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A Note on Perelman's No Shrinking Breather Theorem

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Abstract
As an application of his entropy formula, Perelman (The entropy formula for the Ricci flow and its geometric applications, 2002) proved that every compact shrinking breather solution to the Ricci flow is a shrinking gradient Ricci soliton. Zhang (Asian J Math 18(4):727–756, 2014) and Lu and Zheng (J Geom Anal, 1–7, 2017) proved no shrinking breather theorems in the noncompact case under additional conditions. It is a natural question to ask whether one can generalize Perelman's no shrinking breather theorem to the noncompact case assuming only bounded curvature. This is the result we prove in this paper. Our proof uses Perelman's $\mathcal{L}$-geometry and an idea of Lu and Zheng (J Geom Anal, 1–7, 2017). The novelty of this paper is that we can remove the technical assumptions in Zhang (Asian J Math 18(4):727–756, 2014) and Lu and Zheng (J Geom Anal, 1–7, 2017).

Keywords Ricci flow · Shrinking breather · Shrinking soliton · Ancient solution

Mathematics Subject Classification 53C44

1 Introduction

The Ricci flow on a manifold M can be regarded as an orbit in the space $\text{Met}(M)\big/(\text{Diff}\oplus\text{Scal})$, where $\text{Met}(M)$ stands for the space of all the Riemannian metrics on M and $\text{Diff}\oplus\text{Scal}$ denotes the group of self-diffeomorphisms of M and scalings (with positive factors) in $\text{Met}(M)$. The breathers are the periodic orbits in this space.

Definition 1 A metric $g(t)$ evolving by the Ricci flow on a Riemannian manifold M is called a *breather*, if for some $t_1 < t_2$, there exists an $\alpha > 0$ and a diffeomorphism

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$\phi : M \to M$, such that $\alpha g(t_1) = \phi^* g(t_2)$. If $\alpha = 1$, $\alpha < 1$, or $\alpha > 1$, then the breather is called steady, shrinking, or expanding, respectively.

As a special case of the periodic orbits, the Ricci solitons, moving by diffeomorphisms and scalings, are the static orbits in the space $\mathrm{Met}(M)/(\mathrm{Diff} \oplus \mathrm{Scal})$.

Definition 2 A *gradient Ricci soliton* is a 3-tuple (M, g, f), where (M, g) is a Riemannian manifold and f is a smooth function on M called the potential function, which satisfies

$$Ric + \nabla^2 f = \frac{\lambda}{2} g,$$

where $\lambda = 0$, $\lambda = 1$, or $\lambda = -1$, corresponding to the cases of steady, shrinking, or expanding solitons, respectively.

It is well understood that when moving by the 1-parameter family of diffeomorphisms generated by the potential function, along with a scaling factor, the pull-back metric of the soliton satisfies the Ricci flow equation, and this Ricci flow is called the *canonical form* of the Ricci soliton; one may refer to [3] for more details.

Perelman proved that on a closed manifold, any periodic orbit in $\mathrm{Met}(M)/\mathrm{Diff}$ must be static.

Theorem 3 (Perelman's no breather theorem) *A steady, shrinking, or expanding breather on a closed manifold is (the canonical form of) a steady, shrinking, or expanding gradient Ricci soliton, respectively. In particular, in the steady or expanding case, the breather is also Einstein. In dimension 3, this was originally proved by Ivey.*

We extend the no shrinking breather theorem to the complete noncompact case.

Theorem 4 *Every complete noncompact shrinking breather with bounded curvature is (the canonical form of) a shrinking gradient Ricci soliton.*

Our main technique is the $\mathcal{L}$-geometry, an important technique for the Ricci flow established by Perelman. In Sect. 2, we give a brief introduction to the $\mathcal{L}$-functional. In Sect. 3, we prove Theorem 4.

2 Perelman's $\mathcal{L}$-Geometry

The definitions and results in this section can be found in Perelman [8] and Naber [7]. We consider a *backward* Ricci flow $(M, g(\tau))$, $\tau \in [0, T]$, satisfying

$$\frac{\partial}{\partial \tau} g(\tau) = 2Ric(g(\tau)). \tag{1}$$

Let $\gamma(\tau) : [0, \tau_0] \to M$ be a smooth curve. The $\mathcal{L}$-functional of γ is defined by

$$\mathcal{L}(\gamma) := \int_0^{\tau_0} \sqrt{\tau} \Big(R(\gamma(\tau), \tau) + |\dot{\gamma}(\tau)|^2_{g(\tau)} \Big) d\tau. \tag{2}$$

The *reduced distance* between two space-time points $(x_0, 0)$, (x_1, τ_1), where $\tau_1 > 0$, is defined by

$$l_{(x_0,0)}(x_1, \tau_1) := \frac{1}{2\sqrt{\tau_1}} \inf_{\gamma} \mathcal{L}(\gamma), \tag{3}$$

where the inf is taken among all the (piecewise) smooth curves $\gamma : [0, \tau_1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(\tau_1) = x_1$. When regarded as a function of (x_1, τ_1), $l_{(x_0,0)}(\cdot, \cdot)$ is called the *reduced distance based at* $(x_0, 0)$. When the base point is understood, we also write $l_{(x_0,0)}$ as l. It is well known that the *reduced volume* based at $(x_0, 0)$

$$\mathcal{V}_{(x_0,0)}(\tau) := \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l_{(x_0,0)}(\cdot,\tau)} dg(\tau) \tag{4}$$

is monotonically decreasing in τ. We often write $\mathcal{V}_{(x_0,0)}(\tau)$ as $\mathcal{V}(\tau)$ for simplicity. We also remark here that the integrand $(4\pi\tau)^{-\frac{n}{2}} e^{-l}$ of the reduced volume is a subsolution to the conjugate heat equation

$$\frac{\partial}{\partial\tau} u - \Delta u + Ru = 0,$$

in the barrier sense or in the sense of distributions.

Now we consider an ancient solution $(M, g(\tau))$, where $\tau \in [0, \infty)$ is the backward time. The Type I condition is the following curvature bound.

Definition 5 An ancient solution $(M, g(\tau))$, where $\tau \in [0, \infty)$ is the backward time, is called Type I if there exists $C < \infty$, such that

$$|Rm|(\tau) \leq \frac{C}{\tau},$$

for every $\tau \in (0, \infty)$.

To ensure the existence of a smooth limit, the κ-noncollapsing condition is often required.

Definition 6 A backward Ricci flow is called κ-noncollapsed, where $\kappa > 0$, if for any space-time point (x, τ), any scale $r > 0$, whenever $|Rm| \leq r^{-2}$ on $B_{g(\tau)}(x, r) \times [\tau, \tau + r^2]$, it holds that $\mathrm{Vol}_{g(\tau)}(B_{g(\tau)}(x, r)) \geq \kappa r^n$.

We will use the following theorem of Naber [7].

Theorem 7 (Asymptotic shrinker for Type I ancient solution) *Let* $(M, g(\tau))$*, where* $\tau \in [0, \infty)$ *is the backward time, be a Type I* κ*-noncollapsed ancient solution to the Ricci flow. Fix* $x_0 \in M$*. Let* l *be the reduced distance based at* $(x_0, 0)$*. Let* $\{(x_i, \tau_i)\}_{i=1}^{\infty} \subset M \times (0, \infty)$ *be such that* $\tau_i \nearrow \infty$ *and*

$$\sup_{i=1}^{\infty} l(x_i, \tau_i) < \infty. \tag{5}$$

Then $\{(M, \tau_i^{-1} g(\tau\tau_i), (x_i, 1))_{\tau\in[1,2]}\}_{i=1}^{\infty}$ *converges, after possibly passing to a subsequence, to the canonical form of a shrinking gradient Ricci soliton.*

Remark 1 In Naber's original theorem, he fixes the base points $x_i \equiv x_0$. However, it is easy to observe from his proof that so long as (5) holds, all the estimates of l also hold in the same way as in his case. Hence one may apply the blow-down shrinker part of Theorem 2.1 in [7] to the sequence of space-time base points (x_i, τ_i) and the scaling factors τ_i^{-1}.

Remark 2 The estimates for l and the monotonicity formula for $\mathcal{V}$ in [7] do not depend on the noncollapsing condition. According to Hamilton [4], if the noncollapsing assumption is replaced by

$$\inf_{i=1}^{\infty} \mathrm{inj}_{\tau_i^{-1} g(\tau_i)}(x_i) > \delta, \tag{6}$$

where $\mathrm{inj}_g(x)$ stands for the injectivity radius of the metric g at the point x, and $\delta > 0$ is a constant, then the conclusion of Theorem 7 still holds.

3 Proof of the Main Theorem

Following the argument in Lu and Zheng [6], we construct a Type I ancient solution to the Ricci flow starting from a given shrinking breather. After scaling and translating in time, we consider the *backward* Ricci flow $(M, g_0(\tau))_{\tau\in[0,1]}$, where $g_0(\tau)$ satisfies (1), such that there exists $\alpha \in (0, 1)$ and a diffeomorphism $\phi : M \to M$, satisfying

$$\alpha g_0(1) = \phi^* g_0(0). \tag{7}$$

Furthermore, we let $C < \infty$ be the curvature bound, that is,

$$\sup_{M\times[0,1]} |Rm|(g(\tau)) \leq C. \tag{8}$$

For notational simplicity, we define

$$\tau_i = \sum_{j=0}^{i} \alpha^{-j},$$

where $i = 0, 1, 2, \ldots$ Evidently, $\tau_i \nearrow \infty$ since $\alpha \in (0, 1)$, and we can find a $C_0 < \infty$ depending only on α (for instance, one may let $C_0 = (1 - \alpha)^{-1}$) such that

$$\alpha^{-i} \leq \tau_i \leq C_0 \alpha^{-i}, \text{ for every } i \geq 0. \tag{9}$$

For each $i \geq 1$, we define a Ricci flow

$$g_i(\tau) := \alpha^{-i} (\phi^i)^* g_0 \left(\alpha^i (\tau - \tau_{i-1})\right), \text{ where } \tau \in [\tau_{i-1}, \tau_i]. \tag{10}$$

To see all these Ricci flows are well-concatenated, we apply (7) to observe that

$$\begin{aligned} g_1(\tau_0) &= \alpha^{-1}\phi^* g_0(0) = g_0(1), \\ g_i(\tau_{i-1}) &= \alpha^{-i}(\phi^i)^* g_0(0) = \alpha^{-(i-1)}(\phi^{i-1})^* g_0(1) \\ &= \alpha^{-(i-1)}(\phi^{i-1})^* g_0\left(\alpha^{i-1}(\tau_{i-1}-\tau_{i-2})\right) = g_{i-1}(\tau_{i-1}). \end{aligned}$$

Therefore, we define an ancient solution

$$g(\tau) = \begin{cases} g_0(\tau) \text{ for } \tau \in [0,1] \\ g_i(\tau) \text{ for } \tau \in [\tau_{i-1}, \tau_i] \text{ and } i \geq 1. \end{cases} \tag{11}$$

It then follows from the uniqueness theorem of Chen and Zhu [2] that the ancient solution $g(\tau)$ is smooth.

Now we proceed to show that $(M, g(\tau))_{\tau\in[0,\infty)}$, where $g(\tau)$ is defined in (11), is Type I. We need only to consider the case when $\tau \geq 1$. Let $i \geq 1$ be such that $\tau \in [\tau_{i-1}, \tau_i]$. Then

$$|Rm(g(\tau))| = |Rm(g_i(\tau))| \leq \alpha^i \sup_{M\times[0,1]} \left|Rm\left((\phi^i)^* g_0(\tau)\right)\right| \leq C\alpha^i,$$

where we have used (8), (10), and (11). Then we have

$$|Rm(g(\tau))| \leq C\alpha^i \leq \frac{C}{\tau}\tau_i\alpha^i \leq \frac{B}{\tau}, \tag{12}$$

where we have used (9), and $B = CC_0$ is independent of i.

With all these preparations, we are ready to prove our main theorem.

***Proof of Theorem* 4** Fix an arbitrary point $y \in M$ as the base point, and for each $i \geq 0$ we define

$$x_i = \phi^{-(i+1)}(y). \tag{13}$$

In Lu and Zheng [6], they made the assumption that the $\{x_i\}_{i=1}^{\infty}$ do not drift away to spatial infinity so that they may apply Theorem 4.1 in [1] to show that $\{(M, \tau_i^{-1} g(\tau\tau_i), (x_i, 1))_{\tau=[1,2]}\}_{i=1}^{\infty}$ converges, after passing to a subsequence, to the canonical form of a shrinking gradient Ricci soliton. Instead we will show that $l(x_i, \tau_i)$, where $i \geq 0$ and l is the reduced distance based at $(y, 0)$, is a bounded sequence. To see this, we let $\sigma : [0,1] \to M$ be a smooth curve such that $\sigma(0) = y$ and $\sigma(1) = x_0$. Let $A < \infty$ be such that

$$|\dot{\sigma}(\tau)|_{g_0(\tau)} \leq A, \text{ for all } \tau \in [0,1]. \tag{14}$$

For each $i \geq 0$, we define

$$\sigma_i(\tau) := \phi^{-(i+1)} \circ \sigma(\alpha^{i+1}(\tau - \tau_i)), \text{ where } \tau \in [\tau_i, \tau_{i+1}]. \tag{15}$$

We observe that these σ_i's and σ altogether define a continuous curve in M:

$$\begin{aligned}\sigma_0(\tau_0) &= \phi^{-1}\circ\sigma(0) = \phi^{-1}(y) = x_0 = \sigma(1),\\ \sigma_i(\tau_i) &= \phi^{-(i+1)}\circ\sigma(0) = \phi^{-i}\circ\sigma(1)\\ &= \phi^{-i}\circ\sigma(\alpha^i(\tau_i-\tau_{i-1})) = \sigma_{i-1}(\tau_i).\end{aligned}$$

We then define $\gamma_i : [0, \tau_{i+1}] \to M$, where $i \geq 0$, as

$$\gamma_i(\tau) := \begin{cases}\sigma(\tau) & \text{when } \tau\in[0,1],\\ \sigma_j(\tau) & \text{when } \tau\in[\tau_j,\tau_{j+1}] \text{ and } 0\leq j\leq i.\end{cases}$$

Evidently $\gamma_i(\tau)$ is piecewise smooth, and $\gamma_i(0) = y$, $\gamma_i(\tau_{i+1}) = \phi^{-(i+2)}(y) = x_{i+1}$. We compute for $i \geq 0$

$$\begin{aligned}\mathcal{L}(\gamma_i) &= \mathcal{L}(\sigma) + \sum_{j=0}^{i}\int_{\tau_j}^{\tau_{j+1}}\sqrt{\tau}\Big(R(\sigma_j(\tau),\tau) + |\dot{\sigma_j}(\tau)|^2_{g_{j+1}(\tau)}\Big)d\tau\\ &\leq D + \sum_{j=0}^{i}\int_{\tau_j}^{\tau_{j+1}}\sqrt{\tau}\Big(\frac{B}{\tau} + A\alpha^{j+1}\Big)d\tau,\end{aligned}$$

where in the last inequality we have used D, a constant independent of i, to represent $\mathcal{L}(\sigma)$, and we have used the Type I condition (12), the definition (15) of σ_j, and the assumption (14). Continuing the computation using (9), we have

$$\mathcal{L}(\gamma_i) \leq D + C_1\sum_{j=0}^{i}\alpha^{-\frac{j+1}{2}},$$

where C_1 is a constant independent of i. It follows from the definition (3) that

$$\begin{aligned}l(x_{i+1},\tau_{i+1}) &\leq \frac{1}{2\sqrt{\tau_{i+1}}}\mathcal{L}(\gamma_i)\\ &\leq \frac{1}{2}D\alpha^{\frac{i+1}{2}} + \frac{1}{2}C_1\sum_{j=0}^{i}\alpha^{\frac{j}{2}} \leq C_2 < \infty,\end{aligned}$$

where C_2 is a constant independent of i and where we have used $\alpha^{\frac{1}{2}} \in (0, 1)$.

Now we consider the sequence

$$\{(M, \tau_i^{-1}g(\tau\tau_i), (x_i, 1))_{\tau\in[1,\alpha^{-1}]}\}_{i=1}^{\infty}. \tag{16}$$

We observe that

$$\tau_i^{-1}g(\tau_i) = \tau_i^{-1}\alpha^{-(i+1)}\Big(\phi^{i+1}\Big)^*g_0(0),$$

where $\tau_i^{-1}\alpha^{-(i+1)}$ is bounded from above and below by constants independent of i, because of (9). Taking into account the definition (13) of x_i, we can use

$$\mathrm{inj}_{g_0(0)}(y) > 0$$

to verify the condition (6). It follows from Theorem 7 that (16) converges smoothly to the canonical form of a shrinking gradient Ricci soliton. Furthermore, since $(M, \tau_i^{-1}g(\tau_i), x_i)$ and $(M, g_0(0), y)$ differ only by bounded scaling constants and diffeomorphisms that preserve the base points, by the definition of Cheeger–Gromov convergence, such diffeomorphisms do not affect the limit. In other words, there exists a constant $C_3 > 0$ such that

$$(M, \tau_i^{-1}g(\tau_i), x_i) \to (M, C_3 g_0(0), y)$$

in the pointed smooth Cheeger–Gromov sense. Therefore, $(M, g_0(0), y)$ also has a shrinker structure up to scaling. It then follows from the backward uniqueness theorem of Kotschwar [5] that the shrinking breather $(M, g_0(\tau))_{\tau\in[0,1]}$ is the canonical form of a shrinking gradient Ricci soliton. □

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