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Yamabe Flow on Non-compact Manifolds with Unbounded Initial Curvature

Mario B. Schulz[1]

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Abstract

We prove global existence of Yamabe flows on non-compact manifolds M of dimension $m \geq 3$ under the assumption that the initial metric $g_0 = u_0 g_M$ is conformally equivalent to a complete background metric g_M of bounded, non-positive scalar curvature and positive Yamabe invariant with conformal factor u_0 bounded from above and below. We do not require initial curvature bounds. In particular, the scalar curvature of (M, g_0) can be unbounded from above and below without growth condition.

Keywords Yamabe flow · Non-compact · Unbounded scalar curvature · Global existence

Mathematics Subject Classification 53C44 · 35K55 · 35A01

Richard Hamilton's [12] Yamabe flow describes a family of Riemannian metrics $g(t)$ subject to the evolution equation $\frac{\partial}{\partial t} g = -\mathrm{R}_g\, g$, where R_g denotes the scalar curvature corresponding to the metric g. This equation tends to conformally deform a given initial metric towards a metric of vanishing scalar curvature. Hamilton proved existence of Yamabe flows on compact manifolds without boundary. Their asymptotic behaviour was subsequently analysed by Chow [7], Ye [21], Schwetlick and Struwe [17] and Brendle [4,5]. The theory of Yamabe flows on non-compact manifolds is not as developed as in the compact case. Daskalopoulos and Sesum [8] analysed the profiles of self-similar solutions (Yamabe solitons). The question of existence in general was addressed by Ma and An who obtained the following results on complete, non-compact Riemannian manifolds (M, g_0) satisfying certain curvature assumptions:

- If (M, g_0) has Ricci curvature bounded from below and uniformly bounded, non-positive scalar curvature, then there exists a global Yamabe flow on M with g_0 as initial metric [15].

✉ Mario B. Schulz
mario.schulz@math.ethz.ch

1 ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

- If (M, g_0) is locally conformally flat with Ricci curvature bounded from below and uniformly bounded scalar curvature, then there exists a short-time solution to the Yamabe flow on M with g_0 as initial metric [15].
- If (M, g_0) has non-negative scalar curvature R_{g_0} (possibly unbounded from above) and if the equation $-\Delta_{g_0} w = \mathrm{R}_{g_0}$ has a non-negative solution w in M, then there exists a global Yamabe flow on M with g_0 as initial metric [14].

More recently, Bahuaud and Vertman [1,2] constructed Yamabe flows starting from spaces with incomplete edge singularities such that the singular structure is preserved along the flow. Choi, Daskalopoulos and King [6] were able to find solutions to the Yamabe flow on $\mathbb{R}^m$ which develop a type II singularity in finite time.

In dimension $m = 2$, where the Yamabe flow coincides with the Ricci flow, Giesen and Topping [10,18,19] introduced the notion of instantaneous completeness and obtained existence and uniqueness of instantaneously complete Ricci flows on arbitrary surfaces. In particular, they do not require any assumptions on the curvature of the initial surface in order to prove existence of solutions. It is natural to ask whether Giesen and Topping's results generalise to non-compact manifolds of higher dimension.

In [16], the author obtained existence of instantaneously complete Yamabe flows on hyperbolic space of arbitrary dimension $m \geq 3$ provided the initial metric is conformally hyperbolic with conformal factor and scalar curvature bounded from above. The goal of this paper is to construct complete Yamabe flows on non-compact manifolds M of dimension $m \geq 3$ starting from some complete initial metric g_0 but *without* any curvature assumption on (M, g_0). In particular, the initial scalar curvature $\mathrm{R}_{g_0} : M \to \mathbb{R}$ is allowed to be unbounded from above and below. Instead we assume g_0 to be conformally equivalent to some "well-behaved" background metric g_M on M and only require pointwise bounds on the conformal factor u_0 characterising $g_0 = u_0 g_M$. More precisely, we prove the following statement.

Theorem 1 *Let (M, g_M) be a complete, non-compact Riemannian manifold of dimension $m \geq 3$ with positive Yamabe invariant and non-positive, bounded scalar curvature $-\kappa_2 \leq \mathrm{R}_{g_M} \leq -\kappa_1 \leq 0$. Let $g_0 = u_0 g_M$ be any conformal metric on M with conformal factor $u_0 \in C^{2,\alpha}(M)$ for some $0 < \alpha < 1$ allowing constants c_1, c_2 such that*

$$0 < c_1 \leq u_0 \leq c_2 < \infty.$$

Then, there exists a global Yamabe flow $(g(t))_{t\in[0,\infty[}$ on M satisfying

(1) *$g(0) = g_0$ in the sense that $g(t) \to g_0$ locally in C^2 as $t \searrow 0$,*
(2) *$(\kappa_1 t + c_1)\, g_M \leq g(t) \leq (\kappa_2 t + c_2)\, g_M$ for every $t > 0$,*
(3) *$\mathrm{R}_{g(t)} \geq -\frac{1}{t}$ for every $t > 0$.*

Remark Typical examples of background manifolds (M, g_M) satisfying the hypothesis of Theorem 1 are Euclidean space $(\mathbb{R}^m, g_{\mathbb{R}^m})$ with $\mathrm{R}_{g_{\mathbb{R}^m}} \equiv 0$ and hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ with $\mathrm{R}_{g_{\mathbb{H}}} \equiv -m(m-1)$. We verify positivity of their Yamabe invariant in Lemma 2.2.

1 Local Existence

Let $g_0 = u_0 g_M$ be any conformal metric on (M, g_M). Since the Yamabe flow preserves the conformal class of the initial metric, any Yamabe flow $(g(t))_{t\in[0,T]}$ on M with $g(0) = g_0$ is given by $g(t) = u(\cdot, t)\, g_M$ with some function $u\colon M \times [0, T] \to\,]0, \infty[$. Let

$$\eta := \frac{m-2}{4}$$

where $m = \dim M \geq 3$ and $U = u^\eta$. Then, the metric $g = u g_M$ has scalar curvature

$$\mathrm{R}_g = U^{-\frac{m+2}{m-2}}\big(\mathrm{R}_{g_M} U - 4\tfrac{m-1}{m-2}\Delta_{g_M} U\big).$$

Hence, the conformal factor u is subject to the evolution equation

$$\begin{aligned}\frac{\partial u}{\partial t} = -u\mathrm{R}_g &= -u^{1-\frac{m+2}{4}}\big(\mathrm{R}_{g_M} u^\eta - 4\tfrac{m-1}{m-2}\Delta_{g_M} u^\eta\big)\\ &= -\mathrm{R}_{g_M} + \tfrac{m-1}{\eta} u^{-\eta}\Delta_{g_M} u^\eta\\ &= -\mathrm{R}_{g_M} + (m-1)\big(u^{-1}\Delta_{g_M} u + (\eta-1)u^{-2}|\nabla u|^2_{g_M}\big)\end{aligned}$$

which can also be expressed in the form

$$\frac{1}{\eta+1}\frac{\partial U^{1+\frac{1}{\eta}}}{\partial t} = -\mathrm{R}_{g_M} U + \frac{m-1}{\eta}\Delta_{g_M} U. \tag{1}$$

Let $B_r \subset M$ denote the open metric ball of radius r around some origin $p_0 \in M$ and let $M_1 \subset M_2 \subset \ldots \subset M$ be an exhaustion of M with smooth, open, connected, bounded domains such that $B_k \subset M_k \subset B_{k+1}$ for every $k \in \mathbb{N}$. Fix any radius $4 < k \in \mathbb{N}$. Let $\varphi\colon M \to [0, 1]$ be smooth with compact support in B_k such that $\varphi|_{B_{k-1}} \equiv 1$. Under the assumption $u_0 \in C^{2,\alpha}(\overline{M_k})$ for some $0 < \alpha < 1$ and $c_1 := \inf_M u_0 > 0$ we consider

$$\mathring{u}_0 := (1-\varphi)c_1 + \varphi\, u_0 \tag{2}$$

as initial data for the Yamabe flow equation on M_k. In particular, $0 < \mathring{u}_0 \in C^{2,\alpha}(M_k)$ coincides with u_0 in B_{k-1} and takes the constant value c_1 in some neighbourhood of ∂M_k as illustrated in Fig. 1. As parabolic boundary data, we choose

$$\phi(x, t) := c_1 - t\, \mathrm{R}_{g_M}(x). \tag{3}$$

Then, since $\mathring{u}_0$ and ϕ satisfy the first-order compatibility conditions, there exists some $T > 0$ (which a priori depends on k) and a solution $0 < u \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{M_k} \times [0, T])$ of

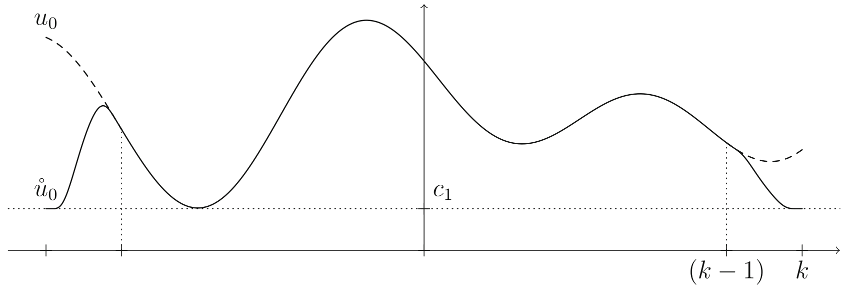

Fig. 1 Initial data $\mathring{u}_0$ for problem (4)

$$\begin{cases} \frac{1}{m-1}\frac{\partial u}{\partial t} = \frac{-\mathrm{R}_{g_M}}{m-1} + \frac{\Delta_{g_M} u}{u} + \frac{(m-6)}{4}\frac{|\nabla u|^2_{g_M}}{u^2} & \text{in } M_k \times [0, T], \\ u = \phi & \text{on } \partial M_k \times [0, T], \\ u = \mathring{u}_0 & \text{on } M_k \times \{0\}. \end{cases} \tag{4}$$

In [16, Lemma 1.1], the author gives a detailed proof of this short-time existence result for bounded domains in hyperbolic space using the inverse function theorem on Banach spaces. This approach generalises to smooth, bounded domains M_k in any Riemannian manifold.

Lemma 1.1 (Upper and lower bound) *Let* $0 < u \in C^{2;1}(M_k \times [0, T])$ *be a solution to problem* (4) *with boundary data* (3) *and initial data* (2). *If the background scalar curvature satisfies* $0 \leq \kappa_1 \leq -\mathrm{R}_{g_M} \leq \kappa_2$ *in* M_k, *then*

$$c_1 + \kappa_1 t \leq u(\cdot, t) \leq c_2 + \kappa_2 t$$

for every $0 \leq t \leq T$, *where we recall* $c_1 = \inf u_0$ *and* $c_2 = \sup u_0$.

Proof Equation (4) implies that the function $w(\cdot, t) = u(\cdot, t) - c_1 - \kappa_1 t$ satisfies

$$\frac{1}{m-1}\frac{\partial w}{\partial t} - \frac{\Delta_{g_M} w}{u} - \frac{(m-6)}{4u^2}\langle \nabla u, \nabla w\rangle_{g_M} = \frac{-\mathrm{R}_{g_M} - \kappa_1}{m-1} \geq 0. \tag{5}$$

Since $u > 0$, equation (5) is uniformly parabolic. Moreover, we have $w \geq 0$ on $(\partial M_k \times [0, T]) \cup (M_k \times \{0\})$. Hence, the linear parabolic maximum principle (see [16, Prop. A.2]) implies $w \geq 0$ in $M_k \times [0, T]$. The proof of the upper bound is analogous. □

Lemma 1.2 (Global existence on bounded domains) *If the background scalar curvature satisfies* $0 \leq \kappa_1 \leq -\mathrm{R}_{g_M} \leq \kappa_2$ *in* M_k, *then there exists a unique solution* $0 < u \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{M_k} \times [0, \infty[)$ *to problem* (4) *with boundary data* (3) *and initial data* (2).

Proof We invoke the same argument as in [16]. First we show that two positive solutions $u, v \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{M_k} \times [0, T])$ of problem (4) with equal initial and boundary data must agree. We have

$$\begin{aligned}
\frac{1}{m-1}\frac{\partial}{\partial t}(u-v) &= \frac{\Delta_{g_M} u}{u} - \frac{\Delta_{g_M} v}{v} + \frac{m-6}{4}\Big(\frac{|\nabla u|^2_{g_M}}{u^2} - \frac{|\nabla v|^2_{g_M}}{v^2}\Big) \\
&= \frac{\Delta_{g_M}(u-v)}{u} + \frac{m-6}{4u^2}\langle \nabla(u+v), \nabla(u-v)\rangle_{g_M} \\
&\quad - \frac{\Delta_{g_M} v}{uv}(u-v) - \frac{m-6}{4}\frac{|\nabla v|^2_{g_M}}{u^2 v^2}(u+v)(u-v)
\end{aligned}$$

which can be considered as linear parabolic equation for $u - v$ with bounded coefficients because $u, v \in C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{M_k} \times [0, T])$ are uniformly bounded away from zero and from above by Lemma 1.1 and because $|\nabla u|$, $|\nabla v|$, Δv are bounded functions in $\overline{M_k}$. Since $(u-v)$ vanishes along $(M_k \times \{0\}) \cup (\partial M_k \times [0, T])$, the parabolic maximum principle (see [16, Prop A.2]) implies $u - v = 0$ in $\overline{M_k} \times [0, T]$ as claimed.

It remains to show that the solution can be extended globally in time. Let $T_* > 0$ be the supremum over all $T > 0$ such that problem (4) has a solution defined in $M_k \times [0, T]$. As shown above, two such solutions agree on their common domain. Therefore, there exists a solution u defined on $M_k \times [0, T_*[$. Suppose $T_* < \infty$. The function $U = u^\eta$ for $\eta = \frac{m-2}{4}$ satisfies equation (1) which can be written in divergence form

$$\frac{1}{m-1}\frac{\partial u^{\eta+1}}{\partial t} = -\frac{(\eta+1)\mathrm{R}_{g_M}}{(m-1)u} u^{\eta+1} + \mathrm{div}_{g_M}\Big(\frac{1}{u}\nabla u^{\eta+1}\Big). \tag{6}$$

Since $|\mathrm{R}_{g_M}| \le \kappa_2$ and since $0 < c_1 \le u \le c_2 + \kappa_2 T_*$ by Lemma 1.1, equation (6) can be interpreted as linear parabolic equation with uniformly bounded coefficients. Hence, parabolic DeGiorgi–Nash–Moser theory [20, §4] applies and yields

$$\|u^{\eta+1}\|_{C^{0,\alpha;0,\frac{\alpha}{2}}(\overline{M_k} \times [0,T_*])} \le C(m, c_1, c_2, \kappa_2, T_*)$$

for some Hölder exponent $0 < \alpha < 1$ and some constant C depending only on the indicated constants. Together with Lemma 1.1 we conclude, that $\frac{1}{u}$ is Hölder continuous and apply linear parabolic theory [13, § IV.5, Theorem 5.2] to obtain

$$\|u^{\eta+1}\|_{C^{2,\alpha;1,\frac{\alpha}{2}}(\overline{M_k} \times [0,T_*])} \le C'(m, c_1, c_2, \kappa_2, T_*).$$

Hence, $u(\cdot, T_*) \in C^{2,\alpha}(\overline{M_k})$ are suitable initial data for problem (4). The boundary data (3) are defined also for $t \ge T_*$ and are compatible with $u(\cdot, T_*)$ at time T_*. Therefore, we can extend the solution in contradiction to the maximality of T_*. □

2 Scalar Curvature Estimates

We assume the background scalar curvature to satisfy $0 \leq \kappa_1 \leq -\mathrm{R}_{g_M} \leq \kappa_2$ in M. From section 1 we recall the exhaustion of M with smooth, connected, bounded domains $M_1 \subset M_2 \subset \ldots \subset M$ such that $B_k \subset M_k \subset B_{k+1}$ for every $k \in \mathbb{N}$. By Lemma 1.2 there exists a unique solution $u_k\colon M_k \times [0,\infty[\to]0,\infty[$ of problem (4) for every $k \in \mathbb{N}$. The goal of this section is to estimate the corresponding scalar curvature R_{ug_M} independently of the index k.

Lemma 2.1 *Let $u \in C^{2,\alpha;1,\frac{\alpha}{2}}(M_k \times [0,T])$ be a solution to problem (4). For every $t \in [0,T]$, the scalar curvature $\mathrm{R}_{g(t)}$ of the Riemannian metric $g(t) = u(\cdot,t)g_M$ satisfies*

$$\mathrm{R}_{g(t)} \geq -\frac{1}{t} \quad \text{in } M_k.$$

Proof Let $w(t) = \frac{1}{\varepsilon+t}$, where $0 < \varepsilon < \frac{c_1}{\kappa_2}$ is chosen such that $\mathrm{R}_{g(0)} > -w(0)$ in M_k. In $M_k \times [0,T]$ we can express the scalar curvature in the form $\mathrm{R}_g = -\frac{1}{u}\frac{\partial u}{\partial t}$. In particular, the choice of boundary data (3) implies

$$\mathrm{R}_g|_{\partial M_k \times [0,T]} = -\frac{1}{\phi}\frac{\partial \phi}{\partial t} = -\frac{-\mathrm{R}_{g_M}}{c_1 - t\,\mathrm{R}_{g_M}} \geq -\frac{1}{\frac{c_1}{\kappa_2} + t} \geq -w(t) \quad \text{on } \partial M_k \times [0,T].$$

Scalar curvature evolves by (see [7, Lemma 2.2])

$$\tfrac{\partial}{\partial t}\mathrm{R}_g = (m-1)\Delta_g \mathrm{R}_g + \mathrm{R}_g^2 \quad \text{in } M_k \times [0,T]. \tag{7}$$

Combined with $\frac{dw}{dt} = -w^2$, we obtain

$$\tfrac{\partial}{\partial t}(\mathrm{R}_g + w) - (m-1)\Delta_g(\mathrm{R}_g + w) - (\mathrm{R}_g - w)(\mathrm{R}_g + w) = 0.$$

Since $(\mathrm{R}_g - w) \leq \mathrm{R}_g$ is bounded from above in $M_k \times [0,T]$ and since the operator $\Delta_g = \frac{1}{u}\Delta_{g_M} + \frac{m-2}{2}u^{-2}\langle \nabla u, \nabla \cdot \rangle_{g_M}$ is uniformly elliptic, the inequality $\mathrm{R}_g \geq -w$ in $M_k \times [0,T]$ follows from the parabolic maximum principle (see [16, Prop. A.2]). □

Integral estimates for the positive part of the scalar curvature can be obtained with the help of Sobolev-type inequalities. This technique requires the Yamabe invariant of (M, g_M) to be positive.

Lemma 2.2 (Yamabe invariant) *Let (M,g) be a Riemannian manifold with scalar curvature R_g. Then the Yamabe invariant Y of (M,g) is defined by*

$$\mathrm{Y}(M,g) := \inf\left\{ \frac{\int_M |\nabla^g f|_g^2 \, d\mu_g + \frac{m-2}{4(m-1)}\int_M \mathrm{R}_g f^2 \, d\mu_g}{\left(\int_M |f|^{\frac{2m}{m-2}} \, d\mu_g\right)^{\frac{m-2}{m}}} \; ; \; f \in C_c^\infty(M) \right\}.$$

The Yamabe invariants of hyperbolic space $(\mathbb{H}^m, g_{\mathbb{H}^m})$, *Euclidean space* $(\mathbb{R}^m, g_{\mathbb{R}^m})$ *and the round sphere* $(\mathbb{S}^m, g_{\mathbb{S}^m})$ *coincide. Their value is*

$$\mathrm{Y}(\mathbb{H}^m, g_{\mathbb{H}^m}) = \mathrm{Y}(\mathbb{R}^m, g_{\mathbb{R}^m}) = \mathrm{Y}(\mathbb{S}^m, g_{\mathbb{S}^m}) = \frac{m(m-2)}{4}|\mathbb{S}^m|^{\frac{2}{m}} > 0. \tag{8}$$

Proof The Yamabe invariant is a conformal invariant. Hyperbolic space is conformally equivalent to the Euclidean ball B_r of any given radius $r > 0$. Via stereographic projection, the sphere minus a point is conformally equivalent to $(\mathbb{R}^m, g_{\mathbb{R}^m})$. The H^1-capacity of a point vanishes. Hence, a minimising sequence for $\mathrm{Y}(\mathbb{H}^m, g_{\mathbb{H}^m})$ yields competitors for $\mathrm{Y}(\mathbb{S}^m, g_{\mathbb{S}^m})$ and vice versa. Since $\mathrm{R}_{g_{\mathbb{S}^m}} = m(m-1)$, the claim follows if we use that $\mathrm{Y}(\mathbb{S}^m, g_{\mathbb{S}^m})$ is attained for constant functions f (see [3,9]). □

If the Yamabe invariant is positive, then its definition leads to a Sobolev-type inequality. Let $g = ug_M$ be any conformal metric on (M, g_M). Let $B \subset M$ be open and $f \in C^1_c(B)$. Computing

$$|\nabla^g f|_g^2 = \tfrac{1}{u}|\nabla^{g_M} f|_{g_M}^2 \tag{9}$$

and denoting $\mathrm{Y} := \mathrm{Y}(M, g_M)$, we obtain

$$\begin{aligned}\int_B |\nabla^g f|_g^2 \, d\mu_g &= \int_B |\nabla^{g_M} f|_{g_M}^2 u^{\frac{m}{2}-1} \, d\mu_{g_M} \\ &\geq \inf_B u^{\frac{m}{2}-1} \int_M |\nabla^{g_M} f|_{g_M}^2 \, d\mu_{g_M} \\ &\geq \inf_B u^{\frac{m}{2}-1} \Bigg(\mathrm{Y} \left(\int_M |f|^{\frac{2m}{m-2}} \, d\mu_{g_M} \right)^{\frac{m-2}{m}} \\ &\quad - \frac{(m-2)}{4(m-1)} \int_M \mathrm{R}_{g_M} f^2 \, d\mu_{g_M} \Bigg).\end{aligned} \tag{10}$$

As in Lemma 1.1 we will assume henceforth that the background manifold (M, g_M) has scalar curvature R_{g_M} satisfying

$$0 \leq \kappa_1 \leq -\mathrm{R}_{g_M} \leq \kappa_2 < \infty. \tag{11}$$

Lemma 1.1 is the main reason for assumption (11) but it also allows us to drop the second term in (10) because of its sign to obtain the Sobolev inequality

$$\int_B |\nabla^g f|_g^2 \, d\mu_g \geq \left(\frac{\inf_B u}{\sup_B u} \right)^{\frac{m-2}{2}} \mathrm{Y} \left(\int_B |f|^{\frac{2m}{m-2}} \, d\mu_g \right)^{\frac{m-2}{m}}. \tag{12}$$

Lemma 2.3 *Given* $k > 4$, *let* u *be the solution to problem* (4) *in* $M_k \times [0, \infty[$ *and let* $\mathrm{R}_{g(t)}$ *be the scalar curvature of the Riemannian metric* $g(t) = u(\cdot, t)g_M$ *in* M_k. *Let* $0 < r_0 < k - 4$ *and* $1 < T < \infty$ *be fixed and let* $p > 1$ *be any exponent. Then, for every* $t \in [0, T]$, *the positive part* $\mathrm{R}_+(x, t) = \max\{0, \mathrm{R}_{g(t)}(x)\}$ *satisfies*

$$\int_{B_{r_0}} \mathrm{R}_+^p(\cdot,t)\,d\mu_{g(t)} \le C$$

where the constant C depends on r_0, T and p but not on k.

Proof For any exponent $p > 1$ and any $\psi \in C_c^\infty(M_k)$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_k} \psi \mathrm{R}_+^p\,d\mu_g \\
&\quad = p\int_{M_k} \psi \mathrm{R}_+^{p-1}\big(\mathrm{R}^2 + (m-1)\Delta_g \mathrm{R}\big)\,d\mu_g - \frac{m}{2}\int_{M_k} \psi \mathrm{R}_+^p \mathrm{R}\,d\mu_g \\
&\quad = (p - \tfrac{m}{2})\int_{M_k} \psi \mathrm{R}_+^{p+1}\,d\mu_g + p(m-1)\int_{M_k} \psi \mathrm{R}_+^{p-1}\Delta_g \mathrm{R}\,d\mu_g \\
&\quad = (p - \tfrac{m}{2})\int_{M_k} \psi \mathrm{R}_+^{p+1}\,d\mu_g \qquad (13)\\
&\qquad + (m-1)\Big(-\int_{M_k}\langle\nabla\psi, \nabla \mathrm{R}_+^p\rangle_g\,d\mu_g - \frac{4(p-1)}{p}\int_{M_k}\psi|\nabla \mathrm{R}_+^{\frac{p}{2}}|_g^2\,d\mu_g\Big). \qquad (14)
\end{aligned}$$

The strategy of the proof is as follows.

Step 1. If $1 < p < \frac{m}{2}$ then the negative sign of (13) leads to a bound uniformly in k.
Step 2. We use the estimate obtained in the first step to deal with the case $p = \frac{m}{2}$.
Step 3. The bound from the second step can be extended to $p = \beta\frac{m}{2}$ for some $\beta > 1$.
Step 4. Using the estimate for $p = \beta\frac{m}{2}$ we obtain a bound with any exponent $p > \frac{m}{2}$.

In each step we choose a different cutoff function ψ and apply different estimates to control the terms in (13) and (14). Young's inequality appears frequently in either of the following forms.

$$\forall a, b \ge 0 \qquad \forall 0 < s < 1: \qquad a^s b^{1-s} \le sa + (1-s)b, \qquad (15)$$

$$\forall b, c \ge 0 \qquad \forall x \in \mathbb{R}: \qquad bx - cx^2 \le \tfrac{b^2}{4c}. \qquad (16)$$

Step 1. Let $\varphi_1 \in C_c^\infty(B_{r_0+4})$ be a cutoff function such that $0 \le \varphi_1 \le 1$, such that $\varphi_1|_{B_{r_0+3}} \equiv 1$ and such that $|\nabla\varphi_1|_{g_M} \le 2$. Given $1 < p < \frac{m}{2}$, we set

$$\psi_1 = \varphi_1^{2(p+1)}.$$

This cutoff function has the property that

$$|\nabla\psi_1|_{g_M}^2 \le \big(4(p+1)\big)^2 \varphi_1^{4p+2} = \big(4(p+1)\big)^2 \psi_1^{1+\frac{p}{p+1}}.$$

Choosing $\psi = \psi_1$ and recalling

$$|\nabla\psi_1|_g^2 = \frac{1}{u}|\nabla\psi_1|_{g_M}^2$$

the terms in (14) can be estimated as follows using Young's inequality and Hölder's inequality:

$$\begin{aligned}
&-\int_{M_k}\langle\nabla\psi_1,\nabla \mathrm{R}_+^p\rangle_g\,d\mu_g-\frac{4(p-1)}{p}\int_{M_k}\psi_1|\nabla \mathrm{R}_+^{\frac{p}{2}}|_g^2\,d\mu_g\\
&\le\frac{p}{4(p-1)}\int_{M_k}\frac{|\nabla\psi_1|_g^2}{\psi_1}\mathrm{R}_+^p\,d\mu_g\\
&\le\frac{4p(p+1)^2}{(p-1)}\int_{B_{r_0+4}}\frac{\psi_1^{\frac{p}{p+1}}}{u}\mathrm{R}_+^p\,d\mu_g\\
&\le\frac{4p(p+1)^2}{(p-1)}\Big(\int_{B_{r_0+4}}u^{\frac{m}{2}-p-1}\,d\mu_{g_M}\Big)^{\frac{1}{p+1}}\Big(\int_{M_k}\psi_1\mathrm{R}_+^{p+1}\,d\mu_g\Big)^{\frac{p}{p+1}}\\
&\le\frac{(4p(p+1)^2)^{p+1}}{(p+1)\lambda^p(p-1)^{p+1}}\Big(\int_{B_{r_0+4}}u^{\frac{m}{2}-p-1}\,d\mu_{g_M}\Big)+\frac{\lambda p}{p+1}\Big(\int_{M_k}\psi_1\mathrm{R}_+^{p+1}\,d\mu_g\Big)
\end{aligned}\tag{17}$$

where the parameter $\lambda>0$ is arbitrary. If we choose $1<p<\frac{m}{2}$ and $\lambda=\frac{(\frac{m}{2}-p)(p+1)}{(m-1)p}$, then we obtain

$$\frac{\partial}{\partial t}\int_{M_k}\psi_1\mathrm{R}_+^p\,d\mu_g\le C_{m,p}\Big(\int_{B_{r_0+4}}u^{\frac{m}{2}-p-1}\,d\mu_{g_M}\Big)\tag{18}$$

with some constant $C_{m,p}$ depending only on m and p. Moreover, since $u(\cdot,t)\ge c_1+\kappa_1 t$ by Lemma 1.1, the right hand side of (18) is integrable in $t\in[0,T]$ if $1<p<\frac{m}{2}$. In particular, for $p=\frac{2m}{5}\in\,]1,\frac{m}{2}[$ we obtain

$$\int_{B_{r_0+3}}\mathrm{R}_+^{\frac{2m}{5}}\,d\mu_g\le\int_{B_{r_0+4}}|\mathrm{R}_{g(0)}|^{\frac{2m}{5}}\,d\mu_{g(0)}+C_{m,r_0,T}.\tag{19}$$

Step 2. For any regular, non-negative functions ψ, F and any exponent p, there holds

$$\begin{aligned}
|\nabla(\psi F^p)^{\frac{1}{2}}|^2&=|\tfrac{1}{2}\psi^{-\frac{1}{2}}F^{\frac{p}{2}}\nabla\psi+\psi^{\frac{1}{2}}\nabla F^{\frac{p}{2}}|^2\\
&=\frac{F^p}{4\psi}|\nabla\psi|^2+F^{\frac{p}{2}}\langle\nabla\psi,\nabla F^{\frac{p}{2}}\rangle+\psi|\nabla F^{\frac{p}{2}}|^2.
\end{aligned}$$

Therefore,

$$-\frac{1}{2}\langle\nabla\psi,\nabla\mathrm{R}_+^p\rangle_g=\frac{|\nabla\psi|_g^2}{4\psi}\mathrm{R}_+^p-|\nabla(\psi\mathrm{R}_+^p)^{\frac{1}{2}}|_g^2+\psi|\nabla\mathrm{R}_+^{\frac{p}{2}}|_g^2,\tag{20}$$

where ψ is a new cutoff function to be chosen. Given $p \geq \frac{m}{2} \geq \frac{3}{2}$, we use (20) to estimate the terms in (14) by

$$
\begin{aligned}
&-\int_{M_k} \langle \nabla\psi, \nabla \mathrm{R}_+^p\rangle_g \, d\mu_g - \frac{4(p-1)}{p}\int_{M_k} \psi |\nabla \mathrm{R}_+^{\frac{p}{2}}|_g^2 \, d\mu_g \\
&= -\frac{1}{2}\int_{M_k} \langle \nabla\psi, \nabla \mathrm{R}_+^p\rangle_g \, d\mu_g + \Big(1 - \frac{4(p-1)}{p}\Big)\int_{M_k} \psi |\nabla \mathrm{R}_+^{\frac{p}{2}}|_g^2 \, d\mu_g \\
&- \int_{M_k} |\nabla(\psi \mathrm{R}_+^p)^{\frac{1}{2}}|_g^2 \, d\mu_g + \frac{1}{4}\int_{M_k} \frac{|\nabla\psi|_g^2}{\psi}\mathrm{R}_+^p \, d\mu_g \\
&\leq -\int_{M_k} |\nabla(\psi \mathrm{R}_+^p)^{\frac{1}{2}}|_g^2 \, d\mu_g + \int_{M_k} \frac{|\nabla\psi|_g^2}{\psi}\mathrm{R}_+^p \, d\mu_g
\end{aligned}
$$

where we used $(1 - \frac{4(p-1)}{p}) \leq -\frac{1}{3}$ and applied Young's inequality in the form (16). Consequently, for any $p \geq \frac{m}{2}$

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_k} \psi \mathrm{R}_+^p \, d\mu_g \leq (p - \tfrac{m}{2})\int_{M_k} \psi \mathrm{R}_+^{p+1} \, d\mu_g - (m-1)\int_{M_k} |\nabla(\psi \mathrm{R}_+^p)^{\frac{1}{2}}|^2 \, d\mu_g \\
+ (m-1)\int_{M_k} \frac{|\nabla\psi|_g^2}{\psi}\mathrm{R}_+^p \, d\mu_g. \qquad (21)
\end{aligned}
$$

Let $\varphi_2 \in C_c^\infty(B_{r_0+3})$ be a cutoff function such that $0 \leq \varphi_2 \leq 1$, such that $\varphi_2|_{B_{r_0+2}} \equiv 1$ and such that $|\nabla\varphi_2|_{g_M} \leq 2$. Given $0 < \alpha < 1$, we set $\psi_2 = \varphi_2^{\frac{2}{1-\alpha}}$. Then,

$$
|\nabla\psi_2|_g^2 = \frac{1}{u}|\nabla\psi_2|_{g_M}^2 \leq \frac{1}{c_1}\Big(\frac{4}{1-\alpha}\Big)^2 \varphi_2^{\frac{4}{1-\alpha}-2} = \frac{1}{c_1}\Big(\frac{4}{1-\alpha}\Big)^2 \psi_2^{1+\alpha}.
$$

With this choice of ψ_2 and any $\lambda > 0$, Hölder's inequality and Young's inequality in the form (15) yield

$$
\begin{aligned}
&\int_{M_k} \frac{|\nabla\psi_2|_g^2}{\psi_2}\mathrm{R}_+^p \, d\mu_g \\
&\leq \frac{1}{c_1}\Big(\frac{4}{1-\alpha}\Big)^2 \int_{M_k} \psi_2^\alpha \mathrm{R}_+^p \, d\mu_g \\
&\leq \frac{1}{c_1}\Big(\frac{4}{1-\alpha}\Big)^2 \Big(\int_{M_k} (\psi_2 \mathrm{R}_+^p)^{\frac{m}{m-2}} \, d\mu_g\Big)^{\frac{m-2}{m}\alpha} \Big(\int_{B_{r_0+3}} \mathrm{R}_+^{p\frac{1-\alpha}{1-\frac{m-2}{m}\alpha}} \, d\mu_g\Big)^{1-\frac{m-2}{m}\alpha} \\
&\leq \alpha\lambda \Big(\int_{M_k} (\psi_2 \mathrm{R}_+^p)^{\frac{m}{m-2}} \, d\mu_g\Big)^{\frac{m-2}{m}} \\
&\quad + (1-\alpha)\lambda^{-\frac{\alpha}{1-\alpha}} c_1^{-\frac{1}{1-\alpha}} \Big(\frac{4}{1-\alpha}\Big)^{\frac{2}{1-\alpha}} \Big(\int_{B_{r_0+3}} \mathrm{R}_+^{p\frac{1-\alpha}{1-\frac{m-2}{m}\alpha}} \, d\mu_g\Big)^{\frac{1-\frac{m-2}{m}\alpha}{1-\alpha}}. \qquad (22)
\end{aligned}
$$

Restricting to the case $p = \frac{m}{2}$, we choose $\alpha = \frac{m}{m+8}$ such that $\frac{1-\alpha}{1-\frac{m-2}{m}\alpha} = \frac{4}{5}$. Then we choose $\lambda > 0$ such that

$$\alpha\lambda \le \left(\frac{\inf u}{\sup u}\right)^{\frac{m-2}{2}} \mathrm{Y},$$

where we again depend on the uniform upper and lower bound on u from Lemma 1.1. With these choices and the Sobolev estimate (12), we obtain

$$\frac{\partial}{\partial t}\int_{M_k} \psi_2 \mathrm{R}_+^{\frac{m}{2}}\, d\mu_g \le C\Big(\int_{B_{r_0+3}} \mathrm{R}_+^{\frac{2m}{5}}\, d\mu_g\Big)^{\frac{5}{4}}.$$

In particular, using (19) from step 1, we conclude

$$\int_{B_{r_0+2}} \mathrm{R}_+^{\frac{m}{2}}\, d\mu_g \le C_{m,r_0,T}. \tag{23}$$

Step 3. Let $\psi_3 \in C_c^\infty(B_{r_0+2})$ be a cutoff function such that $\psi_3|_{B_{r_0+1}} \equiv 1$. This step is based on the estimate

$$(p - \tfrac{m}{2})\int_{M_k} \psi_3 \mathrm{R}_+^{p+1}\, d\mu_g \le (p - \tfrac{m}{2})\Big(\int_{B_{r_0+2}} \mathrm{R}_+^{\frac{m}{2}}\, d\mu_g\Big)^{\frac{2}{m}}\Big(\int (\psi_3 \mathrm{R}_+^{p})^{\frac{m}{m-2}}\, d\mu_g\Big)^{\frac{m-2}{m}}.$$

By step 2, $\|\mathrm{R}_+\|_{L^{\frac{m}{2}}(B_{r_0+2})}$ is bounded uniformly in k. If $p = \beta\frac{m}{2}$ with $\beta > 1$ sufficiently close to 1, then

$$(p - \tfrac{m}{2})\|\mathrm{R}_+\|_{L^{\frac{m}{2}}(B_{r_0+2})} \le \frac{m-1}{2}\left(\frac{\inf u}{\sup u}\right)^{\frac{m-2}{2}} \mathrm{Y}$$

and we can conclude as in step 2.

Step 4. Let $\varphi_4 \in C_c^\infty(B_{r_0+1})$ be a cutoff function such that $0 \le \varphi_4 \le 1$, such that $\varphi_4|_{B_{r_0}} \equiv 1$ and such that $|\nabla\varphi_4|_{g_M} \le 2$. As in step 2, we set $\psi_4 = \varphi_4^{\frac{2}{1-\alpha}}$ with $0 < \alpha < 1$.

We apply Lemma 2.2 to estimate (21) and obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_k} \psi_4 \mathrm{R}_+^{p}\, d\mu_g \\
&\quad\le \Big(p - \tfrac{m}{2} + \tfrac{m-2}{4}\Big)\int_{M_k} \psi_4 \mathrm{R}_+^{p+1}\, d\mu_g - (m-1)\,\mathrm{Y}\Big(\int_{M_k} (\psi_4 \mathrm{R}_+^{p})^{\frac{m}{m-2}}\, d\mu_g\Big)^{\frac{m-2}{m}} \\
&\qquad + (m-1)\int_{M_k} \frac{|\nabla\psi_4|_g^2}{\psi_4}\mathrm{R}_+^{p}\, d\mu_g.
\end{aligned} \tag{24}$$

As in shown in (22) we have

$$\int_{M_k} \frac{|\nabla \psi_4|_g^2}{\psi_4} \mathrm{R}_+^p \, d\mu_g \leq \alpha\lambda \Big(\int_{M_k} (\psi_4 \mathrm{R}_+^p)^{\frac{m}{m-2}} \, d\mu_g \Big)^{\frac{m-2}{m}}$$
$$+ (1-\alpha)\lambda^{-\frac{\alpha}{1-\alpha}} c_1^{-\frac{1}{1-\alpha}} \Big(\frac{4}{1-\alpha} \Big)^{\frac{2}{1-\alpha}} \Big(\int_{B_{r_0+1}} \mathrm{R}_+^{p \frac{1-\alpha}{1-\frac{m-2}{m}\alpha}} \, d\mu_g \Big)^{\frac{1-\frac{m-2}{m}\alpha}{1-\alpha}} \tag{25}$$

for any $\lambda > 0$. This time we choose $0 < \alpha < 1$ depending on m and p such that

$$p \frac{1-\alpha}{1-\frac{m-2}{m}\alpha} = \frac{m}{2}.$$

Then we choose $\lambda = \frac{1}{2\alpha} \mathrm{Y} > 0$. Let $\beta > 1$ as in step 3. Hölder's inequality and Young's inequality yield

$$\int_{M_k} \psi \mathrm{R}_+^{p+1} \, d\mu_g$$
$$\leq \Big(\int_{M_k} (\psi \mathrm{R}_+^p)^{\frac{m}{m-2}} \, d\mu_g \Big)^{\frac{m-2}{m}\gamma} \Big(\int_{M_k} \psi^{\frac{1-\gamma}{1-\frac{m-2}{m}\gamma}} \mathrm{R}_+^{\beta \frac{m}{2}} \, d\mu_g \Big)^{1-\frac{m-2}{m}\gamma}$$
$$\leq \gamma\delta \Big(\int_{M_k} (\psi \mathrm{R}_+^p)^{\frac{m}{m-2}} \, d\mu_g \Big)^{\frac{m-2}{m}} + (1-\gamma)\delta^{-\frac{\gamma}{1-\gamma}} \Big(\int_{B_{r_0+1}} \mathrm{R}_+^{\beta \frac{m}{2}} \, d\mu_g \Big)^{\frac{1-\frac{m-2}{m}\gamma}{1-\gamma}} \tag{26}$$

where $\delta > 0$ is arbitrary but $0 < \gamma < 1$ must satisfy $p + 1 = p\gamma + \beta(\frac{m}{2} - \frac{m-2}{2}\gamma)$. We solve the equation for

$$\gamma = \frac{p - \beta\frac{m}{2} + 1}{p - \beta\frac{m}{2} + \beta}$$

which indeed satisfies $0 < \gamma < 1$ since $\beta > 1$, and compute

$$\frac{1 - \frac{m-2}{m}\gamma}{1-\gamma} = \frac{2(p - \frac{m}{2} + 1)}{m(\beta - 1)}.$$

Finally we chose $\delta > 0$ such that $(p - \frac{m}{2} - \frac{m-2}{4})\gamma\delta < \frac{1}{2}(m-1)\,\mathrm{Y}$ and combine (24), (25) and (26) to

$$\frac{\partial}{\partial t} \int_{M_k} \psi_4 \mathrm{R}_+^p \, d\mu_g$$
$$\leq C_{m,p,c_1} \Big(\int_{B_{r_0+1}} \mathrm{R}_+^{\frac{m}{2}} \, d\mu_g \Big)^{\frac{2p}{m}} + C_{m,p,r_0} \Big(\int_{B_{r_0+1}} \mathrm{R}_+^{\beta \frac{m}{2}} \, d\mu_g \Big)^{\frac{2(p-\frac{m}{2}+1)}{m(\beta-1)}}. \tag{27}$$

With the estimates from steps 2 and 3, the claim follows. □

Lemma 2.4 *Given $k > 4$, let u be the solution to problem (4) in $M_k \times [0, \infty[$ and let $\mathrm{R}_{g(t)}$ be the scalar curvature of the Riemannian metric $g(t) = u(\cdot, t) g_M$ in M_k. Let $0 < r_0 < k - 4$ and $1 < T < \infty$ be fixed and let $p > \frac{m}{2}$ be any exponent. Then, for every $t \in [0, T]$*

$$\int_{B_{r_0}} |\mathrm{R}_{g(t)}|^p \, d\mu_{g(t)} \leq C$$

where the constant C depends on r_0, T and p but not on k.

Proof In view of Lemma 2.3, it remains to prove a similar estimate for the negative part $\mathrm{R}_-(x, t) = \max\{0, -\mathrm{R}_{g(t)}(x)\}$. Since $\mathrm{R} = \mathrm{R}_+ - \mathrm{R}_-$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{M_k} \psi \mathrm{R}_-^p \, d\mu_g \\
&\quad = p \int_{M_k} \psi \mathrm{R}_-^{p-1} \big(-\mathrm{R}^2 - (m-1) \Delta_g \mathrm{R} \big) \, d\mu_g - \frac{m}{2} \int_{M_k} \psi \mathrm{R}_-^p \mathrm{R} \, d\mu_g \\
&\quad = (-p + \tfrac{m}{2}) \int_{M_k} \psi \mathrm{R}_-^{p+1} \, d\mu_g - p(m-1) \int_{M_k} \psi \mathrm{R}_-^{p-1} \Delta_g \mathrm{R} \, d\mu_g \\
&\quad = (-p + \tfrac{m}{2}) \int_{M_k} \psi \mathrm{R}_-^{p+1} \, d\mu_g \\
&\qquad + p(m-1) \Big(\int_{M_k} \mathrm{R}_-^{p-1} \langle \nabla \psi, \nabla \mathrm{R} \rangle_g + \psi \langle \nabla \mathrm{R}_-^{p-1}, \nabla \mathrm{R} \rangle \, d\mu_g \Big) \\
&\quad = (-p + \tfrac{m}{2}) \int_{M_k} \psi \mathrm{R}_-^{p+1} \, d\mu_g \\
&\qquad + (m-1) \Big(- \int_{M_k} \langle \nabla \psi, \nabla \mathrm{R}_-^p \rangle_g \, d\mu_g - \frac{4(p-1)}{p} \int_{M_k} \psi |\nabla \mathrm{R}_-^{\frac{p}{2}}|_g^2 \, d\mu_g \Big).
\end{aligned} \tag{28}$$

We choose $\psi = \varphi^{2(p+1)}$, where $\varphi \in C_c^\infty(B_{r_0+4})$ satisfies $0 \leq \varphi \leq 1$, $\varphi|_{B_{r_0}} \equiv 1$ and $|\nabla \varphi|_{g_M} \leq 2$ as in step 1 of the proof of Lemma 2.3 and estimate (28) as in (17). Thus,

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{M_k} \psi \mathrm{R}_-^p \, d\mu_g \\
&\quad \leq \Big(\frac{(m-1)\lambda p}{p+1} - (p - \tfrac{m}{2}) \Big) \int_{M_k} \psi \mathrm{R}_-^{p+1} \, d\mu_g + \frac{C_{m,p}}{\lambda^p} \Big(\int_{B_{r_0+4}} u^{\frac{m}{2} - p - 1} \, d\mu_{g_M} \Big)
\end{aligned}$$

where the parameter $\lambda > 0$ is arbitrary. Since $p > \frac{m}{2}$ we may choose $\lambda = \frac{(p - \frac{m}{2})(p+1)}{(m-1)p}$ to obtain

$$\frac{\partial}{\partial t} \int_{M_k} \psi \mathrm{R}_-^p \, d\mu_g \leq C_{m,p} \Big(\int_{B_{r_0+4}} u^{\frac{m}{2} - p - 1} \, d\mu_{g_M} \Big).$$

Since $u \geq c_1 > 0$ by Lemma 1.1, we conclude

$$\int_{B_{r_0}} \mathrm{R}_-^p(\cdot, t)\, d\mu_{g(t)} \leq \int_{B_{r_0+4}} |\mathrm{R}_{g(0)}|^p\, d\mu_{g_0} + tC_{m,r_0,p,c_1}.$$

□

Remark We prove the $L^p(B_{r_0})$-estimates for R_+ and R_- separately rather than directly estimating $\|\mathrm{R}\|_{L^p(B_{r_0})}$ because it is interesting to see that the estimate for the negative part of scalar curvature is simpler than the estimate for the positive part if $p > \frac{m}{2}$. The sign of the non-linear term $(p - \frac{m}{2})\mathrm{R}_+^{p+1}$ respectively $(-p + \frac{m}{2})\mathrm{R}_-^{p+1}$ makes the difference.

Proof of Theorem 1 For every $k \in \mathbb{N}$, let u_k be the solution to problem (4) in $M_k \times [0, \infty[$. Let $U_k = u_k^\eta$ for $\eta = \frac{m-2}{4}$. Lemmata 1.1 and 2.4 imply that for every fixed $t \geq 0$, the sequence $\{U_k(\cdot, t)\}_{4<k\in\mathbb{N}}$ is bounded in the Sobolev space $W^{2,p}(M_1)$ for any $p > \frac{m}{2}$. In fact, we may apply the Calderon–Zygmund Inequality [11, Theorem 9.11] to the elliptic equation

$$\Delta_{g_M} U_k = \frac{\eta}{(m-1)}\left(\mathrm{R}_{g_M} U_k - U_k^{1+\frac{1}{\eta}} \mathrm{R}_{u_k g_M}\right)$$

in $M_1 \subset B_2$. Let $T \geq 1$ be fixed. Since

$$\frac{\partial U_k}{\partial t} = -\eta U_k \mathrm{R}_{u_k g_M},$$

we obtain that the sequence $\{U_k\}_{4\leq k\in\mathbb{N}}$ is bounded in $W^{1,p}(M_1 \times [0, T])$ for any fixed $p > \frac{m}{2}$. If we choose $p = 2(m+1)$, then Sobolev's embedding $W^{1,p}(M_1 \times [0, T]) \hookrightarrow C^{0,\alpha}(M_1 \times [0, T])$ is compact for any $0 < \alpha < \frac{1}{2}$ (recall that $M_1 \times [0, T]$ is bounded with Lipschitz boundary) and we obtain a subsequence $\Lambda_1 \subset \mathbb{N}$ such that

$$\{U_k|_{M_1\times[0,T]}\}_{4<k\in\Lambda_1}$$

converges in $C^{0,\alpha}(M_1 \times [0, T])$ to some V_1. In particular, $\{U_k(\cdot, t)\}_{4<k\in\Lambda_1}$ converges to $V_1(\cdot, t)$ in $C^{0,\alpha}(M_1)$ for every fixed $t \in [0, T]$. As observed above, $\{U_k(\cdot, t)\}_{4<k\in\Lambda_1}$ is bounded in $W^{2,p}(M_1)$ which compactly embeds into $C^{1,\alpha}(M_1)$. Hence, a subsequence converges in $C^{1,\alpha}(M_1)$ and its limit must be $V_1(\cdot, t)$. Thus, $V_1 \in C^{1,\alpha;0,\alpha}(M_1 \times [0, T])$. Passing to the limit in the weak formulation of equation (1), we conclude that V_1 is a weak solution to equation (1) in $M_1 \times [0, T]$. By parabolic regularity theory, V_1 is actually regular and a classical solution.

We repeat this argument to obtain a subsequence $\Lambda_2 \subset \Lambda_1$ such that

$$\{U_k|_{M_2\times[0,2T]}\}_{5<k\in\Lambda_2}$$

converges in $C^{0,\alpha}(M_2 \times [0, 2T])$ to some solution V_2 of equation (1) in $M_2 \times [0, 2T]$. Iterating this procedure leads to a diagonal subsequence of $\{U_k\}_{4<k}$ which converges to

a limit U satisfying the Yamabe flow equation (1) in $M \times [0, \infty[$. Moreover, the uniform bounds from Lemmata 1.1 and 2.1 are preserved in the limit and by construction, the initial condition is satisfied. □

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