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A Sharp Inequality for Harmonic Diffeomorphisms of the Unit Disk

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Abstract We extend the classical Schwarz–Pick inequality to the class of harmonic mappings between the unit disk and a Jordan domain with given perimeter. It is intriguing that the extremals in this case are certain harmonic diffeomorphisms between the unit disk and a convex domain that solve the Beltrami equation of second order.

Keywords Harmonic functions · Convex domain · Hardy spaces

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1 Introduction

Let $\mathbf{U}$ be the unit disk in the complex plane $\mathbf{C}$ and denote by $\mathbf{T}$ its boundary. A harmonic mapping f of the unit disk into the complex plane can be written by $f(z) = g(z)+\overline{h(z)}$, where g and h are holomorphic functions defined on the unit disk. Two of essential properties of harmonic mappings are given by Lewy theorem, and Rado–Kneser–Choquet theorem. Lewy theorem states that a injective harmonic mapping f is indeed a diffeomorphism, or what is the same its Jacobian $J_f := |\partial f|^2 - |\bar\partial f|^2 = |g'(z)|^2 - |h'(z)|^2 \neq 0$. Rado–Kneser–Choquet theorem states that a Poisson extension of a homeomorphism of the unit circle $\mathbf{T}$ onto a convex Jordan curve γ is a diffeomorphism on the unit disk onto the inner part of γ. For those and many more important properties of harmonic mappings, we refer to the book of Duren [2].

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The standard Schwarz lemma states that if f is a holomorphic mapping of the unit disk $\mathbf{U}$ into, itself such that $f(0) = 0$ then $|f(z)| \leq |z|$.

Its counter-part for harmonic mappings states the following ([2, Sect. 4.6]). Let f be a complex-valued function harmonic in the unit disk $\mathbf{U}$ into itself, with $f(0) = 0$. Then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|,$$

and this inequality is sharp for each point $z \in \mathbf{U}$. Furthermore, the bound is sharp everywhere (but is attained only at the origin) for univalent harmonic mappings f of $\mathbf{U}$ onto itself with $f(0) = 0$.

The standard Schwarz–Pick lemma for holomorphic mappings states that every holomorphic mapping f of the unit disk onto itself satisfies the inequality:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}. \tag{1.1}$$

If the equality is attained in (1.1) for a fixed $z = a \in \mathbf{U}$, then f is a Möbius transformation of the unit disk.

It follows from (1.1) the weaker inequality:

$$|f'(z)| \leq \frac{1}{1 - |z|^2} \tag{1.2}$$

with the equality in (1.2) for some fixed $z = a$ if and only if $f(z) = e^{it} \frac{z-a}{1-z\bar{a}}$. We will extend this result to harmonic mappings.

2 Main Result

Theorem 2.1 *If f is a harmonic orientation preserving diffeomorphism of the unit disk $\mathbf{U}$ onto a Jordan domain Ω with rectifiable boundary of length $2\pi R$, then the sharp inequality*

$$|\partial f(z)| \leq \frac{R}{1 - |z|^2}, \quad z \in \mathbf{U} \tag{2.1}$$

holds. If the equality in (2.1) is attained for some a, then Ω is convex, and there is a holomorphic function $\mu : \mathbf{U} \to \mathbf{U}$ and a constant $\theta \in [0, 2\pi]$, such that

$$F(z) := e^{-i\theta} f\left(\frac{z+a}{1+z\bar{a}}\right) = R\left(\int_0^z \frac{dt}{1+t^2\mu(t)} + \overline{\int_0^z \frac{\mu(t)dt}{1+t^2\mu(t)}}\right). \tag{2.2}$$

Moreover, every function f defined by (2.2) is a harmonic diffeomorphism and maps the unit disk to a Jordan domain bounded by a convex curve of length $2\pi R$ and the inequality (2.1) is attained for $z = a$.

Corollary 2.2 *Under the conditions of Theorem 2.1, if $R = 1$ and $|\mu|_\infty = k < 1$, then the mapping F is $K = \frac{1+k}{1-k}$ bi-Lipschitz, and K—quasi-conformal.*

Proof We have that

$$F_z(z) = \frac{1}{1 + z^2\mu(z)}$$

and

$$\overline{F_{\bar{z}}(z)} = \frac{\mu(z)}{1 + z^2\mu(z)}.$$

Thus

$$\frac{1-k}{1+k} \le |F_z| - |F_{\bar{z}}| := |lF| \le |dF| := |F_z| + |F_{\bar{z}}| \le \frac{1+k}{1-k}.$$

Thus, F is K—bi-Lipschitz. Furthermore, we have

$$\frac{|F_{\bar{z}}(z)|}{|F_z(z)|} = |\mu(z)| \le k,$$

and so

$$\frac{(|F_z| + |F_{\bar{z}}|)^2}{|F_z|^2 - |F_{\bar{z}}|^2} = \frac{|F_z| + |F_{\bar{z}}|}{|F_z| - |F_{\bar{z}}|} \le \frac{1+k}{1-k} = K.$$

Therefore, f is K—quasi-conformal. □

Corollary 2.3 *If $\Omega = \mathbf{U}$, then the equality is attained in (2.1) for some a if only if f is a Möbius transformation of the unit disk onto a disk.*

Proof of Corollary 2.3 Under conditions of Theorem 2.1, the function (2.2) can be written as

$$F(z) := e^{-i\theta} f\left(\frac{z+a}{1+z\bar{a}}\right) = R\left(\int_0^z \left(1 - t^2h'(t)\right)dt + \overline{h(z)}\right) \tag{2.3}$$

where $h(z) = \sum_{k=0}^{\infty} a_k z^k$ is defined on the unit disk and satisfies the condition:

$$\frac{|h'(z)|}{|1 - z^2h'(z)|} < 1, \quad z \in \mathbf{U}. \tag{2.4}$$

Further

$$J_F(z) = |1 - z^2h'(z)|^2 - |h'(z)|^2.$$

Since

$$h'(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k,$$

and

$$1 - z^2 h'(z) = \sum_{k=0}^{\infty} c_k z^k = 1 - \sum_{k=2}^{\infty} (k-1) a_{k-1} z^k,$$

it follows that

$$\begin{aligned}
|f(\mathbf{U})| &= \int_{\mathbf{U}} J_F(z) dx dy \\
&= \int_0^{2\pi} \int_0^1 r J_F(re^{it}) dr dt \\
&= 2\pi R^2 \sum_{k=0}^{\infty} \frac{|b_k|^2}{2k+2} - 2\pi R^2 \sum_{k=0}^{\infty} \frac{|c_k|^2}{2k+2} \\
&= 2\pi R^2 \left(1 + \sum_{k=2}^{\infty} \frac{|(k-1)a_{k-1}|^2}{2k+2} - \sum_{k=0}^{\infty} \frac{|(k+1)a_{k+1}|^2}{2k+2}\right) \\
&= 2\pi R^2 \left(1 + \sum_{k=0}^{\infty} \frac{|(k+1)a_{k+1}|^2}{2k+6} - \sum_{k=0}^{\infty} \frac{|(k+1)a_{k+1}|^2}{2k+2}\right) \\
&= \pi R^2 \left(1 - 2 \sum_{k=0}^{\infty} \frac{(1+k)|a_{k+1}|^2}{(3+k)}\right).
\end{aligned}$$

If $R = 1$, this implies that $\Omega = \mathbf{U} + a_0$ if and only if $h \equiv a_0$. This concludes the proof. □

Using the corresponding result in [1] and Theorem 2.1, we have

Corollary 2.4 *If as in (2.3), $F(z) = g + \overline{h}$, then $F(z) = g(z) - h(z)$ is univalent and convex in direction of real axis.*

Using Theorem 2.1, we obtain

Corollary 2.5 *For every positive constant R and every holomorphic function μ of the unit disk into itself, there is a unique convex Jordan domain $\Omega = \Omega_{\mu,R}$, with the perimeter $2\pi R$, such that the initial boundary problem (Beltrami equation)*

$$\begin{cases} \overline{f_{\bar{z}}(z)} = \mu(z) f_z(z), \\ f_z(0) = R, \\ f(0) = 0, \end{cases} \tag{2.5}$$

admits a unique univalent harmonic solution $f = f_{\mu,R} : \mathbf{U} \xrightarrow{\text{onto}} \Omega$.

Remark 2.6 If instead of boundary problem (2.5), we observe

$$\begin{cases} \overline{g_{\bar z}(z)} = \mu(z) g_z(z), \\ g_z(a) = \frac{R}{1-|a|^2}, \\ g(a) = 0, \end{cases} \tag{2.6}$$

then the solution g is given by

$$g(z) = e^{i\theta} f\left(\frac{z-a}{1-z\bar a}\right)$$

and thus, $g(\mathbf{U}) = e^{i\theta} \cdot \Omega_{\mu,R}$. Here, f is a solution of (2.5).

3 Proof of the Main Result

Proof of Theorem 2.1 Assume first that $f(z) = g(z) + \overline{h(z)}$ has C^1 extension to the boundary and assume without loss of generality that $R = 1$. Then, we have

$$\partial_t \left(g(z) + \overline{h(z)}\right) = i g'(z) z + \overline{i h'(z) z} \tag{3.1}$$

Therefore, for $z = e^{it}$

$$|i g'(z) z + \overline{i h'(z) z}| = |g'(z) - \overline{h'(z) z^2}|.$$

Thus

$$2\pi = \int_{\mathbf{T}} \left|\partial_t \left(g(z) + \overline{h(z)}\right)\right| |dz| = \int_{\mathbf{T}} |g'(z) - \overline{h'(z) z^2}| |dz|.$$

As $|g'(z) - \overline{h'(z)z^2}|$ is subharmonic, it follows that

$$|g'(0)| \le \frac{1}{2\pi} \int_{\mathbf{T}} |g'(z) - \overline{h'(z) z^2}| |dz|.$$

Thus, we have that $|g'(0)| \le 1$. Now, if $m(z) = \frac{z+a}{1+z\bar a}$, then $m(0) = a$, and thus, $F(z) = f(m(z))$ is a harmonic diffeomorphism of the unit disk onto itself. Furthermore

$$\partial F(0) = f'(a) m'(0) = \partial f(a)(1 - |a|^2).$$

Therefore, by applying the previous case to F, we obtain

$$|\partial f(a)| \le \frac{1}{1 - |a|^2}.$$

Assume now that the equality is attained for $z = 0$. Then

$$|g'(0)| = \frac{1}{2\pi r}\int_{\mathbf{rT}} |g'(z) - \overline{h'(z)z^2}||dz|,$$

or what is the same

$$|g'(0)| = \frac{1}{2\pi}\int_{\mathbf{T}} |g'(zr) - \overline{h'(zr)r^2z^2}||dz|.$$

Thus, for $0 \le r \le 1$, we have

$$\frac{1}{2\pi}\int_{\mathbf{T}} |g'(zr) - \overline{h'(rz)r^2z^2}||dz| - |g'(0)| \equiv 0. \tag{3.2}$$

To continue recall the definition of the Riesz measure μ of a subharmonic function u. Namely, there exists a unique positive Borel measure μ, so that

$$\int_{\mathbf{U}} \varphi(z)d\mu(z) = \int_{\mathbf{U}} u\Delta\varphi(z)dm(z), \quad \varphi \in C_0^2(\mathbf{U}).$$

Here, dm is the Lebesgue measure defined on the complex plane $\mathbf{C}$. If $u \in C^2$, then

$$d\mu = \Delta u dm.$$

□

We need the following proposition.

Proposition 3.1 [5, Theorem 2.6 (Riesz representation theorem)]. *If u is a subharmonic function defined on the unit disk then for $r < 1$, we have*

$$\frac{1}{2\pi}\int_{\mathbf{T}} u(rz)|dz| - u(0) = \frac{1}{2\pi}\int_{|z|<r} \log\frac{r}{|z|}d\mu(z)$$

where μ is the Riesz measure of u.

By applying Proposition 3.1 to the subharmonic function

$$u(z) = |g'(z) - \overline{h'(z)z^2}|$$

in view of (3.2), we obtain that

$$\frac{1}{2\pi}\int_{|z|<r} \log\frac{r}{|z|}d\mu(z) \equiv 0.$$

Thus, in particular, we infer that $\mu = 0$, or what is the same $\Delta u = 0$. As $u = |w|$, where $w = |u|e^{i\theta}$ is harmonic, it follows that

$$\Delta u = u|\nabla\theta|^2 = 0.$$

Therefore, $\nabla\theta \equiv 0$, and hence, $\theta = \text{const}$.

Therefore

$$e^{-i\theta}(g'(z) - \overline{h'(z)z^2}) = G(z) + \overline{H(z)},$$

is a real harmonic function. Here

$$G(z) = e^{-i\theta} g'(z)$$

and

$$H(z) = -e^{i\theta} h'(z) z^2$$

are analytic functions satisfying the condition $|H(z)| < |G(z)|$ in view of Lewy theorem. Thus

$$G(z) + \overline{H(z)} = \overline{G(z)} + H(z)$$

or what is the same

$$G(z) - H(z) = \overline{G(z) - H(z)}.$$

Thus, $G(z) - H(z)$ is a real holomorphic function, and therefore, it is a constant function. Furthermore

$$e^{-i\theta} g'(z) + e^{i\theta} h'(z) z^2 = G(z) - H(z) = G(0) - H(0) = e^{-i\theta} g'(0).$$

Hence

$$G(z) + \overline{H(z)} = G(z) + \overline{G(z)} - e^{-i\theta} g'(0) = 2\Re\left[e^{-i\theta} g'(z)\right] - e^{-i\theta} g'(0).$$

Assume without loss of the generality that $\theta = 0$ and $g'(0) = 1$. Then

$$g'(z) = 1 - h'(z) z^2. \tag{3.3}$$

From (2.4), we infer that

$$(1 - 2\Re(h'(z)z^2)) > |h'(z)|^2(1 - |z|^2). \tag{3.4}$$

Further for $z = e^{it}$, from (3.1) and (3.3), we have

$$\partial_t f(z) = iz(1 - 2\Re(h'(z)z^2)). \tag{3.5}$$

To get the representation (2.2), by Lewy theorem, we have that the holomorphic mapping $\mu(z) = \frac{h'(z)}{g'(z)}$ maps the unit disk into itself. By (3.3), we deduce that

$$g(z) = \int_0^z \frac{dt}{1 + t^2\mu(t)}$$

and

$$h(z) = \int_0^z \frac{\mu(t)dt}{1 + t^2\mu(t)}.$$

It follows by (3.5) and (3.4) that $\partial_t \arg \partial_t f(z) = 1 > 0$, and this implies that the image of $\mathbf{U}$ under f is a convex domain.

To prove that, every mapping f defined by (2.2) is a diffeomorphism of the unit disk onto a convex Jordan domain, we use Choquet–Kneser–Rado theorem. First of all, we have

$$\arg \partial_t F(z) = (\pi/2 + t).$$

Therefore

$$\partial_t \arg \partial_t F(z) = 1 > 0$$

which means that $F(\mathbf{T})$ is a convex curve.

As

$$\frac{\partial_t F(z)}{|\partial_t F(z)|} = iz,$$

if $z_1, z_2 \in \mathbf{T}$ with $f(z_1) = f(z_2)$, then

$$\frac{\partial_t F(z_1)}{|\partial_t F(z_1)|} = \frac{\partial_t F(z_2)}{|\partial_t F(z_2)|}$$

and so $z_1 = z_2$. Thus by Choquet–Kneser–Rado theorem, F is a diffeomorphism.

If f is not C^1 up to the boundary, then we apply the approximating sequence. Let Ω be a fixed Jordan domain and assume that ϕ is a conformal mapping of the unit disk onto Ω, with $\phi(0) = 0$. For $r_n = \frac{n}{n+1}$, let $\Omega_n = \phi(r_n\mathbf{U})$, and let $U_n = f^{-1}\Omega_n$. Let $\phi_n : \mathbf{U} \to U_n$ be a conformal mapping satisfying the condition $\phi_n(0) = 0$. Then, $f_n = f \circ \phi_n$ is a conformal mapping of the unit disk onto the Jordan domain Ω_n. Furthermore, by subharmonic property of $|\phi'(z)|$, we conclude that

$$R_n = |\partial\Omega_n| = \int_{\mathbf{T}} |\phi'(r_n z)|dz| < \int_{\mathbf{T}} |\phi'(z)|dz| = |\partial\Omega| = R = 1.$$

Then, we have that

$$|\partial f_n(z)| \le \frac{R_n}{1 - |z|^2}, \quad z \in \mathbf{U}. \tag{3.6}$$

As ϕ_n converges in compacts to the identity mapping, and thus, ϕ_n' converges in compacts to the constant 1, we conclude that the inequality (2.1) is true for non-smooth domains.

It remains to consider the equality statement in this case. However, we know that $\partial\Omega$ is rectifiable if and only if $\partial_t f \in h_1(\mathbf{U})$ (see, e.g., [4, Theorem 2.7]). Here, h_1 stands for the Hardy class of harmonic mappings. Now, the proof is just repetition of the previous approach, and we omit the details. □

Example 3.2 If $\mu(z) = z^n$, then F defined in (2.2), maps the unit disk to $n+2-$regular polygon of perimeter $2\pi R$ and centered at 0. Namely, we have that

$$\partial_z F(z) = \frac{R}{1+z^{n+2}}, \quad \partial_{\bar z} F(z) = \frac{R z^n}{1+z^{n+2}}.$$

The rest follows from the similar statement obtained by Duren in [2, p. 62].

Remark 3.3 If μ is a holomorphic mapping of the unit disk onto itself and F is defined by (2.2), then $F(0) = 0$ and

$$|DF|^2 := |F_z|^2 + |F_{\bar z}|^2 \geq \frac{R^2}{2}.$$

Indeed, we have that

$$|DF|^2 = R^2 \frac{1+|\mu|^2}{|1+z^2\mu|^2} \geq \frac{R^2}{2} = \frac{L^2}{8\pi^2} \geq \frac{\rho^2}{2}.$$

Here, $\rho = \operatorname{dist}(0, \partial\Omega)$. Thus, we have the sharp inequality:

$$|DF|^2 \geq \frac{\rho^2}{2}. \tag{3.7}$$

In [3], it is proved that we have the general inequality

$$|Df|^2 \geq \frac{\rho^2}{16}, \tag{3.8}$$

for every harmonic diffeomorphism of the unit disk onto a convex domain Ω with $f(0) = 0$. Some examples suggest that the best inequality in this context is

$$|Df|^2 \geq \frac{\rho^2}{8}, \tag{3.9}$$

The last conjectured inequality is not proved. The gap between $\frac{\rho^2}{2}$ and $\frac{\rho^2}{8}$ in (3.7) and (3.9) appears as the mappings F are special extremal mappings which for the case of Ω being the unit disk are just rotations.

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