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Closed Range Estimates for $\bar{\partial}_b$ on CR Manifolds of Hypersurface Type

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Abstract

The purpose of this paper is to establish sufficient conditions for closed range estimates on $(0, q)$-forms, for some *fixed* q, $1 \le q \le n-1$, for $\bar{\partial}_b$ in both L^2 and L^2-Sobolev spaces in embedded, not necessarily pseudoconvex CR manifolds of hypersurface type. The condition, named weak $Y(q)$, is both more general than previously established sufficient conditions and easier to check. Applications of our estimates include estimates for the Szegö projection as well as an argument that the harmonic forms have the same regularity as the complex Green operator. We use a microlocal argument and carefully construct a norm that is well suited for a microlocal decomposition of form. We do not require that the CR manifold is the boundary of a domain. Finally, we provide an example that demonstrates that weak $Y(q)$ is an easier condition to verify than earlier, less general conditions.

Keywords Weak $Z(q)$ · Weak $Y(q)$ · Tangential Cauchy–Riemann operator · $\bar{\partial}_b$ · Closed range · Microlocal analysis

Mathematics Subject Classification Primary 32W10; Secondary 32F17 · 32V20 · 35A27 · 35N15

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1 Introduction

In this paper, we show that the tangential Cauchy–Riemann operator has closed range on $(0, q)$-forms, for a *fixed* q, $1 \le q \le n-1$, in L^2 and L^2-Sobolev spaces on a general class of embedded CR manifolds of hypersurface type that satisfy a general geometric condition called *weak* $Y(q)$. We work on a smooth CR submanifold $M \subset \mathbb{C}^n$ that may be neither pseudoconvex nor the boundary of a domain. The weak $Y(q)$ condition, first written down by Harrington and Raich [10] and applied to boundaries of domains in Stein manifolds, is the most general known condition that ensures closed range of the tangential Cauchy–Riemann operator on $(0, q)$-forms. We also provide an example that shows that the generality provided by the definition makes it easier to verify than previous and more restrictive conditions. Additionally, we show that for any Sobolev level, there is a weight such that the (weighted) complex Green operator (inverse to the weighted Kohn Laplacian) is continuous and the harmonic forms in this weighted space are elements of the prescribed Sobolev space.

This paper generalizes both [9] and [10] in the following ways. We do not require our CR manifold to be the boundary of a domain. In effect, we translate the $\bar\partial$-techniques of [10] to the microlocal setting. In [9], they prove results akin to our main results, but the "weak $Y(q)$" condition they define is more restrictive than the weak $Y(q)$ condition here. Additionally, we use a reengineered elliptic regularization argument to show that (weighted) harmonic $(0, q)$-forms are smooth, a fact not mentioned in [9,10]. Additionally, we are careful to monitor the regularized operators and the fact that they preserve orthogonality with the space of (weighted) harmonic forms, a fact that has not been observed before (in part because we prove smoothness of harmonic forms early in regularization process).

Throughout this paper, we will consider $M \subset \mathbb{C}^N$ being a $2n - 1$ real dimension, C^∞, compact, orientable CR manifold, $N \ge n$ of hypersurface type. This last condition means that the CR dimension of M is $n - 1$ so that the complex tangent bundle splits into a complex subbundle of dimension $n-1$, the conjugate subbundle, and one totally real direction. An appropriate restriction of the $\bar\partial$-complex to M yields the $\bar\partial_b$-complex.

The $\bar\partial_b$-operator was introduced by Kohn and Rossi [15] to study the boundary values of holomorphic functions on domains in $\mathbb{C}^n$, and it was soon realized that the $\bar\partial_b$-complex was deeply intertwined with the geometry and potential theory of such domains and their boundaries. The story of the L^2-theory of the $\bar\partial_b$-operator begins with Shaw [19] and Boas and Shaw [2] (in the top degree) on boundaries of pseudoconvex domains in $\mathbb{C}^n$ and with Kohn [13] on the boundaries of pseudoconvex domains in Stein manifolds. Nicoara [16] established closed range for $\bar\partial_b$ (at all form levels) on smooth, embedded, compact, orientable CR manifolds of hypersurface dimension in the case that $n \ge 3$ and Baracco [1] established the $n = 2$ case. Thus, from the point of view closed range, the pseudoconvex case is completely understood.

Harrington and Raich [9] began an investigation of the $\bar\partial_b$-problem on nonpseudoconvex CR manifolds of hypersurface type. Specifically, they fixed a level q, $1 \le q \le n-2$, and sought a general condition that sufficed to prove closed range of $\bar\partial_b$ on $(0, q)$-forms (and in L^2-Sobolev spaces in suitably weighted spaces). They worked on CR manifolds of hypersurface type, and our results generalize theirs by showing that the conclusions they draw are still true with a weaker hypothesis, namely, the weak $Y(q)$ condition from [10]. The analysis in [10] is loosely based on the ideas of Shaw and does not use a microlocal argument, but rather $\bar{\partial}$-methods. This requires the CR manifold to be the boundary of a domain, a hypothesis that we relax. The name weak $Y(q)$ stems from the fact that it is a weakening of the classical $Y(q)$ condition, a geometric condition that is equivalent to the complex Green operator satisfying $1/2$-estimates on $(0, q)$-forms. The *complex Green operator*, when it exists, is the name for the (relative) inverse to $\Box_b$ in $L^2_{0,q}(M)$ and denoted by G_q.

Our methods involve a microlocal argument in the spirit of [9,16,17] and a recently reengineered elliptic regularization that not only allows for a weighted complex Green operator to solve the $\bar{\partial}_b$-problem in a given L^2-Sobolev space, but also shows that the weighted L^2-harmonic forms reside in that Sobolev space [7,14]. This last fact is not clear from the elliptic regularization methods used in [9,16]. For a discussion of the weak $Y(q)$ condition and its related, nonsymmetrized version, weak $Z(q)$, please see [6,8–11] and for discussion on the elliptic regularization method, [7,14].

The outline of the argument is as follows: we start by proving a basic identity that is well suited to the geometry of M. The problem with basic identities for $\bar{\partial}_b$ is that the Levi form appears with in a term that also contains the derivative in the totally real direction. The microlocal argument is used to control this term—specifically, we construct a norm based on a microlocal decomposition of our form which allows us to use a version of the sharp Gårding's inequality and eliminate the T from the inner product term. This allows us to prove a basic estimate (Proposition 4.1) from the basic identity and the main results are due to careful applications of the basic estimate.

The outline of the paper is the following. We conclude this section with statements of our main theorems. In Sect. 2, we define our notation. In Sect. 3, we give some computations in local coordinates and the microlocal decomposition. In Sect. 4, we prove the basic estimate, Proposition 4.1. In Sect. 5, we prove the Theorem 1.2. Many of the consequences of Theorem 1.2 use identical proofs to [9, Theorem 1.2], once we have completed the elliptic regularization argument, established the continuity of $G_{q,t}$ on $H^s_{0,q}(M)$, and proved the regularity of the weighted harmonic forms. In Sect. 6, we outline how to pass from Theorem 1.2 to Theorem 1.1. We conclude the paper in Sect. 7 with an example.

Theorem 1.1 *Let M^{2n-1} be an embedded C^∞, compact, orientable CR manifold of hypersurface type that satisfies weak $Y(q)$ for some fixed q, $1 \le q \le n-2$. Then the following hold:*

(1) *The operators $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ have closed range;*
(2) *The operators $\bar{\partial}^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ and $\bar{\partial}^*_b : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ have closed range;*
(3) *The Kohn Laplacian $\Box_b := \bar{\partial}_b\bar{\partial}^*_b + \bar{\partial}^*_b\bar{\partial}_b$ has closed range on $L^2_{0,q}(M)$;*
(4) *The complex Green operator G_q exists and is continuous on $L^2_{0,q}(M)$;*
(5) *The canonical solution operators, $\bar{\partial}^*_b G_q : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ and $G_q\bar{\partial}^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ are continuous;*
(6) *The canonical solution operators, $\bar{\partial}_b G_q : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $G_q\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ are continuous;*

(7) *The space of the harmonic forms $\mathcal{H}_{0,q}(M)$, defined to be the (0,q)-forms annihilated by $\bar{\partial}_b$ and $\bar{\partial}_b^*$, is finite dimensional;*

(8) *If $\tilde{q} = q$ or $q+1$ and $\alpha \in L^2_{0,\tilde{q}}$, then there exists $u \in L^2_{0,\tilde{q}-1}$ so that*

$$\bar{\partial}_b u = \alpha$$

and $\|u\|_0 \le C\|\alpha\|_0$ for some constant C independent of α;

(9) *The Szegö projections $S_q = I - \bar{\partial}_b^* \bar{\partial}_b G_q$ and $S_{q-1} = I - \bar{\partial}_b^* G_q \bar{\partial}_b$ are continuous on $L^2_{0,q}(M)$.*

In fact, Theorem 1.1 follows immediately from Theorem 1.2 using standard techniques and the fact that the constructed norm $|||\cdot|||_t$ is equivalent to the unweighted norm $\|\cdot\|_0$. We denote the L^2 space with respect to $|||\cdot|||_t$ by $L^2(M, |||\cdot|||_t)$. Additionally, we use the (equivalent) norm $|||\Lambda^s \cdot|||_t$ on $H^s(M)$ because with it, we can obtain better constants and denote the $H^s(M)$ with respect to this measurement by $H^s(M, |||\cdot|||_t)$.

Theorem 1.2 *Let M^{2n-1} be a C^∞ compact, orientable, weakly $Y(q)$ CR manifold of hypersurface type embedded in $\mathbb{C}^N$, $N \ge n$, and $1 \le q \le n-2$. For each $s \ge 0$ there exists $T_s \ge 0$ so that the following hold:*

i. *The operators $\bar{\partial}_b : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q+1}(M, |||\cdot|||_t)$ and $\bar{\partial}_b : L^2_{0,q-1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ have closed range. Additionally, for any $s > 0$ if $t \ge T_s$, then $\bar{\partial}_b : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q+1}(M, |||\cdot|||_t)$ and $\bar{\partial}_b : H^s_{0,q-1}(M, |||\cdot|||_t) \to H^s_q(M, |||\cdot|||_t)$ have closed range.*

ii. *The operators $\bar{\partial}^*_{b,t} : L^2_{0,q+1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ and $\bar{\partial}^*_{b,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q-1}(M, |||\cdot|||_t)$ have closed range. Additionally, if $t \ge T_s$, then $\bar{\partial}^*_{b,t} : H^s_{0,q+1}(M, |||\cdot|||_t) \to H^s_{0,q}(M, |||\cdot|||_t)$ and $\bar{\partial}^*_{b,t} : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q-1}(M, |||\cdot|||_t)$ have closed range.*

iii. *The Kohn Laplacian $\Box_{b,t} := \bar{\partial}_b \bar{\partial}^*_{b,t} + \bar{\partial}^*_{b,t} \bar{\partial}_b$ has closed range on $L^2_{0,q}(M, |||\cdot|||_t)$, and if $t \ge T_s$, $\Box_{b,t}$ also has closed range on $H^s_{0,q}(M, |||\cdot|||_t)$.*

iv. *The space of (weighted) harmonic forms $\mathcal{H}^q_t(M)$, defined to be the $(0,q)$-forms annihilated by $\bar{\partial}_b$ and $\bar{\partial}^*_{b,t}$, is finite dimensional.*

v. *The complex Green operator $G_{q,t}$ exists and is continuous on $L^2_{0,q}(M, |||\cdot|||_t)$ and also on $H^s_{0,q}(M, |||\cdot|||_t)$ if $t \ge T_s$.*

vi. *The canonical solution operators for $\bar{\partial}_b$, $\bar{\partial}^*_{b,t} G_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q-1}(M, |||\cdot|||_t)$ and $G_{q,t} \bar{\partial}^*_{b,t} : L^2_{0,q+1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ are continuous. Additionally, $\bar{\partial}^*_{b,t} G_{q,t} : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q-1}(M, |||\cdot|||_t)$ and $G_{q,t} \bar{\partial}^*_{b,t} : H^s_{0,q+1}(M, |||\cdot|||_t) \to H^s_{0,q}(M, |||\cdot|||_t)$ are continuous if $t \ge T_s$.*

vii. *The canonical solution operators for $\bar{\partial}^*_{b,t}$, $\bar{\partial}_b G_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q+1}(M, |||\cdot|||_t)$ and $G_{q,t} \bar{\partial}_b : L^2_{0,q-1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ are continuous. Additionally, $\bar{\partial}_b G_{q,t} : H^s_{0,q}(M, |||\cdot|||_t) \to H^s_{0,q+1}(M, |||\cdot|||_t)$ and $G_{q,t} \bar{\partial}_b : H^s_{0,q-1}(M, |||\cdot|||_t) \to H^s_{0,q}(M, |||\cdot|||_t)$ are continuous if $t \ge T_s$.*

viii. *The Szegö projections* $S_{q,t} = I - \bar\partial^*_{b,t}\bar\partial_b G_{q,t}$ *and* $S_{q-1,t} = I - \bar\partial^*_{b,t} G_{q,t}\bar\partial_b$ *are continuous on* $L^2_{0,q}(M, |||\cdot|||_t)$ *and* $L^2_{0,q-1}(M, |||\cdot|||_t)$*, respectively. Additionally, if* $t \geq T_s$ *then* $S_{q,t}$ *and* $S_{q-1,t}$ *are continuous on* $H^s_{0,q}(M, |||\cdot|||_t)$ *and* $H^s_{0,q-1}(M, |||\cdot|||_t)$*, respectively.*

2 Definitions and Notation

2.1 CR Manifolds

Definition 2.1 Let M a smooth manifold of real dimensional $2n - 1$. M is called a *CR manifold of hypersurface type* if M is equipped with a subbundle of the complexified tangent bundle $\mathbb{C}T(M)$ denoted by $\mathbb{L}$ satisfying:

(i) $\dim_{\mathbb{C}} \mathbb{L}_x = n - 1$ where $\mathbb{L}_x$ is the fiber over $x \in M$.
(ii) $\mathbb{L}_x \cap \overline{\mathbb{L}_x} = \{0\}$ where $\overline{\mathbb{L}_x}$ is the complex conjugate of $\mathbb{L}_x$.
(iii) If $L, L' \in \mathbb{L}$, then $\left[L, L'\right] := LL' - L'L$ is in $\mathbb{L}$.

$\mathbb{L}$ is called the CR structure of M. Since M is embedded in $\mathbb{C}^N$, we define $T^{1,0}_z(M) = T^{1,0}_z(\mathbb{C}^N) \cap T_z(M) \otimes \mathbb{C}$ (under the natural inclusion). Since the complex dimension of the CR structure is $n - 1$ for all $z \in M$, we can set $\mathbb{L} = T^{1,0}(M) = \bigcup_{z\in M} T^{1,0}_z(M)$, and this defines a CR structure on M called the *induced CR structure* on M.

For this paper, we consider only smooth, orientable CR manifolds of hypersurface type embedded in a complex space $\mathbb{C}^N$, though our techniques should generalize to Stein manifolds, a topic that we do not pursue here to notational simplicity and clarity. Let $T^{p,q}(M)$ denote the space of exterior algebra generated by $T^{1,0}(M)$ and $T^{0,1}(M)$. Let $\Lambda^{p,q}(M)$ denote the bundle of (p, q)-forms on $T^{p,q}(M)$, that is, $\Lambda^{p,q}(M)$ consists of skew-symmetric multilinear maps of $T^{p,q}(M)$ into $\mathbb{C}$. Because we are in $\mathbb{C}^N$, our calculations do not depend on p, and we therefore set $p = 0$ for the remainder of the manuscript.

2.2 $\bar\partial_b$ on Embedded Manifolds

Since $M \subset \mathbb{C}^N$ for some $N \geq n$, and our CR structure is the induced one, it is natural to use the induced metric on $\mathbb{C}T(M)$, denoted by $\langle\cdot,\cdot\rangle_x$ for each $x \in M$. The metric $\langle\cdot,\cdot\rangle_x$ is compatible with the induced CR structure in the sense that the vector spaces $T^{1,0}_x$ and $T^{0,1}_x$ are orthogonal. We use the inner product on $\Lambda^{0,q}(M)$ given by

$$(\varphi, \psi)_0 = \int_M \langle\varphi, \psi\rangle_x \, \mathrm{d}V$$

where dV is the volume element on M. The involution condition (iii) in Definition 2.1 means that $\bar\partial_b$ can be defined as the restriction of the de Rham exterior derivative d to $\Lambda^{0,q}(M)$.

The Hermitian inner product above gives rise to an L^2-norm $\|\cdot\|_0$, and we also denote the closure of $\bar\partial_b$ in this norm by $\bar\partial_b$ (by an abuse of notation). In this way,

$\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ is a well-defined, closed, densely defined operator, and we define $\bar{\partial}^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ to be its L^2 adjoint. The Kohn Laplacian $\Box_b : L^2_{0,q}(M) \to L^2_{0,q}(M)$ is defined as

$$\Box_b := \bar{\partial}^*_b \bar{\partial}_b + \bar{\partial}_b \bar{\partial}^*_b.$$

2.3 The Levi Form

From the CR structure on M, there is a local orthonormal basis $L_1, ..., L_{n-1}$ of the $(1,0)$-vector fields in a neighborhood U of a point $x \in M$. Let $\omega_1, \dots, \omega_{n-1}$ be the dual basis of $(1,0)$-forms so that $\langle \omega_j, L_k \rangle = \delta_{jk}$. This means $\bar{L}_1, \dots, \bar{L}_{n-1}$ is an orthonormal basis of $T^{0,1}(U)$ with dual basis $\bar{\omega}_1, \dots, \bar{\omega}_{n-1}$ in U. Finally, there is a vector T, taken purely imaginary, so that $\{L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, T\}$ is an orthonormal basis of $T(U)$. Since M is oriented, there exists a globally defined 1-form γ that annihilates $T^{1,0}(M) \oplus T^{0,1}(M)$ and is normalized so that $\langle \gamma, T \rangle = -1$.

Definition 2.2 The *Levi form* at a point $x \in M$ is the Hermitian form given by $\langle d\gamma_x, L \wedge \bar{L}' \rangle$ for any $L, L' \in T^{1,0}_x(U)$, and U is a neighborhood of $x \in M$.

Cartan's formula implies that for any $L, L' \in T^{1,0}(M)$, we have

$$\langle d\gamma, L \wedge \bar{L}' \rangle = -\langle \gamma, [L, \bar{L}'] \rangle. \tag{2.1}$$

In local coordinates, for any $1 \le j, k \le n-1$,

$$[L_j, \overline{L}_k] = c_{jk} T \ mod \ T^{1,0}(U) \oplus T^{0,1}(U)$$

so that $\langle d\gamma, L_j \wedge \overline{L}_k \rangle = c_{jk}$. We will call $[c_{jk}]_{1 \le j,k \le n-1}$ the *Levi matrix* with respect to $L_1, ..., L_{n-1}, T$.

Let $\mu_1, ..., \mu_{n-1}$ be the eigenvalues of $[c_{jk}]$ such that $\mu_1 \le \mu_2 \le ... \le \mu_{n-1}$. The CR structure is called (strictly) pseudoconvex in some point $p \in M$ if the matrix $[c_{jk}(p)]$ is positive (definite) semidefinite. If the CR structure is (strictly) pseudoconvex in every point, then it is called (strictly) pseudoconvex.

Now, we introduce the main geometric condition for our CR manifolds, given by Harrington and Raich in [10].

Definition 2.3 For $1 \le q \le n-1$ we say M satisfies $Z(q)$-*weakly* if there exists a real $\Upsilon \in T^{1,1}(M)$ satisfying

(A) $|\theta|^2 \ge (i\theta \wedge \overline{\theta})(\Upsilon) \ge 0$ for all $\theta \in \Lambda^{1,0}(M)$.
(B) $\mu_1 + \mu_2 + \cdots + \mu_q - i \langle d\gamma_x, \Upsilon \rangle \ge 0$ where $\mu_1, ..., \mu_{n-1}$ are the eigenvalues of the Levi form at x in increasing order.
(C) $\omega(\Upsilon) \ne q$ where ω is the $(1,1)$-form associated to the induced metric on $\mathbb{C}T(M)$.

We say that M satisfies *weak* $Y(q)$ if M satisfies both $Z(q)$-weakly and $Z(n-q-1)$-weakly.

For example, it is easy to see that if M is pseudoconvex, then M satisfies weak $Z(q)$ for any $1 \le q \le n-1$ with $\Upsilon = 0$. Please see [6,10,11] for a discussion of the weak $Z(q)$ property. The symmetric hypotheses on form levels on q and $n-1-q$ are necessary due a Hodge-* operator [3,18].

Remark 2.4 If M is a CR manifold satisfying $Y(q)$ weakly, then Υ corresponding to weak $Z(q)$, which we denote by Υ_q, may be unrelated to the Υ that corresponds to weak $Z(n-q-1)$ (similarly denoted by Υ_{n-1-q}).

Given a function φ defined near M, we define the two form

$$\Theta^{\varphi} = \frac{1}{2}\Big(\partial_b\bar{\partial}_b\varphi - \bar{\partial}_b\partial_b\varphi\Big) + \frac{1}{2}\nu(\varphi)\, d\gamma$$

where ν is the real part of the complex normal to M. When we work locally, we often associate Θ^{φ} with the matrix $\Theta^{\varphi}_{jk} = \langle \Theta^{\varphi}, L_j \wedge \bar{L}_k \rangle$. We know that for such φ

$$\Big\langle \frac{1}{2}\big(\partial\bar{\partial}\varphi - \bar{\partial}\partial\varphi\big), L \wedge \bar{L} \Big\rangle = \big\langle \Theta^{\varphi}, L \wedge \bar{L} \big\rangle$$

which means $\Theta^{|z|^2} = \partial\bar{\partial}|z|^2 = -i\omega$ [9, Proposition 3.1].

3 Local Coordinates and Pseudodifferential Operators

3.1 Pseudodifferential Operators

We follow the setup from [17]. By the compactness of M, there exists a finite cover $\{U_\mu\}_\mu$, so each U_μ has a special boundary system and can be parameterized by a hypersurface in $\mathbb{C}^n$ (U_μ may be shrunk as necessary).

Let $\xi = (\xi_1, \ldots, \xi_{2n-2}, \xi_{2n-1}) = (\xi', \xi_{2n-1})$ be the coordinates in Fourier space so that ξ' is the dual variable to the variables in the maximal complex tangent space and ξ_{2n-1} is dual to the totally real part of $T(M)$, i.e., the "bad" direction T. Define

$$\mathcal{C}^+ = \left\{\xi : \xi_{2n-1} \ge \frac{1}{2}\left|\xi'\right| \text{ and } |\xi| \ge 1\right\}; \quad \mathcal{C}^- = \left\{\xi : -\xi \in \mathcal{C}^+\right\};$$
$$\mathcal{C}^0 = \left\{\xi : -\frac{3}{4}\left|\xi'\right| \le \xi_{2n-1} \ge \frac{3}{4}\left|\xi'\right|\right\} \cup \{\xi : |\xi| \le 1\}.$$

$\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint, but both intersect $\mathcal{C}^0$ nontrivially. Next, let ψ^+, ψ^- and ψ^0 be smooth functions on the unit sphere so that

$$\psi^+(\xi) = 1 \text{ when } \xi_{2n-1} \ge \frac{3}{4}\left|\xi'\right| \text{ and } \operatorname{supp}\psi^+ \subset \left\{\xi : \xi_{2n-1} \ge \frac{1}{2}\left|\xi'\right|\right\};$$
$$\psi^-(\xi) = \psi^+(-\xi);\ \psi^0(\xi) \text{ satisfies } \psi^0(\xi)^2 = 1 - \psi^+(\xi)^2 - \psi^-(\xi)^2.$$

Extend ψ^+, ψ^-, and ψ^0 homogeneously outside of the unit ball, i.e., if $|\xi| \ge 1$, then

$$\psi^+(\xi) = \psi^+(\xi/|\xi|), \quad \psi^-(\xi) = \psi^-(\xi/|\xi|), \quad \text{and} \quad \psi^0(\xi) = \psi^0(\xi/|\xi|).$$

Finally, extend ψ^+, ψ^-, and ψ^0 smoothly inside the unit ball so that $(\psi^+)^2+(\psi^-)^2+(\psi^0)^2 = 1$ and ψ^+ and ψ^- are supported away from $B(0, \frac{1}{2})$. For a fixed constant $A > 0$ to be chosen later, define for any $t > 0$,

$$\psi_t^+(\xi) = \psi^+(\xi/(tA)), \psi_t^-(\xi) = \psi^-(\xi/(tA)), \text{ and } \psi^0(\xi) = \psi^0(\xi/(tA)).$$

Let Ψ_t^+, Ψ_t^-, and Ψ_t^0 be the pseudodifferential operators of order zero with symbols ψ_t^+, ψ_t^-, and ψ_t^0, respectively. The equality $(\psi_t^+)^2 + (\psi_t^-)^2 + (\psi_t^0)^2 = 1$ implies that

$$(\Psi_t^+)^*\Psi_t^+ + (\Psi_t^-)^*\Psi_t^- + (\Psi_t^0)^*\Psi_t^0 = I.$$

Suppose ψ and $\tilde\psi$ are cut-off functions so that $\tilde\psi|_{\mathrm{supp}\psi} \equiv 1$. If Ψ and $\tilde\Psi$ are pseudodifferential operators with symbols ψ and $\tilde\psi$, respectively, then we say that $\tilde\Psi$ *dominates* Ψ.

For each μ, let $\Psi_{\mu,t}^+$, $\Psi_{\mu,t}^-$, and $\Psi_{\mu,t}^0$ be the operators Ψ_t^+, Ψ_t^-, and Ψ_t^0, respectively, defined on U_μ, where $\mathcal{C}_\mu^+$, $\mathcal{C}_\mu^-$ are $\mathcal{C}_\mu^0$ be the corresponding regions of ξ-space dual to U_μ. It follows that

$$(\Psi_{\mu,t}^+)^*\Psi_{\mu,t}^+ + (\Psi_{\mu,t}^-)^*\Psi_{\mu,t}^- + (\Psi_{\mu,t}^0)^*\Psi_{\mu,t}^0 = I.$$

Additionally, let $\tilde\Psi_{\mu,t}^+$ and $\tilde\Psi_{\mu,t}^-$ be pseudodifferential operators that dominate $\Psi_{\mu,t}^+$ and $\Psi_{\mu,t}^-$, respectively (where $\Psi_{\mu,t}^+$ and $\Psi_{\mu,t}^-$ are defined on some U_μ). If $\tilde{\mathcal{C}}_\mu^+$ and $\tilde{\mathcal{C}}_\mu^-$ are the supports of the symbols of $\tilde\Psi_{\mu,t}^+$ and $\tilde\Psi_{\mu,t}^-$, respectively, then we can choose $\{U_\mu\}$, $\tilde\psi_{\mu,t}^+$, and $\tilde\psi_{\mu,t}^-$ so that the following result holds [16].

Lemma 3.1 (Lemma 4.3, [16]) *Let M be a compact, orientable, embedded CR manifold. There is a finite open covering $\{U_\mu\}_\mu$ of M so that if $U_\mu, U_{\mu'} \in \{U_\mu\}$ have nonempty intersection, then there exits a diffeomorphism ϑ between U_μ and $U_{\mu'}$ with Jacobian $\mathcal{J}_\vartheta$ such that*

(i) *${}^t\mathcal{J}_\vartheta(\mathcal{C}_\mu^+) \cap \mathcal{C}_{\mu'}^- = \emptyset$ and $\mathcal{C}_{\mu'}^+ \cap {}^t\mathcal{J}_\vartheta(\mathcal{C}_\mu^-) = \emptyset$ where ${}^t\mathcal{J}_\theta$ is the inverse of the transpose of the Jacobian of ϑ;*
(ii) *let ${}^\vartheta\Psi_{t,\mu}^+$, ${}^\vartheta\Psi_{t,\mu}^-$ and ${}^\vartheta\Psi_{t,\mu}^0$ be the transfer of $\Psi_{t,\mu}^+$, $\Psi_{t,\mu}^-$, and $\Psi_{t,\mu}^0$, respectively, via ϑ, then on $\{\xi : \xi_{2n-1} \ge \frac{4}{5}|\xi'| \text{ and } |\xi| \ge (1+\varepsilon)tA\}$, the principal symbol of ${}^\vartheta\Psi_{t,\mu}^+$ is identically equal to 1, on $\{\xi : \xi_{2n-1} \le -\frac{4}{5}|\xi'| \text{ and } |\xi| \ge (1+\varepsilon)tA\}$, the principal symbol of ${}^\vartheta\Psi_{t,\mu}^-$ is identically equal to 1, and on $\{\xi : -\frac{1}{3}|\xi'| \le \xi_{2n-1} \le \frac{1}{3}|\xi'| \text{ and } |\xi| \ge (1+\varepsilon)tA\}$, the principal symbol of ${}^\vartheta\Psi_{t,\mu}^0$ is identically equal to 1, where $\varepsilon > 0$ and can be very small.*
(iii) *Let ${}^\vartheta\tilde\Psi_{t,\mu}^+$, ${}^\vartheta\tilde\Psi_{t,\mu}^-$ be the transfer via ϑ of $\tilde\Psi_{t,\mu}^+$, $\tilde\Psi_{t,\mu}^-$ respectively. Then the principal symbol of ${}^\vartheta\tilde\Psi_{t,\mu}^+$ is identically 1 on $\mathcal{C}_{\mu'}^+$ and the principal symbol of ${}^\vartheta\tilde\Psi_{t,\mu}^-$ is identically 1 on $\mathcal{C}_{\mu'}^-$;*
(iv) *$\tilde{\mathcal{C}}_{\mu'}^+ \cap \tilde{\mathcal{C}}_{\mu'}^- = \emptyset$.*

We will suppress the left superscript ϑ as it should be clear from the context which pseudodifferential operator must be transferred. If P is any of the operators $\Psi^+_{t,\mu}$, $\Psi^-_{t,\mu}$, or $\Psi^0_{t,\mu}$, then it is immediate that

$$D^\alpha_\xi \sigma(P) = \frac{1}{|t|^\alpha} q_\alpha(x,\xi)$$

for $|\alpha| \geq 0$, where $q_\alpha(x,\xi)$ is bounded independently of t.

3.2 Norms

If ϕ is a real function defined on M, then define the weighted Hermitian inner for $(0,q)$-forms f and g, denoted by $(f,g)_\phi$ by $(f,g)_\phi = \left(e^{-\phi} f, g\right)_0$. For example, if $f = \sum_{J\in\mathcal{I}_q} f_J \bar{\omega}^J$ is a (0,q)-form supported on neighborhood U, where $\mathcal{I}_q = \{J = (j_1,\ldots,j_q) : 1 \leq j_1 < j_2 < \cdots < j_q\}$ and $\omega^J = \omega_{j_1} \wedge \cdots \wedge \omega_{j_q}$. The weighted L^2-norm on $(0,q)$-forms is $\|f\|^2_\phi := \sum_{J\in\mathcal{I}_q} \|f_J\|^2_\phi$ where $\|f_J\|^2_\phi = \int_M |f_J|^2 e^{-\phi} \mathrm{d}V$, and we denote the corresponding weighted L^2 space by $L^2_{0,q}(M, e^{-\phi})$.

We now construct a norm that is well adapted to the microlocal analysis. Let $\{U_\mu\}_\mu$ be a covering of M that admits the family of pseudodifferential operators $\{\Psi^+_{\mu,t},\ \Psi^-_{\mu,t},\ \Psi^0_{\mu,t}\}$ and a partition of unity $\left\{\zeta_\mu\right\}_\mu$ subordinate to the cover satisfying $\sum_\mu \zeta^2_\mu = 1$. For each μ let $\tilde{\zeta}_\mu$ be a cut-off function that dominates ζ_μ such that $\operatorname{supp}\tilde{\zeta}_\mu \subset U_\mu$, and ϕ^+, ϕ^- smooth functions defined on M. We define the global inner product and norm as follows:

$$\begin{aligned}(f,g)_{\phi^+,\phi^-} := (f,g)_t := \sum_\mu \Big[&\left(\tilde{\zeta}_\mu \Psi^+_{\mu,t} \zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^+_{\mu,t} \zeta_\mu g^\mu\right)_{\phi^+} \\ &+ \left(\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu g^\mu\right)_0 \\ &+ \left(\tilde{\zeta}_\mu \Psi^-_{\mu,t} \zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^-_{\mu,t} \zeta_\mu g^\mu\right)_{\phi^-} \Big]\end{aligned}$$

and

$$|||f|||^2_{\phi^+,\phi^-} := \sum_\mu \left[\|\tilde{\zeta}_\mu \Psi^+_{\mu,t} \zeta_\mu f^\mu\|^2_{\phi^+} + \|\tilde{\zeta}_\mu \Psi^0_{\mu,t} \zeta_\mu f^\mu\|^2_0 + \|\tilde{\zeta}_\mu \Psi^-_{\mu,t} \zeta_\mu f^\mu\|^2_{\phi^-} \right]$$

where f^μ and g^μ are the forms f and g, respectively, expressed in the local coordinates on U_μ. The superscript μ will often omitted. In the case that $\phi^+(z) = t|z|^2$ or $-t|z|^2$ and $\phi^-(z) = -t|z|^2$ or $t|z|^2$, we denote the norm by $|||\cdot|||_t$ and in general replace the subscript with t (e.g., we write c_t for c_{ϕ^+,ϕ^-}).

For a form f on M, the Sobolev norm of order s is given by the following:

$$\|f\|^2_{H^s} = \sum_\mu \|\tilde{\zeta}_\mu \Lambda^s \zeta_\mu f^\mu\|^2_0$$

where Λ is the pseudodifferential operator with symbol $(1+|\xi|^2)^{1/2}$. In [16], Nicoara shows that there exist constants c_{ϕ^+,ϕ^-} and C_{ϕ^+,ϕ^-} so that

$$c_{\phi^+,\phi^-}\|f\|_0^2 \le |||f|||_{\phi^+,\phi^-}^2 \le C_{\phi^+,\phi^-}\|f\|_0^2. \tag{3.1}$$

Additionally, there exists a invertible self-adjoint operator E_{ϕ^+,ϕ^-} so that $(f,g)_0 = \big(f, E_{\phi^+,\phi^-}g\big)_{\phi^+,\phi^-}$, where E_{ϕ^+,ϕ^-} is the inverse of

$$\sum_\mu \Big(\zeta_\mu(\Psi^+_{\mu,t})^*\tilde\zeta_\mu e^{-\phi^+}\tilde\zeta_\mu\Psi^+_{\mu,t}\zeta_\mu + \zeta_\mu(\Psi^0_{\mu,t})^*\tilde\zeta_\mu^2\Psi^0_{\mu,t}\zeta_\mu + \zeta_\mu(\Psi^-_{\mu,t})^*\tilde\zeta_\mu e^{-\phi^-}\tilde\zeta_\mu\Psi^-_{\mu,t}\zeta_\mu\Big)$$

and this operator is bounded in $L^2(M)$ independently of $tA \ge 1$ (see Corollary 4.6 in [16]).

3.3 $\bar\partial_b$ and its Adjoints

If f is a function on M, then in a local coordinates

$$\bar\partial_b f = \sum_{j=1}^{n-1} \bar L_j f\,\bar\omega_j$$

and if $f = \sum_{J\in\mathcal{I}_q} f_J\,\bar\omega^J$ is a $(0,q)$-form, then there exist functions m^J_K such that

$$\bar\partial_b f = \sum_{J\in\mathcal{I}_q, K\in\mathcal{I}_{q+1}} \sum_{j=1}^{n-1} \epsilon^{jJ}_K \bar L_j f_J\,\bar\omega^K + \sum_{J\in\mathcal{I}_q, K\in\mathcal{I}_{q+1}} f_J m^J_K\,\bar\omega^K$$

where ϵ^{jJ}_K is equal to 0 if $\{K\} \ne \{j\}\cup J$ and is the sign of the permutation that reorders jJ to K otherwise. We also define

$$f_{jI} = \sum_{J\in\mathcal{I}_q} \epsilon^{jI}_J f_J \tag{3.2}$$

(in this case, $I \in \mathcal{I}_{q-1}$). Let $\bar L_j^*$ be the adjoint of $\bar L_j$ in $(\ ,\ )_0$, $\bar L_j^{*,\phi}$ be the adjoint of $\bar L_j$ in $(\ ,\ )_\phi$. Then on a small neighborhood U we will have $\bar L_j^* = -L_j + \sigma_j$ and $\bar L_j^{*,\phi} = -L_j + L_j\phi + \sigma_j$ where σ_j is smooth function on U. Because we will need it later, we observe that there are smooth functions d^ℓ_{sr} and σ_s so that

$$\big[\bar L_r, \bar L_s^{*,\phi}\big] = c_{sr}T + \bar L_r L_s\phi + \sum_{\ell=1}^{n-1}(d^\ell_{sr}L_\ell - \bar d^\ell_{rs}\bar L_\ell) + \bar L_r\sigma_s. \tag{3.3}$$

We denote the L^2 adjoint of $\bar\partial_b$ in $L^2_{0,q}(M, e^{-\phi})$ by $\bar\partial_b^{*,\phi}$. For the remainder of the paper, ϕ stands for either ϕ^+ or ϕ^- and

$$|\phi^+(z)| = |\phi^-(z)| = |t||z|^2,$$

though virtually all of our calculations hold for general ϕ, up to the point when our calculation require an analysis of the eigenvalues of the Levi form.

To keep track of the terms that arise in our integration by parts, we use the following shorthand for forms f supported in a neighborhood U_μ (recognizing that these operators depend on our choice of neighborhoods $\{U_\mu\}$):

$$\nabla_{\bar{L}^{*,\phi}} f = \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1} \bar{L}_j^{*,\phi} f_J\,\bar{\omega}^J; \quad \nabla_{\bar{L}} f = \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1} \bar{L}_j f_J\,\bar{\omega}^J;$$

$$\|\overline{\nabla}_\Upsilon f\|_\phi^2 = \sum_{J\in\mathcal{I}_q}\sum_{j,k=1}^{n-1}\left(b^{\bar{k}j}\bar{L}_k f_J, \bar{L}_j f_J\right)_\phi = \sum_{j,k=1}^{n-1}\left(b^{\bar{k}j}\bar{L}_k f, \bar{L}_j f\right)_\phi$$

$$\|\nabla_\Upsilon f\|_\phi^2 = \sum_{J\in\mathcal{I}_q}\sum_{j,k=1}^{n-1}\left(b^{\bar{k}j}\bar{L}_j^{*,\phi} f_J, \bar{L}_k^{*,\phi} f_J\right)_\phi = \sum_{j,k=1}^{n-1}\left(b^{\bar{k}j}\bar{L}_j^{*,\phi} f, \bar{L}_k^{*,\phi} f\right)_\phi$$

where $\Upsilon = i\sum_{j,k=1}^{n-1} b^{\bar{k}j}\bar{L}_k \wedge L_j$ is a real (1, 1) vector defined on U_μ initially satisfying (A) in Definition 2.3. Again, if $f = \sum_{J\in\mathcal{I}_q} f_J\,\bar{\omega}^J$ is defined locally, then

$$\begin{aligned}\bar{\partial}_b^* f &= \sum_{I\in\mathcal{I}_{q-1}, J\in I_q}\sum_{j=1}^{n-1}\epsilon_J^{jI}\bar{L}_j^* f_J\,\bar{\omega}^I + \sum_{I\in\mathcal{I}_{q-1}, J\in I_q} f_J \overline{m}_J^I\,\bar{\omega}^I \\ &= \sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n-1}\bar{L}_j^* f_{jI}\,\bar{\omega}^I + \sum_{I\in\mathcal{I}_{q-1}, J\in I_q} f_J\overline{m}_J^I\,\bar{\omega}^I\end{aligned}$$

and

$$\bar{\partial}_b^{*,\phi} f = \sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n-1}\bar{L}_j^{*,\phi} f_{jI}\,\bar{\omega}^I + \sum_{I\in\mathcal{I}_{q-1}, J\in\mathcal{I}_q} f_J\overline{m}_J^I\,\bar{\omega}^I$$

Note that a consequence of the compactness of M and the boundedness of ϕ, the domains of $\bar{\partial}_b^*$ and $\bar{\partial}_b^{*,\phi}$ are equal. Also we have $\bar{\partial}_b^{*,\phi} = \bar{\partial}_b^* - \left[\bar{\partial}_b^*, \phi\right]$. Let $\bar{\partial}_{b,t}^*$ be the adjoint of $\bar{\partial}_b$ with respect to the inner product $(\cdot,\cdot)_t$. We also define the weighted Kohn Laplacian $\Box_b$ by $\Box_{b,t} := \bar{\partial}_b\bar{\partial}_{b,t}^* + \bar{\partial}_{b,t}^*\bar{\partial}_b$ where

$$\begin{aligned}\mathrm{Dom}(\Box_{b,t}) := \Big\{&\phi\in L_{0,q}^2(M) : \phi\in\mathrm{Dom}(\bar{\partial}_b)\cap\mathrm{Dom}(\bar{\partial}_{b,t}^*), \\ &\bar{\partial}_b\phi\in\mathrm{Dom}(\bar{\partial}_{b,t}^*),\ \text{and } \bar{\partial}_{b,t}^*\phi\in\mathrm{Dom}(\bar{\partial}_b)\Big\}.\end{aligned}$$

The computations proving Lemmas 4.8 and 4.9 and equation (4.4) in [16] can be applied here with only a change of notation, so we have the following two results, recorded here as Lemmas 3.2 and 3.3. The consequence is that $\bar{\partial}^*_{b,t}$ acts like $\bar{\partial}^{*,\phi^+}_b$ (denoted just by $\bar{\partial}^{*,+}_b$) for forms whose support is basically $\mathcal{C}^+$ and $\bar{\partial}^{*,\phi^-}_b$ (denoted just by $\bar{\partial}^{*,-}_b$) on forms whose support is basically $\mathcal{C}^-$.

Lemma 3.2 *On smooth (0,q)-forms,*

$$\begin{aligned}
\bar{\partial}^*_{b,t} &= \bar{\partial}^*_b - \sum_\mu \zeta_\mu^2 \tilde{\Psi}^+_{\mu,t}\left[\bar{\partial}^*_b, \phi^+\right] + \sum_\mu \zeta_\mu^2 \tilde{\Psi}^-_{\mu,t}\left[\bar{\partial}^*_b, \phi^-\right] \\
&\quad + \sum_\mu \Big(\tilde{\zeta}_\mu \left[\tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu, \bar{\partial}_b\right]^* \tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu + \zeta_\mu (\Psi^+_{\mu,t})^* \tilde{\zeta}_\mu \left[\bar{\partial}^{*,+}_b, \tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu\right]\tilde{\zeta}_\mu \\
&\quad + \tilde{\zeta}_\mu \left[\tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu, \bar{\partial}_b\right]^* \tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu + \zeta_\mu (\Psi^-_{\mu,t})^* \tilde{\zeta}_\mu \left[\bar{\partial}^{*,-}_b, \tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu\right]\tilde{\zeta}_\mu + E_A \Big)
\end{aligned}$$

where the error term E_A is a sum of order zero terms and "lower order" terms. Also, the symbol of E_A is supported in $\mathcal{C}^0_\mu$ for each μ.

We use the following energy forms in our calculations:

$$\begin{aligned}
Q_{b,t}(f,g) &= \left(\bar{\partial}_b f, \bar{\partial}_b g\right)_t + \left(\bar{\partial}^*_{b,t} f, \bar{\partial}^*_{b,t} g\right)_t \\
Q_{b,+}(f,g) &= \left(\bar{\partial}_b f, \bar{\partial}_b g\right)_{\phi^+} + \left(\bar{\partial}^{*,+}_b f, \bar{\partial}^{*,+}_b g\right)_{\phi^+} \\
Q_{b,0}(f,g) &= \left(\bar{\partial}_b f, \bar{\partial}_b g\right)_0 + \left(\bar{\partial}^*_b f, \bar{\partial}^*_b g\right)_0 \\
Q_{b,-}(f,g) &= \left(\bar{\partial}_b f, \bar{\partial}_b g\right)_{\phi^-} + \left(\bar{\partial}^{*,-}_b f, \bar{\partial}^{*,-}_b g\right)_{\phi^-}.
\end{aligned}$$

The space of weighted harmonic forms $\mathcal{H}^q_t$ is defined by

$$\begin{aligned}
\mathcal{H}^q_t &:= \left\{ f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : \bar{\partial}_b f = 0, \bar{\partial}^*_{b,t} f = 0 \right\} \\
&= \left\{ f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : Q_{b,t}(f,f) = 0 \right\}.
\end{aligned}$$

We have the following relationship between the energy forms. See [9, Lemma 3.4] or [16, Lemma 4.9].

Lemma 3.3 *If f is a smooth (0,q)-form on M, then there exist constants K, K_t, and K' with $K \geq 1$ so that*

$$\begin{aligned}
&K\, Q_{b,t}(f,f) + K_t \sum_\nu \|\tilde{\zeta}_\mu \tilde{\Psi}^0_{\mu,t}\zeta_\mu f^\mu\|_0^2 + K' \,|||f|||_t^2 + O_t(\|f\|^2_{-1}) \\
&\geq \sum_\mu \Big[Q_{b,+}(\tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^+_{\mu,t}\zeta_\mu f^\mu) \\
&\quad Q_{b,0}(\tilde{\zeta}_\mu \Psi^0_{\mu,t}\zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^0_{\mu,t}\zeta_\mu f^\mu) + Q_{b,-}(\tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu f^\mu, \tilde{\zeta}_\mu \Psi^-_{\mu,t}\zeta_\mu f^\mu) \Big]
\end{aligned}$$

K and K' do not depend on t, ϕ^- or ϕ^+.

4 The Basic Estimate

In this section, we compile the technical pieces that will allows us to establish a basic estimate the ground level L^2 estimates for Theorem 1.2 in Sect. 5.

Proposition 4.1 *Let $M^{2n-1} \subset \mathbb{C}^N$ be a smooth, compact, orientable CR manifold of hypersurface type that satisfies weak $Y(q)$ for some fixed $1 \le q \le n-2$. Set*

$$\phi^+(z) = \begin{cases} t|z|^2 & \text{if } \omega(\Upsilon_q) < q \\ -t|z|^2 & \text{if } \omega(\Upsilon_q) > q \end{cases} \quad \text{and} \quad \phi^-(z) = \begin{cases} -t|z|^2 & \text{if } \omega(\Upsilon_{n-1-q}) < n-1-q \\ t|z|^2 & \text{if } \omega(\Upsilon_{n-1-q}) > n-1-q. \end{cases} \tag{4.1}$$

There exist constants K and K_t where K does not depend on t so that

$$t\,|||f|||_t^2 \le K Q_{b,t}(f,f) + K_t \|f\|_{-1}^2, \tag{4.2}$$

for t sufficiently large.

The main work in establishing (4.2) is to prove the following:

$$t\,|||f|||_t^2 \le K Q_{b,t}(f,f) + K\,|||f|||_t^2 + K_t \sum_{\mu} \sum_{J\in\mathcal{I}_q} \|\tilde\zeta_\mu \tilde\Psi^0_{\mu,t} \zeta_\mu f^\mu_J\|_0^2 + K'_t \|f\|_{-1}^2. \tag{4.3}$$

In order to prove (4.3), we estimate a $(0,q)$-form f with support in neighborhood U in a generic energy form $Q_{b,\phi}(f,g) := (\bar\partial_b f, \bar\partial_b g)_\phi + (\bar\partial_b^{*,\phi} f, \bar\partial_b^{*,\phi} g)_\phi$. Throughout the estimate, we will make use of three terms, $E_0(f)$, $\tilde E_1(f)$, and $\tilde E_2(f)$ to collect the error terms that we will bound later. We want $E_0(f) = O(\|f\|_\phi^2)$ and

$$\tilde E_1(f) = \sum_{J,J'\in\mathcal{I}_q} \sum_{j=1}^{n-1} \left(\bar L_j f_J, a_{JJ'} f_{J'}\right)_\phi \quad \text{and} \quad \tilde E_2(f) = \sum_{J,J'\in\mathcal{I}_q} \sum_{j=1}^{n-1} \left(\bar L_j^{*,\phi} f_J, \tilde a_{JJ'} f_{J'}\right)_\phi$$

for some collection of smooth functions $a_{JJ'}$ and $\tilde a_{JJ'}$ that may change line to line.

Integration by parts (see, e.g., [17, Lemma 4.2]) shows that

$$\begin{aligned} Q_{b,\phi}(f,f) &= \|\nabla_{\bar L} f\|_\phi^2 + \sum_{J,J'\in I_q} \sum_{\substack{j,k=1 \\ j\ne k}}^{n-1} \epsilon^{kJ}_{jJ'} \left(\left[\bar L_j^{*,\phi}, \bar L_k\right] f_J, f_{J'}\right)_\phi \\ &\quad + \sum_{J\in\mathcal{I}_q} \sum_{j\in J} \left(\left[\bar L_j, \bar L_j^{*,\phi}\right] f_J, f_J\right)_\phi + 2\,\mathrm{Re}\left(\tilde E_2(f) + \tilde E_1(f)\right) + E_0(f). \end{aligned}$$

Developing the commutator terms as in [17, Lemma 4.2] and using the fact that $L_j = -\bar{L}_j^{*,\phi} + L_j\phi + \sigma_j$, we have the equality

$$\begin{aligned} Q_{b,\phi}(f,f) &= \|\nabla_{\bar{L}} f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \operatorname{Re}\left(c_{jk} T f_{jI}, f_{kI}\right)_\phi \\ &\quad + \operatorname{Re} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left[\left((\bar{L}_k L_j \phi) f_{jI}, f_{kI}\right)_\phi + \left(\sum_{l=1}^{n-1} d_{jk}^l L_l \phi f_{jI}, f_{kI} \right)_\phi \right] \\ &\quad + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f). \end{aligned}$$

Since

$$\begin{aligned} \operatorname{Re} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\bar{L}_k L_j \phi f_{jI}, f_{kI}\right)_\phi &= \frac{1}{2} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\bar{L}_k L_j \phi + L_j \bar{L}_k \phi) f_{jI}, f_{kI}\right)_\phi \\ \operatorname{Re} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\sum_{l=1}^{n-1} d_{jk}^l L_l \phi f_{jI}, f_{kI} \right)_\phi &= \frac{1}{2} \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\sum_{l=1}^{n-1} (d_{jk}^l L_l \phi + \bar{d}_{kj}^l \bar{L}_l \phi) f_{jI}, f_{kI} \right)_\phi \end{aligned} \tag{4.4}$$

and

$$\frac{1}{2}\left(\bar{L}_k L_j \phi + L_j \bar{L}_k \phi\right) + \frac{1}{2} \sum_{l=1}^{n-1} (d_{jk}^l L_l \phi + \bar{d}_{kj}^l \bar{L}_l \phi) = \Theta_{jk}^\phi - \frac{1}{2}\nu(\phi) c_{jk}$$

it follows that

$$\begin{aligned} Q_{b,\phi}(f,f) &= \|\nabla_{\bar{L}} f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \operatorname{Re}\left(c_{jk} T f_{jI}, f_{kI}\right)_\phi \\ &\quad + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\Theta_{jk}^\phi - \frac{1}{2}\nu(\phi) c_{jk}) f_{jI}, f_{kI} \right)_\phi \\ &\quad + \tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f). \end{aligned} \tag{4.5}$$

On the other hand, integration by parts, expanding the commutator terms, and using (4.4), we will have

$$\begin{aligned} \|\overline{\nabla}_\Upsilon f\|_\phi^2 &= \sum_{j,k=1}^{n-1} \left[\left(b^{\bar{k}j} \bar{L}_j^{*,\phi} f, \bar{L}_k^{*,\phi} f \right)_\phi + \left(\left[\bar{L}_j^{*,\phi}, \bar{L}_k \right] f, b^{\bar{j}k} f \right)_\phi + \left(\bar{L}_j^{*,\phi} (b^{\bar{k}j}) \bar{L}_k f, f \right)_\phi \right] \\ &\quad + \sum_{j,k=1}^{n-1} \left(\bar{L}_j^{*,\phi} f, \bar{L}_k^{*,\phi} (b^{\bar{j}k}) f \right)_\phi \end{aligned}$$

$$= \|\nabla_\Upsilon f\|_\phi^2 - \sum_{j,k=1}^{n-1}\left[\left(b^{\bar{k}j}c_{jk}Tf, f\right)_\phi + \left(b^{\bar{k}j}(\Theta_{jk}^\phi - \frac{1}{2}\nu(\phi)c_{jk})f, f\right)_\phi\right]$$
$$+\tilde{E}_2(f) + \tilde{E}_1(f) + E_0(f). \tag{4.6}$$

Motivated by [10, p. 1725], we write $\|\nabla_{\bar{L}} f\|_\phi^2 = \left(\|\nabla_{\bar{L}} f\|_\phi^2 - \|\overline{\nabla}_\Upsilon f\|_\phi^2\right) + \|\overline{\nabla}_\Upsilon f\|_\phi^2$ and use (4.6) to obtain

$$Q_{b,\phi}(f,f) = \left(\|\nabla_{\bar{L}} f\|_\phi^2 - \|\overline{\nabla}_\Upsilon f\|_\phi^2\right) + \|\nabla_\Upsilon f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(c_{jk}Tf_{jI}, f_{kI}\right)_\phi$$
$$-\left(i\left\langle d\gamma, \Upsilon\right\rangle Tf, f\right)_\phi + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left((\Theta_{jk}^\phi - \frac{1}{2}\nu(\phi)c_{jk})f_{jJ}, f_{kI}\right)_\phi$$
$$-\left(i\left\langle\Theta^\phi, \Upsilon\right\rangle f, f\right)_\phi + \left(\frac{1}{2}\nu(\phi)i\left\langle d\gamma, \Upsilon\right\rangle f, f\right)_\phi$$
$$+\tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f)$$

Since

$$\sum_{J\in\mathcal{I}_q}(af_J, f_J)_\phi = \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{a\delta_{jk}}{q}f_{jI}, f_{kI}\right)_\phi$$

where (δ_{jk}) is the identity matrix I_{n-1}, we have

$$Q_{b,\phi}(f,f) = \left(\|\nabla_{\bar{L}} f\|_\phi^2 - \|\overline{\nabla}_\Upsilon f\|_\phi^2\right) + \|\nabla_\Upsilon f\|_\phi^2$$
$$+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(\left(c_{jk} - \frac{i\left\langle d\gamma, \Upsilon\right\rangle\delta_{jk}}{q}\right)Tf_{jI}, f_{kI}\right)_\phi$$
$$+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\left(\Theta_{jk}^\phi - \frac{i\left\langle\Theta, \Upsilon\right\rangle\delta_{jk}}{q}\right)f_{jI}, f_{kI}\right)_\phi$$
$$-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi)\left(c_{jk} - \frac{i\left\langle d\gamma, \Upsilon\right\rangle\delta_{jk}}{q}\right)f_{jI}, f_{kI}\right)_\phi$$
$$+\tilde{E}_1(f) + \tilde{E}_2(f) + E_0(f).$$

Bounding the error terms $\tilde{E}_1(f)$ and $\tilde{E}_2(f)$ uses the same argument, and we demonstrate the bound for $\tilde{E}_1(f)$. Terms of the form $\sum_{j=1}^{n-1}\left(a_j\bar{L}_j g, h\right)_\phi$ comprise $\tilde{E}_1$ for various functions g and h, and we compute

$$\sum_{j=1}^{n-1}\left(a_j\bar L_j g,h\right)_\phi=\sum_{j,k=1}^{n-1}\left((\delta_{jk}-b^{\bar jk})\bar L_j g,\bar a_k h\right)_\phi+\sum_{j,k=1}^{n-1}\left(b^{\bar jk}\bar L_j g,\bar a_k h\right)_\phi. \quad (4.7)$$

To estimate the first terms, observe that for $\varepsilon>0$, a small constant/large constant argument shows that

$$\left|\sum_{j,k=1}^{n-1}\left((\delta_{jk}-b^{\bar jk})\bar L_j g,\bar a_k h\right)_\phi\right|\le\varepsilon\sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1}(\delta_{jk}-b^{\bar jk})\bar L_j g\right\|_\phi^2+O_{\frac{1}{\varepsilon}}(\|h\|_\phi^2).$$

Stepping away from the integration (momentarily), suppose that at some point in U, A is a unitary matrix that diagonalizes the Hermitian matrix $\bar B=(b^{\bar jk})$ of Υ such that $\bar B=A^*\Lambda A$, where $\Lambda=\operatorname{diag}\{\lambda_1,\dots,\lambda_{n-1}\}$ and $\lambda_1,\cdots,\lambda_{n-1}$ are the eigenvalues of $\bar B$. Consider $[\bar L_j g]$ as a column vector with components $[\bar L_j g]_k$. Then since $(1-\lambda_j)^2\le(1-\lambda_j)$ for all j,

$$\begin{aligned}\textstyle\sum_{k=1}^{n-1}\left|\sum_{j=1}^{n-1}(\delta_{jk}-b^{\bar jk})(\bar L_j g)\right|^2&=\left|[Id-B]\left[\bar L_j g\right]\right|^2=\textstyle\sum_{j=1}^{n-1}(1-\lambda_j)^2\left|\left[A\left[\overline{\bar L_j g}\right]\right]_j\right|^2\\&\le\textstyle\sum_{j=1}^{n-1}(1-\lambda_j)\left|\left[A\left[\overline{\bar L_j g}\right]\right]_j\right|^2=\sum_{j=1}^{n-1}\left|\bar L_j g\right|^2-\sum_{j,k=1}^{n-1}b^{\bar kj}\overline{\bar L_j g}\bar L_k g.\end{aligned}$$

Returning to the integration, we now observe,

$$\sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1}(\delta_{jk}-b^{\bar jk})\bar L_j g\right\|_\phi^2\le\|\nabla_{\bar L}g\|_\phi^2-\|\overline{\nabla}_\Upsilon g\|_\phi.$$

For the second term in (4.7), a similar small constant/large constant argument shows

$$\left|\sum_{j,k=1}^{n-1}(a_k g,b^{\bar kj}\bar L_j^{*,\phi}h)_\phi\right|\le O_{\frac{1}{\varepsilon}}(\|g\|_\phi^2)+\varepsilon\sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1}b^{\bar kj}\bar L_j^{*,\phi}h\right\|_\phi^2,$$

and linear algebra (as above) helps to establish

$$\sum_{k=1}^{n-1}\left\|\sum_{j=1}^{n-1}b^{\bar kj}\bar L_j^{*,\phi}h\right\|_\phi^2\le\sum_{j,k}\left(b^{\bar kj}\bar L_j^{*,\phi}h,\bar L_k^{*,\phi}h\right)_\phi=\|\nabla_\Upsilon h\|_\phi^2.$$

Summarizing the above, for ε sufficiently small and f supported in a small neighborhood, we have

$$\begin{aligned}Q_{b,\phi}(f,f)\ge&\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\operatorname{Re}\left(\left(c_{jk}-\frac{i\langle d\gamma,\Upsilon\rangle\delta_{jk}}{q}\right)Tf_{jI},f_{kI}\right)_\phi\\&+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\left(\Theta_{jk}^\phi-\frac{i\left\langle\Theta^\phi,\Upsilon\right\rangle\delta_{jk}}{q}\right)f_{jI},f_{kI}\right)_\phi\end{aligned}$$

$$-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi)\left(\left(c_{jk}-\frac{i\langle d\gamma,\Upsilon\rangle\,\delta_{jk}}{q}\right)\right)f_{jI},f_{kI}\right)_{\phi}+O(\|f\|_{\phi}^2) \tag{4.8}$$

To handle the T terms, we recall the following results. The first is a well-known multilinear algebra result that appears (among other places) in Straube [20]:

Lemma 4.2 *Let* $B=\left(b_{jk}\right)_{1\le j,k\le n-1}$ *be a Hermitian matrix and* $1\le q\le n-1$. *The following are equivalent:*

i. *If* $u\in\Lambda^{0,q}$, *then* $\sum_{K\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}b_{jk}u_{jK}\overline{u_{kK}}\ge M\,|u|^2$.
ii. *The sum of any* q *eigenvalues of* B *is at least* M.
iii. $\sum_{s=1}^{q}\sum_{j,k=1}^{n-1}b_{jk}t_j^s\overline{t_k^s}\ge M$ *for any orthonormal vectors* $\{t^s\}_{1\le s\le q}\subset\mathbb{C}^{n-1}$.

The next two results are consequences of the sharp Gårding Inequality and appear as [17, Lemma 4.6, Lemma 4.7].

Lemma 4.3 *Let* f *a (0,q)-form supported on* U *so that up to a smooth term* $\hat{f}$ *is supported in* $\mathcal{C}^+$, *and let* $\left[h_{jk}\right]$ *a Hermitian matrix such that the sum of any q eigenvalues is* ≥ 0. *Then*

$$\begin{aligned}\operatorname{Re}\Big\{\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}Tf_{jI},f_{kI}\right)_{\phi}\Big\}\\ \ge tA\operatorname{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}f_{jI},f_{kI}\right)_{\phi}-O(\|f\|_{\phi}^2)-O_t(\|\tilde{\zeta}\widetilde{\Psi}_t^0f\|_0^2).\end{aligned}$$

Lemma 4.4 *Let* f *a (0,q)-form supported on* U *so that up to a smooth term* $\hat{f}$ *is supported in* $\mathcal{C}^-$, *and let* $\left[h_{jk}\right]$ *a Hermitian matrix such that the sum of any n-1-q eigenvalues is* ≥ 0. *Then*

$$\begin{aligned}&\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\left(h_{jj}(-T)f_J,f_J\right)_{\phi}-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}(-T)f_{jI},f_{kI}\right)_{\phi}\right\}\\ &\ge tA\operatorname{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\left(h_{jj}f_J,f_J\right)_{\phi}-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h_{jk}f_{jI},f_{kI}\right)_{\phi}\right\}\\ &\quad-O(\|f\|_{\phi}^2)-O_t(\|\tilde{\zeta}\tilde{\Psi}_t^0f\|_0^2).\end{aligned}$$

Now, we are ready to estimate $Q_{b,+}(\cdot,\cdot)$ and $Q_{b,-}(\cdot,\cdot)$.

Proposition 4.5 *Let* $f\in\operatorname{Dom}\bar{\partial}_b\cap\operatorname{Dom}\bar{\partial}_b^*$ *be a* $(0,q)$*-form supported in* U *and let* ϕ^+ *be as in* (4.1). *Then there exists a constant* C *so that*

$$Q_{b,+}\left(\tilde{\zeta}\Psi_t^+f,\tilde{\zeta}\Psi_t^+f\right)+C\|\tilde{\zeta}\Psi_t^+f\|_{\phi^+}+O_t(\|\tilde{\zeta}\tilde{\Psi}_t^0f\|_0^2)\ge tB_{\phi^+}\|\tilde{\zeta}\Psi_t^+f\|_{\phi^+}^2.$$

Proof By (4.8), the fact that the Fourier transform of $\tilde{\zeta}\Psi_t^+ f$ is supported in $\mathcal{C}^+$ up to smooth term, and Proposition 4.3, we have

$$\begin{aligned}
Q_{b,+}(\tilde{\zeta}\Psi_t^+ f, \tilde{\zeta}\Psi_t^+ f) \geq tA \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \mathrm{Re}\left(\left(c_{jk} - \frac{i\langle d\gamma, \Upsilon_q\rangle \delta_{jk}}{q}\right)\tilde{\zeta}\Psi_t^+ f_{jI}, \tilde{\zeta}\Psi_t^+ f_{kI}\right)_{\phi^+} \\
+ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\left(\Theta_{jk}^{\phi^+} - \frac{i\left\langle \Theta^{\phi^+}, \Upsilon_q\right\rangle \delta_{jk}}{q}\right)\tilde{\zeta}\Psi_t^+ f_{jI}, \tilde{\zeta}\Psi_t^+ f_{kI}\right)_{\phi^+} \\
- \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(\frac{1}{2}\nu(\phi^+)\left(\left(c_{jk} - \frac{i\langle d\gamma, \Upsilon_q\rangle \delta_{jk}}{q}\right)\right)\tilde{\zeta}\Psi_t^+ f_{jI}, \tilde{\zeta}\Psi_t^+ f_{kI}\right)_{\phi^+} \\
- O(\|\tilde{\zeta}\Psi_t^+ f\|_{\phi^+}^2) - O_t(\|\tilde{\zeta}\tilde{\Psi}_t^0 f\|_0^2)
\end{aligned}$$

By choosing $A \geq \sup_{z\in M} \frac{1}{2}\left|\nu(|z|^2)\right|$, Lemma 4.2 implies that

$$Q_{b,+}(\tilde{\zeta}\Psi_t^+ f, \tilde{\zeta}\Psi_t^+ f) + C\|\tilde{\zeta}\Psi_t^+ f\|_{\phi^+}^2 + O_t(\|\tilde{\zeta}\tilde{\Psi}_t^0 f\|_0^2) \geq tB_{\phi^+}\|\tilde{\zeta}\Psi_t^+ f\|_{\phi^+}^2$$

for some constants C and B_{ϕ^+} where B_{ϕ^+} satisfies $\left|q - \omega(\Upsilon_q)\right| > B_{\phi^+}$ on M. □

In order to estimate the terms $Q_{b,-}(\tilde{\zeta}\Psi_t^- f, \tilde{\zeta}\Psi_t^- f)$ we have to modify the analysis slightly from the $Q_{b,+}$ case. Similarly to (4.5), we have

$$\begin{aligned}
Q_{b,\phi}(f, f) = \|\nabla_{\bar{L}^*,\phi} f\|_\phi^2 + \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(c_{jk}Tf_{jI}, f_{kI}\right)_\phi - \sum_{j=1}^{n-1}\left(c_{jj}Tf, f\right)_\phi \\
+ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left((\Theta_{jk}^\phi - \frac{1}{2}\nu(\phi)c_{jk})f_{jI}, f_{kI}\right)_\phi \\
- \sum_{j=1}^{n-1}\left((\Theta_{jj}^\phi - \frac{1}{2}\nu(\phi)c_{jj})f, f\right)_\phi \\
- O_\epsilon(\|\nabla_{\bar{L}^*,\phi} f\|_\phi^2 - \|\nabla_\Upsilon f\|_\phi^2) - O_\epsilon(\|\overline{\nabla}_\Upsilon f\|_\phi^2) \\
- O_{\frac{1}{\epsilon}}(\|f\|_\phi^2) - O(\|f\|_\phi^2).
\end{aligned} \tag{4.9}$$

Analogously to (4.6), we have

$$\begin{aligned}
\|\nabla_\Upsilon f\|_\phi^2 = \sum_{j,k=1}^{n-1}\left[\left(b^{\bar{k}j}\bar{L}_k f, \bar{L}_j f\right)_\phi + \left(b^{\bar{k}j}c_{jk}Tf, f\right)_\phi + \left(b^{\bar{k}j}(\Theta_{jk}^\phi - \frac{1}{2}\nu(\phi)c_{jk})f, f\right)_\phi\right] \\
- O_\epsilon(\|\nabla_{\bar{L}^*,\phi} f\|_\phi^2 - \|\nabla_\Upsilon f\|_\phi^2) - O_\epsilon(\|\overline{\nabla}_\Upsilon f\|_\phi^2) - O_{\frac{1}{\epsilon}}(\|f\|_\phi^2) - O(\|f\|_\phi^2).
\end{aligned} \tag{4.10}$$

It now follows from (4.9) and (4.10) that

$$
\begin{aligned}
Q_{b,\phi}(f,f) \geq& \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1} \operatorname{Re}\left(c_{jk}Tf_{jI}, f_{kI}\right)_\phi - \operatorname{Re}\left(\sum_{j=1}^{n-1} c_{jj}Tf, f\right)_\phi - O(\|f\|_\phi^2)\\
&+ \operatorname{Re}\left(i\langle d\gamma, \Upsilon\rangle Tf, f\right)_\phi + \left(i\left\langle \Theta^\phi, \Upsilon\right\rangle f, f\right)\\
&+ \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\Theta^\phi_{jk}f_{jI}, f_{kI}\right)_\phi - \left(\sum_{j=1}^{n-1}\Theta^\phi_{jj}f, f\right)\\
&- \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi)c_{jk}f_{jI}, f_{kI}\right)_\phi + \left(\frac{1}{2}\nu(\phi)\sum_{j=1}^{n-1}c_{jj}f, f\right)\\
&- \left(\frac{1}{2}\nu(\phi)i\langle d\gamma, \Upsilon\rangle f, f\right). \qquad (4.11)
\end{aligned}
$$

If we set

$$
h^-_{jk} = c_{jk} - \delta_{jk}\frac{i\langle d\gamma, \Upsilon\rangle}{n-1-q}, \quad \text{and} \quad h^\Theta_{jk} = \Theta^\phi_{jk} - \delta_{jk}\frac{i\left\langle\Theta^\phi, \Upsilon\right\rangle}{n-1-q},
$$

then we can rewrite (4.11) by

$$
\begin{aligned}
Q_{b,\phi}(f,f) \geq& -\operatorname{Re}\left(\sum_{j=1}^{n-1}h^-_{jj}Tf, f\right) + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\operatorname{Re}\left(h^-_{jk}Tf_{jI}, f_{kI}\right)\\
&- \left(\sum_{j=1}^{n-1}h^\Theta_{jj}f, f\right) + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(h^\Theta_{jk}f_{jI}, f_{kI}\right)\\
&+ \left(\frac{1}{2}\nu(\phi)\sum_{j=1}^{n-1}h^-_{jj}f, f\right) - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left(\frac{1}{2}\nu(\phi)h^-_{jk}f_{jI}, f_{kI}\right)\\
&- O(\|f\|^2_\phi)
\end{aligned}
$$

Since the sum of q eigenvalues of the matrix $\frac{Tr(H)}{q}Id - H$ is equal to sum of $(n-1-q)$ eigenvalues of the matrix H, we may now proceed as in the proof of (4.5) to obtain the following proposition.

Proposition 4.6 *Let* $f \in \operatorname{Dom}\bar\partial_b \cap \operatorname{Dom}\bar\partial_b^*$ *be a* $(0,q)$*-form supported in* U *and let* ϕ *be as in* (4.1)*. Then there exists a constant* C *so that*

$$
Q_{b,-}\left(\tilde\zeta\Psi^-_t f, \tilde\zeta\Psi^-_t f\right) + C\|\tilde\zeta\Psi^-_t f\|_{\phi^-} + O_t(\|\tilde\zeta\tilde\Psi^0_t f\|_0^2) \geq tB_{\phi^-}\|\tilde\zeta\Psi^-_t f\|^2_{\phi^-}.
$$

In contrast with the estimates in Lemmas (4.5) and (4.6) for forms supported on $\mathcal{C}^+$ and $\mathcal{C}^-$ up to smooth terms, we have better estimates for forms supported on $\mathcal{C}^0$ up to smooth terms. The next lemma can be proved like using the same process done in Lemma 4.17 and Lemma 4.18 on [16].

Lemma 4.7 *Let f be a (0,q)-form supported in U_μ for some μ such that up to smooth term, $\hat{f}$ is supported in $\widetilde{\mathcal{C}}^0_\mu$. There exist positive constants $C > 1$ and Γ independent of t for which*

$$C Q_{b,t}(f, E_t f) + \Gamma \|f\|_0^2 \geq \|f\|_1^2 \tag{4.12}$$

The other term appearing in our main estimate, $O\big(\|\tilde{\zeta}\widetilde{\Psi}^0_t\cdot\|_0^2\big)$, can be handled with [17, Proposition 4.11].

Proposition 4.8 *For any $\epsilon > 0$, there exists $C_{\epsilon,t} > 0$ so that*

$$\|\tilde{\zeta}\Psi^0_t\zeta\varphi_0^2\| \leq \epsilon Q_{b,t}(\varphi,\varphi) + C_{\epsilon,t}\|\varphi\|^2_{-1}.$$

We are finally ready to proof Proposition 4.1.

Proof of the Proposition 4.1 We only need to set the value of the constant K, K', and K_t in Lemma 3.3 according to the Propositions 4.5 and 4.6. From the definition of $|||\cdot|||_t$, the estimate (4.3) follows.

The passage from (4.3) to the basic estimate (4.2) follows immediately from Lemma 4.7 and Proposition 4.8. □

5 The Proof of Theorem 1.2

Now that we have the tools of Sect. 4, we can prove strong closed range estimates using many of the arguments of [9]. We do, however, use a substantially different elliptic regularization to pay particular attention to the regularity of the weighted harmonic forms, the relationship of the harmonic forms with the regularized operators, and an especially detailed look at the induction base case.

Lemma 5.1 (Lemma 5.1, [9]) *Let M be a smooth, embedded CR manifold of hypersurface type that satisfies $Y(q)$ weakly. If $t > 0$ is suitably large and the functions ϕ^+, ϕ^- are as in (4.1), then*

(i) *$\mathcal{H}^q_t$ is finite dimensional;*
(ii) *There exists C that does not depend on ϕ^+ and ϕ^- so that for all $(0,q)$-forms $u \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$ satisfying $u \perp \mathcal{H}^q_t$ (with respect to $(\cdot,\cdot)_t$) we have*

$$|||u|||_t^2 \leq C Q_{b,t}(u,u). \tag{5.1}$$

By [5, Theorem 1.1.2], $\bar{\partial}_b : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q+1}(M, |||\cdot|||_t)$ and $\bar{\partial}^*_{b,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q-1}(M, |||\cdot|||_t)$ have closed range. Consequently, their adjoints $\bar{\partial}_b : L^2_{0,q-1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ and $\bar{\partial}^*_{b,t} : L^2_{0,q+1}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t)$ have closed range as well [5, Theorem 1.1.1].

5.1 Continuity of the Green operator $G_{q,t}$

The complex Green operator $G_{q,t}$ is the inverse to $\Box_{b,t}$ on $\mathcal{H}_{q,t}^{\perp}(M)$ (and is defined to be 0 on $\mathcal{H}_{q,t}(M)$). Recall the following well-known lemma. See, e.g., [4,16].

Lemma 5.2 *Let H be a Hilbert space equipped with the inner product $(\cdot,\cdot)$, corresponding norm $\|\cdot\|$, and a positive definite Hermitian form Q defined on a dense subset $D \subset H$ satisfying*

$$\|\varphi\|^2 \le C Q(\varphi,\varphi) \tag{5.2}$$

for all $\varphi \in D$. Furthermore, D and Q are such that D is a Hilbert space under the inner product $Q(\cdot,\cdot)$. Then there exists a unique self-adjoint injective operator F with $\mathrm{Dom}(F) \subset D$ *satisfying*

$$Q(\varphi,\phi) = (F\varphi,\phi)$$

for all $\varphi \in \mathrm{Dom}(F)$ and $\phi \in D$. F is called the Friedrich's representative.

In order to use the result above, we prove a density result on ${}^{\perp}\mathcal{H}_t^q(M)$.

Lemma 5.3 $\big(\mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M), Q_{b,t}(\cdot,\cdot)^{1/2}\big)$ *is a Hilbert space (for $(0,q)$-forms), and* $\mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M)$ *is dense in ${}^{\perp}\mathcal{H}_t^q$.*

Proof Suppose $\{u_\ell\} \subset \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M)$ is a Cauchy sequence with respect to the norm $Q_{b,t}(\cdot,\cdot)^{1/2}$. Then $\bar\partial_b u_\ell$ and $\bar\partial_{b,t}^* u_\ell$ are Cauchy sequences in $L^2_{0,q+1}(M, |||\cdot|||_t)$ and $L^2_{0,q-1}(M, |||\cdot|||_t)$, respectively, so they converge to $v_1 \in L^2_{0,q+1}(M, |||\cdot|||_t)$ and $v_2 \in L^2_{0,q-1}(M, |||\cdot|||_t)$, respectively. By (5.1), this means $\{u_\ell\}$ is a Cauchy sequence in $L^2_{0,q}(M, |||\cdot|||_t)$, and hence converges to some $u \in L^2_{0,q}(M, |||\cdot|||_t)$. Thus $u \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$, $\bar\partial_b u = v_1$, and $\bar\partial_{b,t}^* u = v_2$ since $\bar\partial_b$ and $\bar\partial_{b,t}^*$ are closed operators. Since $0 = (u_\ell, w)_t$ for all $w \in \mathcal{H}_t^q$ and $|||u_\ell - u|||_t \to 0$, $u \in {}^{\perp}\mathcal{H}_t^q(M)$. Thus $u \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) \cap {}^{\perp}\mathcal{H}_t^q$.

Next, suppose $u \in {}^{\perp}\mathcal{H}_t^q(M)$ is nonzero and $u_\ell \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$ satisfies $u_\ell \to u$ on $L^2_{0,q}(M, |||\cdot|||_t)$. Let $v_\ell = (I - H_t^q)u_\ell$, with H_t^q the orthogonal projection onto $\mathcal{H}_t^q$. The forms $v_\ell \in {}^{\perp}\mathcal{H}_t^q(M) \cap \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$. Since $u \neq 0$, it cannot be the case that $v_\ell = 0$ for every ℓ. Since $|||u_\ell|||_t^2 = \big|\big|\big|H_t^q u_\ell\big|\big|\big|_t^2 + |||v_\ell|||_t^2$, and the forms $H_t^q u_\ell$ and v_ℓ are orthogonal, $H_t^q u_\ell$ and v_ℓ both converge in $L^2_{0,q}(M, |||\cdot|||_t)$. Let $\alpha = \lim_{\ell\to\infty} H_t^q u_\ell$, $v = \lim_{\ell\to\infty} v_\ell$, and since $H_t^q u_\ell = u_\ell - v_\ell$, $\alpha = u - v \in {}^{\perp}\mathcal{H}_t^q(M)$. However, $\alpha \in \mathcal{H}_t^q$ since $\mathcal{H}_t^q$ is closed, forcing $\alpha = 0$. Thus, $|||u - v_\ell|||_t \le |||u - u_\ell|||_t + \big|\big|\big|H_t^q u_\ell\big|\big|\big|_t \to 0$. Consequently $\mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) \cap {}^{\perp}\mathcal{H}_t^q(M)$ is dense in ${}^{\perp}\mathcal{H}_t^q(M)$. □

We now can establish the existence and L^2-continuity of the complex Green operator $G_{q,t}$ using the following well-known result (we adapt the presentation and argument in [16, Corollary 5.5]).

Corollary 5.4 *Let M be a smooth compact, orientable embedded CR manifold of hypersurface type that satisfies weak $Y(q)$. If $t > 0$ is suitable large, ϕ^+, ϕ^- are as in (4.1),* *and $\alpha \in {}^\perp\mathcal{H}_t^q$, then there exists a unique $\varphi_t \in {}^\perp\mathcal{H}_t^q \cap \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$ such that*

$$Q_{b,t}(\varphi_t, \phi) = (\alpha, \phi)_t\,, \qquad \text{for all } \phi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*).$$

We define the Green operator $G_{q,t}$ to be the operator that maps α into φ_t. $G_{q,t}$ is a bounded operator, and if additionally α is closed, then $u_t = \bar\partial_{b,t}^* G_{q,t}\alpha$ satisfies $\bar\partial_b u_t = \alpha$. We define $G_{q,t}$ to be identically 0 on $\mathcal{H}_t^q$.

5.2 Smoothness of Harmonic Forms

Here we will prove that $\mathcal{H}_t^q \subset H^s_{0,q}(M, |||\cdot|||_t)$ for t sufficiently large. We adapt the arguments of [7,14]. See also [12,16].

Fix $s \ge 1$. For forms $f, g \in H^1_{0,q}(M, |||\cdot|||_t)$, set

$$Q^{\delta,\nu}_{b,t}(f,g) = Q_{b,t}(f,g) + \delta Q_{d_b}(f,g) + \nu\,(f,g)_t$$

where $Q_{d_b}(\cdot,\cdot)$ is the Hermitian inner product associated to the de Rham exterior derivative d_b, i.e., $Q_{d_b}(u,v) = (d_b u, d_b v)_t + \left(d^*_{b,t}u, d^*_{b,t}v\right)_t$, and $\delta, \nu \ge 0$. Also note that $Q^{0,\nu}_{b,t}(f,g) = Q_{b,t}(f,g) + \nu\,(f,g)_t$ for $f, g \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$. Then

$$|||\varphi|||_t^2 \le \frac{1}{\nu} Q^{\delta,\nu}_{b,t}(\varphi,\varphi).$$

for all $\varphi \in H^1_{0,q}(M, |||\cdot|||_t)$ if $\delta > 0$ and all $\varphi \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*)$ if $\delta = 0$. By the Lemma 5.2 there exist self-adjoint operators (for $0 \le \delta \le 1$ and $0 < \nu \le 1$) $\Box^{\delta,\nu}_{b,t} : \mathrm{Dom}(\Box^{\delta,\nu}_{b,t}) \to L^2_{0,q}(M, |||\cdot|||_t)$, with inverses $G^{\delta,\nu}_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to \mathrm{Dom}(\Box^{\delta,\nu}_{b,t})$ satisfying

$$\left|\left|\left|G^{\delta,\nu}_{q,t}\varphi\right|\right|\right|_t^2 \le \frac{1}{\nu}\,|||\varphi|||_t^2 \tag{5.3}$$

for all $\varphi \in L^2_{0,q}(M, |||\cdot|||_t)$ and all $\delta \in [0,1]$.

Our goal is to prove

$$\|G^{0,\nu}_{q,t}\varphi\|_{H^s} \le K_t \|\varphi\|_{H^s} + C_{t,s}\|G^{0,\nu}_{q,t}\varphi\|_0. \tag{5.4}$$

In fact, (5.4) is the main tool that we need to prove that $\mathcal{H}_t^q(M) \subset H^s_{0,q}(M, |||\cdot|||_t)$, for t sufficiently large. Given (5.4), the argument for regularity of the harmonic forms follows nearly verbatim from [12, Proposition 5.2], from equation (5.20) onwards. Equation (5.4) plays the role of [12, (5.20)].

We now prove (5.4). The operator $\Box^{\delta,\nu}_{b,t}$ is elliptic when $\delta > 0$ which means that $G^{\delta,\nu}_{q,t} : H^s_{0,q}(M, |||\cdot|||_t) \to H^{s+2}_{0,q}(M, |||\cdot|||_t)$.

If $\varphi \in H^s_{0,q}(M, |||\cdot|||_t)$, then

$$\|G^{\delta,\nu}_{q,t}\varphi\|^2_{H^s} = \|\Lambda^s G^{\delta,\nu}_{q,t}\varphi\|_0^2 \le C_t \left|\left|\left|\Lambda^s G^{\delta,\nu}_{q,t}\varphi\right|\right|\right|_t^2.$$

Since $G_{q,t}^{\delta,\nu}\varphi \in H_{0,q}^{s+2}(M, |||\cdot|||_t)$, the basic estimate yields

$$\left|\left|\left|\Lambda^s G_{q,t}^{\delta,\nu}\varphi\right|\right|\right|_t^2 \leq \frac{K}{t} Q_{b,t}^{\delta,\nu}(\Lambda^s G_{q,t}^{\delta,\nu}\varphi, \Lambda^s G_{q,t}^{\delta,\nu}\varphi) + C_{t,s}\|G_{q,t}^{\delta,\nu}\varphi\|_{H^{s-1}}^2 \tag{5.5}$$

A careful integration by parts shows that

$$\begin{aligned}
&\left|\left|\left|\bar\partial_b \Lambda^s G_{q,t}^{\delta,\nu}\varphi\right|\right|\right|_t^2 \\
&\quad = \left\langle \Lambda^s \bar\partial_{b,t}^* \bar\partial_b G_{q,t}^{\delta,\nu}\varphi, \Lambda^s G_{q,t}^{\delta,\nu}\varphi\right\rangle + \left\langle \bar\partial_b \Lambda^s G_{q,t}^{\delta,\nu}\varphi, \left([\Lambda^s, \bar\partial_b] + \Lambda^{-s}\left[[\Lambda^s, \bar\partial_b], \Lambda^s\right]\right) G_{q,t}^{\delta,\nu}\varphi\right\rangle \\
&\qquad + \left\langle [\Lambda^s, \bar\partial_b] G_{q,t}^{\delta,\nu}\varphi, \left([\Lambda^s, \bar\partial_b] + \Lambda^{-s}\left[[\Lambda^s, \bar\partial_b], \Lambda^s\right]\right) G_{q,t}^{\delta,\nu}\varphi\right\rangle + \left\langle [\bar\partial_b, \Lambda^s] G_{q,t}^{\delta,\nu}\varphi, \bar\partial_b \Lambda^s G_{q,t}^{\delta,\nu}\varphi\right\rangle.
\end{aligned}$$

We next apply the same sequence of integration by parts and commutators to the other terms in $Q_{b,t}^{\delta,\nu}(\Lambda^s G_{q,t}^{\delta,\nu}\varphi, \Lambda^s G_{q,t}^{\delta,\nu}\varphi)$. Using a small constant/large constant argument and the fact that $\bar\partial_{b,t}^* = \bar\partial_b^* + tP_0$ where P_0 is a (pseudo)differential operator of order 0, we can absorb terms to obtain

$$Q_{b,t}^{\delta,\nu}(\Lambda^s G_{q,t}^{\delta,\nu}\varphi, \Lambda^s G_{q,t}^{\delta,\nu}\varphi) \leq C\, \left|\left|\left|\Lambda^s\varphi\right|\right|\right|_t^2 + C_s \left|\left|\left|\Lambda^s G_{q,t}^{\delta,\nu}\varphi\right|\right|\right|_t^2 + C_{t,s}\|G_{q,t}^{\delta,\nu}\varphi\|_{H^{s-1}}^2 \tag{5.6}$$

where C does not depend t, s, δ, or ν, and C_s does not depend on t, δ, or ν. By (5.5), for t sufficiently large

$$\|G_{q,t}^{\delta,\nu}\varphi\|_{H^s}^2 \leq K_t\|\varphi\|_{H^s}^2 + C_{t,s}\|G_{q,t}^{\delta,\nu}\varphi\|_{H^{s-1}}^2.$$

By induction, we can reduce the H^{s-1}-norm to an L^2-norm, and by (5.3), we observe

$$\|G_{q,t}^{\delta,\nu}\varphi\|_{H^s}^2 \leq K_t\|\varphi\|_{H^s}^2 + C_{t,s,\nu}\|\varphi\|_0^2,$$

uniformly in $\delta > 0$. Then there exists a sequence $\{G_{q,t}^{\delta_k,\nu}\varphi\}_k$ converging weakly to an element u_ν in $H_{0,q}^s(M, |||\cdot|||_t)$ when $\delta_k \to 0$, and satisfying both

$$\|u_\nu\|_{H^s} \leq K_t\|\varphi\|_{H^s} + C_{t,s,\nu}\|\varphi\|_0 \quad \text{and} \quad \|u_\nu\|_{H^s} \leq K_t\|\varphi\|_{H^s} + C_{t,s}\|u_\nu\|_0. \tag{5.7}$$

Since $H_{0,q}^s(M, |||\cdot|||_t)$ embeds compactly in $H_{0,q}^{s'}(M, |||\cdot|||_t)$, it follows that $G_{q,t}^{\delta_k,\nu}\varphi \to u_\nu$ strongly in $H_{0,q}^{s'}(M, |||\cdot|||_t)$ for $0 \leq s' < s$. Also, observe that the next conclusion is not automatic in the $s = 1$ case.

$$\left\|\!\left|\bar\partial_b G^{\delta,\nu}_{q,t}\varphi\right|\!\right\|_t^2 + \left\|\!\left|\bar\partial^*_{b,t} G^{\delta,\nu}_{q,t}\varphi\right|\!\right\|_t^2 \le Q^{\delta,\nu}_{q,t}(G^{\delta,\nu}_{q,t}\varphi, G^{\delta,\nu}_{q,t}\varphi)$$
$$= \left(\varphi, G^{\delta,\nu}_{q,t}\varphi\right)_t \le |||\varphi|||_t \left\|\!\left|G^{\delta,\nu}_{q,t}\varphi\right|\!\right\|_t \le C_\nu\, |||\varphi|||_t^2\,, \tag{5.8}$$

and, moreover, $\bar\partial_b G^{\delta_k,\nu}_{q,t}\varphi$ and $\bar\partial^*_{b,t} G^{\delta_k,\nu}_{q,t}\varphi$ are Cauchy sequences in L^2. Indeed, assuming $\delta_k \le \delta_j$ we have

$$\begin{aligned}
&\left\|\!\left|\bar\partial_b G^{\delta_k,\nu}_{q,t}\varphi - \bar\partial_b G^{\delta_j,\nu}_{q,t}\varphi\right|\!\right\|_t^2 + \left\|\!\left|\bar\partial^*_{b,t} G^{\delta_k,\nu}_{q,t}\varphi - \bar\partial^*_{b,t} G^{\delta_j,\nu}_{q,t}\varphi\right|\!\right\|_t^2 \\
&\le Q^{\delta_k,\nu}_{b,t}(G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi) \\
&= \left(\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right)_t - Q^{\delta_k,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_k,\nu}_{q,t}\varphi) + Q^{\delta_k,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_j,\nu}_{q,t}\varphi) \\
&\le \left(\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right)_t - Q^{\delta_k,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_k,\nu}_{q,t}\varphi) + Q^{\delta_j,\nu}_{q,t}(G^{\delta_j,\nu}_{q,t}\varphi, G^{\delta_j,\nu}_{q,t}\varphi) \\
&= \left(\varphi, G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right)_t - \left(G^{\delta_j,\nu}_{q,t}\varphi, \varphi\right)_t + \left(\varphi, G^{\delta_j,\nu}_{q,t}\varphi\right)_t \\
&\le |||\varphi|||_t \left\|\!\left|G^{\delta_k,\nu}_{q,t}\varphi - G^{\delta_j,\nu}_{q,t}\varphi\right|\!\right\|_t.
\end{aligned}$$

Since $\bar\partial_b$ and $\bar\partial^*_{b,t}$ are closed operators it follows that $u_\nu \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial^*_b)$, $\bar\partial_b G^{\delta_k,\nu}_{q,t}\varphi \to \bar\partial_b u_\nu$ and $\bar\partial^*_{b,t} G^{\delta_k,\nu}_{q,t}\varphi \to \bar\partial^*_{b,t} u_\nu$ in L^2. This means $G^{\delta_k,\nu}_{q,t}\varphi$ converges strongly to u_ν in the $Q^{0,\nu}_{b,t}(\cdot,\cdot)^{1/2}$-norm. Thus, we will have, for any $v \in H^2_{0,q}(M, |||\cdot|||_t)$, by (5.3),

$$\begin{aligned}
\left|Q^{0,\nu}_{b,t}(G^{\delta_k,\nu}_{q,t}\varphi - G^{0,\nu}_{q,t}\varphi, v)\right| &= \Big|Q^{\delta_k,\nu}_{b,t}(G^{\delta_k,\nu}_{q,t}\varphi, v) - \delta_k\left(d_b G^{\delta_k,\nu}_{q,t}\varphi, d_b v\right)_t \\
&\quad -\delta_k\left(d^*_{b,t} G^{\delta_k,\nu}_{q,t}\varphi, d^*_{b,t} v\right)_t - (\varphi, v)_t\Big| \\
&= \delta_k\left|\left(G^{\delta_k,\nu}_{q,t}\varphi, (d^*_{b,t}d_b + d_b d^*_{b,t})v\right)_t\right| \le \delta_k C_{\nu,t}\, |||\varphi|||_t\, \|v\|_2.
\end{aligned}$$

It now follows that $G^{0,\nu}_{q,t}\varphi = u_\nu$ and by (5.7), (5.4) now follows.

5.3 Regularity of the Green Operator and the Canonical Solutions

In this section we assume t is sufficiently large and the weighted harmonic $(0,q)$-forms, if they exist, are elements of $H^1_{0,q}(M) \neq \{0\}$. We use an elliptic regularization argument. The operator $G_{q,t} : L^2_{0,q}(M, |||\cdot|||_t) \to L^2_{0,q}(M, |||\cdot|||_t) \cap {}^\perp\mathcal{H}^q_t(M)$. Consequently, the regularity result for $G_{q,t}$ must be on ${}^\perp\mathcal{H}^q_t(M) \cap H^s_{0,q}(M)$ for $s \ge 0$. Continuity on all of $H^s_{0,q}(M)$ then follows because we already established that harmonic forms are elements of $H^s_{0,q}(M)$.

The quadratic form $Q^\delta_{q,t}(\cdot,\cdot) := Q^{\delta,0}_{q,t}(\cdot,\cdot)$ is an inner product on $H^1_{0,q}(M)$. By (5.1),

$$|||u|||_t^2 \le C Q_{b,t}(u,u) \le C Q^\delta_{b,t}(u,u) \tag{5.9}$$

for all $u \in H^1_{0,q}(M) \cap {}^\perp\mathcal{H}^q_t(M)$. If $f \in L^2_{0,q}(M)$, then

$$\left|(f, g)_t\right| \leq |||f|||_t \, |||g|||_t \leq |||f|||_t \, C^{1/2} Q^\delta_{b,t}(g, g)$$

for all $g \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$. This means the mapping $g \mapsto (f, g)_t$ is a bounded conjugate linear functional on ${}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$. By the Riesz Representation Theorem, there exists an element $G^\delta_{q,t} f \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$ such that $(f, g)_t = Q^\delta_{b,t}(G^\delta_{q,t} f, g)$ for all $g \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$. Moreover, by (5.9)

$$C^{-1} \left|\left|\left| G^\delta_{q,t} f \right|\right|\right|^2_t \leq Q^\delta_{b,t}(G^\delta_{q,t} f, G^\delta_{q,t} f) = \left(f, G^\delta_{q,t} f\right)_t \leq |||f|||_t \, |||G^\delta_{q,t} f|||_t$$

where C is independent of δ. Consequently,

$$\left|\left|\left| G^\delta_{q,t} f \right|\right|\right|_t \leq C \, |||f|||_t \tag{5.10}$$

Since $Q^\delta_{b,t}(\cdot, \cdot)$ satisfies $Q^\delta_{b,t}(f, f) \geq \delta \left|\left|\left|\Lambda^1 f\right|\right|\right|^2_t$ for every $f \in H^1_{0,q}(M)$, the bilinear form $Q^\delta_{b,t}(\cdot, \cdot)$ is elliptic on $H^1_{0,q}(M)$. This means that $\varphi \in H^s_{0,q}(M)$ implies $G^\delta_{q,t}\varphi \in H^{s+2}_{0,q}(M)$ (before, we only knew that $G^\delta_{q,t}\varphi \in {}^\perp\mathcal{H}^q_t(M) \cap H^1_{0,q}(M)$).

Let $\varphi \in H^s_{0,q}(M)$, then

$$\|G^\delta_{q,t}\varphi\|^2_{H^s} = \|\Lambda^s G^\delta_{q,t}\varphi\|^2_0 \leq C_t \left|\left|\left|\Lambda^s G^\delta_{q,t}\varphi\right|\right|\right|^2_t. \tag{5.11}$$

We apply the basic estimate to $G^\delta_{q,t}\varphi \in H^{s+2}_{0,q}(M)$ and observe

$$\left|\left|\left|\Lambda^s G^\delta_{q,t}\varphi\right|\right|\right|^2_t \leq \frac{K}{t} Q_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}}. \tag{5.12}$$

Using the argument of (5.6), we can establish

$$\begin{aligned} Q_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) &\leq Q^\delta_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) \\ &\leq C \left|\left|\left|\Lambda^s\varphi\right|\right|\right|^2_t + C_s \left|\left|\left|\Lambda^s G^\delta_{q,t}\varphi\right|\right|\right|^2_t + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}} \end{aligned} \tag{5.13}$$

where C is independent of t, s, δ, and ν and C_s is independent of t, δ, and ν.

Plugging (5.13) into (5.12) and choosing t sufficiently large to absorb terms, we have

$$\left|\left|\left|\Lambda^s G^\delta_{q,t}\varphi\right|\right|\right|^2_t \leq K_t\|\varphi\|^2_{H^s} + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}}, \tag{5.14}$$

since $\left|\left|\left|\Lambda^s G^\delta_{q,t}\varphi\right|\right|\right|_t < \infty$. Plugging (5.14) into (5.11), it follows that

$$\|G^\delta_{q,t}\varphi\|^2_{H^s} \leq K_t\|\varphi\|^2_{H^s} + C_{t,s}\|G^\delta_{q,t}\varphi\|^2_{H^{s-1}}.$$

Using (5.10) and induction, we estimate

$$\|G^{\delta}_{q,t}\varphi\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|\varphi\|^2_0. \tag{5.15}$$

With (5.15) in hand, we now turn to sending $\delta \to 0$, in a similar manner to [9]. If $\varphi \in H^s_{0,q}(M)$, then $\left\{G^{\delta}_{q,t}\varphi : 0 < \delta < 1\right\}$ is bounded in $H^s_{0,q}(M)$, so there exists $\delta_k \to 0$ and $\tilde u \in H^s_{0,q}(M)$ so that $G^{\delta_k}_{q,t}\varphi \to \tilde u$ weakly in $H^s_{0,q}(M)$. Since the inclusion of $H^s_{0,q}(M)$ in $L^2_{0,q}(M)$ is compact, we have $G^{\delta_k}_{q,t}\varphi \to \tilde u$ strongly in $L^2_{0,q}(M)$ and $\tilde u \in {}^{\perp}\mathcal{H}^q_t(M)$. Also

$$\|\tilde u\|^2_{H^s} \le K_t\|\varphi\|^2_{H^s} + C_{t,s}\|\varphi\|^2_0. \tag{5.16}$$

Also,

$$\begin{aligned}\left|\!\left|\!\left|\bar\partial_b G^{\delta}_{q,t}\varphi\right|\!\right|\!\right|^2_t + \left|\!\left|\!\left|\bar\partial^*_{b,t} G^{\delta}_{q,t}\varphi\right|\!\right|\!\right|^2_t &\le Q^{\delta}_{b,t}(G^{\delta}_{q,t}\varphi, G^{\delta}_{q,t}\varphi)\\ &= \left(\varphi, G^{\delta}_{q,t}\varphi\right)_t \le |\!|\!|\varphi|\!|\!|_t \left|\!\left|\!\left|G^{\delta}_{q,t}\varphi\right|\!\right|\!\right| \le C_t\,|\!|\!|\varphi|\!|\!|^2_t,\end{aligned}$$

and, as in the previous section, we can prove $\bar\partial_b G^{\delta_k}_{q,t}\varphi$ and $\bar\partial^*_{b,t}G^{\delta_k}_{q,t}\varphi$ are Cauchy sequences in $L^2_{0,q}(M)$. Since $\bar\partial_b$ and $\bar\partial^*_{b,t}$ are closed operators we will have $\tilde u \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial^*_b)$, $\bar\partial_b G^{\delta}_{q,t}\varphi \to \bar\partial_b\tilde u$ and $\bar\partial^*_{b,t}G^{\delta}_{q,t}\varphi \to \bar\partial^*_{b,t}\tilde u$ in $L^2_{0,q}(M)$, and

$$|\!|\!|\bar\partial_b\tilde u|\!|\!|^2_t + |\!|\!|\bar\partial^*_{b,t}\tilde u|\!|\!|^2_t \le C_t\,|\!|\!|\varphi|\!|\!|^2_t\,. \tag{5.17}$$

Consequently if $v \in H^{s+2}_{0,q}(M)$, then $\lim Q^{\delta_k}_{b,t}(G^{\delta_k}_{q,t}\varphi, v) = Q_{b,t}(\tilde u, v)$. However, $Q^{\delta_k}_{b,t}(G^{\delta_k}_{q,t}\varphi, v) = (\varphi, v)_t = Q_{b,t}(G_{q,t}\varphi, v)$. So by uniqueness $G_{q,t}\varphi = \tilde u$ and (5.16) we have

$$\|G_{q,t}\varphi\|^2_{H^s} \le K_t\,|\!|\!|\varphi|\!|\!|^2_{H^s} + C_{t,s}\|\varphi\|^2_0, \tag{5.18}$$

and by (5.17)

$$|\!|\!|\bar\partial_b G_{q,t}\varphi|\!|\!|^2_t + |\!|\!|\bar\partial^*_{b,t}G_{q,t}\varphi|\!|\!|^2_t \le C_t\,|\!|\!|\varphi|\!|\!|^2_t\,. \tag{5.19}$$

These two last equations prove the continuity of $G_{q,t}$ on $H^s_{0,q}(M)$ and as well as $\bar\partial_b G_{q,t}$ and $\bar\partial^*_{b,t}G_{q,t}$ on $L^2_{0,q}(M)$.

The remainder of the proof of Theorem 1.2 follows from (by now) standard arguments. See, e.g., the proof of [9, Theorem 1.2], and Sect. 6, in particular.

6 Proof of the Theorem 1.1

Since the $L^2(M, |\!|\!|\cdot|\!|\!|_t)$ and $L^2(M)$ are equivalent spaces, it is immediate that $\bar\partial_b : L^2_{0,\tilde q-1}(M) \to L^2_{0,\tilde q}(M)$ has closed range for $\tilde q = q$ or $q+1$. Moreover, by [5, Theorem 1.1.1], their adjoints $\bar\partial^*_b : L^2_{0,\tilde q}(M) \to L^2_{0,\tilde q-1}(M)$, $\tilde q = q$ or $q+1$ have closed range as well. Moreover, the dimension of the space of harmonic $(0,q)$-forms is independent of the weight and is therefore finite (see, e.g., [18, p.772] or [12]). Standard arguments now establish the rest of Theorem 1.1.

7 Examples

In this section, we modify the main example of [10] and show how the flexibility of choosing Υ makes it easier to verify than the older weak $Y(q)$ condition of [9].

Let $M \subset \mathbb{C}^5$ be the boundary of a domain Ω so that on neighborhood U of the origin so that

$$M \cap U = \{z = (z_1, \dots, z_5) \in \mathbb{C}^5 : \operatorname{Im} z_5 = P(z_1, z_2, z_3, z_4)\}.$$

We set

$$\rho(z) = P(z_1, z_2, z_3, z_4) - \operatorname{Im} z_5$$

where the polynomial

$$P(z_1, z_2, z_3, z_4) = 2x_1|z_2|^2 - x_1 y_1^4 + |z_3|^2 + |z_4|^2.$$

Observe that

$$\bar\partial\rho = \left(|z_2|^2 - \frac{1}{2}y_1^4 - 2ix_1y_1^3\right) d\bar z_1 + 2x_1 z_2\, d\bar z_2 + \bar z_3\, d\bar z_3 + \bar z_4\, d\bar z_4 - \frac{i}{2}\, d\bar z_5$$

and

$$\begin{aligned}\partial\bar\partial\rho = &-3x_1y_1^2\, dz_1 \wedge d\bar z_1 + z_2\, dz_1 \wedge d\bar z_2 + \bar z_2\, dz_2 \wedge d\bar z_1 \\ &+2x_1\, dz_2 \wedge d\bar z_2 + dz_3 \wedge d\bar z_3 + dz_4 \wedge d\bar z_4.\end{aligned}$$

We choose a basis for $T^{1,0}(M \cap U)$ by setting

$$L_j = \frac{\partial}{\partial z_j} + 2i\frac{\partial P}{\partial z_j}\frac{\partial}{\partial z_5}, \quad 1 \le j \le 4.$$

In this basis, we can represent the Levi form by the 4×4 matrix

$$(c_{j\bar k}) = \mathcal{L}_{\rho_1}(i\bar L_k \wedge L_j) = i\partial\bar\partial\rho\left(i\frac{\partial}{\partial\bar z_k} \wedge \frac{\partial}{\partial z_j}\right) = \begin{pmatrix} -3x_1y_1^2 & z_2 & 0 & 0 \\ \bar z_2 & 2x_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\rho_{j\bar k}). \tag{7.1}$$

Since (c_{jk}) has three positive eigenvalues whenever either $z_2 \neq 0$ or both $x \neq 0$ and $y \neq 0$, it follows that $Z(2)$ is satisfied on a dense subset of $M \cap U$.

Proposition 7.1 *The CR manifold M satisfies weak $Y(2)$ on $M \cap U$.*

Proof The construction of Υ in the proof of [10, pp. 1747–1748] works here as well. Moreover, since $\mu_3 > 0$, it is immediate that we can use the same form Υ for both the weak $Z(2) = Z(5-2-1)$ and weak $Z(3)$ cases. □

Showing that the older weak $Z(2)$ condition fails is quite difficult—showing that the condition fails in *all* choices of coordinates amounts to solving a nonlinear problem. Specifically, we know that the signature of the Levi form does not change, but the eigenvalues certainly can. Computing eigenvalues after coordinate changes or changes of metric is nonlinear and is already quite difficult in the 4×4 case. We also point out that none of the weak $Y(q)$ conditions are invariant under the metric as an example from [10] shows (no condition that depends on sums of eigenvalues is likely to be invariant under changes of metric).

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