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Amoeba-Shaped Polyhedral Complex of an Algebraic Hypersurface

Mounir Nisse[1] · **Timur Sadykov**[2]

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Abstract Given a complex algebraic hypersurface H, we introduce a subset of the Newton polytope of the defining polynomial for H which is a polyhedral complex and enjoys the key topological and combinatorial properties of the amoeba of H for a large class of hypersurfaces. We provide an explicit formula for this polyhedral complex in the case when the spine of the amoeba is dual to a triangulation of the Newton polytope of the defining polynomial. In particular, this yields a description of the polyhedral complex when the hypersurface is optimal (Forsberg et al. in Adv Math 151:45–70, 2000). We conjecture that a polyhedral complex with these properties exists in general.

Keywords Amoebas · Newton polytope · Tropical geometry · Polyhedral complex

Mathematics Subject Classification 32A60 · 52B55

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✉ Timur Sadykov
Sadykov.TM@rea.ru

Mounir Nisse
mounir.nisse@gmail.com

1 School of Mathematics, Korea Institue for Advanced Study, 87 Hoegiro Dongdaemun-gu, Seoul 130-722, Republic of Korea

2 Department of Mathematics and Computer Science, Plekhanov Russian University, Moscow, Russia 115054

1 Introduction

Amoebas of complex algebraic varieties have attracted substantial attention in the recent years after their inception in [5]. Being a semi-analytic subset of the real space, the amoeba carries a lot of geometric, algebraic, topological, and combinatorial information on the corresponding algebraic variety [10]. The degeneration of the amoeba of a complex algebraic variety leads to the concept of a tropical algebraic variety providing an important link between complex analysis and enumerative algebraic geometry [11,12]. Amoebas of algebraic hypersurfaces possess rich analytic and combinatorial structure reflected in their spines, contours, and tentacles. They appear in numerous applications in real algebraic geometry, complex analysis, mirror symmetry and in several other areas. Moreover, they are naturally linked to the geometry of Newton polytopes, which can be seen in particular with Viro's patchworking principle (i.e., tropical localization) based on the combinatorics of subdivisions of convex lattice polytopes. Also, certain tropical varieties can be seen as a limiting aspect (or "degeneration") of amoebas of algebraic varieties. For example, complex curves viewed as Riemann surfaces turn to metric graphs (one-dimensional combinatorial objects), and n-dimensional complex varieties turn to n-dimensional polyhedral complexes with some properties (see [7] and [9]). Amoebas have their similar objects in the real torus called *coamoebas*, which are the projection of algebraic varieties onto the real torus which have many interesting and nice properties (see [13]).

Despite the simple definition, efficient computation of the amoeba of a given algebraic variety represents a task of formidable computational complexity. Various approaches have been recently tried to compute the shape of an amoeba [2,16] or approximate it by simpler geometric objects [4,15,17]. The fundamental problems addressed in numerous papers are the detection of the topological type of an amoeba, the membership problem for a given connected component of an amoeba complement, and detection of the order [3] of such a component in the case of hypersurface amoebas.

Alongside with the definition of unbounded affine amoeba of an algebraic hypersurface, a competing definition of compactified amoeba has been introduced in [5]. While the affine amoeba of a hypersurface is its Reinhardt diagram in the logarithmic scale, its compactified amoeba is defined to be the image of the hypersurface under the moment map [6] providing a homeomorphism between the Newton polytope of the defining polynomial of that hypersurface and the positive orthant of the real vector space. Being topologically equivalent to the standard affine amoeba, its compactified counterpart often has the substantial disadvantage of exhibiting complement components of very different relative size (see Examples 3.6 and 5.3). This makes it difficult to work with compactified amoebas in a computationally reliable way and probably explains the focus of research on affine amoebas.

In the present paper we introduce the definition of an amoeba-shaped polyhedral complex of an algebraic hypersurface satisfying certain technical assumption on the Newton polytope of its defining polynomial and its tropical counterpart (see Theorem 3.3). Like the compactified amoeba, this polyhedral complex is a subset of the Newton polytope of the defining polynomial of the hypersurface. An explicit formula for this polyhedral complex is provided in the case when the hypersurface H is optimal [1,3]. Furthermore, we give a topological description of the complement of the affine amoebas of a class of algebraic hypersurfaces in combinatorial terms naturally and strongly related to their Newton polytopes and the coefficients of their defining polynomials. We conjecture that such an amoeba-shaped polyhedral complex exists in general and that the order of a connected component in its complement is the lattice point in this component.

Pictures of amoebas in the paper have been created in MATLAB R2017a. The authors thank D. Bogdanov for providing a helpful online tool for automated generation of MATLAB code which is available for free public use at http://dvbogdanov.ru/?page=amoeba.

2 Notation and Preliminaries

Throughout the paper, we denote by n the number of $x \in \mathbb{C}^n$ variables. For $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, we denote by x^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. By the *support* of a polynomial $p(x)$ we will mean the set of the vectors of exponents of the monomials which appear in $p(x)$. The *Newton polytope* $\mathcal{N}_{p(x)}$ of a Laurent polynomial $p(x)$ is defined to be the convex hull in $\mathbb{R}^n$ of its support. We will often drop some of the subindices to simplify the notation, e.g. we will denote the Newton polytope of a polynomial p by $\mathcal{N}_p$ or even by $\mathcal{N}$ instead of $\mathcal{N}_p(x)$ if there is no confusion.

Definition 2.1 The *amoeba* $\mathcal{A}_f$ of a Laurent polynomial $f(x)$ (or of the algebraic hypersurface $\{f(x) = 0\}$) is defined to be the image of the hypersurface $f^{-1}(0)$ under the map $\mathrm{Log} : (x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|)$.

For $n > 1$, the amoeba of a polynomial is a closed unbounded semi-analytic subset of the real vector space $\mathbb{R}^n$. Throughout the paper we will often call such amoebas *affine* in order to distinguish them from the compactified and the weighted compactified counterparts.

The following result shows that the Newton polytope $\mathcal{N}_{p(x)}$ reflects the structure of the amoeba $\mathcal{A}_{p(x)}$ [3, Theorem 2.8 and Proposition 2.6].

Theorem 2.2 (see [3]) *Let $p(x)$ be a Laurent polynomial and let $\{M\}$ denote the family of connected components of the amoeba complement ${}^c\mathcal{A}_{p(x)}$. There exists an injective function $\nu : \{M\} \to \mathbb{Z}^n \cap \mathcal{N}_{p(x)}$ such that the cone which is dual to $\mathcal{N}_{p(x)}$ at the point $\nu(M)$ coincides with the recession cone of M. In particular, the number of connected components of ${}^c\mathcal{A}_{p(x)}$ cannot be smaller than the number of vertices of $\mathcal{N}_{p(x)}$ and cannot exceed the number of integer points in $\mathcal{N}_{p(x)}$.*

Throughout the paper, the vector $\nu(M)$ will be called the *order* of the connected component M in the amoeba complement.

Definition 2.3 (*see* [5, *Chapter* 6]) The *compactified amoeba* $\overline{\mathcal{A}}_f$ of a Laurent polynomial $f(x) = \sum_{s\in S} a_s x^s$ (or, equivalently, of the algebraic hypersurface $\{f(x) = 0\}$) is defined to be the image of the hypersurface $f^{-1}(0)$ under the *moment map*

$$\mu_S(x) := \frac{\sum_{s\in S} s \cdot |x^s|}{\sum_{s\in S} |x^s|}.$$

By [5, Chapter 6], the compactified amoeba of a polynomial is a closed subset of its Newton polytope. The amoeba and the compactified amoeba of a polynomial are homeomorphic. From the computational point of view both have advantages and shortcomings. It is in general difficult to locate the position of an affine amoeba in the real space while the integer convex polytope represents a computationally much more manageable ambient space. On the other hand, some of the connected components of the complement to a compactified amoeba can be elusively small as illustrated by Examples 3.6 and 5.3. Besides, the connected components of the complement to the compactified amoeba of a polynomial are in general not convex.

Definition 2.4 (*cf.* [3, *Definition* 2.9]) An algebraic hypersurface $H \subset (\mathbb{C}^*)^n$, $n \geq 2$, is called *optimal* if the number of connected components of its amoeba complement ${}^c\mathcal{A}_H$ equals the number of integer points in the Newton polytope of the defining polynomial of H. We will say that a polynomial (as well as its amoeba) is optimal if its zero locus is an optimal algebraic hypersurface.

Since the amoeba of a polynomial does not carry any information on the multiplicities of its roots, any one-dimensional amoeba (which is just a finite set of distinct points in $a_1, \ldots, a_k \in \mathbb{R}$) can be treated as the amoeba of the polynomial $\prod_{j=1}^{k}(x - e^{a_k})$ all of whose roots are positive and distinct. Thus Definition 2.4 is trivial in the univariate case. The correct extension of Definition 2.4 to one dimension is to say that all the roots of the polynomial in question have different absolute values.

Definition 2.5 An algebraic hypersurface $H \subset (\mathbb{C}^*)^n$, $n \geq 2$, is called *solid* if the number of connected components of its amoeba complement ${}^c\mathcal{A}_H$ equals the number of vertices of the Newton polytope of the defining polynomial of H. We will say that a polynomial (as well as its amoeba) is solid if its zero locus is a solid algebraic hypersurface.

3 Explicit Analytic Formula for the Amoeba-Shaped Polyhedral Complex

To transform the (compactified) amoeba of an algebraic hypersurface into a combinatorial object, one needs to take into account the relative size of the coefficients of its defining polynomial. The next definition is central to the paper.

Definition 3.1 Following the ideas of [19], we define the *weighted moment map* associated with the algebraic hypersurface $\{x \in \mathbb{C}^n : f(x) := \sum_{s\in S} a_s x^s = 0\}$ through

$$\mu_f(x) := \frac{\sum_{s\in S} s \cdot |a_s||x^s|}{\sum_{s\in S} |a_s||x^s|}.$$

It follows from the general theory of moment maps [6] that $\mu_f(\mathbb{C}^n) \subseteq \mathcal{N}_f$.

Definition 3.2 By the *weighted compactified amoeba* of an algebraic hypersurface $H = \{x \in \mathbb{C}^n : f(x) = 0\}$ we will mean the set $\mu_f(H)$. We denote it by $\mathcal{WCA}(f)$.

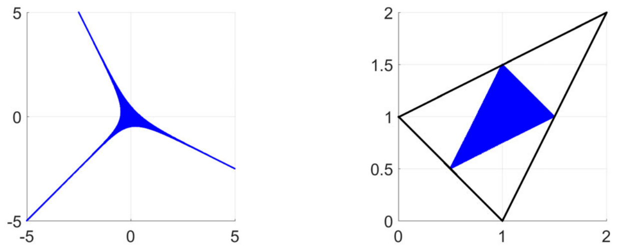


Fig. 1 The affine and the compactified amoebas of the polynomial $x + y + x^2y^2 + xy/2$. The polyhedral complex coincides with the compactified amoeba of this polynomial

Recall that the *Hadamard power* of order $r \in \mathbb{R}$ of a polynomial $f(x) = \sum_{s\in S} a_s x^s$ is defined to be $f^{[r]}(x) := \sum_{s\in S} a_s^r x^s$.

Theorem 3.3 *Let f be a polynomial in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with the Newton polytope $\mathcal{N}$ such that $|a_\alpha| \geq 1$ for every $\alpha \in \mathrm{Vert}(\mathcal{N})$. Assume that the function which assigns to each $\alpha \in \mathcal{N} \cap \mathbb{Z}^n$ the real number $\log|a_\alpha|$ is concave, and the subdivision of $\mathcal{N}$ dual to the tropical hypersurface Γ associated to the tropical polynomial f_{trop} defined by:*

$$f_{trop}(\zeta) = \max_{\alpha\in\mathcal{N}\cap\mathbb{Z}^n} \{\log|a_\alpha| + \langle\alpha, \zeta\rangle\}$$

is a triangulation. Then the set-theoretical limit

$$\mathcal{P}_f^\infty := \lim_{r\to\infty} WC\mathcal{A}(f^{[r]}) \tag{3.1}$$

is a polyhedral complex. Moreover, its complement in $\mathcal{N}$ has the same topology of the complement of the amoeba $\mathcal{A}$ of f, i.e. $\pi_0(\mathbb{R}^n \setminus \mathcal{A}) = \pi_0(\mathcal{N} \setminus \mathcal{P}_f^\infty)$. In particular, if $n = 2$ then $\mathcal{P}_f^\infty$ is a simplicial complex.

The assumptions in Theorem 3.3 are sufficient for the right-hand side of (3.1) to be a polyhedral complex. However, they are far from being necessary, which is illustrated by the following examples.

Example 3.4 The connected components of the complement of $\mathcal{P}_f^\infty$ in the Newton polytope are not necessarily convex. The amoeba, the compactified amoeba and the associated polyhedral complex for the polynomial $x + y + x^2y^2 + cxy$ are depicted in Figs. 1 and 2 for $c = 1/2$ and $c = 2$, respectively.

Example 3.5 The amoeba, the compactified amoeba and the associated polyhedral complex for the polynomial $x + y + xy^2 + x^2y + cxy$ are depicted in Figs. 3 and 4 for $c = 1/2$ and $c = 5$, and respectively.

Example 3.6 The zero locus of the polynomial $x + 30xy + 20x^2y + x^3y + y^2$ is an optimal hypersurface. Its amoeba, compactified amoeba and the associated polyhedral complex are depicted in Fig. 5. Although the compactified amoeba is optimal, the

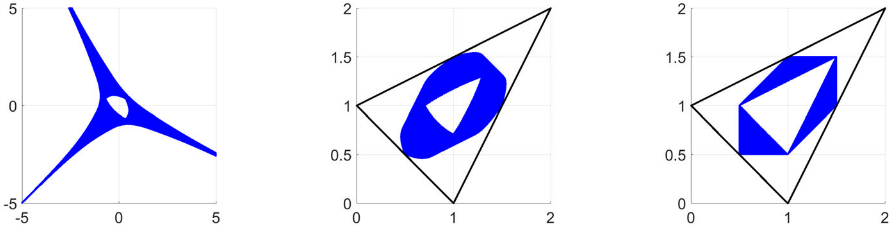


Fig. 2 The affine amoeba, the compactified amoeba and the polyhedral complex of the polynomial $x + y + x^2y^2 + 2xy$

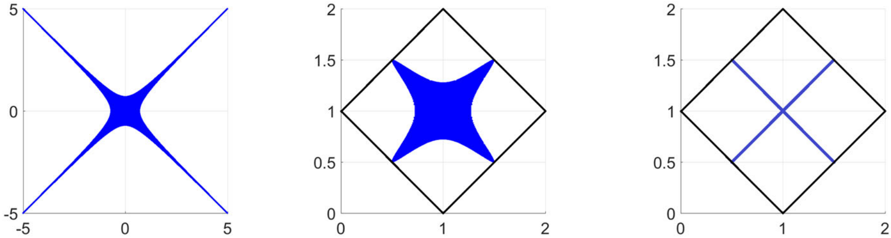


Fig. 3 The affine amoeba, the compactified amoeba and the polyhedral complex of the polynomial $x + y + xy^2 + x^2y + xy/2$

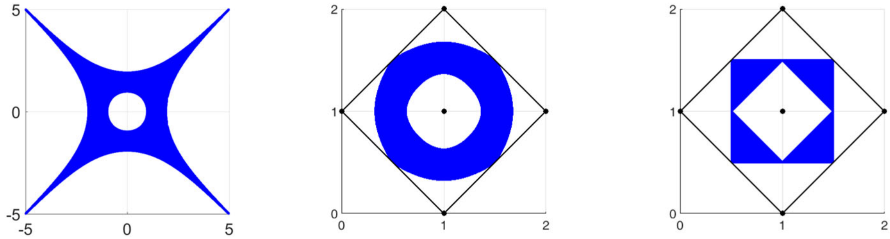


Fig. 4 The affine amoeba, the compactified amoeba and the polyhedral complex of the polynomial $x + y + xy^2 + x^2y + 5xy$

three connected components of its complement that correspond to the vertices of the Newton polytope are very small in comparison with the two bounded components that fill almost all of the Newton polygon. In fact, these three components are so small that it is difficult (yet not impossible) to distinguish them by eye on the presented picture. This and similar examples motivate the search for a better compact counterpart of an affine amoeba pursued in the present paper.

Example 5.2 shows that the sufficient condition of the theorem is not necessary for the Hadamard power approach to work. Indeed, the polynomial in this example is neither dense nor optimal.

Proposition 3.7 *If $f(x)$ is a polynomial whose Newton polytope $\mathcal{N}_f$ is a simplex and with nonzero coefficients only on the vertices of $\mathcal{N}_f$ then $\mathcal{P}_f^{\infty}$ is given by the convex hull of the middle points of all 1-dimensional faces of $\mathcal{N}_f$.*

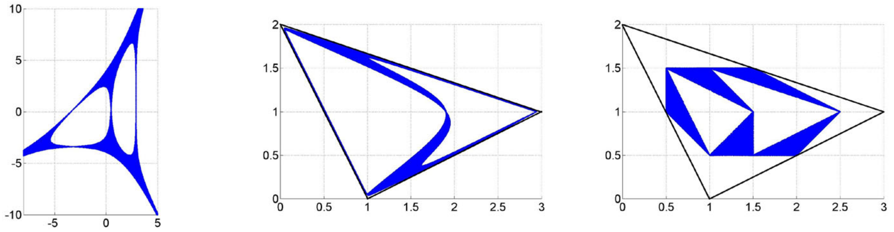


Fig. 5 The affine amoeba, the compactified amoeba and the polyhedral complex of the polynomial $x + 30xy + 20x^2y + x^3y + y^2$

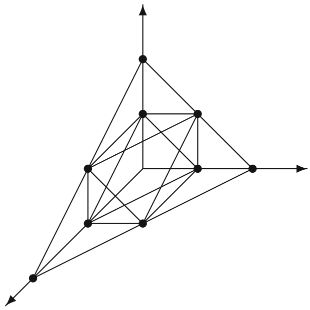


Fig. 6 The polyhedral complex $\mathcal{P}_f^\infty$ in the three-dimensional hyperplane case $f(x, y, z) = 1 + x + y + z$

Proof Use a monomial change of variables to reduce to the hyperplane case and employ an argument parallel to that in the proof of [3, Proposition 4.2]. □

Example 3.8 A hyperplane. The polyhedron associated with the hyperplane $\{1 + x + y + z = 0\}$ is depicted in Fig. 6 inside the unit simplex. It is combinatorially equivalent to an octahedron.

4 Proof of the Main Theorem

Let t be a strictly positive real number and H_t be the self-diffeomorphism of $(\mathbb{C}^*)^n$ defined by

$$\begin{array}{rccl} H_t: & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n, \\ & (x_1, \ldots, x_n) & \longmapsto & \left(|x_1|^{\frac{1}{\log t}} \frac{x_1}{|x_1|}, \ldots, |x_n|^{\frac{1}{\log t}} \frac{x_n}{|x_n|} \right). \end{array}$$

which defines a new complex structure on $(\mathbb{C}^*)^n$ denoted by $J_t = (dH_t) \circ J \circ (dH_t)^{-1}$ where J is the standard complex structure.

A J_t-holomorphic hypersurface V_t is a hypersurface holomorphic with respect to the J_t complex structure on $(\mathbb{C}^*)^n$. It is equivalent to say that $V_t = H_t(V)$ where $V \subset (\mathbb{C}^*)^n$ is a holomorphic hypersurface for the standard complex structure J on $(\mathbb{C}^*)^n$.

Recall that the Hausdorff distance between two closed subsets A, B of a metric space (E, d) is defined by:

$$d_{\mathcal{H}}(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}.$$

Here we take $E = \mathbb{R}^n \times (S^1)^n$, with the distance defined as the product of the Euclidean metric on $\mathbb{R}^n$ and the flat metric on $(S^1)^n$.

Definition 4.1 A complex tropical hypersurface $V_\infty \subset (\mathbb{C}^*)^n$ is the limit (with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$) of a sequence of a J_t-holomorphic hypersurfaces $V_t \subset (\mathbb{C}^*)^n$ when t tends to ∞.

The argument map is the map defined as follows:

$$\begin{array}{rrcl}\widetilde{\text{Arg}}: & (\mathbb{C}^*)^n & \longrightarrow & (S^1)^n, \\ & (x_1, \ldots, x_n) & \longmapsto & (\widetilde{\arg}(x_1), \ldots, \widetilde{\arg}(x_n)).\end{array}$$

We use the following notations: if $z = (x_1, x_2, \ldots, x_n) \in (\mathbb{C}^*)^n$ and $x_j = \rho_j e^{i\gamma_j}$, then $\widetilde{\text{Arg}}(z) = (\widetilde{\arg}(x_1), \widetilde{\arg}(x_2), \ldots, \widetilde{\arg}(x_n)) := (e^{i\gamma_1}, e^{i\gamma_2}, \ldots, e^{i\gamma_n})$ and $\text{Arg}(z) = (\arg(x_1), \arg(x_2), \ldots, \arg(x_n)) := (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Let $\mathbb{K}$ be the field of the Puiseux series with real exponents, which is the field of series $a(t) = \sum_{j\in A_a} \xi_j t^j$ with $\xi_j \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is a well-ordered set (which means that any subset has a smallest element). It is well known that the field $\mathbb{K}$ is algebraically closed and of characteristic zero, and it has a non-Archimedean valuation $\text{val}(a) = -\min A_a$:

$$\begin{cases} \text{val}(ab) & = \text{val}(a) + \text{val}(b) \\ \text{val}(a+b) & \leq \max\{\text{val}(a),\ \text{val}(b)\}, \end{cases}$$

and we put $\text{val}(0) = -\infty$.

We complexify the valuation map as follows:

$$\begin{array}{rcl} w: \mathbb{K}^* & \longrightarrow & \mathbb{C}^*, \\ a & \longmapsto & w(a) = e^{\text{val}(a) + i\arg(\xi_{-\text{val}(a)})}. \end{array}$$

Let $\widetilde{\text{Arg}}$ be the argument map $\mathbb{K}^* \to S^1$ defined by: for any $a \in \mathbb{K}$ with $a(t) = \sum_{j\in A_a} \xi_j t^j$, $\widetilde{\text{Arg}}(a) = e^{i\arg(\xi_{-\text{val}(a)})}$ (this map extends the map $\widetilde{\text{Arg}}: \mathbb{C}^* \to S^1$ defined by $\rho e^{i\theta} \mapsto e^{i\theta}$).

Applying this map coordinate-wise we obtain the map

$$W: (\mathbb{K}^*)^n \longrightarrow (\mathbb{C}^*)^n.$$

Using Kapranov's theorem [8] and degeneration of a complex structure, Mikhalkin gives an algebraic definition of a complex tropical hypersurface (see [12]) as follows:

Theorem 4.2 (see [11]) *The set $V_\infty \subset (\mathbb{C}^*)^n$ is a complex tropical hypersurface if and only if there exists an algebraic hypersurface $V_{\mathbb{K}} \subset (\mathbb{K}^*)^n$ such that $W(V_{\mathbb{K}}) = V_\infty$.*

Let $\mathrm{Log}_{\mathbb{K}}(x_1, \ldots, x_n) = (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))$, which means that $\mathbb{K}$ is equipped with the norm defined by $\|z\|_{\mathbb{K}} = e^{\mathrm{val}(z)}$ for any $z \in \mathbb{K}^*$. Then we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{K}^*)^n & \xrightarrow{\;W\;} & (\mathbb{C}^*)^n \\ & {\scriptstyle \mathrm{Log}_{\mathbb{K}}} \searrow \quad \swarrow {\scriptstyle \mathrm{Log}} & \\ & \mathbb{R}^n & \end{array}$$

Theorem 4.3 (see [11,12]) *Let f be a polynomial in $\mathbb{K}[x_1, \ldots, x_n]$. Then we have the following:*

$$\lim_{t\to\infty} H_t\left(V_{f_{\frac{1}{t}}}\right) = W(V_f)$$

with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$, and where $f_{\frac{1}{t}}$ is the polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ where we fixed the variable $t \gg 1$ of the coefficients of f.

It was shown by Mikhalkin that if f and f' are polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ such that the coefficients of f' are the leading monomials of the coefficients of f (the coefficients are elements in the field of Puiseux series $\mathbb{K}$ and every element in $\mathbb{K}$ has a leading term by definition of the field of Puiseux series) then $W(V_f) = W(V_{f'})$. Also, if ζ is a point in $\mathrm{Log}_{\mathbb{K}}(V_f)$ and U_ζ is an open neighborhood of ζ, and $U_\zeta \cap \mathrm{Log}_{\mathbb{K}}(V_f)$ is a cone centered at ζ (i.e., for every $x \in U_\zeta \cap \mathrm{Log}_{\mathbb{K}}(V_f)$ we have $[\zeta, x] \subset \mathrm{Log}_{\mathbb{K}}(V_f)$), then the dual $\mathcal{N}_\zeta$ of the stratum containing ζ is well defined and $W(V_f) \cap \mathrm{Log}^{-1}(U_\zeta)$ is equal to $W(V_{f^{\mathcal{N}_\zeta}}) \cap \mathrm{Log}^{-1}(U_\zeta)$, where $f^{\mathcal{N}_\zeta}$ denotes the truncation of f to $\mathcal{N}_\zeta$ (in case of plane curves see Proposition 6.1 of [12] for more detail, in case of hypersurface of higher dimension the proof is the same).

By Viro's Theorem 4.4 there exists $U \in \mathbb{R}^n$ such that for t sufficiently large $\mathrm{Log}^{-1}(U_\zeta) \cap H_t(V_{f_{\frac{1}{t}}})$ is isotopic to $\mathrm{Log}^{-1}(U_\zeta) \cap e^U H_t(V_{f^{\mathcal{N}_\zeta}})$ and we have the following diagram:

$$\begin{array}{ccc} \mathrm{Log}^{-1}(U_\zeta) \cap H_t(V_{f_{\frac{1}{t}}}) & \xrightarrow{\;\simeq\;} & \mathrm{Log}^{-1}(U_\zeta) \cap e^U H_t(V_{f^{\mathcal{N}_\zeta}}) \\ {\scriptstyle t\to\infty} \downarrow & & \downarrow {\scriptstyle t\to\infty} \\ \mathrm{Log}^{-1}(U_\zeta) \cap W(V_f) & \xrightarrow{\;=\;} & \mathrm{Log}^{-1}(U_\zeta) \cap e^U W(V_{f^{\mathcal{N}_\zeta}}). \end{array}$$

Theorem 4.4 (see [18]) *Let $f = \sum a_j z^j$ be a polynomial in $\mathbb{K}[x_1, \ldots, x_n]$ where $a_j(t) = \sum \xi_r^j t^r$. Let ζ be a point in $\mathrm{Log}_{\mathbb{K}}(V_f)$ and U_ζ be an open neighborhood of ζ such that $U_\zeta \cap \mathrm{Log}_{\mathbb{K}}(V_f)$ is a cone centered at ζ. Let $\mathcal{N}_\zeta$ be a cell of the subdivision that is dual to $\mathrm{Log}_{\mathbb{K}}(V_f)$ and contains ζ, and let $f^{\mathcal{N}_\zeta}(z)$ be the polynomial in $\mathbb{C}[x_1, \ldots, x_n]$*

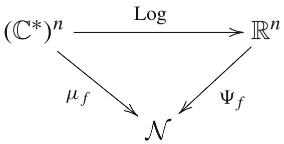
defined as follows:

$$f^{\mathcal{N}_\zeta}(z) = \sum_{j \in \mathcal{N}_\zeta} \xi^j_{-val(a_j)} z^j.$$

Then there exists $U \in \mathbb{R}^n$ *such that for any sufficiently large* t, $\mathrm{Log}^{-1}(U_\zeta) \cap H_t(V_{f_{\frac{1}{t}}})$ *is isotopic to* $\mathrm{Log}^{-1}(U_\zeta) \cap e^U H_t(V_{f^{\mathcal{N}_\zeta}})$.

Let now f be a polynomial in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with the Newton polytope $\mathcal{N}$ and amoeba $\mathcal{A}$ with the spine Γ_f. Moreover, assume that the subdivision τ_f of $\mathcal{N}$ dual to Γ_f is a triangulation.

The moment map μ_f is an embedding whose image is the interior of $\mathcal{N}$, and we have the following commutative diagram:



The maps Log and μ_f both have orbits $(S^1)^n$ as fibers, and we obtain a reparametrization of $\mathbb{R}^n$ which we denote by Ψ_f (see [5]).

We can now finish the proof of main Theorem 3.3. The fact that the function $\alpha \longmapsto \log|a_\alpha|$ is concave implies that the same hypothesis holds for any Hadamard product $f^{[r]}$ and any positive number r. Moreover, the subdivisions of $\mathcal{N}$ dual to the tropical hypersurfaces $\Gamma^{[r]}$ associated to $f^{[r]}$ are all the same (because the $\Gamma^{[r]}$'s have all the same combinatorial type). More precisely, this means that the complement in $\mathbb{R}^n$ of $\mathcal{A}$ and $\mathcal{A}_r$ have the same number of connected components. In particular, it means that if the amoeba $\mathcal{A}$ of f is optimal then all amoebas $\mathcal{A}_r$ of $f^{[r]}$ are optimal. Moreover, we know that any lattice simplex can be identified with the standard simplex by an element of the group $AGL_n(\mathbb{Z})$ of affine linear transformations of $\mathbb{R}^n$ whose linear part belongs to $GL_n(\mathbb{Z})$. This means that when we pass to the limit as r tends to infinity of (3.1), and we take the truncation of the polynomial to an element of the subdivision τ (which is a simplex by our hypothesis) dual to Γ we obtain, up to a linear transformation, the polyhedron corresponding to the standard hyperplane. For simplicity, let ζ be a vertex of $\Gamma^{[r]}$ for sufficiently large r, and U_ζ be an open neighborhood of ζ. Then by Viro's Theorem 4.4 we have $\mathrm{Log}^{-1}(U_\zeta) \cap V_{f^{[r]}}$ is isotopic to $\mathrm{Log}^{-1}(U_\zeta) \cap e^U V_{f^{[r];\mathcal{N}_\zeta}}$ for some $U \in \mathbb{R}^n$ where $f^{[r];\mathcal{N}_\zeta}$ is the truncation of $f^{[r]}$ to the dual $\mathcal{N}_\zeta$ of the vertex ζ, which is a simplex by hypothesis. But this last set is, up to an affine transformation, the polyhedral complex corresponding to a hyperplane.

When r tends to ∞, the Newton polytopes stay the same because $|a_\alpha| > 1$ for every α a vertex of the Newton polytope, and then Viro's patchwork is used to obtain a polyhedral complex.

In particular, if $n = 2$, we obtain a simplicial complex, because the polyhedron corresponding to the line is a 2-dimensional simplex (i.e. a triangle in this case). □

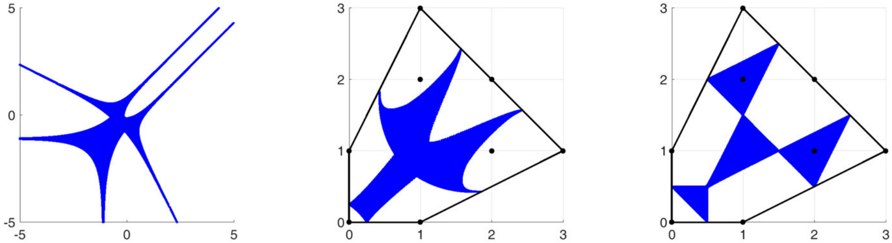


Fig. 7 The affine amoeba, the compactified amoeba and the polyhedral complex of the polynomial $1 + 3x + 3y + x^2y + 4x^3y + xy^2 + 10x^2y^2 + 4xy^3$

5 The Polyhedral Complex of a General Algebraic Hypersurface

The explicit analytic formula (3.1) for the polyhedral complex of an algebraic hypersurface cannot work in the general case. Indeed, if (the absolute values of) all coefficients of its defining polynomial f are equal to 1, then the Hadamard power in (3.1) yields nothing new and $\mathcal{P}_f^\infty$ will in general remain curved. Example 5.2 shows however that the condition in Theorem 3.3 is far from being necessary. Numerous computer experiments suggest the following conjecture.

Conjecture 5.1 *Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_1^{\pm 1}]$ be a Laurent polynomial. Denote by $\overline{\mathcal{A}}_f \subset \mathcal{N}_f$ its compactified amoeba and by $\{M\}$ the set of (nonempty) connected components of the complement of $\overline{\mathcal{A}}_f$ in the Newton polytope $\mathcal{N}_f$. We furthermore denote by $\nu(M) \in \mathcal{N}_f \cap \mathbb{Z}^n$ the order [3] of such a component.*

There exists a polyhedral complex $\mathcal{P}_f \subset \mathcal{N}_f$ with the following properties:

1. *The polyhedral complex $\mathcal{P}_f$ is a deformation retract of the compactified amoeba $\overline{\mathcal{A}}_f$.*
2. *For any complement component M of $\mathcal{N}_f \setminus \overline{\mathcal{A}}_f$ the only integer point that belongs to this component is its order: $M \cap \mathbb{Z}^n = \nu(M)$.*

Example 5.2 Consider the bivariate polynomial $f(x, y) = 1 + 3x + 3y + x^2y + 4x^3y + xy^2 + 10x^2y^2 + 4xy^3$. The amoeba $\mathcal{A}_f$, the compactified amoeba $\overline{\mathcal{A}}_f$, and the polyhedral complex $\mathcal{P}_f^\infty$ are shown in Fig. 7. We observe that this polynomial is not dense as the integer point $(1, 1)$ does to belong to its support. Moreover, this polynomial is not optimal since the complement of its amoeba lacks bounded components of orders $(1, 1)$, $(1, 2)$ and $(2, 1)$. The set of integer points in the Newton polygon $\mathcal{N}_f$ that do not belong to the polyhedral complex $\mathcal{P}_f^\infty$ consists of the orders of connected components in $\mathcal{N}_f \setminus \overline{\mathcal{A}}_f$. In this example, the polyhedral complex $\mathcal{P}_f^\infty$ has been computed as the limit of the weighted compactified amoebas of the Hadamard powers of $f(x, y)$.

Another deformation retract of an amoeba, its spine, has been defined and studied in [14]. The spine of the (affine) amoeba of a polynomial can be defined as the set where certain piecewise linear approximation of the Ronkin function [14] associated with this polynomial is nonsmooth. The spine of an amoeba is a proper subset of the amoeba itself and thus inherits all the problems that arise when one needs to determine the topological type of an amoeba or its position in the ambient affine space.

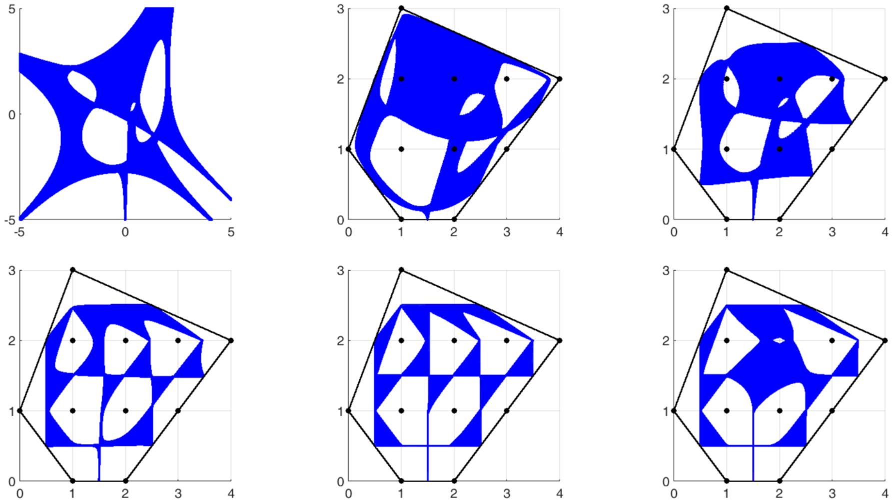


Fig. 8 The affine amoeba, the compactified amoeba, the weighted compactified amoebas of the 1st, 2nd and 3rd Hadamard powers of the polynomial $x + x^2 + y + xy^3 + x^4y^2 + 3x^3y + 10xy + 10x^2y + 10xy^2 + 15x^2y^2 + 10x^3y^2$, and a bounded component of its deformation vanishing at the lattice point (2, 2)

Polyhedral complexes related to amoebas have also been used in [11] for the analysis of topology of nonsingular algebraic hypersurfaces in projective spaces. The approach developed in [11] is based on Viro's patchworking and allows one to treat a complex algebraic hypersurface as a singular fibration over a polyhedral complex, the generic fiber being isomorphic to a smooth torus. This polyhedral complex is a subset of the Newton polytope of the defining polynomial of the algebraic hypersurface and is dual to a certain lattice subdivision of this polytope. However, it is generally different from (3.1) and has a different dimension (see Example 1 and Section 6.7 in [11]). Besides, it is defined through patchworking rather than the explicit formula (3.1).

Example 5.3 The zero locus of the polynomial $p(x, y) = x + x^2 + y + xy^3 + x^4y^2 + 3x^3y + 10xy + 10x^2y + 10xy^2 + 15x^2y^2 + 10x^3y^2$ is an optimal hypersurface. The computation of its affine amoeba requires considerable accuracy due to the very different relative size of the bounded connected components of its complement (see Fig. 8). The compactified amoeba of this polynomial is also a difficult set to compute since the vertex components are mapped by the moment map to very small regions inside the Newton polygon. Figure 8 features the evolution of the weighted compactified amoeba of the Hadamard powers of the polynomial $p(x, y)$ as the exponent ranges from 1 to 3. The rightmost down picture in Fig. 8 shows the weighted compactified amoeba of the 3rd Hadamard power of a deformation of $p(x, y)$. The small bounded component of the complement to this set shrinks and vanishes precisely at its order, that is, at the point $(2, 2) \in \mathcal{N}_p$.

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