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Geometric Pluripotential Theory on Sasaki Manifolds

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Abstract
We extend profound results in pluripotential theory on Kähler manifolds (Darvas in arXiv:1902.01982, 2019) to Sasaki setting via its transverse Kähler structure. As in Kähler case, these results form a very important piece to solve the existence of Sasaki metrics with constant scalar curvature in terms of properness of $\mathcal{K}$-energy, considered by the first named author in He (arXiv:1802.03841, 2019). One main result is to generalize Darvas' theory on the geometric structure of the space of Kähler potentials in Sasaki setting. Along the way we extend most of corresponding results in pluripotential theory to Sasaki setting via its transverse Kähler structure.

Keywords Sasaki structure · Transverse Kähler potential · Orlicz–Finsler geometry · Pythagorean formula

Mathematics Subject Classification 53C25 · 32U15

1 Introduction

Sasaki manifolds have gained their prominence in physics, algebraic geometry, and Riemannian geometry [13]. There are tremendous work in the last two decades in Sasaki geometry, in particular on Sasaki–Einstein manifolds, see [13,14,27,37,39,50, 54] and reference therein. On the other hand, Sasaki geometry is an odd- dimensional analog of Kähler geometry and almost all results in Kähler geometry have their counterparts in Sasaki geometry. Calabi's extremal metric [17,18] (and csck) has played a very important role in Kähler geometry and it has a direct adaption in Sasaki set-

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ting [16]. In 1997, Donaldson [34] proposed an extremely fruitful program to approach existence of csck (extremal metrics) on a compact Kähler manifold with a fixed Kähler class. Donaldson's program has also been extended to Sasaki setting, see [42,46] for example.

A major problem in Kähler geometry is to characterize exactly when a Kähler class contains a csck (extremal). The analytic part for existence of csck is to solve a fourth-order highly non-linear elliptic equation, the scalar curvature-type equation. This problem is regarded as a very hard problem in the field. Recently Chen and Cheng [22–24] have solved a major conjecture that existence of csck is equivalent to well-studied conditions such as properness of Mabuchi's K-energy, or geodesic stability. The first named author [49] proved the following counterpart in Sasaki setting,

Theorem 1 [49] *There exists a Sasaki metric with constant scalar curvature if and only if the $\mathcal{K}$-energy is reduced proper with respect to $Aut_0(\xi, J)$, the identity component of automorphism group which preserves the Reeb vector field and transverse complex structure.*

The proof of Theorem 1 is an adaption of recent breakthrough of Chen–Cheng [24] on the existence of csck in Kähler setting to Sasaki setting. Technically, the arguments consist of two major parts: a priori estimates of non-linear PDE and pluripotential theory. Building up on previous development of pluripotential theory, Darvas [28,29] has developed profound theory to study the geometric structure of space of Kähler potentials. Among others, he introduced a Finsler metric d_1, and proved very effective estimates of distance function d_1 in terms of well-studied energy functionals such as Aubin's I-functional. Darvas's results turn out to be very useful to understand the geometric structure of space of Kähler potentials, in particular in the study of csck [6,24,32]. In this paper, we extend many results in pluripotential theory on Kähler manifolds, notably in [28,29,44] to Sasaki setting. These results play an important role in the proof of Theorem 1. To prove these results, we need to explore the geometric structures of Sasaki manifolds, in particular the Kähler cone structure and transverse Kähler structure.

Let (M, g) be a compact Riemannian manifold of dimension $2n+1$, with a Riemannian metric g. Sasaki manifolds have very rich geometric structures and have many equivalent descriptions. A probably most straightforward formulation is as follows: its metric cone

$$X = M \times \mathbb{R}_+, g_X = \mathrm{d}r^2 + r^2 g$$

is a Kähler cone. Hence there exists a complex structure J on X such that (g_X, J) defines a Kähler structure. We identify M with its natural embedding $M \to \{r = 1\} \subset X$. The 1-form η is given by $\eta = J(r^{-1}\mathrm{d}r)$ and it defines a contact structure on M. The vector field $\xi := J(r\partial_r)$ is a nowhere vanishing, holomorphic Killing vector field and it is called the *Reeb vector field* when it is restricted on M. The integral curves of ξ are geodesics, and give rise to a foliation on M, called the *Reeb foliation*. Then there is a Kähler structure on the local leaf space of the Reeb foliation, called the *transverse Kähler structure*. A standard example of a Sasaki manifold is the odd-dimensional round sphere S^{2n+1}. The corresponding Kähler cone is $\mathbb{C}^{n+1}\backslash\{0\}$ with the flat metric and its transverse Kähler structure descends to $\mathbb{CP}^n$ with the Fubini-Study metric.

We can also formulate Sasaki geometry, in particular the transverse Kähler structure via its contact bundle $\mathcal{D} = \mathrm{Ker}(\eta) \subset TM$. The complex structure J on the cone descends to the contact bundle via $\Phi := J|_{\mathcal{D}}$. The Sasaki metric can be written as follows,

$$g = \eta \otimes \eta + g^T,$$

where g^T is the transverse Kähler metric, given by $g^T := 2^{-1}\mathrm{d}\eta(\Phi \otimes \mathbb{I})$. The transverse Kähler form is denoted by $\omega^T = 2^{-1}\mathrm{d}\eta$. We shall study the transverse Kähler geometry of Sasaki metrics, with the Reeb vector field ξ and transverse complex structure (equivalently the complex structure J on the cone) both fixed. This means that we fix the basic Kähler class $[\omega^T]$ with $\omega^T = 2^{-1}\mathrm{d}\eta$ and study the Sasaki structures induced by the space of transverse Kähler potentials,

$$\mathcal{H} = \{\phi \in C^\infty_B(M) : \omega_\phi = \omega^T + \mathrm{d}_B\mathrm{d}^c_B\phi > 0\},$$

where $C^\infty_B(M)$ is the space of smooth *basic functions*. The main result in the paper is:

Theorem 2 $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$ *is a complete geodesic metric space for* $p \in [1, \infty)$*, which is the metric completion of* $(\mathcal{H}, \mathrm{d}_p)$*. For any* $u, v \in \mathcal{E}_p(M, \xi, \omega^T)$*,* $\mathrm{d}_p(u, v)$ *is realized by a unique finite-energy geodesic in* $\mathcal{E}_p(M, \xi, \omega^T)$ *connecting* u *and* v*. There exists a uniform constant* $C = C(n, p) > 1$ *such that*

$$C^{-1} I_p(u, v) \le \mathrm{d}_p(u, v) \le C I_p(u, v),$$

where the energy functional I_p *is given by*

$$I_p(u, v) = \|u - v\|_{p,u} + \|u - v\|_{p,v}.$$

Moreover, we have

$$\mathrm{d}_p\left(u, \frac{u+v}{2}\right) \le C\mathrm{d}_p(u, v).$$

We refer to Sect. 3 for notions such as $\mathcal{E}_p(M, \xi, \omega^T)$, d_p. Theorem 2 is the counterpart of main results in [28] in Sasaki setting. An important notion in the study of csck is the convexity of $\mathcal{K}$-energy along $C^{1,\bar{1}}$ geodesics [3] (see also [25]), which was generalized to Sasaki setting by [51,58]. Given the results above, one can then extend $\mathcal{K}$-energy to $\mathcal{E}_1$-class and keep its convexity along finite energy geodesics as in [7]. Moreover, this allows to define precisely the properness of $\mathcal{K}$-energy in terms of the distance d_1. One can then prove Theorem 1 using a priori estimates of scalar curvature-type equation together with properness assumption, where the effective estimates of d_1 in Theorem 2 play an important role; for details, see [49].

Along the way to prove Theorem 2, it is necessary to extend results as in [30,44] to Sasaki setting. Certainly the essential ideas lie in results in Kähler setting and many results are rather straightforward extensions from Kahler setting; we refer to Darvas' lecture notes [30] for an excellent reference. However, we should also emphasize that in

Sasaki setting, there are several substantial new difficulties when the Reeb foliation is irregular. To overcome these difficulties, the Sasaki structure (the Kähler cone structure and transverse Kähler structure) plays an essential role. Lemma 3.1 is an extension of Blocki–Kolodziej's approximation of plurisubharmonic functions by smooth decreasing sequence. For this proof we construct explicit holomorphic charts on the Kähler cone out of its transverse Kähler structure, see Lemma 2.1. This very explicit relation between the holomorphic charts and foliation charts of transverse Kähler structure seems to appear in literature for the first time, to the authors' knowledge. This explicit construction of holomorphic charts builds a very straightforward relation between plurisubharmonic functions on cone and (transverse) plurisubharmonic functions via transverse Kähler structure. Lemmas 3.2 and 3.3 give Darvas' volume partition formula for rooftop construction. This decomposition is a very important technical result for Darvas' theory and the proof in Kähler setting does not carry over for irregular Sasaki structures. We overcome this difficulty using Type-I deformation (see Theorem 6.1, Lemmas 6.1 and 6.2). (Of course there are many other places that there are substantial new difficulties; for example, the geodesic equation solved by Guan–Zhang is harder.) For completeness we include the details of almost all arguments, even in the case when the proof follows rather straightforwardly from the Kähler setting. The pluripotential theory in Sasaki setting has few references (see [51,58] for example) and we hope that our presentation is helpful.

We organize the paper as follows. In Sect. 2 we introduce basic notations and concepts of Sasaki geometry. We study the geometric structure of the space of transverse Kähler potentials using geodesic equation and pluripotential theory in Sect. 3. In Sect. 4 we prove the main theorem. We include a brief discussion of Sasaki-extremal metric in Sect. 5. Appendix contains various topics in pluripotential theory, including complex Monge–Ampere operator and various energy functionals on $\mathcal{E}_1$; we prove various results which are stated in [49, Section 2.2].

2 Preliminary on Sasaki Geometry

A good reference on Sasaki geometry can be found in the monograph [13] by Boyer–Galicki. Let M be a compact differentiable manifold of dimension $2n+1 (n \geq 1)$. A Sasaki structure on M is defined to be a Kähler cone structure on $X = M \times \mathbb{R}_+$, i.e., a Kähler metric (g_X, J) on X of the form

$$g_X = \mathrm{d}r^2 + r^2 g,$$

where $r > 0$ is a coordinate on $\mathbb{R}_+$, and g is a Riemannian metric on M. We call (X, g_X, J) the *Kähler cone* of M. We also identify M with the link $\{r = 1\}$ in X if there is no ambiguity. Because of the cone structure, the Kähler form on X can be expressed as

$$\omega_X = \frac{1}{2}\sqrt{-1}\partial\overline{\partial}r^2 = \frac{1}{2}\mathrm{d}\mathrm{d}^c r^2.$$

We denote by $r\partial_r$ the homothetic vector field on the cone, which is easily seen to be a real holomorphic vector field. A tensor α on X is said to be of homothetic degree k if

$$\mathcal{L}_{r\partial_r}\alpha = k\alpha.$$

In particular, ω and g have homothetic degree two, while J and $r\partial_r$ has homothetic degree zero. We define the *Reeb vector field*

$$\xi = J(r\partial_r).$$

Then ξ is a holomorphic Killing field on X with homothetic degree zero. Let η be the dual one-form to ξ:

$$\eta(\cdot) = r^{-2}g_X(\xi, \cdot) = 2\mathrm{d}^c \log r = \sqrt{-1}(\bar{\partial} - \partial)\log r \ .$$

We also use (ξ, η) to denote the restriction of them on (M, g). Then we have

- η is a contact form on M, and ξ is a Killing vector field on M which we also call the Reeb vector field;
- $\eta(\xi) = 1, \iota_\xi \mathrm{d}\eta(\cdot) = \mathrm{d}\eta(\xi, \cdot) = 0$;
- the integral curves of ξ are geodesics.

The Reeb vector field ξ defines a foliation $\mathcal{F}_\xi$ of M by geodesics. There is a classification of Sasaki structures according to the global property of the leaves. If all the leaves are compact, then ξ generates a circle action on M, and the Sasaki structure is called *quasiregular*. In general, this action is only locally free, and we get a polarized orbifold structure on the leaf space. If the circle action is globally free, then the Sasaki structure is called *regular*, and the leaf space is a polarized Kähler manifold. If ξ has a non-compact leaf, the Sasaki structure is called *irregular*.

One can also understand Sasaki structure through contact metric structure. There is an orthogonal decomposition of the tangent bundle

$$TM = L\xi \oplus \mathcal{D},$$

where $L\xi$ is the trivial bundle generated by ξ, and $\mathcal{D} = \mathrm{Ker}(\eta)$. The metric g and the contact form η determine a $(1, 1)$ tensor field Φ on M by

$$g(Y, Z) = \frac{1}{2}\mathrm{d}\eta(Y, \Phi Z), Y, Z \in \Gamma(\mathcal{D})$$

Φ restricts to an almost complex structure on $\mathcal{D}$:

$$\Phi^2 = -\mathbb{I} + \eta \otimes \xi.$$

Since both g and η are invariant under ξ, there is a well-defined Kähler structure (g^T, ω^T, J^T) on the local leaf space of the Reeb foliation. We call this a *transverse Kähler structure*. In the quasiregular case, this is the same as the Kähler structure on the quotient. Clearly, $\omega^T = 2^{-1}\mathrm{d}\eta$. The upper script T is used to denote both the transverse geometric quantity, and the corresponding quantity on the bundle $\mathcal{D}$. For example, we have on M

$$g = \eta \otimes \eta + g^T.$$

From the above discussion it is not hard to see that there is an intrinsic formulation of a Sasaki structure as a compatible integrable pair (η, Φ), where η is a contact one-form and Φ is an almost CR structure on $\mathcal{D} = \mathrm{Ker}\eta$. Here "compatible" means first that $\mathrm{d}\eta(\Phi U, \Phi V) = \mathrm{d}\eta(U, V)$ for any $U, V \in \mathcal{D}$, and $\mathrm{d}\eta(U, \Phi U) > 0$ for any non-zero $U \in \mathcal{D}$. Further, we require $\mathcal{L}_\xi \Phi = 0$, where ξ is the unique vector field with $\eta(\xi) = 1$, and $\mathrm{d}\eta(\xi, \cdot) = 0$. Φ induces a splitting

$$\mathcal{D} \otimes \mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1},$$

with $\overline{\mathcal{D}^{1,0}} = \mathcal{D}^{0,1}$. "Integrable" means that $[\mathcal{D}^{0,1}, \mathcal{D}^{0,1}] \subset \mathcal{D}^{0,1}$. This is equivalent to that the induced almost complex structure on the local leaf space of the foliation by ξ is integrable. For more discussions on this, see [13, Chapter 6].

Definition 2.1 A p-form θ on M is called basic if

$$\iota_\xi \theta = 0, L_\xi \theta = 0.$$

Let Λ_B^p be the bundle of basic p-forms and $\Omega_B^p = \Gamma(S, \Lambda_B^p)$ the set of sections of Λ_B^p.

The exterior differential preserves basic forms. We set $\mathrm{d}_B = d|_{\Omega_B^p}$. Thus the subalgebra $\Omega_B(\mathcal{F}_\xi)$ forms a subcomplex of the de Rham complex, and its cohomology ring $H_B^*(\mathcal{F}_\xi)$ is called the *basic cohomology ring*. When (M, ξ, η, g) is a Sasaki structure, there is a natural splitting of $\Lambda_B^p \otimes \mathbb{C}$ such that

$$\Lambda_B^p \otimes \mathbb{C} = \oplus \Lambda_B^{i,j},$$

where $\Lambda_B^{i,j}$ is the bundle of type (i, j) basic forms. We thus have the well-defined operators

$$\partial_B : \Omega_B^{i,j} \to \Omega_B^{i+1,j},$$
$$\bar{\partial}_B : \Omega_B^{i,j} \to \Omega_B^{i,j+1}.$$

Then we have $\mathrm{d}_B = \partial_B + \bar{\partial}_B$. Set $\mathrm{d}_B^c = \frac{1}{2}\sqrt{-1}\left(\bar{\partial}_B - \partial_B\right)$. It is clear that

$$\mathrm{d}_B \mathrm{d}_B^c = \sqrt{-1}\partial_B \bar{\partial}_B, \mathrm{d}_B^2 = (\mathrm{d}_B^c)^2 = 0.$$

We shall recall the transverse complex (Kähler) structure on local coordinates. Let U_α be an open covering of M and $\pi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ submersions such that

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

is biholomorphic when $U_\alpha \cap U_\beta$ is not empty. One can choose local coordinate charts $(z_1, \ldots, z_n)$ on V_α and local coordinate charts $(x, z_1, \ldots, z_n)$ on $U_\alpha \subset M$ such that $\xi = \partial_x$, where we use the notations

$$\partial_x = \frac{\partial}{\partial x}, \partial_i = \frac{\partial}{\partial z_i}, \bar{\partial}_j = \partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial z_{\bar{j}}}.$$

The map $\pi_\alpha : (x, z_1, \ldots, z_n) \to (z_1, \ldots, z_n)$ is then the natural projection. There is an isomorphism, for any $p \in U_\alpha$,

$$\mathrm{d}\pi_\alpha : D_p \to T_{\pi_\alpha(p)} V_\alpha.$$

Hence the restriction of g on $\mathcal{D}$ gives an Hermitian metric g_α^T on V_α since ξ generates isometries of g. One can verify that there is a well-defined Kähler metric g_α^T on each V_α and

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

gives an isometry of Kähler manifolds (V_α, g_α^T). The collection of Kähler metrics $\{g_\alpha^T\}$ on $\{V_\alpha\}$ can be used as an alternative definition of the transverse Kähler metric. The (local) transverse holomorphic (Kähler) structure is essential for us and we shall use these charts enormously. We summarize as follows:

Definition 2.2 (*Local foliation charts*) We can choose the open covering $\{U_\alpha\}$ of M such that there exists a local product structure for each α, determined by its foliation structure and transverse complex structure. That is, there are charts

$$\Psi_\alpha : U_\alpha \to W_\alpha \subset \mathbb{R} \times \mathbb{C}^n,$$

where $W_\alpha = (-\delta, \delta) \times V_\alpha$. For a point $p \in W_\alpha$, we write $p = (x, z_1, \ldots, z_n)$ with $\xi = \partial_x$ and $V_\alpha = B_r(0) \subset \mathbb{C}^n$ for $0 < r$. We assume that δ, r are sufficiently small depending only on (M, ξ, η, g), and ω_α^T is uniformly equivalent to an Euclidean metric on each $V_\alpha = B_r \subset \mathbb{C}^n$,

$$\frac{1}{2}\delta_{i\bar{j}} \leq \omega_\alpha^T \leq 2\delta_{i\bar{j}}.$$

In Sasaki geometry, it is often mostly convenient to work with these charts when we need to consider the Sasaki structure locally. For each U_α, we assume it is contained in the geodesic normal neighborhood of its "center," $\Psi_\alpha^{-1}(0, 0, \ldots, 0)$, by choosing δ, r small enough. We call these charts *foliation charts*. The existence of foliation charts is well known in the subject, see [40]; in particular, any Sasaki metric g can be locally expressed in terms of a real function of $2n$ variables. Given a foliation chart $W_\alpha = (-\delta, \delta) \times V_\alpha$, for $(x, z_1, \ldots, z_n) \in U_\alpha$, locally there exists a strictly plurisubharmonic function $h : V_\alpha \to \mathbb{R}$, and the Sasaki structure reads

$$\xi = \partial_x; \ \eta = \mathrm{d}x - \sqrt{-1}\sum_i (h_i \mathrm{d}z^i - h_{\bar{i}}\mathrm{d}z^{\bar{i}})$$
$$\omega^T = \sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}z^{\bar{j}}; \ g = \eta \otimes \eta + 2h_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}z^{\bar{j}}. \tag{2.1}$$

If we consider a Sasaki structure induced by a transverse Kähler potential ϕ, then locally we have $h \to h + \phi$. In particular, we have

$$\eta_\phi = \eta + \sqrt{-1}(\bar{\partial} - \partial)\phi, \omega_\phi = \omega^T + \sqrt{-1}\partial\bar{\partial}\phi.$$

We shall also use holomorphic charts on its Kähler cone X. There exist indeed holomorphic charts on the Kähler cone X which are closely related to foliation charts on M. This seems to be much less well known and we shall describe them now.

Lemma 2.1 (Holomorphic coordinates on the Kähler cone) *For a Sasaki structure locally generated by a plurisubharmonic function $h : V_\alpha \to \mathbb{R}$ in foliation charts on M, then the following gives a local holomorphic structure on its Kähler cone X, for $w = (w_0, \dots, w_n) \in \tilde{U}_\alpha \subset \mathbb{C} \times V_\alpha$,*

$$w_0 = \log r - h(z, \bar{z}) + \sqrt{-1}x, w_i = z_i, i = 1, \dots, n, z = (z_1, \dots, z_n). \tag{2.2}$$

The holomorphic structure J is given by the holomorphic coordinates $w = (w_0, \dots, w_n)$,

$$J\frac{\partial}{\partial w_i} = \sqrt{-1}\frac{\partial}{\partial w_i}, i = 0, \dots, n. \tag{2.3}$$

Proof Given (2.1), it is straightforward to check that (2.2) gives a holomorphic chart satisfying (2.3). □

Remark 2.1 These holomorphic charts would be very useful for us later; in particular, when we consider plurisubharmonic functions on X and transverse plurisubharmonic functions on M. The explicit holomorphic charts given above seem to appear in literature first time to our knowledge, while the foliation charts are well known.

When the Reeb vector field ξ is irregular, the local foliation charts satisfy cocycle condition but they do not give a manifold (or orbifold) structure of the quotient $M/\mathcal{F}_\xi$. We shall recall *Type-I deformation* defined in [15]. Let (M, ξ_0, η_0, g_0) be a compact Sasaki manifold, denote its automorphism group by $\mathrm{Aut}(M, \xi_0, \eta_0, g_0)$. We fix a torus

$$T \subset \mathrm{Aut}(M, \xi_0, \eta_0, g_0) \text{ such that } \xi_0 \in \mathfrak{t} = \text{Lie algebra}(T).$$

Definition 2.3 (Type-I deformation) Let (M, ξ_0, η_0, g_0) be a T-invariant Sasaki structure. For any $\xi \in \mathfrak{t}$ such that $\eta_0(\xi) > 0$. We define a new Sasaki structure on M explicitly as

$$\eta = \frac{\eta_0}{\eta_0(\xi)}, \Phi = \Phi_0 - \Phi_0\xi \otimes \eta, g = \eta \otimes \eta + \frac{1}{2}\mathrm{d}\eta(\mathbb{I} \otimes \Phi).$$

Note that under Type-I deformation, the essential change is the Reeb vector field $\xi_0 \leftrightarrow \xi$ and this construction can be done vice versa.

3 The Space of Transverse Kähler Potentials

In this section, we consider the space of transverse Kähler potentials on a compact Sasaki manifold through its transverse Kähler structure. It turns out to be necessary to consider these objects not only from point of view of PDE, but also from the point of view of pluripotential theory. Geometric pluripotential theory on Kähler manifolds turns out to be one crucial piece in the proof of properness conjecture [6,24]. We refer [30,44] and references therein for details of pluripotential theory. We extend these results to Sasaki manifolds. These results would form a crucial piece for existence of constant scalar curvature (cscs) on Sasaki manifolds as well, see [49] for details.

Using the transverse Kähler structure of a Sasaki structure, many of the extensions of pluripotential theory on Kähler manifolds to Sasaki manifolds are rather straightforward, and the proofs are a direct adaption of Kähler setting with some necessary modifications. On the other hand, there are several exceptions that would need essential inputs from the Sasaki structure. And the proofs are new and substantially different, compared with the Kähler setting. We summarize the main differences as follows. The first is Lemma 3.1, where we will prove a counterpart of an approximation result of plurisubharmonic functions as in Kähler setting by Blocki–Kolodziej [10]. One can apply Blocki–Kolodziej approximation locally to transverse Kähler structure and obtain a local approximation, but such construction has trouble to patch together when the Sasaki structure is irregular. Instead, we need to do the construction on the Kähler cone, and the holomorphic chart on the cone (Lemma 2.1) plays a substantial role in our construction. The second main difference is Lemma 3.2, where we will prove an important property of the rooftop envelop construction $P(u_0, u_1)$ on the non-contact set; this result plays a very important role in Darvas's theory. The proof as in Kähler setting again does not work directly to Sasaki setting when the Sasaki structure is irregular. Instead, we need to apply a Type-I deformation carefully (Theorem 6.1, Lemma 6.1) to bypass the difficulty.

3.1 The Quasiplurisubharmonic Functions on Sasaki Manifolds

Denote $\mathcal{H} = \{\phi \in C^\infty_B(M) : \omega_\phi = \omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\phi > 0\}$, the space of transverse Kähler potentials on a Sasaki manifold (M, ξ, η, g). Given $\phi \in \mathcal{H}$, it defines a new Sasaki structure, $(M, \xi, \eta_\phi, g_{\eta_\phi})$ as follows,

$$\eta_\phi = \eta + 2d^c_B\phi, \omega_\phi = \omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\phi, g_{\eta_\phi} = \eta_\phi \otimes \eta_\phi + \omega_\phi.$$

The most relevant results in pluripotential theory for us lie in [44], [5, Section 2], [45] and [30]. Part of them has been done by van Covering [58, Section 2], including the Monge–Ampere operator and weak convergence, with main focus on L^∞ and C^0 potentials. We shall need most of the results on the energy classes $\mathcal{E}$ and $\mathcal{E}_p$ (defined below).

Given a Sasaki structure (M, ξ, η, g), we recall the following definition,

Definition 3.1 An L^1, upper semicontinuous (usc) function $u : M \to \mathbb{R} \cup \{-\infty\}$ is called a transverse ω^T-plurisubharmonic (TPSH for short) if u is invariant under the Reeb flow, and u is ω^T-plurisubharmonic on each local foliation chart V_α, that is $\omega_\alpha^T + \sqrt{-1}\partial_B\bar{\partial}_B u \geq 0$ as a (1, 1)-current on V_α.

It is apparent that the definition above does not depend on the choice of foliation charts. Indeed, u is invariant along the flow of ξ and we extend u trivially in the cone direction to a function on cone. Using the holomorphic structure on the cone (see Lemma 2.1), u is a TPSH if and only if $\omega^T + \sqrt{-1}\partial\bar{\partial}u \geq 0$ is a closed, positive (1,1) current on the cone X. We use the notation,

$$\mathrm{PSH}(M, \xi, \omega^T) = \{u \in L^1(M) : u \text{ is usc and invariant under the Reeb flow; } \omega_u \geq 0\}$$

One of the cornerstones of Bedford–Taylor theory [2] is to associate a complex Monge–Ampere measure to a bounded psh function. Their construction generalizes to bounded Kähler potentials in a straightforward manner [44] and it has direct adaption to Sasaki setting. We refer to [58, Section 2] and Sect. 1 for definition of complex Monge–Ampere measures $\omega_u^n \wedge \eta$ for $u \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$ on Sasaki manifolds, which is a direct adaption of Bedford–Taylor theory [2].

Proposition 3.1 *Suppose that the sequence $u_j \in PSH(M, \xi, \omega^T) \cap L^\infty$ decreases to $u \in PSH(M, \xi, \omega^T) \cap L^\infty$. Then for $k = 1, \ldots, n$, we have the following weak convergences of complex Monge–Ampere measures,*

$$\omega_{u_j}^k \wedge (\omega^T)^{n-k} \wedge \eta \to \omega_u^k \wedge (\omega^T)^{n-k} \wedge \eta. \tag{3.1}$$

Proof By applying a partition of unity subordinated to covering by foliation charts, we need to show that for $f \in C^\infty$, supported on a foliation chart $W_\alpha = (-\delta, \delta) \times V_\alpha$

$$\int_M f\omega_{u_j}^k \wedge (\omega^T)^{n-k} \wedge \eta \to \int_M f\omega_u^k \wedge (\omega^T)^{n-k} \wedge \eta. \tag{3.2}$$

We should emphasize that f is not a basic function in general. The weak convergence in Kähler setting implies that for each $x \in (-\delta, \delta)$

$$\int_{V_\alpha} f(x, z, \bar{z})\omega_{u_j}^k \wedge (\omega^T)^{n-k} \to \int_{V_\alpha} f(x, z, \bar{z})\omega_u^k \wedge (\omega^T)^{n-k}.$$

Note that for each x, f is supported on V_α. Taking integration with respect to dx, this leads to (3.2), since on W_α, $\omega_u^k \wedge (\omega^T)^{n-k} \wedge \eta = \omega_u^k \wedge (\omega^T)^{n-k} \wedge \mathrm{d}x$ as a product measure. □

The following Bedford–Taylor identity in Sasaki setting would be used numerously:

Proposition 3.2 *For $u, v \in PSH(M, \xi, \omega^T) \cap L^\infty$,*

$$\chi_{\{u>v\}}\omega_{\max(u,v)}^n \wedge \eta = \chi_{\{u>v\}}\omega_u^n \wedge \eta. \tag{3.3}$$

Proof We only need to prove this in foliation charts. Recall for each foliation chart $W_\alpha = (-\delta, \delta) \times V_\alpha$, $V_\alpha = B_r(0) \subset \mathbb{C}^n$ gives the local transverse complex structure. For a point $p \in W_\alpha$, we write $p = (x, z)$ with $\xi = \partial_x$. Given $u \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$ it defines a Kähler current ω_u^n on V_α. Since both u and v are basic functions, u, v are independent of x in W_α. Hence on $W_\alpha \cap \{u > v\} = (-\delta, \delta) \times \{z \in V_\alpha : u > v\}$. Note that $\omega_u^T \wedge \eta$ is invariant along the Reeb direction, and it coincides with the product measure $\mathrm{d}x \wedge \omega_u^n$ on $W_\alpha = (-\delta, \delta) \times V_\alpha$. On each W_α, we have

$$\chi_{\{(x,z)\in W_\alpha : u>v\}} \omega^n_{\max(u,v)} \wedge \eta = \chi_{\{z\in V_\alpha : u>v\}} \omega^n_{\max(u,v)} \wedge \mathrm{d}x$$
$$\chi_{\{(x,z)\in W_\alpha : u>v\}} \omega^n_u \wedge \eta = \chi_{\{z\in V_\alpha : u>v\}} \omega^n_u \wedge \mathrm{d}x.$$

To prove (3.3), it reduces to show that

$$\chi_{\{z\in V_\alpha : u>v\}} \omega^n_{\max(u,v)} = \chi_{\{z\in V_\alpha : u>v\}} \omega^n_u.$$

This is just the Bedford–Taylor identity [2]. □

It is possible to generalize the Bedford–Taylor constructions to a much larger class on a compact Kähler manifold, see Guedj–Zeriahi [44]. The reference [30, Section 2] is sufficient for our purpose. These constructions in Kähler setting have a direct extension to Sasaki setting, where Proposition 3.2 plays an important role. First we prove the following well-known result in pluripotential theory.

Proposition 3.3 *There exists $C = C(M, g)$ such that for any $u \in PSH(M, \xi, \omega^T)$,*

$$\sup_M u \le \frac{1}{Vol(M)} \int_M u \,\mathrm{d}\mu_g + C.$$

Proof When u is C^2 this is obvious by the fact that $\Delta_g u + n \ge 0$. In general, we can prove this using sub-mean value property of plurisubharmonic functions, similar as in [30, Lemma 3.45]. In this proof, we can either use foliation charts on M or Kähler cone structure on $X = C(M)$. We use foliation charts in this argument.

We assume $\sup_M u = 0$ and want to show that the integration of u is uniformly bounded below. We can cover M by finitely many nested foliation charts $U_k \subset W_k \subset M (1 \le k \le N)$ such that there exist diffeomorphisms $\varphi_k : B(0, 4) \times (-2\delta, 2\delta) \to W_k$ with $\varphi_k(B(0, 1) \times (-\delta, \delta)) = U_k$, where δ is a fixed positive constant and $B(0, 1) \subset B(0, 4) \subset \mathbb{C}^n$ are Euclidean balls centered at the origin in $\mathbb{C}^n$. We assume that $(z, x) \in B(0, 4) \times (-2\delta, 2\delta)$ such that $z \in B(0, 4)$ represents transverse holomorphic charts and $x \in (-2\delta, 2\delta)$ represents the Reeb direction (i.e., $\xi = \partial_x$). On each W_k, there exists a smooth basic function $\psi_k = \psi_k(z)$ such that $\omega^T = \sqrt{-1}\partial_z \bar{\partial}_z \psi_k$. Note that we only need to show that, there exists a uniformly bounded constant $C > 0$, such that

$$\int_{U_k} u \,\mathrm{d}\mu_g \ge -C, k \in \{1, \ldots, N\}.$$

Note that u is basic, we have

$$\int_{B(0,1)\times(-\delta,\delta)} u\circ\varphi_k \mathrm{d}\mu_{x,z} = 2\delta \int_{B(0,1)} u\circ\varphi_k(z,x_0)\mathrm{d}\mu_z,\, x_0\in(-\delta,\delta)$$

where $\mathrm{d}\mu_{x,z}$ and $\mathrm{d}\mu_z$ are Euclidean measure on $\mathbb{C}^n\times\mathbb{R}$ and $\mathbb{C}^n$, respectively. Hence we only need to show that

$$\int_{B(0,1)} u\circ\varphi_k(z,x_0)\mathrm{d}\mu_z \geq -C,\, k\in\{1,\cdots,N\}. \tag{3.4}$$

Note that by our construction, $(\psi_k+u)\circ\varphi_k$ is independent of x and is plurisubharmonic on $B(0,4)$ for each k. As u is usc, its supremum is realized at some point $p_1\in M$ such that $u\leq u(p_1)=0$. Since U_k covers M, we can assume $p_1\in U_1$ with the coordinate $\varphi_1(z_1,x_1)=p_1$ for some $(z_1,x_1)\in B(0,1)\times(-\delta,\delta)$. Note that since u is basic, hence it is independent of x-coordinate we can also take $x_1=0$. Since $B(z_1,2)\subset B(0,4)$, we have the following sub-mean value property for $(\psi_1+u)\circ\varphi_1$,

$$\psi_1\circ\varphi_1(z_1,0)=(\psi_1+u)\circ\varphi_1(z_1,0)\leq\frac{1}{\mu(B(z_1,2))}\int_{B(z_1,2)}(\psi_1+u)\circ\varphi_1(z,0)\mathrm{d}\mu_z.$$

Since $u\leq 0$ and $B(0,1)\subset B(z_1,2)$, there exists $C_1>0$, independent of u, such that

$$\int_{B(0,1)} u\circ\varphi_1\mathrm{d}\mu_z\geq -C_1. \tag{3.5}$$

Since $\{U_k\}_k$ covers M, we can assume U_1 intersects U_2. We can choose $r_2>0$, such that $\varphi_2(B(z_2,r_2)\times(\delta_1,\delta_2))\subset U_1\cap U_2$ for some $B(z_2,r_2)\subset B(0,1)$ and $-\delta<\delta_1<\delta_2<\delta$. Since $u\leq 0$, it follows that there exists $\tilde{C}_1>0$, independent of u ($\tilde{C}_1$ depends only on C_1, r_2, and ψ_2), such that

$$\frac{1}{\mu(B(z_2,r_2))}\int_{B(z_2,r_2)}(u+\psi_2)\circ\varphi_2\mathrm{d}\mu_z\geq-\tilde{C}_1.$$

Since $(u+\psi_2)\circ\varphi_2$ is plurisubharmonic in $B(0,4)$, we can obtain that

$$\begin{aligned}&\frac{1}{\mu(B(z_2,2))}\int_{B(z_2,2)}(u+\psi_2)\circ\varphi_2\mathrm{d}\mu_z\\ &\quad\geq\frac{1}{\mu(B(z_2,r_2))}\int_{B(z_2,r_2)}(u+\psi_2)\circ\varphi_2\mathrm{d}\mu_z\geq-\tilde{C}_1.\end{aligned}$$

Since $u\leq 0$ and $B(0,1)\subset B(z_2,2)$, we obtain for some $C_2>0$

$$\int_{B(0,1)} u\circ\varphi_2\mathrm{d}\mu_z\geq -C_2.$$

We continue this process to consider that $U_1 \cup U_2$ intersects a member, say U_3. After at most $N-2$ step, we prove (3.4). □

As a direct consequence, we know the following (see [33, Proposition I.5.9]):

Proposition 3.4 *The set* $\mathcal{C} = \{u \in PSH(M, \xi, \omega^T) : \sup_M u \leq C\}$ *is bounded in* L^1 *and it is precompact in* $L^1(\mathrm{d}\mu_g)$ *topology.*

Proof By the above-mentioned proposition, we know that $\sup_M u$ bounded above implies that $\int_M |u| \mathrm{d}\mu_g$ is uniformly bounded. By the Motel property of subharmonic functions and plurisubharmonic functionals [33, Propositions I.4.21, I.5.9], $\mathcal{C}$ is precompact with respect to $L^1(\mathrm{d}\mu_g)$ topology. Note that in Sasaki setting we apply the compactness of plurisubharmonic functions to nested foliations charts $U_k \subset W_k$ as above for ω_k^T-plurisubharmonic functions locally, that $\mathcal{C}$ is precompact in L^1 topology in each U_k. After passing by subsequence if necessary, we can then get weak compactness of $\mathcal{C}$ with respect to $L^1(\mathrm{d}\mu_g)$ topology. □

Let $v \in \mathrm{PSH}(M, \xi, \omega^T)$. For $h \in \mathbb{R}$, we denote $v_h = \max\{v, -h\}$ to be the *canonical cutoffs* of v. It is evident that v_h is invariant under the Reeb flow and hence $v_h \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$. If $h_1 < h_2$, then Proposition 3.2 implies that

$$\chi_{\{v>-h_1\}}\omega^n_{v_{h_1}} \wedge \eta = \chi_{\{v>-h_1\}}\omega^n_{v_{h_2}} \wedge \eta \leq \chi_{\{v>-h_2\}}\omega^n_{v_{h_2}} \wedge \eta.$$

Hence $\chi_{\{v>-h\}}\omega^n_{v_h} \wedge \eta$ is an increasing sequence of Borel measure on M with respect to h. This leads to the following definition:

Definition 3.2 We define

$$\omega^n_v \wedge \eta := \lim_{h\to\infty} \chi_{\{v>-h\}}\omega^n_{v_h} \wedge \eta. \tag{3.6}$$

We shall emphasize that by the definition above, we have for any Borel set $B \subset M$,

$$\int_B \omega^n_v \wedge \eta = \lim_{h\to\infty} \int_B \chi_{\{v>-h\}}\omega^n_{v_h} \wedge \eta. \tag{3.7}$$

Hence the convergence in (3.6) is a stronger than the weak convergence of measures.

To proceed, we need the following approximation of TPSH functions. Our proof uses the Kähler cone structure and builds up on Blocki–Kolodziej [10].

Lemma 3.1 *Given* $u \in PSH(M, \xi, \omega^T)$*, there exists a decreasing sequence* $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ *such that* u_k *converges to* u*.*

Proof First we assume that u has zero Lelong number. Recall X is the Kähler cone and we identify M with the link $\{r = 1\} \subset X$. For $u \in \mathrm{PSH}(M, \xi, \omega^T)$, we extend u to be a function on X such that $u(r, p) = u(p)$, for any $r > 0$. We recall that $\omega^T = \frac{1}{2}\mathrm{d}\eta = \mathrm{dd}^c(\log r) = \sqrt{-1}\partial\bar{\partial}(\log r)$. Hence for $u \in \mathrm{PSH}(M, \xi, \omega^T)$, we have the following,

$$\sqrt{-1}\partial\bar{\partial}(\log r + u) \geq 0$$

In other words, $v = u + \log r$ is a plurisubharmonic function on X. This is transparent in foliations charts and corresponding holomorphic charts as in Lemma 2.1. Let h_α be a local potential for ω^T in a foliation chart V_α, and we write $h = h(w_1, \bar{w}_1, \dots, w_n, \bar{w}_n)$ in the holomorphic chart on cone, then $\log r = h_\alpha + \mathrm{Re}(w_0)$. Denote ω_X to be the Kähler form on X. Since u has zero Lelong number, applying Blocki–Kolodziej [10, Theorem 2], we get a sequence of smooth functions v_k converges to u, decreasing in k, such that on $X^{'} = \{2^{-1} \leq r \leq 2\} \subset X$

$$\sqrt{-1}\partial\bar{\partial}(v_k) + \omega^T + k^{-1}\omega_X \geq 0. \tag{3.8}$$

We can assume in addition that v_k is invariant under the flow of ξ, by taking average with respect to the torus action generated by $\xi \in \mathrm{Aut}(\xi, \eta, g)$. We define a basic function u_k on M such that, by taking $r = 1$, $u_k = v_k|_{r=1}$.

Now for any point on $X^{'}$, we choose holomorphic charts $\tilde{U}_\alpha$ as in Lemma 2.1 to cover $X^{'}$. We write the function in a holomorphic chart as

$$v_k = v_k(\mathrm{Re}(w_0), x, w_1, \bar{w}_1 \dots, w_n, \bar{w}_n).$$

We recall the relation between the holomorphic charts and the foliation charts,

$$w_0 = \log(r) + \sqrt{-1}x - h_\alpha(z, \bar{z}), w_i = z_i, i = 1, \dots, n. \tag{3.9}$$

Note we assume that v_k is invariant under the flow of ξ, hence v_k is independent of $x = \mathrm{Im}(w_0)$. We write v_k as follows, using (3.9),

$$v_k(\mathrm{Re}(w_0), w_1, \bar{w}_1, \dots, w_n, \bar{w}_n) = v_k(\log r - h(z, \bar{z}), z, \bar{z}).$$

Locally, this gives

$$u_k(z, \bar{z}) = v_k(-h_\alpha(z, \bar{z}), z, \bar{z}). \tag{3.10}$$

The tangent space T_pX is given by, in terms of coordinate $(r, x, z_1, \dots, z_n)$,

$$T_pX \otimes \mathbb{C} = \mathrm{span}\left\{\frac{\partial}{\partial r}, r^{-1}\frac{\partial}{\partial x}, X_i = \frac{\partial}{\partial z_i} + \sqrt{-1}h_i\frac{\partial}{\partial x}, \bar{X}_j = \frac{\partial}{\partial \bar{z}_j} - \sqrt{-1}h_{\bar{j}}\frac{\partial}{\partial x}\right\}.$$

Note that the contact bundle $D_p = \mathrm{span}\{X_i, X_{\bar{i}}, i = 1, \dots, n\}$. For $p \in M \subset X$, we can assume that $h(z, \bar{z}) = \partial h = \bar{\partial} h = 0$, $h_{i\bar{j}} = \delta_{i\bar{j}}$ at p, and hence

$$T_pX = T_pM \oplus \left\{\frac{\partial}{\partial r}\right\} = \mathrm{span}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, r^{-1}\frac{\partial}{\partial x}, \frac{\partial}{\partial r}\right\}.$$

By (3.8), we compute (at p),

$$\left(\sqrt{-1}\partial\bar{\partial}v_k + \omega^T + k^{-1}\omega_X\right)\left(\frac{\partial}{\partial z_i}, -\sqrt{-1}\frac{\partial}{\partial \bar{z}_i}\right) = -\partial_t v_k + 1 + k^{-1} + (v_k)_{i\bar{i}} \geq 0, \tag{3.11}$$

where t stands for the first argument of v_k. This is equivalent to the following, on M we have,

$$\sqrt{-1}\partial_B\bar{\partial}_B u_k + (1+k^{-1})\omega^T \geq 0.$$

It is clear that u_k converges to u, deceasing in k. Without loss of generality, we can assume that $u \leq -1$ and $u_k \leq 0$. It follows that $k(k+2)^{-1}u_k \in \mathcal{H}$ such that $k(k+2)^{-1}u_k$ converges to u, decreasing in k. This completes the proof when u has zero Lelong number.

Now suppose $u \in \mathrm{PSH}(M, \xi, \omega^T)$. We consider the canonical cutoffs $u_j = \max\{u, -j\} \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$. By the above statements, we know that for each j, there exists a sequence of smooth functions $\{v_j^k\}_k \subset \mathcal{H}$ which decreases to u_j. By adding a small constant k^{-1} to each v_j^k, we can assume that $\{v_j^k\}_k$ strictly decreases (for each j). Then for each k, we can find k_{j+1} such that

$$v_{j+1}^{k_{j+1}} < v_j^k. \tag{3.12}$$

Indeed we consider the open set $U^l := \{x \in M : v_{j+1}^l < v_j^k\}$. Clearly $\{U^l\}_l$ is an increasing sequence of open sets such that $\cup_l U^l = M$, since

$$\lim_{l\to\infty} v_{j+1}^l = u_{j+1} \leq u_j < v_j^k.$$

Since M is compact, there exists k_{j+1} such that $U^{k_{j+1}} = M$. By (3.12), we can find a sequence $\{v_j^{k_j}\}_j \subset \mathcal{H}$ inductively such that $v_j^{k_j} \searrow u$. This completes the proof. □

Remark 3.1 The Kähler cone structure, in particular, the relation between holomorphic charts and foliation charts as in Lemma 2.1, plays a very important role in Sasaki setting. If the Reeb vector field is irregular, the approximation from transverse Kähler structure can produce local approximation. But it seems to be hard to patch such a local construction together when the Reeb vector field is irregular. Instead we do approximation on the Kähler cone. We shall mention that in (3.12), the assumption that each sequence $\{v_j^k\}_k$ strictly decreases is necessary. For example, we can take $u = 1$ over $[0, 1]$, $v = 0$ over $[0, 1)$ and $v(1) = 1$. We can choose $u_k = 1$ for each k, and $v_k(x) = x^k + k^{-1}$. Then $v \leq u$ and $\{u_k\}_k$ decreases to u and v_k (strictly) decreases to v. But for $\{u_k\}_k$ and $\{v_k\}_k$, (3.12) does not hold: given u_k, there does not exist l such that $v_l \leq u_k$ since $v_l(1) > 1$ for all l.

As a direct consequence, we have the following (just as in Kähler setting, see [30, Lemma 2.2]),

Proposition 3.5 *For $u \in PSH(M, \xi, \omega^T) \cap L^\infty$,*

$$Vol(M) := \int_M \omega_u^n \wedge \eta = \int_M \omega_T^n \wedge \eta. \tag{3.13}$$

Proof By Lemma 3.1, we can choose a smooth sequence u_k converges to u as a decreasing sequence. It then follows from Bedford–Taylor theory (see Proposition 3.1) that $\omega_{u_k}^n \wedge \eta$ converges to $\omega_u^n \wedge \eta$ weakly, we obtain (3.13). □

It is then clear that, given (3.6), we have only $\int_M \omega_v^n \wedge \eta \le \mathrm{Vol}(M)$ for $v \in \mathrm{PSH}(M, \xi, \omega^T)$.

Definition 3.3 We define the full-mass elements in $\mathrm{PSH}(M, \xi, \omega^T)$ as

$$\mathcal{E}(M, \xi, \omega^T) := \{v \in \mathrm{PSH}(M, \xi, \omega^T) : \int_M \omega_v^n \wedge \eta = \mathrm{Vol}(M)\} \tag{3.14}$$

As in Kähler case, many of the properties that hold for bounded TPSH functions hold for elements of $\mathcal{E}(M, \xi, \omega^T)$ as well. We include the *comparison principle, monotonicity property*, and *generalized Bedford–Taylor identity* as follows. These properties are proved in [44] for Kähler setting. Given (3.3) and (3.13), our proof follows almost identical as in Kähler setting (see [44, Theorem 1.5, Proposition 1.6, Corollary 1.7]). Nevertheless, we include the details.

Proposition 3.6 (Comparison principle) *Suppose* $u, v \in \mathcal{E}(M, \xi, \omega^T)$. *Then*

$$\int_{\{v<u\}} \omega_u^n \wedge \eta \le \int_{\{v<u\}} \omega_v^n \wedge \eta. \tag{3.15}$$

Proof Our proof is similar to Kähler case, see [30, Proposition 2.3]. First we show (3.15) for u, v bounded. Using Propositions 3.2 and 3.5, we write

$$\begin{aligned}
\int_{\{v<u\}} \omega_u^n \wedge \eta = \int_{\{v<u\}} \omega_{\max\{u,v\}}^n \wedge \eta &= \int_M \omega_{\max\{u,v\}}^n \wedge \eta - \int_{\{u\le v\}} \omega_{\max\{u,v\}}^n \wedge \eta \\
&\le \mathrm{Vol}(M) - \int_{\{u<v\}} \omega_{\max\{u,v\}}^n \wedge \eta \\
&= \int_M \omega_v^n \wedge \eta - \int_{\{u<v\}} \omega_v^n \wedge \eta \\
&= \int_{\{v\le u\}} \omega_v^n \wedge \eta.
\end{aligned}$$

Replacing v by $v + \epsilon$, we have

$$\int_{\{v+\epsilon<u\}} \omega_u^n \wedge \eta \le \int_{\{v+\epsilon\le u\}} \omega_v^n \wedge \eta.$$

Recall that

$$\{v < u\} = \cup_{\epsilon>0}\{v + \epsilon < u\} = \cup_{\epsilon>0}\{v + \epsilon \le u\}.$$

Hence (3.15) for bounded potentials follows immediately by letting $\epsilon \to 0$.

In general, let $u_l = \max\{u, -l\}, v_k = \max\{v, -k\}, l, k \in \mathbb{N}$ be the canonical cutoffs of u, v respectively. We apply (3.15) for bounded potentials to get

$$\int_{\{v_l<u_k\}} \omega_{u_k}^n \wedge \eta \leq \int_{\{v_l<u_k\}} \omega_{v_l}^n \wedge \eta.$$

Together with the inclusions $\{v_l < u\} \subset \{v_l < u_k\} \subset \{v < u_k\}$, we have

$$\int_{\{v_l<u\}} \omega_{u_k}^n \wedge \eta \leq \int_{\{v<u_k\}} \omega_{v_l}^n \wedge \eta. \tag{3.16}$$

Letting $l \to \infty$, using the definition (3.6) on $\omega_{v_l}^n \wedge \eta$ and $\{v < u\} = \cup_{l\in\mathbb{N}}\{v_l < u\}$, (3.16) gives

$$\int_{\{v<u\}} \omega_{u_k}^n \wedge \eta \leq \int_{\{v<u_k\}} \omega_{v}^n \wedge \eta.$$

Letting $k \to \infty$, using the definition (3.6) on $\omega_{u_k}^n \wedge \eta$ and $\{v \leq u\} = \cap_{k\in\mathbb{N}}\{v < u_k\}$, we get

$$\int_{\{v<u\}} \omega_{u}^n \wedge \eta \leq \int_{\{v\leq u\}} \omega_{v}^n \wedge \eta.$$

The replacing v by $v + \epsilon$ in the above inequality, we can then argue as in the bounded case, taking the limit $\epsilon \to 0$ yields (3.15). □

Proposition 3.7 (Monotonicity property) *Suppose $u \in \mathcal{E}(M, \xi, \omega^T)$ and $v \in PSH(M, \xi, \omega^T)$. If $u \leq v$, then $v \in \mathcal{E}(M, \xi, \omega^T)$.*

Proof This is proved in [44, Proposition 1.6] in Kähler case and the Sasaki case is almost identical. First we show that $\psi = v/2 \in \mathcal{E}(M, \xi, \omega^T)$. We can assume that $u \leq v < -2$, hence $\psi < -1$. This normalization gives the following inclusions for the canonical cutoffs u_j, v_j, ψ_j,

$$\{\psi \leq -j\} = \{\psi_j \leq -j\} \subset \{u_{2j} < \psi_j - j + 1\} \subset \{u_{2j} \leq -j\}.$$

By Proposition 3.15 and the inclusions above, we have

$$\begin{aligned}\int_{\{\psi_j\leq -j\}} \omega_{\psi_j}^n \wedge \eta &\leq \int_{\{u_{2j}<\psi_j-j+1\}} \omega_{\psi_j}^n \wedge \eta \leq \int_{\{u_{2j}<\psi_j-j+1\}} \omega_{u_{2j}}^n \wedge \eta \\ &\leq \int_{\{u_{2j}\leq -j\}} \omega_{u_{2j}}^n \wedge \eta.\end{aligned}$$

Note that we have

$$\int_{\{u_{2j}\leq -j\}} \omega_{u_{2j}}^n \wedge \eta = \mathrm{Vol}(M) - \int_{\{u_{2j}>-j\}} \omega_{u_{2j}}^n \wedge \eta.$$

Applying Proposition 3.2 to $\max\{u_{2j}, -j\} = u_j$ on the set $\{u_{2j} > -j\} = \{u_j > -j\}$, we have

$$\int_{\{u_{2j}>-j\}} \omega_{u_{2j}}^n \wedge \eta = \int_{\{u_j>-j\}} \omega_{u_j}^n \wedge \eta.$$

It then follows that

$$\int_{\{u_{2j}\le -j\}} \omega_{u_{2j}}^n \wedge \eta = \int_{\{u_j\le -j\}} \omega_{u_j}^n \wedge \eta = \int_{\{u\le -j\}} \omega_{u_j}^n \wedge \eta.$$

By definition of $u \in \mathcal{E}(M, \xi, \omega^T)$, it follows that, as $j \to \infty$,

$$\int_{\{\psi_j\le -j\}} \omega_{\psi_j}^n \wedge \eta \le \int_{\{u\le -j\}} \omega_{u_j}^n \wedge \eta \to 0.$$

Hence $\psi = v/2 \in \mathcal{E}(M, \xi, \omega^T)$. To show that $v \in \mathcal{E}(M, \xi, \omega^T)$, we observe that $\{v \le -2j\} = \{\psi \le -j\}$ and $\omega_{\psi_j} \ge \omega_{v_{2j}}/2$, hence

$$\int_{\{v\le -2j\}} \omega_{v_{2j}}^n \wedge \eta \le 2^n \int_{\{v\le -2j\}} \omega_{\psi_j}^n \wedge \eta \le 2^n \int_{\{\psi\le -j\}} \omega_{\psi_j}^n \wedge \eta.$$

By letting $j \to \infty$, we can then conclude that $v \in \mathcal{E}(M, \xi, \omega^T)$. □

Proposition 3.8 (Generalized Bedford–Taylor identity) *For* $u \in \mathcal{E}(M, \xi, \omega^T)$*,* $v \in$ *PSH*(M, ξ, ω^T)*, then* $\max\{u, v\} \in \mathcal{E}(M, \xi, \omega^T)$ *and*

$$\chi_{\{u>v\}}\omega_{\max(u,v)}^n \wedge \eta = \chi_{\{u>v\}}\omega_u^n \wedge \eta. \tag{3.17}$$

Proof Our argument is identical to the Kähler setting; see [44, Corollary 1.7] and [30, Lemma 2.5]. Proposition 3.7 implies that $w := \max\{u, v\} \in \mathcal{E}(M, \xi, \omega^T)$. Now observe that $\max\{u_j, v_{j+1}\} = \max\{u, v, -j\} = w_j$. Since the cutoffs are bounded we have

$$\chi_{\{u_j>v_{j+1}\}}\omega_{w_j}^n \wedge \eta = \chi_{\{u_j>v_{j+1}\}}\omega_{u_j}^n \wedge \eta. \tag{3.18}$$

By 3.7, we know that $\chi_{\{u>v\}}\omega_{u_j}^n\wedge\eta \to \chi_{\{u>v\}}\omega_u^n\wedge\eta$ and $\chi_{\{u>v\}}\omega_{w_j}^n\wedge\eta \to \chi_{\{u>v\}}\omega_w^n\wedge \eta$ as $j \to \infty$ (we also use the fact that $u, w \in \mathcal{E}(M, \xi, \omega^T)$). Since

$$\{u > v\} \subset \{u_j > v_{j+1}\} \text{ and } \{u_j > v_{j+1}\}\backslash\{u > v\} \subset \{u \le -j\},$$

it follows that

$$0 \le (\chi_{\{u_j>v_{j+1}\}} - \chi_{\{u>v\}})\omega_{u_j}^n \wedge \eta \le \chi_{\{u\le -j\}}\omega_{u_j}^n \wedge \eta \to 0.$$

Similarly since

$$\{u_j > v_{j+1}\}\backslash\{u > v\} \subset \{w \le -j\}$$

we also obtain that

$$0 \leq (\chi_{\{u_j > v_{j+1}\}} - \chi_{\{u>v\}})\omega_{w_j}^n \wedge \eta \leq \chi_{\{w \leq -j\}}\omega_{w_j}^n \wedge \eta \to 0.$$

By taking limit in (3.18) together with the limit facts above, we get the desired result. □

Next we introduce finite-energy class on Sasaki manifolds, following [44]. By considering Young weights $\chi \in \mathcal{W}_p^+$ (see [30, Chapter 1] for a short introduction to Young weights), one can introduce various finite-energy subclasses of $\mathcal{E}(M, \xi, \omega^T)$,

$$\mathcal{E}_\chi(M, \xi, \omega^T) := \{u \in \mathcal{E}(M, \xi, \omega^T) : E_\chi(u) < \infty\},$$

where E_χ is the χ-energy defined by

$$E_\chi(u) := \int_M \chi(u)\omega_u^n \wedge \eta.$$

Of special importance are the weights $\chi^p(t) = |t|^p/p$ and the associated classes $\mathcal{E}_p(M, \xi, \omega^T)$. For theses weights it is clear that $\mathcal{E}_p(M, \xi, \omega^T) \subset \mathcal{E}_1(M, \xi, \omega^T)$ for $p \geq 1$. We will need the following straightforward fact:

Proposition 3.9 *For any $u \in \mathcal{E}_1(M, \xi, \omega^T)$, u has Lelong number zero at every point.*

Proof For similar results in Kähler case, see [44, Corollary 1.8]. This is straightforward. We can assume $\sup u = 0$. For $u \in \mathcal{E}_1(M, \xi, \omega^T)$, we have

$$\int_M (-u)\omega_u^n \wedge \eta < \infty.$$

We consider locally $(0, 0) \in W_\alpha = (-\delta, \delta) \times V_\alpha$ in a foliation chart. Then we have

$$2\delta \int_{V_\alpha} (-u)\omega_u^n < \int_M (-u)\omega_u^n \wedge \eta < \infty.$$

This implies that u has Lelong number zero at $(0, 0)$. □

The following result implies that to test membership in $\mathcal{E}_\chi(M, \xi, \omega^T)$ it is enough to test the finiteness condition $E_\chi(u) < \infty$ on canonical cutoffs.

Proposition 3.10 *Suppose $u \in \mathcal{E}(M, \xi, \omega^T)$ with canonical cutoffs $\{u_k\}_{k\in\mathbb{N}}$. If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and increasing, then*

$$\int_M h(|u|)\omega_u^n \wedge \eta < \infty \Longleftrightarrow \limsup_{k\to\infty} \int_M h(|u_k|)\omega_{u_k}^n \wedge \eta < \infty.$$

Moreover, if the above condition holds, then

$$\int_M h(|u|)\omega_u^n \wedge \eta = \lim_{k\to\infty} \int_M h(|u_k|)\omega_{u_k}^n \wedge \eta.$$

Proof Our proof is similar to the Kähler case, see [30, Proposition 2.6]. Without loss of generality we can assume that $u \le 0$. If $\limsup_{k\to\infty} \int_M h(|u_k|)\omega_{u_k}^n \wedge \eta < \infty$, we obtain that the sequence of Radon measures $h(|u_k|)\omega_{u_k}^n \wedge \eta$ is weakly compact. Hence there exists a subsequence $h(|u_{k_j}|)\omega_{u_{k_j}}^n \wedge \eta$ converging weakly to a Radon measure μ. Recall that $h(|u_{k_j}|)$ is an increasing sequence of lower semicontinuous functions converging to $h(|u|)$ and $\omega_{u_{k_j}}^n \wedge \eta \xrightarrow{w} \omega_u^n \wedge \eta$, this yields that $h(|u|)\omega_u^n \wedge \eta \le \mu$ as measure. In particular $\int_M \omega_u^n \wedge \eta \le \mu(M) < \infty$.

Now assume $\int_M h(|u|)\omega_u^n \wedge \eta < \infty$. If $\lim_{t\to+\infty} h(t) = +\infty$, we have

$$\lim_{k\to\infty} \int_{\{u\le -k\}} h(|u|)\omega_u^n \wedge \eta = \lim_{l\to+\infty} \int_{\{h(|u|)>l\}} h(|u|)\omega_u^n \wedge \eta = 0.$$

It follows from Propositions 3.5, 3.8 and Definition 3.3 that

$$\int_{\{u\le -k\}} \omega_{u_k}^n \wedge \eta = \int_{\{u\le -k\}} \omega_u^n \wedge \eta.$$

Then by Propositions 3.5, 3.8 and Definition 3.3 again we have

$$\begin{aligned}
&\left|\int_M h(|u_k|)\omega_{u_k}^n \wedge \eta - \int_M h(|u|)\omega_u^n \wedge \eta\right| \\
&\le \int_{\{u\le -k\}} h(k)\omega_{u_k}^n \wedge \eta + \int_{\{u\le -k\}} h(|u|)\omega_u^n \wedge \eta \\
&= h(k)\int_{\{u\le -k\}} \omega_u^n \wedge \eta + \int_{\{u\le -k\}} h(|u|)\omega_u^n \wedge \eta \\
&\le 2\int_{\{u\le -k\}} h(|u|)\omega_u^n \wedge \eta.
\end{aligned}$$

It follows that $\int_M h(|u_k|)\omega_{u_k}^n \wedge \eta$ is bounded and $\int_M h(|u|)\omega_u^n \wedge \eta = \lim_{k\to\infty} \int_M h(|u_k|)\omega_{u_k}^n \wedge \eta$.

If $\lim_{t\to+\infty} h(t) = L < \infty$, it follows from Proposition 3.5 that $\int_M h(|u_k|)\omega_{u_k}^n \wedge \eta$ is bounded. Moreover for any $\epsilon > 0$ there exists $N > 0$ such that $0 < L - h(t) < \epsilon$ for all $t > N$. By Propositions 3.5, 3.8 and Definition 3.3 we have

$$\begin{aligned}
&\left|\int_M h(|u_k|)\omega_{u_k}^n \wedge \eta - \int_M h(|u|)\omega_u^n \wedge \eta\right| \\
&= \left|\int_M (L - h(|u_k|))\omega_{u_k}^n \wedge \eta - \int_M (L - h(|u|))\omega_u^n \wedge \eta\right| \\
&\le \int_{\{u\le -k\}} (L - h(|u_k|))\omega_{u_k}^n \wedge \eta + \int_{\{u\le -k\}} (L - h(|u|))\omega_u^n \wedge \eta \\
&\le 2\epsilon \mathrm{Vol}(M)
\end{aligned}$$

for $k > N$. It yields that $\int_M h(|u|)\omega_u^n \wedge \eta = \lim_{k\to\infty} \int_M h(|u_k|)\omega_{u_k}^n \wedge \eta$. $\square$

With the proposition above, we can then prove the so-called *fundamental estimate*.

Proposition 3.11 (Fundamental estimate) *Suppose* $\chi \in \mathcal{W}_p^+$ *and* $u, v \in \mathcal{E}_\chi(M, \xi, \omega^T)$ *such that* $u \le v \le 0$*. Then*

$$E_\chi(v) \le (p+1)^n E_\chi(u). \tag{3.19}$$

Proof The proof is similar to the Kähler case, see [44, Lemma 3.5]. First of all we assume that $u, v \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$. For $0 \le j \le n-1$, we have

$$\begin{aligned}\int_M \chi(u)\omega_v^{j+1} \wedge \omega_u^{n-j-1} \wedge \eta = &\int_M \chi(u)\omega^T \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &+ \int_M \sqrt{-1}\chi(u)\partial_B\overline{\partial}_B v \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta.\end{aligned}$$

Recall that $\chi'(l) \le 0$ for $l < 0$. Using integration by parts, we have

$$\begin{aligned}\int_M \chi(u)\omega^T \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta &= \int_M \chi(u) \wedge \omega_v^j \wedge \omega_u^{n-j} \wedge \eta\\ &\quad - \int_M \sqrt{-1}\chi(u)\partial_B\overline{\partial}_B u \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &= \int_M \chi(u) \wedge \omega_v^j \wedge \omega_u^{n-j} \wedge \eta\\ &\quad + \int_M \sqrt{-1}\chi'(u)\partial_B u \wedge \overline{\partial}_B u \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &\le \int_M \chi(u) \wedge \omega_v^j \wedge \omega_u^{n-j} \wedge \eta.\end{aligned}$$

Recall that $\chi'(l) \le 0$ for $l < 0$ and $l\chi'(l) \le p\chi(l)$ for $l \ge 0$. Using the integration by parts repeatedly, we have

$$\begin{aligned}&\int_M \sqrt{-1}\chi(u)\partial_B\overline{\partial}_B v \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &= \int_M \sqrt{-1}v\chi''(u)\partial_B u \wedge \overline{\partial}_B u \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &\quad + \int_M \sqrt{-1}v\chi'(u)\partial_B\overline{\partial}_B u \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &\le \int_M \sqrt{-1}v\chi'(u)\partial_B\overline{\partial}_B u \wedge \omega_v^j \wedge \omega_u^{n-j-1} \wedge \eta\\ &\le \int_M v\chi'(u)\omega_v^j \wedge \omega_u^{n-j} \wedge \eta = \int_M |v|\chi'(|u|)\omega_v^j \wedge \omega_u^{n-j} \wedge \eta\\ &\le \int_M |u|\chi'(|u|)\omega_v^j \wedge \omega_u^{n-j} \wedge \eta \le p\int_M \chi(|u|)\omega_v^j \wedge \omega_u^{n-j} \wedge \eta.\end{aligned}$$

Combine the inequalities above we obtain

$$\int_M \chi(u)\omega_v^{j+1}\wedge\omega_u^{n-j-1}\wedge\eta \le (p+1)\int_M \chi(u)\omega_v^j\wedge\omega_u^{n-j}\wedge\eta.$$

It follows that

$$E_\chi(v)\le \int_M \chi(u)\omega_v^n\wedge\eta \le (p+1)^n E_\chi(u).$$

In the general case $u, v\in \mathcal{E}_\chi(M,\xi,\omega^T)$, we have $E_\chi(v_k)\le (p+1)^n E_\chi(u_k)$ for the canonical cutoffs u_k, v_k of u, v. It follows from Proposition 3.10 that $E_\chi(v)\le (p+1)^n E_\chi(u)$. □

As a direct consequence, we obtain the *monotonicity property* for $\mathcal{E}_\chi(M,\xi,\omega^T)$.

Proposition 3.12 *Suppose $u\in\mathcal{E}_\chi(M,\xi,\omega^T)$ and $v\in PSH(M,\xi,\omega^T)$. If $u\le v$, then $v\in\mathcal{E}_\chi(M,\xi,\omega^T)$.*

Proof Without loss of generality we can assume that $u\le v\le 0$. The monotonicity property implies that $v\in\mathcal{E}(M,\xi,\omega^T)$. We have $u\le v_k$ for the canonical cutoffs v_k of v, then $E_\chi(v_k)\le(p+1)^n E_\chi(u)$ according to the Proposition 3.11. It follows from Proposition 3.10 that $E_\chi(v)\le(p+1)^n E_\chi(u)$ and $v\in\mathcal{E}_\chi(M,\xi,\omega^T)$. □

We also have the following,

Proposition 3.13 *Suppose $u, v\in\mathcal{E}_\chi(M,\xi,\omega^T)$ for $\chi\in\mathcal{W}_p^+$. If $u, v\le 0$, then*

$$\int_M \chi(u)\omega_v^n\wedge\eta \le p2^p(E_\chi(u)+E_\chi(v)).$$

Proof For similar result is Kähler case, see [44, Proposition 3.6]. For $\delta>0$, we have $\tilde\chi(t)=\chi(t)+\delta|t|\in\mathcal{W}_p^+$. Assume that $t>0$, it is obvious $\tilde\chi(t), \tilde\chi'(t)>0$. Recall that $\epsilon^p\tilde\chi(t)\le\tilde\chi(\epsilon t)$ and $t\tilde\chi'(t)\le p\tilde\chi(t)$ for $\tilde\chi\in\mathcal{W}_p^+$ and $0<\epsilon<1$, hence we have $\tilde\chi(2t)\le 2^p\tilde\chi(t)$. It follows from the convexity of the function $\tilde\chi(t)$ that $\frac{\tilde\chi(t)}{t}\le\tilde\chi'(t)$. Then

$$\tilde\chi'(2t)=\frac{1}{2}\frac{2t\tilde\chi'(2t)}{\tilde\chi(2t)}\frac{\tilde\chi(2t)}{\tilde\chi(t)}\frac{\tilde\chi(t)}{t}\le p2^{p-1}\tilde\chi'(t).$$

Then $\delta\to 0$ implies that $\chi'(2t)\le p2^{p-1}\chi'(t)$ for $t>0$.

By Proposition 3.6 and $\{|u|>2t\}\subset\{u<v-t\}\cup\{v<-t\}$, we have

$$\begin{aligned}\int_M \chi(u)\omega_v^n\wedge\eta &= \int_0^\infty \chi'(t)\omega_v^n\wedge\eta\{|u|>t\}\mathrm{d}t\\ &\le p2^p\int_0^\infty \chi'(t)\omega_v^n\wedge\eta\{|u|>2t\}\mathrm{d}t\\ &\le p2^p\left(\int_0^\infty \chi'(t)\omega_v^n\wedge\eta\{u<v-t\}\mathrm{d}t\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \chi'(t)\omega_v^n \wedge \eta\{v < -t\}\mathrm{d}t\Big) \\
&\leq p2^p\left(\int_0^\infty \chi'(t)\omega_u^n \wedge \eta\{u < v - t\}\mathrm{d}t + E_\chi(v)\right) \\
&\leq p2^p\left(\int_0^\infty \chi'(t)\omega_u^n \wedge \eta\{u < -t\}\mathrm{d}t + E_\chi(v)\right) \\
&= p2^p(E_\chi(u) + E_\chi(v)).
\end{aligned}
$$

□

Proposition 3.14 *Suppose $u \in \mathcal{E}_\chi(M, \xi, \omega^T)$, $\chi \in \mathcal{W}_p^+$. Then there exists $\tilde{\chi} \in \mathcal{W}_{2p+1}^+$ such that $\chi(t) \leq \tilde{\chi}(t)$, $\chi(t)/\tilde{\chi}(t) \to 0$ as $t \to \infty$ and $u \in \mathcal{E}_{\tilde{\chi}}(M, \xi, \omega^T)$.*

Proof This construction borrows from similar results in Kähler case, see [30, Lemma 2.10]. Take $\chi_0 = \chi$, recall that $\lim_{t\to\infty} \chi_0(t) = \infty$ and $u \in \mathcal{E}_\chi(M, \xi, \omega^T)$, we have

$$
\lim_{t\to\infty} \int_{\{|u|>t\}} \chi(|u|)\omega_u^n \wedge \eta = \lim_{s\to\infty} \int_{\{\chi(u)>s\}} \chi(|u|)\omega_u^n \wedge \eta = 0.
$$

Then one can choose $t_1 > 0$ such that $\int_{\{|u|>t_1\}} \chi(|u|)\omega_u^n \wedge \eta < \frac{1}{2^2}$. We define $\chi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ by the formula:

$$
\chi_1(t) = \begin{cases} \chi_0(t) & \text{if} \quad t \leq t_1 \\ \chi_0(t_1) + 2(\chi_0(t) - \chi_0(t_1)) & \text{if} \quad t > t_1. \end{cases}
$$

Then it is easy to verify that

(1) $\chi_0(t) \leq \chi_1(t)$;
(2) $\lim_{t\to\infty} \frac{\chi_1(t)}{\chi_0(t)} = 2$;
(3) $E_{\chi_1}(u) \leq E_{\chi_0}(u) + \frac{1}{2}$;
(4) $\sup_{t>0} \frac{|t\chi_1'(t)|}{|\chi_1(t)|} \leq \sup_{t>0} \frac{2|t\chi_0'(t)|}{|\chi_0(t)|} < 2p + 1$;
(5) $\lim_{t\to\infty} \frac{t\chi_1'(t)}{\chi_1(t)} \leq p$.

These properties imply that for $t_2 > t_1$ big enough, the function $\chi_2 : \mathbb{R}^+ \to \mathbb{R}^+$

$$
\chi_2(t) = \begin{cases} \chi_1(t) & \text{if} \quad t \leq t_2 \\ \chi_1(t_2) + 2(\chi_1(t) - \chi_1(t_2)) & \text{if} \quad t > t_2 \end{cases}
$$

satisfies

(1) $\chi_1(t) \leq \chi_2(t)$;
(2) $\lim_{t\to\infty} \frac{\chi_2(t)}{\chi_1(t)} = 2$;
(3) $E_{\chi_2}(u) \leq E_{\chi_1}(u) + \frac{1}{2^2}$;

(4) $\sup_{t>0} \frac{|t\chi_2'(t)|}{|\chi_2(t)|} < 2p+1$;

(5) $\lim_{t\to\infty} \frac{t\chi_2'(t)}{\chi_2(t)} \leq p$.

Continuing the above construction we can obtain an increasing sequence $\{\chi_k\}_k$ and the limit weight $\tilde{\chi}(t) = \lim_{k\to\infty} \chi_k(t)$ will satisfy the requirements of the proposition. □

Proposition 3.15 *Assume that* $\{\psi_k\}_{k\in\mathbb{N}}, \{\phi_k\}_{k\in\mathbb{N}}, \{v_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_\chi(M,\xi,\omega^T)$ *decrease (increase a. e) to* $\phi, \psi, v \in \mathcal{E}_\chi(M,\xi,\omega^T)$, *respectively. Suppose*

(1) $\psi_k \leq \phi_k$ *and* $\psi_k \leq v_k$.

(2) $h : \mathbb{R} \to \mathbb{R}$ *is continuous with* $\limsup_{|l|\to\infty} |h(l)|/\chi(l) \leq C$ *for some* $C \geq 0$.

Then we have the weak convergence of

$$h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta \to h(\phi - \psi)\omega_v^n \wedge \eta.$$

Proof For similar results in Kähler case, see [30, Proposition 2.11]. Without loss of generality one can assume all the functions $\phi_k, \phi, \psi_k, \psi, v, v_k$ are negative. We will only prove the proposition for decreasing sequences, the case of increasing sequences can be proved similarly.

First of all we suppose that the functions involved are uniformly bounded below, that is, there exists $L > 1$ such that $-L \leq \phi_k, \phi, \psi_k, \psi, v_k, v \leq 0$. Given $\epsilon > 0$, it follows from Theorem 6.3 that there exists an open subset $O_\epsilon \subset M$ such that $\mathrm{cap}(O_\epsilon) < \epsilon$ and $\phi_k, \phi, \psi_k, \psi, v_k, v$ are continuous on $M - O_\epsilon$. Then $\phi_k \to \phi$ and $\psi_k \to \psi$ uniformly on $M - O_\epsilon$. Hence there exists N such that for $k > N$ we have $|h(\phi_k - \psi_k) - h(\phi - \psi)| < \epsilon$ on $M - O_\epsilon$ and the term

$$\begin{aligned}&\int_M h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta - \int_M h(\phi - \psi)\omega_{v_k}^n \wedge \eta \\ &\quad = \left(\int_{O_\epsilon} + \int_{M-O_\epsilon}\right)[h(\phi_k - \psi_k) - h(\phi - \psi)]\omega_{v_k}^n \wedge \eta\end{aligned}$$

is bounded by $2\epsilon L^n \max_{|l|\leq L} |h(l)| + \epsilon \mathrm{Vol}(M)$. Hence

$$\int_M h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta - \int_M h(\phi - \psi)\omega_{v_k}^n \wedge \eta \to 0. \tag{3.20}$$

Given $\epsilon > 0$, it follows from Theorem 6.3 that there exists an open subset $\tilde{O}_\epsilon$ such that $\mathrm{cap}(\tilde{O}_\epsilon) < \epsilon$ and ϕ, ψ are continuous on $M - \tilde{O}_\epsilon$. By the Tietze's extension theorem, the function $h(\phi - \psi)|_{M-\tilde{O}_\epsilon}$ can be extended to a continuous function α on M bounded by $\max_{|l|\leq L} |h(l)|$. By Proposition 3.1 we have $\omega_{v_k}^n \wedge \eta \to \omega_v^n \wedge \eta$ weakly. Then there exists a constant N such that for $k > N$ we have $|\int_M \alpha\omega_{v_k}^n \wedge \eta - \int_M \alpha\omega_v^n \wedge \eta| < \epsilon$ and the term

$$\int_M h(\phi - \psi)\omega_{v_k}^n \wedge \eta - \int_M h(\phi - \psi)\omega_v^n \wedge \eta$$

$$= \int_{O_\epsilon} (h(\phi - \psi) - \alpha)\omega_{v_k}^n \wedge \eta - \int_{O_\epsilon} (h(\phi - \psi) - \alpha)\omega_v^n \wedge \eta$$
$$+ \left(\int_M \alpha\omega_{v_k}^n \wedge \eta - \int_M \alpha\omega_v^n \wedge \eta \right)$$

is bounded by $4\epsilon L^n \max_{|l| \le L} |h(l)| + \epsilon$. Hence

$$\int_M h(\phi - \psi)\omega_{v_k}^n \wedge \eta - \int_M h(\phi - \psi)\omega_v^n \wedge \eta \to 0. \tag{3.21}$$

It follows from (3.20) and (3.21) that $h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta \to h(\phi - \psi)\omega_v^n \wedge \eta$.

Now consider the general case when $\phi_k, \phi, \psi_k, \psi, v_k, v$ are unbounded. Let $\phi_k^l, \phi^l, \psi_k^l, \psi^l, v_k^l, v^l$ be the canonical cutoffs of the corresponding potentials, then we only have to show that

$$\int_M h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta - \int_M h(\phi_k^l - \psi_k^l)\omega_{v_k^l}^n \wedge \eta \to 0 \tag{3.22}$$

and

$$\int_M h(\phi - \psi)\omega_v^n \wedge \eta - \int_M h(\phi^l - \psi^l)\omega_{v^l}^n \wedge \eta \to 0 \tag{3.23}$$

as $l \to \infty$ uniformly with respect to k.

By Proposition 3.14 there exists $\tilde{\chi} \in \mathcal{W}_{2p+1}^+$ such that $\chi \le \tilde{\chi}$, $\lim_{t\to\infty} \frac{\chi(t)}{\tilde{\chi}(t)} = 0$ and $\psi \in \mathcal{E}_{\tilde{\chi}}(M, \xi, \omega^T)$. Then $\psi_k, \phi_k, \phi, v_k, v \in \mathcal{E}_{\tilde{\chi}}(M, \xi, \omega^T)$ according to Proposition 3.12.

Recall that there exists $L > 0$ such that $\chi(L) \ge 1$ and $|h(t)| \le (C+1)\chi(t)$ for $|t| > L$. Take $\tilde{C} = \max\{C + 1, \frac{\max_{0\le l\le L} |h(l)|}{\chi(L)}\}$, then we have

$$|h(l_1 - l_2)| \le \tilde{C}\chi(l_2)$$

for $l_2 \le -L$ and $l_2 \le l_1 \le 0$. Using Propositions 3.8, 3.11, and 3.13, we have

$$\left| \int_M h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta - \int_M h(\phi_k^l - \psi_k^l)\omega_{v_k^l}^n \wedge \eta \right|$$
$$= \left| \int_{\{\psi_k \le -l\}} h(\phi_k - \psi_k)\omega_{v_k}^n \wedge \eta - \int_{\{\psi_k \le -l\}} h(\phi_k^l - \psi_k^l)\omega_{v_k^l}^n \wedge \eta \right|$$
$$\le \int_{\{\psi_k \le -l\}} |h(\phi_k - \psi_k)|\omega_{v_k}^n \wedge \eta + \int_{\{\psi_k \le -l\}} |h(\phi_k^l - \psi_k^l)|\omega_{v_k^l}^n \wedge \eta$$
$$\le \tilde{C} \left(\int_{\{\psi_k \le -l\}} \chi(\psi_k)\omega_{v_k}^n \wedge \eta + \int_{\{\psi_k \le -l\}} \chi(\psi_k^l)\omega_{v_k^l}^n \wedge \eta \right)$$
$$\le \tilde{C} \sup_{s \le -l} \frac{\chi(s)}{\tilde{\chi}(s)} \left(\int_{\{\psi_k \le -l\}} \tilde{\chi}(\psi_k)\omega_{v_k}^n \wedge \eta + \int_{\{\psi_k \le -l\}} \tilde{\chi}(\psi_k^l)\omega_{v_k^l}^n \wedge \eta \right)$$

$$
\begin{aligned}
&\leq \tilde{C} \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)} \left(\int_M \tilde{\chi}(\psi_k)\omega_{v_k}^n \wedge \eta + \int_M \tilde{\chi}(\psi_k^l)\omega_{v_k^l}^n \wedge \eta \right) \\
&\leq (2p+1)2^{2p+1}\tilde{C} \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)} (E_{\tilde{\chi}}(\psi_k) + E_{\tilde{\chi}}(v_k) + E_{\tilde{\chi}}(\psi_k^l) + E_{\tilde{\chi}}(v_k^l)) \\
&\leq 4(2p+1)(2p+2)^n 2^{2p+1}\tilde{C} E_{\tilde{\chi}}(\psi) \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)}.
\end{aligned}
$$

for $l > L$ and the statement (3.22) follows. We also have

$$
\begin{aligned}
&\left| \int_M h(\phi-\psi)\omega_v^n \wedge \eta - \int_M h(\phi^l - \psi^l)\omega_{v^l}^n \wedge \eta \right| \\
&= \left| \int_{\{\psi\leq -l\}} h(\phi-\psi)\omega_v^n \wedge \eta - \int_{\{\psi\leq -l\}} h(\phi^l - \psi^l)\omega_{v^l}^n \wedge \eta \right| \\
&\leq \int_{\{\psi\leq -l\}} |h(\phi-\psi)|\omega_v^n \wedge \eta + \int_{\{\psi\leq -l\}} |h(\phi^l - \psi^l)|\omega_{v^l}^n \wedge \eta \\
&\leq \tilde{C} \left(\int_{\{\psi\leq -l\}} \chi(\psi)\omega_v^n \wedge \eta + \int_{\{\psi\leq -l\}} \chi(\psi^l)\omega_{v^l}^n \wedge \eta \right) \\
&\leq \tilde{C} \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)} \left(\int_{\{\psi\leq -l\}} \tilde{\chi}(\psi)\omega_v^n \wedge \eta + \int_{\{\psi\leq -l\}} \tilde{\chi}(\psi^l)\omega_{v^l}^n \wedge \eta \right) \\
&\leq \tilde{C} \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)} \left(\int_M \tilde{\chi}(\psi)\omega_v^n \wedge \eta + \int_M \tilde{\chi}(\psi^l)\omega_{v^l}^n \wedge \eta \right) \\
&\leq (2p+1)2^{2p+1}\tilde{C} \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)} (E_{\tilde{\chi}}(\psi) + E_{\tilde{\chi}}(v) + E_{\tilde{\chi}}(\psi^l) + E_{\tilde{\chi}}(v^l)) \\
&\leq 4(2p+1)(2p+2)^n 2^{2p+1}\tilde{C} E_{\tilde{\chi}}(\psi) \sup_{s\leq -l} \frac{\chi(s)}{\tilde{\chi}(s)}.
\end{aligned}
$$

for $l > L$ and the statement (3.23) follows. This completes the proof. □

Proposition 3.16 *Suppose $\chi \in \mathcal{W}_p^+$ and $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_\chi(M, \xi, \omega^T)$ is a decreasing sequence converging to $u \in PSH(M, \xi, \omega^T)$. If $\sup_k E_\chi(u_k) < \infty$, then $u \in \mathcal{E}_\chi(M, \xi, \omega^T)$ and*

$$E_\chi(u) = \lim_{k\to\infty} E_\chi(u_k).$$

Proof For similar results in Kähler case, see [44, Proposition 5.6]. Without loss of generality, we assume that $u_1 \leq 0$. The canonical cutoffs $u_k^l = \max\{u_k, -l\}$ decreases to the canonical cutoff $u^l = \max\{u, -l\}$. As $-l \leq u^l \leq u_k^l \leq 0$, Propositions 3.15 and 3.11 imply that

$$E_\chi(u^l) = \lim_{k\to\infty} E_\chi(u_k^l) \leq (p+1)^n \sup_k E_\chi(u_k).$$

By Proposition 3.10, $u \in \mathcal{E}_\chi(M, \xi, \omega^T)$. Applying the previous proposition in the case $\psi_k = v_k = u_k, \phi_k = 0$ gives that $E_\chi(u) = \lim_{k\to\infty} E_\chi(u_k)$. □

A very important notion in pluripotential theory is the *envelop construction*, which we shall describe below. In our setting on a compact Sasaki manifold, given a usc function $f \in M \to [-\infty, \infty)$ such that f is invariant under the Reeb flow, we consider the envelop

$$P(f) := \sup\{u \in \mathrm{PSH}(M, \xi, \omega^T) \text{ such that } u \leq f\}. \tag{3.24}$$

As in Kähler setting, we have the following

Proposition 3.17 *The envelop construction $P(f) \in PSH(M, \xi, \omega^T)$.*

Proof This statement is local in nature, hence we only need to argue in foliations charts $W_\alpha = (-\delta, \delta) \times V_\alpha$, where $V_\alpha \subset \mathbb{C}^n$ give a transverse holomorphic charts. Since $P(f)$ is invariant under the Reeb flow, its usc regularization $P(f)^*$ is invariant under the Reeb flow. Hence by $P(f)^*$ is ω^T_α-psh on each V_α, see [12, Theorem 1.2.3 (viii)]. Since f is usc, hence $P(f)^* \leq f^* = f$. Hence $P(f)^*$ is a candidate in the definition of $P(f)$, gives that $P(f)^* \leq P(f)$. This implies that $P(f) = P(f)^*$ and $P(f) \in \mathrm{PSH}(M, \xi, \omega^T)$. □

We also introduce the notion *rooftop envelop*, for usc functions $f_1, \ldots, f_n$ which are invariant under the Reeb flow,

$$P(f_1, \ldots, f_n) := P(\min\{f_1, \ldots, f_n\}).$$

We have the following,

Theorem 3.1 *Given $f \in C^\infty_B(M)$, then we have the following estimate*

$$\|P(f)\|_{C^{1,\bar{1}}} \leq C(M, \omega^T, g, \|f\|_{C^{1,\bar{1}}}).$$

Moreover, if $u_1, \ldots, u_k \in \mathcal{H}_\Delta$, where we use the notation

$$\mathcal{H}_\Delta = \{u \in PSH(M, \xi, \omega^T) : \|u\|_{C^{1,\bar{1}}} < \infty\}$$

then $P(u_1, \ldots, u_k) \in \mathcal{H}_\Delta$.

We shall prove Theorem 3.1 in Appendix. The following result (for similar result in Kähler case, see [2, Corollary 9.2]) would be very essential for the rooftop envelop $P(u_0, u_1)$:

Lemma 3.2 *For $u_0, u_1 \in \mathcal{H}_\Delta$, then*

$$\omega^n_{P(u_0,u_1)} \wedge \eta = 0 \tag{3.25}$$

on the non-contact set $\Gamma = \{P(u_0, u_1) < \min(u_0, u_1)\}$.

Proof First we assume ξ is regular or quasiregular, then the proof follows similarly as in Kähler setting. We sketch the proof briefly. We consider the quotient Kähler manifold (orbifold) $(Z = M/\mathcal{F}_\xi, \omega_Z)$ such that $\omega^T = \pi^*\omega_Z$, where $\pi : M \to Z$ is the natural quotient map. Since u_0, u_1, and $P(u_0, u_1)$ are all basic functions, and they descend to Z to define the functions on Z, which we still denote as u_0, u_1, and $P(u_0, u_1)$. We only need to show that $(\omega_Z + \sqrt{-1}\partial\bar{\partial}P(u_0, u_1))^n = 0$ on $\Gamma_Z := \{z \in Z : P(u_0, u_1) < \min(u_0, u_1)\}$. Note that $\Gamma_Z = \pi(\Gamma)$. This simply follows from [2, Corollary 9.2].

Now we deal with the case when ξ is irregular. We need to use a Type-I deformation to approximate (M, ξ, η, g, Φ), as in Theorem 6.1. Denote T^k to be the torus in $\mathrm{Aut}(\xi, \eta, g)$ with the Lie algebra $\mathfrak{t}$. Take $\rho_i \in \mathfrak{t}$ such that $\rho_i \to 0$ (convergence is smooth with respect to a fixed metric g). We can take ρ_i such that $\xi_i = \xi + \rho_i$ is quasiregular. Consider the Type-I deformation $(M, \xi_i, \eta_i, g_i, \Phi_i)$ as in Definition 2.3. Given $u_0, u_1 \in \mathcal{H}_\Delta$ and we know that $P(u_0, u_1) \in \mathcal{H}_\Delta$ (see Theorem 3.1), by Lemma 6.1, there exists $\epsilon_i \to 0$ such that $(1-\epsilon_i)u_0, (1-\epsilon_i)u_1, (1-\epsilon_i)P(u_0, u_1) \in \mathrm{PSH}(M, \xi_i, \omega_i^T)$. Define

$$P_i = P_i((1-\epsilon_i)u_0, (1-\epsilon_i)u_1) = \sup\{v \in \mathrm{PSH}(M, \xi_i, \omega_i^T), v \le (1-\epsilon_i)u_0, (1-\epsilon_i)u_1\}. \tag{3.26}$$

Since $(1-\epsilon_i)P(u_0, u_1) \in \mathrm{PSH}(M, \xi_i, \omega_i^T)$ and $(1-\epsilon_i)P(u_0, u_1) \le (1-\epsilon_i)u_0, (1-\epsilon_i)u_1$, hence $(1-\epsilon_i)P(u_0, u_1) \le P_i$. On the other hand, we apply Lemma 6.1 and we know there exists $\varepsilon_i \to 0$, such that $(1-\varepsilon_i)P_i \in \mathrm{PSH}(M, \xi, \omega^T)$. It follows that

$$(1-\varepsilon_i)P_i \le P(u_0, u_1) \le P_i(1-\epsilon_i)^{-1}.$$

By Theorem 3.1, we know that $|\mathrm{d}\Phi dP_i|$ is uniformly bounded and hence $P_i \to P(u_0, u_1)$ in $C^{1,\alpha}$. For any compact subset $K \subset \Gamma = \{P(u_0, u_1) < \min(u_0, u_1)\}$, we can choose i sufficiently large, such that $P_i < \min\{(1-\epsilon_i)u_0, (1-\epsilon_i)u_1\}$. Since ξ_i is quasiregular, by (3.26), we can then get that

$$\left(\omega_i^T + \frac{1}{2}\mathrm{d}\Phi_i dP_i\right)^n \wedge \eta_i = 0, \text{ on } K.$$

Taking $i \to \infty$, by Lemma 6.2, we get that

$$\left(\omega^T + \frac{1}{2}\mathrm{d}\Phi dP(u_0, u_1)\right)^n \wedge \eta = 0, \text{ on } K.$$

This completes the proof. □

As a consequence, we get a volume partition formula for $\omega^n_{P(u_0,u_1)} \wedge \eta$ as follows:

Lemma 3.3 *For $u_0, u_1 \in \mathcal{H}_\Delta$, denote $\Lambda_{u_0} = \{P(u_0, u_1) = u_0\}$ and $\Lambda_{u_1} = \{P(u_0, u_1) = u_1\}$. Then we have the following*

$$\omega^n_{P(u_0,u_1)} \wedge \eta = \chi_{\Lambda_{u_0}}\omega^n_{u_0} \wedge \eta + \chi_{\Lambda_{u_1}\setminus\Lambda_{u_0}}\omega^n_{u_1} \wedge \eta. \tag{3.27}$$

Proof The proof is similar to the Kähler case, see [29, Proposition 2.2]. The previous lemma implies that the measure $\omega^n_{P(u_0,u_1)} \wedge \eta$ is supported on the set $\Lambda_{u_0} \cup \Lambda_{u_1}$. It follows from Theorem 3.1 that $P(u_0, u_1)$ has bounded Laplacian, hence all second partial derivatives of $P(u_0, u_1)$ are in $L^p(M)$ for all $p > 1$. Then all the second-order partial derivatives of $P(u_0, u_1)$ and u_0 coincide on Λ_{u_0} almost everywhere, all the second-order partial derivatives of $P(u_0, u_1)$ and u_1 coincide on Λ_{u_1} almost everywhere. Recall the definition of Monge–Ampere operators on psh functions belong to $W^{2,n}$, we can write:

$$\omega^n_{P(u_0,u_1)} \wedge \eta = \chi_{\Lambda_{u_0}} \omega^n_{u_0} \wedge \eta + \chi_{\Lambda_{u_1} \setminus \Lambda_{u_0}} \omega^n_{u_1} \wedge \eta.$$

□

Lemma 3.4 *Suppose $\chi \in \mathcal{W}^+_p$ and $u_0, u_1 \in \mathcal{E}_\chi(M, \xi, \omega^T)$. Then $P(u_0, u_1) \in \mathcal{E}_\chi(M, \xi, \omega^T)$. If $u_0, u_1 \leq 0$, then the following estimates hold*

$$E_\chi(P(u_0, u_1)) \leq (p + 1)^n (E_\chi(u_0) + E_\chi(u_1)). \tag{3.28}$$

Proof The proof is similar to the Kähler case, see [29, Lemma 3.4]. Without loss of generality we can assume $u_0, u_1 < 0$. It follows from Lemma 3.1 that there exist negative transverse Kähler potentials $u^k_0, u^k_1 \in \mathcal{H}$ deceasing to u_0, u_1 respectively. By Theorem 3.1, the rooftop envelopes $P(u^k_0, u^k_1) \in \mathcal{H}_\Delta$ decreases to $P(u_0, u_1)$. And we have the following inequality by Lemma 3.3:

$$\omega^n_{P(u^k_0,u^k_1)} \wedge \eta \leq \chi_{\Lambda_{u_0}} \omega^n_{u_0} \wedge \eta + \chi_{\Lambda_{u_1}} \omega^n_{u_1} \wedge \eta.$$

Then

$$\begin{aligned} E_\chi(P(u^k_0, u^k_1)) &= \int_M \chi(P(u^k_0, u^k_1)) \omega^n_{P(u^k_0,u^k_1)} \wedge \eta \\ &\leq \int_{P(u^k_0,u^k_1)=u^k_0} \chi(u^k_0) \omega^n_{u^k_0} \wedge \eta + \int_{P(u^k_0,u^k_1)=u^k_1} \chi(u^k_1) \omega^n_{u^k_1} \wedge \eta \\ &\leq E_\chi(u^k_0) + E_\chi(u^k_1) \\ &\leq (p + 1)^n (E_\chi(u_0) + E_\chi(u_1)). \end{aligned}$$

By Proposition 3.16 we have $P((u_0, u_1)) \in \mathcal{E}_\chi(M, \xi, \omega^T)$ and the required inequality holds. □

As a corollary we know that $\mathcal{E}_\chi(M, \xi, \omega^T)$ is convex.

Corollary 3.1 *If $u_0, u_1 \in \mathcal{E}_\chi(M, \xi, \omega^T)$, then $tu_0 + (1-t)u_1 \in \mathcal{E}_\chi(M, \xi, \omega^T)$ for any $t \in [0, 1]$.*

Proof By the previous lemma we have $P(u_0, u_1) \in \mathcal{E}_\chi(M, \xi, \omega^T)$. Notice that $P(u_0, u_1) \leq tu_0 + (1 - t)u_1$ for $t \in [0, 1]$, then the monotonicity property of $\mathcal{E}_\chi(M, \xi, \omega^T)$ implies that $tu_0 + (1 - t)u_1 \in \mathcal{E}_\chi(M, \xi, \omega^T)$. □

To finish this subsection, we establish a domination principle which will be needed later.

Lemma 3.5 *Let $U \subset M$ be a Borel set with $(\omega^T)^n \wedge \eta(U) > 0$ and $u \in \mathcal{E}_1(M, \xi, \omega^T)$. Then there exists $\varphi \in \mathcal{E}_1(M, \xi, \omega^T)$ with $\varphi \leq u$ and $\omega_\varphi^n \wedge \eta(U) > 0$.*

Proof The proof is similar to the Kähler case, see [30, Lemma 2.22]. Without loss of generality we can assume that $u < 0$. Then we can choose a sequence $u_k \in \mathcal{H}$ decreasing to u with $u_k < 0$. For a constant $\tau > 0$, we have $\{P(u_k + \tau, 0) = u_k + \tau\} \subset \{u_k \leq -\tau\}$. It follows from Proposition 3.3 that

$$\omega^n_{P(u_k+\tau,0)} \wedge \eta \leq \chi_{\{u_k \leq -\tau\}} \omega^n_{u_k} \wedge \eta + (\omega^T)^n \wedge \eta \leq -\frac{u_k}{\tau} \omega^n_{u_k} \wedge \eta + (\omega^T)^n \wedge \eta.$$

The sequence $P(u_k+\tau, 0) \in \mathcal{E}_1(M, \xi, \omega^T)$ decreases to $P(u+\tau, 0) \in \mathcal{E}_1(M, \xi, \omega^T)$. It follows from Proposition 3.15 that

$$\omega^n_{P(u+\tau,0)} \wedge \eta \leq -\frac{u}{\tau} \omega^n_u \wedge \eta + (\omega^T)^n \wedge \eta.$$

Hence we have

$$\begin{aligned}
\omega^n_{P(u+\tau,0)} \wedge \eta(M - U) &\leq \frac{1}{\tau} \int_{M-U} |u| \omega^n_u \wedge \eta + (\omega^T)^n \wedge \eta(M - U) \\
&\leq \frac{1}{\tau} \int_M |u| \omega^n_u \wedge \eta + (\omega^T)^n \wedge \eta(M - U).
\end{aligned}$$

It follows from $\omega^n_{P(u+\tau,0)} \wedge \eta(M) = (\omega^T)^n \wedge \eta(M) = \mathrm{Vol}(M)$ that

$$\omega^n_{P(u+\tau,0)} \wedge \eta(U) \geq (\omega^T)^n \wedge \eta(U) - \frac{1}{\tau} \int_M |u| \omega^n_u \wedge \eta$$

and $\omega^n_{P(u+\tau,0)} \wedge \eta(U) > 0$ for τ big enough. Then $\varphi = P(u + \tau, 0) - \tau$ satisfies the requirements. □

Lemma 3.6 (The domination principle) *If $u, v \in \mathcal{E}_1(M, \xi, \omega^T)$ and $u \leq v$ almost everywhere with respect to the measure $\omega_v^n \wedge \eta$. Then $u \leq v$.*

Proof The proof is similar to the Kähler case, see [30, Proposition 2.21]. We only have to prove $u \leq v$ almost everywhere with respect to $(\omega^T)^n \wedge \eta$ for $u, v < 0$.

Suppose that $(\omega^T)^n \wedge \eta(\{u > v\}) > 0$. The previous lemma implies that there exists $\varphi \in \mathcal{E}_1(M, \xi, \omega^T)$ with $\varphi \leq u$ and $\omega^n_\varphi \wedge \eta(\{u > v\}) > 0$. It follows from Corollary 3.1 that $t\varphi + (1-t)u \in \mathcal{E}_1(M, \xi, \omega^T)$ for $t \in [0, 1]$. Using the fact $\omega^n_{t\varphi+(1-t)u} \wedge \eta \geq t^n \omega^n_\varphi \wedge \eta$, the Comparison principle (3.15) and $\{v < t\varphi + (1-t)u\} \subset \{v < u\}$, we have

$$t^n \int_{\{v<t\varphi+(1-t)u\}} \omega^n_\varphi \wedge \eta \leq \int_{\{v<t\varphi+(1-t)u\}} \omega^n_{t\varphi+(1-t)u} \wedge \eta$$

$$
\begin{aligned}
&\le \int_{\{v<t\varphi+(1-t)u\}} \omega_v^n \wedge \eta \\
&\le \int_{\{v<u\}} \omega_v^n \wedge \eta \\
&= 0
\end{aligned}
$$

and $\omega_\varphi^n \wedge \eta(\{v < t\varphi + (1-t)u\}) = 0$ for $t \in (0, 1]$. Then

$$
\omega_\varphi^n \wedge \eta(\{v < u\}) = \lim_{k\to\infty} \omega_\varphi^n \wedge \eta\left(\left\{v < \frac{1}{k}\varphi + (1-\frac{1}{k})u\right\}\right) = 0.
$$

This leads to a contradiction. □

3.2 The Space of Transverse Kähler Potentials and $(\mathcal{H}, d_2)$

The Riemannian structure on $\mathcal{H}$ has been studied extensively, notably by Guan–Zhang [42]. Guan–Zhang proved that for any two points $\phi_1, \phi_2 \in \mathcal{H}$, there exists a unique $C_B^{1,\bar{1}}$ geodesic which realizes the distance of $(\mathcal{H}, d_2)$ and $(\mathcal{H}, d_2)$ is a metric space. The Riemannian structure would play a very central role, as in Chen's result [20] in Kähler setting.

We shall recall these results. For $\psi_1, \psi_2 \in T_\phi\mathcal{H} = C_B^\infty(M)$, define a L^2 inner product on this tangent space

$$
(\psi_1, \psi_2)_\phi = \int_M \psi_1\psi_2 \mathrm{d}\mu_\phi
$$

and the length $||\psi||_\phi$ of a vector $\psi \in T_\phi\mathcal{H}$ is

$$
||\psi||_{2,\phi} = \left(\int_M \psi_1\psi_2 \mathrm{d}\mu_\phi\right)^{\frac{1}{2}},
$$

where we use the notation

$$
\mathrm{d}\mu_\phi = \omega_\phi^n \wedge \eta_\phi = \omega_\phi^n \wedge \eta. \tag{3.29}
$$

For a smooth path $\phi_t \in \mathcal{H}$, the length of the path is defined to be

$$
l(\phi_t) = \int_0^1 ||\dot{\phi}_t||_{2,\phi_t} \mathrm{d}t.
$$

This is a direct adaption of Mabuchi's metric [53] on the space of Kähler potentials to Sasaki setting. The Levi-Civita connection ∇ is torsion free and satisfies

$$
\frac{\mathrm{d}}{\mathrm{d}t}(u_t, v_t)_{\phi_t} = (\nabla_{\dot{\phi}_t} u_t, v_t)_{\phi_t} + (u_t, \nabla_{\dot{\phi}_t} v_t)_{\phi_t}
$$

for any smooth vector fields u_t, v_t along the path ϕ_t in $\mathcal{H}$. Let $u_t \in C^\infty_B(M)$ be smooth vector fields along a smooth curve ϕ_t in $\mathcal{H}$, then

$$\nabla_{\dot{\phi}_t} u_t = \dot{u}_t - \frac{1}{4} < \nabla\dot{\phi}_t, \nabla u_t >_{\phi_t} . \tag{3.30}$$

The geodesic equation can be written as

$$\nabla_{\dot{\phi}_t}(\dot{\phi}_t) = \ddot{\phi}_t - \frac{1}{4}|\nabla\dot{\phi}_t|^2_{\phi_t} = 0. \tag{3.31}$$

Given $\phi_0, \phi_1 \in \mathcal{H}$, to solve the geodesic equation, Guan–Zhang [42] introduced the following perturbation equation, for a path $\phi_t : M \times [0, 1] \to \mathbb{R}$,

$$\begin{cases} \left(\ddot{\phi}_t - \frac{1}{4}|\nabla\dot{\phi}_t|^2_{\omega_{\phi_t}}\right)\omega^n_\phi \wedge \eta = \epsilon(\omega^T)^n \wedge \eta, M \times (0,1) \\ \phi|_{t=0} = \phi_0 \\ \phi|_{t=1} = \phi_1. \end{cases} \tag{3.32}$$

Define a function ψ on $M \times [1, 3/2]$, as a subset of the cone X,

$$\psi(\cdot, r) = \phi_t(\cdot) + 4\log r, \quad t = 2r - 2.$$

Set a $(1, 1)$ form by,

$$\Omega_\psi = \omega_X + \frac{r^2}{2}\sqrt{-1}\left(\partial\bar{\partial}\psi - \frac{\partial\psi}{\partial r}\partial\bar{\partial}r\right).$$

Guan–Zhang wrote an equivalent form of (3.32) in terms of a complex Monge–Ampere equation on ψ of the following form (with $f = r^2, \epsilon \in (0, 1]$),

$$\begin{cases} (\Omega_\psi)^{n+1} = \epsilon f(\omega_X)^{n+1}, M \times \left(1, \frac{3}{2}\right) \\ \psi|_{M\times\{r=1\}} = \phi_0, \psi|_{M\times\{r=3/2\}} = \psi_1 + 4\log(3/2). \end{cases} \tag{3.33}$$

Guan–Zhang proved the following results regarding (3.33):

Theorem 3.2 (Guan–Zhang) *Fix a Sasaki structure (M, ξ, η, g) on a compact manifold M. For any positive basic function f and any two points $\phi_0, \phi_1 \in \mathcal{H}$, there exists a unique smooth solution of ψ to (3.33), satisfying the following estimates: ψ is basic and there exists a constant $C > 0$, depending only on $\|f^{\frac{1}{n}}\|_{C^2(M\times[1,\frac{3}{2}])}$, $\|\phi_0\|_{C^{2,1}}$, $\|\phi_1\|_{C^{2,1}}$ such that*

$$\|\psi\|_{C^2_w} := \|\psi\|_{C^1} + \sup|\Delta\psi| \le C. \tag{3.34}$$

Denote the corresponding solution of (3.32) by ϕ^ϵ_t, then ϕ^ϵ_t is called a ϵ-geodesic (smooth) connecting ϕ_0, ϕ_1 satisfying

$$\|\phi^\epsilon_t\|_{C^1} + \sup(\ddot{\phi}^\epsilon + |\nabla\dot{\phi}^\epsilon_t|_g + \Delta_g\phi^\epsilon_t) \le C. \tag{3.35}$$

When $\epsilon \to 0$, there exists a unique (weak C^2_w) limit ϕ_t of $\phi^\epsilon_t : M \times [0, 1] \to \mathbb{R}$ connecting ϕ_0, ϕ_1 such that $\Omega_{\phi^\epsilon+4logr}$ is positive. The later is equivalent to

$$\omega_{\phi^\epsilon_t} > 0, \ddot{\phi}^\epsilon_t - \frac{1}{4}|\nabla\dot{\phi}^\epsilon_t|^2_{\omega_{\phi^\epsilon_t}} > 0.$$

As a consequence, $(\mathcal{H}, d_2)$ is a metric space.

Remark 3.2 The constant $1/4$ appears in the geodesic equation

$$\ddot{\phi}_t - \frac{1}{4}|\nabla\dot{\phi}_t|^2_{\omega_{\phi_t}} = 0.$$

This constant is insignificant. In Kähler setting, some authors write the constant as $1/2$ and some write as 1, depending on the gradient ∇ is interpreted as real or complex; they differ by a constant 2. The constant $1/4$ appears in Sasaki setting in [42] since the authors use the real gradient and use the space of Sasaki potentials (transverse Kähler potentials) defined as

$$\{\phi : \mathrm{d}\eta + \sqrt{-1}\partial_B\bar{\partial}_B\phi > 0.\}$$

In the following, we shall write the geodesic equation as

$$\ddot{\phi}_t - |\nabla\dot{\phi}_t|^2_{\omega_{\phi_t}} = 0,$$

where we use complex gradient, and our choice space of transverse Kähler potentials is as

$$\mathcal{H} = \{\phi \in C^\infty_B(M) : \omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\phi > 0\}.$$

To prove $(\mathcal{H}, d_2)$ is a metric space, Guan–Zhang [42, Lemma 14, Proof of Theorem 2] proved the following triangle inequality,

Lemma 3.7 (Guan–Zhang) *Let $\psi(s) : [0, 1] \to \mathcal{H}$ be a smooth curve, $\phi \in \mathcal{H}\backslash\psi([0, 1])$. Fix $\epsilon \in (0, 1]$. Let $u^\epsilon \in C^\infty_B([0, 1] \times [0, 1] \times M)$ be the function such that $u^\epsilon_t(\cdot, s)$ is the ϵ-geodesic connecting ϕ and ψ_s, for $t \in [0, 1]$. Then the following estimate holds:*

$$l(u^\epsilon_t(\cdot, 0)) \le l(\psi) + l(u^\epsilon_t(\cdot, 1)) + \epsilon C, \tag{3.36}$$

where $C = C(\phi, \psi, g)$ is a uniform constant, independent of ϵ.

There are several estimates which are not explicitly stated or not proved in [42]. We include these estimates below since we shall need them below. Regarding (3.32), first we have the following comparison principle,

Lemma 3.8 *Suppose we have two solutions φ, ϕ with boundary datum φ_0, φ_1 and ϕ_0, ϕ_1, respectively,*

$$\left(\ddot{\phi}_t - |\nabla\dot{\phi}_t|^2_{\omega_{\phi_t}}\right)\omega_\phi^n \wedge \eta = \epsilon(\omega^T)^n \wedge \eta = \left(\ddot{\varphi}_t - |\nabla\dot{\varphi}_t|^2_{\omega_{\varphi_t}}\right)\omega_\varphi^n \wedge \eta, \tag{3.37}$$

then we have the following

$$\max|\phi - \varphi| \le \max|\phi_0 - \varphi_0| + \max|\phi_1 - \varphi_1|. \tag{3.38}$$

Proof This is a standard comparison principle. We sketch the proof for completeness. Denote the operator

$$F(D^2\phi) = \log\det\begin{pmatrix}\ddot{\phi} & \nabla\dot{\phi}\\ (\nabla\dot{\phi})^t & g^T_{i\bar{j}} + \phi_{i\bar{j}}\end{pmatrix} - \log\det(g^T_{i\bar{j}}) = \log\left(\ddot{\phi}_t - |\nabla\dot{\phi}_t|^2_{\omega_{\phi_t}}\right) + \log\frac{\det(g^T_{i\bar{j}} + \phi_{i\bar{j}})}{\det(g^T_{i\bar{j}})}.$$

The ϵ-geodesic equation can be written as $F(D^2\phi) = \epsilon$. Now suppose $F(D^2\phi) = F(D^2\varphi) = \epsilon > 0$, then (3.38) holds. Otherwise suppose at some interior point

$$\phi - \varphi > \max|\phi_0 - \varphi_0| + \max|\phi_1 - \varphi_1|.$$

Hence $\phi - \varphi + at(1-t)$ obtains its maximum at an interior point p for some $a > 0$. Denote $v = \phi + at(t-1)$. Then on one hand,

$$F(D^2 v) > F(D^2\phi) = \epsilon.$$

On the other hand at p, $D^2 v \le D^2\varphi$. It follows from the concavity of F, we have at p,

$$F(D^2 v) - F(D^2\varphi) \le \mathcal{L}_F(v - \varphi) \le 0,$$

where $\mathcal{L}_{F_v}$ is the linearized operator of F at v. Contradiction. □

One can actually be more precise about the estimate (3.35) [and (3.34)]. For simplicity, we state the result for (3.32).

Lemma 3.9 *The ϵ geodesic ϕ_t^ϵ connecting $\phi_0, \phi_1 \in \mathcal{H}$ satisfies the following estimate,*

$$\max|\dot{\phi}_t^\epsilon| \le \max|\phi_1 - \phi_0| + C\max|\nabla(\phi_1 - \phi_0)|_g^2 + \epsilon, \tag{3.39}$$

where C depends only on ϕ_0, ϕ_1. Moreover, we have

$$|\nabla\phi_t^\epsilon|_g + \sup\Delta_g\phi^\epsilon \le C(\|\phi_0\|_{C^1}, \|\phi_1\|_{C^1}, \sup\Delta_g\phi_0, \Delta_g\phi_1, g). \tag{3.40}$$

Proof The first estimate follows from $\ddot{\phi}^\epsilon_t > 0$ and the following C^0 estimate (3.41), which can be proved similarly using the concavity of F. First there exists $a > 0$ such that

$$at(t-1) + (1-t)\phi_0 + t\phi_1 \leq \phi^\epsilon_t \leq (1-t)\phi_0 + t\phi_1. \tag{3.41}$$

The right-hand side is a direction consequence of $\ddot{\phi}^\epsilon_t > 0$, while the left-hand side can be argued as follows. Denote $U^a = at(t-1) + (1-t)\phi_0 + t\phi_1$; we know ϕ^ϵ_t agrees with U^a on the boundary. Hence if $\phi^\epsilon_t < U^a$, then $\phi^\epsilon_t - U^a$ takes its minimum at some interior point p. At p, we know $D^2\phi^\epsilon \geq D^2U^a$. By concavity of F, we get (at p)

$$0 \leq \mathcal{L}_{F_a}(\phi^\epsilon_t - U^a) \leq F(D^2\phi^\epsilon_t) - F(D^2U^a).$$

That is $F(D^2U^a) \leq \log\epsilon$. This is a contradiction when $a > 0$ is sufficiently large. Indeed, a direct computation shows that if $a \geq C\max|\nabla(\phi_1 - \phi_0)|^2 + \epsilon$, then $F(D^2U^a) > \log\epsilon$. Hence for such choice of a, (3.41) holds. By convexity in t direction, we know that

$$\dot{\phi}^\epsilon_t(\cdot, 0) \leq \dot{\phi}^\epsilon_t \leq \dot{\phi}^\epsilon_t(\cdot, 1).$$

It is evident to show that

$$-a + \phi_1 - \phi_0 \leq \dot{\phi}^\epsilon_t(\cdot, 0) \leq \phi_1 - \phi_0 \leq \dot{\phi}^\epsilon_t(\cdot, 1) \leq a + \phi_1 - \phi_0.$$

Hence (3.39) follows. The gradient estimate $|\nabla\phi^\epsilon_t|$ is given by [42, Proposition 2]. The estimate on $\Delta_g\phi^\epsilon_t$, depending only on ϕ_0, ϕ_1 up to second-order derivative, was proved for Kähler setting by the first named author [47, Theorem 1.1] (for $\epsilon = 0$, it was proved earlier in [8] using pluripotential theory). The method in [47] is to deal with Eq. (3.32) directly, and it can be carried over to prove the interior estimate of $\Delta_g\phi^\epsilon$ word by word (since in Sasaki setting, this estimate only involves transverse Kähler structure and basic functions). For completeness, we sketch the proof. Denote $\phi^\epsilon = \phi$ for simplicity. We write the equations as

$$\log(\ddot{\phi} - |\nabla\dot{\phi}|^2_\phi) + \log\det\left(g^T_{i\bar{j}} + \phi_{i\bar{j}}\right) = \log\epsilon + \log\det\left(g^T_{i\bar{j}}\right), \tag{3.42}$$

using the transverse Kähler metric $g^T_{i\bar{j}}$. For any basic function h, we denote

$$\begin{aligned} Dh =& \Delta_\phi h + \frac{h_{tt} + g^{i\bar{l}}_\phi g^{k\bar{j}}_\phi h_{k\bar{l}}\phi_{ti}\phi_{t\bar{j}}}{\phi_{tt} - |\nabla\phi_t|^2_\phi} \\ & - \frac{g^{i\bar{j}}_\phi(h_{ti}\phi_{t\bar{j}} + h_{t\bar{j}}\phi_{ti})}{\phi_{tt} - |\nabla\phi_t|^2_\phi}, \end{aligned} \tag{3.43}$$

where $(g_\phi^{i\bar{j}})$ is the inverse of the transverse Kähler metric $(g^T_{i\bar{j}} + \phi_{i\bar{j}})$. Proceeding exactly as in the computation in the Kähler setting (see [47, (2.4)–(2.19)]), we compute

$$D(\log(n+\Delta\phi) - C\phi + t^2) > g_\phi^{i\bar{j}} g^T_{i\bar{j}} + (\ddot{\phi} - |\nabla\dot{\phi}|^2_\phi)^{-1} - (n+1)C, \tag{3.44}$$

where C depends only on the background transverse Kähler metric g^T and n. Hence we have

$$D(\log(n+\Delta\phi) - C\phi + t^2) > (n + \Delta\phi + \ddot{\phi} - |\nabla\dot{\phi}|^2_\phi)^{\frac{1}{n}} \epsilon^{-\frac{1}{n}} - (n+1)C, \tag{3.45}$$

where we have used the elementary inequality

$$\left(\sum_{i=0}^n a_i^{-1}\right)^n \geq \frac{\sum_{i=0}^n a_i}{\Pi_{i=0}^n a_i}.$$

Hence it follows that either $\log(n+\Delta\phi) - C\phi + t^2$ achieves its maximum on the boundary, or at an interior maximum point P,

$$(n + \Delta\phi + \ddot{\phi} - |\nabla\dot{\phi}|^2_\phi)^{\frac{1}{n}} \epsilon^{-\frac{1}{n}} - (n+1)C \leq D(\log(n+\Delta\phi) - C\phi + t^2)(P) \leq 0.$$

This gives the desired bound

$$-n < \Delta\phi^\epsilon \leq C(\|\phi_0\|_{C^1}, \|\phi_1\|_{C^1}, \Delta_g\phi_0, \Delta_g\phi_1, g).$$

□

By taking $\epsilon \to 0$, we have the following,

Lemma 3.10 *Suppose ϕ is the weak geodesic connecting $\phi_0, \phi_1 \in \mathcal{H}$, then for some positive constant $C = C(M, g, \|\phi_0\|_{C^2}, \|\phi_1\|_{C^2})$, we have*

$$|\dot{\phi}| \leq \max|\phi_1 - \phi_0| + C \max|\nabla\phi_1 - \nabla\phi_0|^2_g$$

As a consequence, when $\phi_0 \to \phi_1$ in $\mathcal{H}$, then $d_2(\phi_0, \phi_1) \to 0$.

Remark 3.3 One can get a much sharper estimate,

$$|\dot{\phi}| \leq \max|\phi_1 - \phi_0|$$

using the uniqueness and comparison for the generalized solutions of complex Monge–Ampere in the sense of Bedford–Taylor, see [30, Lemma 3.5] for Kähler setting. We shall prove this sharper version below.

Using Lemmas 3.7 and 3.10, it follows that the distance function $d_2(\phi_0, \phi_1)$ is realized by the weak geodesic ϕ connecting ϕ_0, ϕ_1. In particular,

Lemma 3.11 *Given $\phi_0, \phi_1 \in \mathcal{H}$, we have,*

$$d_2(\phi_0, \phi_1) = \|\dot{\phi}\|_{2,\phi_t}, \forall t \in [0, 1] \tag{3.46}$$

Proof Let ϕ_t^ϵ be the ϵ geodesic connecting ϕ_0, ϕ_1. Then we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_M |\dot{\phi}_t^\epsilon|^2 (\omega_{\phi_t^\epsilon})^n \wedge \eta &= 2 \int_M \dot{\phi}_t^\epsilon (\ddot{\phi}_t^\epsilon - |\nabla \dot{\phi}_t^\epsilon|_{\phi_t^\epsilon}) (\omega_{\phi_t^\epsilon})^n \wedge \eta \\ &= 2\epsilon \int_M \dot{\phi}_t^\epsilon (\omega^T)^n \wedge \eta. \end{aligned} \tag{3.47}$$

Since $|\dot{\phi}_t^\epsilon|$ is uniformly bounded, letting $\epsilon \to 0$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M |\dot{\phi}_t|^2 (\omega_{\phi_t})^n \wedge \eta = 0.$$

This proves (3.46). In particular if $\phi_0 \neq \phi_1$, $\dot{\phi}_t$ is not identically zero for any t. Moreover, if ϵ is small enough, depending on $\phi_0 \neq \phi_1$, then $\dot{\phi}_t^\epsilon$ is not identically zero for any $t \in [0, 1]$. This follows from (3.47) and it is easy to see that $\int_M |\dot{\phi}_t^\epsilon|^2 (\omega_{\phi_t^\epsilon})^n \wedge \eta$ has a positive lower bound for any t (say $l(\phi_t^\epsilon)/2$), if ϵ is sufficiently small. □

We also have the following

Theorem 3.3 (Guan–Zhang, Theorem 2). *For $u, v, w \in \mathcal{H}$,*

$$d_2(u, w) \le d_2(u, v) + d_2(v, w).$$

3.3 The Orlicz–Finsler Geometry on Sasaki Manifolds

The Orlicz–Finsler geometry on the space of Kähler potentials was introduced by Darvas [28] and it has played an important role in problems regarding csck and Calabi's extremal metric in Kähler geometry. In particular, the Finsler metric d_1 will play an important role and it is used to define the properness of $\mathcal{K}$-energy. In this section, we discuss the Orlicz–Finsler geometry on Sasaki manifolds. We prove the following theorem, which is the counterpart of Darvas's [28, Theorem 1] in Sasaki setting.

Theorem 3.4 *If $\chi \in \mathcal{W}_p^+$, $p \ge 1$, then $(\mathcal{H}, \mathrm{d}_\chi)$ is a metric space and for any $u_0, u_1 \in \mathcal{H}$, the $C_B^{1,\bar{1}}$ geodesic $t \to u_t$ connecting u_0, u_1 satisfies*

$$\mathrm{d}_\chi(u_0, u_1) = \|\dot{u}_t\|_{\chi, u_t}, t \in [0, 1]. \tag{3.48}$$

Theorem 3.4 is the generalization for d_2 to general Young weights. This important result in Darvas's theory says that, the same $C_B^{1,\bar{1}}$ geodesic (with respect to d_2) is "length minimizing" for all d_χ metric structures and this holds in Sasaki setting. The proof of Theorem 3.4 pretty much follows Darvas's proof [30, Theorem 3.4], with minor modifications adapted to Sasaki setting. The main point is that only transverse

Kähler structure is involved, and hence this is essentially the same as in Kähler setting. We include the details for completeness.

Following Darvas (see [30, Chapter 3]), we define the Orlicz–Finsler length of $v \in T_u\mathcal{H} = C^\infty_B(M)$ for any weight $\chi \in \mathcal{W}^+_p$:

$$\|v\|_{\chi,u} = \inf\left\{r > 0 : \frac{1}{\mathrm{Vol}(M)}\int_M \chi\left(\frac{v}{r}\right)\omega_u^n \wedge \mathrm{d}\eta \leq \chi(1)\right\}. \tag{3.49}$$

For simplicity, we shall assume $\mathrm{Vol}(M) = 1$ in this section. Given a smooth curve $\gamma : t \in [0, 1] \to \mathcal{H}$, its length is computed by the formula

$$l_\chi(\gamma_t) = \int_0^1 \|\dot{\gamma}_t\|_{\chi,\gamma_t}\mathrm{d}t. \tag{3.50}$$

Furthermore, the distance $\mathrm{d}_\chi(u_0, u_1)$ between $u_0, u_1 \in \mathcal{H}$ is defined to be

$$\mathrm{d}_\chi(u_0, u_1) = \inf\{l_\chi(\gamma_t) : \gamma_t \text{ is a smooth curve with } \gamma_0 = u_0, \gamma_1 = u_1\}. \tag{3.51}$$

First we have the following,

Proposition 3.18 *Suppose $\chi \in \mathcal{W}^+_p \cap C^\infty(\mathbb{R})$. For a smooth curve $u_t (t \in [0, 1])$ in $\mathcal{H}$ and a vector field $f_t \in C^\infty_B(M)$ along this curve with $f_t \not\equiv 0$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}||f_t||_{\chi,u_t} = \frac{\int_M \chi'\left(\frac{f_t}{||f_t||_{\chi,\phi_t}}\right)\nabla_{\dot{u}_t} f_t \mathrm{d}\mu_{u_t}}{\int_M \chi'\left(\frac{f_t}{||f_t||_{\chi,u_t}}\right)\frac{f_t}{||f_t||_{\chi,u_t}}\mathrm{d}\mu_{u_t}}. \tag{3.52}$$

Proof This works as in [28, Proposition 3.1] word by word. We skip the details. □

Lemma 3.12 *Suppose $\chi \in \mathcal{W}^+_p \cap C^\infty(\mathbb{R})$ and $u_0, u_1 \in \mathcal{H}, u_0 \neq u_1$. Then the ϵ-geodesics $[0, 1] \ni t \to u^\epsilon_t \in \mathcal{H}$ connecting u_0, u_1 satisfies the following estimate:*

$$\begin{aligned}\int_M \chi(\dot{u}^\epsilon_t)\omega^n_{u^\epsilon_t} \wedge \eta \geq \max\Bigg(&\int_M \chi(\min(u_1 - u_0, 0))\omega^n_{u_0} \wedge \eta,\\ &\int_M \chi(\min(u_0 - u_1, 0))\omega^n_{u_1} \wedge \eta\Bigg) - \epsilon C\end{aligned} \tag{3.53}$$

for all $t \in [0, 1]$, where $C := C(\chi, ||u_0||_{C^2(M)}, ||u_1||_{C^2(M)})$.

Proof This follows exactly as in Kähler setting [30, Lemma 3.8], by a direct computation and the convexity of χ. □

Lemma 3.13 *Suppose $\chi \in \mathcal{W}^+_p \cap C^\infty(\mathbb{R})$ and $u_0, u_1 \in \mathcal{H}, u_0 \neq u_1$. Then there exists a constant ϵ_0 that depends on u_0, u_1 such that for all $\epsilon \in (0, \epsilon_0]$ the ϵ-geodesic $[0, 1] \ni t \to u^\epsilon_t \in \mathcal{H}$ connecting u_0, u_1 satisfies:*

$$\frac{\mathrm{d}}{\mathrm{d}t}||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon} = \epsilon \frac{\int_M \chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)(\omega^T)^n \wedge \eta}{\int_M \frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)\omega_{u_t^\epsilon} \wedge \eta_{u_t^\epsilon}}, \quad t \in [0, 1]. \tag{3.54}$$

Proof If we choose $\epsilon_0 > 0$ sufficiently small, then $\dot{u}_t^\epsilon$ is not identically zero for any $t \in [0, 1]$, if $\epsilon \in (0, \epsilon_0]$, given $u_0 \neq u_1$, see (3.46). Then the results follows from Proposition 3.18. □

We have the following, similar to (3.46) (for d_2),

Proposition 3.19 *Suppose $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$ and $u_0, u_1 \in \mathcal{H}$, $u_0 \neq u_1$. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ the ϵ-geodesic $[0, 1] \ni t \to u_t^\epsilon \in \mathcal{H}$ connecting u_0, u_1 satisfies*

(i) $||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon} > R_0, t \in [0, 1]$;
(ii) $|\frac{\mathrm{d}}{\mathrm{d}t}||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}| \leq \epsilon R_1, t \in [0, 1]$,

where ϵ_0, R_0, R_1 depend on upper bounds for $||u_0||_{C^2(M)}, ||u_1||_{C^2(M)}$ and lower bounds for $||\chi(u_1 - u_0)||_{L^1((\omega^T)^n \wedge \eta)}$, $\frac{\omega_{u_0}^n \wedge \eta_{u_0}}{(\omega^T)^n \wedge \eta}$ and $\frac{\omega_{u_1}^n \wedge \eta_{u_1}}{(\omega^T)^n \wedge \eta}$.

Proof (i) Recall Eq. (1.11) in [30]

$$||f||_{\chi,\mu} \geq \min\left\{\frac{\int_\Omega \chi(f)\mathrm{d}\mu}{\chi(1)}, \left(\frac{\int_\Omega \chi(f)\mathrm{d}\mu}{\chi(1)}\right)^{\frac{1}{p}}\right\}$$

and Lemma 3.12, the estimate in (i) follows immediately.

(ii) Choose ϵ_0 small so that Lemma 3.13 applies. Recall the Young identity

$$\chi(a) + \chi^*(\chi'(a)) = a\chi'(a), a, b \in \mathbb{R}, \chi'(a) \in \partial\chi(a)$$

Then we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\right| |\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}| &= \epsilon \frac{\left|\int_M \chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)(\omega^T)^n \wedge \eta\right|}{\int_M \frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)\omega_{u_t^\epsilon} \wedge \eta_{u_t^\epsilon}} \\
&= \epsilon \frac{\left|\int_M \chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)(\omega^T)^n \wedge \eta\right|}{\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)\right)\omega_{u_t^\epsilon} \wedge \eta_{u_t^\epsilon}} \\
&\leq \frac{\epsilon}{\chi(1)}\left|\int_M \chi'\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,\dot{u}_t^\epsilon}}\right)(\omega^T)^n \wedge \eta\right|.
\end{aligned} \tag{3.55}$$

Then the estimates (ii) follows from (i) and the fact that $\dot{u}_t^\epsilon$ is uniformly bounded in terms of $||u_0||_{C^2(M)}, ||u_1||_{C^2(M)}$.

□

Remark 3.4 The estimate (i) in Proposition 3.19 holds for general weights $\chi \in \mathcal{W}_p^+$. Recall that $\dot{u}_t^\epsilon$ is uniformly bounded in terms of $||u_0||_{C^2(M)}, ||u_1||_{C^2(M)}$. We can choose smooth weights $\chi_k \in \mathcal{W}_{p_k}^+ \cap C^\infty(\mathbb{R})$ which approximate χ uniformly on compact subsets of $\mathbb{R}$. Moreover we have $\lim_{k\to\infty} ||\dot{u}_t^\epsilon||_{\chi_k,u_t^\epsilon} = ||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}$ [30, Section 1]. It follows that the estimates (i) hold for χ.

Next we are ready to prove the triangle inequality, as in Lemma 3.7 for d_2 and [28, Proposition 3.4] in Kähler setting.

Proposition 3.20 *Suppose $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$, $\psi_s \in \mathcal{H}$ is a smooth curve, $\phi \in \mathcal{H}\backslash\psi([0,1])$, and $\epsilon > 0$. $u^\epsilon \in C^\infty([0,1]\times[0,1]\times M)$ is the smooth function for which $t \to u_t^\epsilon(\cdot,s) = u^\epsilon(t,s,\cdot)$ is the ϵ-geodesic connecting ϕ and ψ_s. Then there exists $\epsilon_0(\phi,\psi) > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ the following holds:*

$$l_\chi(u_t^\epsilon(\cdot,0)) \le l_\chi(\psi_s) + l_\chi(u_t^\epsilon(\cdot,1)) + \epsilon R$$

for some $R(\phi,\psi,\chi,\epsilon_0) > 0$ independent of ϵ.

Proof Fix $s \in [0,1]$. By Propositions 3.18 and 3.19, there exists a constant $\epsilon_0(\phi,\psi) > 0$ such that for $\epsilon \in (0,\epsilon_0)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} l_\chi(u_t(\cdot,s)) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}||\dot{u}(t,s,\cdot)||_{\chi,u(t,s,\cdot)}\mathrm{d}t \\
&= \int_0^1 \frac{\int_M \chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\nabla_{\frac{\mathrm{d}u}{\mathrm{d}s}}\dot{u}\,\mathrm{d}\mu_{u_t}}{\int_M \chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\mathrm{d}\mu_{u_t}}\mathrm{d}t \\
&= \int_0^1 \frac{\int_M \chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\nabla_{\frac{\mathrm{d}u}{\mathrm{d}s}}\dot{u}\,\mathrm{d}\mu_{u_t}}{\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t}}\mathrm{d}t \\
&= \int_0^1 \frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_M \chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\frac{\mathrm{d}u}{\mathrm{d}s}\mathrm{d}\mu_{u_t} - \int_M \frac{\mathrm{d}u}{\mathrm{d}s}\nabla_{\dot{u}}\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t}}{\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t}}\mathrm{d}t.
\end{aligned}$$

Moreover we have

$$\nabla_{\dot{u}}\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t} = \chi''\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\left(\frac{\nabla_{\dot{u}}\dot{u}}{||\dot{u}||_{\chi,u}} - \frac{\dot{u}}{||\dot{u}||^2_{\chi,u}}\frac{\mathrm{d}}{\mathrm{d}t}||\dot{u}||_{\chi,u}\right)\mathrm{d}\mu_{u_t}. \tag{3.56}$$

It follows from Proposition 3.19 that $||\dot{u}||_{\chi,t}$ is uniformly bounded away from zero and both $\nabla_{\dot{u}}\dot{u}\mathrm{d}\mu_{u_t}$ and $\frac{\mathrm{d}}{\mathrm{d}t}||\dot{u}||_{\chi,u}$ are uniformly bounded by the form ϵR, where R

is uniformly bounded. Moreover $\dot{u}$, $\frac{\mathrm{d}u}{\mathrm{d}s}$ are uniformly bounded independent of ϵ [42, Lemma 14]. Hence

$$\frac{\mathrm{d}}{\mathrm{d}s} l_\chi(u_t(\cdot, s)) = \int_0^1 \frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_M \chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\frac{\mathrm{d}u}{\mathrm{d}s}\mathrm{d}\mu_{u_t}}{\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t}}\mathrm{d}t + \epsilon R,$$

where R is uniform bounded independent of ϵ.

Recall that $\chi^{*'}(\chi'(l)) = l$ for $l \in \mathbb{R}$, the expression

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t}\right) \\
&\quad = \int_M \frac{\dot{u}}{||\dot{u}||_{\chi,u}}\chi''\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\nabla_{\dot{u}}\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\mathrm{d}\mu_{u_t}
\end{aligned}$$

is a term of type ϵR. Hence we can write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} l_\chi(u_t(\cdot, s)) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\frac{\int_M \chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\frac{\mathrm{d}u}{\mathrm{d}s}\mathrm{d}\mu_{u_t}}{\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}}{||\dot{u}||_{\chi,u}}\right)\right)\mathrm{d}\mu_{u_t}}\mathrm{d}t + \epsilon R \\
&= \frac{\int_M \chi'\left(\frac{\dot{u}(1,s)}{||\dot{u}(1,s)||_{\chi,\psi}}\right)\frac{d\psi}{\mathrm{d}s}\mathrm{d}\mu_\psi}{\chi(1) + \int_M \chi^*\left(\chi'\left(\frac{\dot{u}(1,s)}{||\dot{u}(1,s)||_{\chi,\psi}}\right)\right)\mathrm{d}\mu_\psi} \\
&\geq -\left|\left|\frac{d\psi}{\mathrm{d}s}\right|\right|_{\chi,\psi} + \epsilon R,
\end{aligned}$$

where the last line follows from the Young inequality

$$\chi(a) + \chi^*(b) \geq ab, a, b \in \mathbb{R}.$$

The integration of the above inequality with respect to $s \in [0, 1]$ yields the desired inequality. □

Now we are ready to prove Theorem 3.4. Certainly the proof follows closely Darvas's result in Kähler setting [28, Section 3].

Proof First we show that for $u_0, u_1 \in \mathcal{H}$ and the weak $C_B^{1,\bar{1}}$-geodesic u_t connecting u_0, u_1

$$\mathrm{d}_\chi(u_0, u_1) = l_\chi(u_t). \tag{3.57}$$

We assume $u_0 \neq u_1$. Recall that, by Guan–Zhang [42], ϵ-geodesics u_t^ϵ connecting u_0, u_1 converge to the weak $C_B^{1,\bar{1}}$ geodesic u_t in $C^{1,\alpha}$. Hence $\dot{u}_t^\epsilon$ converges uniformly to $\dot{u}_t$.

Recall that $\dot{u}_t^\epsilon$ is uniformly bounded in terms of $||u_0||_{C^2(M)}, ||u_1||_{C^2(M)}$. Combine with the remark after Proposition 3.19, there exist constants $0 < C_1 < C_2$ such that for sufficiently small $\epsilon > 0$

$$C_1 \leq ||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon} \leq C_2.$$

Take a cluster point N of $\{||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}\}_{\epsilon>0}$, after taking a subsequence, we can assume that $||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon} \to N$ as $\epsilon \to 0$. Then $\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}}$ converges to $\frac{\dot{u}_t}{N}$ uniformly. Moreover, we have $\omega_{u_t^\epsilon}^n \wedge \eta$ converges to $\omega_{u_t}^n \wedge \eta$ weakly.

Recall $||f||_{\chi,\mu} = \alpha > 0$ if and only if $\int_\Omega \chi(\frac{f}{\alpha})\mathrm{d}\mu = \chi(1)$[30, Section 1]. We have

$$\chi(1) = \int_M \chi\left(\frac{\dot{u}_t^\epsilon}{||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}}\right)\omega_{u_t^\epsilon}^n \wedge \eta \to \int_M \chi\left(\frac{\dot{u}_t}{N}\right)\omega_{u_t}^n \wedge \eta$$

and $N = ||\dot{u}_t||_{\chi,u_t}$. Hence $||\dot{u}_t||_{\chi,u_t}$ is the only possible cluster point of $\{||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon}\}_{\epsilon>0}$. It means that

$$||\dot{u}_t^\epsilon||_{\chi,u_t^\epsilon} \to ||\dot{u}_t||_{\chi,u_t}$$

as $\epsilon \to 0$. Then by the dominated convergence theorem we have

$$\lim_{\epsilon\to 0} l_\chi(u_t^\epsilon) = l_\chi(u_t) \tag{3.58}$$

and $\mathrm{d}_\chi(u_0, u_1) \leq l_\chi(u_t)$.

Then Eq. (3.57) follows if we can prove

$$l_\chi(\phi_t) \geq l_\chi(u_t) \tag{3.59}$$

for all smooth curves ϕ_t in $\mathcal{H}$ connecting u_0, u_1.

First we consider the case $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$. We can assume that $u_1 \notin \phi([0, 1))$ and take $h \in [0, 1)$. Applying Proposition 3.20 to the case $\phi = u_1$ and $\psi_s = \phi|_{[0,h]}$, letting $\epsilon \to 0$, we can obtain

$$l_\chi(u_t) \leq l_\chi(\phi_t|_{[0,h]}) + l_\chi(w_t^h),$$

where u_t is the $C_B^{1,\bar{1}}$ geodesic connecting u_1, u_0, and w_t^h is the $C_B^{1,\bar{1}}$ geodesic connecting u_1, ϕ_h. By Lemma 3.9, $l_\chi(w_t^h) \to 0$ as $h \to 1$. Hence $l_\chi(\phi_t) \geq l_\chi(u_t)$.

For the general weight $\chi \in \mathcal{W}_p^+$, we need to do approximation as in [28, Proposition 2.4]. There exists sequence $\chi_k \in \mathcal{W}_{p_k}^+ \cap C^\infty(\mathbb{R})$ such that χ_k converges to χ uniformly on compact subsets. Then we have

$$\int_0^1 ||\dot{\phi}_t||_{\chi_k,\phi_t}\mathrm{d}t = l_{\chi_k}(\phi_t) \geq l_{\chi_k}(u_t) = \int_0^1 ||\dot{u}_t||_{\chi_k,u_t}\mathrm{d}t$$

and $||\dot{\phi}_t||_{\chi_k,\phi_t} \to ||\dot{\phi}_t||_{\chi,\phi_t}$, $||\dot{u}_t||_{\chi_k,u_t} \to ||\dot{u}_t||_{\chi,u_t}$ [30, Section 1]. Moreover,$\dot{u}_t$, $\dot{\phi}_t$ are uniformly bounded. By the dominated convergence theorem, $l_\chi(\phi_t) \geq l_\chi(u_t)$. This completes the proof of 3.57.

Recall $l_\chi(u_t) = \int_0^1 ||\dot{u}_t||_{\chi,u_t} \mathrm{d}t$ and by Lemma 3.14, we have

$$\mathrm{d}_\chi(u_0, u_1) = \|\dot{u}_t\|_{\chi,u_t}, t \in [0, 1].$$

Suppose $u_0 \neq u_1 \in \mathcal{H}$, take $\epsilon \to 0$ in the estimate Lemma 3.12 we obtain $\dot{u}_0 \not\equiv 0$ and $\mathrm{d}_\chi(u_0, u_1) = ||\dot{u}_0||_{\chi,u_0} > 0$. This implies that $(\mathcal{H}, \mathrm{d}_\chi)$ is a metric space. □

Lemma 3.14 *Let u_t be the weak $C_B^{1,\bar{1}}$ geodesic connecting u_0, u_1. Then for any $\chi \in \mathcal{W}_p^+$ and $t_0, t_1 \in [0, 1]$, the following holds*

$$\mathrm{d}_\chi(u_0, u_1) = ||\dot{u}_{t_0}||_{\chi,u_{t_0}} = ||\dot{u}_{t_1}||_{\chi,u_{t_1}}. \tag{3.60}$$

Proof It had been shown that for ϵ-geodesics u_t^ϵ joining u_0, u_1, we have

$$||\dot{u}^\epsilon_{t_0}||_{\chi,u^\epsilon_{t_0}} \to ||\dot{u}_{t_0}||_{\chi,u_{t_0}}, ||\dot{u}^\epsilon_{t_1}||_{\chi,u^\epsilon_{t_1}} \to ||\dot{u}_{t_1}||_{\chi,u_{t_1}}$$

as $\epsilon \to 0$. Proposition 3.19 implies that

$$|||\dot{u}^\epsilon_{t_0}||_{\chi,u^\epsilon_{t_0}} - ||\dot{u}^\epsilon_{t_1}||_{\chi,u^\epsilon_{t_1}}| \leq |t_0 - t_1|\epsilon R_1.$$

Then taking $\epsilon \to 0$ we have $||\dot{u}_{t_0}||_{\chi,u_{t_0}} = ||\dot{u}_{t_1}||_{\chi,u_{t_1}}$. □

Finally, we have the following triangle inequality,

Lemma 3.15 *For $u, v, w \in \mathcal{H}$, $\chi \in \mathcal{W}_p^+$, $p \geq 1$,*

$$\mathrm{d}_\chi(u, w) \leq \mathrm{d}_\chi(u, v) + \mathrm{d}_\chi(v, w).$$

4 The Metric Space $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$

In this section, we prove Theorem 2. We shall follow the Kähler setting closely as in [28, Section 4], but we shall only consider d_p distance. Given $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, $p \geq 1$, by Lemma 3.1 there exists decreasing sequences $\{u_0^k\}_{k\in\mathbb{N}}, \{u_1^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that $u_0^k \searrow u_0$ and $u_1^k \searrow u_1$. We shall prove that the following formula for distance d_p is well defined,

$$\mathrm{d}_p(u_0, u_1) = \lim_{k\to\infty} \mathrm{d}_p(u_0^k, u_1^k) \tag{4.1}$$

and the definition in (4.1) coincides with (3.51) (we only consider $\chi(l) = |l|^p/p$). We will prove that

Theorem 4.1 $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$ *is a complete geodesic metric space extending* $(\mathcal{H}, \mathrm{d}_p)$.

We start with the notion of generalized solution of complex Monge–Ampere in the sense of Bedford–Taylor in Sasaki setting, which was considered by van Coevering in [58], by adapting the complex Monge–Ampere operator for basic functions in $\mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$ to Sasaki setting. van Coevering discussed in particular weak solution in $\mathrm{PSH}(M, \xi, \omega^T) \cap C^0(M)$ [58, Section 2.4]. Let $S = [0, 1] \times S^1$ be the cylinder and $N = M \times S$. Then N is a manifold of dimension $2n+3$ with boundary and N has a transverse holomorphic structure, simply the product structure of transverse holomorphic structure on M, and holomorphic structure on S. A path $\phi : [0, 1] \to C^\infty_B(M)$ corresponds to an S^1-invariant function Φ_w on N. If ϕ_t is a smooth path in $\mathcal{H}$ then a direct computation gives

$$(\pi^*\omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\Phi)^{n+1} = c_m(\ddot{\phi} - |\nabla\dot{\phi}|^2_{\omega^T_{\phi_t}})(\omega^T_{\phi_t})^n \wedge dw \wedge d\bar{w}. \tag{4.2}$$

Note that this choice of complexification [see van Coevering (4.2)] is different with the choice of Guan–Zhang (3.33). It seems that (4.2) would be more natural to discuss weak solutions. By (4.2), a smooth geodesic then corresponds to a solution of homogeneous complex Monge–Ampere for basic function $\Phi : N \to \mathbb{R}$,

$$(\pi^*\omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\Phi)^{n+1} \wedge \eta = 0.$$

We define a *weak geodesic* between $u_0, u_1 \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$ as follows, for $\Phi(\cdot, w) = \Phi(\cdot, t) \in \mathrm{PSH}(N^\circ, \xi, \pi^*\omega^T) \cap L^\infty$, $(t = \mathrm{Re}(w))$, it satisfies

$$\begin{cases} (\pi^*\omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\Phi)^{n+1} \wedge \eta = 0 \\ \lim_{t\to 0}\Phi(\cdot, t) = u_0,\ \ \lim_{t\to 1}\Phi(\cdot, t) = u_1. \end{cases} \tag{4.3}$$

We have the following *strong maximum principle*, see [58, Theorem 2.5.3], [11, Theorem 21], and [30, Theorem 3.2].

Lemma 4.1 *Let* $u, v \in PSH(N^\circ, \xi, \pi^*\omega^T) \cap L^\infty(N)$. *Suppose that*

$$(\pi^*\omega^T + \sqrt{-1}\partial_B\bar{\partial}_B u)^{n+1} \wedge \eta \le (\pi^*\omega^T + \sqrt{-1}\partial_B\bar{\partial}_B v)^{n+1} \wedge \eta$$

and $\lim_{x\to\partial N}(u - v)(x) \ge 0$, *then* $u \ge v$ *on* N.

Proof Our proof is similar to Kähler case, see [30, Theorem 3.2]. Fix $\epsilon > 0$ and $v_\epsilon := \max\{u, v - \epsilon\} \in \mathrm{PSH}(N^\circ, \xi, \omega^T) \cap L^\infty$. Then $v_\epsilon = u$ near the boundary $\partial N = M \times S^1 \times \{t = 0\} \cup M \times S^1 \times \{t = 1\}$. Hence it is enough to show that $u = v_\epsilon$ on N.

We write $N = M \times S$ and $\omega_u = \pi^*\omega^T + dd^c_B u$, etc. Note that on each foliation chart $W_\alpha = (-\delta, \delta) \times V_\alpha$ of M, we have the following inequality on $V_\alpha \times S$ for complex Monge–Ampere measure [12, Theorem 2.2.10]

$$\omega^{n+1}_{v_\epsilon} \ge \chi_{\{u \ge v-\epsilon\}\cap V_\alpha}\omega^{n+1}_u + \chi_{\{u<v-\epsilon\}\cap V_\alpha}\omega^{n+1}_v \ge \omega^{n+1}_u.$$

It follows that on N, we have

$$\omega_{v_\epsilon}^{n+1} \wedge \eta \geq \omega_u^{n+1} \wedge \eta.$$

Then we have the following

$$0 \leq \int_N (v_\epsilon - u)(\omega_{v_\epsilon}^{n+1} - \omega_u^{n+1}) \wedge \eta. \tag{4.4}$$

Using integration by parts, we obtain that

$$\int_N \mathrm{d}(u - v_\epsilon) \wedge \mathrm{d}_B^c(u - v_\epsilon) \wedge \omega_u^k \wedge \omega_{v_\epsilon}^{n-k} \wedge \eta = 0, 0 \leq k \leq n.$$

By an induction argument as in [30, Theorem 3.2], we can prove that

$$\int_N \mathrm{d}(u - v_\epsilon) \wedge \mathrm{d}_B^c(u - v_\epsilon) \wedge \omega_u^k \wedge (\pi^*\omega^T)^{n-k} \wedge \eta = 0, 0 \leq k \leq n.$$

For $k = n$, this shows that

$$\int_{M\times S} \mathrm{d}(u - v_\epsilon) \wedge \mathrm{d}_B^c(u - v_\epsilon) \wedge (\pi^*\omega^T)^n \wedge \eta = 0.$$

Writing $\rho = u - v_\epsilon$, this reads

$$\int_{M\times S} |\partial_t \rho|^2 \mathrm{d}t \wedge \mathrm{d}s \wedge (\pi^*\omega^T)^n \wedge \eta = 0.$$

Hence $\partial_t \rho = 0$. Since $\rho = 0$ near the boundary $\partial N = M \times S^1 \times \{t = 0\} \cup M \times S^1 \times \{t = 1\}$, this shows that $\rho = 0$. It completes the proof. □

Remark 4.1 One can certainly formulate a general version of comparison principle as in [30, Theorem 3.2]. But one would need certainly a (transverse) Kähler form. Note that $\pi^*\omega^T$ is not transverse Kähler (it is zero along S-direction). Here we use the product structure of N.

With this maximum principle for bounded TPSH, we have the following,

Lemma 4.2 *Given $u_0, u_1 \in \mathcal{H}$, let $u_t : [0, 1] \to \mathcal{H}$ be the unique $C_B^{1,\bar{1}}$ geodesic connecting u_0, u_1. Then we have the following,*

$$\|\dot{u}_t\|_{C^0} \leq \|u_0 - u_1\|_{C^0}, \forall t \in [0, 1].$$

Proof Note that this gives a much sharper estimate than Lemma 3.10. The proof follows the Kähler setting [30, Lemma 3.5]. Denote $C = \max |u_0 - u_1|$. By the convexity of u in t-variable, we know that

$$\dot{u}_0 \leq \dot{u}_t \leq \dot{u}_1.$$

Note that $v_t = u_0 - Ct$ is a smooth geodesic connecting u_0 and $u_0 - C$. Hence its complexification gives a solution to (4.3). By Lemma 4.1, we know that $v_t \leq u_t$, for $t \in [0, 1]$, since $u_0 - C \leq u_1$. It follows that $-C \leq \dot{u}_0$. Similarly, one can prove that $\dot{u}_1 \leq C$, by considering $\tilde{v}_t = u_0 + Ct$. □

Remark 4.2 The upper envelop construction was used to construct bounded weak geodesic segment in Kähler setting by Berndtsson [9], where he proved that Lemma 4.2 holds for $u_0, u_1 \in \mathrm{PSH}(M, \omega)$ (when (M, ω) is Kähler). A direct adaption to Sasaki setting using Lemma 4.1 would lead to an extension of Berndtsson's result to Sasaki setting.

In general, $\Phi(\cdot, w) \in \mathrm{PSH}(N^\circ, \xi, \pi^*\omega^T)$ will be called *weak subgeodesic*, if $\Phi(\cdot,) = \Phi(\cdot, \mathrm{Re}(w))$, $(t = \mathrm{Re}(w))$. For $u_0, u_1 \in \mathrm{PSH}(M, \xi, \omega^T)$, we define

$$u = \sup\left\{\Phi : \Phi(\cdot, t) \in \mathrm{PSH}(N^\circ, \xi, \pi^*\omega^T),\ \lim_{t\to 0,1} \Phi(\cdot, t) \leq u_{0,1}\right\}. \tag{4.5}$$

We have the following:

Proposition 4.1 *$u \in PSH(N^\circ, \xi, \pi^*\omega^T)$. Denote $u_t = u(\cdot, t)$. We refer $t \to u_t$ to the weak geodesic segment connecting u_0, u_1.*

Proof Note that usc u^* is basic, and $u^* \in \mathrm{PSH}(N^\circ, \xi, \pi^*\omega^T)$. Since Φ is convex in t direction, it follows that $\Phi(\cdot, t) \leq (1-t)u_0 + tu_1$. Hence $u_t \leq (1-t)u_0 + tu_1$. It follows that

$$u^* \leq (1-t)u_0 + tu_1.$$

In other words, $u^* \leq u$ by definition. It follows that $u^* = u$. □

Proposition 4.2 *If $u_0, u_1 \in PSH(M, \xi, \omega^T) \cap L^\infty$, u is defined by (4.5) and $u_t = u(\cdot, t)$ is the weak geodesic. Let C be a constant $\geq ||u_1 - u_0||_{L^\infty}$.*

(1) *We have*

$$\max(u_0 - Ct, u_1 - C(1-t)) \leq u_t \leq (1-t)u_0 + tu_1. \tag{4.6}$$

(2) *$u_t \in PSH(M, \xi, \omega^T) \cap L^\infty$ and u is the unique solution of* (4.3).
(3) *u_t is uniformly Lipschitz continuous with respect to t:*

$$|u_t - u_s| \leq C|s - t|.$$

for $s, t \in [0, 1]$.
(4) *The derivatives $\dot{u}_0, \dot{u}_1$ exists and*

$$|\dot{u}_0| \leq C, \quad |\dot{u}_1| \leq C.$$

Proof (1) It is obvious that $u_0 - Ct, u_1 - C(1-t)$ are weak subgeodesics. It follows from the definition of u_t (4.5) that

$$\max(u_0 - Ct, u_1 - C(1-t)) \leq u_t.$$

The other half of the inequality comes from the convexity of u_t with respect to t.

(2) By the inequality (4.6) we have $u_t \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$ and $\lim_{t\to 0,1} u_t = u_{0,1}$. Then $u \in \mathrm{PSH}(N^\circ, \xi, \pi^*\omega^T) \cap L^\infty$. Using the classical Perron-Bremmerman argument, we have $(\pi^*\omega^T + \sqrt{-1}\partial_B\overline{\partial}_B u)^{n+1} \wedge \eta = 0$. Hence u is a solution of (4.3). The uniqueness of the solution of (4.3) follows from the strong maximum principle.

(3) If one of s, t equals to 0 or 1, the required inequality is a direct consequence of (4.6). If $0 < s < t < 1$, by the convexity of u_t with respect to t we have

$$\frac{t-s}{s}(u_s - u_0) \leq u_t - u_s \leq \frac{t-s}{1-s}(u_1 - u_s)$$

and the inequality follows from the case $t = 0, 1$ we have proved.

(4) By the convexity of u_t, we have

$$\frac{u_{t_1} - u_0}{t_1} \leq \frac{u_{t_2} - u_0}{t_2}$$

for $0 < t_1 < t_2$. These quantities are uniformly bounded by C. Hence $\dot{u}_0$ exists and $|\dot{u}_0| \leq C$. The case of $\dot{u}_1$ follows by a similar argument.

□

Remark 4.3 If $u_0, u_1 \in \mathcal{H}$, the weak geodesic u_t coincides with the $C_B^{1,\bar{1}}$ geodesic.

The weak geodesic u_t connecting $u_0, u_1 \in \mathrm{PSH}(M, \xi, \omega^T)$ has the advantage of admitting some homogeneous structures and offering a new interpretation of rooftop envelope. Moreover, it is closed for class $\mathcal{E}_p(M, \xi, \omega^T)$: the weak geodesic u_t connecting $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$ stays in the same class. It is called the finite-energy geodesic in $\mathcal{E}_p(M, \xi, \omega^T)$.

Proposition 4.3 *Let $u_0^k, u_1^k \in PSH(M, \xi, \omega^T)$ be sequences decreasing to $u_0, u_1 \in PSH(M, \xi, \omega^T)$, respectively. Suppose that $u_t^k, u_t \in PSH(M, \xi, \omega^T)$ be the weak geodesic connecting u_0^k, u_1^k and u_0, u_1, respectively. Then*

(1) *u_t^k decreases to u_t for $t \in [0, 1]$;*
(2) *For any $t_1, t_2 \in [0, 1]$, $[0, 1] \ni t \to u_{(1-t)t_1+tt_2} \in PSH(M, \xi, \omega^T)$ is the weak geodesic connecting u_{t_1} and u_{t_2}.*

Proof (1) By the definition of u_t^k (4.5) it is obvious that $\{u_t^k\}_{k\in\mathbb{N}}$ is decreasing and $v_t = \lim_{k\to\infty} u_t^k \in \mathrm{PSH}(M, \xi, \omega^T)$. Again by the definition of u_t^k, u_t (4.5) we have $u_t^k \geq u_t$, hence $v_t \geq u_t$.
Recall that u_t^k is convex with respect to t. Then $u_t^k \leq (1-t)u_0^k + tu_1^k$ and $v_t \leq (1-t)u_0 + tu_1$. It follows from the definition of u_t (4.5) that $v_t \leq u_t$. Consequently the sequence $\{u_t^k\}_{k\in\mathbb{N}}$ decreases to u_t for $t \in [0, 1]$.

(2) Recall that u_0, u_1 are the decreasing limits of their canonical cutoffs, it follows from part (1) that we only have to prove the proposition for u_0, u_1 in L^∞. $v_t = u_{(1-t)t_1+tt_2}$ is a path connecting u_{t_1}, u_{t_2}. By Proposition 4.2 we have $\lim_{t\to 0,1} v_t = u_{t_1,t_2}$ and $\Phi(\cdot, t) = v_t$ is a solution of Eq. ((4.3)) with initial data u_{t_1}, u_{t_2}. Then it follows from Proposition 4.2(2) that $v_t = u_{(1-t)t_1+tt_2}$ is the weak geodesic connecting u_{t_1}, u_{t_2}.

□

Lemma 4.3 *Suppose $u_0, u_1 \in PSH(M, \xi, \omega^T)$ and $t \to u_t$ is the weak geodesic segment connecting u_0, u_1.*

(1) *For any $\tau \in \mathbb{R}$ we have*

$$\inf_{t\in(0,1)} (u_t - t\tau) = P(u_0, u_1 - \tau), \tau \in \mathbb{R}. \tag{4.7}$$

(2) *If $u_0, u_1 \in PSH(M, \xi, \omega^T) \cap L^\infty$, then*

$$\{\dot{u}_0 \geq \tau\} = \{P(u_0, u_1 - \tau) = u_0\}. \tag{4.8}$$

(3) *If $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, then $u_t \in \mathcal{E}_p(M, \xi, \omega^T)$ for $t \in [0, 1]$.*

Proof (1) First note that $t \to v_t = u_t - \tau t$ is the weak geodesic connecting $u_0, u_1 - \tau$, hence the proof can be reduced to the particular case $\tau = 0$. By definition $P(u_0, u_1) \leq u_0, u_1$. As a result, the constant weak subgeodesic $t \to h_t := P(u_0, u_1)$ is a candidate for definition of u_t, hence $h_t \leq u_t, t \in [0, 1]$. It follows that $P(u_0, u_1) \leq \inf_{t\in[0,1]} u_t$.

For the other direction, we use Kiselman minimum principle [33, Chapter I, Theorem 7.5], which asserts that $w := \inf_{t\in[0,1]} u_t \in \mathrm{PSH}(M, \xi, \omega^T)$ (note that u_t is a genuine plurisubharmonic function on foliation charts, for each t and u_t is convex in t variable; hence Kiselman minimum principle applies, as in Kähler setting). Note that $u_t \leq (1-t)u_0 + tu_1$, it follows that w is a candidate for $P(u_0, u_1)$ and hence $w \leq P(u_0, u_1)$. This completes the proof.

(2) For $x \in M$ we have $P(u_0, u_1 - \tau)(x) = u_0(x)$ if and only if $\inf_{t\in[0,1]} (u_t(x) - t\tau) = u_0(x)$. By the convexity of u_t in the t variable, it is equivalent to $\dot{u}_0(x) \geq \tau$.

(3) By Lemma 3.4, we have $P(u_0, u_1) \in \mathcal{E}_p(M, \xi, \omega^T)$. Notice that $P(u_0, u_1) \leq u_0, u_1$. It follows from (1) that $P(u_0, u_1) \leq u_t$. By Proposition 3.11 we have $u_t \in \mathcal{E}_p(M, \xi, \omega^T)$ for $t \in [0, 1]$.

□

Now we prove Theorem 4.1, through a series of propositions and lemmas, following [28, Section 4] (and in particular [30, Section 3]).

First of all, the d_p distance between comparable smooth potentials behaves well.

Lemma 4.4 *Suppose $u, v \in \mathcal{H}$ with $u \leq v$. We have*

$$\max\left\{\frac{1}{2^{n+p}} \int_M |u - v|^p \omega_u^n \wedge \eta, \int_M |u - v|^p \omega_v^n \wedge \eta\right\} \leq \mathrm{d}_p(u, v)^p \leq \int_M |u - v|^p \omega_u^n \wedge \eta. \tag{4.9}$$

Proof Let $w_t : [0, 1] \to \mathcal{H}$ be the $C_B^{1,\bar{1}}$ geodesic connecting u and v. By Theorem 3.4, we have

$$\mathrm{d}_p(u, v)^p = \int_M |\dot{w}_0|^p \omega_u^n \wedge \eta = \int_M |\dot{w}_1|^p \omega_v^n \wedge \eta. \tag{4.10}$$

By Lemma 4.1, we have $u \leq w_t$ given $u \leq v$. Since w_t is convex in t, it follows that

$$0 \leq \dot{w}_0 \leq v - u \leq \dot{w}_1. \tag{4.11}$$

It then follows that, by (4.10) and (4.11),

$$\int_M |u - v|^p \omega_v^n \wedge \eta \leq \mathrm{d}_p(u, v)^p \leq \int_M |v - u|^p \omega_u^n \wedge \eta. \tag{4.12}$$

Next we use $\omega_u^n \wedge \eta \leq 2^n \omega_{\frac{u+v}{2}}^n \wedge \eta$ to obtain that

$$2^{-n} \int_M |u - v|^p \omega_u^n \wedge \eta \leq \int_M |u - v|^p \omega_{\frac{u+v}{2}}^n \wedge \eta.$$

We write the right-hand side above as follows and apply (4.12) for $u \leq (u + v)/2$ to obtain

$$2^{-p} \int_M |u - v|^p \omega_{\frac{u+v}{2}}^n \wedge \eta = \int_M \left| u - \frac{u+v}{2} \right|^p \omega_{\frac{u+v}{2}}^n \wedge \eta \leq \mathrm{d}_p \left(u, \frac{u+v}{2} \right)^p.$$

The lemma below implies that $\mathrm{d}_p(u, (u+v)/2) \leq \mathrm{d}_p(u, v)$, completing the proof. $\square$

Lemma 4.5 *Suppose $u, v, w \in \mathcal{H}$ and $u \leq v \leq w$. Then we have*

$$\mathrm{d}_p(u, v) \leq \mathrm{d}_p(u, w), \mathrm{d}_p(v, w) \leq \mathrm{d}_p(u, w).$$

Proof Let α_t, β_t be the $C_B^{1,\bar{1}}$ geodesic segments connecting u, v and u, w, respectively. Since $u \leq v \leq w$, by Lemma 4.1 we have $u \leq \alpha_t \leq v$ and $u \leq \beta_t \leq w$; moreover, $\alpha_t \leq \beta_t$. Since $\alpha_0 = \beta_0$, this gives that $0 \leq \dot{\alpha}_0 \leq \dot{\beta}_0$. Theorem 3.4 then implies that $\mathrm{d}_p(u, v) \leq \mathrm{d}_p(u, w)$. Similarly we can prove $\mathrm{d}_p(v, w) \leq \mathrm{d}_p(u, w)$. $\square$

Next we prove that the distance formula (4.1) is well defined and extends the original definition (3.51).

Lemma 4.6 *Given $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, the limit (4.1) is finite and independent of the approximating sequences $u_0^k, u_1^k \in \mathcal{H}$.*

Proof First we show that given $u \in \mathcal{E}_p(M, \xi, \omega^T)$ and a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ decreasing to u. We have $\mathrm{d}_p(u_l, u_k) \to 0$ as $l, k \to \infty$. We can assume that $l \leq k$ and hence $u_k \leq u_l$. Then Lemma 4.4 implies that

$$\mathrm{d}_p(u_l, u_k)^p \leq \int_M |u_l - u_k|^p \omega_{u_k}^n \wedge \eta.$$

Clearly, we have $u - u_l \le u_k - u_l \le 0$ and $u - u_l, u_k - u_l \in \mathcal{E}_p(M, \xi, \omega_{u_l})$. Hence applying Proposition 3.11 for the class $\mathcal{E}_p(M, \xi, \omega_{u_l})$, we obtain that

$$\mathrm{d}_p(u_l, u_k)^p \le \int_M |u_l - u_k|^p \omega_{u_k}^n \wedge \eta \le (p+1)^n \int_M |u - u_l|^p \omega_u^n \wedge \eta. \tag{4.13}$$

As u_l decreases to $u \in \mathcal{E}_p(M, \xi, \omega^T)$, the monotone convergence theorem implies that the right-hand side above converges to zero as $l \to \infty$, hence $\mathrm{d}_p(u_l, u_k) \to 0$ as $l, k \to \infty$.

Now by Lemma 3.15, we know that

$$|\mathrm{d}_p(u_0^l, u_1^l) - \mathrm{d}_p(u_0^k, u_1^k)| \le \mathrm{d}_p(u_0^l, u_0^k) + \mathrm{d}_p(u_1^l, u_1^k) \to 0, l, k \to \infty.$$

Hence this proved that the limit (4.1) is convergent and finite.

Next we show that the limit is independent of the choice of approximating sequences. Let v_0^l, v_1^l be other approximating sequences. Certainly we can assume the sequences are strictly decreasing, by adding small constants if necessary. Fix k and consider the sequence $\{\max\{u_0^{k+1}, v_0^j\}_{j\in\mathbb{N}}\}$ decreases pointwise to u_0^{k+1}. By Dini's lemma, the convergence is uniform (for fixed k) and hence we can choose j_k sufficiently large such that $v_0^j < u_0^k$, $j \ge j_k$. Repeating the argument we can assume $v_1^j < u_1^k$, for $j \ge j_k$. By triangle inequality again, we have

$$|\mathrm{d}_p(v_0^j, v_1^j) - \mathrm{d}_p(u_0^k, u_1^k)| \le \mathrm{d}_p(v_0^j, u_0^k) + \mathrm{d}_p(v_1^j, u_1^k), \ j \ge j_k.$$

By (4.13) we know that if k is sufficiently large, $\mathrm{d}_p(v_0^j, u_0^k) + \mathrm{d}_p(v_1^j, u_1^k)$ is sufficiently small. Hence the distance $\mathrm{d}_p(u_0, u_1)$ is independent of the choice of approximating sequence. □

For $u_0, u_1 \in \mathcal{H}$, we can approximate u_0, u_1 by constant sequences. The previous lemma indicates that the distance (4.1) on $\mathcal{E}_p(M, \xi, \omega^T)$ is an extension of the distance (3.51) on $\mathcal{H}$ for weight $\chi(l) = \frac{|l|^p}{p}$.

For $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, we choose a decreasing sequence $\{u_0^k\}_{k\in\mathbb{N}}, \{u_1^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that $u_0^k \searrow u_0, u_1^k \searrow u_1$. We connect u_0^k, u_1^k by the unique $C_B^{1,\bar{1}}$ geodesic segment u_t^k. By Lemma 4.1, it follows that u_t^k decreases in k. Hence the limit $\lim_{k\to\infty} u_t^k$ exists. Using Dini's lemma as above, one can show that the limit does not depend on the choice of approximating sequence. By Proposition 4.3 and the remark before it, the limit indeed coincides with the weak geodesic u_t connecting u_0, u_1:

$$u_t = \lim_{k\to\infty} u_t^k.$$

Lemma 4.7 *For $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, the weak geodesic u_t connecting them is a d_p-geodesic in the sense that*

$$\mathrm{d}_p(u_{t_1}, u_{t_2}) = |t_1 - t_2| \mathrm{d}_p(u_0, u_1)$$

for $t_1, t_2 \in [0, 1]$.

Proof Let $\{u_0^k\}_{k\in\mathbb{N}}, \{u_1^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ be sequences strictly decreasing to u_0, u_1, respectively, and $u_t^k \in \mathcal{H}_\triangle$ the unique $C_B^{1,\bar{1}}$ geodesic connecting u_0^k, u_1^k. By Theorem 3.4, we have

$$\mathrm{d}_p(u_0, u_1)^p = \lim_{k\to\infty} \mathrm{d}_p(u_0^k, u_1^k)^p = \lim_{k\to\infty} \int_M |\dot{u}_0^k|^p \omega_{u_0^k}^n \wedge \eta. \tag{4.14}$$

For $l \in (0, 1)$, Lemma 4.1 implies that u_l^k strictly decreases to u_l. Then one can choose a sequence $\{w_l^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that

(1) $u_l^{k+1} \leq w_l^k \leq u_l^k$;
(2) For the $C_B^{1,\bar{1}}$ geodesic v_t^k connecting u_0^k and w_l^k with $v_0^k = u_0^k$, $v_1^k = w_l^k$, we have

$$\left| \int_M |\dot{v}_0^k|^p \omega_{u_0^k}^n \wedge \eta - l^p \int_M |\dot{u}_0^k|^p \omega_{u_0^k} \wedge \eta \right| < \frac{1}{k}.$$

In fact there exists a sequence $\{\varphi^j\}_{j\in\mathbb{N}} \subset \mathcal{H}$ decreasing to u_l^k. By Dini's lemma, φ^j converges to u_k^l uniformly. It follows from Lemma 4.8 and Proposition 4.3 that for j big enough, $w_l^k = \varphi^j$ will satisfy our requirements. By Theorems 3.4 and (4.14),

$$\mathrm{d}_p(u_0, u_l)^p = \lim_{k\to\infty} \mathrm{d}_p(u_0^k, w_l^k)^p = \lim_{k\to\infty} \int_M ||\dot{v}_0^k||\omega_{u_0^k}^n \wedge \eta = l^p \mathrm{d}_p(u_0, u_1)^p.$$

Hence $\mathrm{d}_p(u_0, u_l) = l\mathrm{d}_p(u_0, u_1)$ for $l \in [0, 1]$.

Without loss of generality, we assume that $0 \leq t_1 \leq t_2 \leq 1$. By Proposition 4.3, $h_t = u_{(1-t)t_2}$ is the weak geodesic connecting u_{t_2} and u_0. By following from the results above, we have

$$\mathrm{d}_p(u_{t_2}, u_{t_1}) = \left(1 - \frac{t_1}{t_2}\right) \mathrm{d}_p(u_{t_2}, u_0) = (t_2 - t_1)\mathrm{d}_p(u_1, u_0).$$

This completes the proof. □

Lemma 4.8 *Suppose* $u_0, u_1 \in PSH(M, \xi, \omega^T) \cap L^\infty$. *Let* $\{u_1^k\}_{k\in\mathbb{N}} \subset PSH(M, \xi, \omega^T) \cap L^\infty$ *be a sequence decreasing to* u_1 *and* $u_t, u_t^k \in PSH(M, \xi, \omega^T) \cap L^\infty$ *the weak geodesic connecting* u_0, u_1 *and* u_0, u_1^k, *respectively. Then*

$$\lim_{k\to\infty} \int_M |\dot{u}_0^k|^p \omega_{u_0}^n \wedge \eta = \int_M |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta.$$

Proof Denote by $C = \max(||u_1^1 - u_0||_{L^\infty}, ||u_1 - u_0||_{L^\infty})$. It follows Proposition 4.2 that $||\dot{u}_0||_{L^\infty} \leq C$, $||\dot{u}_0^k||_{L^\infty} \leq C$. By Proposition 4.3 the sequence $\{u_t^k\}_{k\in\mathbb{N}}$ decreases to u_t hence the sequence $\{\dot{u}_0^k\}_{k\in\mathbb{N}}$ is decreasing with $\dot{u}_0^k \geq \dot{u}_0$.

Moreover, we have $\dot{u}_0^k$ decreases to $\dot{u}_0$. If this is not true, we can find $x_0 \in M$, $a \in \mathbb{R}$ such that $\dot{u}_0^k > a > \dot{u}_0$. Then there exists $0 < t_0 < 1$ such that $u_t^k(x_0) > u_0 + at > u_t(x_0)$ for $t \in [0, t_0]$. It contradicts with the fact that u_t^k decreases to u_t.

Then the lemma follows from Lebesgue's dominated convergence theorem. □

Pythagorean formula about d_p distance involves that rooftop envelope plays an essential role in Darvas's results [28,29] and we have a similar formula in Sasaki setting:

Theorem 4.2 (Pythagorean formula) *Given $u_0, u_1 \in \mathcal{E}_p(M, \xi, \omega^T)$, we have $P(u_0, u_1) \in \mathcal{E}_p(M, \xi, \omega^T)$ and*

$$\mathrm{d}_p(u_0, u_1)^p = \mathrm{d}(u_0, P(u_0, u_1))^p + \mathrm{d}_p(u_1, P(u_0, u_1))^p. \tag{4.15}$$

Proof First we prove the formula for $u_0, u_1 \in \mathcal{H}$. It follows from Theorem 3.1 that $P(u_0, u_1) \in \mathcal{H}_\Delta$. Let u_t be the $C_B^{1,\bar{1}}$ geodesic connecting u_0, u_1. Let v_t be the weak geodesic connecting $P(u_0, u_1), u_1$. It follows from Lemma 4.1 that $P(u_0, u_1) \le v_t$ for $t \in [0, 1]$. Hence we have $\dot{v}_0 \ge 0$. By Lemmas 4.9, Lemma 4.3, the definition of rooftop envelope, and Lemma 3.3, we have

$$\begin{aligned}
\mathrm{d}_p(P(u_0, u_1), u_1)^p &= \int_M |\dot{v}_0|^p \omega^p_{P(u_0,u_1)} \wedge \eta \\
&= \int_{\{\dot{v}_0 > 0\}} |\dot{v}_0|^p \omega^n_{P(u_0,u_1)} \wedge \eta \\
&= p \int_0^\infty s^{p-1} \omega^n_{P(u_0,u_1)} \wedge \eta(\{\dot{v}_0 \ge s\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega^n_{P(u_0,u_1)} \wedge \eta(\{P(P(u_0, u_1), u_1 - s) \\
&= P(u_0, u_1)\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega^n_{P(u_0,u_1)} \wedge \eta(\{P(u_0, u_1 - s) = P(u_0, u_1)\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega^n_{u_0} \wedge \eta(\{P(u_0, u_1 - s) = P(u_0, u_1) = u_0\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega^n_{u_0} \wedge \eta(\{P(u_0, u_1 - s) = u_0\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega^n_{u_0} \wedge \eta(\{\dot{u}_0 \ge s\}) \mathrm{d}s \\
&= \int_{\{\dot{u}_0 > 0\}} |\dot{u}_0|^p \omega^n_{u_0} \wedge \eta.
\end{aligned}$$

By a similar argument we also have

$$\mathrm{d}_p(u_0, P(u_0, u_1))^p = \int_{\{\dot{u}_0 < 0\}} |\dot{u}_0|^p \omega^n_{u_0} \wedge \eta.$$

Now using Theorem 3.4 we have

$$\begin{aligned}
\mathrm{d}_p(u_0, u_1)^p &= \int_M |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta \\
&= \int_{\{\dot{u}_0<0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta + \int_{\{\dot{u}_0>0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta \\
&= \mathrm{d}_p(u_0, P(u_0, u_1))^p + \mathrm{d}_p(P(u_0, u_1), u_1)^p
\end{aligned}$$

and the Pythagorean formula holds for smooth potentials $u_0, u_1 \in \mathcal{H}$.

For the general case we can choose sequences $\{u_0^k\}_{k\in\mathbb{N}}, \{u_1^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ decreases to u_0, u_1, respectively. Then the sequence $P(u_0^k, u_1^k) \in \mathcal{H}_\Delta$ decreases to $P(u_0, u_1)$ and the Pythagorean formula follows from Lemma 4.11. □

Lemma 4.9 *Let u_t be the weak geodesic connecting $u_0, u_1 \in \mathcal{H}_\Delta$. Then the following holds:*

$$\mathrm{d}_p(u_0, u_1)^p = \int_M |\dot{u}_0|^p \omega_{u_0}^p \wedge \eta = \int_M |\dot{u}_1|^p \omega_{u_1}^n \wedge \eta.$$

Proof $v_t = u_{1-t}$ is the weak geodesic connecting u_1, u_0. By Lemma 4.3, we have

$$\begin{aligned}
\{P(u_0+s, u_1) < u_1\} &= M - \{P(u_0+s, u_1) = u_1\} \\
&= M - \{\dot{v}_0 \geq -s\} \\
&= \{\dot{u}_1 > s\}.
\end{aligned}$$

Recall that $\omega_{u_1}^n \wedge \eta$ has total finite measure $\mathrm{Vol}(M)$, hence except for a countably many $s \in \mathbb{R}$ we have $\omega_{u_1}^n \wedge \eta(\{u_0 = u_1 - s\}) = 0$ and $\omega_{u_1}^n \wedge \eta(\{\dot{u}_1 \geq s\}) = \omega_{u_1}^n \wedge \eta(\{\dot{u}_1 > s\})$. For such real number s, it follows from Lemma 3.3 that

$$\omega_{P(u_0, u_1-s)}^n \wedge \eta = \chi_{\{P(u_0,u_1-s)=u_0\}} \omega_{u_0}^n \wedge \eta + \chi_{\{P(u_0,u_1-s)=u_1-s\}} \omega_{u_1}^n \wedge \eta$$

and

$$\mathrm{Vol}(M) = \omega_{u_0}^n \wedge \eta(\{P(u_0, u_1-s) = u_0\}) + \omega_{u_1}^n \wedge \eta(\{P(u_0, u_1-s) = u_1 - s\}).$$

It follows from Lemma 4.3, the definition of rooftop envelope, that

$$\begin{aligned}
\int_{\{\dot{u}_0>0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta &= p \int_0^\infty s^{p-1} \omega_{u_0}^n \wedge \eta(\{\dot{u}_0 \geq s\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega_{u_0}^n \wedge \eta(\{P(u_0, u_1-s) = u_0\}) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} (\mathrm{Vol}(M) - \omega_{u_1}^n \wedge \eta(\{P(u_0, u_1-s) = u_1 - s\})) \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \omega_{u_1}^n \wedge \eta(\{P(u_0, u_1-s) < u_1 - s\}) \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}&= p\int_0^\infty s^{p-1}\omega_{u_1}^n \wedge \eta(\{P(u_0+s,u_1)<u_1\})\mathrm{d}s\\&= p\int_0^\infty s^{p-1}\omega_{u_1}^n \wedge \eta(\{\dot u_1>s\})\mathrm{d}s\\&= p\int_0^\infty s^{p-1}\omega_{u_1}^n \wedge \eta(\{\dot u_1\geq s\})\mathrm{d}s\\&= \int_{\{\dot u_1>0\}}|\dot u_1|^p\omega_{u_1}^n\wedge\eta.\end{aligned}$$

A similar argument gives that

$$\int_{\{\dot u_0<0\}}|\dot u_0|^p\omega_{u_0}^n\wedge\eta=\int_{\{\dot u_1<0\}}|\dot u_1|^p\omega_{u_1}^n\wedge\eta.$$

It follows that

$$\int_M|\dot u_0|^p\omega_{u_0}^n\wedge\eta=\int_M|\dot u_1|^p\omega_{u_1}^n\wedge\eta.$$

Now choose sequence $\{u_0^k\}_{k\in\mathbb{N}}, \{u_1^k\}_{k\in\mathbb{N}}\subset\mathcal{H}$ decreasing to u_0, u_1, respectively. Let u_t^{kl}, u_t be the $C_B^{1,\bar 1}$ geodesic connecting u_0^k, u_1^l and u_0, u_1, respectively. Let u_t^k be the $C_B^{1,\bar 1}$ geodesic connecting u_0^k, u_1. It follows from Lemmas 4.11, 4.8 and the above results that

$$\begin{aligned}\mathrm{d}_p(u_0^k,u_1)^p &= \lim_{l\to\infty}\mathrm{d}_p(u_0^k,u_1^l)^p=\lim_{l\to\infty}\int_M|\dot u_0^{kl}|^p\omega_{u_0^k}^n\wedge\eta=\int_M|\dot u_0^k|^p\omega_{u_0^k}^n\wedge\eta\\&=\int_M|\dot u_1^k|^p\omega_{u_1}^n\wedge\eta.\end{aligned}$$

Then using Lemmas 4.11, 4.8, and Proposition 4.3, we have

$$\mathrm{d}_p(u_0,u_1)^p=\lim_{k\to\infty}\mathrm{d}_p(u_0^k,u_1)^p=\lim_{k\to\infty}\int_M|\dot u_1^k|^p\omega_{u_1}^n\wedge\eta=\int_M|\dot u_1|^p\omega_{u_1}^n\wedge\eta.$$

This completes the proof. □

Lemma 4.10 *Assume that $u, v\in\mathcal{E}_p(M,\xi,\omega^T)$ with $u\leq v$.Then we have*

$$\max\left(\frac{1}{2^{n+p}}\int_M|v-u|^p\omega_u^n\wedge\eta,\int_M|u-v|^p\omega_v^n\wedge\eta\right)\leq\mathrm{d}_p(u,v)^p\leq\int_M|v-u|^p\omega_u^p\wedge\eta.$$

Proof First we can choose $u_k, w_k\in\mathcal{H}$ strictly decreasing to u, v, respectively. Then $\max(u_k,w_k)\in\mathrm{PSH}(M,\xi,\omega^T)$ are continuous and strictly decreases to v. By Dini's lemma there exists $v_k\in\mathcal{H}$ such that $\max(u_{k-1},v_{k-1})\geq v_k\geq\max(u_k,v_k)$. Then v_k decreases to v and $u_k\leq v_k$. It follows from Lemma 4.4 that

$$\max\left(\frac{1}{2^{n+p}}\int_M |v_k-u_k|^p\omega_{u_k}^n\wedge\eta,\int_M |u_k-v_k|^p\omega_{v_k}^n\wedge\eta\right)\le \mathrm{d}_p(u_k,v_k)^p$$
$$\le\int_M |v_k-u_k|^p\omega_{u_k}^p\wedge\eta.$$

By Proposition 3.15, the required inequality follows as $k\to\infty$. □

Lemma 4.11 *If the sequence $\{u_k\}_{k\in\mathbb{N}},\{v_k\}_{k\in\mathbb{N}}\subset\mathcal{E}_p(M,\xi,\omega^T)$ decreases (increases) to $u,v\in\mathcal{E}_p(M,\xi,\omega^T)$, respectively, then $\mathrm{d}_p(u_k,v_k)\to\mathrm{d}_p(u,v)$ as $k\to\infty$. In particular, $\mathrm{d}_p(u_k,u)\to 0$.*

Proof If the sequence $\{u_k\}_{k\in\mathbb{N}}$ is decreasing, using the triangle inequality and Lemma 4.10, we have

$$|\mathrm{d}_p(u_k,v_k)-\mathrm{d}_p(u,v)|\le\mathrm{d}_p(u_k,u)+\mathrm{d}_p(v,v_k)$$
$$\le\left(\int_M|u_k-u|^p\omega_u^n\wedge\eta\right)^{\frac{1}{p}}+\left(\int_M|v_k-v|^p\omega_v^n\wedge\eta\right)^{\frac{1}{p}}$$

and the lemma follows from Lemma 3.15.

If the sequence $\{u_k\}_{k\in\mathbb{N}}$ is increasing, using the triangle inequality and Lemma 4.10, we have

$$|\mathrm{d}_p(u_k,v_k)-\mathrm{d}_p(u,v)|\le\mathrm{d}_p(u_k,u)+\mathrm{d}_p(v,v_k)$$
$$\le\left(\int_M|u_k-u|^p\omega_{u_k}^n\wedge\eta\right)^{\frac{1}{p}}+\left(\int_M|v_k-v|^p\omega_{v_k}^n\wedge\eta\right)^{\frac{1}{p}}$$

and the lemma follows from Lemma 3.15. □

Next we proceed to prove that $(\mathcal{E}_p(M,\xi,\omega^T),\mathrm{d}_p)$ is a complete metric space.

Lemma 4.12 *Suppose $u_0,u_1\in\mathcal{E}_p(M,\xi,\omega^T)$. Then we have*

$$\mathrm{d}_p\left(u_0,\frac{u_0+u_1}{2}\right)^p\le C\mathrm{d}_p(u_0,u_1)^p.$$

Proof It is obvious that $P(u_0,u_1)\le P(u_0,\frac{u_0+u_1}{2})\le u_0$ and $P(u_0,u_1)\le P(u_0,\frac{u_0+u_1}{2})\le\frac{u_0+u_1}{2}$. By the Pythagorean theorem 4.2, Lemmas 4.5, and 4.10, we have

$$\mathrm{d}_p\left(u_0,\frac{u_0+u_1}{2}\right)^p=\mathrm{d}_p\left(u_0,P\left(u_0,\frac{u_0+u_1}{2}\right)\right)^p$$
$$+\mathrm{d}_p\left(\frac{u_0+u_1}{2},P\left(u_0,\frac{u_0+u_1}{2}\right)\right)^p$$
$$\le\mathrm{d}_p(u_0,P(u_0,u_1))^p+\mathrm{d}_p\left(\frac{u_0+u_1}{2},P(u_0,u_1)\right)^p$$

$$
\begin{aligned}
&\leq \int_M |u_0 - P(u_0,u_1)|^p \omega^n_{P(u_0,u_1)} \wedge \eta \\
&\quad + \int_M \left|\frac{u_0+u_1}{2} - P(u_0,u_1)\right|^p \omega^n_{P(u_0,u_1)} \wedge \eta \\
&\leq 2\left(\int_M |u_0 - P(u_0,u_1)|^p \omega^n_{P(u_0,u_1)} \wedge \eta \right.\\
&\quad \left. + \int_M |u_1 - P(u_0,u_1)|^p \omega^n_{P(u_0,u_1)} \wedge \eta\right) \\
&\leq 2^{n+p+1}(\mathrm{d}_p(u_0, P(u_0,u_1))^p + \mathrm{d}_p(u_1, P(u_0,u_1))^p) \\
&= 2^{n+p+1}\mathrm{d}_p(u_0,u_1)^p.
\end{aligned}
$$

This completes the proof. ◻

Theorem 4.3 *For any $u_0, u_1 \in \mathcal{E}_p(M,\xi,\omega^T)$, we have*

$$
C^{-1}\mathrm{d}_p(u_0,u_1)^p \leq \int_M |u_0-u_1|^p(\omega^n_{u_0}\wedge\eta + \omega^n_{u_1}\wedge\eta) \leq C\mathrm{d}_p(u_0,u_1)^p. \tag{4.16}
$$

Proof Using the triangle inequality, arithmetic–geometric mean inequality, and Lemma 4.10, we have:

$$
\begin{aligned}
\mathrm{d}_p(u_0,u_1)^p &\leq (\mathrm{d}_p(u_0,\max(u_0,u_1)) + \mathrm{d}_p(u_1,\max(u_0,u_1)))^p \\
&\leq 2^{p-1}(\mathrm{d}_p(u_0,\max(u_0,u_1))^p + \mathrm{d}_p(u_1,\max(u_0,u_1))^p) \\
&\leq 2^{p-1}\left(\int_M |u_0 - \max(u_0,u_1)|^p \omega^n_{u_0}\wedge\eta \right.\\
&\quad \left. + \int_M |u_1 - \max(u_0,u_1)|^p\omega^n_{u_1}\wedge\eta\right) \\
&= 2^{p-1}\left(\int_{\{u_0<u_1\}} |u_0-u_1|^p\omega^n_{u_0}\wedge\eta + \int_{\{u_1<u_0\}}|u_1-u_0|^p\omega^n_{u_1}\wedge\eta\right) \\
&\leq 2^{p-1}\int_M |u_0-u_1|^p(\omega^n_{u_0}\wedge\eta + \omega^n_{u_1}\wedge\eta).
\end{aligned}
$$

By the previous lemma, the Pythagorean formula, and Lemma 4.10, there exists a constant C such that

$$
\begin{aligned}
C\mathrm{d}_p(u_0,u_1)^p &\geq \mathrm{d}_p\left(u_0, \frac{u_0+u_1}{2}\right)^p \\
&\geq \mathrm{d}_p(u_0, P\left(u_0, \frac{u_0+u_1}{2}\right))^p \\
&\geq \int_M \left|u_0 - P\left(u_0, \frac{u_0+u_1}{2}\right)\right| \omega^n_{u_0}\wedge\eta.
\end{aligned}
$$

Recall that $\omega_{u_0}^n \wedge \eta \leq 2^n \omega_{\frac{u_0+u_1}{2}}^n \wedge \eta$. Similarly, we also have:

$$\begin{aligned}
C\mathrm{d}_p(u_0, u_1)^p &\geq \mathrm{d}_p\left(u_0, \frac{u_0+u_1}{2}\right)^p \\
&\geq \mathrm{d}_p\left(\frac{u_0+u_1}{2}, P\left(u_0, \frac{u_0+u_1}{2}\right)\right)^p \\
&\geq \int_M \left|\frac{u_0+u_1}{2} - P\left(u_0, \frac{u_0+u_1}{2}\right)\right|^p \omega_{\frac{u_0+u_1}{2}}^n \wedge \eta \\
&\geq \frac{1}{2^n}\int_M \left|\frac{u_0+u_1}{2} - P\left(u_0, \frac{u_0+u_1}{2}\right)\right|^p \omega_{u_0}^n \wedge \eta.
\end{aligned}$$

Hence by the Holder inequality, we have:

$$\begin{aligned}
(2^n+1)C\mathrm{d}_p(u_0, u_1)^p \geq \int_M \Bigg(&\left|u_0 - P\left(u_0, \frac{u_0+u_1}{2}\right)\right|^p \\
&+ \left|\frac{u_0+u_1}{2} - P\left(u_0, \frac{u_0+u_1}{2}\right)\right|^p\Bigg)\omega_{u_0}^n \wedge \eta \\
\geq \frac{1}{2^{2p-1}}&\int_M |u_0 - u_1|^p \omega_{u_0}^n \wedge \eta.
\end{aligned}$$

By symmetry of u_0, u_1, we also have:

$$(2^n+1)C\mathrm{d}_p(u_0, u_1)^p \geq \frac{1}{2^{2p-1}}\int_M |u_0 - u_1| \omega_{u_1}^n \wedge \eta.$$

Adding the last two inequalities, we obtain:

$$2^{2p+1}(2^n+1)C\mathrm{d}_p(u_0, u_1)^p \geq \int_M |u_0 - u_1|^p (\omega_{u_0}^n \wedge \eta + \omega_{u_1}^p \wedge \eta).$$

This completes the proof. ◻

Lemma 4.13 *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_p(M, \xi, \omega^T)$ be a d_p-bounded sequence decreasing (increasing) to u. Then $u \in \mathcal{E}(M, \xi, \omega^T)$ and $\mathrm{d}_p(u_k, u) \to 0$.*

Proof If $\{u_k\}_{k\in\mathbb{N}}$ is decreasing, we can assume that $u_k < 0$. It follows from Lemma 4.10 that

$$\max\left(\frac{1}{2^{n+p}}\int_M |u_k|^p \omega_{u_k}^n \wedge \eta, \int_M |u_k|^p (\omega^T)^n \wedge \eta\right) \leq \mathrm{d}_p(u_k, 0)^p$$

are uniformly bounded. $\int_M |u_k|^p (\omega^T)^n \wedge \eta$ is uniformly bounded; the monotone convergence theorem and the dominated convergence theorem imply that $u_k \to u$ in L^1 and $u \in \mathrm{PSH}(M, \xi, \omega^T)$. $E_p(u_k) = \int_M |u_k|^p \omega_{u_k}^n \wedge \eta$ is uniformly bounded; it follows from Proposition 3.16 and Lemma 4.11 that $u \in \mathcal{E}_p(M, \xi, \omega^T)$ and $\mathrm{d}_p(u_k, u) \to 0$.

If $\{u_k\}_{k\in\mathbb{N}}$ is increasing, it follows from Theorem 4.3 that there exists a constant C such that

$$\int_M |u_k|^p(\omega_{u_k}^n \wedge \eta + (\omega^T)^n \wedge \eta) \leq C\mathrm{d}_p(u_k, 0)^p$$

is uniformly bounded. By Propositions 3.3 and 3.4, we have $u_k \to u$ in L^1 and $u \in \mathrm{PSH}(M, \xi, \omega^T)$. By Proposition 3.16 and Lemma 4.11, we have $u \in \mathcal{E}_p(M, \xi, \omega^T)$ and $\mathrm{d}_p(u_k, u) \to 0$. □

Proposition 4.4 *Given $u_0, u_1, v \in \mathcal{E}_p(M, \xi, \omega^T)$,*

$$\mathrm{d}_p(P(u_0, v), P(u_1, v)) \leq \mathrm{d}_p(u_0, u_1).$$

Proof By Theorem 3.1 and Lemma 4.11, we only have to prove the inequality for $u_0, u_1, v \in \mathcal{H}_\triangle$. In this case, $P(u_0, v), P(u_1, v) \in \mathcal{H}_\triangle$ according to Theorem 3.1.

First we assume that $u_0 \leq u_1$. Let u_t, v_t be the $C_B^{1,\bar{1}}$ geodesic connecting u_0, u_1 and $P(u_0, v), P(u_1, v)$, respectively. Then $P(u_0, v) \leq P(u_1, v) \leq v$ and Proposition 4.1 imply that $P(u_0, v) \leq v_t \leq v$. Hence for $x \in \{P(u_0, v) = v\}$, $v_t(x)$ is independent of t and $\dot{v}_0(x) = 0$. Then we have

$$\int_{\{P(u_0,v)=v\}} |\dot{v}_0|^p \omega_v^n \wedge \eta = 0.$$

$P(u_0, v) \leq P(u_1, v)$, $P(u_0, v) \leq u_0$, $P(u_1, v) \leq u_1$, and Proposition 4.1 imply that $P(u_0, v) \leq v_t \leq u_t$ for $t \in [0, 1]$ and $\dot{v}_0 \geq 0$. Moreover, for $x \in \{P(u_0, v) = u_0\}$ we have

$$\dot{v}_0(x) = \lim_{t\to 0+} \frac{v_t(x) - v_0(x)}{t} \leq \lim_{t\to 0+} \frac{u_t(x) - u_0(x)}{t} = \dot{u}_0(x).$$

Then it follows from Lemmas 4.9 and 3.3 that

$$\begin{aligned}
\mathrm{d}_p(P(u_0, v), P(u_1, v))^p &= \int_M |\dot{v}_0| \omega_{P(u_0,v)}^n \wedge \eta \\
&\leq \int_{\{P(u_0,v)=u_0\}} |\dot{v}_0|^p \omega_{u_0}^n \wedge \eta + \int_{\{P(u_0,v)=v\}} |\dot{v}_0|^p \omega_v^n \wedge \eta \\
&\leq \int_{\{P(u_0,v)=u_0\}} |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta \\
&\leq \int_M |\dot{u}_0|^p \omega_{u_0}^n \wedge \eta \\
&= \mathrm{d}_p(u_0, u_1)^p.
\end{aligned}$$

For the general case, using the Pythagoreans formula we have

$$\begin{aligned}
\mathrm{d}_p(P(u_0, v), P(u_1, v))^p = {} & \mathrm{d}_p(P(u_0, v), P(u_0, u_1, v))^p \\
& + \mathrm{d}_p(P(u_1, v), P(u_0, u_1, v))^p
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{d}_p(P(u_0, v), P(P(u_0, u_1), v))^p \\
&\quad + \mathrm{d}_p(P(u_1, v), P(P(u_0, u_1), v))^p \\
&\leq \mathrm{d}_p(u_0, P(u_0, u_1))^p + \mathrm{d}_p(u_1, P(u_0, u_1))^p \\
&= \mathrm{d}_p(u_0, u_1)^p.
\end{aligned}
$$

This completes the proof. □

Proposition 4.5 $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$ *is a complete metric space.*

Proof First we show that $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$ is a metric space. The symmetry of d_p is obvious and the triangle inequality is inherited from the triangle inequality for smooth potentials. We only have to check the non-degeneracy of d_p. Suppose $w_1, w_2 \in \mathcal{E}_p(M, \xi, \omega^T)$ and $\mathrm{d}_p(w_1, w_2) = 0$. It follows from the Pythagorean formula that $\mathrm{d}_p(w_1, P(w_1, w_2)) = 0$ and $\mathrm{d}_p(P(w_1, w_2), w_2) = 0$. Then Lemma 4.10 implies that $w_1 = P(w_1, w_2) = w_2$ with respect to the measure $\omega^n_{P(w_1, w_2)} \wedge \eta$. Then the domination principle Lemma 3.6 implies that $w_1 \leq P(w_1, w_2)$ and $w_2 \leq P(W_1, w_2)$. It follows that $w_1 = P(w_1, w_2) = w_2$. Hence $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$ is a metric space.

Next we show that the metric space $(\mathcal{E}_p(M, \xi, \omega^T), \mathrm{d}_p)$ is complete. Suppose $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_p(M, \xi, \omega^T)$ is a d_p Cauchy sequence. We will prove that there exists $u \in \mathcal{E}_p(M, \xi, \omega^T)$ such that $\mathrm{d}_p(u_k, u) \to 0$.

Without loss of generality we can assume that

$$
\mathrm{d}_p(u_k, u_{k+1}) \leq \frac{1}{2^k}
$$

for $k \in \mathbb{N}$. Denote by $u_k^l = P(u_k, u_{k+1}, \ldots, u_{k+l})$ for $k, l \in \mathbb{N}$ and $u_k^0 = u_k$. It follows from the definition of rooftop envelope and Proposition 4.4 that

$$
\mathrm{d}_p(u_k^l, u_k^{l+1}) = \mathrm{d}_p(P(u_k^l, u_{k+l}), P(u_k^l, u_{k+l+1})) \leq \mathrm{d}_p(u_{k+l}, u_{k+l+1}) \leq \frac{1}{2^{k+l}}
$$

and the sequence $\{u_k^l\}_{l\in\mathbb{N}} \subset \mathcal{E}_p(M, \xi, \omega^T)$ is d_p bounded and decreasing. According to Lemma 4.13, $\tilde{u}_k = \lim_{l\to\infty} u_k^l \in \mathcal{E}_p(M, \xi, \omega^T)$ and $\mathrm{d}_p(u_k^l, \tilde{u}_k) \to 0$ as $l \to \infty$. Moreover, $u_k^{l+1} \leq u_{k+1}^l$ implies that $\tilde{u}_k \leq \tilde{u}_{k+1}$ and $\{\tilde{u}_k\}_{k\in\mathbb{N}}$ is a increasing sequence in $\mathcal{E}_p(M, \xi, \omega^T)$.

It follows from Lemma 4.11, the definition of rooftop envelope, and Proposition 4.4 that

$$
\begin{aligned}
\mathrm{d}_p(\tilde{u}_k, \tilde{u}_{k+1}) &= \lim_{l\to\infty} \mathrm{d}_p(u_k^{l+1}, u_{k+1}^l) \\
&= \lim_{l\to\infty} \mathrm{d}_p(P(u_{k+1}^l, u_k), P(u_{k+1}^l, u_{k+1})) \\
&\leq \lim_{l\to\infty} \mathrm{d}_p(u_k, u_{k+1}) \\
&\leq \frac{1}{2^k}
\end{aligned}
$$

and the sequence $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_p(M,\xi,\omega^T)$ is d_p bounded and increasing. By Lemma 4.13, $u = \lim_{k\to\infty} \tilde{u}_k \in \mathcal{E}_p(M,\xi,\omega^T)$ and $\lim_{k\to\infty} \mathrm{d}_p(\tilde{u}_k, u) = 0$. Moreover, by Proposition 4.4 we have

$$\begin{aligned}\mathrm{d}_p(u_k^l, u_k) &= \mathrm{d}_p(P(u_k, u_{k+1}^{l-1}), P(u_k, u_k)) \leq \mathrm{d}_p(u_{k+1}^{l-1}, u_k) \leq \mathrm{d}_p(u_{k+1}^{l-1}, u_{k+1}) \\ &\quad + \mathrm{d}_p(u_k, u_{k+1})\end{aligned}$$

and

$$\mathrm{d}_p(u_k^l, u_k) \leq \mathrm{d}_p(u_{k+l}^0, u_{k+l}) + \sum_{j=1}^{l} \mathrm{d}_p(u_{k+j-1}, u_{k+j}) = \sum_{j=1}^{l} \mathrm{d}_p(u_{k+j-1}, u_{k+j}).$$

It follows from Lemma 4.11 that

$$\mathrm{d}_p(\tilde{u}_k, u_k) \leq \sum_{j=1}^{\infty} \frac{1}{2^{k+j-1}} = \frac{1}{2^{k-1}}.$$

By the triangle inequality

$$\mathrm{d}_p(u_k, u) \leq \mathrm{d}_p(\tilde{u}_k, u_k) + \mathrm{d}_p(\tilde{u}_k, u),$$

we have $\mathrm{d}_p(u_k, u) \to 0$. This completes the proof. □

To end this section, we remark that Theorem 4.1 follows from Lemmas 4.6, 4.7, and Proposition 4.5. Our main Theorem 2 follows from Theorem 4.1, Lemmas 4.7, 4.3(3), and 4.12, and Theorem 4.3.

5 Sasaki-Extremal Metric

We give a brief discussion of existence of Sasaki-extremal metric and properness of modified $\mathcal{K}$-energy. Calabi's extremal metric was extended to Sasaki setting by Boyer–Galicki–Simanca [16]. A Sasaki metric is called Sasaki-extremal if its transverse Kähler metric is extremal in the sense of Calabi [17]. As in Kähler setting, given a priori estimates [49] and the pluripotential theory developed in the paper, we have the following:

Theorem 5.1 *A compact Sasaki manifold (M, ξ, η, g) admits a Sasaki-extremal metric in the transverse Kähler class $[\omega^T]$ if and only if the modified $\mathcal{K}$-energy is reduced proper.*

We recall some basic notions [16,17,35,36,52]. We use the group $\mathrm{Aut}_0(\xi, J)$ to denote the subgroup of diffeomorphism group of M which preserves both ξ and transverse holomorphic structure. Its Lie algebra is the Lie algebra of all *Hamiltonian holomorphic vector fields* in the sense of [37, Definition 4.4].

First one can define Sasaki–Futaki invariant as follows, given $X \in \mathfrak{aut}$, the Lie algebra of $\mathrm{Aut}_0(\xi, J)$,

$$\mathcal{F}_X(\omega^T) = \int_M X(f)\omega_T^n \wedge \eta, \tag{5.1}$$

where f is the potential of transverse scalar curvature,

$$\Delta f = R^T - \underline{R}.$$

The first step is certainly to verify that (5.1) does not depend on a particular choice of transverse Kähler form in $[\omega^T]$ (see [16, Proposition 5.1]). We are interested in the reduced part $\mathfrak{h}_0$ of $\mathfrak{aut}$, which consists of *Hamiltonian holomorphic vector fields* such that $\eta(Y)$ has non-empty zero. When (M, ξ, η, g) is a Sasaki-extremal metric, then similar as in Calabi's decomposition, we have [16, Theorem 4.8] the decomposition

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}_0,$$

where $\mathfrak{a}$ consists of parallel vector fields of the transverse Kähler metric g^T. Moreover, the reduced part $\mathfrak{h}_0$ has the decomposition

$$\mathfrak{h}_0 = \mathfrak{z}_0 \oplus J\mathfrak{z}_0 \oplus (\oplus_{\lambda>0}\mathfrak{h}^\lambda),$$

where $\mathfrak{z}_0 = \mathrm{aut}(\xi, \eta, g)/\{\xi\}$ and

$$\mathfrak{h}^\lambda = \{Y \in \mathfrak{h} : \mathcal{L}_X Y = \lambda Y, X = (\bar\partial R)^\#, \}$$

where $X := (\bar\partial R)^\#$ is the dual vector and it is the extremal vector field in $\mathfrak{h}_0$. In general, we can define Futaki–Mabuchi bilinear form [36] on $\mathfrak{h}_0$ as in Kähler setting (in Sasaki setting this is well defined on $\mathfrak{aut}$ since every Hamiltonian vector field has a potential, simply given by $\eta(Y)$; for example, ξ has potential 1). Given $Y, Z \in \mathfrak{aut}$, define

$$B(Y, Z) = \int_M \eta(Y)\eta(Z)(\omega^T)^n \wedge \eta. \tag{5.2}$$

It is straightforward to check that (5.2) remains unchanged if $\eta \to \eta + d_B^c\phi$ for $\phi \in \mathcal{H}$. If we restrict us on the *real Hamiltonian holomorphic vector fields* such that $\eta(Y)$ is real, then there exists a unique vector field V such that

$$\mathcal{F}_{\mathrm{Re}(Y)} = B(\mathrm{Re}(Y), V). \tag{5.3}$$

We call such V and its corresponding $X = V - \sqrt{-1}JV$ *the extremal vector field.* As in Kähler setting, for JV-invariant metrics in $\mathcal{H}$, we define the modified $\mathcal{K}$-energy [41,56] as

$$\delta\mathcal{K}_V = -\int_M \delta\phi(R_\phi - \underline{R} - \eta_\phi(V))\omega_\phi^n \wedge \eta. \tag{5.4}$$

Let $\mathrm{Aut}_0(\xi, J, V)$ be the subgroup of $\mathrm{Aut}_0(\xi, J)$ which commutes with the flow of JV.

Proposition 5.1 *The $\mathcal{K}_V$ energy is invariant under the action of $Aut_0(\xi, J, V)$*

Proof The proof is similar to Kähler setting [48, Lemma 2.1] and it follows in a tautologic way from Futaki invariant and definition of extremal vector field through Futaki–Mabuchi bilinear form. We fix a background transverse Kähler structure ω^T such that it is JV invariant. For $\sigma \in \mathrm{Aut}_0(\xi, J, V)$, let σ_t be one parameter subgroup generated by the flow of $Y_{\mathbb{R}} := \mathrm{Re}(Y)$ for some $Y \in \mathfrak{aut}$. Since Y commutes with V, hence $\sigma_t^*\omega_0$ is invariant with respect to JV if $\omega_0 \in [\omega^T]$ is invariant. We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\sigma_t^*\omega_0) &= -\int_M \sigma_t^*(\eta_0(\mathrm{Re}(Y))(R_0 - \underline{R} - \eta_0(V))\omega_0^n \wedge \eta_0) \\ &= -\int_M \eta_0(Y_{\mathbb{R}})(R_0 - \underline{R})\omega_0^n \wedge \eta_0 + \int_M \eta_0(Y_{\mathbb{R}})\eta_0(V)\omega_0^n \wedge \eta_0.\end{aligned}$$

The right-hand side is zero by (5.3). □

We define the distance d_1 modulo the group action $G_0 := \mathrm{Aut}_0(\xi, J, V)$. Fix a compact subgroup K of G_0 such that K contains the flow of JV (and ξ of course). Denote

$$\mathcal{H}_0^K = \{\phi \in \mathcal{H}_0, \phi \text{ is invariant under the flow of } K\}.$$

Note that G_0 acts on $\mathcal{H}_0$ through $\omega_\phi \to \sigma^*\omega_\phi = \omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\sigma[\phi]$. Given any $\phi, \psi \in \mathcal{H}_0$, we can consider the distance modulo G_0 as follows [26]

$$d_{1,G_0}(\phi,\psi) = \inf_{\sigma_1,\sigma_2 \in G_0} d_1(\sigma_1[\phi], \sigma_2[\psi]) = \inf_{\sigma \in G_0} d_1(\phi, \sigma[\psi]).$$

Definition 5.1 We say $\mathcal{K}_V$ is reduced proper for K-invariant metrics with respect to d_{1,G_0}, if the following conditions hold

(1) $\mathcal{K}_V$ is bounded below over $\mathcal{H}^K$.
(2) There exists constant $C, D > 0$ such that for $\phi \in \mathcal{H}^K$

$$\mathcal{K}_V(\phi) \geq C d_{1,G_0}(0, \phi) - D.$$

To prove Theorem 5.1, we proceed exactly as in [48], to consider the modified Chen's continuity path [21], for a K-invariant transverse Kähler metric ω^T,

$$t(R_\phi - \underline{R} - \eta_\phi(V)) + (1-t)(\Lambda_{\omega_\phi}\omega^T - n) = 0. \tag{5.5}$$

Given a priori estimates as in [49] and the pluripotential theory on Sasaki manifolds developed in this paper, we can then follow [48,49] to prove Theorem 5.1. Since the argument is almost identical, we only sketch the process and skip the details.

(1) The openness of (5.5) is proved similarly [48, Theorem 3.4]; note that we assume transverse Kähler metrics and potentials are K-invariant.

(2) For $0 < t < 1$, $\mathcal{K}_V$ bounded below over $\mathcal{H}^K$ implies that the distance $d(0, \phi_t)$ is uniformly bounded by a constant in the order $C((1-t)^{-1} + 1)$, where ϕ_t is the solution of (5.5) at t. This together with the fact that ϕ_t minimizes $t\mathcal{K}_V + (1-t)\mathbb{J}$, gives the uniform upper bound of entropy of $H(\phi_t)$ (depending on $(1-t)^{-1}$). Hence estimates in [49, Theorem 2] apply to get the solution for any $t < 1$.

(3) Choose an increasing sequence $t_i \to 1$; first using the properness assumption, we can assume that there are $\sigma_i \in G$ such that $\psi_i := \sigma_i[\phi_{t_i}]$ ($\omega_{\psi_i} = \sigma_i^* \omega_{\phi_{t_i}}$) satisfies that $d(0, \psi_i)$ is uniformly bounded above. Then ψ_i satisfies a scalar curvature-type equation

$$\omega_{\psi_i}^n = e^{F_i} (\omega^T)^n$$
$$\Delta_{\psi_i} F_i = h_i + \mathrm{tr}_{\psi_i} \left(Ric(\omega^T) - \frac{1-t_i}{t_i} \omega_i \right),$$

where h_i is uniformly bounded and $\omega_i = \sigma_i^*(\omega^T)$. One can use [49, Theorem 3] and arguments as in [48, Theorem 3.5] to conclude the convergence of ψ_i, F_i to a smooth Sasaki-extremal structure.

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Appendix

Approximation Through Type-I Deformation and Regularity of Rooftop Envelop

Using Type-I deformation, we can obtain the following approximation of irregular Sasaki structure (M, ξ, η, g), which would be important for us; see [55] and in particular [13, Theorem 7.1.10] for the approximation. Suppose ξ is irregular, then the Reeb flow generates an isometry in $\mathrm{Aut}(M, \xi, \eta, g)$. Let $T^k \subset \mathrm{Aut}(M, \xi, \eta, g)$ $(k \geq 2)$ be the torus generated by ξ and denote $\mathfrak{t}$ to be its Lie algebra. We can then choose $\rho_i \to 0$, $\rho_i \in \mathfrak{t}$ such that $\xi_i = \xi + \rho_i$ is quasiregular. Define

$$\eta_i = \frac{\eta}{1 + \eta(\rho_i)}, \Phi_i = \Phi - \frac{1}{1 + \eta(\rho_i)} \Phi \rho_i \otimes \eta, \omega_i^T = \frac{1}{2} \mathrm{d}\eta_i, g_i = \eta_i \otimes \eta_i + \omega_i^T (\mathbb{I} \otimes \Phi_i), \tag{6.1}$$

where Φ is the $(1, 1)$ tensor field defined on the contact bundle $\mathcal{D} = \mathrm{Ker}(\eta)$. We recall the following:

Theorem 6.1 (Approximation of irregular Sasaki structure) *Let (M, ξ, η, g) be an irregular Sasaki structure on a compact manifold M. Then we can choose $\rho_i \to 0$ such that ξ_i is quasiregular and (6.1) define a quasiregular Sasaki structure which is invariant under the action of T^k, the torus generated by ξ in $\mathrm{Aut}(M, \xi, \eta, g)$.*

Lemma 6.1 *Let (M, ξ, η, g) be a Sasaki structure on a compact manifold M. Consider a torus $T \subset Aut(M, \xi, \eta, g)$ and $\xi_i \in \mathfrak{t}$. Choose $\xi_i = \xi + \rho_i$ for ρ_i sufficiently small. Consider two Sasaki structures $(\xi, \eta, \Phi, g) \leftrightarrow (\xi_i, \eta_i, \Phi_i, g_j)$ via Type-I deformation. Then we have the following. Suppose u is T invariant and $u \in PSH(M, \xi, \omega^T)$ with $|d\Phi du| \leq C_0$. Then for ρ_i sufficiently small, there exists positive constant $\epsilon_i \to 0$ (as $\rho_i \to 0$) such that,*

$$(1-\epsilon_i)u \in PSH(M, \xi_i, \omega_i^T). \tag{6.2}$$

Similarly, suppose $|d\Phi \mathrm{d}u| \leq C_0$ and $u \in PSH(M, \xi_i, \omega_i^T)$, then there exists positive constant $\epsilon_i \to 0$ as $i \to \infty$, such that

$$(1-\epsilon_i)u \in PSH(M, \xi, \omega^T). \tag{6.3}$$

Proof Since u is T^k-invariant, hence u is a basic function with respect to both ξ and ξ_i. We write

$$\omega_i^T + \sqrt{-1}\partial_B^i \bar{\partial}_B^i u = \omega_i^T + \frac{1}{2}\mathrm{d}\Phi_i du.$$

Using (6.1), we compute

$$\begin{aligned}\omega_i^T + \frac{1}{2}\mathrm{d}\Phi_i du =& \frac{\omega^T}{1+\eta(\rho_i)} + \eta \wedge d\left(\frac{1-\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\right) + \frac{1}{2}\mathrm{d}\Phi\mathrm{d}u + 2\omega^T \frac{\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\\ =& \frac{1+2\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\omega^T + \frac{1}{2}\mathrm{d}\Phi\mathrm{d}u + \eta \wedge d\left(\frac{1-\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\right)\\ =& \,\omega^T + \frac{1}{2}\mathrm{d}\Phi\mathrm{d}u + \left(\frac{1+2\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)} - 1\right)\omega^T + \eta \wedge d\left(\frac{1-\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\right).\end{aligned} \tag{6.4}$$

If $|\mathrm{d}\Phi du| \leq C_0$, then (6.4) implies that $|\mathrm{d}\Phi_i du| \leq C_1$ (vice versa). Moreover, when $\rho_i \to 0$,

$$\frac{1+2\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)} \to 1, \quad d\left(\frac{1-\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\right) \to 0.$$

We can then choose $\epsilon_i \to 0$ as $\rho_i \to 0$, such that

$$\omega_i^T + \frac{1}{2}\mathrm{d}\Phi_i \mathrm{d}(u(1-\epsilon_i)) \geq 0.$$

This proves (6.2). Note that given the relation of Φ and Φ_i, then $|d\Phi \mathrm{d}u| \leq C_0$ implies that $|\mathrm{d}\Phi_i du|$ is uniformly bounded (we suppose ρ_i is uniformly small in smooth topology). Interchanging ξ and ξ_i, this proves (6.3). □

Remark 6.1 Note that the complex structure on the cone remains unchanged under Type-I deformation [50, Lemma 2.2]. The transverse holomorphic structure is changed since the foliation is changed, due to the change of Reeb vector foliation; on the other hand, the contact bundle $\mathcal{D}$ remains unchanged. Note that $(\mathcal{D}, \Phi)$ and $(\mathcal{D}, \Phi_i)$ can be identified to transverse holomorphic tangent bundle $T^{1,0}(\mathcal{F}_\xi)$ and $T^{1,0}(\mathcal{F}_{\xi_i})$ (the foliations are different). Since the term $\eta \wedge d\left(\frac{1-\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)}\right)$ vanishes on $\mathcal{D}$ and $\left(\frac{1+2\mathrm{d}u(\Phi\rho_i)}{1+\eta(\rho_i)} - 1\right)\omega^T$ involves with only $\mathrm{d}u$, hence the above statement holds if we only assume that $|\mathrm{d}u|$ is uniformly bounded. Since we shall not need this, we skip the argument. However, it seems that assumption like $|\mathrm{d}u| \le C$ is necessary and we are not able to extend this to $\mathrm{PSH}(M, \xi, \omega^T)$.

As mentioned above, we fix a torus $T \subset \mathrm{Aut}(N, \xi, \eta, g)$ and consider $\rho_i \in \mathfrak{t}$ sufficiently small. Let $\xi_i = \xi + \rho_i$ and let $(\xi_i, \eta_i, g_i, \Phi_i)$ be the Type-I deformation of (ξ, η, g, Φ).

Lemma 6.2 *Let $\rho_i \to 0$. Suppose a sequence of T-invariant functions $u_i \in PSH(M, \xi_i, \omega_i^T)$ with $|\mathrm{d}\Phi du_i|_{\omega^T} \le C_0$ converges to $u \in PSH(M, \xi, \omega^T)$. Then $|\mathrm{d}\Phi \mathrm{d}u|_{\omega^T} \le C_0$ and we have the following weak convergence of the measure*

$$\left(\omega_i^T + \frac{1}{2}\mathrm{d}\Phi_i du_i\right)^n \wedge \eta_i \to \left(\omega^T + \frac{1}{2}\mathrm{d}\Phi du\right)^n \wedge \eta.$$

Proof By (6.4) and $|\mathrm{d}\Phi du_i|_{\omega^T} \le C_0$, $\omega_i^T + \frac{1}{2}\mathrm{d}\Phi_i du_i$ and $\omega^T + \frac{1}{2}\mathrm{d}\Phi du_i$ differ by a term with small L^∞ norm, hence we only need to prove that

$$\left(\omega^T + \frac{1}{2}\mathrm{d}\Phi du_i\right)^n \wedge \eta_i \to \left(\omega^T + \frac{1}{2}\mathrm{d}\Phi du\right)^n \wedge \eta.$$

Note that $\eta_i = \eta/(1 + \eta(\rho_i))$ converges smoothly to η, then the above follows from the weak convergence of $(\omega^T + \frac{1}{2}\mathrm{d}\Phi du_i)^n \wedge \eta$. $\square$

Next we give a proof of Theorem 3.1 in Sasaki setting, regarding the regularity of envelop construction.

Lemma 6.3 *Assume $\beta > 0$ and $u, v \in PSH(M, \xi, \omega^T) \cap L^\infty$. If*

$$(\omega_v^T)^n \wedge \eta \ge e^{\beta(v-f)}(\omega^T)^n \wedge \eta,\ (\omega_u^T)^n \wedge \eta \le e^{\beta(u-f)}(\omega^T)^n \wedge \eta$$

then $v \le u$.

Proof By the comparison principle (3.6)

$$\int_{\{u<v\}} (\omega_v^T)^n \wedge \eta \le \int_{\{u<v\}} (\omega_u^T)^n \wedge \eta.$$

Then we have

$$\int_{\{u<v\}} e^{\beta(v-f)}(\omega^T)^n \wedge \eta \le \int_{\{u<v\}} e^{\beta(u-f)}(\omega^T)^n \wedge \eta.$$

It follows that $\{u < v\}$ has zero Lebesgue measure and $v \le u$ almost everywhere with respect to Lebesgue measure. Moreover, we have $v \le u$ everywhere on M since they are ω^T-plurisubharmonic. □

Using the same computation to the transverse complex Monge–Ampere equation (as in the complex Monge–Ampere equation [57, p. 99]), we can obtain the following Laplacian estimate.

Lemma 6.4 *Suppose $u \in \mathcal{H}$ is a solution for the equation*

$$(\omega_u^T)^n \wedge \eta = e^g (\omega^T)^n \wedge \eta.$$

Then

$$\triangle^{\omega_u^T} \log Tr_{\omega^T} \omega_u^T \ge \frac{\triangle^{\omega^T} g}{Tr_{\omega^T} \omega_u^T} - 2B Tr_{\omega_u^T} \omega^T$$

where $B > 0$ is a constant which depends on ω.

Theorem 6.2 *Given $f \in C_B^\infty(M)$, then we have the following estimate*

$$\|P(f)\|_{C^{1,\bar{1}}} \le C(M, \omega^T, g, \|f\|_{C^{1,\bar{1}}}).$$

Moreover, if $u_1, \ldots, u_k \in \mathcal{H}_\Delta$, where we use the notation

$$\mathcal{H}_\Delta = \{u \in PSH(M, \xi, \omega^T) : \|u\|_{C^{1,\bar{1}}} < \infty\}$$

then $P(u_1, \ldots, u_k) \in \mathcal{H}_\Delta$.

Proof The first result was proved by Berman–Demailly [8] in Kähler setting. Since all quantities are basic and only transverse Kähler structure is involved, the argument as in Kähler setting has a direct adaption; see [30, Theorem A.7] for details in Kähler setting.

For each $\beta > 0$, consider the equation

$$\left(\omega_{u_\beta}^T\right)^n \wedge \eta = e^{\beta(u_\beta - f)} (\omega^T)^n \wedge \eta. \tag{6.5}$$

This reads locally

$$\frac{\det(g_{i\bar{j}}^T + u_{\beta i\bar{j}})}{\det(g_{i\bar{j}}^T)} = e^{\beta(u_\beta - f)}.$$

The transverse version of Aubin–Yau theorem implies that there exists a unique solution u_β for any $\beta > 0$ and a smooth function f. The unique solution u_β satisfies the following:

$$\|u_\beta - P(f)\|_{C^0} \to 0, \beta \to \infty \tag{6.6}$$

and there exists $\beta_0 > 0$ and a uniform constant C such that $\beta \geq \beta_0$,

$$-n < \Delta^{\omega^T} u_\beta \leq C. \tag{6.7}$$

To prove (6.6), we choose $x_0 \in M$ such that $u_\beta - f$ obtains its maximum at x_0. Combining with Eq. (6.5), we have

$$\sqrt{-1}\partial_B\overline{\partial}_B(f - u_\beta) \geq 0$$

and

$$u_\beta - f = \frac{1}{\beta}\log\frac{(\omega^T_{u_\beta})^n \wedge \eta}{(\omega^T)^n \wedge \eta} \leq \frac{1}{\beta}\log\frac{(\omega^T_f)^n \wedge \eta}{(\omega^T)^n \wedge \eta}$$

at x_0. It follows that

$$u_\beta - \frac{C}{\beta} \leq f$$

on M where $C = \sup_M \log\frac{(\omega^T_f)^n \wedge \eta}{(\omega^T)^n \wedge \eta}$. By the definition (3.24) we have

$$u_\beta - P(f) \leq \frac{C}{\beta}. \tag{6.8}$$

On the other hand, we choose $v \in \mathcal{H}$ and $L > 0$ such that

$$\omega^T_v \geq L\omega^T \text{ and } v \leq f.$$

One can choose $\beta_1 > 2$ such that $\epsilon = \frac{2n\log\beta}{\beta} < 1$ for all $\beta \geq \beta_1$. Take $\beta_2 = \max\{\frac{1}{L}, \beta_1\}$, then for any $\beta \geq \beta_2$, we have

$$0 < \delta, \epsilon < 1 \text{ and } e^{-\beta\epsilon} \leq \delta^n L^n$$

where $\delta = \frac{1}{\beta}$. It follows that

$$u_{\delta,\epsilon} := (1-\delta)P(f) + \delta v - \epsilon \leq f - \epsilon$$

and

$$(\omega^T_{u_{\delta,\epsilon}})^n \wedge \eta \geq \delta^n(\omega^T_v)^n \geq \delta^n L^n(\omega^T)^n \wedge \eta \geq e^{-\beta\epsilon}(\omega^T)^n \wedge \eta \geq e^{\beta(u_{\delta,\epsilon} - f)}(\omega^T)^n \wedge \eta.$$

By Eq. ((6.5)) and Lemma (6.3), we have

$$u_{\delta,\epsilon} = \left(1 - \frac{1}{\beta}\right)P(f) + \frac{v}{\beta} - \frac{2n\log\beta}{\beta} \leq u_\beta$$

and

$$P(f) \leq \frac{\beta}{\beta-1}u_\beta + \frac{2n\log\beta}{\beta-1} - \frac{1}{\beta-1}\inf_M v.$$

Combined with Eq. ((6.8)) we can derive that

$$\frac{1}{\beta}\inf_M v - \frac{1}{\beta}\sup_M f - \frac{2n\log\beta}{\beta} \leq u_\beta - P(f) \leq \frac{C}{\beta} \tag{6.9}$$

for $\beta \geq \beta_2$. Then (6.6) follows immediately.

It is standard to deduce the lower bound in (6.7) from the fact $\omega^T + \sqrt{-1}\partial_B\overline{\partial}_B u_\beta \geq 0$. By Eq. ((6.5)) and Lemma 6.4, we have

$$2B\mathrm{Tr}_{\omega^T_{u_\beta}}\omega^T + \triangle^{\omega^T_{u_\beta}}\log\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta} \geq \beta\frac{\triangle^{\omega^T}(u_\beta - f)}{\mathrm{Tr}_{\omega^T}\omega^T_\beta}.$$

It follows that

$$2nB + \triangle^{\omega^T_{u_\beta}}(\log\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta} - Bu_\beta) \geq \beta\frac{\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta} - 2n - \triangle^{\omega^T}f}{\mathrm{Tr}_{\omega^T}\omega^T_\beta}$$

and

$$\begin{aligned}\beta\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta}e^{-Bu_\beta} &\leq \beta(2n + \triangle^{\omega^T}f)e^{-Bu_\beta}\\ &\quad + [2nB + \triangle^{\omega^T_{u_\beta}}\log(\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta}e^{-Bu_\beta})]\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta}e^{-Bu_\beta}.\end{aligned}$$

Assume that $\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta}e^{-Bu_\beta}$ obtains its maximum s at $x_1 \in M$ and $C_1 = \sup_M(2n + \triangle^{\omega^T}f)$, then we have

$$\beta s \leq \beta C_1 e^{-Bu_\beta(x_1)} + 2nBs.$$

By the inequality (6.9) and $P(f) \leq f$, u_β is uniformly bounded. Hence we obtain an upper bound for $\mathrm{Tr}_{\omega^T}\omega^T_{u_\beta}e^{-Bu_\beta}$ if $\beta \geq \beta_0 = \max\{3nB, \beta_2\}$. We conclude that $\triangle^{\omega^T}u_\beta \leq C$ for $\beta \geq \beta_0$.

The first statement follows from (6.6) and (6.7).

For the second statement, first note that we only need to show that if $u_0, u_1 \in \mathcal{H}_\Delta$, then $P(u_0, u_1) \in \mathcal{H}_\Delta$. Let u_t be the geodesic segment connecting u_0, u_1, then by Lemma 3.9, we know that $u_t \in \mathcal{H}_\Delta$ (see [8] and [47] for Kähler setting). Now we have already known $P(u_0, u_1) = \inf_{t\in[0,1]} u_t$, then by [31, Proposition 4.4] (applied to each foliation chart), Δu_t is uniformly bounded. This shows that $P(u_0, u_1) \in \mathcal{H}_\Delta$. □

More generally, one can obtain results as in [31] that $P(f_1, \ldots, f_n) \in C_B^{1,\bar{1}}$ given $f_1, \ldots, f_n \in C_B^{1,\bar{1}}$. The point is that given two functions $f_1, f_2, h = \min\{f_1, f_2\}$ satisfy $\Delta h \leq \max\{\Delta f_1, \Delta f_2\}$ in viscosity sense, writing $h = \frac{f_1+f_2}{2} - \frac{|f_1-f_2|}{2}$. The argument as in [30, Theorem A.7] applies using the maximum principle in viscosity sense. Since we do not need this, we shall skip the details.

Complex Monge–Ampere Operator and Intrinsic Capacity on Compact Sasaki Manifolds

We discuss briefly the Bedford–Taylor theory on Sasaki manifolds. For details for complex Monge–Ampere operator, see Bedford–Taylor [2]. We also extend intrinsic Monge–Ampere capacity to Sasaki setting, see [43] for Kähler setting.

Given a Sasaki structure, there is a splitting of tangent bundle $TM = L\xi \otimes \mathcal{D}$, where $\mathcal{D} = \mathrm{Ker}(\eta)$, with $\Phi : \mathcal{D} \to \mathcal{D}$ inducing a splitting $\mathcal{D} \otimes \mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Hence the subbundle $\Lambda^{2p}(\mathcal{D}^*)$ of $\Lambda^{2p}M$ is well defined and Φ induces a splitting to give bidegree of forms in $\Lambda^{2p}(\mathcal{D}^*)$. Note that we have the following,

$$\Lambda^{2p}(\mathcal{D}^*) = \{\theta : \theta \in \Lambda^{2p}M, \iota_\xi\theta = 0\}.$$

We do not assume that $\theta \in \Lambda^{2p}(\mathcal{D}^*)$ is basic. That is, the coefficients of θ might not be invariant under the Reeb flow. A simple observation shows that if $\theta \in \Lambda^{2p}(\mathcal{D}^*)$, then θ is basic if it is closed, $d\theta = 0$ (since $\iota_\xi\theta = 0$). Hence a closed $2p$-form in $\Lambda^{2p}(\mathcal{D}^*)$ is basic and can be regarded as a *transverse* closed $2p$-form, defined as in [58]. In general, $d\Lambda^{2p}(\mathcal{D}^*)$ is not in $\Lambda^{2p+1}(\mathcal{D}^*)$.

Next, we give a very brief discussion of *transverse* positive closed currents of bidegree of (p, p) on M, $0 \leq p \leq n$; see [58] for similar treatment. We simply treat them as closed differential forms of bidegree (p, p) in $\Lambda^{2p}(\mathcal{D}^*)$ with measurable coefficients which are invariant under the Reeb flow. Its total variation is controlled by

$$\|T\| := \int_M T \wedge (\omega^T)^{n-p} \wedge \eta.$$

Given $\phi \in \mathrm{PSH}(M, \xi, \omega^T)$, we write $\phi \in L^1(T)$ if ϕ is integrable with respect to the measure $T \wedge (\omega^T)^{n-p} \wedge \eta$. In this case, the current ϕT is well defined and we write

$$\begin{aligned}
&\omega_\phi \wedge T := \omega^T \wedge T + \mathrm{dd}^c_B(\phi T)\\
&\omega_\phi \wedge T \wedge (\omega^T)^{n-p-1} \wedge \eta = T \wedge (\omega^T)^{n-p} \wedge \eta + \mathrm{dd}^c_B(\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta.
\end{aligned}$$

The positivity is a local notion and we simply think T as a positive closed (p, p)-form on each foliation chart. Hence $\omega_\phi \wedge T$ is also a transverse closed positive $(p+1, p+1)$ form. Note that we think transverse positive closed currents of bidegree of (p, p)-type as a linear functional on $\Lambda^{n-p,n-p}(\mathcal{D}^*)$, hence the test forms are of bidegree $(n-p, n-p)$. A main point is that test forms are not restricted to basic forms. In other words, given such a current T and $\gamma \in \Lambda^{n-p,n-p}(\mathcal{D}^*)$, we have the following pairing:

$$\gamma \to \int_M \gamma \wedge T \wedge \eta.$$

When $\phi \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$, it follows that $\phi \in L^1(T)$ for any transverse positive closed current T of bidegree (p, p) and hence one can define inductively $\omega_\phi^k \wedge (\omega^T)^{n-k}$; in particular, this leads to the definition of transverse complex Monge–Ampere operator ω_ϕ^n of bidegree (n, n). Moreover, the cocycle condition on transverse holomorphic structure ensures that $\omega_\phi^k \wedge (\omega^T)^{n-k}$ is well defined on M. In particular, $\omega_\phi^n \wedge \eta$ defines a positive Borel measure on M.

It is more convenient to consider this construction locally in foliations charts $W_\alpha = (-\delta, \delta) \times V_\alpha$. By taking test forms $\gamma \in \Lambda^{n-p,n-p}(\mathcal{D}^*)$ with compact support, we can consider $T \wedge \eta$ on a foliation chart for a transverse positive closed (p, p) current T. In particular, this give a local description of the complex Monge–Ampere measures $\omega_\phi^k \wedge (\omega^T)^{n-k} \wedge \eta$. By taking test functions f supported in a foliation chart, the measure $\omega_\phi^k \wedge (\omega^T)^{n-k} \wedge \eta$ for each k is regarded as the product measure $\omega_\phi^k \wedge (\omega^T)^{n-k} \wedge \mathrm{d}x$ on W_α, where $\xi = \partial_x$ is the Reeb direction. Note that $\omega_\phi^k \wedge (\omega^T)^{n-k}$ is defined on V_α as the usual way in Kähler setting, and the cocycle condition on transverse holomorphic structure ensures that $\omega_\phi^k \wedge (\omega^T)^{n-k}$ is well defined as a transverse positive closed current of bidegree (n, n). On each foliation chart, we have $\omega_\phi^k \wedge (\omega^T)^{n-k} \wedge \eta = \omega_\phi^k \wedge (\omega^T)^{n-k} \wedge \mathrm{d}x$ as a product measure. This coincides with the local description given by van Coevering [58, Section 2].

Moreover, when $u, v \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$, $\mathrm{d}u \wedge \mathrm{d}_B^c v \wedge T$ can also be defined, where T is a transverse closed positive current of bidegree $(n-1, n-1)$. By the polarization formula we only need to define $\mathrm{d}u \wedge \mathrm{d}_B^c u \wedge T$. By adding a positive constant if necessary, we assume $u \geq 0$. Then we define

$$\mathrm{d}u \wedge \mathrm{d}_B^c u \wedge T := \frac{1}{2}\mathrm{dd}_B^c(u^2) \wedge T - u\mathrm{dd}_B^c u \wedge T. \tag{6.10}$$

In particular, $\mathrm{d}u \wedge \mathrm{d}_B^c u \wedge T$ is positive if T is a transverse closed positive current of bidegree $(n-1, n-1)$. We can then define $\mathrm{d}u \wedge \mathrm{d}_B^c u \wedge T \wedge \eta$ as a positive Borel measure. Using the polarization formula, we have the following Cauchy–Schwarz inequality, for $u, v \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$,

$$\left|\int_M \mathrm{d}u \wedge \mathrm{d}_B^c v \wedge T \wedge \eta\right|^2 \leq \left(\int_M \mathrm{d}u \wedge \mathrm{d}_B^c u \wedge T \wedge \eta\right)\left(\int_M dv \wedge \mathrm{d}_B^c v \wedge T \wedge \eta\right). \tag{6.11}$$

We also record the following Stokes' theorem in Sasaki setting, and its proof follows the Bedford–Taylor theory as in Kähler setting via approximation (Lemma 3.1); see [58, Theorem 2.3.1, Proposition 2.3.2].

Lemma 6.5 *Let $u, v, \phi \in PSH(M, \xi, \omega^T) \cap L^\infty$, then for each $0 \le k \le n-1$, we have*

$$\begin{aligned}\int_M u dd^c_B v \wedge \omega_\phi^k \wedge (\omega^T)^{n-k-1} \wedge \eta &= \int_M v dd^c_B u \wedge \omega_\phi^k \wedge (\omega^T)^{n-k-1} \wedge \eta \\ &= -\int_M du \wedge d^c_B v \wedge \omega_\phi^k \wedge (\omega^T)^{n-k-1} \wedge \eta.\end{aligned} \tag{6.12}$$

We record a basic inequality in Sasaki setting, usually referred to Chern–Levine–Nirenberg inequality.

Proposition 6.1 (Chern–Levine–Nirenberg inequalities) *Let T be a positive closed current of bidegree (p, p) on M and $\phi \in PSH(M, \xi, \omega^T) \cap L^\infty$. Then $\|\omega_\phi \wedge T\| = \|T\|$. Moreover, if $\psi \in PSH(M, \xi, \omega^T) \cap L^1(T)$, then $\psi \in L^1(\omega_\phi \wedge T)$ and*

$$\|\psi\|_{L^1(T\wedge\omega_\phi)} \le \|\psi\|_{L^1(T)} + (2\max\{\sup\psi, 0\} + \sup\phi - \inf\phi)\|T\|. \tag{6.13}$$

Proof By Stokes' theorem, we have $\int_M dd^c_B(\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta = 0$, hence

$$\|\omega_\phi \wedge T\| = \int_M \omega^T \wedge T \wedge (\omega^T)^{n-p-1} \wedge \eta = \|T\|.$$

To prove (6.13), we first assume $\psi \le 0, \phi \ge 0$. By assumption, $\psi \in L^1(T)$, then

$$\begin{aligned}\|\psi\|_{L^1(T\wedge\omega_\phi)} := \int_M -\psi T \wedge \omega_\phi \wedge (\omega^T)^{n-p-1} \wedge \eta &= \|\psi\|_{L^1(T)} \\ &+ \int_M -\psi dd^c_B(\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta.\end{aligned}$$

By Stokes' theorem, we compute

$$\begin{aligned}\int_M -\psi dd^c_B(\phi T) \wedge (\omega^T)^{n-p-1} \wedge \eta &= \int_M dd^c_B(-\psi) \wedge \phi T \wedge (\omega^T)^{n-p-1} \wedge \eta \\ &\le \int_M \phi T \wedge (\omega^T)^{n-p} \wedge \eta \\ &\le \sup_M \phi \int_M T \wedge (\omega^T)^{n-p} \wedge \eta = (\sup_M \phi)\|T\|.\end{aligned}$$

Now suppose $\sup\psi > 0$. Replacing ϕ by $\phi - \inf\phi$, we compute

$$\|\psi\|_{L^1(T\wedge\omega_\phi)} \le \int_M (2\sup\psi - \psi) T \wedge \omega_\phi \wedge (\omega^T)^{n-p-1} \wedge \eta.$$

The same argument as above leads to (6.13) for the general case. □

For a Borel subset E on a Sasaki manifold (M, ξ, ω^T), we define the capacity as

$$\mathrm{cap}_{\omega^T}(E) := \sup\left\{\int_E \omega_\varphi^n \wedge \eta : \varphi \in \mathrm{PSH}(M, \xi, \omega^T), 0 \le \varphi \le 1\right\}.$$

It is obvious that $\mathrm{cap}_{\omega^T}(\cup_{k=1}^{\infty} E_k) \le \sum_{k=1}^{\infty} \mathrm{cap}_{\omega^T}(E_k)$ for a sequence of Borel sets E_k. We have the following:

Proposition 6.2 *Let $\phi \in PSH(M, \xi, \omega^T)$ with $0 \le \phi \le 1$ and $\psi \in PSH(M, \xi, \omega^T)$ such that $\psi \le 0$. Then*

$$\int_M -\psi\omega_\phi^n \wedge \eta \le \int_M (-\psi)(\omega^T)^n \wedge \eta + n\int_M (\omega^T)^n \wedge \eta. \tag{6.14}$$

Proof We only need to prove (6.14) for canonical cutoffs $\psi_k = \max\{\psi, -k\}$ ($-\psi_k$ increases to $-\psi$ and we can apply monotone convergence theorem). We have the following:

$$\begin{aligned}
\int_M -\psi_k\omega_\phi^n \wedge \eta &= \int_M -\psi_k\omega_\phi^{n-1} \wedge (\omega^T + \sqrt{-1}\partial_B\bar{\partial}_B\phi) \wedge \eta \\
&= \int_M -\psi_k\omega_\phi^{n-1} \wedge \omega^T \wedge \eta + \int_M -\psi_k\omega_\phi^{n-1} \wedge \sqrt{-1}\partial_B\bar{\partial}_B\phi \wedge \eta \\
&= \int_M -\psi_k\omega_\phi^{n-1} \wedge \omega^T \wedge \eta + \int_M \phi\omega_\phi^{n-1} \wedge (-\sqrt{-1}\partial_B\bar{\partial}_B\psi_k) \wedge \eta \\
&\le \int_M -\psi_k\omega_\phi^{n-1} \wedge \omega^T \wedge \eta + \int_M (\omega_\phi)^{n-1} \wedge \omega^T \wedge \eta \\
&\le \int_M -\psi_k\omega_\phi^{n-1} \wedge \omega^T \wedge \eta + \int_M (\omega^T)^n \wedge \eta.
\end{aligned}$$

We can then proceed inductively to obtain (6.14). Note that the argument above is a special case of (6.13). □

Proposition 6.3 *Suppose that $u \in PSH(M, \xi, \omega^T)$ and $u \le 0$. Then for $t > 0$ we have*

$$cap_{\omega^T}(\{u < -t\}) \le \frac{1}{t}\left(\int_M (-u)(\omega^T)^n \wedge \eta + n\int_M (\omega^T)^n \wedge \eta\right).$$

Proof This is a direct consequence of Proposition 6.2. Denote $K_t = \{u < -t\}$, then

$$\begin{aligned}
\int_{K_t} \omega_\phi^n \wedge \eta &\le \frac{1}{t}\int_M -\psi\omega_\phi^n \wedge \eta \\
&\le \frac{1}{t}\left(\int_M -\psi(\omega^T)^n \wedge \eta + n\int_M (\omega^T)^n \wedge \eta\right).
\end{aligned}$$

□

Proposition 6.4 *Suppose that $u_k, u \in PSH(M, \xi, \omega^T) \cap L^\infty$ and u_k decreases to u. Then for $\delta > 0$ we have*

$$cap_{\omega^T}(\{u_k > u + \delta\}) \to 0, k \to \infty.$$

Proof This proceeds exactly the same as in [43, Proposition 3.7]. We sketch the argument briefly. We assume $\mathrm{Vol}(M) = 1$ for simplicity. Fix $\delta > 0$ and $\phi \in \mathrm{PSH}(M, \xi, \omega^T)$ such that $0 \le \phi \le 1$. We have

$$\int_{\{u_k > u+\delta\}} \omega_\phi^n \wedge \eta \le \delta^{-1} \int_M (u_k - u)\omega_\phi^n \wedge \eta.$$

By Stokes' theorem, we write

$$\begin{aligned}\int_M (u_k - u)\omega_\phi^n \wedge \eta &= \int_M (u_k - u) \wedge \omega^T \wedge \omega_\phi^{n-1} \wedge \eta \\ &\quad + \int_M (u_k - u) \wedge \mathrm{dd}_B^c\phi \wedge \omega_\phi^{n-1} \wedge \eta \\ &= \int_M (u_k - u) \wedge \omega^T \wedge \omega_\phi^{n-1} \wedge \eta \\ &\quad - \int_M \mathrm{d}(u_k - u) \wedge \mathrm{d}_B^c\phi \wedge \omega_\phi^{n-1} \wedge \eta.\end{aligned}$$

By the Cauchy–Schwartz inequality, setting $f_k = u_k - u \ge 0$,

$$\begin{aligned}\left|\int_M \mathrm{d}(u_k - u) \wedge \mathrm{d}_B^c\phi \wedge \omega_\phi^{n-1} \wedge \eta\right|^2 &\le \int_M df_k \wedge \mathrm{d}_B^c f_k \wedge \wedge\omega_\phi^{n-1} \\ &\quad \wedge\eta \int_M d\phi \wedge \mathrm{d}_B^c\phi \wedge \wedge\omega_\phi^{n-1} \wedge \eta.\end{aligned}$$

We compute

$$\begin{aligned}\int_M d\phi \wedge \mathrm{d}_B^c\phi \wedge \wedge\omega_\phi^{n-1} \wedge \eta &= \int_M \phi(-\mathrm{dd}_B^c\phi) \wedge \omega_\phi^{n-1} \wedge \eta \\ &\le \int_M \phi\omega^T \wedge \omega_\phi^{n-1} \wedge \eta \le 1.\end{aligned}$$

Similarly, we compute

$$\begin{aligned}\int_M df_k \wedge \mathrm{d}_B^c f_k \wedge \wedge\omega_\phi^{n-1} \wedge \eta &= \int_M f_k(\mathrm{dd}_B^c u - d\mathrm{d}_B^c u_k) \wedge \omega_\phi^{n-1} \wedge \eta \\ &\le \int_M f_k\omega_u \wedge \omega_\phi^{n-1} \wedge \eta.\end{aligned}$$

Combining all these together gives

$$\int_M (u_k - u)\omega_\phi^n \wedge \eta \le \int_M (u_k - u) \wedge \omega^T \wedge \omega_\phi^{n-1} \wedge \eta + \left(\int_M (u_k - u)\omega_u \wedge \omega_\phi^{n-1} \wedge \eta \right)^{1/2}.$$

Suppose $u_k - u \le c_0$ for a fixed positive constant $c_0 \ge 1$. Then we have

$$\int_M (u_k - u)\omega_\phi^n \wedge \eta \le \sqrt{c_0} \left(\int_M (u_k - u) \wedge \omega^T \wedge \omega_\phi^{n-1} \wedge \eta \right)^{1/2} + \left(\int_M (u_k - u)\omega_u \wedge \omega_\phi^{n-1} \wedge \eta \right)^{1/2}.$$

Hence we have

$$\int_M (u_k - u)\omega_\phi^n \wedge \eta \le \sqrt{2c_0} \left(\int_M (u_k - u) \wedge (\omega^T + \omega_u) \wedge \omega_\phi^{n-1} \wedge \eta \right)^{1/2}.$$

We can proceed inductively by replacing ω_ϕ by $\omega^T + \omega_u$ to obtain

$$\int_M (u_k - u)\omega_\phi^n \wedge \eta \le (\sqrt{2c_0})^n \left(\int_M (u_k - u) \wedge (\omega^T + \omega_u)^n \wedge \eta \right)^{1/2^n}.$$

The dominated convergence theorem implies the right-hand side goes to zero, independent of ϕ. This completes the proof. □

As a consequence, we have the following:

Theorem 6.3 *Let $\varphi \in PSH(M, \xi, \omega^T)$, then for any $\epsilon > 0$ there exists an open subset $O_\epsilon \subset M$ such that $cap_{\omega^T}(O_\epsilon) < \epsilon$ and φ is continuous on $M - O_\epsilon$.*

Proof By Proposition 6.3 there exists $t_0 > 0$ such that $\mathrm{cap}_{\omega^t}(O_0) < \frac{\epsilon}{2}$ for the open subset $O_0 = \{u < -t_0\}$. Take the cutoff $u_{t_0} = \max\{u, -t_0\} \in \mathrm{PSH}(M, \xi, \omega^T)$, then there exists a sequence $u_k \in \mathcal{H}$ decreasing to u. By Proposition 6.4, we can choose a subsequence u_{k_j} such that $\mathrm{cap}_{\omega^T}(O_j) < \frac{\epsilon}{2^{j+1}}$ for the open subset $O_j = \{u_{k_j} > u + \frac{1}{j}\}$. Then for the open subset $O_\epsilon = \cup_{j=0}^\infty O_j$ we have $\mathrm{cap}_{\omega^T}(O) < \epsilon$. Moreover u_{K_j} converges uniformly to u on $M - O_\epsilon$, hence u is continuous on $M - O_\epsilon$. □

Remark 6.2 The discussions above are taken from Kähler setting [43, Section 3]. Note that in (6.13) it is necessary to replace $\sup \psi$ by $\max\{\sup \psi, 0\}$ (similarly one needs to replace $\sup_X \psi$ by $\max\{\sup_X \psi, 0\}$ in [43, Proposition 3.1]).

We also need the following uniqueness in Sasaki setting, see [44, Theorem 3.3].

Theorem 6.4 *Suppose $u, v \in \mathcal{E}_1(M, \xi, \omega^T)$ such that*

$$\omega_u^n \wedge \eta = \omega_v^n \wedge \eta,$$

then $u - v = const.$

Proof This follows exactly as in [44, Theorem 3.3] and we sketch the argument. The first step is that for $u \in \mathcal{E}_1(M, \xi, \omega^T)$ and its canonical cutoffs $u_j = \max\{u, -j\}$, then $\nabla u_j \in L^2(\mathrm{d}\mu_g)$ and has uniformly bounded L^2 norm (see [44, Proposition 3.2]). We can assume that $u \leq 0$ and hence $u_j \leq 0$. Then for $\phi \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$ such that $\phi \leq 0$, we know that, for any basic positive closed of $(n-1, n-1)$ type.

$$\begin{aligned}\int_M (-\phi)\omega \wedge T &= \int_M (-\phi)(\omega_\phi - \mathrm{d}\mathrm{d}_B^c\phi) \wedge T = \int_M (-\phi)\omega_\phi \wedge T \\ &\quad + \int_M \mathrm{d}\Phi \wedge \mathrm{d}_B^c\phi \wedge T \leq \int_M (-\phi)\omega_\phi \wedge T.\end{aligned}$$

An inductive argument applies to $T = \omega_\phi^k \wedge (\omega^T)^{n-k-1}$, we get that

$$0 \leq \int_M \mathrm{d}\Phi \wedge \mathrm{d}_B^c\phi \wedge T \leq \int_M (-\phi)\omega_\phi^n \wedge \eta. \tag{6.15}$$

Taking $\phi = u_j$ in (6.15) and noting that the right-hand side is uniformly bounded, we get ∇u_j is uniformly bounded in $L^2(\mathrm{d}\mu_g)$, hence $\nabla u \in L^2(\mathrm{d}\mu_g)$.

We assume that $u, v \leq -1$ and $\mathrm{Vol}(M) = 1$. Set $f = (u-v)/2$ and $h = (u+v)/2$. We need to establish that $\nabla f = 0$ by showing that $\int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge (\omega^T)^{n-1} \wedge \eta = 0$. If we assume u, v are bounded, then we have

$$\int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge \omega_h^{n-1} \wedge \eta \leq \sum \int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge \omega_u^k \wedge \omega_v^{n-1-k} \wedge \eta = -\int_M \frac{f}{2}(\omega_u^n - \omega_v^n) \wedge \eta, \tag{6.16}$$

where we use the fact that $\mathrm{d}\mathrm{d}_B^c f = (\omega_u - \omega_v)/2$. We shall also establish the following a priori bound, when u, v are bounded,

$$\int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge (\omega^T)^{n-1} \wedge \eta \leq 3^n \left(\int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge \omega_h^{n-1} \wedge \eta\right)^{1/2^{n-1}}. \tag{6.17}$$

We apply (6.16) and (6.17) to the canonical cutoffs u_j, v_j (writing f_j, h_j correspondingly and using Proposition 3.15),

$$\lim \int_M \mathrm{d}f_j \wedge \mathrm{d}_B^c f_j \wedge (\omega^T)^{n-1} \wedge \eta = 0.$$

We can then conclude that

$$\int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge (\omega^T)^{n-1} \wedge \eta = 0.$$

This implies that $u - v$ is a constant. To establish (6.17), we need several observations as follows. First observe that for $l = n-2, \ldots, 0$,

$$\int_M (-h)\omega_h^{2+l} \wedge (\omega^T)^{n-2-l} \wedge \eta \leq \int_M (-h)(\omega^T)^n \wedge \eta \leq 1,$$

where the last inequality follows from $-h \leq 1$ and the normalization of the volume. We can then apply the following inequality inductively for $T = \omega_h^l \wedge (\omega^T)^{n-l-1}$ such that

$$\int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega^T \wedge T \wedge \eta \leq 3 \left(\int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge T \wedge \eta \right)^{1/2}, \tag{6.18}$$

which proves (6.17). Now we establish (6.18). We write

$$\mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega^T = \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h - \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \mathrm{d}\mathrm{d}_B^c h$$

hence we obtain, integrating by parts,

$$\begin{aligned} \int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega^T \wedge T \wedge \eta = & \int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge T \wedge \eta \\ & + \int_M \mathrm{d} f \wedge \mathrm{d}_B^c h \wedge \frac{\omega_u - \omega_v}{2} \wedge T \wedge \eta. \end{aligned}$$

By Cauchy–Schwartz inequality, we have

$$\begin{aligned} \left| \int_M \mathrm{d} f \wedge \mathrm{d}_B^c h \wedge \omega_u \wedge T \wedge \eta \right|^2 \leq & \, 4 \int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge T \wedge \eta \int_M \mathrm{d} h \\ & \wedge \mathrm{d}_B^c h \wedge \omega_h \wedge T \wedge \eta. \end{aligned}$$

We can get a similar control

$$\begin{aligned} \left| \int_M \mathrm{d} f \wedge \mathrm{d}_B^c h \wedge \omega_v \wedge T \wedge \eta \right|^2 \leq & \, 4 \int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge T \wedge \eta \int_M \mathrm{d} h \\ & \wedge \mathrm{d}_B^c h \wedge \omega_h \wedge T \wedge \eta. \end{aligned}$$

Clearly, we have the following ($h \leq 0$, $S = \omega_h^l \wedge (\omega^T)^{n-l-2}$)

$$\int_M \mathrm{d} h \wedge \mathrm{d}_B^c h \wedge \omega_h \wedge S \wedge \eta \leq \int_M (-h) \omega_h^2 \wedge S \wedge \eta \leq 1.$$

Combining these estimate altogether we conclude that,

$$\begin{aligned} \int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega^T \wedge S \wedge \eta \leq & \int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge T \wedge \eta \\ & + 2 \left(\int_M \mathrm{d} f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge T \wedge \eta \right)^{1/2}. \end{aligned}$$

The last observation is that

$$\int_M \mathrm{d}f \wedge \mathrm{d}_B^c f \wedge \omega_h \wedge S \wedge \eta = \frac{1}{4}\int_M (u-v)(\omega_v - \omega_u) \wedge \omega_h \wedge S \wedge \eta$$
$$\leq \int_M (-h)\omega_h^2 \wedge S \wedge \eta \leq 1.$$

This completes the proof of (6.18) by combining two inequalities above. □

Functionals in Finite-Energy Class $\mathcal{E}_1$ and Compactness

We discuss briefly well-known functionals in Kähler geometry and their properties over finite-energy class $\mathcal{E}_1$, see [30, Section 3.8]. The energy functionals include Monge–Ampere energy $\mathbb{I}$ and Aubin's I-functional on $\mathcal{E}_1$, see [1,4,5,30] for Kähler setting. These results have a direct adaption in Sasaki setting. Recall Aubin's I-functional in Sasaki setting, for $u, v \in \mathcal{H}$

$$I(u, v) := I(\omega_u, \omega_v) = \frac{1}{n!}\int_M (v-u)(\omega_u^n - \omega_v^n) \wedge \eta. \tag{6.19}$$

We also recall the J-functional

$$J(u, v) := J(\omega_u, \omega_v) = \frac{1}{n!}\int_M (v-u)\omega_u^n \wedge \eta - \mathbb{I}_{\omega_u}(v), \tag{6.20}$$

where the $\mathbb{I}_{\omega_u}(v)$-functional is given by

$$\mathbb{I}_{\omega_u}(v) = \frac{1}{(n+1)!}\int_M (v-u)\sum_{k=0}^{n} \omega_u^k \wedge \omega_v^{n-k} \wedge \eta. \tag{6.21}$$

We define the $\mathbb{I}$-functional (with the base ω^T) on $\mathcal{H}$,

$$\mathbb{I}_{\omega^T}(u) = \frac{1}{(n+1)!}\int_M u \sum_{k=0}^{n} \omega_u^k \wedge \omega_T^{n-k} \wedge \eta. \tag{6.22}$$

The $\mathbb{I}$-functional is also called the Monge–Ampère energy, since if $t \to v_t \in \mathcal{H}$ is smooth, then we have (as in Kähler setting),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{I}(v_t) = \frac{1}{n!}\int_M \dot{v}_t \omega_{v_t}^n \wedge \eta. \tag{6.23}$$

We mention that I is symmetric with respect to u, v but J is not. I, J are both defined on the metric level, independent of the choice of normalization of potentials u, v; while

$\mathbb{I}_{\omega_u}(v)$ depends on the normalization of u, v. When u, v are bounded, then Bedford–Taylor theory allows to integrate by parts and the I-functional takes the formula

$$I(\omega_u, \omega_v) = \frac{1}{(n+1)!} \sum_{j=0}^{n-1} \int_M \mathrm{d}(u - v) \wedge \mathrm{d}_B^c(u - v) \wedge \omega_u^j \wedge \omega_v^{n-1-j} \wedge \eta. \quad (6.24)$$

Hence it is non-negative.

We need more information about $\mathbb{I}$-functional, see [30, Section 3.7] for Kähler setting. These properties in Sasaki setting follow in a rather straightforward way given pluripotential theory extended to Sasaki setting. We include these facts here for completeness.

Proposition 6.5 *Given $u, v \in PSH(M, \xi, \omega^T) \cap L^\infty$, the following cocycle condition holds*

$$\mathbb{I}(u) - \mathbb{I}(v) = \frac{1}{(n+1)!} \sum_{k=0}^{n} \int_M (u - v)\omega_u^k \wedge \omega_v^{n-k} \wedge \eta = \mathbb{I}_{\omega_u}(v). \quad (6.25)$$

Moreover, we have $\mathbb{I}(u)$ is concave in u in the sense that,

$$\frac{1}{n!} \int_M (u - v)\omega_u^n \wedge \eta \le \mathbb{I}(u) - \mathbb{I}(v) \le \frac{1}{n!} \int_M (u - v)\omega_v^n \wedge \eta. \quad (6.26)$$

As a direct consequence, if $u, v \in PSH(M, \xi, \omega^T) \cap L^\infty$ such that $u \ge v$. Then $\mathbb{I}(u) \ge \mathbb{I}(v)$.

Proof This follows almost identical as in [30, Proposition 3.8], given the pluripotential theory established in Sasaki setting in the paper. We sketch the argument. When $u, v \in \mathcal{H}$, this follows exactly the same as in Kähler setting, by taking $h_t = (1-t)u + tv$ and then use (6.23) to compute directly. When $u, v \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$, we then use $u_k, v_k \in \mathcal{H}$ decreasing to u, v (Lemma 3.1), respectively. Using Bedford–Taylor's theorem in Sasaki setting [58, Theorem 2.3.1], we proceed exactly as in Kähler setting to conclude that $\mathbb{I}(u_k) \to \mathbb{I}(u)$, etc. For the estimate (6.26), we compute

$$\begin{aligned}
\int_M (u - v)\omega_u^k \wedge \omega_v^{n-k} \wedge \eta &= \int_M (u - v)\omega_u^{k-1} \wedge \omega_v^{n-k+1} \wedge \eta \\
&\quad + \int_M (u - v)\sqrt{-1}\partial\bar{\partial}(u - v) \wedge \omega_u^{k-1} \wedge \omega_v^{n-k} \wedge \eta \\
&= \int_M (u - v)\omega_u^{k-1} \wedge \omega_v^{n-k+1} \wedge \eta \\
&\quad - \int_M \sqrt{-1}\partial(u - v) \wedge \bar{\partial}(u - v) \wedge \omega_u^{k-1} \wedge \omega_v^{n-k} \wedge \eta \\
&\le \int_M (u - v)\omega_u^{k-1} \wedge \omega_v^{n-k+1} \wedge \eta.
\end{aligned}$$

Using the estimate inductively for the terms in (6.25) leads to (6.26). Clearly, $\mathbb{I}(u)$ is concave in u given (6.26). ◻

The monotonicity property allows to define $\mathbb{I}(u)$ for $u \in \text{PSH}(M, \xi, \omega^T)$ through the limit process, using the canonical cutoffs $u_k = \max\{u, -k\}$

$$\mathbb{I}(u) = \lim_{k\to\infty} \mathbb{I}(\max\{u, -k\}).$$

Though the above limit is well defined, it may equal $-\infty$. It turns out $\mathbb{I}(u)$ is finite exactly on $\mathcal{E}_1(M, \xi, \omega^T)$. We record some further properties of $\mathbb{I}(u)$ for $u \in \mathcal{E}_1(M, \xi, \omega^T)$. The proofs are almost identical and we shall skip the details, see [30, Propositions 3.40, 3.42, 3.43; Lemma 3.41].

Proposition 6.6 *Let $u \in PSH(M, \xi, \omega^T)$. Then $-\infty < \mathbb{I}(u)$ if and only if $u \in \mathcal{E}_1(M, \xi, \omega^T)$. Moreover,*

$$|\mathbb{I}(u_0) - \mathbb{I}(u_1)| \le d_1(u_0, u_1), u_0, u_1 \in \mathcal{E}_1(M, \xi, \omega^T). \tag{6.27}$$

Proposition 6.7 *Suppose $u_0, u_1 \in \mathcal{E}_1(M, \xi, \omega^T)$ and $t \to u_t$ is the finite-energy geodesic connecting u_0, u_1. Then $t \to \mathbb{I}(u_t)$ is linear in t. We also have the following distance formula:*

$$d_1(u_0, u_1) = \mathbb{I}(u_0) + \mathbb{I}(u_1) - 2\mathbb{I}(P(u_0, u_1)).$$

In particular, $d_1(u_0, u_1) = \mathbb{I}(u_0) - \mathbb{I}(u_1)$ if $u_0 \ge u_1$.

We have the following (see [30, Lemma 3.47])

Lemma 6.6 *Suppose $u, u^j, v, v^j \in \mathcal{E}_1(M, \xi, \omega^T)$ and $u^j \searrow u$ and $v^j \searrow v$. Then the following hold:*

$$I(u, v) = I(u, \max\{u, v\}) + I(\max\{u, v\}, v). \tag{6.28}$$

Moreover, $\lim_{j\to\infty} I(u^j, v^j) = I(u, v)$.

Proof By Proposition 3.8, we have

$$\chi_{\{v>u\}}\omega^n_{\max\{u,v\}} \wedge \eta = \chi_{\{v>u\}}\omega^n_v \wedge \eta.$$

Hence it follows that

$$I(u, \max\{u, v\}) = \frac{1}{(n+1)!}\int_{\{v>u\}} (u - v)(\omega^n_v - \omega^n_u) \wedge \eta.$$

Interchange $u \leftrightarrow v$, we get $I(v, \max\{u, v\}) = \int_{\{u>v\}}(u - v)(\omega^n_v - \omega^n_u) \wedge \eta$. This proves (6.28). We write

$$I(u^j, v^j) = I(u^j, \max\{u^j, v^j\}) + I(v^j, \max\{u^j, v^j\}).$$

Since $u^j, v^j \leq \max\{u^j, v^j\}$, we can apply Proposition 3.15 to conclude $I(u^j, \max\{u^j, v^j\}) \to I(u, \max\{u, v\})$ and $I(v^j, \max\{u^j, v^j\}) \to I(v, \max\{u, v\})$, using the formula (6.19). This completes the proof. □

We have the following well-known inequalities:

Proposition 6.8 *For $u, v \in PSH(M, \xi, \omega^T) \cap L^\infty$, we have*

$$\frac{1}{n+1} I(u, v) \leq J(u, v) \leq \frac{n}{n+1} I(u, v).$$

Moreover, $J(u, v)$ is convex in v since $\mathbb{I}_{\omega^T}(v)$ is concave in v.

Proof This is well known, by direct computation [38, Proposition 4.2.1] for $u, v \in \mathcal{H}$. A direct approximation argument using Lemma 3.1 shows that this can be generalized for $u, v \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$. □

The functionals $(I, J, \mathbb{I})$ are well defined for $u, v \in \mathcal{E}_1(M, \xi, \omega^T)$ [see Proposition (3.16)]. Note that (6.26) and Proposition 6.8 both hold in $\mathcal{E}_1(M, \xi, \omega^T)$ (see [4] for Kähler setting). This follows by an approximation argument applying Proposition 3.15. Next we prove the following, as a direct adaption of [5, Theorem 1.8],

Lemma 6.7 *There exists a positive $C = C(n)$ such that for $u, v, w \in \mathcal{E}_1(M, \xi, \omega^T)$, then*

$$I(u, v) \leq C(I(u, w) + I(v, w)). \tag{6.29}$$

Proof With Lemma 6.6, we only need to argue (6.29) holds for bounded potentials, with u, v, w replaced by canonical cutoffs u_k, v_k, w_k. The proof follows exactly as in [5, Theorem 1.8, Lemma 1.9]. and we include the proof for completeness. For $u, v, \psi \in \mathrm{PSH}(M, \xi, \omega^T) \cap L^\infty$, set

$$\|\mathrm{d}(u - v)\|_\psi := \left(\int_M \mathrm{d}(u - v) \wedge \mathrm{d}^c_B(u - v) \wedge \omega_\psi^{n-1} \wedge \eta \right)^{\frac{1}{2}}.$$

Using (6.24), it is straightforward to see that

$$\|\mathrm{d}(u - v)\|^2_{\frac{u+v}{2}} \leq I(u, v) \leq 2^{n-1} \|\mathrm{d}(u - v)\|^2_{\frac{u+v}{2}}. \tag{6.30}$$

We need the following, there exists a constant $C = C(n)$ for $u, v, \psi \in \mathrm{PSH} M, \xi, \omega^T \cap L^\infty$, we have the following (see [5, Lemma 1.9]),

$$\|\mathrm{d}(u - v)\|^2_\psi \leq C I(u, v)^{1/2^{n-1}} \left(I(u, \psi)^{1-1/2^{n-1}} + I(v, \psi)^{1-1/2^{n-1}} \right). \tag{6.31}$$

With (6.31) we prove (6.29). Taking $\phi = \frac{u+v}{2}$, the triangle inequality gives,

$$\|\mathrm{d}(u - v)\|_\phi \leq \|\mathrm{d}(u - w)\|_\phi + \|d(v - w)\|_\phi.$$

Using (6.30) and (6.31), we have

$$I(u, v) \leq 2^{n-1} \|\mathrm{d}(u-v)\|_\phi^2 \leq C(\|\mathrm{d}(u-w)\|_\phi^2 + \|d(v-w)\|_\phi^2)$$
$$\leq CI(u, w)^{1/2^{n-1}} \left(I(u, \phi)^{1-1/2^{n-1}} + I(w, \phi)^{1-1/2^{n-1}} \right)$$
$$+ CI(v, w)^{1/2^{n-1}} \left(I(v, \phi)^{1-1/2^{n-1}} + I(w, \phi)^{1-1/2^{n-1}} \right).$$

By Proposition 6.8, we have

$$I(u, \phi) \leq nI(u, v), I(v, \phi) \leq nI(v, u), I(w, \phi) \leq n(I(w, u) + I(w, v)).$$

It follows that

$$I(u, v) \leq C \left(I(u, w)^{\frac{1}{2^{n-1}}} + I(v, w)^{\frac{1}{2^{n-1}}} \right) \Big(I(u, v)^{1-1/2^{n-1}}$$
$$+ I(u, w)^{1-1/2^{n-1}} + I(v, w)^{1-1/2^{n-1}} \Big).$$

We assume $I(u, v) \geq \max\{I(u, w), I(v, w)\}$ (otherwise we are done). Hence it follows

$$I(u, v)^{1/2^{n-1}} \leq C \left(I(u, w)^{\frac{1}{2^{n-1}}} + I(v, w)^{\frac{1}{2^{n-1}}} \right).$$

This is sufficient to prove that

$$I(u, v) \leq C(I(u, w) + I(v, w)).$$

Now we establish (6.31) (see [5, Lemma 1.9]). First observe that

$$\|\mathrm{d}(u-v)\|_\psi \leq \|\mathrm{d}(u-\psi)\|_\psi + \|d(v-\psi)\|_\psi \leq I(u, \psi)^{1/2} + I(v, \psi)^{1/2}.$$

Hence we have

$$\|\mathrm{d}(u-v)\|_\psi^2 \leq 2(I(u, \psi) + I(v, \psi)).$$

Hence if $I(u, v) \geq I(u, \psi) + I(v, \psi)$, clearly we have

$$\|\mathrm{d}(u-v)\|_\psi^2 \leq 2(I(u, \psi) + I(v, \psi))$$
$$\leq CI(u, v)^{1/2^{n-1}} \left(I(u, \psi)^{1-\frac{1}{2^{n-1}}} + I(v, \psi)^{1-\frac{1}{2^{n-1}}} \right). \quad (6.32)$$

Now we suppose $I(u, v) \leq I(u, \psi) + I(v, \psi)$. Taking $\phi = \frac{u+v}{2}$, we consider

$$b_p := \int_M \mathrm{d}(u-v) \wedge \mathrm{d}_B^c(u-v) \wedge \omega_\psi^p \wedge \omega_\phi^{n-p-1} \wedge \eta.$$

By (6.30), $b_0 \leq I(u, v)$ and $b_{n-1} = \|\mathrm{d}(u-v)\|^2_{\psi}$. We claim that, $p = 0, \cdot, n-2$,

$$b_{p+1} \leq b_p + 4\sqrt{b_p I(\psi, \phi)}. \tag{6.33}$$

We compute

$$\begin{aligned} b_{p+1} - b_p &= \int_M \mathrm{d}(u-v) \wedge \mathrm{d}^c_B(u-v) \wedge \mathrm{dd}^c_B(\psi - \phi)\omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta \\ &= -\int_M \mathrm{d}(u-v) \wedge \mathrm{dd}^c_B(u-v) \wedge \mathrm{d}^c_B(\psi - \phi)\omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta \\ &= -\int_M \mathrm{d}(u-v) \wedge (\omega_u - \omega_v) \wedge \mathrm{d}^c_B(\psi - \phi)\omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta. \end{aligned}$$

Using Cauchy–Schwarz inequality, we compute

$$\begin{aligned} &\left| \int_M \mathrm{d}(u-v) \wedge \omega_u \wedge d(\psi - \phi)\omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta \right| \\ &\leq \left(\int_M \mathrm{d}(u-v) \wedge \mathrm{d}^c_B(u-v) \wedge \omega_u \wedge \omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta \right)^{1/2} \\ &\quad \times \left(\int_M d(\psi - \phi) \wedge \mathrm{d}^c_B(\psi - \phi) \wedge \omega_u \wedge \omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta \right)^{1/2} \leq 2\sqrt{b_p I(\psi, \phi)}, \end{aligned}$$

where we have used that $\omega_u \leq 2\omega_{\phi}$ and (6.24). We can get the same estimate for

$$\left| \int_M \mathrm{d}(u-v) \wedge \omega_v \wedge d(\psi - \phi)\omega^p_{\psi} \wedge \omega^{n-p-2}_{\phi} \wedge \eta \right|.$$

This establishes (6.33). By Proposition 6.8, we know that

$$I(\psi, \phi) \leq (n+1)J(\psi, \phi) \leq \frac{n}{2}(I(\psi, u) + I(\psi, v)).$$

Denote $a = (I(\psi, u) + I(\psi, v))$. We write (6.33) as

$$b_{p+1} \leq b_p + 4\sqrt{b_p a},\ p = 0, \ldots, n-2.$$

Note that $b_0 = I(u, v) \leq a$, hence it is evident that $b_p \leq Ca$. Hence it follows that, for $p = 0, \ldots, n-2$,

$$b_{p+1} \leq C\sqrt{b_p a}.$$

A direct computation gives that,

$$b_{n-1} \leq C b_0^{1/2^{n-1}} a^{1-\frac{1}{2^{n-1}}}.$$

This completes the proof. ◻

More generally, we have the following [30, Proposition 3.48]

Proposition 6.9 *Suppose* $C > 0$ *and* $\phi, \psi, u, v \in \mathcal{E}_1(M, \xi, \omega^T)$ *satisfies*

$$-C \leq \mathbb{I}(\phi), \mathbb{I}(\psi), \mathbb{I}(u), \mathbb{I}(v), \sup_M \phi, \sup_M \psi, \sup_M u, \sup_M v \leq C.$$

Then there exists a continuous function $f_C : \mathbb{R}^+ \to \mathbb{R}^+$ *depending only on* C *with* $f_C(0) = 0$ *such that*

$$\left|\int_M \phi(\omega_u^n - \omega_v^n) \wedge \eta\right| \leq f_C(I(u, v))$$
$$\left|\int_M (u - v)(\omega_\phi^n - \omega_\psi^n) \wedge \eta\right| \leq f_C(I(u, v)). \tag{6.34}$$

Proof The proof is similar in philosophy as Lemma 6.7 and follows almost identically as in Kähler setting, see [30, Proposition 3.48]. Hence we skip the details. ◻

As a consequence, we have the following [30, Theorem 3.46]:

Theorem 6.5 *Suppose* $u_k, u \in \mathcal{E}_1(M, \xi, \omega^T)$. *Then the following holds:*

(1) $d_1(u_k, u) \to 0$ *if and only if* $\int_M |u_k - u|\omega_T^n \wedge \eta \to 0$ *and* $\mathbb{I}(u_k) \to \mathbb{I}(u)$.
(2) *If* $d_1(u_k, u) \to 0$, *then* $\omega_{u_k}^n \wedge \eta \to \omega_u^n \wedge \eta$ *weakly and* $\int_M |u_k - u|\omega_v^n \wedge \eta \to 0$ *for* $v \in \mathcal{E}_1(M, \xi, \omega^T)$.

Proof If $d_1(u_k, u) \to 0$, then Propositions 6.6 and 6.9 imply (1) and (2). For the reverse direction in (1), it follows almost identically as in Kähler setting, see [30, Proposition 3.52], using Proposition 6.9 and approximation argument. We sketch the process. First we have

$$\int_M u_k \omega_u^n \wedge \eta \to \int_M u\omega_u^n \wedge \eta.$$

And then one argues that

$$I(u, u_k) \leq (n+1)\left(\mathbb{I}(u_k) - \mathbb{I}(u) - \int_M (u - u_k)\omega_u^n \wedge \eta\right)$$

Hence this shows that $I(u, u_k) \to 0$. Using Proposition 6.9 and Lemma 6.6, one can then show

$$\int_M |u_k - u|\omega_u^n \wedge \eta, \int_M |u_k - u|\omega_{u_k}^n \wedge \eta \to 0, k \to \infty.$$

This gives the desired convergence $d_1(u_k, u) \to 0$. ◻

As an application of results established above, we have the following compactness result in Sasaki setting, following [30, Theorem 4.45].

Theorem 6.6 *Let $u_j \in \mathcal{E}_1(M, \xi, \omega^T)$ be a d_1-bounded sequence for which the entropy*

$$\sup_j H(u_j) < \infty.$$

Then $\{u_j\}$ contains a d_1-convergence sequence.

Proof We sketch the proof for completeness; for details see [30, Theorem 4.45]. First d_1 bounded implies that $\mathbb{I}$ and $\sup u$ are both bounded. Together with Proposition 3.4, this implies that d_1 bounded set is precompact in L^1. That is, there exists $u \in \mathcal{E}_1(M, \xi, \omega^T)$ such that after passing by subsequence,

$$\int_M |u_k - u|(\omega^T)^n \wedge \eta \to 0.$$

Moreover, we have (see [30, Proposition 4.14, Corollary 4.15])

$$\limsup \mathbb{I}(u_k) \le \mathbb{I}(u).$$

Since all elements in $\mathcal{E}_1(M, \xi, \omega^T)$ have zero Lelong number, we apply Zeriahi's uniform version of the famous Skoda integrability theorem [59] (we apply Zeriahi's theorem in each foliation chart) to obtain: for any $p \ge 1$, there exists $C = C(p)$ such that

$$\int_M e^{-pu_j}(\omega^T)^n \wedge \eta \le C.$$

Since $\sup u_j \le C$, we have

$$\int_M e^{p|u_j|}(\omega^T)^n \wedge \eta \le C.$$

Now we need to use the assumption that $H(u_j)$ is uniformly bounded above. We proceed as in the proof of [30, Theorem 4.45] to conclude

$$\int_M |u_j - u|\omega_{u_j}^n \wedge \eta \to 0.$$

By Proposition 6.26 (which holds for $\mathcal{E}_1$), we can then conclude that $\liminf \mathbb{I}(u_j) \ge \mathbb{I}(u)$. This gives $\lim \mathbb{I}(u_j) = \mathbb{I}(u)$. Hence $d_1(u_j, u) \to 0$, as a consequence of Theorem 6.5. □

Finally we have the extension of $\mathcal{K}$-energy, see [7, Theorem 1.2] for Kähler setting.

Theorem 6.7 *The $\mathcal{K}$-energy can be extended to a functional $\mathcal{K} : \mathcal{E}_1 \to \mathbb{R} \cup \{+\infty\}$. Such a $\mathcal{K}$-energy in $\mathcal{E}^1$ is the greatest d_1-lsc extension of $\mathcal{K}$-energy on $\mathcal{H}$. Moreover, $\mathcal{K}$-energy is convex along the finite-energy geodesics of $\mathcal{E}^1$.*

Proof As in Kähler setting [19], we can write the $\mathcal{K}$-energy as the following:

$$\mathcal{K}(\phi) = H(\phi) + \mathbb{J}_{\omega^T, -Ric}(\phi),$$

where $H(\phi)$ is the entropy part and $\mathbb{J}$ is the entropy part, taking the formula, respectively,

$$H(\phi) = \int_M \log \frac{\omega_\phi^n \wedge \eta}{\omega_T^n \wedge \eta} dv_\phi$$

$$\mathbb{J}_{-Ric}(\phi) = \frac{n\underline{R}}{(n+1)!} \int_M \phi \sum_{k=0}^{n} \omega_T^k \wedge \omega_\phi^{n-k} \wedge \eta - \frac{1}{n!} \int_M \phi \sum_{k=0}^{n-1} Ric \wedge \omega_T^k \wedge \omega_\phi^{n-1-k} \wedge \eta.$$

As a direct consequence of this formula, $\mathcal{K}(\phi)$ is well defined for $\phi \in \mathcal{H}_\Delta$. More importantly, for $\phi_0, \phi_1 \in \mathcal{H}$, and $\phi_t \in \mathcal{H}_\Delta$ being the geodesic connecting ϕ_0, ϕ_1, $\mathcal{K}(\phi_t)$ is convex with respect to $t \in [0, 1]$.

Now we extend $H(\phi)$ and $\mathbb{J}_{-Ric}$ to $\mathcal{E}_1$ separately. As in [7], the extension of $\mathbb{J}_{-Ric}$ to $\mathcal{E}_1$ is d_1-continuous, while since $d_1(u_k, u) \to 0$ implies that $\omega_{u_k}^n \wedge \eta \to \omega_u^n \wedge \eta$ weakly (Theorem 6.5), this implies that the extension of $\phi \to H(\phi)$ to $\mathcal{E}_1$ is d_1 lsc. Moreover, by [49, Lemma 5.4], the extension of $\mathcal{K}$ is the greatest lsc extension. In the end, the convexity of the extended $\mathcal{K}$-energy along the finite-energy geodesic segments follows exactly as in [7, Theorem 4.7]. □

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