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A Criterion for Uniqueness of Tangent Cones at Infinity for Minimal Surfaces

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Abstract We partially resolve a conjecture of Meeks on the asymptotic behavior of minimal surfaces in $\mathbb{R}^3$ with quadratic area growth.

Keywords Differential geometry · Minimal surfaces · Geometric analysis · Tangent cones · Uniqueness · Scherk

Mathematics Subject Classification 49Q05 · 53A10

1 Introduction

Let Σ be an embedded minimal surface in $\mathbb{R}^3$. One of the fundamental properties of minimal surfaces is the following:

Theorem 1.1 (Monotonicity) [1] *Let $r > s$. Then*

$$\frac{A(\Sigma \cap B_r)}{r^2} - \frac{A(\Sigma \cap B_s)}{s^2} = \int_{\Sigma \cap B_r \setminus B_s} \frac{|x^N|^2}{|x|^4} \geq 0$$

Note that if we define the *area density* as

$$\Theta(r) := \frac{A(\Sigma \cap B_r)}{\pi r^2},$$

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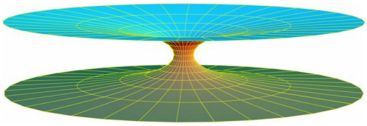

Fig. 1 Catenoid (from http://www.indiana.edu/~minimal)

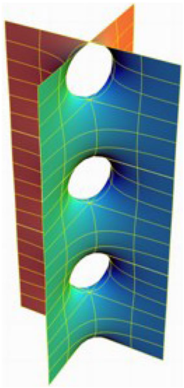

Fig. 2 Scherk singly periodic (from http://www.indiana.edu/~minimal)

then the monotonicity formula implies that Θ is nondecreasing. If

$$\lim_{r\to\infty} \Theta(r) = \Theta(\infty) = k < \infty,$$

we say that Σ has quadratic area growth, or the area growth of k planes.

For surfaces with the growth of 2 planes, there are two canonical examples: the catenoid (Fig. 1), and Scherk's singly periodic surfaces, which occur in a one parameter family (Figs. 2 and 3), where the parameter is the angle between the two leaves. As the angle goes to zero, the Scherk surfaces approach a catenoid on compact sets after an appropriate rescaling. In 2005, Meeks and Wolf proved the following theorem:

Theorem 1.2 [4] *Suppose that Σ is an embedded minimal surface in $\mathbb{R}^3$ which has infinite symmetry group and $\Theta(\infty) < 3$. Then Σ is either a catenoid or a Scherk example.*

Meeks has conjectured that the symmetry condition in the above may be removed:

Conjecture 1.3 [3] *Let Σ be an embedded minimal surface in $\mathbb{R}^3$ with area growth of 2 planes. Then Σ is either a catenoid or a Scherk example.*

However, an initial difficulty with the above is that it is not yet known that a minimal surface with quadratic growth even needs to be asymptotic to a catenoid or a Scherk example. By the compactness results from Geometric Measure Theory, it is known that if Σ is an embedded minimal surface with quadratic area growth, then for any sequence $r_i \to \infty$, there exists a subsequence ρ_i such that $\Sigma/\rho_i \cap B_1$ converges to

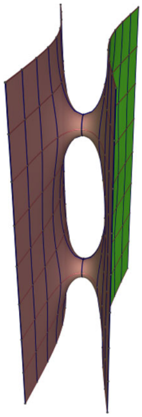

Fig. 3 Non-orthogonal Scherk (from http://www.indiana.edu/~minimal)

a minimal cone $\mathcal{C}$ in the varifold topology. Such a cone $\mathcal{C}$ is called a *tangent cone at infinity*. A priori, there may be many tangent cones at infinity.

This leads to the following conjecture, also due to Meeks:

Conjecture 1.4 [3] *Let Σ be an embedded minimal surface in $\mathbb{R}^3$ with quadratic area growth. Then Σ has a unique tangent cone at infinity.*

In the case of finite genus, this had already been resolved by Collin [2], who proved that any minimal surface with finite genus and quadratic area growth must be asymptotic to a single multiplicity k plane. In particular, when combined with a result of Schoen [5], this resolves Meeks' full conjecture in the case of finite genus—that is, the only minimal surface with the area growth of two planes and finite genus is the catenoid.

In this paper, we prove that Meeks' Conjecture 1.4 holds true under additional assumptions:

Theorem 1.5 *Let Σ be an embedded minimal surface with the area growth of k planes. Suppose that there exists $\alpha < 1$ such that for all R sufficiently large, there exists a line l_R*

$$\Sigma \cap B_R \cap \{d(x, l_R) > R^\alpha\}$$

is a union of at least $2k$ disks Σ_i and such that $\partial\Sigma_i$ is homotopically nontrivial in $\partial(B_R \cap \{d(x, l_R) > R^\alpha\})$. Then Σ has a unique tangent cone at infinity.

This leads to the following:

Theorem 1.6 *Let Σ be an embedded minimal surface with quadratic area growth. Let*

$$\mathcal{C}_\alpha = \{x_1^2 + x_2^2 \leq R^{2\alpha}\}.$$

Then if for some R_0, $\Sigma\backslash(B_{R_0} \cup \mathcal{C}_\alpha)$ is a union of $2k$ topological disks Σ_i each with finitely many boundary components, then Σ has a unique tangent cone at infinity.

Note that the corollary substitutes the homotopy requirement from the theorem for the existence of a single line around which we can base our sublinearly growing set. To the author's knowledge, these two theorems are the first progress towards proving Meeks' conjecture.

1.1 Summary of Proofs

Both of the above theorems are proved by first showing a lower area bound for the area of Σ inside large balls. This, combined with the upper area bound coming from the monotonicity formula and quadratic area growth, along with a projection argument due to Brian White, leads to uniqueness of tangent cones.

Both theorems prove their lower area bound by working on each leaf of Σ separately. The lower area bound used in Theorem 1.5 is rather straightforward to prove using the homotopy requirement. However, bounding the area from below in Theorem 1.6 is slightly more detailed, and relies on arguments made in the proof of Lemma 2.1, as well as a case by case analysis of the possible shapes of the leaves of Σ.

2 Proof of Theorem 1.5

The proof of this begins with the following:

Lemma 2.1 (Lower Area Bound) *Suppose that Σ satisfies the conditions of Theorem 1.5. Then for some $C = C(\Sigma)$*

$$Area(B_R \cap \Sigma) > k\pi R^2 - CR^{\alpha+1}.$$

Proof We will work on each leaf Σ_i separately, and the lemma will come from adding the area of all the leaves together.

First note that $B_R \cap \{d(x, l_R) > R^\alpha\} = T_R$ is a rotationally symmetric solid torus and (since Σ_i is a disk), $\partial\Sigma_i$ is contractible in T_R. However, since T_R is rotationally symmetric, the smallest spanning disk for any such curve has area at least that of a vertical cross section C. Any such vertical cross section consists of a half-circle of radius R minus a strip of length $2R$ and width CR^α. Thus, we have

$$A(\Sigma_i) \geq A(C) \geq \frac{\pi}{2}R^2 - CR^{\alpha+1}.$$

□

Remark 2.2 Note that Lemma 2.1 implies that there are in fact exactly $2k$ disks in the statement of Theorem 1.5.

We make a definition:

Definition 2.3 The *error* at scale r of a minimal surfaces with area growth of k planes is defined as

$$e(r) = \pi k - \frac{\text{Area}(\Sigma \cap B_r)}{r^2}.$$

Thus, Lemma 2.1 is equivalent to the statement:

$$e(r) \leq Cr^{\alpha-1}. \tag{1}$$

We now apply an argument of Brian White [6] to prove uniqueness of the tangent cone.

Lemma 2.4 *Let Σ satisfy the following: $\exists R_0, \alpha < 1$ such that for $R_0 < r < \infty$,*

$$e(r) < Cr^{1-\alpha} \tag{2}$$

Then Σ has a unique tangent cone at infinity.

Proof Define $F(z) = z/|z|$. Then note that $A(F(\Sigma \cap (B_r \backslash B_s)))$ is equal to the area of the projection of $\Sigma \cap (B_r \backslash B_s))$ onto the unit sphere. We will bound this area. We have

$$\begin{aligned} A(F(\Sigma \cap (B_r \backslash B_s))) &= \int_{\Sigma \cap B_r \backslash B_s} \frac{|x^N|}{|x|^3} \mathrm{d}\Sigma \\ &\leq \left[\int_{\Sigma \cap B_r \backslash B_s} \frac{|x^N|^2}{|x|^4} \mathrm{d}\Sigma \right]^{1/2} \left[\int_{\Sigma \cap B_r \backslash B_s} \frac{1}{|x|^2} \mathrm{d}\Sigma \right]^{1/2}. \end{aligned}$$

By the monotonicity formula, 1.1 and the fact that the area density of Σ is uniformly bounded by k, we can bound the term inside the first bracket:

$$\begin{aligned} \int_{\Sigma \cap B_r \backslash B_s} \frac{|x^N|^2}{|x|^4} \mathrm{d}\Sigma &\leq \frac{A(\Sigma \cap B_r)}{r^2} - \frac{A(\Sigma \cap B_s)}{s^2} \\ &\leq k\pi - \frac{A(\Sigma \cap B_s)}{s^2} = e(s). \end{aligned}$$

For the term in the second bracket, we have

$$\int_{\Sigma \cap B_r \backslash B_s} \frac{1}{|x|^2} \mathrm{d}\Sigma \leq \int_{\Sigma \cap B_r \backslash B_s} \frac{1}{s^2} \mathrm{d}\Sigma \leq A(B_r \cap \Sigma) s^{-2}.$$

Thus, we get that

$$A(F(\Sigma \cap (B_r \backslash B_s))) \leq e(s)^{1/2} (s^{-2} A(B_r \cap \Sigma))^{1/2}.$$

Now, by Eq. (2), along with the fact that $A(B_r \cap \Sigma) < k\pi r^2$, we have that this is bounded by

$$Cs^{(\alpha-1)/2} \left[\left(\frac{r}{s}\right)^2 (r^{-2} A(B_r \cap \Sigma)) \right]^{1/2} \leq C \frac{r}{s^{(1-\alpha)/2+1}}.$$

Pick s and r such that $s \leq r \leq 2s$. Then

$$A(F(\Sigma \cap (B_r \backslash B_s))) \leq Cs^{(\alpha-1)/2}.$$

We then sum the above bound to see

$$\begin{aligned} A(F(\Sigma \cap (B_{2^n r} \backslash B_r))) &= \sum_{k=1}^{n} A(F(\Sigma \cap (B_{2^k r} \backslash B_{2^{k-1} r}))) \\ &\leq C \sum_{k=1}^{n} (2^k r)^{(\alpha-1)/2} \\ &\leq \frac{C}{r^{(1-\alpha)/2}} \frac{1}{1 - 2^{(1-\alpha)/2}}. \end{aligned}$$

As $r \to \infty$, this term goes to zero. Thus, the area of the projection of $\Sigma \backslash B_r$ approaches zero as r gets large, which means that the tangent cone must be unique. □

3 Proof of Theorem 1.6

For the reader's convenience, we restate the assumptions: that there exists α, R_0 such that if

$$\mathcal{C}_\alpha = \{x_1^2 + x_2^2 \leq R^{2\alpha}\}$$

and $\Sigma \backslash (B_{R_0} \cup \mathcal{C}_\alpha$ is a union of $2k$ disks Σ_i, each with finitely many boundary components.

Note that the closure of Σ_i in $\mathbb{R}^3$ must be conformally equivalent to $\overline{\mathbb{D}}^2$ with finitely many boundary points removed. Take a neighborhood N of one of these missing boundary points which does not come close to any other missing boundary points. Then $N \subset \Sigma_i$ has exactly one boundary component. There are two options for the shape of ∂N.

(1) The function $x_3|_{\partial N}$ is unbounded in both directions.
(2) $x_3|_{\partial N}$ is bounded in one direction.

Note that x_3 cannot be bounded in both directions, as then ∂N would be compact, which it is not.

We temporarily assume that Option 1 occurs (see Fig. 4). Let γ be the portion of ∂N which is not on the boundary of $\mathcal{C}_\alpha \cup B_{R_0}$. Note that we can take R_0 to be large enough so that ∂B_{R_0} is arbitrarily close to the missing point of $\partial \overline{\mathbb{D}}^2$, and thus in particular, $\gamma \subset B_{R_0}$. Redefine N to be $N \cap B^c_{R_0}$, and let $R >> R_0$.

Lemma 3.1 *$\partial B_R \cap N$ has a component which starts at the $x_3 \to +\infty$ side of $\partial N \cap \partial \mathcal{C}_\alpha$ and ends at the $x_3 \to -\infty$ side.*

Proof Suppose not. Then every component of $\partial B_R \cap N$ starts and ends on the same side of the missing point. In particular, there are an even number of points on each side. Consider moving along $\partial \mathcal{C}_\alpha$ towards the missing point. Each point of $\partial B_R \cap N \cap \partial \mathcal{C}_\alpha$

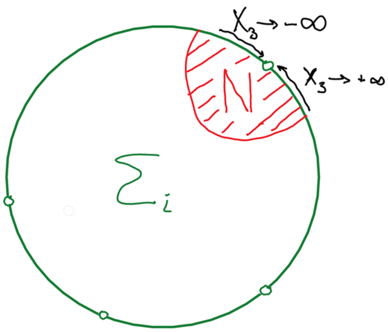

Fig. 4 N and Σ_i for option 1 (conformal picture)

represents a change from radius smaller than R to radius larger than R. However, since the radius started at $R_0 < R$, there cannot be an even number of these points. □

The above lemma implies that some component of $N \cap B_R \cap C_\alpha^c$ will satisfy the homotopy conditions of Theorem 1.5. This implies that it is possible to prove the Lower Area Bound lemma for this component, and in particular, the area must be asymptotic to $\pi R^2/2$.

The following lemma will complete our proof:

Lemma 3.2 *Under our assumptions, Option 2 is not possible.*

Proof Suppose that Option 2 occurs. WLOG, let $x_3|_{\partial N}$ be bounded below by 0, and let $(x_1, x_2, 0) \in \partial N$ be the point at which that minimum is achieved. Let $\rho = (x_1^2+x_2^2)^{1/2}$. Let C be a catenoid where the radius of the center geodesic is strictly larger than 2ρ. Then by a simple application of the maximum principle, N must intersect C. In particular, this implies that $\inf_{\partial B_R} x_3|_N < C_0 + \log R$.

Now, consider a sequence of R_i such that $\Sigma \cap B_{R_i}$ converges to a tangent cone at infinity. By compactness, $R_i^{-1} N \cap \partial B_{R_i}$ must either converge to a union of geodesics on B_1 or must disappear at infinity. However, due to the discussion of the previous paragraph, N cannot disappear at infinity, and so must converge to a nontrivial union of geodesics Γ_j, possibly with endpoints at the north or south poles. We aim to show that these Γ_j are all great circles.

Let p be a nonsmooth point on $\cup\Gamma_j$. Then there must exist a neighborhood S of p such that $|A|$ restricted to $S \cap R_i^{-1} N$ is unbounded as $i \to \infty$. However, since N is a minimal disk with quadratic area growth bounds, $|A|(x)$ must be bounded by $C/d(x)$, where $d(x)$ is the distance of x from the boundary of N.

Suppose that our nonsmooth p is not equal to the south pole. Then we can choose our neighborhood S of p to stay away from the x_3 axis, so we will have that $|A| < C$ uniformly on $S \cap R_i^{-1} N$. Suppose that p is equal to the south pole. Then by the assumption of Option 2, ∂N is only contained in the region $x_3 \geq 0$. So, we can choose $S = B_{1/2}(p)$, and this implies the same uniform $|A|$ bound.

Therefore, there will be no nonsmooth points of $\cup\Gamma_j$, which implies that Γ_j consists of a single great circle passing through the north pole.

In particular, this implies that there are some $\epsilon(R_i) \to 0$ such that the area of $R_i^{-1}N \cap B_1$ is greater than $\pi - \epsilon(R_i)$, where $\epsilon \to 0$ as $R_i \to \infty$. Thus, we have at least $2k$ components of $\Sigma \backslash \mathcal{C}_\alpha$, each of which has area growth at least $\pi R^2/2$ by the discussion of Option 1. However, since the global area growth is $k\pi R^2$, no component can have growth πR^2. $\square$

4 Future Directions

There are several potential extensions of the work above. Theorem 1.5 and Corollary 1.6 effectively assume that all tangent cones of Σ are unions of planes *with a common axis*. It is likely not significantly more difficult to show that the same result holds in the case when the one-dimensional singular set is more complicated, as long as away from a sublinearly growing neighborhood, Σ is a union of disks. That is, we have the following as another potential step towards the resolution of Meeks' Conjecture:

Conjecture 4.1 *Let Σ have the area growth of k planes, and suppose that there exists a uniform $\alpha < 1$ such that for each $R > R_0 >> 1$, the following is true: There exist line segments $L_i(R)$, $1 \le i \le m(R) < M$ such that outside of an $\alpha-$sublinearly growing neighborhood of $\cup L_i(R)$, $\Sigma \cap B_R$ is a union of disks. Then Σ has a unique tangent cone at infinity.*

There are likely other simple conditions which can be put on Σ to force Lemma 2.1 to hold. However, it may be possible to prove theorems approaching Conjecture 1.4 without factoring through some kind of lower area bound.

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