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Looping Directions and Integrals of Eigenfunctions over Submanifolds

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Abstract Let (M, g) be a compact n-dimensional Riemannian manifold without boundary and e_λ be an L^2-normalized eigenfunction of the Laplace–Beltrami operator with respect to the metric g, i.e.,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda \quad \text{and} \quad \|e_\lambda\|_{L^2(M)} = 1.$$

Let Σ be a d-dimensional submanifold and $\mathrm{d}\mu$ a smooth, compactly supported measure on Σ. It is well known (e.g., proved by Zelditch, Commun Partial Differ Equ 17(1–2):221–260, 1992 in far greater generality) that

$$\int_\Sigma e_\lambda \, \mathrm{d}\mu = O\left(\lambda^{\frac{n-d-1}{2}}\right).$$

We show this bound improves to $o\left(\lambda^{\frac{n-d-1}{2}}\right)$ provided the set of looping directions,

$$\mathcal{L}_\Sigma = \{(x, \xi) \in \mathrm{SN}^*\Sigma : \Phi_t(x, \xi) \in \mathrm{SN}^*\Sigma \text{ for some } t > 0\}$$

has measure zero as a subset of $\mathrm{SN}^*\Sigma$, where here Φ_t is the geodesic flow on the cosphere bundle S^*M and $\mathrm{SN}^*\Sigma$ is the unit conormal bundle over Σ.

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1 Introduction

In what follows, (M, g) will denote a compact, boundaryless, n-dimensional Riemannian manifold. Let Δ_g denote the Laplace–Beltrami operator and e_λ an L^2-normalized eigenfunction of Δ_g on M, i.e.,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda \quad \text{and} \quad \|e_\lambda\|_{L^2(M)} = 1.$$

In [7], Sogge and Zelditch investigate which manifolds have a sequence of eigenfunctions e_λ with $\lambda \to \infty$ which saturate the standard sup-norm bound

$$\|e_\lambda\|_{L^\infty(M)} = O\left(\lambda^{\frac{n-1}{2}}\right).$$

They show this bound necessarily improves to $o\left(\lambda^{\frac{n-1}{2}}\right)$ if at each x, the set of looping directions through x,

$$\mathcal{L}_x = \left\{\xi \in S_x^*M : \Phi_t(x, \xi) \in S_x^*M \text{ for some } t > 0\right\}$$

has measure zero[1] as a subset of S_x^*M for each $x \in M$. Here, Φ_t denotes the geodesic flow on the unit cosphere bundle S^*M after time t. The hypotheses were later weakened by Sogge et al. [8], where they showed

$$\|e_\lambda\|_{L^\infty(M)} = o\left(\lambda^{\frac{n-1}{2}}\right)$$

provided the set of *recurrent* directions at x has measure zero for each $x \in M$.

We are interested in extending the result in [7] to integrals of eigenfunctions over submanifolds. Let Σ be a submanifold of dimension d with $d < n$ and a measure $d\mu(x) = h(x)d\sigma(x)$ where $d\sigma$ is the surface measure on Σ and h is a smooth function supported on a compact subset of Σ. In his 1992 paper [12], Zelditch proves, among other things, a Kuznecov asymptotic formula

$$\sum_{\lambda_j \le \lambda} \left| \int_\Sigma e_j \, d\mu \right|^2 \sim \lambda^{n-d} + O\left(\lambda^{n-d-1}\right), \tag{1.1}$$

where e_j for $j = 0, 1, 2, \ldots$ form a Hilbert basis of eigenfunctions on M with corresponding eigenvalues λ_j. From (1.1) follows the standard bound

$$\int_\Sigma e_\lambda \, d\mu = O\left(\lambda^{\frac{n-d-1}{2}}\right) \tag{1.2}$$

[1] Let $\psi_j : U_j \subset \mathbb{R}^n \to M$ be coordinate charts of a general manifold M. We say a set $E \subset M$ has measure zero if the preimage $\psi_j^{-1}(E)$ has Lebesgue measure 0 in $\mathbb{R}^n$ for each chart ψ_j. Sets of Lebesgue measure zero are preserved under transition maps, ensuring this definition is intrinsic to the C^∞ structure of M.

which is sharp[2] in general. However, it should be noted that (1.1) implies that generic eigenfunctions satisfy much better bounds. Indeed for any function $R(\lambda) \to +\infty$, an extraction argument shows there exists a density one sequence of eigenfunctions satisfying

$$\int_\Sigma e_\lambda \, \mathrm{d}\mu = O\left(\lambda^{-\frac{d}{2}} R(\lambda)\right).$$

Though (1.2) is already well known and has been proven in stronger terms, we state it here as a theorem. We do this for two reasons. First, it provides a baseline with which to compare our main result. Second, we end up providing a direct proof in the form of Proposition 2.1, which we will need for our main argument anyway.

Theorem 1.1 *Let Σ be a d-dimensional submanifold with $0 \le d < n$, and $\mathrm{d}\mu(x) = h(x)\mathrm{d}\sigma(x)$ where h is a smooth, real-valued function supported on a compact neighborhood in Σ. Then,*

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} \left| \int_\Sigma e_j \, \mathrm{d}\mu \right|^2 = O\left(\lambda^{n-d-1}\right).$$

(1.2) *follows.*

We let $\mathrm{SN}^*\Sigma$ denote the unit conormal bundle over Σ. We define the set of looping directions through Σ by

$$\mathcal{L}_\Sigma = \{(x, \xi) \in \mathrm{SN}^*\Sigma : \Phi_t(x, \xi) \in \mathrm{SN}^*\Sigma \text{ for some } t > 0\}.$$

A covector in $\mathcal{L}_\Sigma$ is the initial data of a geodesic which departs Σ conormally and eventually arrives again at Σ conormally. Our main result is as follows.

Theorem 1.2 *Assume the hypotheses of Theorem 1.1 and additionally that $\mathcal{L}_\Sigma$ has measure zero as a subset of $\mathrm{SN}^*\Sigma$. Then,*

$$\sum_{\lambda_j \in [\lambda, \lambda+\delta]} \left| \int_\Sigma e_\lambda \, \mathrm{d}\mu \right|^2 \le C\delta\lambda^{n-d-1} + C_\delta \lambda^{n-d-2},$$

where C is a constant independent of δ and λ, and C_δ is a constant depending on δ but not λ.

Sogge and Zelditch's result [7] implies the theorem for $d = 0$. We adapt their strategy to provide the proof for the remaining cases $d \ge 1$. The following theorem is an immediate corollary of Theorem 1.2 and shows (1.2) cannot be saturated whenever $\mathcal{L}_\Sigma$ has measure zero in $\mathrm{SN}^*\Sigma$.

[2] The standard examples take place on the sphere S^n. In the $d = 0$ case, this bound is saturated by the highest weight spherical harmonics. In the $d = n - 1$ situation, the bound is saturated by zonal functions around the 'equator.' In this section we show that, for *any* submanifold Σ in S^n, there exists some sequence of eigenfunctions saturating (1.2).

Theorem 1.3 *Assume the hypotheses of Theorem 1.2. Then,*

$$\int_{\Sigma} e_\lambda \, \mathrm{d}\mu = o\left(\lambda^{\frac{n-d-1}{2}}\right).$$

We illustrate Theorem 1.3 in three settings: the torus, the sphere, and surfaces with negative sectional curvature.

The torus Let $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ denote the flat, n-dimensional torus. Let Σ be a small patch of a sphere centered at 0 in $\mathbb{T}^n$. Since all geodesics passing through Σ in the conormal direction intersect the origin, the set of looping directions $\mathcal{L}_\Sigma$ is countable by the countability of $\mathbb{Z}^n$ and hence has measure 0. The conclusion of Theorem 1.3 is easily verified by the result [3, Proposition 3.5], which provides an essentially optimal bound on the integrals of eigenfunctions over hypersurfaces in the torus.

Proposition 1.4 ([3]) *Suppose Σ has nonvanishing Gaussian curvature in the torus. Then,*

$$\left| \int_{\Sigma} e_\lambda \mathrm{d}\mu(x) \right| = O\left(\lambda^{-1/2+\varepsilon}\right)$$

for all $\varepsilon > 0$, where we may set $\varepsilon = 0$ if $n \geq 5$.

Applying this result to spheres yields a much better bound than suggested by Theorem 1.3.

On the other hand if Σ is a closed hyperplane in $\mathbb{T}^n$, neither the hypotheses nor the conclusion of Theorem 1.3 are satisfied. All geodesics departing Σ conormally arrive again conormally after some fixed, uniform time. At the same time, one can construct a sequence of exponentials which are all identically 1 along Σ.

The sphere Let S^n denote the n-dimensional sphere equipped with the standard metric. Every geodesic in S^n is periodic, so $\mathcal{L}_\Sigma = \mathrm{SN}^*\Sigma$ for *every* submanifold Σ. Hence, no submanifold of S^n satisfies the hypotheses of Theorem 1.3. As we will find, no submanifold of S^n enjoys the little-o improvement of Theorem 1.3 either.

The functional ideas here are Zelditch's–Kuznecov formula (1.1) and the fact that all eigenvalues are of the form

$$\lambda = \sqrt{k(k+n-1)} \quad \text{for some } k = 0,\ 1,\ 2, \ldots$$

(see [5, Sect. 3.4] or [2, Theorem 3.1]). Let e_j for $j = 0,\ 1,\ 2, \ldots$ denote some Hilbert basis of eigenfunctions on M with corresponding eigenvalues λ_j. For each distinct eigenvalue λ, we construct an eigenfunction e_λ by

$$e_\lambda = \frac{\sum_{\lambda_j = \lambda} \left(\int_{\Sigma} \overline{e_j} \, \mathrm{d}\sigma \right) e_j}{\left(\sum_{\lambda_j = \lambda} \left| \int_{\Sigma} \overline{e_j} \, \mathrm{d}\sigma \right|^2 \right)^{1/2}}.$$

Note $\|e_\lambda\|_{L^2(M)} = 1$ and

$$\int_\Sigma e_\lambda \, d\sigma = \left(\sum_{\lambda_j = \lambda} \left| \int_\Sigma e_j \, d\sigma \right|^2 \right)^{1/2}. \tag{1.3}$$

By Zelditch's–Kuznecov formula (1.1), we have

$$\sum_{\lambda_j \in [\lambda, \lambda + C]} \left| \int_\Sigma e_j \, d\sigma \right|^2 \geq \lambda^{n-d-1}$$

for some large enough constant C. Moreover there are no more than $C + 1$ distinct eigenvalues in the interval $[\lambda, \lambda + C]$ for each λ. Hence in every interval of length C, there exists λ such that the right-hand side of (1.3) is bounded below by $\lambda^{\frac{n-d-1}{2}} / \sqrt{C+1}$, i.e., the bound in (1.2) is saturated.

Negatively curved surfaces We can use recent results to verify Theorem 1.3 for some examples where M is a surface ($n = 2$) with negative sectional curvature. Chen and Sogge [1] proved that if Σ is a geodesic in such a manifold M,

$$\int_\Sigma e_\lambda \, d\mu = o(1).$$

They consider a lift $\tilde{\Sigma}$ of Σ to the universal cover of M. Using the Gauss–Bonnet theorem, they show for each deck transformation α, there is at most one geodesic which intersects both $\tilde{\Sigma}$ and $\alpha(\tilde{\Sigma})$ perpendicularly. Since there are only countably many deck transformations, $\mathcal{L}_\Sigma$ is at most a countable subset of $SN^*\Sigma$ and so satisfies the hypotheses of Theorem 1.3. Since [1], Sogge et al. [9] have obtained an explicit decay of $O(1/\sqrt{\log \lambda})$ while also allowing the sectional curvature of M to vanish of finite order. Recently the author [10,11] obtained Chen and Sogge's $o(1)$ bound, and more recently the explicit bound $O(1/\sqrt{\log \lambda})$, if M has nonpositive sectional curvature and the geodesic curvature of Σ avoids that of circles of infinite radius. These curves similarly have countable $\mathcal{L}_\Sigma$ provided they are sufficiently short.

2 Microlocal Tools

The hypotheses on the looping directions in Theorem 1.2 ensure that the wavefront sets of μ and $e^{it\sqrt{-\Delta_g}}\mu$ have minimal intersection for any given t away from 0. We can then use pseudodifferential operators to break the measure μ into two parts, the first which has small essential support and the second whose wavefront set is disjoint from that of $e^{it\sqrt{-\Delta_g}}\mu$. The following propositions will allow us to handle these cases, respectively. The first proposition generalizes both [5, Lemma 5.2.2] and the standard Theorem 1.1 and is our main technical result.

Before we proceed, we lay out some Fermi local coordinates which we will return to repeatedly. Fix $p \in \Sigma$, and consider local coordinates $x = (x_1, \ldots, x_n) = (x', \bar{x})$

centered about p, where x' denotes the first d coordinates and $\bar{x}$ the remaining $n-d$ coordinates. We let $(x', 0)$ parametrize Σ on a neighborhood of p in such a way that $\mathrm{d}x'$ agrees with the surface measure on Σ. Let g denote the metric tensor with respect to our local coordinates. We require

$$g = \begin{bmatrix} * & 0 \\ 0 & I \end{bmatrix} \quad \text{wherever } \bar{x} = 0,$$

where I here is the $(n-d) \times (n-d)$ identity matrix. This is ensured after inductively picking smooth sections $v_j(x')$ of $\mathrm{SN}\Sigma$ for $j = d+1, \ldots, n$ with $\langle v_i, v_j \rangle = \delta_{ij}$, and then using

$$(x_1, \ldots, x_n) \mapsto \exp\left(x_{d+1} v_{d+1}(x') + \cdots + x_n v_n(x')\right) \tag{2.1}$$

as our coordinate map. In these coordinates we write $\mathrm{d}\mu(x) = h(x')\,\mathrm{d}x'$ where h is a smooth, compactly supported function on $\mathbb{R}^d$.

Proposition 2.1 *Let $b(x, \xi)$ be smooth for $\xi \neq 0$ and homogeneous of degree 0 in the ξ variable. We define $b \in \Psi^0_{\mathrm{cl}}(M)$ by*

$$b(x, D) f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x-y, \xi\rangle} b(x, \xi)\,\mathrm{d}y\,\mathrm{d}\xi$$

for x, y, and ξ expressed locally according to our coordinates (2.1). Then,

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} \left| \int_\Sigma b e_j \,\mathrm{d}\mu \right|^2 \le C \left(\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x', \omega)|^2 h(x')^2 \,\mathrm{d}\omega\,\mathrm{d}x' \right) \lambda^{n-d-1} + C_b \lambda^{n-d-2},$$

where C is a constant independent of b and λ and C_b is a constant independent of λ but which depends on b.

Note Theorem 1.1 follows by setting $b \equiv 1$. The proof is based largely on that of [5, Lemma 5.2.2]. We will come to a point in our argument where it seems like we may have to perform a stationary phase argument involving an eight-by-eight Hessian matrix. Instead, we appeal to [4, Theorem 7.7.6] to break this process into two steps involving two four-by-four Hessian matrices.

Proof For simplicity, we assume without loss of generality that $\mathrm{d}\mu$ is a real measure. Let χ be a nonnegative Schwartz-class function on $\mathbb{R}$ with $\chi(0) = 1$ and $\hat{\chi}$ supported on a small neighborhood of 0.[3] It suffices to show

[3] This reduction is standard and appears in [1,7], proofs of sup-norm estimates of eigenfunctions and the sharp Weyl law as presented in [5,6], and in many other related problems.

$$\begin{aligned}&\sum_j \chi\left(\lambda_j-\lambda\right)\left|\int_\Sigma b e_j \,\mathrm{d}\mu\right|^2\\&\quad\sim\left(\int_{\mathbb{R}^d}\int_{S^{n-d-1}}|b(y', \omega)|^2 h(y')^2 \,\mathrm{d}\omega\,\mathrm{d}y'\right)\lambda^{n-d-1}+O_b\left(\lambda^{n-d-2}\right).\end{aligned}$$

We may by a partition of unity assume that $b(x, D)$ has small x-support. The left-hand side is

$$\begin{aligned}&=\sum_j\int_\Sigma\int_\Sigma \chi\left(\lambda_j-\lambda\right) b(x, D)e_j(x)\overline{b(y, D)e_j(y)}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\&=\frac{1}{2\pi}\sum_j\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}\hat{\chi}(t)\mathrm{e}^{it(\lambda_j-\lambda)}b(x, D)e_j(x)\overline{b(y, D)e_j(y)}\,\mathrm{d}t\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\&=\frac{1}{2\pi}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}\hat{\chi}(t)\mathrm{e}^{-it\lambda}b\mathrm{e}^{it\sqrt{-\Delta_g}}b^*(x, y)\,\mathrm{d}t\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y), \qquad (2.2)\end{aligned}$$

where here $b\mathrm{e}^{it\sqrt{-\Delta_g}}b^*(x, y)$ is the kernel

$$\sum_j \mathrm{e}^{it\lambda_j} b(x, D)e_j(x)\overline{b(y, D)e_j(y)}$$

of the half-wave operator $\mathrm{e}^{it\sqrt{-\Delta_g}}$ conjugated by b. Set $\beta \in C_0^\infty(\mathbb{R})$ with small support and where $\beta \equiv 1$ near 0. Then,

$$\begin{aligned}&\int_{\mathbb{R}^d} b(x', D)f(x')h(x')\,\mathrm{d}x'\\&\quad=\frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\mathrm{e}^{i\lambda\langle x'-w,\eta\rangle}b(x', \eta)f(w)h(x')\,\mathrm{d}x'\,\mathrm{d}w\,\mathrm{d}\eta\\&\quad=\frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\mathrm{e}^{i\lambda\langle x'-w,\eta\rangle}\beta(\log|\eta|)b(x', \eta)f(w)h(x')\,\mathrm{d}x'\,\mathrm{d}w\,\mathrm{d}\eta\\&\qquad+O\left(\lambda^{-N}\right), \qquad (2.3)\end{aligned}$$

where the second line is obtained by a change of variables $\eta \mapsto \lambda\eta$, and the third line is obtained after multiplying in the cutoff $\beta(\log|\eta|)$ and bounding the discrepancy by $O(\lambda^{-N})$ by integrating by parts in x'. Additionally,

$$\begin{aligned}b^*(z, D)d\mu(z)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\mathrm{e}^{i\langle z-y',\zeta\rangle}b(y', \zeta)h(y')\,\mathrm{d}y'\,\mathrm{d}\zeta\\&=\frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\mathrm{e}^{i\lambda\langle z-y',\zeta\rangle}b(y', \zeta)h(y')\,\mathrm{d}y'\,\mathrm{d}\zeta\\&=\frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\mathrm{e}^{i\lambda\langle z-y',\zeta\rangle}\beta(\log|\zeta|)b(y', \zeta)h(y')\,\mathrm{d}y'\,\mathrm{d}\zeta+O\left(\lambda^{-N}\right)\end{aligned}$$

$$= \frac{\lambda^n}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} e^{i\lambda\langle z-y',\zeta\rangle} \beta(\log|\zeta|)\beta(|z-y'|)b(y',\zeta)h(y')\,dy'\,d\zeta$$
$$+ O\left(\lambda^{-N}\right), \tag{2.4}$$

where the second and third lines are obtained similarly as before and the fourth line is obtained after multiplying by $\beta(\log|z-y'|)$ and integrating the remainder by parts in ζ.

We will use Hörmander's parametrix of the half-wave operator $e^{it\sqrt{-\Delta_g}}$ on M to treat the integral in t. In local coordinates (2.1), we have

$$e^{it\sqrt{-\Delta_g}}(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\varphi(x,y,\xi)+tp(y,\xi))} q(t, x, y, \xi)\,d\xi$$

modulo a smooth kernel where

$$p(y, \xi) = \sqrt{\sum_{j,k} g^{ij}(y)\xi_i\xi_j},$$

where q is a symbol in ξ satisfying bounds

$$\left|\partial^\alpha_{t,x,y}\partial^\beta_\xi q(t, x, y, \xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{-|\beta|}$$

for all multiindices α and β, and where φ is homogeneous of degree 1 in ξ and smooth for $\xi \neq 0$ and satisfies

$$\varphi(x, y, \xi) = \langle x-y, \xi\rangle + O\left(|x-y|^2|\xi|\right). \tag{2.5}$$

This parametrix is valid only when $|t|$ and $|x-y|$ are small. In fact, we may take q to be supported on an arbitrarily small neighborhood of the diagonal $x = y$ of our choosing provided we only consider times t on a correspondingly small neighborhood of 0. (See [6, Chap. 4] for a treatment of Hörmander's parametrix.)

Using Hörmander's parametrix,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\chi}(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}(w, z)\,dt$$
$$= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i(\varphi(w,z,\xi)+tp(z,\xi)-t\lambda)}\hat{\chi}(t)q(t, w, z, \xi)\,d\xi\,dt$$
$$= \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i\lambda(\varphi(w,z,\xi)+t(p(z,\xi)-1))}\hat{\chi}(t)q(t, w, z, \lambda\xi)\,d\xi\,dt$$
$$= \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i\lambda(\varphi(w,z,\xi)+t(p(z,\xi)-1))}\beta(\log p(z,\xi))\hat{\chi}(t)q(t, w, z, \lambda\xi)\,d\xi$$
$$+ O\left(\lambda^{-N}\right). \tag{2.6}$$

Here the third line comes from a change of coordinates $\xi \mapsto \lambda\xi$. The fourth line follows after applying the cutoff $\beta(\log p(z, \zeta))$ and integrating the discrepancy by parts in t. Combining (2.3), (2.4), and (2.6), we write (2.2) as

$$\lambda^{3n} \int \cdots \int e^{i\Phi(t,x',y',w,z,\eta,\zeta,\xi)} a(\lambda;\, t,\, x',\, y',\, w,\, z,\, \eta,\, \zeta,\, \xi) \\ dx'\, dy'\, dw\, dz\, d\eta\, d\zeta\, d\xi + O\left(\lambda^{-N}\right) \tag{2.7}$$

with amplitude

$$a(\lambda;\, t,\, x',\, y',\, w,\, z,\, \eta,\, \zeta,\, \xi) = \frac{1}{(2\pi)^{3n+1}} \hat{\chi}(t) q(t,\, w,\, z,\, \lambda\xi) \beta(\log p(z,\, \xi)) \beta(\log|\eta|) \\ \beta(\log|\zeta|) \beta(|z - y'|) b(x',\, \eta) b(y',\, \zeta) h(x') h(y')$$

and phase

$$\Phi(t,\, x',\, y',\, w,\, z,\, \eta,\, \zeta,\, \xi) = \langle x' - w,\, \eta\rangle + \varphi(w,\, z,\, \xi) + t(p(z,\, \xi) - 1) + \langle z - y',\, \zeta\rangle.$$

We pause here to make a couple of observations. First, a has compact support in all variables, support which we may adjust to be smaller by controlling the supports of $\hat{\chi}$, β, b, and the support of q near the diagonal. Second, the derivatives of a are bounded independently of $\lambda \geq 1$. We are now in a position to use the method of stationary phase—not in all variables at once, though. First, we fix t, x', y', and ξ, and use stationary phase in w, z, η, and ζ. We have

$$\begin{aligned} \nabla_w \Phi &= -\eta + \nabla_w \varphi(w,\, z,\, \xi), \\ \nabla_z \Phi &= \nabla_z \varphi(w,\, z,\, \xi) + t\nabla_z p(z,\, \xi) + \zeta, \\ \nabla_\eta \Phi &= x' - w, \\ \nabla_\zeta \Phi &= z - y' \end{aligned}$$

which all simultaneously vanish if and only if

$$(w,\, z,\, \eta,\, \zeta) = \left(x',\, y',\, \nabla_x \varphi(x',\, y',\, \xi),\, -\nabla_y \varphi(x',\, y',\, \xi) - t\nabla_y p(y',\, \xi)\right). \tag{2.8}$$

At such a critical point we have the Hessian matrix

$$\nabla^2_{w,z,\eta,\zeta} \Phi = \begin{bmatrix} * & * & -I & 0 \\ * & * & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

which has determinant -1. By [4, Theorem 7.7.6], (2.7) is equal to a complex constant times

$$\begin{aligned}\lambda^n &\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\Phi(t,x',y',\xi)} a(\lambda;\, t,\, x',\, y',\, \xi)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}\xi'\,\mathrm{d}t\\
&+\lambda^{n-1}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\Phi(t,x',y',\xi)} R_N(\lambda;\, t,\, x',\, y',\, \xi)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}\xi'\,\mathrm{d}t\\
&+O\left(\lambda^{-N}\right) \end{aligned} \tag{2.9}$$

where we have phase

$$\Phi(t,\, x',\, y',\, \xi) = \varphi(x',\, y',\, \xi) + t(p(y',\, \xi) - 1),$$

amplitude

$$a(\lambda;\, t,\, x',\, y',\, \xi) = a(\lambda;\, t,\, x',\, y',\, w,\, z,\, \eta,\, \zeta,\, \xi)$$

with $w,\ z,\ \eta$, and ζ subject to the constraints (2.8), where R_N is a compactly supported smooth function in $t,\ x',\ y'$, and ξ, whose derivatives are bounded uniformly with respect to λ, and where N can be taken to be as large as desired.

Write $\xi = (\xi',\, \bar{\xi})$ and write $\bar{\xi} = r\omega$ in polar coordinates with $r \geq 0$ and $\omega \in S^{n-d-1}$. The first integral in (2.9) is then written

$$\begin{aligned}\lambda^n &\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{S^{n-d-1}}\int_0^{\infty} \mathrm{e}^{i\lambda\Phi(t,x',y',\xi)} a(\lambda;\, t,\, x',\, y',\, \xi)\\
&r^{n-d-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\xi'\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t.\end{aligned}$$

We will fix y' and ω and use the method of stationary phase in the remaining variables $t,\ x',\ \xi'$, and r (a total of $2d+2$ dimensions). We assert that, for fixed y' and ω, there is a nondegenerate stationary point at $(t,\, x',\, \xi',\, r) = (0,\, y',\, 0,\, 1)$. $\Phi = 0$ at such a stationary point, and after perhaps shrinking the support of a we apply [4, Theorem 7.7.6] again to write the first integral in (2.9) as constant times

$$\lambda^{n-d-1}\int_{\mathbb{R}^d}\int_{S^{n-d-1}} a(\lambda;\, 0,\, y',\, y',\, \omega)\,\mathrm{d}y'\,\mathrm{d}\omega + O\left(\lambda^{n-d-2}\right).$$

The proposition will follow after noting $a(\lambda;\, 0,\, y',\, y',\, \omega) = |b(y',\, \omega)|^2 h(y')^2$ and applying the same stationary phase argument to the second integral in (2.9).

We have

$$\begin{aligned}\partial_t\Phi &= p(y',\, \xi) - 1,\\
\nabla_{x'}\Phi &= \nabla_{x'}\varphi(x',\, y',\, \xi),\\
\nabla_{\xi'}\Phi &= \nabla_{\xi'}\varphi(x',\, y',\, \xi) + t\nabla_{\xi'}p(y',\, \xi),\\
\partial_r\Phi &= \partial_r\varphi(x',\, y',\, \xi) + t\partial_r p(y',\, \xi).\end{aligned}$$

Note for fixed y' and ω, $(t,\, x',\, \xi',\, r) = (0,\, y',\, 0,\, 1)$ is a critical point of Φ. Now we compute the second derivatives at this point. We immediately see that $\partial_t^2\Phi,\ \partial_t\nabla_{x'}\Phi$,

$\nabla^2_{\xi'}\Phi$, $\partial_r \nabla_{\xi'}\Phi$, and $\partial_r^2\Phi$ all vanish. Moreover, $\partial_r\partial_r\Phi = 1$ since $p(y', \xi) = r$, where $\xi' = 0$. By our coordinates (2.1) and the fact that $[g^{ij}]_{i,j\le d}$ is necessarily positive definite,

$$p(y', \xi) = \sqrt{\sum_{j,k} g^{jk}\xi_j\xi_k} = \sqrt{r^2 + \sum_{j,k\le d} g^{jk}\xi'_j\xi'_k} \ge r = p(y', r\omega).$$

Hence, $\partial_t \nabla_{\xi'}\Phi = \nabla_{\xi'} p(y', \xi) = 0$. Since φ is homogeneous of degree 1 in ξ, at $\xi' = 0$ and $t = 0$,

$$\nabla_{x'}\partial_r\Phi = \nabla_{x'}\partial_r\varphi(x', y', \xi) = \nabla_{x'}\varphi(x', y', \omega) = 0$$

since $\varphi(x', y', \omega) = O(|x' - y'|^2)$ by (2.5) and the fact that $\langle x' - y', \omega\rangle = 0$. Finally by (2.5),

$$\nabla_{\xi'}\varphi(x', y', \xi' + \omega) = x' + O\left(|x' - y'|^2\right)$$

whence at the critical point

$$\nabla_{x'}\nabla_{\xi'}\Phi = I,$$

the $d \times d$ identity matrix. In summary, the Hessian matrix of Φ at the critical point $(t, x', \xi', r) = (0, y', 0, 1)$ is

$$\nabla^2_{t,x',\xi',r}\Phi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & * & I & 0 \\ 0 & I & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which has full rank. This concludes the proof of Proposition 2.1. $\square$

The second proposition, below, allows us to deal with the partition of μ whose wavefront set is disjoint from that of $e^{it\sqrt{-\Delta_g}}\mu$ for $t > 0$.

Proposition 2.2 *Let u and v be distributions on M for which*

$$\mathrm{WF}(u) \cap \mathrm{WF}(v) = \emptyset.$$

Then

$$t \mapsto \int_M e^{it\sqrt{-\Delta_g}} u(x)\overline{v(x)}\, dx$$

is a smooth function of t on some neighborhood of 0.

Proof Using a partition of unity, we write

$$I = \sum_j A_j$$

modulo a smoothing operator where $A_j \in \Psi^0_{\mathrm{cl}}(M)$ with essential supports in small conic neighborhoods. We then write, formally,

$$\int_M \mathrm{e}^{it\sqrt{-\Delta_g}} u(x)\overline{v(x)}\,\mathrm{d}x = \sum_{j,k} \int_M A_j \mathrm{e}^{it\sqrt{-\Delta_g}} u(x)\overline{A_k v(x)}\,\mathrm{d}x.$$

We are done if for each i and j,

$$\int_M A_j \mathrm{e}^{it\sqrt{-\Delta_g}} u(x)\overline{A_k v(x)}\,\mathrm{d}x \quad \text{is smooth for } |t| \ll 1. \tag{2.10}$$

If the essential supports of A_j and A_k are disjoint, then $A_j^* A_k$ is a smoothing operator, and so $A_j^* A_k v$ is a smooth function and the contributing term

$$\int_M u(x)\overline{\mathrm{e}^{it\sqrt{-\Delta_g}} A_j^* A_k v(x)}\,\mathrm{d}x$$

is smooth is t. Assume the essential support of A_j is small enough so that for each j there exists a small conic neighborhood Γ_j which fully contains the essential support of A_k if it intersects the essential support of A_j. We in turn take Γ_j small enough so that for each j, $\overline{\Gamma_j}$ either does not intersect $\mathrm{WF}(u)$ or does not intersect $\mathrm{WF}(v)$. In the latter case, $A_k v$ is smooth and we have (2.10) as before. In the former case,

$$\overline{\Gamma_j} \cap \mathrm{WF}\left(\mathrm{e}^{it\sqrt{-\Delta_g}} u\right) = \emptyset \quad \text{for } |t| \ll 1$$

since both sets above are closed and the geodesic flow is continuous. Then $A_j \mathrm{e}^{it\sqrt{-\Delta_g}} u(x)$ is smooth as a function of t and x, and we have (2.10). □

3 Proof of Theorem 1.2

We make a few convenient assumptions. First, we take the injectivity radius of M to be at least 1 by scaling the metric g. Second, we assume the support of $\mathrm{d}\mu$ has diameter less than $1/2$ by a partition of unity. We reserve the right to further scale the metric g and restrict the support of $\mathrm{d}\mu$ as needed, finitely many times.

As before, we set $\chi \in C^\infty(\mathbb{R})$ with $\chi(0) = 1$, $\chi \geq 0$, and $\operatorname{supp} \hat{\chi} \subset [-1, 1]$. It suffices to show

$$\sum_j \chi\left(T\left(\lambda_j - \lambda\right)\right) \left| \int_\Sigma e_\lambda\,\mathrm{d}\mu \right|^2 \leq C T^{-1} \lambda^{n-d-1} + C_T \lambda^{n-d-2}$$

for $T > 1$. Similar to the reduction in the proof of Proposition 2.1, the left-hand side is equal to

$$\begin{aligned}
&\sum_j \int_\Sigma \int_\Sigma \chi\left(T\left(\lambda_j - \lambda\right)\right) e_j(x)\overline{e_j(y)}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \\
&= \frac{1}{2\pi} \sum_j \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t) \mathrm{e}^{itT(\lambda_j - \lambda)} e_j(x)\overline{e_j(y)}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\, \mathrm{d}t \\
&= \frac{1}{2\pi T} \sum_j \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t/T) \mathrm{e}^{-it\lambda} \mathrm{e}^{it\lambda_j} e_j(x)\overline{e_j(y)}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\, \mathrm{d}t \\
&= \frac{1}{2\pi T} \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t/T) \mathrm{e}^{-it\lambda} \mathrm{e}^{it\sqrt{-\Delta_g}}(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\, \mathrm{d}t.
\end{aligned}$$

Hence, it suffices to show

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t/T) \mathrm{e}^{-it\lambda} \mathrm{e}^{it\sqrt{-\Delta_g}}(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\, \mathrm{d}t \right| \\
&\quad \leq C\lambda^{n-d-1} + C_T \lambda^{n-d-2}.
\end{aligned} \tag{3.1}$$

Let $\beta \in C_0^\infty(\mathbb{R})$ be supported on a small interval about 0 with $\beta \equiv 1$ near 0. We cut the integral in (3.1) into $\beta(t)$ and $1 - \beta(t)$ parts. Since $\beta(t)\hat{\chi}(t/T)$ and its derivatives are all bounded independently of $T \geq 1$,

$$\left| \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \beta(t)\hat{\chi}(t/T) \mathrm{e}^{-it\lambda} \mathrm{e}^{it\sqrt{-\Delta_g}}(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\, \mathrm{d}t \right| \leq C\lambda^{n-d-1}$$

by the arguments in Proposition 2.1. Hence, it suffices to show

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma (1 - \beta(t))\hat{\chi}(t/T) \mathrm{e}^{-it\lambda} \mathrm{e}^{it\sqrt{-\Delta_g}}(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\, \mathrm{d}t \right| \\
&\quad \leq C\lambda^{n-d-1} + C_T \lambda^{n-d-2}.
\end{aligned} \tag{3.2}$$

Here we shrink the support of μ so that $\beta(d_g(x, y)) = 1$ for $x, y \in \operatorname{supp} \mu$. We now state and prove a useful decomposition based off of those in [7,8], and [5, Chap. 5]. We let $\mathcal{L}_\Sigma(\operatorname{supp} \mu, T)$ denote the subset of $\mathcal{L}_\Sigma$ relevant to the support of μ and the timespan $[1, T]$, specifically

$$\begin{aligned}
\mathcal{L}_\Sigma(\operatorname{supp} \mu, T) = \big\{&(x, \xi) \in \mathrm{SN}^*\Sigma : \Phi_t(x, \xi) = (y, \eta) \in \mathrm{SN}^*\Sigma \\
&\text{for some } t \in [1, T] \quad \text{and where } x, y \in \operatorname{supp} \mu\big\}.
\end{aligned}$$

Lemma 3.1 *Fix $T > 1$ and $\varepsilon > 0$. There exists $b, B \in \Psi_{\mathrm{cl}}^0(M)$ supported on a neighborhood of* $\operatorname{supp} \mu$ *with the following properties.*

(1) $b(x, D) + B(x, D) = I$ *on* $\operatorname{supp} \mu$.

(2) *Using coordinates* (2.1),

$$\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x', \omega)|^2 \, d\omega \, dx' < \varepsilon,$$

where $b(x, \xi)$ *is the principal symbol of* $b(x, D)$.

(3) *The essential support of* $B(x, D)$ *contains no elements of* $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$.

Proof As shorthand, we write

$$\mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma = \{(x, \xi) \in \mathrm{SN}^*\Sigma : x \in \operatorname{supp}\mu\}.$$

We first argue that $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ is closed for each $T > 1$. However, $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ is the projection of the set

$$\left\{(t, x, \xi) \in [1, T] \times \mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma : \Phi_t(x, \xi) \in \mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma\right\} \tag{3.3}$$

onto $\mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$, and since $[1, T]$ is compact it suffices to show that (3.3) is closed. However, (3.3) is the intersection of $[1, T] \times \mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$ with the preimage of $\mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$ under the continuous map

$$(t, x, \xi) \mapsto \Phi_t(x, \xi).$$

Since $\mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$ is closed, (3.3) is closed.

Since $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ is closed and has measure zero, there is $\tilde{b} \in C^\infty(S^*M)$ supported on a neighborhood of $\mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$ with $0 \le \tilde{b}(x, \xi) \le 1$, $\tilde{b}(x, \xi) \equiv 1$ on an open neighborhood of $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$, and

$$\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |\tilde{b}(x', \omega)|^2 \, d\omega \, dx' < \varepsilon. \tag{3.4}$$

We set $\psi \in C_0^\infty(\Sigma)$ to be a cutoff function supported on a neighborhood of $\operatorname{supp}\mu$ in M with $\psi \equiv 1$ on $\operatorname{supp}\mu$. We use the coordinates in (2.1) and define symbols

$$b(x, \xi) = \psi(x)\tilde{b}(x, \xi/|\xi|)$$

and

$$B(x, \xi) = \psi(x)(1 - \tilde{b}(x', \xi/|\xi|)),$$

along with their associated operators

$$b(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi\rangle} b(x, \xi) f(y) \, dy \, d\xi$$

and

$$B(x,\ D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x-y,\xi\rangle} B(x,\ \xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

By construction,

$$B(x,\ D) + b(x,\ D) = \psi(x),$$

whose restriction to supp μ is 1, yielding (1). (2) Follows from the definition of $b(x,\ D)$ and (3.4). We have (3) since the support of $1 - \tilde{b}(x,\ \xi)$ contains no elements of $\mathcal{L}_\Sigma(\operatorname{supp}\mu,\ T)$. □

Returning to the proof of Theorem 1.2, let X_T denote the function with

$$\hat{X}_T(t) = (1 - \beta(t))\hat{\chi}(t/T),$$

and let $X_{T,\lambda}$ denote the operator with kernel

$$X_{T,\lambda}(x,\ y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_T(t) \mathrm{e}^{-it\lambda} \mathrm{e}^{it\sqrt{-\Delta_g}}(x,\ y)\,\mathrm{d}t.$$

We use part (1) of Lemma 3.1 to write the integral in (3.2) as

$$\begin{aligned}\int_\Sigma \int_\Sigma X_{T,\lambda}(x,\ y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) &= \int_\Sigma \int_\Sigma BX_{T,\lambda}B^*(x,\ y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &\quad + \int_\Sigma \int_\Sigma BX_{T,\lambda}b^*(x,\ y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &\quad + \int_\Sigma \int_\Sigma bX_{T,\lambda}B^*(x,\ y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &\quad + \int_\Sigma \int_\Sigma bX_{T,\lambda}b^*(x,\ y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x).\end{aligned}$$

We claim the first three terms on the right are $O_T(\lambda^{-N})$ for $N = 1, 2, \ldots$ We will only prove this for the first term—the argument is the same for the second term and the bound for the third term follows since $X_{T,\lambda}$ is self-adjoint. Interpreting μ as a distribution on M, we write formally

$$\begin{aligned}&\int_\Sigma \int_\Sigma BX_{T,\lambda}B^*(x,\ y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &\quad = \int_M \int_M X_{T,\lambda}(x,\ y) B^*\mu(y)\overline{B^*\mu(x)}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_T(t)\mathrm{e}^{-it\lambda} \int_M \mathrm{e}^{it\sqrt{-\Delta_g}}(B^*\mu)(x)\overline{B^*\mu(x)}\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.5)\end{aligned}$$

Once we show

$$\mathrm{WF}\left(\mathrm{e}^{it\sqrt{-\Delta_g}}B^*\mu\right)\cap \mathrm{WF}(B^*\mu)=\emptyset \quad \text{for all } t\in \operatorname{supp}\hat{X}_T, \tag{3.6}$$

the integral over M will be smooth in t by Proposition 2.2. Integration by parts in t then gives the desired bound of $O_T(\lambda^{-N})$. By the calculus of wavefront sets and pseudodifferential operators,

$$\mathrm{WF}(B^*\mu)\subset \operatorname{esssupp} B\cap N^*_{\operatorname{supp}\mu}\Sigma.$$

To prove (3.6), suppose $(x,\ \xi)$ is a unit covector in $\mathrm{WF}(B^*\mu)$. By part (3) of Lemma 3.1, $\Phi_t(x,\ \xi)$ is not in $\mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$ for any $1\le |t|\le T$. By propagation of singularities,

$$\mathrm{WF}\left(\mathrm{e}^{it\sqrt{-\Delta_g}}B^*\mu\right)=\Phi_t\,\mathrm{WF}(B^*\mu),$$

hence

$$\mathrm{WF}\left(\mathrm{e}^{it\sqrt{-\Delta_g}}B^*\mu\right)\cap \mathrm{WF}(B^*\mu)=\emptyset \quad \text{for } 1\le |t|\le T. \tag{3.7}$$

Since the support of μ has been made small, if there is $(x,\ \xi)\in \mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$ and some $t>0$ in the support of $(1-\beta(t))\hat{\chi}(t/T)$ for which $\Phi_t(x,\ \xi)\in \mathrm{SN}^*_{\operatorname{supp}\mu}\Sigma$, then $t\ge 1$ since the diameter of $\operatorname{supp}\mu$ is small and the injectivity radius of M is at least 1. We now have (3.6), from which follows (3.5) as promised.

What remains is to bound

$$\left|\int_\Sigma\int_\Sigma bX_{T,\lambda}b^*(x,\ y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right|\le \lambda^{n-d-1}+C_{T,b}\lambda^{n-d-2}. \tag{3.8}$$

We have

$$bX_{T,\lambda}b^*(x,\ y)=\sum_j X_T\left(\lambda_j-\lambda\right)be_j(x)\overline{be_j(y)},$$

and so we write the integral in (3.8) as

$$\sum_j X_T\left(\lambda_j-\lambda\right)\left|\int_\Sigma b(x,\ D)e_j(x)\,\mathrm{d}\mu(x)\right|^2. \tag{3.9}$$

Note X_T satisfies bounds

$$|X_T(\tau)|\le C_{T,N}(1+|\tau|)^{-N}\quad \text{for } N=1,\ 2,\ldots \tag{3.10}$$

We dominate $|X_T|$ by a step function

$$\sum_{k\in\mathbb{Z}} a_{T,k}\chi_{[k,k+1)}$$

satisfying similar bounds as $|X_T|$ with coefficients

$$a_{T,k} = \sup_{[k,k+1]} |X_T| .$$

Now,

$$\begin{aligned}
&\left|\sum_j X_T\left(\lambda_j - \lambda\right)\left|\int_\Sigma b(x,\ D)e_j(x)\,\mathrm{d}\mu(x)\right|^2\right| \\
&\quad \le \sum_{k\in\mathbb{Z}} a_{T,k} \sum_{\lambda_j-\lambda\in[k,k+1)} \left|\int_\Sigma b(x,\ D)e_j(x)\,\mathrm{d}\mu(x)\right|^2 . \qquad (3.11)
\end{aligned}$$

Using Proposition 2.1 and part (2) of Lemma 3.1, we write

$$\begin{aligned}
&\sum_{\lambda_j-\lambda\in[k,k+1)} \left|\int_\Sigma b(x,\ D)e_j(x)\,\mathrm{d}\mu(x)\right|^2 \\
&\quad \le \varepsilon(|\lambda+k|+1)^{n-d-1} + C_b(|\lambda+k|+1)^{n-d-2}.
\end{aligned}$$

Hence, (3.11) is bounded by

$$\begin{aligned}
&\le C_T \sum_{k\in\mathbb{Z}} a_{T,k}\left(\varepsilon(|\lambda+k|+1)^{n-d-1} + C_b(|\lambda+k|+1)^{n-d-2}\right) \\
&\le \varepsilon C_T \lambda^{n-d-1} + C_{T,b}\lambda^{n-d-2} \quad \text{for } \lambda \ge 1
\end{aligned}$$

by the bounds (3.10). Taking ε in part (2) of Lemma 3.1 small enough so that $\varepsilon C_T \le 1$ yields (3.8). This concludes the proof of Theorem 1.2.

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