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Lipschitz Homotopy Convergence of Alexandrov Spaces

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Abstract

We introduce the notion of good coverings of metric spaces, and prove that if a metric space admits a good covering, then it has the same locally Lipschitz homotopy type as the nerve complex of the covering. As an application, we obtain a Lipschitz homotopy stability result for a moduli space of compact Alexandrov spaces without collapsing.

Keywords Alexandrov space · Lipschitz homotopy · Good covering

Mathematics Subject Classification 53C20

1 Introduction

For given n and $D, v_0 > 0$, let $\mathcal{A}(n, D, v_0)$ denote the set of isometry classes of compact n-dimensional Alexandrov spaces with curvature ≥ -1, diameter $\leq D$, and volume $\geq v_0$. Perelman's stability theorem has played important roles in the geometry of Alexandrov spaces with curvature bounded below. This theorem implies that the set of homeomorphism classes of spaces in $\mathcal{A}(n, D, v_0)$ is finite. Although he also claimed the Lipschitz version of the stability theorem is true, it has not yet been appeared.

We formulate our results for general metric spaces having good coverings. We say that a locally finite open covering of a metric space is *good* if any non-empty intersection in the covering has a Lipschitz strong deformation retraction to a point (see Definition 2.6 for the detail).

We use a symbol $\tau(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ to denote a positive continuous function satisfying $\lim_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k \to 0} \tau(\epsilon_1, \epsilon_2, \ldots, \epsilon_k) = 0$.

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The main theorems of the present paper are stated as follows.

Theorem 1.1 *Let M be a σ-compact metric space having a good covering $\mathcal{U}$. Then M has the same locally Lipschitz homotopy type as the nerve of $\mathcal{U}$.*

We remark that in Theorem 1.1 if M is compact, it has the same Lipschitz homotopy type as the nerve of $\mathcal{U}$.

Theorem 1.2 *There exists a positive number $\epsilon = \epsilon_n(D, v_0)$ such that if $M, M' \in \mathcal{A}(n, D, v_0)$ have the Gromov–Hausdorff distance $d_{GH}(M, M') < \epsilon$, then M has the same Lipschitz homotopy type as M'. More precisely if $\theta : M \to M'$ is an ϵ-approximation, then there is a Lipschitz homotopy equivalence $f : M \to M'$ satisfying that $|f(x), \theta(x)| < \tau(\epsilon)$ for all $x \in M$.*

As a direct consequence of Theorem 1.2, we have.

Corollary 1.3 *The set of Lipschitz homotopy types of Alexandrov spaces in $\mathcal{A}(n, D, v_0)$ is finite.*

This provides a weaker version of "the finiteness of bi-Lipschitz homeomorphism classes" mentioned above.

In Corollary 1.3 we prove that every M and M' in $\mathcal{A}(n, D, v_0)$ with small Gromov–Hausdorff distance have the same Lipschitz homotopy type through isomorphic nerves of some good coverings on them. However it was shown in [1] and [14] that there is an almost isometric map from a closed domain of an almost regular part of M to a closed domain of an almost regular part of M'. John Lott asked us if one can extend such an almost isometric map to a Lipschitz homotopy equivalence $M \to M'$. The answer is yes:

Theorem 1.4 *Let δ be a sufficiently small positive number with respect to n. For given compact n-dimensional Alexandrov space M with curvature ≥ -1 and a closed domain D in the δ-regular part of M, there exists an $\epsilon = \epsilon_{M,D} > 0$ satisfying the following: Let M' be a compact n-dimensional Alexandrov space with curvature ≥ -1 and with $d_{GH}(M, M') < \epsilon$, and let $\theta : M \to M'$ be an ϵ-approximation. Then there is a Lipschitz homotopy equivalence $f : M \to M'$ such that*

(1) *the restriction of f to D is $\tau(\epsilon)$-almost isometric;*
(2) *$|f(x), \theta(x)| < \tau(\epsilon)$ for all $x \in M$.*

Theorem 1.1 has an application to the set of homotopies of mapping between two metric spaces. Let $[X, Y]$ denote the set of all homotopy classes of continuous maps from X to Y, and $[X, Y]_{\text{loc-Lip}}$ the set of all locally Lipschitz homotopy classes of locally Lipschitz maps from X to Y. In Corollary 1.3 of [7], we proved that if K is a simplicial complex and Y is a locally Lipschitz contractible metric space, then the natural map $[K, Y]_{\text{loc-Lip}} \to [K, Y]$ is bijective.

Using Theorem 1.1 and Corollary 1.3 of [7], we obtain the following.

Corollary 1.5 *Let X be a σ-compact metric space admitting a good covering, and Y a locally Lipschitz contractible metric space. Then, the natural map $[X, Y]_{\text{loc-Lip}} \to [X, Y]$ is bijective.*

In particular, every continuous map from X to Y is homotopic to a locally Lipschitz one.

As an immediate consequence of Corollary 1.5, we have the following for instance.

Corollary 1.6 *Let X be a finite-dimensional compact Alexandrov space with curvature bounded below, and Y a locally Lipschitz contractible metric space. Then every continuous map from X to Y is homotopic to a Lipschitz map.*

Organization The rest of the present paper consists of Sects. 2–6. In Sect. 2, we recall the notions of Lipschitz homotopies, Alexandrov spaces and good coverings needed in this paper. Sections 3 and 4 are devoted to prove Theorem 1.1, where we employ a basic strategy in the proof of Theorem 9.4.15 of [13]. Since the argument in [13] is only topological, we need to proceed in the category of (locally) Lipschitz maps. In Sect. 3, we consider the case when metric spaces are compact, and deal with the non-compact case in Sect. 4. Using Theorem 1.1 and a stability result of nerves of good coverings in [9], we prove Theorem 1.2 and Corollaries 1.3 and 1.5 in Sect. 5. In Sect. 6, we prove Theorem 1.4 by developing a gluing method in [1].

2 Preliminaries

In this paper, the distance between two points x, y in a metric space is denoted by $|xy|$ or $|x, y|$. The open metric ball around x of radius r is denoted by $B(x, r)$. To prove the main result, we prepare several terminologies.

2.1 Homotopies in the Category of (Locally) Lipschitz Maps

Let X and Y be metric spaces.

Definition 2.1 We say that a subset A of X is a *locally Lipschitz strong deformation retract* of X if there is a Lipschitz map $F : X \times [0, 1] \to X$ such that $F(x, 0) = x$, $F(x, 1) \in A$ and $F(a, t) = a$ for any $x \in X$, $a \in A$ and $t \in [0, 1]$. Then, the map F is called a locally Lipschitz strong deformation retraction of X to A.

Definition 2.2 Two maps $h_0, h_1 : X \to Y$ are said to be *locally Lipschitz homotopic* if there exists a locally Lipschitz map $h : X \times [0, 1] \to Y$ such that $h_i = h(\cdot, i)$ $(i = 0, 1)$.

We say that X and Y are *locally Lipschitz homotopy equivalent* if there are locally Lipschitz maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are locally Lipschitz homotopic to 1_X and 1_Y, respectively. In this case, f and g are called *locally Lipschitz homotopy equivalences*.

In the above definition, if a locally Lipschitz homotopy can be chosen to be a Lipschitz one, then it is called a Lipschitz homotopy. For other notions appeared in Definitions 2.1 and 2.2, we use similar terminologies. From definition, if Y is a (locally) Lipschitz strong deformation retract of X, then X and Y are (locally) Lipschitz homotopy equivalent.

Let X be an unbounded metric space, and $f : X \to X$ a Lipschitz map whose image is a bounded subset. Then it follows from definition that f is not Lipschitz homotopic to 1_X. In particular if a metric space X is Lipschitz homotopy equivalent to a bounded metric space, then X is also bounded.

2.2 The Gromov–Hausdorff Distance

A map $f : X \to Y$ between metric spaces is called an ϵ-*approximation* if it satisfies

- $||f(x), f(y)| - |x, y|| < \epsilon$ for all $x, y \in X$;
- for any $y \in Y$, there is an $x \in X$ such that $|f(x), y| < \epsilon$.

The *Gromov–Hausdorff distance* $d_{\mathrm{GH}}(X, Y)$ between X and Y is defined as

$$d_{\mathrm{GH}}(X, Y) := \inf \left\{ \epsilon > 0 \,\middle|\, \begin{array}{l} \text{there exist } \epsilon\text{-approximations} \\ X \to Y \text{ and } Y \to X \end{array} \right\}.$$

A bijective map $f : X \to Y$ is called an ϵ-*almost isometry* if both f and f^{-1} are Lipschitz with Lipschitz constants at most $1 + \epsilon$.

2.3 Alexandrov Spaces and Good Coverings

We briefly recall the definition of Alexandrov spaces and their properties. For details, we refer to [1]. A complete metric space X is called an *Alexandrov space* if it is a length space and for any $p \in X$, there exist $\kappa \in \mathbb{R}$ and a neighborhood U of p such that for any $x, y, z \in U \setminus \{p\}$, we have

$$\tilde{\angle}_\kappa xpy + \tilde{\angle}_\kappa ypz + \tilde{\angle}_\kappa zpx \le 2\pi,$$

where $\tilde{\angle}_\kappa xpy$ is defined as the angle of a comparison triangle $\tilde{\Delta}xpy = \Delta \tilde{x} \tilde{p} \tilde{y}$ at $\tilde{p}$ in the complete simply connected surface M_κ of constant curvature κ. It is known that the Hausdorff dimension of X coincides with its Lebesgue covering dimension [1,12], which is called the dimension of X. When κ is chosen to be independent of the choice of points $p \in X$, we say that X is of curvature $\ge \kappa$. When X is of dimension n, its volume is measured by the n-dimensional Hausdorff measure.

Complete Riemannian manifolds and orbifolds, the quotient spaces of complete Riemannian manifolds by isometric actions, and the Gromov–Hausdorff limits of sequences of complete Riemannian manifolds with a uniform lower sectional curvature bound are typical examples of Alexandrov spaces.

For $m \in \mathbb{N}$ and $\delta > 0$, a point p in an Alexandrov space X of curvature $\ge \kappa$ is called (m, δ)-*strained* if there exist pairs of points $\{(a_i, b_i)\}_{i=1}^m$ such that

$$\begin{aligned} \tilde{\angle}_\kappa a_i p b_i &> \pi - \delta, & \tilde{\angle}_\kappa a_i p b_j &> \pi/2 - \delta, \\ \tilde{\angle}_\kappa a_i p a_j &> \pi/2 - \delta, & \tilde{\angle}_\kappa b_i p b_j &> \pi/2 - \delta \end{aligned}$$

for all $1 \le i \ne j \le m$. The set $\{(a_i, b_i)\}$ is called an (m, δ)-*strainer at* p. The *length* ℓ of the strainer $\{(a_i, b_i)\}$ at p is defined as

$$\ell := \min\{|p, a_i|, |p, b_i| \mid 1 \le i \le m\}.$$

From now on, we shall use the convention

$$\tilde{\angle} xyz := \tilde{\angle}_\kappa xyz,$$

when the curvature lower bound is understood.

In an n-dimensional Alexandrov space X, a point $p \in X$ is called δ-*regular* if it is (n, δ)-strained and $\delta \ll 1/n$. The set of all δ-regular points is called a δ-*regular part*, and is denoted by $\mathcal{R}_X(\delta)$.

In the present paper, we are concerned with a moduli space of Alexandrov spaces with curvature bounded from below by a uniform constant, say κ. Rescaling the metric, we assume $\kappa = -1$ without loss of generality. Thus we deal with the moduli space $\mathcal{A}(n, D, v_0)$ as explained in the introduction.

Theorem 2.3 [1] *Suppose that X is n-dimensional and δ is sufficiently small with $\delta \ll 1/n$. If p is (n, δ)-strained by an (n, δ)-strainer $\{(a_i, b_i)\}_{i=1}^n$ with length ℓ, then the map φ defined by*

$$\varphi(x) := (|a_1, x|, \ldots, |a_n, x|)$$

is a $\tau(\delta, \sigma/\ell)$-almost isometry from $B(p, \sigma)$ to an open subset of $\mathbb{R}^n$.

We will use Theorem 2.3 in Sect. 6.

Perelman proved the following theorem, called the topological stability theorem.

Theorem 2.4 ([10], see also [5]) *Let $D > 0$ and $n \in \mathbb{N}$ be fixed. Let M_j be a sequence of n-dimensional compact Alexandrov spaces of diameter $\leq D$ and curvature ≥ -1 which converges to an n-dimensional compact Alexandrov space M as $j \to \infty$. Then, there is j_0 such that M_j and M are homeomorphic for all $j \geq j_0$.*

In particular, the set of homeomorphism types of spaces in the moduli space $\mathcal{A}(n, D, v_0)$ is finite.

The last statement follows from the fact that $\mathcal{A}(n, D, v_0)$ is compact with respect to the Gromov–Hausdorff distance.

We shall define a new notion of good coverings for *metric spaces*. In [9], we have proved that any Alexandrov space has a covering with geometrically and topologically good properties:

Theorem 2.5 [9] *For any open covering of a finite-dimensional Alexandrov space X, there is an open covering $\mathcal{U}$ of X which is a refinement of the original covering, satisfying the following: Let $V = \bigcap_{i=0}^k U_{j_i}$ be any non-empty intersection of finitely many elements of $\mathcal{U}$. Then*

(1) *V is convex in the sense that every minimal geodesic joining any two points in V is contained in V;*
(2) *there exists a point $p \in V$ such that (V, p) is homeomorphic to a cone (C, v), where v is the apex of C;*
(3) *there exists a Lipschitz strong deformation retraction $h : V \times [0, 1] \to V$ of V to the point p as above (2) such that $|h_t(x), p|$ is non-increasing in $t \in [0, 1]$ for each $x \in V$.*

By extracting some fundamental properties that the covering $\mathcal{U}$ in Theorem 2.5 posses, we define good coverings for general metric spaces as follows.

Definition 2.6 A locally finite open covering $\mathcal{U} = \{U_j\}_{j \in J}$ of a metric space X is *good* if it satisfies the following:

(1) the closure of each element of $\mathcal{U}$ is compact;
(2) every non-empty intersection $\bigcap_{i=0}^{k} U_{j_i}$ of finitely many elements of $\mathcal{U}$ has a Lipschitz strong deformation retraction to a point p.

Any such a point p as in (2) is called a *center* of $\bigcap_{i=0}^{k} U_{j_i}$. We also say that $\mathcal{U}$ is a good r-cover if diam $(U_j) < r$ for any $j \in J$.

3 Proof of Theorem 1.1 (Compact Case)

In this section, we prove Theorem 1.1 in the case when M is compact. We deal with the non-compact case in the next section. For the proof of Theorem 1.1, we employ a basic strategy in the proof of Theorem 9.4.15 of [13], where it is proved that if a topological space has a locally finite covering all of whose non-empty intersection are contractible, then it has the same homotopy type as the nerve of the covering. Since the argument there is only topological, we have to proceed in the category of (locally) Lipschitz maps.

Setting and Strategy

Let M be a compact metric space having a good cover $\mathcal{U} = \{U_j\}_{j \in J}$. Note that J is a finite set since $\mathcal{U}$ is locally finite and M is compact. Let $J = \{1, 2, \ldots, N\}$. Let $\mathcal{N}_{\mathcal{U}}$ denote the *nerve* of $\mathcal{U}$, which is a simplicial complex with the set of vertexes $\{U \in \mathcal{U} \mid U \neq \emptyset\}$, and whose k-simplices are unordered $(k + 1)$-tuples $\langle U_{j_0}, U_{j_1}, \ldots, U_{j_k} \rangle$ of elements in $\mathcal{U}$ so that $\bigcap_{i=0}^{k} U_{j_i} \neq \emptyset$. We denote by $|\mathcal{N}_{\mathcal{U}}| \subset \mathbb{R}^N$ its *geometric realization*, where we assume that jth vertex $v_j := \langle U_j \rangle$ of $\mathcal{N}_{\mathcal{U}}$ is given by

$$v_j = (0, \ldots, \overset{j}{1}, \ldots, 0) \in \mathbb{R}^N.$$

Let $\theta : V(\mathcal{N}_{\mathcal{U}}) \to [0, 1]$ be a function defined on the set of vertices of $\mathcal{N}_{\mathcal{U}}$ satisfying

(1) $\sum_{j \in J} \theta(v_j) = 1$;
(2) supp(θ) defines a simplex σ_θ of $\mathcal{N}_{\mathcal{U}}$.

Since θ defines the point $\sum_{j \in J} \theta(v_j) v_j$ of σ_θ, it can be considered as an element of $|\mathcal{N}_{\mathcal{U}}|$. From now on, we identify a function θ satisfying (1), (2) with an element $\sum_{j \in J} \theta(v_j) v_j \in |\mathcal{N}_{\mathcal{U}}|$. That is,

$$|\mathcal{N}_{\mathcal{U}}| = \left\{ \theta \equiv \sum_{j \in J} \theta(v_j) v_j \,\middle|\, \theta \text{ satisfies above (1), (2)} \right\}.$$

This will be useful later on (see the proof of Lemma 3.1, for instance). For any subset $A \subset J$, we put

$$U_A := \bigcap_{j \in A} U_j.$$

Each simplex $\sigma \in \mathcal{N}_{\mathcal{U}}$ defines a subset $A(\sigma) \subset J$, and we also use the symbol

$$U_\sigma = U_{A(\sigma)}.$$

By definition, there is a Lipschitz contraction $\varphi : U_\sigma \times [0, 1] \to U_\sigma$ to a point p_σ of U_σ.

We define a function f_j on M by

$$f_j(x) = \frac{|x, U_j^c|}{|x, U_j^c| + |x, p_j|},$$

where U_j^c denotes the complement of U_j and p_j is a center of U_j to which U_j has a Lipschitz strong deformation retraction. Since $|x, U_j^c| + |x, p_j| \geq |p_j, U_j^c|/2 > 0$, it is straightforward to check that f_j is Lipschitz. Set

$$\xi_j(x) = \frac{f_j(x)}{\sum_i f_i(x)}.$$

Then $\{\xi_j\}_{j \in J}$ defines a partition of unity dominated with $\mathcal{U}$ satisfying

(1) $\mathrm{supp}(\xi_j) = \bar{U}_j$;
(2) each ξ_j is Lipschitz;
(3) $\sum_j \xi_j = 1$.

The polyhedron $|\mathcal{N}_{\mathcal{U}}|$ has the distance induced from the metric of $\mathbb{R}^N$ defined as

$$d(x, y) = \max_{1 \leq i \leq N} |x_i - y_i|.$$

In the rest of this section, we are going to construct metric spaces $\mathcal{D}(\mathcal{U})$ and $\mathcal{M}(p)$ together with natural bi-Lipschitz embeddings

$$\begin{array}{ccc}
M & \xrightarrow{\tau} & \mathcal{D}(\mathcal{U}) \\
& & \downarrow{\iota} \\
|\mathcal{N}_{\mathcal{U}}| & \xrightarrow{\Psi} & \mathcal{M}(p)
\end{array} \qquad (3.1)$$

and prove that their images are Lipschitz strong deformation retracts of the target spaces. This strategy comes from [13]. The most complicated part is a construction of a Lipschitz strong deformation retraction from $\mathcal{M}(p)$ to $\iota(\mathcal{D}(\mathcal{U}))$, which will be

done simplex-wisely by means of provided Lipschitz strong deformation retractions of U_σ's to their centers.

We divide the proof into three steps.

Step 1 We consider the following subspace of the product metric space $|\mathcal{N}_\mathcal{U}| \times M$ defined as

$$\mathcal{D}(\mathcal{U}) := \{(\theta, x) \in |\mathcal{N}_\mathcal{U}| \times M \mid x \in U_{\operatorname{supp}\theta}\}.$$

Let $p : \mathcal{D}(\mathcal{U}) \to |\mathcal{N}_\mathcal{U}|$ and $q : \mathcal{D}(\mathcal{U}) \to M$ be the projections:

$$p(\theta, x) = \theta, \quad q(\theta, x) = x.$$

Lemma 3.1 *There exists a Lipschitz map $\tau : M \to \mathcal{D}(\mathcal{U})$ such that*

(1) *τ is a section of the map q (i.e., $q \circ \tau = 1_M$);*
(2) *$\tau(M)$ is a Lipschitz strong deformation retract of $\mathcal{D}(\mathcal{U})$.*

Proof Define $\tau : M \to \mathcal{D}(\mathcal{U})$ by

$$\tau(x) = (\Theta(x), x),$$

where $\Theta(x) \in |\mathcal{N}_\mathcal{U}|$ is defined by $\Theta(x)(v_j) = \xi_j(x)$, $j \in J$. Obviously τ and $\Theta : M \to |\mathcal{N}_\mathcal{U}|$ are Lipschitz. For any $x \in M$, let $s(x) := \{j \in J \mid x \in U_j\}$, which forms a simplex of $\mathcal{N}_\mathcal{U}$. For any $(\theta, x) \in \mathcal{D}(\mathcal{U})$, supp$(\theta)$ defines a face of $s(x)$. Thus we can define the Lipschitz map $H : \mathcal{D}(\mathcal{U}) \times [0, 1] \to \mathcal{D}(\mathcal{U})$ by

$$H(\theta, x, s) = (s\Theta(x) + (1 - s)\theta, x) \tag{3.2}$$

satisfying $H(\theta, x, 0) = 1_{\mathcal{D}(\mathcal{U})}(\theta, x)$, $H(\theta, x, 1) = (\Theta(x), x) = \tau(x)$ and $H(\tau(x), s) = \tau(x)$ for every $s \in [0, 1]$. Obviously H is Lipschitz. This completes the proof. $\square$

Corollary 3.2 *M has the same Lipschitz homotopy type as $\mathcal{D}(\mathcal{U})$.*

Proof Let $q' : \tau(M) \to M$ be the restriction of q to $\tau(M)$. Since τ is Lipschitz and q' is 1-Lipschitz with $q' \circ \tau = 1_M$ and $\tau \circ q' = 1_{\tau(M)}$, M and $\tau(M)$ are bi-Lipschitz homeomorphic to each other. The conclusion follows from Lemma 3.1. $\square$

Step 2 For $L > 0$ (see (3.3) for the proper choice of L), consider the mapping cylinder of p:

$$\mathcal{M}(p) := \mathcal{D}(\mathcal{U}) \times [0, L] \amalg |\mathcal{N}_\mathcal{U}|/(\theta, x, L) \sim \theta.$$

Recall that

$$\mathcal{D}(\mathcal{U}) = \bigcup_{\sigma \in \mathcal{N}_\mathcal{U}} \sigma \times U_\sigma \subset |\mathcal{N}_\mathcal{U}| \times M.$$

The canonical correspondence $[(\theta, x, t)] \to (\theta, [(x, t)])$ gives rise to the identification

$$\mathcal{M}(p) = \bigcup_{\sigma \in \mathcal{N}_\mathcal{U}} \sigma \times K(U_\sigma) \subset |\mathcal{N}_\mathcal{U}| \times K(M),$$

where $K(V) = V \times [0, L]/V \times L$ denotes the Euclidean cone. From now on, we consider the metric of $\mathcal{M}(p)$ induced from that of the product metric $|\mathcal{N}_\mathcal{U}| \times K(M)$, where the metric of the Euclidean cone $K(M) = M \times [0, L]/M \times L$ is defined as

$$|[x, t], [x', t']|^2$$
$$= (L - t)^2 + (L - t')^2 - 2(L - t)(L - t') \cos(\min\{\pi, |x, x'|\}),$$

for $[x, t], [x', t'] \in K(M)$.

Note that there is a natural isometric embedding $\Psi : |\mathcal{N}_\mathcal{U}| \to \mathcal{M}(p)$ defined by

$$\Psi(\theta) = (\theta, [x, L]) = [\theta] = (\theta, v_M),$$

where v_M denotes the vertex of $K(M)$.

Lemma 3.3 $|\mathcal{N}_\mathcal{U}|$ *is a Lipschitz strong deformation retract of* $\mathcal{M}(p)$.

Proof Define $\Psi' : \mathcal{M}(p) \to |\mathcal{N}_\mathcal{U}|$ by

$$\Psi'(\theta, [x, t]) = \theta.$$

Since

$$|\Psi'(\theta, [x, t]), \Psi'(\theta', [x', t'])|$$
$$= |\theta, \theta'|$$
$$\leq \sqrt{|\theta, \theta'|^2 + |[x, t], [x', t']|^2}$$
$$= |(\theta, [x, t]), (\theta', [x', t'])|,$$

and since $|\Psi(\theta_1), \Psi(\theta_2)| = |\theta_1, \theta_2|$, both Ψ and Ψ' are 1-Lipschitz.

Note that $\Psi \circ \Psi'(\theta, [x, t]) = [\theta] = (\theta, [x, L])$ and $\Psi' \circ \Psi = 1_{|\mathcal{N}_\mathcal{U}|}$. Define $F : \mathcal{M}(p) \times [0, 1] \to \mathcal{M}(p)$ by

$$F(\theta, [x, t], s) = (\theta, [x, (1 - s)t + sL]).$$

Then $F_0 = 1_{\mathcal{M}(p)}$ and $F_1 = \Psi \circ \Psi'$. We show that F is Lipschitz. Since it suffices to prove that it is locally Lipschitz, let us assume that $(\theta, [x, t], s)$ and $(\theta', [x', t'], s')$ are close to each other. We then have

$$|F(\theta, [x, t], s), F(\theta', [x', t'], s')|^2 = |\theta, \theta'|^2 + |[x, u], [x', u']|^2,$$

where we set $u = (1 - s)t + sL$, $u' = (1 - s')t' + s'L$, and

$$|[x, u], [x', u']|^2$$
$$= (L - u)^2 + (L - u')^2 - 2(L - u)(L - u') \cos |x, x'|$$
$$\leq (u - u')^2 + (L - u)(L - u')|x, x'|^2.$$

Since $|u - u'| \leq (1 - s')|t - t'| + (L - t)|s - s'|$, we have

$$|F(\theta, [x, t], s), F(\theta', [x', t'], s')|^2$$
$$\leq |\theta, \theta'|^2 + 2|t - t'|^2 + 2|s - s'|^2 + (L - u)(L - u')|x, x'|^2.$$

Similarly we have

$$|(\theta, [x, t], s), (\theta', [x', t'], s')|^2$$
$$\geq |\theta, \theta'|^2 + |t - t'|^2 + \frac{1}{2}(L - t)(L - t')|x, x'|^2 + |s - s'|^2.$$

Combining those inequalities, we conclude that F is Lipschitz. $\qquad\square$

Step 3 Let us define $\iota : \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{U}) \times 0 \subset \mathcal{M}(p)$ by $\iota(\theta, x) = (\theta, x, 0)$. In this last step, we prove

Proposition 3.4 *There exists a Lipschitz strong deformation retraction* $\Phi : \mathcal{M}(p) \times [0, 1] \to \mathcal{M}(p)$ *of* $\mathcal{M}(p)$ *to* $\mathcal{D}(\mathcal{U}) \times 0$.

The compact case of Theorem 1.1 now follows from Corollary 3.2, Lemma 3.3 and Proposition 3.4.

Let

$$L > 6. \tag{3.3}$$

Let k_0 denote the dimension of $\mathcal{N}$. For each $0 \leq k \leq k_0$, let $\mathcal{N}^{(k)}$ denote the k-skeleton of $\mathcal{N}_{\mathcal{U}}$, and $\mathcal{D}^k := p^{-1}(|\mathcal{N}^{(k)}|)$ and $p^k := p|_{\mathcal{D}^k} : \mathcal{D}^k \to \mathcal{N}^{(k)}$. Let $\mathcal{M}(p^k)$ denote the mapping cone of p^k:

$$\mathcal{M}(p^k) := \mathcal{D}^k \times [0, L] \amalg |\mathcal{N}^{(k)}|/(\theta, x, L) \sim \theta.$$

As before, we have

$$\mathcal{D}^k = \bigcup_{\sigma \in \mathcal{N}^{(k)}} \sigma \times U_\sigma,$$
$$\mathcal{M}(p^k) = \bigcup_{\sigma \in \mathcal{N}^{(k)}} \sigma \times K(U_\sigma) \subset |\mathcal{N}_{\mathcal{U}}| \times K(M).$$

Lemma 3.5 *For each k, There exists a Lipschitz strong deformation retraction* $\Phi^k : \mathcal{M}(p^k) \times [0, 1] \to \mathcal{M}(p^k)$ *of* $\mathcal{M}(p^k)$ *to* $\mathcal{D}^k \times 0 \bigcup \mathcal{M}(p^{k-1})$.

The construction of the Lipschitz strong deformation retraction Φ^k in Lemma 3.5 will be done simplex-wisely. This is based on the following sublemma.

Sublemma 3.6 *For each k-simplex $\sigma \in \mathcal{N}_\mathcal{U}$, there exists a Lipschitz strong deformation retraction of $\sigma \times K(U_\sigma)$ to $(\sigma \times U_\sigma \times 0) \bigcup \partial\sigma \times K(U_\sigma)$.*

Since $\partial\sigma \times K(U_\sigma) \subset \mathcal{M}(p^{k-1})$, applying Sublemma 3.6 to each k-simplex of $\mathcal{N}_\mathcal{U}$, we obtain Lemma 3.5.

By using Lemma 3.5 repeatedly, we have a finite sequence of Lipschitz retractions:

$$\mathcal{M}(p) = \mathcal{M}(p^{k_0}) \longrightarrow \mathcal{D}(\mathcal{U}) \times 0 \bigcup \mathcal{M}(p^{k_0-1}) \longrightarrow \cdots$$
$$\longrightarrow D(\mathcal{U}) \times 0 \bigcup \mathcal{M}(p^0) \longrightarrow \mathcal{D}(\mathcal{U}) \times 0. \tag{3.4}$$

From (3.4), we conclude that $\mathcal{D}(\mathcal{U}) \times 0$ is a Lipschitz strong deformation retract of $\mathcal{M}(p)$. Thus all we have to do is to prove Sublemma 3.6.

Remark 3.7 From (3.4), one might think that $k_0 = \dim \mathcal{N}_\mathcal{U} < \infty$ is essential in the argument below. However, we can generalize the argument of this section to the general case of $\dim \mathcal{N}_\mathcal{U} = \infty$. This will be verified in Sect. 4.

The following is the important first step in the proof of Sublemma 3.6, which is the case of $k = 0$.

Claim 3.8 *Let U be an element of $\mathcal{U}$. Then there exists a Lipschitz strong deformation retraction $K(U) \times [0, 1] \to K(U)$ of $K(U)$ to $U \times 0$.*

Proof Let $\varphi : U \times [0, L] \to U$ be a Lipschitz strong deformation retraction to $p \in U$. We may assume that $\varphi(x, t) = p$ for all $t \geq L/2$ and $x \in U$. Define the retraction $r : K(U) \to U \times 0 \subset K(U)$ by

$$r([x, t]) := [\varphi(x, t), 0].$$

First we show that r is Lipschitz. Again we way assume that $[x, t]$ and $[x', t']$ are sufficiently close. Note that

$$|r([x, t]), r([x', t'])| \leq |[\varphi(x, t), 0], [\varphi(x', t), 0]| + |[\varphi(x', t), 0], [\varphi(x', t'), 0]|$$
$$\leq L|\varphi(x, t), \varphi(x', t)| + L|\varphi(x', t), \varphi(x', t')|$$
$$\leq CL|x, x'| + CL|t - t'|.$$

From here on, we use the symbols $C, C_1, C_2, \ldots$ to denote some uniform positive constants.

If both t and t' are greater than $L/2$, then $\varphi(x, t) = \varphi(x', t') = p$. Therefore we may assume that $t, t' \leq L/2$. Then we have

$$|[x, t], [x', t']|^2 \geq (t - t')^2 + (L - t)(L - t')|x, x'|^2/2$$
$$\geq (t - t')^2 + (L^2/8)|x, x'|^2.$$

Combining the two inequalities, we have

$$|r([x, t]), r([x', t'])| \leq CL|x, x'| + C_1 L|t - t'|$$
$$\leq C(1 + L)|[x, t], [x', t']|.$$

Now let $g : [0, L] \times [0, 1] \to [0, 1]$ be a Lipschitz function such that

- $g(t, s) = 1$ on $[0, L] \times [0, 1/3]$;
- $g(t, 1) = 0$ for all $0 \leq t \leq L$,

and define $\Phi : K(U) \times [0, 1] \to K(U)$ by

$$\Phi([x, t], s) = [\varphi(x, st), g(t, s)t].$$

Note that $\Phi([x, t], 0) = [x, t]$, $\Phi([x, t], 1) = r([x, t])$.

To show that Φ is Lipschitz, let $([x, t], s)$ and $([x', t'], s')$ be elements of $K(U) \times [0, 1]$ sufficiently close to each other. By triangle inequalities, it suffices to show the following:

(1) $|\Phi([x, t], s), \Phi([x', t], s)| \leq C_1|[x, t], [x', t]|$;
(2) $|\Phi([x, t], s), \Phi([x, t'], s)| \leq C_2 L|[x, t], [x, t']|$;
(3) $|\Phi([x, t], s), \Phi([x, t], s')| \leq C_3 L(1 + L)|s - s'|$.

We show (1)

$$|\Phi([x, t], s), \Phi([x', t], s)| = |[\varphi(x, st), g(t, s)t], [\varphi(x', st), g(t, s)t]|$$
$$\leq |L - g(t, s)t||\varphi(x, st), \varphi(x', st)|$$
$$\leq |L - g(t, s)t|C|x, x'|$$
$$\leq |L - g(t, s)t||x, x'|.$$

If $s \leq 1/3$, then $|L - g(t, s)t||x, x'| = (L - t)|x, x'| \leq 2|[x, t], [x', t]|$. If $s \geq 1/3$ and $t \geq L/2$, then $ts \geq L_0$, and therefore $|\Phi([x, t], s), \Phi([x', t], s)| = 0$. If $s \geq 1/3$ and $t \leq L/2$, then $|L - g(t, s)t||x, x'| \leq L|x, x'| \leq 2(L-t)|x, x'| \leq 3|[x, t], [x', t]|$.

We show (2)

$$|\Phi([x, t], s), \Phi([x, t'], s)|^2 = |[\varphi(x, st), g(t, s)t], [\varphi(x, st'), g(t', s)t']|^2$$
$$\leq |g(t, s)t - g(t', s)t'|^2 + L^2|\varphi(x, st), \varphi(x, st')|^2,$$
$$\leq |g(t, s)t - g(t', s)t'|^2 + L^2 C_1|st - st'|^2,$$

where obviously

$$|g(t, s)t - g(t', s)t'| \leq |g(t, s)t - g(t', s)t| + |g(t', s)t - g(t', s)t'|$$
$$\leq C(1 + L)|t - t'|.$$

Thus we have $|\Phi([x, t], s), \Phi([x, t'], s)| \leq C_2(1 + L)|[x, t], [x, t']|$.

We show (3)

$$
\begin{aligned}
|\Phi([x,t],s),\Phi([x,t],s')|^2 &= |[\varphi(x,st),g(t,s)t],[\varphi(x,s't),g(t,s')t]|^2 \\
&\le |g(t,s)t - g(t,s')t|^2 + C_1 L^2 |st - s't|^2 \\
&\le C_2 L^2 |s-s'|^2 + C_3 L^4 |s-s'|^2 \\
&\le C_4 L^2 (1+L^2)|s-s'|^2.
\end{aligned}
\tag{3.5}
$$

This shows that Φ is Lipschitz, and together with (3.5) this completes the proof of Claim 3.8. $\qquad\qquad\square$

Next we consider the general case.

Proof of Sublemma 3.6 Let σ be any simplex of $\mathcal{N}$. Note that $\sigma \times 0 \cup \partial\sigma \times [0,L]$ is a Lipschitz strong deformation retract of $\sigma \times [0,L]$. Let $r : \sigma \times [0,L] \to \sigma \times 0 \cup \partial\sigma \times [0,L]$ be a Lipschitz strong deformation retraction defined by the radial projection from the point $(x^*, 2L) \in \sigma \times \mathbb{R}$, where x^* is the barycenter of σ. Let us represent r as

$$
r(x,t) = (\psi_0(x,t), u(x,t)) \in \sigma \times 0 \cup \partial\sigma \times [0,L] \subset \sigma \times [0,L].
$$

Define the retraction $f : \sigma \times K(U) \to \sigma \times U \times 0 \cup \partial\sigma \times K(U)$ by

$$
f(x,[y,t]) = (\psi_0(x,t), [\varphi(y,t-u(x,t)), w(x,t)]),
$$

where $w : \sigma \times [0,L] \to [0,L]$ is defined as follows: Let us consider the following closed subsets of $\mathbb{R}^{N+1}$:

$$
\begin{aligned}
\Omega_0 &= \{(x,t) \in \sigma \times [0,L] \,|\, u(x,t) \le L/10\}, \\
\Omega_1 &= \{(x,t) \in \sigma \times [0,L] \,|\, u(x,t) \ge L/2\}.
\end{aligned}
$$

Note that $|\Omega_0, \Omega_1| \ge c > 0$ for some constant $c > 0$. Let $s_i(x,t) = |(x,t), \Omega_i|$, $i = 1,, 2$, and define w by

$$
w(x,t) := \frac{s_1(x,t)}{s_0(x,t)+s_1(x,t)} u(x,t) + \frac{s_0(x,t)}{s_0(x,t)+s_1(x,t)} t.
$$

Note that w is Lipschitz and has the property

$$
w(x,t) = \begin{cases} u(x,t) & \text{if } u(x,t) \le L/10 \\ t & \text{if } u(x,t) \ge L/2. \end{cases}
$$

Note also that f is the identity on $\sigma \times U \times 0 \cup \partial\sigma \times K(U)$, and therefore it defines a retraction of $\sigma \times U \times 0 \cup \partial\sigma \times K(U)$. We show that f is Lipschitz. It suffices to show that the second component

$$
f_2(x,[y,t]) = ([\varphi(y,t-u(x,t)), w(x,t)])
$$

of f is Lipschitz. As before, we may assume that $(x, [y, t])$ and $(x', [y', t'])$ are sufficiently close to each other. Letting $u = u(x, t)$, $u' = u(x', t)$, $w = w(x, t)$, $w' = w(x', t)$ we have

$$
\begin{aligned}
|f_2(x, [y, t]), & f_2(x', [y, t])|^2 \\
&= |[\varphi(y, t - u), w], [\varphi(y, t - u'), w']|^2 \\
&\leq (L - w)^2 + (L - w')^2 - 2(L - w)(L - w') \cos |\varphi(y, t - u), \varphi(y, t - u')| \\
&\leq (w - w')^2 + (L - w)(L - w')|\varphi(y, t - u), \varphi(y, t - u')|^2 \\
&\leq (w - w')^2 + C_1 L^2 (u - u')^2 \\
&\leq C_2(1 + L^2)|x, x'|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|f_2(x, [y, t]), f_2(x, [y', t])| &= |[\varphi(y, t - u), w], [\varphi(y', t - u), w]| \\
&\leq (L - w)|\varphi(y, t - u), \varphi(y', t - u)|,
\end{aligned}
$$

where since $|[y, t], [y', t]| \geq (1/2)(L - t)|y, y'|$, we may assume that $t \geq 9L/10$. If $t \geq 9L/10$ and $u(x, t) \leq L/2$, then $\varphi(\cdot, t - u) = p$. If $t \geq 9L/10$ and $u(x, t) > L/2$, then $w(x, t) = t$, and we have

$$
\begin{aligned}
|f_2(x, [y, t]), f_2(x', [y, t])| &\leq (L - t)C|y, y'| \\
&\leq C|[y, t], [y', t]|.
\end{aligned}
$$

Finally letting $u = u(x, t)$, $u' = u(x, t')$, $w = w(x, t)$, $w' = w(x, t')$ we have

$$
\begin{aligned}
|f_2(x, [y, t]), f_2(x, [y, t'])|^2 &= |[\varphi(y, t - u), w], [\varphi(y, t - u'), w']|^2 \\
&\leq (w - w')^2 + L^2|\varphi(y, t - u), \varphi(y, t - u')|^2 \\
&\leq (w - w')^2 + C_1 L^2 (u - u')^2 \\
&\leq C_2(1 + L^2)|t - t'|^2.
\end{aligned}
$$

Thus f is Lipschitz.

Now define the homotopy $\Phi : \sigma \times K(U) \times [0, 1] \to \sigma \times K(U)$ by

$$
\begin{aligned}
\Phi(x, &[y, t], s) \\
&= ((1 - s)x + s\psi_0(x, t), [\varphi(y, \mu(s)(t - u(x, t)), (1 - v(s))t + v(s)w(x, t)]),
\end{aligned}
$$

where μ and v are Lipschitz functions on $[0, 1]$ satisfying

$$
\mu(s) = \begin{cases} 1 & \text{if } s \geq 2/3 \\ 0 & \text{if } s \leq 1/2, \end{cases} \qquad v(s) = \begin{cases} 1 & \text{if } s \geq 3/4 \\ 0 & \text{if } s \leq 2/3. \end{cases}
$$

Obviously, $\Phi(\cdot, 0) = 1_{\sigma \times K(U)}$, $\Phi(\cdot, 1) = f$ and $\Phi(\cdot, s)$ fixes each point of $\sigma \times U \times 0 \cup \partial\sigma \times K(U)$. We show that Φ is Lipschitz. It suffices to show that the second component

$$\Phi_2(x, [y, t], s) = ([\varphi(y, \mu(s)(t - u(x, t)), (1 - v(s))t + v(s)u(x, t)])$$

of Φ is Lipschitz. As before, we may assume that $(x, [y, t], s)$ and $(x', [y', t'], s')$ are sufficiently close to each other. Letting $u = u(x, t)$, $u' = u(x', t)$, $w = w(x, t)$, $w' = w(x', t)$ and $\mu = \mu(s)$, $v = v(s)$, we have

$$
\begin{aligned}
|\Phi_2(x, &[y, t], s), \Phi_2(x', [y, t], s)|^2 \\
&= |[\varphi(y, \mu(t - u)), (1 - v)t + vw], [\varphi(y, \mu(t - u')), (1 - v)t + vw']|^2 \\
&\leq v^2(w - w')^2 + L^2|\varphi(y, \mu(t - u)), \varphi(y, \mu(t - u')|^2 \\
&\leq v^2(w - w')^2 + C_1 L^2 \mu^2 (u - u')^2 \\
&\leq C_2(1 + L^2)|x, x'|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\Phi_2(x, &[y, t], s), \Phi_2(x, [y', t], s)| \\
&= |[\varphi(y, \mu(t - u)), (1 - v)t + vw], [\varphi(y', \mu(t - u)), (1 - v)t + vw]| \\
&\leq (L - (1 - v)t - vw)|\varphi(y, \mu(t - u), \varphi(y', \mu(t - u)| \\
&\leq (L - (1 - v)t)C|y, y'|,
\end{aligned}
$$

where if $t < 9L/10$, then $L|y, y'| \leq CL|[y, t], [y', t]|$. Hence we may assume that $t \geq 9L/10$. If $u(x, t) > L/2$ then $w(x, t) = t$. If $u(x, t) \leq L/2$ and $s \geq 2/3$, then $\mu(s) = 1$ and $\varphi(\cdot, \mu(t - u)) = p$. If $u(x, t) \leq L/2$ and $s \leq 2/3$, then $v = 0$. Thus we conclude that

$$
\begin{aligned}
|\Phi_2(x, [y, t], s), \Phi_2(x, [y', t], s)| &\leq (L - t)C|y, y'| \\
&\leq C|[y, t], [y', t]|.
\end{aligned}
$$

Next letting $u = u(x, t)$, $u' = u(x, t')$, $w = w(x, t)$, $w' = w(x, t')$, we have

$$
\begin{aligned}
|\Phi_2(x, &[y, t], s), \Phi_2(x, [y, t'], s)|^2 \\
&= |[\varphi(y, \mu(t - u)), (1 - v)t + vw], [\varphi(y, \mu(t' - u')), (1 - v)t' + vw']|^2 \\
&\leq ((1 - v)(t - t') + v(w - w'))^2 + L^2|\varphi(y, \mu(t - u), \varphi(y, \mu(t' - u'))| \\
&\leq C(1 + L^2)(t - t')^2.
\end{aligned}
$$

Finally letting $\mu' = \mu(s')$, $v' = v(s')$, we have

$$
\begin{aligned}
|\Phi_2(x, &[y, t], s), \Phi_2(x, [y, t], s')|^2 \\
&= |[\varphi(y, \mu(t - u)), (1 - v)t + vw], [\varphi(y, \mu'(t - u)), (1 - v')t + v'w]|^2 \\
&\leq (t(v' - v) + w(v - v'))^2 + L^2|\varphi(y, \mu(t - u), \varphi(y, \mu'(t - u))|^2
\end{aligned}
$$

$$\leq L^2(v - v')^2 + C_1 L^2(\mu - \mu')^2$$
$$\leq C_2 L^2(s - s')^2.$$

Thus Φ is Lipschitz. This completes the proof of Sublemma 3.6. $\square$

This completes the proof of Proposition 3.4. We have just proved the compact case of Theorem 1.1.

By the above discussion, we have the following commutative diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\ \tau\ } & \mathcal{D}(\mathcal{U}) \\
\Theta \downarrow & & \downarrow \iota \\
|\mathcal{N}_\mathcal{U}| & \xleftarrow{\ \Psi'\ } & \mathcal{M}(p)
\end{array}
$$

From Lemmas 3.1, 3.3, and Proposition 3.4 together with (3.1), we have the following.

Corollary 3.9 *Let $M, \mathcal{U}, \{\xi_j\}_{j \in J}$ be the same as in this section. Then the natural map*

$$\Theta : M \ni x \mapsto (\xi_j(x))_{j \in J} \in |\mathcal{N}_\mathcal{U}|$$

is a Lipschitz homotopy equivalence.

Corollary 3.10 *Let M, $\mathcal{U} = \{U_j\}_{j=1}^N$ and $\mathcal{N}_\mathcal{U}$ be the same as in this section, and $\zeta : |\mathcal{N}_\mathcal{U}| \to M$ a Lipschitz homotopy inverse to $\Theta : M \to |\mathcal{N}_\mathcal{U}|$. For every $\theta \in |\mathcal{N}_\mathcal{U}|$, let σ be the open simplex of $\mathcal{N}_\mathcal{U}$ containing θ with $\sigma = \langle U_{j_0}, \ldots, U_{j_k} \rangle$. Then we have*

$$\zeta(\theta) \in \bigcup_{i=0}^k U_{j_i}. \tag{3.6}$$

Proof Let $H : \mathcal{D}(\mathcal{U}) \times [0, 1] \to \mathcal{D}(\mathcal{U})$ be a Lipschitz strong deformation retraction of $\mathcal{D}(\mathcal{U})$ to $\tau(M)$ given in (3.2), and set $H_1 := H(\,\cdot\,, 1)$. Let $\Phi : \mathcal{M}(p) \times [0, 1] \to \mathcal{M}(p)$ be a Lipschitz strong deformation retraction of $\mathcal{M}(p)$ to $\mathcal{D}(\mathcal{U}) \times 0$ given in Proposition 3.4, and set $\Phi_1 := \Phi(\,\cdot\,, 1)$. From our argument in this section, we have the following commutative diagram:

$$
\begin{array}{ccc}
M & \xleftarrow{\ \tau^{-1} \circ H_1\ } & \mathcal{D}(\mathcal{U}) \\
\zeta \uparrow & & \uparrow \Phi_1 \\
|\mathcal{N}_\mathcal{U}| & \xrightarrow{\ \Psi\ } & \mathcal{M}(p)
\end{array}
$$

Note that $\tau^{-1} \circ H_1(\mu, x) = x$ for every $(\mu, x) \in \mathcal{D}(\mathcal{U})$. Therefore, we can write

$$\Phi_1 \circ \Psi(\theta) = (\eta(\theta), \zeta(\theta)) \in \mathcal{D}(\mathcal{U}),$$

where $\eta(\theta) \in |\mathcal{N}_\mathcal{U}|$ and $\zeta(\theta) \in M$. If σ_1 denotes the open simplex of $\mathcal{N}_\mathcal{U}$ containing $\eta(\theta)$, then it follows from the definition of $\mathcal{D}(\mathcal{U})$ that $\zeta(\theta) \in U_{\sigma_1}$. From Sublemma 3.6 together with (3.4), we see that σ_1 is a face of σ, which yields (3.6). □

4 Non-compact Case

We prove Theorem 1.1 for the general case. Let M be a σ-compact metric space admitting a good covering $\mathcal{U} = \{U_j\}_{j \in J}$. From the local finiteness of $\mathcal{U}$, J is countable, and therefore we may assume $J = \mathbb{N}$. Since $\mathcal{U}$ is locally finite, the number of U_j's meeting each $\bar{U}_i$ is finite. It follows that the nerve $\mathcal{N}_\mathcal{U}$ is locally finite. Note that $|\mathcal{N}_\mathcal{U}| \subset \mathbb{R}^\infty$ in this case. Note that the Lipschitz constant of the strong deformation retraction $U_j \times [0, 1] \to U_j \times [0, 1]$ of U_j to a point of U_j depends on j, and that $\dim \mathcal{N}_\mathcal{U} = \infty$ in general. From the local finiteness of $\mathcal{N}_\mathcal{U}$, basically we can do the same construction as in Sect. 3 to obtain the spaces $\mathcal{D}(\mathcal{U})$, $\mathcal{M}(p)$ in the general case, too. We also have the natural embeddings in a similar manner:

$$
\begin{array}{ccc}
M & \xrightarrow{\ \tau\ } & \mathcal{D}(\mathcal{U}) \\
& & \downarrow{\iota} \\
|\mathcal{N}_\mathcal{U}| & \xrightarrow[\ \Psi\ ]{} & \mathcal{M}(p)
\end{array}
$$

Note that the map $\tau : M \to \mathcal{D}(\mathcal{U})$ defined by

$$\tau(x) = (\Theta(x), x), \quad \Theta(x)(v_j) = \xi_j(x) \quad (j \in J)$$

is locally bi-Lipschitz. In a way similar to Corollary 3.2, we see that $\tau(M)$ has the same locally Lipschitz homotopy type as $\mathcal{D}(\mathcal{U})$. Note also that the natural embedding $\Psi : |\mathcal{N}_\mathcal{U}| \to \mathcal{M}(p)$ defined by

$$\Psi(\theta) = (\theta, [x, L]) = [\theta]$$

is isometric, and we see that $|\mathcal{N}_\mathcal{U}|$ is a locally Lipschitz strong deformation retract of $\mathcal{M}(p)$ in a way similar to Lemma 3.3. Therefore to complete the proof of Theorem 1.1 in the general case, we only have to check the following:

Lemma 4.1 *There exists a locally Lipschitz strong deformation retraction of $\mathcal{M}(p)$ to $\mathcal{D}(\mathcal{U}) \times 0$.*

Proof Recall that in the compact case in Sect. 3, the strong deformation retraction $\Phi : \mathcal{M}(p) \times [0, 1] \to \mathcal{M}(p)$ of $\mathcal{M}(p)$ to $\mathcal{D}(\mathcal{U}) \times 0$ is constructed simplex-wisely from higher dimensions to lower dimensions through Lemma 3.5. Therefore the Lipschitz constant of Φ depends on the dimension k_0 of $\mathcal{N}_\mathcal{U}$.

Since dim $\mathcal{N}_{\mathcal{U}}$ could be infinite in the present case, first of all, we have to verify that the map $\Phi : \mathcal{M}(p) \times [0, 1] \to \mathcal{M}(p)$ is well defined. For any point $(\theta, [x, t]) \in \mathcal{M}(p)$, let σ_θ be the simplex whose interior contains θ. Let L be the star of σ_θ, which is a finite subcomplex because of the local finiteness of $\mathcal{N}_{\mathcal{U}}$. Then

$$\mathcal{M}_L(p) := \bigcup_{\sigma \in L} \sigma \times K(U_\sigma)$$

provides an open subset of $\mathcal{M}(p)$ containing $(\theta, [x, t])$. Set

$$\mathcal{D}_L := \bigcup_{\sigma \in L} \sigma \times U_\sigma.$$

From the argument in Lemma 3.5, we have a Lipschitz strong deformation retraction $\Phi_L : \mathcal{M}_L(p) \times [0, 1] \to \mathcal{M}_L(p)$ of $\mathcal{M}_L(p)$ to $\mathcal{D}_L \times 0$. Since $\mathcal{M}(p)$ is a locally finite union of such open subsets $\mathcal{M}_L(p)$, we can construct a strong deformation retraction $\Phi : \mathcal{M}(p) \times [0, 1] \to \mathcal{M}(p)$ of $\mathcal{M}(p)$ to $\mathcal{D} \times 0$ simplex-wisely in a similar manner. Since $\Phi|_{\mathcal{M}_L(p) \times [0,1]} = \Phi_L$ and the construction is simplex-wise, Φ is locally Lipschitz. This completes the proof. □

5 Proofs of Theorem 1.2 and Corollary 1.5

To prove Theorem 1.2, we need the following result, which follows from the proof of [9, Theorem 1.2].

Theorem 5.1 [9] *For every $M \in \mathcal{A}(n, D, v_0)$, let M_i be a sequence in $\mathcal{A}(n, D, v_0)$ converging to M as $i \to \infty$. Then for any $\mu > 0$, there exists a good μ-covering $\mathcal{U} = \{U_j\}_{j \in J}$ of M satisfying the following. For every ϵ_i-approximations $\phi_i : M \to M_i$ with $\epsilon_i \to 0$, there exist a good 2μ-covering $\mathcal{U}_i = \{U_{ij}\}_{j \in J}$ of M_i and v_i-approximations $\varphi_i : M \to M_i$ with $v_i \to 0$ such that for sufficiently large i*

(1) *$\varphi_i(p_j)$ is a center of U_{ij} in the sense of Definition 2.6 for every $j \in J$;*
(2) *the corresponding $U_j \mapsto U_{ij}$ induces an isomorphism $\mathcal{N}_{\mathcal{U}} \to \mathcal{N}_{\mathcal{U}_i}$ between the nerves of $\mathcal{U}$ and $\mathcal{U}_i$;*
(3) *$\lim_{i \to \infty} \sup_{x \in M} |\phi_i(x), \varphi_i(x)| = 0$.*

Definition 5.2 We call such a $\mathcal{U}_i$ given in Theorem 5.1 a *lift* of $\mathcal{U}$ with respect to φ_i.

Proof of Theorem 1.2 Due to [9, Theorem 1.2], there exist $\epsilon > 0$ and finitely many spaces $M_1, \ldots, M_N \in \mathcal{A}(n, D, v_0)$ and finite simplicial complexes $K_1, \ldots, K_N$ such that

- $\bigcup_{i=1}^N U_\epsilon^{GH}(M_i) = \mathcal{A}(n, D, v_0)$;
- any $M \in U_\epsilon^{GH}(M_i)$ admit a good covering whose nerve complex is isomorphic to K_i.

Here, $U_\epsilon^{GH}(X)$ denotes the ϵ-neighborhood of X in $\mathcal{A}(n, D, v_0)$ with respect to the Gromov–Hausdorff distance. From this and Theorem 1.1, we obtain the first conclusion of Theorem 1.2.

We prove the second conclusion by contradiction. Suppose it does not hold. Then we would have sequences $\{M_i\}, \{M_i'\}$ in $\mathcal{A}(n, D, v_0)$ with $d_{GH}(M_i, M_i') < \delta_i, \lim \delta_i = 0$, together with δ_i-approximation $\theta_i : M_i \to M_i'$ such that

$$\sup_{x \in M_i} |\theta_i(x), k_i(x)| > c > 0, \tag{5.7}$$

for any Lipschitz homotopy equivalence $k_i : M_i \to M_i'$, where c is a constant not depending on i. Passing to a subsequence, we may assume that both M_i and M_i' converge to an Alexandrov space $M \in \mathcal{A}(n, D, v_0)$. We introduce a new positive number $\mu \ll c$. By Theorem 5.1, one can take a good μ-cover $\mathcal{U} = \{U_j\}_{j \in J}$ of M and a good 2μ-cover $\mathcal{U}_i = \{U_{ij}\}_{j \in J}$ of M_i such that $\mathcal{U}_i$ is a lift of $\mathcal{U}$ with respect to some v_i-approximation $\varphi_i : M \to M_i$, where $\lim_{i \to \infty} v_i = 0$. Let $\psi_i : M_i \to M$ be an v_i-approximation, which is an almost inverse of φ_i, in the sense that

$$\sup_{x \in M_i} |\varphi_i \circ \psi_i(x), x| \le v_i, \quad \sup_{x \in M} |\psi_i \circ \varphi_i(x), x| \le v_i.$$

Note that $\theta_i \circ \varphi_i : M \to M_i'$ is a $2(\delta_i + v_i)$-approximation. Applying Theorem 5.1 to $\theta_i \circ \varphi_i$, we also obtain a v_i'-approximation $\varphi_i' : M \to M_i'$ with $\lim_{i \to \infty} v_i' = 0$ and a lift $\mathcal{U}_i' = \{U_{ij}'\}_{j \in J}$ of $\mathcal{U}$ with respect to φ_i' such that

$$\lim_{i \to \infty} \sup_{x \in M} |\theta_i \circ \varphi_i(x), \varphi_i'(x)| = 0. \tag{5.8}$$

Set $p_{ij} := \varphi_i(p_j)$ and $p_{ij}' := \varphi_i'(p_j)$, which are centers of U_{ij} and U_{ij}', respectively. It follows that

$$\sup_{x \in M_i} |\theta_i(x), \varphi_i' \circ \psi_i(x)| \le \mu \tag{5.9}$$

for large i. Let $\alpha_i : \mathcal{N}_{\mathcal{U}} \to \mathcal{N}_{\mathcal{U}_i}$ and $\alpha_i' : \mathcal{N}_{\mathcal{U}} \to \mathcal{N}_{\mathcal{U}_i'}$ be the isomorphisms given by the correspondence $U_j \mapsto U_{ij}$ and $U_j \mapsto U_{ij}'$, respectively. We now consider the following diagram:

$$
\begin{array}{ccccc}
M_i & \underset{h_i}{\overset{\psi_i}{\dashrightarrow}} & M & \overset{\varphi_i'}{\dashrightarrow} & M_i' \\
\Theta_i \downarrow & & \zeta \Vert \Theta \downarrow & & \downarrow \zeta_i' \\
|\mathcal{N}_{\mathcal{U}_i}| & \underset{\alpha_i^{-1}}{\longrightarrow} & |\mathcal{N}_{\mathcal{U}}| & \underset{\alpha_i'}{\longrightarrow} & |\mathcal{N}_{\mathcal{U}_i'}|
\end{array}
$$

Here, $\Theta, \zeta, \Theta_i, \zeta_i'$ are maps given by Corollaries 3.9 and 3.10, for $(M, \mathcal{U}), (M_i, \mathcal{U}_i)$, and $(M_i', \mathcal{U}_i')$, respectively. For instance, $\Theta(x) = (\xi_j(x))_{j \in J}$ for $x \in M$, where $(\xi_j)_{j \in J}$ is a partition of unity by Lipschitz functions subordinate to $\{\bar{U}_j\}_{j \in J}$, and ζ

is a Lipschitz homotopy inverse of Θ given by Corollary 3.10. Now, we consider the compositions

$$h_i := \zeta \circ \alpha_i^{-1} \circ \Theta_i, \quad g_i := \zeta_i' \circ \alpha_i' \circ \Theta,$$

which are Lipschitz homotopy equivalences satisfying

$$\sup_{x \in M_i} |\psi_i(x), h_i(x)| \le 10\mu, \qquad \sup_{x \in M} |\varphi_i'(x), g_i(x)| \le 10\mu. \qquad (5.10)$$

Indeed, for $x \in M_i$, $\Theta_i(x)$ is contained in a unique open simplex $\langle U_{ij_0}, \ldots, U_{ij_k} \rangle \in \mathcal{N}_{\mathcal{U}_i}$. Then, $\alpha_i^{-1} \circ \Theta_i(x)$ is contained in $\langle U_{j_0}, \ldots, U_{j_k} \rangle$. By the property of ζ stated in Corollary 3.10, we have $h_i(x) \in U_{j_\ell}$ for some $0 \le \ell \le k$. On the other hands, since $x \in U_{ij_\ell}$, we have $|x, p_{ij_\ell}| \le 2\mu$ and $|\psi_i(x), p_{j_\ell}| \le 3\mu$. Therefore, we obtain

$$|h_i(x), \psi_i(x)| \le |h_i(x), p_{j_\ell}| + |p_{j_\ell}, \psi_i(x)| \le 4\mu.$$

Thus we obtain (5.10) for h_i. Similarly we obtain (5.10) for g_i. It follows from (5.10) and (5.9) that $\sup_{x \in M_i} |\theta_i(x), g_i \circ h_i(x)| \le 100\mu$, which is a contradiction to (5.7). □

For two metric spaces A and B, let us denote by

$$[A, B]_{\text{loc-Lip}}$$

the set of all locally Lipschitz homotopy classes of locally Lipschitz maps from A to B. Let us denote by

$$[A, B]$$

the set of all homotopy classes of continuous maps from A to B. For another metric space C and a locally Lipschitz map $f : A \to B$, we define a map $f^* : [B, C]_{\text{loc-Lip}} \to [A, C]_{\text{loc-Lip}}$ (and $f^* : [B, C] \to [A, C]$) by $f^*(g) := g \circ f$ up to locally Lipschitz homotopy (and up to homotopy, respectively). From the definition, for a locally Lipschitz map $g : B \to C$, we have

$$(g \circ f)^* = f^* \circ g^*. \qquad (5.11)$$

Proof of Corollary 1.5 Let us fix a good cover $\mathcal{U}$ of a σ-compact metric space X, and let K be the geometric realization of the nerve of $\mathcal{U}$. From [7, Corollary 1.3], the induced map

$$[K, Y]_{\text{loc-Lip}} \to [K, Y]$$

is bijective. By Theorem 1.1, K is locally Lipschitz homotopy equivalent to X. Let $f : X \to K$ and $g : K \to X$ be locally Lipschitz homotopy equivalences such that $g \circ f$ and $f \circ g$ are locally Lipschitz homotopy equivalent to id_X and id_K, respectively. By the

contravariant property (5.11), the induced maps $g^* : [X, Y]_{\text{loc-Lip}} \to [K, Y]_{\text{loc-Lip}}$ and $f^* : [K, Y]_{\text{loc-Lip}} \to [X, Y]_{\text{loc-Lip}}$ are mutually inverse. So are $f^* : [K, Y] \to [X, Y]$ and $g^* : [X, Y] \to [K, Y]$. These imply the conclusion. $\qquad\square$

Remark that in the statement of Corollary 1.5, if X is compact, we obtain a natural bijection between the set of all Lipschitz homotopy classes of Lipschitz maps from X to Y and $[X, Y]$.

A refinement of Corollary 1.5 is the following:

Corollary 5.3 *Let X and Y be as in Corollary 1.5. For any continuous function ϵ : $Y \to (0, \infty)$ and any continuous map $f : X \to Y$, there is a locally Lipschitz map $g : X \to Y$ which is homotopic to f and satisfies*

$$|f(x), g(x)| < \epsilon(f(x))$$

for every $x \in X$.

Proof This follows from [7, Corollary 4.4] and a discussion similar to the proof of Corollary 5.3. $\qquad\square$

6 Gluing with an Almost Isometry

For a small $1/n \gg \delta > 0$, let $\mathcal{R}_M(\delta)$ the open set of M consisting of all (n, δ)-strained points, which is called the δ-regular part of M. In this section we prove Theorem 1.4 by making use of the notion of center of mass developed in [1] (see [2] for the original idea).

Proof of Theorem 1.4 Let $\theta : M \to M'$ be an ϵ-approximation. Take $\mu > 0$ such that the closed 3μ-neighborhood of D is contained in $\mathcal{R}_M(\delta)$. Let D_1 be the closed 2μ-neighborhood of D. We also denote by D_0 the closed μ-neighborhood of D. By [1] and [14], for small enough $\epsilon > 0$ with $\epsilon \ll \mu$, we have a $\tau(\delta)$-almost isometric map

$$g : D_1 \to g(D_1) \subset \mathcal{R}_{M'}(2\delta)$$

such that $d(g(x), \theta(x)) < \tau(\epsilon)$ for all $x \in D_1$. On the other hand from Theorem 1.2, we have a Lipschitz homotopy equivalence

$$f : M \to M'$$

such that $d(f(x), \theta(x)) < \tau(\epsilon)$ for all $x \in M$.

We shall construct a Lipschitz homotopy equivalence $h : M \to M'$ such that $h = g$ on D and $h = f$ on $M \setminus D_0$. Denote by E the closure of $D_1 \setminus D$. Take $R > 0$ such that each point $x \in E$ has an (n, δ)-strainer of length $> R$. Let $\{x_i\}_{i=1}^N \subset E$ be a maximal family with $|x_i, x_j| \geq \delta R/2$ for each $i \neq j$. Then $\{B_i\}_{i=1}^N$ with $B_i := B(x_i, \delta R/2)$

gives a covering of E. By Theorem 2.3, for each $1 \le i \le N$ there are $\tau(\delta)$-almost isometric maps

$$f_i : B(x_i, 2\delta R) \to \mathbb{R}^n, \quad f_i' : B(g(x_i), 2\delta R) \to \mathbb{R}^n.$$

Let

$$d(x) = \min\{|D, x|, \mu\}.$$

Note that the multiplicity of the covering $B(x_i, 2\delta R)$ is uniformly bounded by a constant C_n.

For every $x \in B_i$, let

$$h_i^0(x) := (f_i')^{-1} \left(\frac{d(x)}{\mu} f_i'(f(x)) + \left(1 - \frac{d(x)}{\mu} \right) f_i'(g(x)) \right).$$

This extends to a Lipschitz map $h_i : \overline{M \setminus E} \cup B_i \to M'$ satisfying

$$h_i(x) = \begin{cases} g(x), & x \in D \\ f(x), & x \in \overline{M \setminus D_0}, \end{cases}$$

and $|\theta(x), h_i(x)| < \tau(\epsilon)$ for all $x \in \overline{M \setminus E} \cup B_i$.

Now we are going to glue these Lipschitz maps $\{h_i\}$ to get a Lipschitz map $h : M \to M'$. Define a Lipschitz cut-off function $\varphi_i : M \to \mathbb{R}$ by

$$\varphi_i(x) := \begin{cases} 1 - \frac{|x, x_i|}{\delta R}, & x \in B(x_i, \delta R) \\ 0, & \text{otherwise.} \end{cases}$$

Let $F_i := \overline{M \setminus E} \cup B_1 \cup \cdots \cup B_i$, and set

$$\psi_i(x) := \sum_{j=1}^{i} \varphi_j(x), \ x \in M.$$

Assuming that $h_{1\dots i} : F_i \to M'$ is already defined in such a way that

$$\begin{cases} |\theta(x), h_{1\dots i}(x)| < \tau(\epsilon), \ x \in F_i \\ h_{1\dots i}(x) = \begin{cases} g(x), & x \in D \\ f(x), & x \in \overline{M \setminus D_0}, \end{cases} \end{cases} \tag{6.12}$$

define $h_{1\dots i+1} : F_{i+1} \to M'$ by

$$h_{1\cdots i+1}(x) := \begin{cases} h_{1\cdots i}(x), & x \in F_i \setminus B_{i+1}, \\ (f'_{i+1})^{-1}\left(\left(1 - \frac{\varphi_{i+1}(x)}{\psi_{i+1}(x)}\right) f'_{i+1}(h_{1\cdots i}(x))\right) \\ \quad + \frac{\varphi_{i+1}(x)}{\psi_{i+1}(x)} f'_{i+1}(h_{i+1}(x))), & x \in B_{i+1}. \end{cases}$$

Note that $h_{1\cdots i+1}$ also satisfies (6.12).

Finally we set $h := h_{1\cdots N} : M \to M'$. Note that $|h(x), \theta(x)| < \tau(\epsilon)$ for all $x \in M$, and

$$h(x) = \begin{cases} g(x), & x \in D \\ f(x), & x \in \overline{M \setminus D_0}. \end{cases}$$

Similarly we define $h' : M' \to M$ by using the Lipschitz homotopy inverse f' of f, $g' := g^{-1}$, $D' := g(D)$, $D'_0 := g(D_0)$, $D'_1 := g(D_1)$ and $d' = d \circ g^{-1}$ in place of f, g, D, D_1, and d. Note that every $y \in D'_1$ has $(n, 2\delta)$-strainer of length $> R/2$. Obviously, $|h' \circ h(x), x| < \tau(\epsilon)$ and

$$h' \circ h(x) = \begin{cases} x, & x \in D \\ f' \circ f(x), & x \in \overline{M \setminus D_0}. \end{cases}$$

To construct a Lipschitz homotopy between 1_M and $h' \circ h$, we use a method developed in [3]. We consider the product space $M \times M$ and denote by $\Delta \subset M \times M$ the diagonal. Introduce a positive constant σ with $\epsilon \ll \sigma \ll \mu$ and take a sequence $0 < \sigma_i < \sigma$ with $\lim \sigma_i = 0$. For every $\mathbf{x} := (x_1, x_2) \in D_1 \times D_1 \cap A(\Delta; \sigma_i, \sigma)$, let y denote the midpoint of a minimal geodesic joining x_1 and x_2, where $A(\Delta; \sigma_i, \sigma) := \overline{B(\Delta, \sigma) \setminus B(\Delta, \sigma_i)}$ is the annulus. Note that $\mathbf{y} := (y, y)$ is the foot of a minimal geodesic from $\mathbf{x}$ to Δ. It is possible to take points z_1 and z_2 of M such that

$$\tilde{\angle} y x_i z_i > \pi - \tau(\delta), i = 1, 2,$$
$$|y, z_1| = |y, z_2|, \quad |\Delta, \mathbf{z}| = \sigma,$$

where $\mathbf{z} := (z_1, z_2)$. Then a direct computation shows that

$$|z_i, y| > |z_i, x_i| + (1 - \tau(\delta))|x_i, y|, \quad i = 1, 2,$$

and

$$|\mathbf{z}, \mathbf{y}| > |\mathbf{z}, \mathbf{x}| + (1 - \tau(\delta))|\mathbf{x}, \mathbf{y}|,$$

which yields that

$$\tilde{\angle} \mathbf{z} \mathbf{x} \mathbf{y} > \pi - \tau(\delta).$$

The above argument shows that the distance function d_Δ from Δ is $(1 - \tau(\delta))$-regular on $D_1 \times D_1 \cap A(\Delta; \sigma_i, \sigma)$. Now we consider a smooth approximation of a neighborhood

U_i of $D_1 \times D_1 \cap A(\Delta; \sigma_i, \sigma)$. By [6] and [8], there are a smooth manifold N_i and a bi-Lipschitz homeomorphism $\Phi_i : U_i \to N_i$ together with a gradient like unit vector field X_i for $d_\Delta \circ \Phi_i^{-1}$ defined on N_i such that if $\phi_i(\Phi_i(\mathbf{x}), t)$ denotes the integral curves of $-X_i$ starting at $\Phi_i(\mathbf{x})$, then for each $\mathbf{x} \in D_1 \times D_1 \cap A(\Delta; \sigma_i, \sigma)$ with $|\mathbf{x}, \Delta| = \sigma$,

$$|\Phi_i^{-1} \circ \phi_i(\Phi_i(\mathbf{x}), t_0), \Delta| = \sigma_i,$$

for some $t_0 < 2(\sigma - \sigma_i)$. By combining the flow curves $\{\Phi_i^{-1} \circ \phi_i(\Phi_i(\mathbf{x}), t)\}_i$, we obtain a Lipschitz flow ϕ on $D_1 \times D_1 \cap \overline{B}(\Delta, \sigma)$ such that for each $\mathbf{x} \in D_1 \times D_1 \cap \overline{B}(\Delta, \sigma)$ with $|\mathbf{x}, \Delta| = \sigma$, $\phi(\mathbf{x}, s_0) \in \Delta$ for some $s_0 < 2\sigma$. For $\mathbf{x} = (x, h' \circ h(x))$, if we denote $\phi(\mathbf{x}, t) = (\phi^1(\mathbf{x}, t), \phi^2(\mathbf{x}, t))$, the union of $\phi^1(\mathbf{x}, t)$ and $\phi^2(\mathbf{x}, 1 - t)$ provides the desired Lipschitz homotopy between 1_M and $h' \circ h$ on D_1.

We have just constructed a Lipschitz homotopy $H(x, t)$ between 1_M and $h' \circ h$ on D_1. Recall that we have a Lipschitz homotopy $F(x, t)$ between 1_M and $f' \circ f$. We have to glue F and H to get a Lipschitz homotopy $G(x, t)$ between 1_M and $h' \circ h$ defined on M. Let $\rho : M \times [0, 1] \to [0, 1]$ be a Lipschitz function such that

$$\rho(x, t) = \begin{cases} 0, & \text{on } D \times [0, 1] \cup D_0 \times [1/2, 1], \\ 1, & \text{on } \overline{M \setminus D_1} \times [0, 1]. \end{cases}$$

For every $(x, t) \in B_i \times [0, 1]$, let

$$G_i^0(x, t) := f_i^{-1} \left(\rho(x, t) f_i(F(x, t)) + (1 - \rho(x, t)) f_i(H(x, t)) \right).$$

This extends to a Lipschitz map $G_i : \overline{M \setminus E} \cup B_i \times [0, 1] \to M$ satisfying

$$G_i(x, t) = \begin{cases} x, & \text{on } \overline{M \setminus E} \cup B_i \times 0, \\ f' \circ f(x), & \text{on } \overline{M \setminus E} \cup B_i \times 1, \\ H(x, t), & \text{on } D \times [0, 1], \\ F(x, t), & \text{on } \overline{M \setminus D_1} \times [0, 1]. \end{cases}$$

Assuming that $G_{1\cdots i} : F_i \times [0, 1] \to M$ is already defined in such a way that

$$G_{1\cdots i}(x, t) = \begin{cases} x, & \text{on } F_i \times 0, \\ f' \circ f(x), & \text{on } F_i \times 1, \\ H(x, t), & \text{on } D \times [0, 1], \\ F(x, t), & \text{on } \overline{M \setminus D_1} \times [0, 1], \end{cases} \tag{6.13}$$

define $G_{1\cdots i+1} : F_{i+1} \times [0, 1] \to M$ by

$$G_{1\cdots i+1}(x, t) := \begin{cases} G_{1\cdots i}(x, t), & (x, t) \in (F_i \setminus B_{i+1}) \times [0, 1], \\ (f_{i+1})^{-1} \left(\left(1 - \frac{\varphi_{i+1}(x)}{\psi_{i+1}(x)} \right) f_{i+1}(G_{1\cdots i}(x, t)) \right. \\ \left. + \frac{\varphi_{i+1}(x)}{\psi_{i+1}(x)} f_{i+1}(G_{i+1}(x, t)) \right), & (x, t) \in B_{i+1} \times [0, 1]. \end{cases}$$

Finally set $G := G_{1\cdots N}$. Obviously G is Lipschitz, and $G = H$ on $D \times [0, 1]$ and $G = F$ on $\overline{M \setminus D_1} \times [0, 1]$, and thus G is a required Lipschitz homotopy between 1_M and $h' \circ h$. Similarly we obtain a Lipschitz homotopy between $1_{M'}$ and $h \circ h'$. This completes the proof of Theorem 1.4. □

7 Further Problems

It is quite natural to expect that there should exist uniform Lipschitz constants of the Lipschitz homotopies in Theorems 1.2 and 1.4.

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