\frac{x_0}{4}}(\vec{0})$, so $X^{-1}(\vec{0}) = \emptyset$). Then $u_2(p) = f_1^\epsilon(\frac{x_0}{4})$. On the other hand, since $\Sigma \setminus D_{x_0} \neq \emptyset$, then there exists at least one $q \in \Sigma \setminus D_{x_0}$. Then, as f_1^ϵ is strictly increasing,

$$u_2^\epsilon(q) = u_1^\epsilon(q) = f_1^\epsilon(r(q)) \geq f_1^\epsilon(x_0) > f_1^\epsilon\left(\frac{x_0}{4}\right) = u_2^\epsilon(p).$$

Hence, u_2^ϵ is not constant on Σ. Let us observe now that, as $\delta_0 < f_1^\epsilon(x_0) - f_1^\epsilon(\frac{x_0}{2})$, and $f_2^\epsilon(s) = \bar{h}^\epsilon(s)\ \forall s < \frac{x_0}{2}$, then we have

$$\sup_{D_{\frac{x_0}{2}}} u_2^\epsilon \leq f_1^\epsilon\left(\frac{x_0}{2}\right) + \delta_0 < f_1^\epsilon(x_0).$$

But again since $\emptyset \neq \Sigma \setminus D_\rho \subseteq \Sigma \setminus D_{x_0}$, there exists $q \in \Sigma \setminus D_{x_0}$ with $u_2^\epsilon(q) \geq f_1^\epsilon(x_0)$. Then

$$\sup_\Sigma u_2^\epsilon > \sup_{D_{\frac{x_0}{2}}} u_2^\epsilon.$$

Now, let us suppose that $\sup_{D_{\frac{x_0}{2}}} u_2^\epsilon > \sup_\Sigma u_1^\epsilon$. Then, as $\sup_\Sigma u_1^\epsilon \geq \sup_{\Sigma \setminus D_{\frac{x_0}{2}}} u_1^\epsilon = \sup_{\Sigma \setminus D_{\frac{x_0}{2}}} u_2^\epsilon$, we obtain $\sup_{D_{\frac{x_0}{2}}} u_2^\epsilon > \sup_{\Sigma \setminus D_{\frac{x_0}{2}}} u_2^\epsilon$ and therefore, $\sup_{D_{\frac{x_0}{2}}} u_2^\epsilon \geq \sup_\Sigma u_2^\epsilon$, which is a contradiction. Hence, $\sup_{D_{\frac{x_0}{2}}} u_2^\epsilon \leq \sup_\Sigma u_1^\epsilon$ and therefore, as we know that $\sup_{\Sigma \setminus D_{\frac{x_0}{2}}} u_2^\epsilon = \sup_{\Sigma \setminus D_{\frac{x_0}{2}}} u_1^\epsilon \leq \sup_\Sigma u_1^\epsilon$, then $\sup_\Sigma u_2^\epsilon \leq \sup_\Sigma u_1^\epsilon$. □

We can finish now the proof of the theorem by using as a test function in Theorem 2.3 the smooth function u_2^ϵ given by Lemma 3.2. For any $\epsilon > 0$, since $\sup_\Sigma u_2^\epsilon < \infty$ and Σ is parabolic, we know by using Theorem 2.3 that there exists a sequence $\{x_k\} \subset \Sigma$ such that $u_2^\epsilon(x_k) \geq u_2^{\epsilon^*} - \frac{1}{k}$ and

$$\Delta^\Sigma u_2^\epsilon(x_k) < 0.$$

Then as $\sup_\Sigma u_2^\epsilon > \sup_{D_{\frac{x_0}{2}}} u_2^\epsilon$, there exists $\delta_1 > 0$ such that $\sup_\Sigma u_2^\epsilon - \delta_1 > \sup_{D_{\frac{x_0}{2}}} u_2^\epsilon$. Given the sequence $\{x_k\} \subset \Sigma$, let us consider the numbers k such that $\frac{1}{k} < \delta_1$. Then

$$u_2^\epsilon(x_k) > sup_\Sigma u_2^\epsilon - \frac{1}{k} > \sup_\Sigma u_2^\epsilon - \delta_1 > \sup_{D_{\frac{x_0}{2}}} u_2^\epsilon,$$

so x_k belongs to $\Sigma \setminus D_{\frac{x_0}{2}}$ for k large enough and we have

$$\begin{aligned} 0 > \Delta^\Sigma u_2^\epsilon(x_k) = \Delta^\Sigma u_1^\epsilon(x_k) &= -\frac{2+\epsilon}{r^{4+\epsilon}(x_k)} \|X^T(x_k)\|^2 + \frac{1}{r^{2+\epsilon}(x_k)} (n + \langle H, X \rangle) \\ &\geq -\frac{2+\epsilon}{r^{4+\epsilon}(x_k)} \|X(x_k)\|^2 + \frac{1}{r^{2+\epsilon}(x_k)} (n + \langle H, X \rangle) \\ &= \frac{-2-\epsilon+n-\lambda \|X^\perp(x_k)\|^2}{r^{2+\epsilon}(x_k)} \end{aligned}$$

and we follow the argument as in the first case. □

As a first corollary of Theorem 3.1 we have the following result, which extends one of the results in [26] (namely that complete, non-compact and minimal immersions

$X : \Sigma^n \to \mathbb{R}^{n+m}$ with $n > 2$ are non-parabolic), to self-expanders for the MCF, not necessarily proper.

Corollary 3.3 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete λ-soliton for the MCF, with $\lambda \leq 0$ and $n > 2$. Then Σ is non-parabolic.*

Example 3.4 Let us see some examples, in particular, we must note at this point that the converse in Theorem 3.1 is not true in general:

(1) When $n = 1$, then complete λ-solitons for the MCF are parabolic for all λ.
(2) When $n = 2$, we have examples of complete parabolic and nonparabolic minimal surfaces (e.g., the catenoid is parabolic and the doubly-periodic Scherk's surface is non parabolic). On the other hand, the spheres $S^1(R)$ and the cylinders $S^1(R) \times \mathbb{R}$ are parabolic 2-dimensional MCF-self-shrinkers.
(3) Let us consider the generalized cylinders $C_k(\rho) = S^k(\rho) \times \mathbb{R}^{n-k}$, and its inclusion map $X : C_k(\rho) \to \mathbb{R}^{n+1}$ defined as

$$x = (x_1, \ldots, x_{n+1}) \in C_k(\rho) \to X(x) = (x_1, \ldots, x_{n+1}),$$

which is an immersion of $C_k(\rho)$ in $\mathbb{R}^{n+1}$.
The mean curvature vector field of $X : C_k(\rho) \to \mathbb{R}^n$ is given by

$$\vec{H} = -\frac{k}{\rho^2} X^\perp.$$

Then $X : C_k(\rho) \to \mathbb{R}^{n+1}$ can be considered as λ-self-shrinker for the MCF with $\lambda = \frac{k}{\rho^2}$. Likewise, since

$$\frac{\vec{H}}{\|\vec{H}\|^2} = -\frac{1}{k} X^\perp,$$

the immersion $X : C_k(\rho) \to \mathbb{R}^{n+1}$ is a C-self-expander for the IMCF with $C = \frac{1}{k}$. Hence, $X : C_k(\rho) \to \mathbb{R}^{n+1}$ is at the same time a self-shrinker for the MCF and a self-expander for the IMCF.
We have that the generalized cylinder $S^{n-2}(R) \times \mathbb{R}^2$ is a parabolic self-shrinker/self-expander (its volume growth is at most quadratic), while the generalized cylinders $S^{n-k}(R) \times \mathbb{R}^k$ with $k > 2$ are non-parabolic self-shrinkers/self-expanders ($\mathbb{R}^k$ is non-parabolic when $k \geq 3$, so we can construct easily bounded non-constant and subharmonic functions on $S^{n-k}(R) \times \mathbb{R}^k$ from bounded non-constant and subharmonic functions on $\mathbb{R}^k$).

Concerning the behavior of two dimensional properly immersed and parabolic self-expanders for the MCF, we have the following result:

Corollary 3.5 *Let $X : \Sigma^2 \to \mathbb{R}^{2+m}$ be an immersed, complete and parabolic self-expander for the MCF. Then*

$$\inf_{\Sigma^2} \|\vec{H}\| = 0.$$

Moreover, if X is proper then, for any $R > 0$ and any connected and unbounded component V of $\{p \in \Sigma : \|X(p)\| > R\}$, we have

$$\inf_V \|\vec{H}\| = 0.$$

Proof To prove the first assertion, we are going to apply Theorems 2.3 as in 3.1 with the same family of bounded functions $\{u_1^\epsilon\}_{\epsilon>0}$ depending on $\epsilon > 0$ and constructed using the distance function.

Then, if we assume that $\vec{0} \notin X(\Sigma)$, we have, for all $\epsilon > 0$ and each function $u_1^\epsilon \in \mathcal{C}^\infty(\Sigma)$, a sequence $\{x_k^\epsilon\} \subset \Sigma$ (depending on ϵ) such that $\Delta^\Sigma u_2^\epsilon(x_k) < 0$ and therefore

$$\lambda \|X^\perp(x_k^\epsilon)\|^2 > -\epsilon,$$

so

$$\|X^\perp(x_k^\epsilon)\|^2 < \frac{-\epsilon}{\lambda}.$$

Since $\|\vec{H}\|^2 = \lambda^2 \|X^\perp\|^2$, we have for each sequence $\{x_k^\epsilon\} \subset \Sigma$, depending on ϵ

$$\|\vec{H}(x_k^\epsilon)\|^2 < -\epsilon\lambda,$$

which implies that, for all $\epsilon > 0$,

$$\inf_{\Sigma^n} \|\vec{H}\|^2 \leq -\epsilon\lambda,$$

and hence

$$\inf_{\Sigma^n} \|\vec{H}\|^2 = 0.$$

On the other hand, if we assume that $\vec{0} \in X(\Sigma)$, we argue as in the proof of Theorem 3.1, modifying u_1^ϵ to obtain a new function $u_2^\epsilon \in \mathcal{C}^\infty(\Sigma)$ which satisfies Lemma 3.2. As we have seen before, these new functions cannot be constant, so we apply Lemma 2.2 and Theorem 2.3 again, obtaining, for each $\epsilon > 0$, and each function $u_2^\epsilon \in \mathcal{C}^\infty(\Sigma)$, a sequence $\{x_k\} \subset \Sigma$ (depending on ϵ)

$$\lambda \|X^\perp(x_k)\|^2 > -\epsilon.$$

Now the proof follows as above.

Finally, to prove the second assertion, for any connected and unbounded component V of $\Sigma \setminus D_R$ we define the following function:

$$F_V^\epsilon(x) := \begin{cases} f_1^\epsilon(R) & \text{if } x \in D_R, \\ u_1^\epsilon(x) & \text{if } x \in (\Sigma \setminus D_{2R}) \cap V, \\ f_1^\epsilon(2R) & \text{if } x \in (\Sigma \setminus D_{2R}) \setminus ((\Sigma \setminus D_{2R}) \cap V) . \end{cases}$$

Observe that F_V^ϵ is a smooth function defined on $D_R \cup (\Sigma \setminus D_{2R})$ and has a continuous extension on $D_{2R} \setminus D_R$. Then by using similar arguments as the used in the proof of Lemma 3.2 there exists a smooth extension $\overline{F}_V^\epsilon : \Sigma \to \mathbb{R}$. Since $\overline{F}_V^\epsilon$ is bounded and is non-constant, by Theorem 2.3 there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that

$$\overline{F}_V^\epsilon(x_k) > \sup_\Sigma \overline{F}_V^\epsilon - \frac{1}{k}, \quad \Delta^\Sigma \overline{F}_V^\epsilon(x_k) < 0.$$

This implies that $\{x_k\}$ belongs to V for k large enough, and hence $\overline{F}_V^\epsilon(x_k) = u_1^\epsilon(x_k)$. Furthermore,

$$\Delta^\Sigma \overline{F}_V^\epsilon(x_k) = \Delta^\Sigma u_1^\epsilon(x_k) < 0$$

Then

$$\inf_V \|\vec{H}\|^2 \leq \|\vec{H}(x_k)\|^2 \leq -\epsilon\lambda.$$

Finally the corollary follows letting again ϵ tend to 0. □

Remark h As a consequence of Corollary 3.5, if Σ^2 is a proper self-expander for the MCF and $\|\vec{H}_\Sigma\| > C$ out of a compact set in Σ^2, then Σ^2 is nonparabolic

3.2 Geometric Sufficient Conditions for Parabolicity

We are going to study now sufficient conditions for parabolicity of properly immersed solitons for the MCF. In the paper [36], Rimoldi has shown the following theorem, which shows that proper self-shrinkers for the MCF with mean curvature bounded from below exhibits the opposite behavior than we have pointed out in Remark above for proper self-expanders satisfying the same property. We give the proof here for completeness:

Theorem 3.6 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and non-compact properly immersed λ-self-shrinker for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$. If $\|\vec{H}_\Sigma\| \geq \sqrt{n\lambda}$ outside a compact set, then Σ is a parabolic manifold. In particular, if $\|\vec{H}_\Sigma\| \to \infty$ when $x \to \infty$, then Σ is parabolic.*

Proof Given $r^2 = \|X(p)\|^2$, we have that

$$\Delta^\Sigma r^2 = 2\left(n - \frac{1}{\lambda}\|\vec{H}_\Sigma\|^2\right) \leq 0.$$

As $\vec{H}_\Sigma \to \infty$ when $x \to \infty$ and X is proper, then $\Delta^\Sigma r^2 \leq 0$ outside a compact set. Then, apply Theorem 2.6 to get the conclusion. □

Remark i As a consequence of Corollary 3.5 and Theorem 3.6, we can conclude that, if they exist, *all* complete and non-compact non-parabolic n-dimensional self-shrinkers for the MCF, such that $\|\vec{H}_\Sigma\| \geq \sqrt{n\lambda}$ outside a compact set, *are not* properly immersed.

Respectively, if they exist, *all* complete and non-compact parabolic n-dimensional self-expanders for the MCF ($n > 2$), such that $\|\vec{H}_\Sigma\| \geq C$ outside a compact set, being C any positive constant, *are not* properly immersed.

These affirmations come from the fact that, in case $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a complete and non-compact properly immersed self-shrinker for the MCF (resp. self-expander) satisfying $\|\vec{H}_\Sigma\| \geq \sqrt{n\lambda}$ outside a compact set, (resp. $\|\vec{H}_\Sigma\| \geq C$ outside a compact set being C any positive constant) then Σ must be parabolic (resp., non-parabolic).

To prove our last sufficient condition of parabolicity for properly immersed solitons for MCF, we shall need the following result, which is a consequence of the Euclidean volume growth of properly immersed self-shrinkers, (see [12]). In some sense (see affirmation (3) in the statement of Proposition 3.7) MCF-self-shrinkers behaves in a similar way than minimal immersions in the sphere even when they are not minimal immersions.

Proposition 3.7 *Let $X : \Sigma \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$, then*

(1) $\displaystyle\int_\Sigma e^{-\frac{\lambda}{2}r^2(p)}\mathrm{d}V(p) < \infty.$

(2) $\displaystyle\int_\Sigma r^2(p)e^{-\frac{\lambda}{2}r^2(p)}\mathrm{d}V(p) < \infty.$

(3) $\displaystyle\lambda\int_\Sigma r^2(p)e^{-\frac{\lambda}{2}r^2(p)}\mathrm{d}V(p) = n\int_\Sigma e^{-\frac{\lambda}{2}r^2(p)}\mathrm{d}V(p)$

where, $r(p) := \|X(p)\|$ and $\mathrm{d}V$ stands for the Riemannian volume density of Σ.

The above proposition implies that proper self-shrinkers have finite weighted volume when we consider the density $r^2e^{-\frac{\lambda}{2}r^2}$, this property can be used to obtain a sufficient condition for parabolicity. We shall need the following

Definition 3.8 Let $X : \Sigma \to \mathbb{R}^{n+m}$ be a proper isometric immersion. Let us define the function $\Psi_\Sigma : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\Psi_\Sigma(R) := \int_{\{p\in\Sigma\,:\,\|X(p)\|>R\}} r^2(p)e^{-\frac{\lambda}{2}r^2(p)}\mathrm{d}V(p).$$

Because of Proposition 3.7, if $X : \Sigma \to \mathbb{R}^{n+m}$ is a proper self-shrinker by MCF, then

$$\lim_{R\to\infty} \Psi_\Sigma(R) = 0.$$

The speed of this decay implies in some cases consequences for the parabolicity of Σ as the following theorem shows:

Theorem 3.9 *Let $X : \Sigma \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Suppose that*

$$\int^\infty \frac{te^{-\frac{\lambda}{2}t^2}}{\Psi_\Sigma(t)}\mathrm{d}t = \infty.$$

Then Σ is parabolic.

Proof We are going to apply Theorem 2.5 to the function $u(x) = r(x) = \|X(x)\|$.

Let us consider $\{D_R\}_{R>0}$ an exhaustion of Σ by extrinsic balls. We must take into account that, since the immersion is proper, the extrinsic ball $D_R = \{x \in \Sigma : \|X(x)\| < R\}$ is a precompact set of Σ and by Sard's theorem, its boundary $\partial D_R = \{x \in \Sigma : \|X(x)\| = R\}$ is a smooth submanifold of Σ for almost every R with unit normal vector field $\frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|}$.

Then by applying the divergence theorem on D_R to the vector field $e^{-\frac{\lambda}{2}r^2}\nabla^\Sigma r^2$, we obtain

$$\int_{D_R} \operatorname{div}^\Sigma \left(e^{-\frac{\lambda}{2}r^2}\nabla^\Sigma r^2\right) \mathrm{d}V = 2Re^{-\frac{\lambda}{2}R^2}\int_{\partial D_R} \|\nabla^\Sigma r\| \mathrm{d}A. \tag{3.5}$$

But, using Lemma 2.2 and that $X^T = r\nabla^\Sigma r$, we have

$$\begin{aligned} \operatorname{div}^\Sigma \left(e^{-\frac{\lambda}{2}r^2}\nabla^\Sigma r^2\right) &= \langle \nabla^\Sigma e^{-\frac{\lambda}{2}r^2}, \nabla^\Sigma r^2\rangle + e^{-\frac{\lambda}{2}r^2}\Delta^\Sigma r^2 \\ &= 2e^{-\frac{\lambda}{2}r^2}\left(n - \lambda r^2\right), \end{aligned} \tag{3.6}$$

so Eq. (3.5) can be written as

$$\int_{D_R} e^{-\frac{\lambda}{2}r^2}\left(n - \lambda r^2\right) \mathrm{d}V = Re^{-\frac{\lambda}{2}R^2}\int_{\partial D_R} \|\nabla^\Sigma r\| \mathrm{d}A \geq 0. \tag{3.7}$$

By the equality in (3.7),

$$\begin{aligned} \int_{\{x:r(x)=t\}} \|\nabla^\Sigma r\|\mathrm{d}A &= \int_{\partial D_t} \|\nabla^\Sigma r\|\mathrm{d}A = \frac{e^{\frac{\lambda}{2}t^2}}{t}\int_{D_t} e^{-\frac{\lambda}{2}r^2}\left(n - \lambda r^2\right)\mathrm{d}V \\ &= \frac{e^{\frac{\lambda}{2}t^2}}{t}\left(\int_\Sigma e^{-\frac{\lambda}{2}r^2}\left(n - \lambda r^2\right)\mathrm{d}V \right. \\ &\quad \left. - \int_{\Sigma\setminus D_t} e^{-\frac{\lambda}{2}r^2}\left(n - \lambda r^2\right)\mathrm{d}V\right) \\ &= \frac{e^{\frac{\lambda}{2}t^2}}{t}\int_{\Sigma\setminus D_t} e^{-\frac{\lambda}{2}r^2}\left(\lambda r^2 - n\right)\mathrm{d}V \\ &\leq \lambda\frac{e^{\frac{\lambda}{2}t^2}}{t}\int_{\Sigma\setminus D_t} r^2 e^{-\frac{\lambda}{2}r^2}\mathrm{d}V = \lambda\frac{e^{\frac{\lambda}{2}t^2}}{t}\Psi_\Sigma(t). \end{aligned}$$

By using inequality (2.4) and Theorem 2.5 with $K = D_\rho$ and $G = D_R$ with $R > \rho > 0$ we obtain

$$\begin{aligned}\operatorname{cap}(D_\rho, D_R) &\leq \left(\int_\rho^R \frac{\mathrm{d}t}{\int_{\partial D_t} \|\nabla^\Sigma r\| \mathrm{d}A}\right)^{-1} \\ &\leq \left(\int_\rho^R \frac{t e^{-\frac{\lambda}{2}t^2}}{\lambda \Psi_\Sigma(t)} \mathrm{d}t\right)^{-1}.\end{aligned}$$

Finally the theorem is proved letting R tend to ∞. □

4 A Geometric Description of Parabolicity of IMCF-Solitons

As in the previous section, we start with a necessary condition for parabolicity:

Theorem 4.1 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete soliton for the IMCF, with velocity $C \neq 0$ and $n \geq 1$. Then if Σ is a parabolic manifold, then X is a self-expander for the IMCF and*

$$n - 2 \leq \frac{1}{C} \leq n\cdot$$

Moreover, if $C = \frac{1}{n}$, then $X : \Sigma^n \to S^{n+m-1}(R)$ is minimal for some radius $R > 0$.

Proof Given $\epsilon > 0$, let us consider the test function $u_\epsilon(p) := \frac{1}{\epsilon}(1 - \frac{1}{r^\epsilon(p)})$. We have that $\sup_\Sigma u_\epsilon < \infty$ and $u_\epsilon \in C^2(\Sigma)$ because $\vec{0} \notin X(\Sigma)$. If any of these functions is constant for some $\epsilon > 0$, then all are constant and hence $r = R$ is constant on Σ. Then $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a complete C-soliton for the IMCF such that $x(\Sigma) \subseteq S^{n+m-1}(R)$. Hence, applying Proposition 2.12, $C = \frac{1}{n}$ and Σ is minimal in the sphere $S^{n+m-1}(R)$.

Alternatively, let us suppose that the test functions u_ϵ are nonconstant on Σ for all $\epsilon > 0$. Since $\sup_\Sigma u_\epsilon < \infty$ and Σ is parabolic, we know by using Theorem 2.3 that there exists a sequence $\{x_k\} \subset \Sigma$ such that

$$\Delta^\Sigma u_\epsilon(x_k) < 0.$$

Moreover, by Eq. (2.2)

$$\begin{aligned}0 > \Delta^\Sigma u_\epsilon(x_k) &= -\frac{2+\epsilon}{r^{4+\epsilon}(x_k)}\|X^T\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}\left(n + \langle H, X\rangle\right) \\ &\geq -\frac{2+\epsilon}{r^{4+\epsilon}(x_k)}\|X\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}\left(n + \langle H, X\rangle\right) \\ &= \frac{-2-\epsilon+n-\frac{1}{C}}{r^{2+\epsilon}(x_k)},\end{aligned}$$

where we have used that $\langle \vec{H}, X\rangle = -\dfrac{1}{C}$ because $X : \Sigma \to \mathbb{R}^{n+m}$ is a C-soliton of the IMCF. Therefore,

$$\frac{1}{C} > n - 2 - \epsilon$$

for any $\epsilon > 0$. Then $\dfrac{1}{C} \geq n - 2$.

Now, let us consider the test function $v : \Sigma \to \mathbb{R}$ defined as $v(p) := -\|X(p)\|^2 = -r^2(p)$. If v is constant in Σ (*i.e.*, $v(p) = -R^2$ for all $p \in \Sigma$), then $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a complete C-soliton for the IMCF such that $x(\Sigma) \subseteq S^{n+m-1}(R)$. Hence, applying Proposition 2.12, $C = \frac{1}{n}$ and Σ is minimal in the sphere $S^{n+m-1}(R)$.

On the other hand, if v is non-constant on Σ, as $\sup_\Sigma v < \infty$, $v \in C^\infty(\Sigma)$ and Σ is parabolic, we apply Theorem 2.3 to obtain a sequence $\{x_k\} \subset \Sigma$ such that, using Lemma 2.2:

$$\Delta^\Sigma v_\epsilon(x_k) = -2\left(n - \frac{1}{C}\right) < 0 \;\; \forall k \in N,$$

and hence, $n > \frac{1}{C}$, and the theorem is proved.

Let us suppose now that $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a complete and non-compact, parabolic self-expander for the IMFC with $C = \frac{1}{n}$. Then, using Lemma 2.2:

$$\Delta^\Sigma v(x) = -2\left(n - \frac{1}{C}\right) = 0.$$

As $\sup_\Sigma v < \infty$, $v \in C^\infty(\Sigma)$ and Σ is parabolic, then v, and hence r are constant on Σ. Applying Proposition 2.12, $X : \Sigma^n \to S^{n+m-1}(R)$ is minimal for some radius $R > 0$.

Namely, parabolic self-expanders with velocity $C = \frac{1}{n}$ always realizes as minimal submanifolds of a sphere of some radius. □

As corollaries of Theorem 4.1, we have that 2-dimensional self-shrinkers for IMCF are non-parabolic and that, when $n \geq 3$, self-shrinkers and self-expanders with velocity $C > \frac{1}{n-2}$ are non-parabolic.

Corollary 4.2 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and non-compact soliton for the IMCF. Then*

(1) *If $n = 2$ and $C < 0$, Σ^n is non-parabolic.*
(2) *If $n \geq 3$ and $C < 0$ or $C > \frac{1}{n-2}$, Σ^n is non-parabolic.*

Corollary 4.3 *There are no complete, non-compact and smooth* 1*-dimensional solitons for the IMCF with velocity $C \in (-1, 1)$.*

Proof Let Σ^1 be a complete, non-compact and smooth soliton for the IMCF. Then, Σ is conformally isometric to $\mathbb{R}$ with the standard metric, which is parabolic. Hence, by

Theorem 4.1, we have that $-1 \leq \frac{1}{C} \leq 1$, and hence $C \in (-\infty, -1] \cup [1, \infty)$. This means that $C \in (-1, 1)$ is not an allowed velocity constant for a smooth 1- soliton for IMCF. □

Finally, we shall follow the argument used by Rimoldi in [36] on solitons for the MCF, based in the application of Theorem 2.6 to obtain an extension of previous corollary to solitons for the IMCF with dimension $n > 1$.

Corollary 4.4 *There are no complete, connected and non-compact properly immersed solitons for the IMCF, $X : \Sigma^n \to \mathbb{R}^{n+m}$, with velocity $C \in]0, \frac{1}{n}]$.*

Proof If $C \in]0, \frac{1}{n}]$, then Σ is parabolic because of Theorem 2.6: In fact, given $v(p) := r_0^2(p) = \|X(p)\|^2$, as $C \in]0, \frac{1}{n}]$, then

$$\Delta^{\Sigma} v = 2\left(n - \frac{1}{C}\right) \leq 0.$$

Hence, v is superharmonic outside a compact set and $v(p) \to \infty$ when $p \to \infty$ because Σ is properly immersed. Using Theorem 2.6, Σ is parabolic. Now, we apply Theorem 4.1 to conclude that $C \in [\frac{1}{n}, \frac{1}{n-2}]$. Hence, $C = \frac{1}{n}$, so $X : \Sigma^n \to S^n(R)$ is a spherical and minimal isometric immersion for some radius $R > 0$. Therefore, Σ is compact, which is a contradiction. □

Remark j As a consequence of the proof of Corollary 4.4, if $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a complete and non-compact properly immersed non-parabolic soliton for the IMCF, then $C < 0$ or $C > \frac{1}{n}$, namely, if they exist, *all* complete and non-compact non-parabolic solitons for the IMCF with velocity $C \in (0, \frac{1}{n}]$ *are not* properly immersed.

5 Solitons Confined in a Ball

5.1 Solitons for MCF Confined in a Ball

In this subsection we shall see that stochastically complete λ-self-shrinkers Σ^n only can be confined in an R-ball $B_R^{n+m}(\vec{0})$ if $R \geq \sqrt{\frac{n}{\lambda}}$, where the quantity $\sqrt{\frac{n}{\lambda}}$ is the critical radius that makes the sphere $S^{m+n-1}_{\sqrt{\frac{n}{\lambda}}}$ a λ-self-shrinker for the MCF. We are going to see also, in the spirit of the results in [35] (see Proposition 5) that parabolic self-shrinkers for the MCF, $X : \Sigma^n \to \mathbb{R}^{n+m}$, confined in a ball of radius $\sqrt{\frac{n}{\lambda}}$ realize as minimal submanifolds of the sphere $\mathbb{S}^{n+m-1}(\sqrt{\frac{n}{\lambda}})$.

Theorem 5.1 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete λ-self-shrinker for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$ ($\lambda > 0$). Suppose that Σ is stochastically complete and $X(\Sigma) \subset B_R^{n+m}(\vec{0})$. Then*

$$R \geq \sqrt{\frac{n}{\lambda}}.$$

If moreover Σ *is parabolic and* $X(\Sigma) \subset B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ *, then* $X(\Sigma) \subseteq \mathbb{S}^{n+m-1}(\sqrt{\frac{n}{\lambda}})$*, and*

$$X : \Sigma^n \to \mathbb{S}^{n+m-1}\left(\sqrt{\frac{n}{\lambda}}\right)$$

is a minimal immersion.

Proof Let us consider the function $u : \Sigma \to \mathbb{R}$ defined as $u(p) := \|X(p)\|^2 = r^2(p)$. We assume by hypothesis, $(X(\Sigma) \subseteq B^{n+m}_R(\vec{0}))$, that

$$\sup_\Sigma u \leq R^2 < \infty.$$

Moreover, we have that $\|X^\perp\|^2 \leq \|X\|^2$. Then using Lemma 2.2:

$$\Delta^\Sigma u(x) = 2(n - \lambda\|X^\perp\|^2) \geq 2(n - \lambda u(x)).$$

Since Σ is stochastically complete, there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \in \Sigma$ such that, for all $k \in N$, $u(x_k) \geq \sup_\Sigma u - 1/k$ and $\Delta^\Sigma u(x_k) \leq 1/k$. Therefore,

$$\frac{1}{k} \geq 2(n - \lambda u(x_k)) \quad \forall k \in N.$$

Hence,

$$R^2 \geq r^2(x_k) = u(x_k) \geq \frac{n}{\lambda} - \frac{1}{2\lambda k} \quad \forall k \in N.$$

Letting k tend to infinity we obtain

$$R^2 \geq \frac{n}{\lambda}.$$

Let us suppose now that Σ is parabolic and that $X(\Sigma) \subset B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$. Then, using again Lemma 2.2, and the fact that $\|X^\perp\|^2 \leq \|X\|^2$, we have that $\Delta^\Sigma r^2 \geq 0$. Therefore r^2 is a bounded subharmonic function, and hence, constant.

Then $r^2(x) = R^2\ \forall x \in \Sigma$, for some $R \leq \sqrt{\frac{n}{\lambda}}$ (because $X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$). Hence $X(\Sigma) \subseteq S^{n+m-1}(R)$.

On the other hand, as $X(x) \in T_x S^{n+m-1}(R)^\perp \subseteq T_x\Sigma^\perp\ \forall x \in \Sigma$, then $X = X^\perp$ and $X^T = 0$. But, as u is constant on Σ and $X = X^\perp$, then

$$\Delta^\Sigma u(x) = 2(n - \lambda\|X\|^2) = 0$$

and therefore, $R^2 = r^2(x) = \|X\|^2 = \frac{n}{\lambda}$. Hence $X(\Sigma) \subseteq S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ and by Proposition 2.12 Σ is minimal in $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$. □

Corollary 5.2 *Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a complete and connected self-shrinker for the MCF, with $\lambda > 0$. Suppose that $X(\Sigma) \subseteq B_R^{n+1}(\vec{0})$, with $R < \sqrt{\frac{n}{\lambda}}$. Then Σ is stochastically incomplete.*

Corollary 5.3 *Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a complete and connected self-shrinker for the MCF, with $\lambda > 0$. Let us suppose that Σ^n is parabolic and $X(\Sigma) \subset B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$.*

Then

$$\Sigma^n \equiv S^n\left(\sqrt{\frac{n}{\lambda}}\right).$$

Proof In theorem above, we have proved that $\|X\|^2 = \frac{n}{\lambda}$ on Σ. Hence $X : \Sigma^n \to S^n(\sqrt{\frac{n}{\lambda}})$ is a local isometry and therefore, as Σ is connected and complete and $S^n(\sqrt{\frac{n}{\lambda}})$ is connected, then X is a Riemannian covering (see [37], p. 116). Moreover, as $S^n(\sqrt{\frac{n}{\lambda}})$ is simply connected, then X is an isometry (see [21], Corollary 11.24). □

Remark k If $n > 2$, it is enough to assume that $X : \Sigma^n \to \mathbb{R}^{n+1}$ is a complete and connected soliton for the MCF, by virtue of Theorem 3.1.

Finally, we shall see that it is not possible to find complete and stochastic complete self-expanders confined in a ball.

Theorem 5.4 *There are not complete and stochastically complete self-expanders for MCF $X : \Sigma^n \to \mathbb{R}^{n+m}$ confined in a ball.*

Proof Suppose that Σ is stochastically complete and the immersion $X(\Sigma)$ is bounded. Then, since $\lambda < 0$, we have, on Σ:

$$\Delta^\Sigma r^2 = 2n - 2\lambda\|X^\perp\|^2 \geq 2n.$$

But moreover taking into account that Σ is stochastically complete, there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \in \Sigma$ such that, for all $k \in N$, $r^2(x_k) \geq \sup_\Sigma r^2 - 1/k$ and $\Delta^\Sigma r^2(x_k) \leq 1/k$ Therefore,

$$\frac{1}{k} \geq 2n \ \ \forall k \in N,$$

which is a contradiction. □

5.2 Solitons for IMCF Confined in a Ball

In this subsection we shall see that stochastically complete self-expanders for the IMCF Σ^n confined in any R-ball $B_R^{n+m}(\vec{0})$ have velocity $C = \frac{1}{n}$. Moreover we are going to see that parabolic self-expanders for the IMCF included in an R-ball, $X(\Sigma) \subseteq B_R^{n+m}(\vec{0})$, realize as minimal submanifolds of an r_0-sphere with $r_0 \leq R$, and its velocity (which do not depends on the radii r and R) must be $C = \frac{1}{n}$ in this case.

Theorem 5.5 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete soliton for the IMCF. Suppose that Σ is stochastically complete and $X(\Sigma) \subset B_R^{n+m}(\vec{0})$. Then:*

(1) $C = \frac{1}{n}$.
(2) *If moreover Σ is parabolic, there exists $r_0 \leq R$ such that $X(\Sigma) \subseteq S^{n+m-1}(r_0)$ with and $X : \Sigma^n \to S^{n+m-1}(r_0)$ is minimal.*

Proof Let us suppose that Σ is bounded, *i.e.*, it is confined in a ball $X(\Sigma) \subseteq B_R^{n+m}(\vec{0})$. Let us consider the function

$$u : \Sigma \to \mathbb{R}, \quad x \mapsto u(x) := r^2(x) = \|X(x)\|^2.$$

Since $\sup_\Sigma u < \infty$, and Σ is stochastically complete, there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \in \Sigma$ such that for all $k \in N$, $r^2(x_k) \geq \sup_\Sigma r^2 - 1/k$ and $\Delta^\Sigma r^2(x_k) \leq 1/k$ but by Lemma 2.2 ,

$$\frac{1}{k} \geq \Delta^\Sigma u(x_k) = 2n - \frac{2}{C} \; \forall k \in N. \tag{5.1}$$

Hence, for any $x \in \Sigma^n$

$$\Delta^\Sigma u(x) = 2n - \frac{2}{C} \leq 0.$$

On the other hand, by using the function

$$v(x) := -r^2(x) = -\|X(x)\|^2$$

we can deduce in the same way that

$$2n - \frac{2}{C} \geq 0.$$

Therefore $C = \frac{1}{n}$, and

$$\Delta^\Sigma r^2 = 0.$$

Hence, $u(x) := r^2(x)$ is a bounded harmonic function on Σ.

If, moreover, Σ is parabolic, $u(x) := r^2(x) = r_0^2 \leq R$ is constant on Σ and by applying Proposition 2.12 in the same way than in Theorem 5.1, we have that $X : \Sigma^n \to S^{n+m-1}(r_0)$ is minimal . □

As a corollary, and taking into account that every compact manifold is parabolic, we have the following result due to Castro and Lerma in [4].

Corollary 5.6 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete soliton for the IMCF. Suppose that Σ^n is compact. Then, $C = \frac{1}{n}$ and $X(\Sigma^n)$ is contained in a sphere $S^{n+m-1}(R) \subset \mathbb{R}^{n+m}$ of some radius R centered at the origin of $\mathbb{R}^{n+m}$. Moreover, $X : \Sigma^n :\to S^{n+m-1}(R) \subset \mathbb{R}^{n+m}$ is a minimal immersion into $S^{n+m-1}(R)$.*

Another corollary is the following analogous to Corollary 5.3 for parabolic and confined self-shrinkers for the MCF:

Corollary 5.7 *Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a connected and complete soliton for the IMCF. Let us suppose that Σ^n is parabolic and $X(\Sigma) \subseteq B_R^{n+1}(\vec{0})$ for some $R > 0$. Then*

$$\Sigma^n \equiv S^n(R).$$

Proof As $X(\Sigma) \subseteq B_R^{n+1}(\vec{0})$ for some $R > 0$, we have, applying Theorem 5.5, that $C = \frac{1}{n}$, $X(\Sigma) \subseteq S^n(r_0)$ with $r_0 \leq R$ and $X : \Sigma^n \to \mathbb{S}^n(r_0)$ is minimal.

Hence $X : \Sigma^n \to \mathbb{S}^n(r_0)$ is a local isometry and therefore, as Σ is connected and complete and $\mathbb{S}^n(r_0)$ is connected, then X is a Riemannian covering (see [37], p. 116). Moreover, as $\mathbb{S}^n(r_0)$ is simply connected, then X is an isometry (see [21], Corollary 11.24). □

6 Mean Exit Time, and Volume of MCF-Solitons

The Mean Exit Time function for the Brownian motion defined on a precompact domain of the manifold satisfies a Poisson 2nd order PDE equation with Dirichlet boundary data, which, through the application of the divergence theorem, provides some infomation about the volume growth of the manifold. In the next sections and subsections we will explore these questions for MCF and IMCF solitons.

6.1 Mean Exit time on Solitons for MCF

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be an n-dimensional λ-soliton in $\mathbb{R}^{n+m}$ for the Mean Curvature Flow, (MCF), with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us consider $r : \Sigma \to \mathbb{R}$ the *extrinsic distance function* from $\vec{0}$ in Σ^n. Given the extrinsic ball $D_R(\vec{0}) = X^{-1}(B_R^{n+m}(\vec{0}))$, let us consider the Poisson problem

$$\begin{cases} \Delta^\Sigma E + 1 = 0 & \text{on } D_R, \\ E = 0 & \text{on } \partial D_R. \end{cases} \tag{6.1}$$

The solution of the Poisson problem on a geodesic R-ball $B^n(R)$ in $\mathbb{R}^n$

$$\begin{cases} \Delta E + 1 = 0 & \text{on } B_R^n(R) \\ E = 0 & \text{on } S^{n-1}(R), \end{cases} \tag{6.2}$$

is given by the radial function $E_R^{0,n}(r) = \frac{R^2 - r^2}{2n}$.

Let us denote E_R the solution of (6.1) in $D_R \subseteq \Sigma$. Transplanting the radial solution $E_R^{0,n}(r)$ to the extrinsic ball by mean the extrinsic distance function, we have $\bar{E}_R : D_R \to \mathbb{R}$ defined as $\bar{E}_R(p) := E_R^{0,n}(r(p))$.

Our first result is a comparison for the mean exit time function:

Proposition 6.1 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a properly immersed λ-soliton for the MCF, with respect to $\vec{0} \in \mathbb{R}^{n+m}$ ($\lambda \neq 0$). Let us suppose that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball $D_R(\vec{0})$, we have*

(1) *If $\lambda > 0$,*

$$\bar{E}_R(x) \leq E_R(x), \quad \forall x \in D_R.$$

(2) *Or if $\lambda < 0$,*

$$\bar{E}_R(x) \geq E_R(x), \quad \forall x \in D_R.$$

Proof We have, as $\bar{E}_R(x) := E_R^{0,n}(r(x)) = \frac{R^2 - r(x)^2}{2n}$ and applying Lemma 2.2, that, on D_R

$$\Delta^\Sigma \bar{E}_R = \left(\bar{E}_R''(r) - \bar{E}_R'(r)\frac{1}{r} \right) \|\nabla^\Sigma r\|^2 + \bar{E}_r'(r)\left(\frac{n}{r} + \langle \nabla^{\mathbb{R}^{n+m}} r, \vec{H}_\Sigma \rangle \right) = -1 - \frac{1}{n}\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_\Sigma \rangle. \tag{6.3}$$

On the other hand, $X(p) = r(p)\nabla^{\mathbb{R}^{n+m}} r(p)$ for all $p \in \Sigma$, and, moreover, as we have that $\vec{H}_\Sigma(p) = -\lambda X^\perp(p) \ \ \forall p \in \Sigma$, then

$$\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_\Sigma \rangle = -\lambda \|X^\perp\| = -\frac{\|\vec{H}_\Sigma\|^2}{\lambda}.$$

Therefore, if $\lambda > 0$, we obtain

$$\Delta^\Sigma \bar{E}_R = -1 + \frac{1}{n}\frac{\|\vec{H}_\Sigma\|^2}{\lambda} \geq -1 = \Delta^\Sigma E_R. \tag{6.4}$$

As $\bar{E}_R = E_R$ on ∂D_R, we apply now the Maximum Principle to obtain the inequality

$$\bar{E}_R \leq E_R.$$

Inequality (2) follows in the same way. □

Remark I We assume that $\lambda \neq 0$. When $\lambda = 0$, then Σ is minimal, and we have that, (see [24]):

$$\bar{E}_R(x) = E_R(x), \quad \forall x \in D_R.$$

6.2 Volume of Self-Shrinkers for MCF

As a consequence of Proposition 6.1, and using the Divergence theorem we have the following isoperimetric inequality.

Theorem 6.2 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker in $\mathbb{R}^{n+m}$ for the MCF, with respect to $\vec{0} \in \mathbb{R}^{n+m}$ ($\lambda \neq 0$). Let us suppose that $X(\Sigma) \not\subseteq$*

$S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball $D_R(\vec{0}) = \Sigma \cap B_R^{n+m}(\vec{0})$, we have

$$\frac{\mathrm{Vol}(\partial D_R)}{\mathrm{Vol}(D_R)} \geq \left(1 - \frac{1}{n\lambda}\frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 \mathrm{d}\sigma}{\mathrm{Vol}(D_R)}\right)\frac{\mathrm{Vol}\left(S_R^{n-1}\right)}{\mathrm{Vol}\left(B_R^n\right)} \quad \textit{for all} \ \ R > 0, \tag{6.5}$$

where

$$1 - \frac{1}{n\lambda}\frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{\mathrm{Vol}(D_R)} \geq 0 \quad \forall R > 0. \tag{6.6}$$

Remark m We assume that $\lambda \neq 0$. When $\lambda = 0$, then Σ is minimal, and the extrinsic balls satisfy the following isoperimetric inequality, (see [33]):

$$\frac{\mathrm{Vol}(\partial D_R)}{\mathrm{Vol}(D_R)} \geq \frac{\mathrm{Vol}\left(S_R^{n-1}\right)}{\mathrm{Vol}\left(B_R^n\right)}, \quad \text{for all} \ \ R > 0 \quad .$$

Proof We are going to prove first that

$$1 - \frac{1}{n\lambda}\frac{\int_{D_R} \|\vec{H}_\Sigma\|^2}{\mathrm{Vol}(D_R)} \geq 0 \quad \forall\, R > 0. \tag{6.7}$$

To do that, let us consider the function $r^2 : \Sigma \to \mathbb{R}$, defined as $r^2(p) = \|X(p)\|^2$, where r is the extrinsic distance to $\vec{0}$ in $\Sigma \subseteq \mathbb{R}^{n+m}$. Then, applying Lemma 2.2 to the radial function $F(r) = r^2$

$$\Delta^\Sigma r^2 = 2n + 2\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_\Sigma\rangle. \tag{6.8}$$

Taking into account that $\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_\Sigma\rangle = -\lambda\|X^\perp\| = -\frac{\|\vec{H}_\Sigma\|^2}{\lambda}$ we obtain

$$\Delta^\Sigma r^2 = 2n - 2\frac{\|\vec{H}_\Sigma\|^2}{\lambda}, \tag{6.9}$$

and hence

$$\|\vec{H}_\Sigma\|^2 = n\lambda - \frac{\lambda}{2}\Delta^\Sigma r^2. \tag{6.10}$$

Integrating on D_R equality above, and arranging terms, we have

$$n\lambda\, \mathrm{Vol}(D_R) - \int_{D_R} \|\vec{H}_\Sigma\|^2 \,\mathrm{d}\sigma = \frac{\lambda}{2}\int_{D_R} \Delta^\Sigma r^2 \,\mathrm{d}\sigma. \tag{6.11}$$

Now we apply Divergence theorem taking into account that the unitary normal to ∂D_R in Σ, pointed outward is $\mu = \frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|}$ and the fact that $\nabla^\Sigma r = \frac{X^T}{\|X^T\|}$,

$$\begin{aligned}\int_{D_R} \Delta^\Sigma r^2 \, d\sigma &= \int_{\partial D_R} \left\langle \nabla^\Sigma r^2, \frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|} \right\rangle d\mu \\ &= \int_{\partial D_R} 2r \|\nabla^\Sigma r\| d\mu = 2 \int_{\partial D_R} \|X^T\| d\mu, \end{aligned} \tag{6.12}$$

so Eq. (6.11) becomes

$$n\lambda \operatorname{Vol}(D_R) - \int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma = \lambda \int_{\partial D_R} \|X^T\| d\mu, \tag{6.13}$$

and hence

$$0 \leq \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{\operatorname{Vol}(D_R)} = n\lambda - \frac{\lambda \int_{\partial D_R} \|X^T\| d\mu}{\operatorname{Vol}(D_R)} \leq n\lambda., \tag{6.14}$$

which implies inequality (6.7). On the other hand, integrating on D_R the first equality in (6.4), we obtain

$$\begin{aligned}-\int_{D_R} \Delta^\Sigma \bar{E}_R \, d\sigma &= \int_{D_R} \left(1 - \frac{1}{n} \frac{\|\vec{H}_\Sigma\|^2}{\lambda}\right) d\sigma \\ &= \operatorname{Vol}(D_R) - \frac{1}{n\lambda} \int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma. \end{aligned} \tag{6.15}$$

Now, applying Divergence Theorem, and taking into account, as before, that the unitary normal to ∂D_R in Σ pointed outward is $\mu = \frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|}$, we have

$$-\int_{D_R} \Delta^\Sigma \bar{E}_R d\sigma = -\bar{E}'_R(R) \int_{\partial D_R} \|\nabla^\Sigma r\| d\sigma \leq \frac{\operatorname{Vol}\left(B_R^{0,n}\right)}{\operatorname{Vol}\left(S_R^{0,n-1}\right)} \operatorname{Vol}(\partial D_R). \tag{6.16}$$

Hence

$$\operatorname{Vol}(D_R) - \frac{1}{n\lambda} \int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma \leq \frac{\operatorname{Vol}\left(B_R^{0,n}\right)}{\operatorname{Vol}\left(S_R^{0,n-1}\right)} \operatorname{Vol}(\partial D_R), \tag{6.17}$$

so

$$\frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(\partial D_R)} \leq \frac{\operatorname{Vol}\left(B_R^{0,n}\right)}{\operatorname{Vol}\left(S_R^{0,n-1}\right)} + \frac{1}{n\lambda} \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{\operatorname{Vol}(\partial D_R)} \tag{6.18}$$

and therefore for all $R > 0$,

$$\frac{\mathrm{Vol}(\partial D_R)}{\mathrm{Vol}(D_R)} \geq \left(1 - \frac{1}{n\lambda}\frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 d\sigma}{\mathrm{Vol}(D_R)}\right)\frac{\mathrm{Vol}\left(S_R^{n-1}\right)}{\mathrm{Vol}\left(B_R^n\right)}. \tag{6.19}$$

□

6.3 Proper Self-Shrinkers for MCF and Their Distance to the Origin

We present in this subsection the following theorem which give us a dual description of the behavior of the self-shrinker when we change the hypothesis of parabolicity for the assumption that it is properly immersed.

The key idea is that a properly immersed self-shrinker cannot lie globally on one side of a λ-self-shrinker sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ unless it is a minimal immersion into this sphere.

This result, which describes the position of properly immersed λ-self-shrinkers Σ^n with respect to the critical ball $B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$, is proved using Theorem 6.2, and inequality (6.6). We must remark that the same result has been proved in [19] as a corollary of the fact that properly immersed λ-self-shrinkers of MCF are h-parabolic submanifolds of the Euclidean space $\mathbb{R}^{n+m}$ weighted with the Gaussian density $e^{h(r)}$, $h(r) = -\frac{\lambda}{2}r^2$. We include here this proof in order to show the scope of these purely Riemannian techniques.

Theorem 6.3 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker in $\mathbb{R}^{n+m}$ for the Mean Curvature Flow (MCF), with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us suppose that:*

(1) *Either Σ is confined into the ball $X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$,*
(2) *or Σ yields entirely out the interior of this ball, $X(\Sigma) \subseteq \overline{\mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})}$.*

Then Σ^n is compact and $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.

Proof Let us suppose first that $X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$. Then $\sqrt{\frac{n}{\lambda}} \geq r(p)\ \forall p \in \Sigma$. Hence we have that $\|X^\perp\|^2 \leq \|X\|^2 \leq \frac{n}{\lambda}$. Then, using Lemma 2.2:

$$\triangle^\Sigma r^2(x) = 2(n - \lambda\|X^\perp\|^2) \geq 0.$$

On the other hand, as X is proper and $\Sigma = X^{-1}(B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0}))$, then Σ is compact and hence, parabolic. In conclusion, $r^2(x) = R^2\ \forall x \in \Sigma$, for some $R \leq \sqrt{\frac{n}{\lambda}}$. But as Σ is a λ-soliton for the MCF, then $R = \sqrt{\frac{n}{\lambda}}$ by Proposition 2.12. We can apply too Theorem 5.1.

Let us suppose now that $X(\Sigma) \subseteq \overline{\mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})}$. This means that $\sqrt{\frac{n}{\lambda}} \leq r(p)\ \forall p \in \Sigma$.

Let us assume that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$ and that $\inf_\Sigma r > \sqrt{\frac{n}{\lambda}}$. We will reach a contradiction. First, as $\inf_\Sigma r > \sqrt{\frac{n}{\lambda}}$, we have that, for any $p \in \Sigma$

$$1 - \frac{\lambda}{n} r^2(p) < 0.$$

Hence

$$\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}(R^2 - r^2)} \mathrm{d}\sigma < 0. \tag{6.20}$$

Now, we need the following

Lemma 6.4 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed λ-self-shrinker in $\mathbb{R}^{n+m}$ for the MCF, with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us suppose that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball D_R, if $\mathrm{Vol}(D_R) > 0$, we have*

$$1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 \mathrm{d}\sigma}{n\lambda \,\mathrm{Vol}(D_R)} = \frac{\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}(R^2 - r^2)} \mathrm{d}\sigma}{\mathrm{Vol}(D_R)}. \tag{6.21}$$

Proof By applying the divergence theorem,

$$\begin{aligned} \int_{D_R} \mathrm{div}^\Sigma \left(e^{-\frac{\lambda}{2} r^2} \nabla^\Sigma r^2\right) \mathrm{d}\sigma &= \int_{\partial D_R} e^{-\frac{\lambda}{2} r^2} \left\langle \nabla^\Sigma r^2, \frac{\nabla^\Sigma r}{\|\nabla^\Sigma r\|} \right\rangle \mathrm{d}\mu \\ &= 2R^2 e^{-\frac{\lambda}{2} R^2} \int_{\partial D_R} \|\nabla^\Sigma r\| \mathrm{d}\mu. \end{aligned}$$

By Eq. (6.13) we know that

$$\frac{R}{n \,\mathrm{Vol}(D_R)} \int_{\partial D_R} \|\nabla^\Sigma r\| \mathrm{d}\mu = 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 \mathrm{d}\sigma}{\lambda n \,\mathrm{Vol}(D_R)}$$

Hence,

$$1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 \mathrm{d}\sigma}{\lambda n \,\mathrm{Vol}(D_R)} = \frac{e^{\frac{\lambda}{2} R^2}}{2n \,\mathrm{Vol}(D_R)} \int_{D_R} \mathrm{div} \left(e^{-\frac{\lambda}{2} r^2} \nabla^\Sigma r^2\right) \mathrm{d}\sigma.$$

Finally, the proposition follows taking into account that using Eq. (3.6),

$$\mathrm{div}^\Sigma \left(e^{-\frac{\lambda}{2} r^2} \nabla^\Sigma r^2\right) = 2e^{-\frac{\lambda}{2} r^2} \left(n - \lambda r^2\right).$$

□

Now, applying inequality (6.6) in Theorem 6.2 and Lemma 6.4, we have

$$0 \leq 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 \mathrm{d}\sigma}{n\lambda \operatorname{Vol}(D_R)} = \frac{\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}(R^2 - r^2)} \mathrm{d}\sigma}{\operatorname{Vol}(D_R)} < 0, \tag{6.22}$$

which is a contradiction.

Hence, or $X(\Sigma) \subseteq S^{n+m-1}(R_0)$ for some radius $R_0 > 0$, or $\inf_\Sigma r = \sqrt{\frac{n}{\lambda}}$.

In the first case, we have that $X : \Sigma \to S^{n+m-1}(R_0)$ will be a spherical immersion and, by Proposition 2.12, as Σ is a λ-soliton for the MCF, then X is minimal and $\lambda = \frac{n}{R^2{}_0}$, namely, $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a minimal immersion.

In the second case we shall conclude the same: if $\inf_\Sigma r = \sqrt{\frac{n}{\lambda}}$, then $\sqrt{\frac{n}{\lambda}} \leq r(p)$ for all $p \in \Sigma$ and hence $1 - \frac{\lambda}{n} r^2(p) \leq 0 \; \forall p \in \Sigma$. Then by inequality (6.6) and equality (6.21) we have

$$0 \leq 1 - \frac{\int_{D_R} \|\vec{H}_\Sigma\|^2 \mathrm{d}\sigma}{n\lambda \operatorname{Vol}(D_R)} = \frac{\int_{D_R} \left(1 - \frac{\lambda}{n} r^2\right) e^{\frac{\lambda}{2}(R^2 - r^2)} \mathrm{d}\sigma}{\operatorname{Vol}(D_R)} \leq 0. \tag{6.23}$$

Therefore, $1 - \frac{\lambda}{n} r^2(p) = 0 \; \forall p \in \Sigma$, so $X(\Sigma) \subseteq S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$, and hence $X : \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a complete spherical immersion, and as the radius $R = \sqrt{\frac{n}{\lambda}}$, then by Proposition 2.12, Σ is minimal in the sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$.

Finally, as $X : \Sigma^n \to \mathbb{R}^{n+m}$ is proper, then $\Sigma = X^{-1}(S^{n+m-1}(\sqrt{\frac{n}{\lambda}}))$ is compact. □

As a corollary of Theorem 6.3, we have the following characterization of minimal spherical immersions

Corollary 6.5 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and properly immersed λ-self-shrinker in $\mathbb{R}^{n+m}$ for the MCF, with respect to $\vec{0} \in \mathbb{R}^{n+m}$.*

Then, $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a compact minimal immersion of a round sphere of radius $\sqrt{\frac{n}{\lambda}}$ centered at $\vec{0}$ if and only if $\inf_\Sigma r = \sqrt{\frac{n}{\lambda}}$.

Remark n Note that if either Σ is confined into the ball $X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$, or Σ yields entirely out this ball, $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$, then by Theorem 6.3 we have that $\inf_\Sigma r = \sqrt{\frac{n}{\lambda}}$. Likewise, if either Σ is confined into the ball $X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$, or Σ yields entirely out this ball, $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$, then by Theorem 6.3 we have that $\sup_\Sigma r = \sqrt{\frac{n}{\lambda}}$.

6.4 Comments on a Classification of Proper Self-Shrinkers for the MCF

In [2] it was proved the following classification result for self-shrinkers with polynomial volume growth. We remark again here that in [12] it was proved that properness of the immersion for self-shrinkers implies polynomial volume growth.

Theorem 6.6 *Let $\Sigma^n \to \mathbb{R}^{n+m}$ be a complete λ-self-shrinker without boundary, polynomial volume growth and bounded norm of the second fundamental form by*

$$\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \leq \lambda.$$

Then Σ is one of the following:

(1) *Σ is a round sphere $S^n(\sqrt{\frac{n}{\lambda}})$ (and hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$).*
(2) *Σ is a cylinder $S^k(\sqrt{\frac{k}{\lambda}}) \times \mathbb{R}^{n-k}$ (and hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$).*
(3) *Σ is an hyperplane (and hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = 0$).*

We want to draw attention at this point on the following notion of *separation* of a submanifold:

Definition 6.7 We say that the sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ *separates* the λ-self-shrinker $X : \Sigma \to \mathbb{R}^{n+m}$ if

$$D_{\sqrt{\frac{n}{\lambda}}} = \left\{ p \in \Sigma \,:\, \|X(p)\| < \sqrt{\frac{n}{\lambda}} \right\} \neq \emptyset,$$

and

$$\Sigma \setminus \overline{D_{\sqrt{\frac{n}{\lambda}}}} = \left\{ p \in \Sigma \,:\, \|X(p)\| > \sqrt{\frac{n}{\lambda}} \right\} \neq \emptyset,$$

Remark o When we consider any of the three proper and complete examples with $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \leq \lambda$ in Theorem 6.6, the critical sphere of radius $\sqrt{\frac{n}{\lambda}}$ in $\mathbb{R}^{n+m}$ separates the self-shrinker Σ unless Σ is itself a round sphere $S^n(\sqrt{\frac{n}{\lambda}})$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$. On the other hand, Theorem 6.3 is telling us that *non-separated* λ-self-shrinkers by the critical sphere of radius $\sqrt{\frac{n}{\lambda}}$ must be isometrically immersed in $S^n(\sqrt{\frac{n}{\lambda}})$ as compact and minimal submanifolds.

In Theorem 6.9 of this section we will prove that the fact described in Remark o is still true when the squared norm of the second fundamental form of Σ is bounded above by the greather constant $\frac{5}{3}\lambda$. More precisely, in Theorem 6.9 we will prove that the sphere of radius $\sqrt{\frac{n}{\lambda}}$ separates any λ-self-shrinker properly immersed in $\mathbb{R}^{n+m}$ with $\|A_\Sigma\|^2 < \frac{5}{3}\lambda$ unless the self-shrinker is just the n-sphere of radius $\sqrt{\frac{n}{\lambda}}$.

To prove Theorem 6.9 we will make use of the classification provided by Simon, and Chern, Do Carmo and Kobayashi, for compact minimal immersions in the sphere, (see [8,10,38]), refined later by Li and Li (see [22]). These results can be summarized in the following statement:

Theorem 6.8 (Simon-Do Carmo–Chern–Kobayashi Classification after Li and Li). *Let* $\varphi : (\Sigma^n, \widetilde{g}) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ *be a compact and minimal isometric immersion.*

(1) *If* $m = 1$ *or* $m = 2$*, let us suppose that* $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 \leq \frac{n}{2-\frac{1}{m-1}} = \frac{m-1}{2m-3}n$*. Then*

  (a) *either* $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 = 0$ *and* $(\Sigma^n, \widetilde{g})$ *is isometric to* $S^n(1)$*,*
  (b) *or either (in case* $m = 2$*),* $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 = n$ *and* $(\Sigma^n, \widetilde{g})$ *is isometric to a generalized Clifford torus* $\Sigma^n = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ *immersed as an hypersurface in* $S^{n+1}(1)$*.*

(2) *If* $m \geq 3$*, let us suppose that* $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 \leq \frac{2n}{3}$*. Then*

  (a) *either* $(\Sigma^n, \widetilde{g})$ *is isometric to* $S^n(1)$*, and* $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 = 0$*,*
  (b) *or when* $n = 2$ *and* $m = 3$*, then* $(\Sigma^n, \widetilde{g})$ *is isometric to the Veronese surface* $\Sigma^2 = \mathbb{R}P^2(\sqrt{3})$ *in* $S^4(1)$*, and* $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2 = \frac{4}{3}$*.*

Remark p It is easy to check that the bound for the squared norm of the second fundamental form $\frac{2}{3}n$, used in [22] and which do not depends on the codimension m, is bigger or equal than the bound $\frac{m-1}{2m-3}n$ used in [8,10,38], when $m \geq 3$. In fact, for all $n > 0$, the values are equal when $m = 3$ and $\frac{2}{3}n > \frac{m-1}{2m-3}n$ when $m > 3$.

Let us consider now $X : (\Sigma, g) \to (\mathbb{R}^{n+m}, g_{\text{can}})$ a complete and properly immersed λ-self-shrinker in $\mathbb{R}^{n+m}$. By Theorem 6.3, if the critical sphere of radius $\sqrt{\frac{n}{\lambda}}$ does not separate $X(\Sigma)$, then Σ is therefore compact and is minimally immersed in the round sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ centered at $\vec{0}$.

We are going to present some computations to rescale the immersion X in order to apply Theorem 6.8 to this situation. For that, we are interested in to know what is the relation between the squared norm $\|\widetilde{A}_\Sigma^{S^{n+m-1}(1)}\|^2$ (corresponding to the isometric immersion $\widetilde{X} : (\Sigma, \widetilde{g}) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$) and the squared norm $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2$ (which corresponds to the isometric immersion $X : (\Sigma, g) \to (S^{n+m-1}(\sqrt{\frac{n}{\lambda}}), g_{S^{n+m-1}(\sqrt{\frac{n}{\lambda}})}))$.

The first thing to do that is to relate the metrics on Σ, g and $\widetilde{g}$. Note that, given the immersion $X : (\Sigma, g) \to (\mathbb{R}^{n+m}, g_{\text{can}})$, the rescaled map

$$\widetilde{X} : \Sigma \to \mathbb{R}^{n+m}, \quad p \to \widetilde{X}(p) := \sqrt{\frac{\lambda}{n}} X(p)$$

sends Σ into $S^{n+m-1}(1)$, with codimension $m-1$. Therefore,

$$\widetilde{X} : \left(\Sigma, \frac{\lambda}{n} g\right) \to (\mathbb{R}^{n+m}, g_{\text{can}})$$

is an isometric immersion, and in fact, $\widetilde{X} : (\Sigma, \frac{\lambda}{n} g) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ realizes as a minimal immersion if X is minimal. Hence $\widetilde{g} = \frac{\lambda}{n} g$.

Moreover, it is straightforward to check from this that:

$$\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 = \frac{n}{\lambda} \left\| A_{\Sigma}^{S^{n+m-1}\left(\sqrt{\frac{n}{\lambda}}\right)} \right\|^2$$

and that

$$\left\| A_{\Sigma}^{\mathbb{R}^{n+m}} \right\|^2 = \left\| A_{\Sigma}^{S^{n+m-1}\left(\sqrt{\frac{n}{\lambda}}\right)} \right\|^2 + \lambda.$$

Then we conclude

$$\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 = \frac{n}{\lambda} \left\| A_{\Sigma}^{\mathbb{R}^{n+m}} \right\|^2 - n. \tag{6.24}$$

With this last equation in hand, it is obvious that the bound for the squared norm of the second fundamental form

$$\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 \leq \frac{n}{2 - \frac{1}{m-1}}.$$

is equivalent to the bound $\| A_{\Sigma}^{\mathbb{R}^{n+m}} \|^2 \leq \frac{3m-4}{2m-3}\lambda$.

Moreover, and in the same way, the bound for the squared norm of the second fundamental form given by $\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 \leq \frac{2n}{3}$ is equivalent to the bound $\| A_{\Sigma}^{\mathbb{R}^{n+m}} \|^2 \leq \frac{5}{3}\lambda$.

The previous comments allow us to state the following Theorem.

Theorem 6.9 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete, connected and properly immersed λ-self-shrinker for the MCF with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us suppose that*

$$\| A_{\Sigma}^{\mathbb{R}^{n+m}} \|^2 < \frac{5}{3}\lambda.$$

Then, either

(1) *Σ^n is isometric to $S^n\left(\sqrt{\frac{n}{\lambda}}\right)$ and $\| A_{\Sigma}^{\mathbb{R}^{n+m}} \|^2 = \lambda$,*

(2) *or, the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ of radius $\sqrt{\frac{n}{\lambda}}$ centered at $\vec{0} \in \mathbb{R}^{n+m}$ separates $X(\Sigma)$.*

Remark q The bound $\frac{5}{3}\lambda$ is optimal in the following sense: the Veronese surface $\Sigma^2 = \mathbb{R}P^2(\sqrt{3})$ in $\mathbb{R}^5$ satisfies that $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \frac{5}{3}\lambda$ and it is not separated by sphere $S^4_{\sqrt{\frac{2}{\lambda}}}(\vec{0})$ of radius $\sqrt{\frac{2}{\lambda}}$ centered at $\vec{0} \in \mathbb{R}^5$.

Proof We are going to see first that, if (1) is not satisfied, then it is satisfied (2). Namely, the fact that Σ^n is not isometric to $S^n\left(\frac{n}{\lambda}\right)$ or $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \neq \lambda$, implies that the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ of radius $\sqrt{\frac{n}{\lambda}}$ centered at $\vec{0} \in \mathbb{R}^{n+m}$ separates $X(\Sigma)$.

To see this, let us suppose that this sphere does not separate $X(\Sigma)$. Then, by Theorem 6.3, $X : (\Sigma, g) \to (S^{n+m-1}(\sqrt{\frac{n}{\lambda}}), g_{S^{n+m-1}(\sqrt{\frac{n}{\lambda}})})$ is a compact and minimal immersion. Hence:

(1) If $m = 1$, Σ^n is isometric to $S^n(\sqrt{\frac{n}{\lambda}})$ because X is a Riemannian covering and $S^n(\sqrt{\frac{n}{\lambda}})$ is simply connected, following the same arguments than in Corollaries 5.3 and 5.7. Hence $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$. But this is a contradiction with the assumption that Σ^n is not isometric to $S^n\left(\sqrt{\frac{n}{\lambda}}\right)$ or $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \neq \lambda$.

(2) If $m = 2$, since $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 < \frac{5}{3}\lambda < 2\lambda$ then, applying Theorem 6.8, either

   (a) Σ is isometric to $S^n(\sqrt{\frac{n}{\lambda}})$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$. But this is a contradiction with the assumption that Σ^n is not isometric to $S^n\left(\frac{n}{\lambda}\right)$ or $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \neq \lambda$.

   (b) or, Σ is isometric to the Clifford torus $S^k\left(\sqrt{\frac{k}{n\lambda}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n\lambda}}\right)$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = 2\lambda$. But this is a contradiction with the hypothesis of norm of second fundamental form bounded from above by $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 < 2\lambda$.

(3) If $m = 3$, since $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 < \frac{3}{5}\lambda$ then applying Theorem 6.8, either

   (a) Σ is isometric to $S^n(\sqrt{\frac{n}{\lambda}})$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$. But this is a contradiction with the assumption that Σ^n is not isometric to $S^n\left(\frac{n}{\lambda}\right)$ or $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \neq \lambda$.

   (b) or, Σ is isometric to the Veronese surface in $S^4(\sqrt{\frac{n}{\lambda}})$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \frac{3}{5}\lambda$. But this is a contradiction with the hypothesis of $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 < \frac{3}{5}\lambda$.

(4) If $m > 3$, then, applying Theorem 6.8, Σ should be isometric to $S^n(\sqrt{\frac{n}{\lambda}})$ and $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 = \lambda$. But again this is a contradiction with the assumption that Σ^n is not isometric to $S^n\left(\frac{n}{\lambda}\right)$ or $\|A_\Sigma^{\mathbb{R}^{n+m}}\|^2 \neq \lambda$.

Conversely, if the sphere $S^{n+m-1}_{\sqrt{\frac{n}{\lambda}}}(\vec{0})$ of radius $\sqrt{\frac{n}{\lambda}}$ centered at $\vec{0} \in \mathbb{R}^{n+m}$ does not separate $X(\Sigma)$, then, as we have argumented before, by Theorem 6.3, $X : (\Sigma, g) \to (S^{n+m-1}(\sqrt{\frac{n}{\lambda}}), g_{S^{n+m-1}(\sqrt{\frac{n}{\lambda}})})$ is a compact and minimal immersion, and hence $\widetilde{X} : (\Sigma, \frac{\lambda}{n}g) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ realizes as a minimal immersion, with second fundamental form in the sphere satisfying $\left\| \widetilde{A}_{\Sigma}^{S^{n+m-1}(1)} \right\|^2 < \frac{2n}{3}$ because by hypothesis $\|A_{\Sigma}^{\mathbb{R}^{n+m}}\|^2 < \frac{5}{3}\lambda$. Therefore we apply Theorem 6.8 to conclude that

(1) Σ^n should be isometric to $S^n(\sqrt{\frac{n}{\lambda}})$ and
(2) $\|A_{\Sigma}^{\mathbb{R}^{n+m}}\|^2 = \lambda$.

□

7 Mean Exit Time, and Volume of IMCF-Solitons

7.1 Mean Exit Time on Solitons for IMCF

We start studying the mean exit time function on properly immersed solitons for IMCF $X : \Sigma^n \to \mathbb{R}^{n+m}$.

As in Sect. 6.1, let us consider the Poisson problem defined on extrinsic R-balls $D_R \subseteq \Sigma$

$$\begin{aligned} \Delta^{\Sigma} E + 1 &= 0 \text{ on } D_R, \\ E|_{\partial D_R} &= 0. \end{aligned} \tag{7.1}$$

We saw that the solution of the Poisson problem (6.2) on a geodesic R- ball $B^n(R)$ in $\mathbb{R}^n$ is given by the radial function $E_R^{0,n}(r) = \frac{R^2-r^2}{2n}$.

As in Sect. 6.2, we shall consider the transplanted radial solution of (6.1) $\bar{E}_R(r)$ to the extrinsic ball by mean the extrinsic distance function, so we have $\bar{E}_R : D_R \to \mathbb{R}$ defined as $\bar{E}_R(p) := \bar{E}_R(r(p))\ \forall p \in D_R$. Our first result here is again a comparison for the mean exit time function:

Proposition 7.1 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed soliton in $\mathbb{R}^{n+m}$ for the IMCF, with constant velocity $C \neq \{0, \frac{1}{n}\}$ and with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us suppose that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball $D_R(\vec{0}) = \Sigma \cap B_R^{n+m}(\vec{0})$, we have that the mean exit time function on D_R, E_R, satisfies*

$$E_R(x) = \frac{Cn}{Cn-1}\bar{E}_R(x)\ \ \forall x \in D_R. \tag{7.2}$$

Proof We have, as $\bar{E}_R(x) := E_R^{0,n}(r(x)) = \frac{R^2-r(x)^2}{2n}$ and applying Lema 2.2, that, on D_R

$$\begin{aligned} \Delta^{\Sigma}\bar{E}_R &= \left(\bar{E}_R''(r) - \bar{E}_R'(r)\frac{1}{r}\right)\|\nabla^{\Sigma} r\|^2 \\ &\quad + \bar{E}_r'(r)\left(n\frac{1}{r} + \langle \nabla^{\mathbb{R}^{n+m}} r, \vec{H}_{\Sigma}\rangle\right) = -1 - \frac{1}{n}\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_{\Sigma}\rangle. \end{aligned} \tag{7.3}$$

On the other hand, $X(p) = r(p)\nabla^{\mathbb{R}^{n+m}}(p)$ for all $p \in \Sigma$, being $X(p)$ the position vector of p in $\mathbb{R}^{n+m}$. And, moreover, as we have that $\frac{\vec{H}_\Sigma(p)}{\|\vec{H}_\Sigma(p)\|^2} = -CX^\perp(p)$, then

$$\begin{aligned}\langle r\nabla^{\mathbb{R}^{n+m}} r, \vec{H}_\Sigma\rangle &= \langle X, \vec{H}_\Sigma\rangle = \langle X, -C\|\vec{H}_\Sigma\|^2 X^\perp\rangle \\ &= -C\|\vec{H}_\Sigma\|^2\|X^\perp\|^2 = -\frac{1}{C}.\end{aligned} \tag{7.4}$$

Equation (7.3) becomes

$$\Delta^\Sigma \bar{E}_R = -1 + \frac{1}{Cn} = \frac{1 - Cn}{Cn}. \tag{7.5}$$

Therefore,

$$\begin{aligned}\Delta^\Sigma \frac{Cn}{Cn-1}\bar{E}_R &= \frac{Cn}{Cn-1}\Delta^\Sigma \bar{E}_R = \frac{Cn}{Cn-1}\frac{1-Cn}{Cn} \\ &= -1 = \Delta^\Sigma E_R \ \text{ on } \ D_R\end{aligned} \tag{7.6}$$

and, applying the Maximum Principle,

$$\frac{Cn}{Cn-1}\bar{E}_R = E_R \ \text{ on } \ D_R.$$

□

As a consequence, we obtain again Corollary 4.4:

Corollary 7.2 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and non-compact, properly immersed soliton in $\mathbb{R}^{n+m}$ for the IMCF, with constant velocity $C \neq \{0, \frac{1}{n}\}$ and with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Then*

$$C \notin \left(0, \frac{1}{n}\right).$$

We finalize this subsection with a characterization of solitons for the IMCF in terms of the mean exit time function.

Theorem 7.3 *Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a proper immersion. Let us suppose that $X(\Sigma) \not\subseteq S^n(R)$ for any radius $R > 0$. Then, if for all extrinsic R-balls $D_R(\vec{0})$, we have that $E_R = \alpha\bar{E}_R$, with $\alpha \neq 1$ and $\alpha \neq 0$, then X is a soliton for the IMCF with respect to $\vec{0} \in \mathbb{R}^{n+1}$, with velocity $C = -\frac{\alpha}{\alpha-1}\frac{1}{n}$. Hence, if $\alpha \in (1, \infty)$, then X is a self-shrinker and if $\alpha \in (0, 1)$, then X is a self-expander.*

Proof We have, as $\bar{E}_R(x) := E_R^{0,n}(r(x)) = \frac{R^2 - r(x)^2}{2n}$ and applying Lema 2.2, that, on D_R, for all $R > 0$,

$$\Delta^\Sigma \bar{E}_R = -1 - \frac{1}{n}\langle X, \vec{H}_\Sigma\rangle. \tag{7.7}$$

Hence, as we are assuming that $E_R = \alpha \bar{E}_R$ for all $R > 0$, we have

$$\Delta^\Sigma \alpha \bar{E}_R = -\alpha - \frac{\alpha}{n}\langle X, \vec{H}_\Sigma\rangle = -1. \tag{7.8}$$

Therefore, on Σ,

$$\langle X, \vec{H}_\Sigma\rangle = \langle X^\perp, \vec{H}_\Sigma\rangle = \frac{1-\alpha}{\alpha}n, \tag{7.9}$$

so $\|\vec{H}\| \neq 0$.

But $\vec{H}_\Sigma = h\nu$ where ν is the unit normal vector field pointed outward to Σ, so

$$\langle X, \vec{H}_\Sigma\rangle = \langle X, \nu\rangle h = \frac{1-\alpha}{\alpha}n$$

and therefore, it is straightforward to check that

$$\frac{\vec{H}}{\|\vec{H}\|^2} = \frac{1}{h}\nu = \frac{\alpha}{1-\alpha}\frac{1}{n}\langle X, \nu\rangle\nu = \frac{\alpha}{1-\alpha}\frac{1}{n}X^\perp \tag{7.10}$$

and X is a soliton with $C = -\frac{\alpha}{1-\alpha}\frac{1}{n}$. □

Remark r Note that $\alpha \neq 0, 1$. If $\alpha = 0$, then $E_R = 0$ for all radius $R > 0$, so Σ reduces to a point. On the other hand, if $\alpha = 1$, then Σ is minimal in $\mathbb{R}^{n+1}$, (see [24]), and hence X cannot be a soliton for the IMCF (see Remark g).

7.2 Volume of Solitons for IMCF

As a consequence or the proof above, and using the Divergence theorem we have the following result:

Theorem 7.4 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete properly immersed soliton in $\mathbb{R}^{n+m}$ for the IMCF, with constant velocity $C \neq 0$ and with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us suppose that $X(\Sigma) \nsubseteq S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball $D_R(\vec{0})$, we have*

$$\frac{\operatorname{Vol}(\partial D_R)}{\operatorname{Vol}(D_R)} \geq \frac{Cn-1}{Cn}\frac{\operatorname{Vol}(S^{n-1}(R))}{\operatorname{Vol}(B^n(R))} \quad \textit{for all } \ R > 0 \quad . \tag{7.11}$$

Proof Integrating on the extrinsic ball D_R the equality $\Delta^\Sigma \frac{Cn}{Cn-1}\bar{E}_R = -1$ and applying Divergence theorem as in Theorem 6.2 we obtain, as $C \in \mathbb{R} \sim [0, \frac{1}{n}]$:

$$\begin{aligned}\operatorname{Vol}(D_R) &= \int_{D_R} -\Delta^\Sigma \frac{Cn}{Cn-1}\bar{E}_R = -\frac{Cn}{Cn-1}\bar{E}'_R(R)\int_{\partial D_R}\|\nabla^\Sigma r\|\mathrm{d}\sigma \\ &\leq \frac{Cn}{Cn-1}\frac{\operatorname{Vol}(B^n(R))}{\operatorname{Vol}(S^{n-1}(R))}\operatorname{Vol}(\partial D_R).\end{aligned} \tag{7.12}$$

□

Remark s Equality in inequality (7.11) for all radius $R \leq R_0$ implies that the inequality $\int_{\partial D_R} \|\nabla^{\Sigma} r\| d\sigma \leq \operatorname{Vol}(\partial D_R)$ becomes an equality for all $R \leq R_0$. This implies that $\|\nabla^{\Sigma} r\| = 1 = \|\nabla^{\mathbb{R}^{n+m}} r\|$ in the extrinsic ball D_{R_0}, so $\nabla^{\Sigma} r = \nabla^{\mathbb{R}^{n+m}} r$ in D_{R_0} and Σ is totally geodesic in D_{R_0}. Hence, $\vec{H}_{\Sigma} = \vec{0}$ in D_{R_0}, which is not compatible with the fact that $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a properly immersed soliton in $\mathbb{R}^{n+m}$ for the IMCF. Therefore, if $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a properly immersed soliton in $\mathbb{R}^{n+m}$ for the IMCF, then inequality (7.11) must be strict.

Corollary 7.5 *Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a properly immersed soliton in $\mathbb{R}^{n+m}$ for the IMCF, with constant velocity $C \neq 0$ and with respect to $\vec{0} \in \mathbb{R}^{n+m}$. Let us suppose that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$. Let us define the* volume growth *function*

$$f(t) := \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^n(t))^{\frac{Cn-1}{Cn}}}.$$

Then, given $r_1 > 0$, $f(t)$ is non decreasing for all $t \geq r_1 > 0$.

Proof As $\frac{d}{dt} \operatorname{Vol}(D_t) \geq \operatorname{Vol}(\partial D_t)$ by the co-area formula, we have, applying Theorem 7.4,

$$\frac{d}{dt} \ln f(t) \geq \frac{\operatorname{Vol}(\partial D_t)}{\operatorname{Vol}(D_t)} - \frac{Cn-1}{Cn} \frac{\operatorname{Vol}(\partial S^{n-1}(t))}{\operatorname{Vol}(B^n(t))} \geq 0.$$

□

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