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Comparison Geometry for Integral Bakry–Émery Ricci Tensor Bounds

Jia-Yong Wu[1]

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Abstract We prove mean curvature and volume comparison estimates on smooth metric measure spaces when their integral Bakry–Émery Ricci tensor bounds, extending Wei–Wylie's comparison results to the integral case. We also apply comparison results to get diameter estimates, eigenvalue estimates, and volume growth estimates on smooth metric measure spaces with their normalized integral smallness for Bakry–Émery Ricci tensor. These give generalizations of some work of Petersen–Wei, Aubry, Petersen–Sprouse, Yau and more.

Keywords Bakry–Émery Ricci tensor · Smooth metric measure space · Integral curvature · Comparison theorem · Diameter estimate · Eigenvalue estimate · Volume growth estimate

Mathematics Subject Classification Primary 53C20

1 Introduction and Main Results

In [18], Petersen and Wei generalized the classical relative Bishop–Gromov volume comparison to a situation where one has an integral bound for the Ricci tensor. Let's briefly recall their results. Given an n-dimensional complete Riemannian manifold M, for each $x \in M$ let $\lambda(x)$ be the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_x M \to T_x M$, and

✉ Jia-Yong Wu
jywu81@yahoo.com

1 Department of Mathematics, Shanghai Maritime University, Shanghai 201306, People's Republic of China

$$\mathrm{Ric}_-^H(x) := ((n-1)H - \lambda(x))_+ = \max\{0, (n-1)H - \lambda(x)\},$$

where $H \in \mathbb{R}$, the amount of Ricci tensor below $(n-1)H$. Define

$$\left\|\mathrm{Ric}_-^H\right\|_p(R) := \sup_{x\in M}\left(\int_{B(x,R)} \left(\mathrm{Ric}_-^H\right)^p \mathrm{d}v\right)^{\frac{1}{p}},$$

which measures the amount of Ricci tensor lying below $(n-1)H$, in the L^p sense. Clearly, $\|\mathrm{Ric}_-^H\|_p(R) = 0$ iff $\mathrm{Ric} \geq (n-1)H$. Also let $r(y) = d(y, x)$ be the distance function from x to y, and

$$\varphi(y) := (\Delta r - m_H)_+,$$

where m_H is the mean curvature of the geodesic sphere in M_H^n, the n-dimensional simply connected space with constant sectional curvature H. The classical Laplacian comparison states that if $\mathrm{Ric} \geq (n-1)H$, then $\Delta\, r \leq m_H$. That is to say, if $\mathrm{Ric}_-^H \equiv 0$, then $\varphi \equiv 0$. In fact this comparison result was generalized to integral Ricci tensor lower bound.

Theorem A (Petersen–Wei [18]) *Let M be an n-dimensional complete Riemannian manifold. For any $p > \frac{n}{2}$, $H \in \mathbb{R}$ (assume $r \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$),*

$$\|\varphi\|_{2p}(r) \leq \left[\frac{(n-1)(2p-1)}{2p-n}\left\|\mathrm{Ric}_-^H\right\|_p(r)\right]^{\frac{1}{2}}.$$

Consequently, for any $0 < r \leq R$ (assume $R \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$), there exists a constant $C(n, p, H, R)$ which is non-decreasing in R, such that

$$\left(\frac{V(x,R)}{V_H(R)}\right)^{\frac{1}{2p}} - \left(\frac{V(x,r)}{V_H(r)}\right)^{\frac{1}{2p}} \leq C(n,p,H,R)\left(\left\|\mathrm{Ric}_-^H\right\|_p(R)\right)^{\frac{1}{2}},$$

where $V(x, R)$ denotes the volume of ball $B(x, R)$ in M, and $V_H(R)$ denotes the volume of ball $B(O, R)$ in the model space M_H, where $O \in M_H$.

Petersen and Wei [18,19] used these comparison estimates to extend many classical results of pointwise Ricci tensor condition to the integral curvature condition, such as compactness theorems, Colding's volume convergence, and Cheeger–Colding splitting theorems. Petersen and Sprouse [17] extended Petersen–Wei's comparison results and generalized Myers' theorem to a integral Ricci tensor bound. Aubry [1] used integral comparison estimates on star-shaped domains to improve Petersen–Sprouse's diameter estimate. He also got finite fundamental group theorem in the integral Ricci tensor sense. For more results, see for example [1,2,8–11,19,23].

An n-dimensional smooth metric measure space, denoted by $(M, g, e^{-f}\mathrm{d}v_g)$, is a complete n-dimensional Riemannian manifold (M, g) coupled with a weighted volume $e^{-f}\mathrm{d}v_g$ for some $f \in C^\infty(M)$, where $\mathrm{d}v_g$ is the usual Riemannian volume element on M. It naturally occurs as the collapsed measured Gromov–Hausdorff limit [16]. The f-Laplacian Δ_f associated to $(M, g, e^{-f}\mathrm{d}v_g)$ is given by

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint with respect to $e^{-f}\mathrm{d}v_g$. The associated Bakry–Émery Ricci tensor, introduced by Bakry and Émery [3], is defined as

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f,$$

where Hess is the Hessian with respect to the metric g, which is a natural generalization of the Ricci tensor. In particular, if

$$\mathrm{Ric}_f = \rho\, g$$

for some $\rho \in \mathbb{R}$, then $(M, g, e^{-f}\mathrm{d}v_g)$ is a gradient Ricci soliton. The Ricci soliton is called shrinking, steady, or expanding, if $\rho > 0$, $\rho = 0$, or $\rho < 0$, respectively, which arises as the singularity model of the Ricci flow [12]. When Ric_f is bounded below, many geometrical and topological results were successfully explored provided some condition on f is added. For example, Wei and Wylie [22] proved mean curvature and volume comparisons when Ric_f is bounded below and f or ∇f is (lower) bounded. And they extended many classical theorems, such as Myers' theorem, Cheeger–Gromoll splitting theorem, to the Bakry–Émery Ricci tensor. They also expected volume comparisons to be extended to the case that Ric_f is bounded below in the integral sense, which partly motivates the present paper.

In this paper we not only generalize comparison estimates on manifolds with integral bounds for the Ricci tensor to smooth metric measure spaces, but also extend pointwise comparison estimates on smooth metric measure spaces to the integral setting. In our situation, we consider weighted integral bounds for the Bakry–Émery Ricci tensor instead of usual integral bounds for the Ricci tensor. Our results indicate that Petersen–Wei's and Aubry's type comparison estimates remain true when certain weighted integral Bakry–Émery Ricci tensor bounds and ∇f is lower bounded (even no assumption on f). We also prove a relative weighted (or f-)volume comparison for annular regions under the same curvature integral condition. Some applications, such as diameter estimates, eigenvalue estimates, and volume growth estimates, are discussed.

Fix $H \in \mathbb{R}$, and consider at each point x of an n-dimensional smooth metric measure space $(M, g, e^{-f}\mathrm{d}v_g)$ with the smallest eigenvalue $\lambda(x)$ for the tensor $\mathrm{Ric}_f : T_xM \to T_xM$. We define

$$\mathrm{Ric}^H_{f\,-} := [(n-1)H - \lambda(x)]_+ = \max\{0, (n-1)H - \lambda(x)\},$$

the amount of Ric_f lying below $(n-1)H$. To write our results simply, we introduce a new weighted L^p norm of function ϕ on $(M, g, e^{-f}\mathrm{d}v_g)$:

$$\|\phi\|_{p\ f,a}(r) := \sup_{x\in M}\left(\int_{B(x,r)} |\phi|^p \cdot \mathcal{A}_f e^{-at}\,\mathrm{d}t\mathrm{d}\theta_{n-1}\right)^{\frac{1}{p}},$$

where $\partial_r f \geq -a$ for some constant $a \geq 0$, along a minimal geodesic segment from $x \in M$. Here $\mathcal{A}_f(t,\theta)$ is the volume element of weighted form $e^{-f}\mathrm{d}v_g = \mathcal{A}_f(t,\theta)\mathrm{d}t\wedge\mathrm{d}\theta_{n-1}$ in polar coordinate, and $\mathrm{d}\theta_{n-1}$ is the volume element on unit sphere S^{n-1}. Sometimes it is convenient to work with the normalized curvature quantity

$$\bar{k}(p, H, a, r) := \sup_{x\in M}\left(\frac{1}{V_f(x,r)}\cdot\int_{B(x,r)} \left(\mathrm{Ric}_{f\ -}^{H}\right)^p \mathcal{A}_f e^{-at}\mathrm{d}t\mathrm{d}\theta_{n-1}\right)^{\frac{1}{p}},$$

where $V_f(x,r) := \int_{B(x,r)} e^{-f}\mathrm{d}v$. Obviously, $\|\mathrm{Ric}_{f\ -}^{H}\|_{p\ f,a}(r) = 0$ (or $\bar{k}(p,H,a,r) = 0$) iff $\mathrm{Ric}_f \geq (n-1)H$. When $f = 0$ (and $a = 0$), all above notations recover the usual integral quantities on manifolds.

Motivated by Wei–Wylie's mean curvature comparison [22], we need to consider the error form

$$\varphi := (m_f - m_H - a)_+,$$

where $m_f = m - \partial_r f$ and m is the mean curvature of the geodesic sphere in the outer normal direction; and where m_H is the mean curvature of the geodesic sphere in the model space M_H^n. In [22], Wei and Wylie showed that if $\mathrm{Ric}_{f\ -}^{H} = 0$ and $\partial_r f \geq -a$ $(a \geq 0)$, then $\varphi = 0$. We prove that,

Theorem 1.1 (Mean curvature comparison estimate I) *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$, along a minimal geodesic segment from $x \in M$. For any $p > n/2$, $H \in \mathbb{R}$ (assume $r \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$),

$$\|\varphi\|_{2p\ f,a}(r) \leq \left[\frac{(n-1)(2p-1)}{(2p-n)}\left\|\mathrm{Ric}_{f\ -}^{H}\right\|_{p\ f,a}(r)\right]^{\frac{1}{2}} \tag{1}$$

and

$$\varphi^{2p-1}\mathcal{A}_f\, e^{-ar} \leq (2p-1)^p\left(\frac{n-1}{2p-n}\right)^{p-1}\cdot\int_0^r \left(\mathrm{Ric}_{f\ -}^{H}\right)^p \mathcal{A}_f e^{-at}\mathrm{d}t \tag{2}$$

along that minimal geodesic segment from x.

Moreover, if $H > 0$ *and* $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$*, then we have*

$$\left\| \sin^{\frac{4p-n-1}{2p}}(\sqrt{H}t) \cdot \varphi \right\|_{2p\ f,a}(r) \le \left[\frac{(n-1)(2p-1)}{(2p-n)} \left\| \mathrm{Ric}^H_{f\ -} \right\|_{p\ f,a}(r) \right]^{\frac{1}{2}} \tag{3}$$

and

$$\sin^{4p-n-1}(\sqrt{H}r)\varphi^{2p-1}\mathcal{A}_f e^{-ar} \le (2p-1)^p \left(\frac{n-1}{2p-n} \right)^{p-1} \cdot \int_0^r \left(\mathrm{Ric}^H_{f\ -}\right)^p \mathcal{A}_f e^{-at} \mathrm{d}t \tag{4}$$

along that minimal geodesic segment from x.

Remark 1.2 (1) When f is constant (and $a = 0$), inequality (1) recovers the Petersen–Wei's result [18]; inequalities (2) and (4) recover the Aubry's results [1]. In particular, when $|\nabla f| \le a$ for some constant $a \ge 0$ and the diameter of M is bounded, then f is bounded, and the new weighted norm is equivalent to the usual norm.

(2) When $\mathrm{Ric}^H_{f\ -} \equiv 0$ (i.e., $\mathrm{Ric}_f \ge (n-1)H$), we have $\varphi \equiv 0$ and hence get the Wei–Wylie's comparison result [22].

As in the integral volume comparison for manifolds [18], we can apply Theorem 1.1 to prove weighted volume comparisons in the integral sense. Let $V_f(x, R) := \int_{B(x,R)} e^{-f} \mathrm{d}v$ be the weighted volume of ball $B(x, R)$ in $(M, g, e^{-f}\mathrm{d}v)$. $V^a_H(R)$ denotes the h-volume of the ball $B(O, R)$ in the weighted model space $M^n_{H,a} := (M^n_H, g_H, e^{-h}\mathrm{d}v_{g_H})$, where $O \in M^n_H$ and $h(x) := -a \cdot d(O, x)$. That is,

$$V^a_H(R) := \int_0^R \int_{S^{n-1}} e^{at} \mathcal{A}_H(t, \theta)\, \mathrm{d}\theta_{n-1} \mathrm{d}t = \int_0^R e^{at} A_H(t) \mathrm{d}t,$$

where $\mathcal{A}_H$ denotes the volume element in model space M^n_H, and A_H denotes the volume of the geodesic sphere in M^n_H. For more detailed description about the related notations, see Sect. 3.

Theorem 1.3 (Relative volume comparison estimate I) *Let* $(M, g, e^{-f}\mathrm{d}v)$ *be an* n*-dimensional smooth metric measure space. Assume that*

$$\partial_r f \ge -a$$

for some constant $a \ge 0$*, along all minimal geodesic segments from* $x \in M$*. Let* $H \in \mathbb{R}$ *and* $p > n/2$*. For* $0 < r \le R$ *(assume* $R \le \frac{\pi}{2\sqrt{H}}$ *when* $H > 0$*),*

$$\left(\frac{V_f(x,R)}{V^a_H(R)} \right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r)}{V^a_H(r)} \right)^{\frac{1}{2p-1}} \le C(n,p,H,a,R) \left(\left\| \mathrm{Ric}^H_{f\ -} \right\|^p_{p\ f,a}(R) \right)^{\frac{1}{2p-1}}.$$

Furthermore, when $r = 0$*, we have an absolute volume comparison estimate:*

$$V_f(x, R) \le \left[e^{-\frac{f(x)}{2p-1}} + C(n, p, H, a, R) \left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}^p (R) \right)^{\frac{1}{2p-1}} \right]^{2p-1} V_H^a(R).$$

Here,

$$C(n, p, H, a, R) := \left(\frac{n-1}{(2p-1)(2p-n)} \right)^{\frac{p-1}{2p-1}} \int_0^R A_H(t) \left(\frac{t\, e^{at}}{V_H^a(t)} \right)^{\frac{2p}{2p-1}} \mathrm{d}t.$$

Remark 1.4 (1) The theorem implies a useful volume doubling property, see Corollary 3.3 below. When f is constant (or furthermore $f = 0$) and $a = 0$, the theorem recovers the Petersen–Wei's result [18].

(2) When $\mathrm{Ric}_{f\,-}^H \equiv 0$, i.e., $\mathrm{Ric}_f \ge (n-1)H$, we have the Wei–Wylie's volume comparison result (see (4.10) in [22]).

(3) Integrating along the direction lies in a star-shaped domain at x, we can obtain the same volume comparison estimate for the star-shaped domain at x, where $\mathrm{Ric}_{f\,-}^H$ only needs to integrate on the same star-shaped set.

We can generalize Theorem 1.3 and get an relative weighted volume comparison for two annuluses in the integral sense, which is completely new even in the manifold case. Let $V_f(x, r, R)$ be the f-volume of the annulus $B(x, R) \backslash B(x, r) \subseteq M^n$ for $r \le R$, and $V_H^a(r, R)$ be the h-volume of the annulus $B(O, R) \backslash B(O, r) \subseteq M_{H,a}^n$.

Theorem 1.5 (Relative volume comparison for annulus) *Let* $(M, g, e^{-f}\mathrm{d}v)$ *be an* n*-dimensional smooth metric measure space. Assume that*

$$\partial_r f \ge -a$$

for some constant $a \ge 0$*, along all minimal geodesic segments from* $x \in M$*. Let* $H \in \mathbb{R}$ *and* $p > n/2$*. For* $0 \le r_1 \le r_2 \le R_1 \le R_2$ *(assume* $R_2 \le \frac{\pi}{2\sqrt{H}}$ *when* $H > 0$*),*

$$\left(\frac{V_f(x, r_2, R_2)}{V_H^a(r_2, R_2)} \right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x, r_1, R_1)}{V_H^a(r_1, R_1)} \right)^{\frac{1}{2p-1}} \le \mathcal{C} \cdot \left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}^p (R_2) \right)^{\frac{1}{2p-1}},$$

where $\mathcal{C}$ *is given by*

$$\mathcal{C} = C(n, p, H, a, r_1, r_2, R_1, R_2) := \left(\frac{n-1}{(2p-1)(2p-n)} \right)^{\frac{p-1}{2p-1}}$$
$$\times \left[\int_{r_1}^{r_2} A_H(R_1) \left(\frac{R_1\, e^{aR_1}}{V_H^a(t, R_1)} \right)^{\frac{2p}{2p-1}} \mathrm{d}t + \int_{R_1}^{R_2} A_H(t) \left(\frac{t\, e^{at}}{V_H^a(r_2, t)} \right)^{\frac{2p}{2p-1}} \mathrm{d}t \right].$$

Besides, we are able to prove a general mean curvature comparison estimate, requiring no assumptions on f. Consequently, we get relative volume comparison estimates when f is bounded. See these results in Sect. 4.

The integral comparison estimates have many applications. We start to highlight two extensions of Petersen–Sprouse's results [17] to the weighted case that ∇f is lower bounded. One is the global diameter estimate:

Theorem 1.6 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$, along all minimal geodesic segments from any $x \in M$. Given $p > n/2$, $H > 0$ and $R > 0$, there exist $D = D(n, H, a)$ and $\epsilon = \epsilon(n, p, a, H, R)$ such that if $\bar{k}(p, H, a, R) < \epsilon$, then $diam_M \leq D$.

This theorem shows that a small fluctuation of super gradient shrinking Ricci soliton (i.e., $\mathrm{Ric}_f \geq (n-1)Hg$ for some constant $H > 0$) must be compact provided that the derivative of f has a lower bound. Examples 2.1 and 2.2 in [22] indicate that the assumption of f is necessary. Petersen and Sprouse [17] have proved the case when f is constant. For other Myers' type theorems on smooth metric measure spaces, see [15,20,22].

The other is a generalization of Cheng's eigenvalue upper bounds [6]. For any point $x_0 \in (M, g, e^{-f}\mathrm{d}v)$ and $R > 0$, let $\lambda_1^D(B(x_0, R))$ denote the first eigenvalue of the f-Laplacian Δ_f with the Dirichlet condition in $B(x_0, R)$. Let $\lambda_1^D(n, H, a, R)$ denote the first eigenvalue of the h-Laplacian Δ_h, where $h(x) := -a \cdot d(\bar{x}_0, x)$, with the Dirichlet condition in a metric ball $B(\bar{x}_0, R) \subseteq M^n_{H,a}$, where $R \leq \frac{\pi}{2\sqrt{H}}$. Then, we have a weighted version of Petersen–Sprouse's result [17].

Theorem 1.7 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$, along all minimal geodesic segments from $x_0 \in M$. Given $p > n/2$, for every $\delta > 0$, there exists an $\epsilon = \epsilon(n, p, H, a, R)$ such that if $\bar{k}(p, H, a, R) \leq \epsilon$, then

$$\lambda_1^D(B(x_0, R)) \leq (1+\delta)\,\lambda_1^D(n, H, a, R).$$

Finally we apply Theorem 1.5 to get a weighted volume growth estimate, generalizing Yau's volume growth estimate [24] and Wei–Wylie's result [22].

Theorem 1.8 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq 0$$

along all minimal geodesic segments from any $x \in M$. Given any $p > n/2$ and $R \geq 2$, there is an $\epsilon = \epsilon(n, p, R)$ such that if $\bar{k}(p, 0, 0, R+1) < \epsilon$ (here $H = 0$ and $a = 0$), then for any point $x_0 \in M$, we have

$$V_f(x_0, R) \geq C\, R$$

for some positive constant $C = C(n, p, V_f(x_0, 1))$ depending only on n, p and $V_f(x_0, 1)$.

Constant function f satisfies $\partial_r f \geq 0$ and hence the theorem naturally holds for the ordinary Riemannian manifolds. From Theorem 5.3 in [22], we know that convex function f with the unbounded set of its critical points, also satisfies $\partial_r f \geq 0$. Examples 2.1 and 2.2 in [22] indicate that the hypothesis on f in the theorem is necessary.

The rest of this paper is organized as follows. In Sect. 2, we will prove Theorem 1.1. In Sect. 3, we will apply Theorem 1.1 to prove Theorem 1.3 and further get a volume doubling property when the integral Bakry–Émery Ricci tensor bounds and ∇f is lower bounded. We also prove relative volume comparison estimates for annuluses when the integral of Bakry–Émery Ricci tensor bounds. In Sect. 4, we will discuss a general mean curvature comparison estimates and relative volume comparison estimates for their integral bounds of Bakry–Émery Ricci tensor. In Sect. 5, we will give some applications of new integral comparison estimates. Precisely, we will apply Theorems 1.1 and 1.3 to prove Theorems 1.6 and 1.7. Meanwhile we will apply Theorem 1.5 to prove Theorem 1.8. In Appendix, we give mean curvature and volume comparison estimates on smooth metric measure spaces when only certain integral of m-Bakry–Émery Ricci tensor bounds.

From the work of [1,2,13] we expect Aubry's type diameter estimate, finiteness fundamental group theorem, first Betti number estimate and Gromov's bounds on the volume entropy in the integral sense can be generalized to smooth metric measure spaces. These will be treated in separate paper.

2 Mean Curvature Comparison Estimate I

In this section, we mainly prove Theorem 1.1, a weighted mean curvature comparison estimate on smooth metric measure spaces $(M, g, e^{-f} dv)$ when certain integral Bakry–Émery Ricci tensor bounds and ∇f is lower bounded. The proof first modifies the Bochner formula of Bakry–Émery Ricci tensor to acquire the ODE along geodesics and then integrates the ODE inequality, similar to the arguments of Petersen and Wei [18], and Aubry [1].

Proof of Theorem 1.1 Recall the Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u)$$

for any function $u \in C^\infty(M)$. Letting $u = r(y)$, where $r(y) = d(y, x)$ is the distance function, then we have

$$0 = |\mathrm{Hess}\, r|^2 + \frac{\partial}{\partial r}(\Delta r) + \mathrm{Ric}(\nabla r, \nabla r).$$

Note that Hess r is the second fundamental form of the geodesic sphere and $\Delta r = m$, the mean curvature of the geodesic sphere. By the Schwarz inequality, we have the Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \mathrm{Ric}(\partial r, \partial r).$$

This inequality becomes equality if and only if the radial sectional curvatures are constant. So the mean curvature of the n-dimensional model space m_H satisfies

$$m'_H = -\frac{m_H^2}{n-1} - (n-1)H.$$

Since $m_f := m - \partial_r f$, i.e., $m_f = \Delta_f r$, then $m'_f = m' - \partial_r \partial_r f$, and we have

$$m'_f \leq -\frac{m^2}{n-1} - \mathrm{Ric}_f(\partial r, \partial r).$$

Hence,

$$\begin{aligned}
(m_f - m_H - a)' &= m'_f - m'_H \\
&\leq -\frac{m^2 - m_H^2}{n-1} + (n-1)H - \mathrm{Ric}_f \\
&= -\frac{(m_f + \partial_r f)^2 - m_H^2}{n-1} + (n-1)H - \mathrm{Ric}_f \\
&= -\frac{1}{n-1}\Big[(m_f - m_H + \partial_r f)(m_f + m_H + \partial_r f)\Big] + (n-1)H - \mathrm{Ric}_f \\
&= -\frac{1}{n-1}\Big[(m_f - m_H - a + a + \partial_r f)(m_f - m_H - a + 2m_H + a + \partial_r f)\Big] \\
&\quad + (n-1)H - \mathrm{Ric}_f.
\end{aligned}$$

We recall that $\varphi := (m_f - m_H - a)_+$. Notice that on the interval where $m_f \leq m_H + a$, we have $\varphi = 0$; on the interval where $m_f > m_H + a$, we have $m_f - m_H - a = \varphi$. Moreover, by our assumption of the theorem, we know

$$(n-1)H - \mathrm{Ric}_f \leq \mathrm{Ric}^H_{f\,-}.$$

Therefore, in any case, we have

$$\varphi' + \frac{1}{n-1}\Big[(\varphi + a + \partial_r f)(\varphi + 2m_H + a + \partial_r f)\Big] \leq \mathrm{Ric}^H_{f\,-}.$$

Since $a + \partial_r f \geq 0$, the above inequality implies

$$\varphi' + \frac{\varphi^2}{n-1} + \frac{2m_H\varphi}{n-1} \leq \mathrm{Ric}^H_{f\,-}.$$

Multiplying this inequality by $(2p-1)\varphi^{2p-2} \cdot \mathcal{A}_f$, we have

$$(2p-1)\varphi^{2p-2}\varphi'\mathcal{A}_f + \frac{2p-1}{n-1}\varphi^{2p}\mathcal{A}_f + \frac{4p-2}{n-1}\varphi^{2p-1}m_H\mathcal{A}_f \leq (2p-1)\mathrm{Ric}^H_{f\,-} \cdot \varphi^{2p-2}\mathcal{A}_f.$$

Using

$$\begin{aligned}(\varphi^{2p-1}\mathcal{A}_f)' &= (2p-1)\varphi^{2p-2}\varphi' \cdot \mathcal{A}_f + \varphi^{2p-1} \cdot \mathcal{A}'_f \\ &= (2p-1)\varphi^{2p-2}\varphi' \cdot \mathcal{A}_f + \varphi^{2p-1} \cdot m_f\mathcal{A}_f,\end{aligned}$$

the above integral inequality can be rewritten as

$$(\varphi^{2p-1}\mathcal{A}_f)' - \varphi^{2p-1}(m_f - m_H - a + m_H + a)\mathcal{A}_f + \frac{2p-1}{n-1}\varphi^{2p}\mathcal{A}_f + \frac{4p-2}{n-1}\varphi^{2p-1}m_H \cdot \mathcal{A}_f \leq (2p-1)\mathrm{Ric}^H_{f\,-} \cdot \varphi^{2p-2}\mathcal{A}_f.$$

Rearrange some terms of the above inequality by $\varphi := (m_f - m_H - a)_+$ to get

$$(\varphi^{2p-1}\mathcal{A}_f)' + \left(\frac{2p-1}{n-1} - 1\right)\varphi^{2p}\mathcal{A}_f + \left(\frac{4p-2}{n-1} - 1\right)\varphi^{2p-1} \cdot m_H\mathcal{A}_f - a\varphi^{2p-1}\mathcal{A}_f \leq (2p-1)\mathrm{Ric}^H_{f\,-} \cdot \varphi^{2p-2}\mathcal{A}_f.$$

Notice that the term $-a\varphi^{2p-1}\mathcal{A}_f$ of the above inequality is negative. To deal with this bad term, we multiply the inequality by the integrating factor e^{-ar}, and get that

$$(\varphi^{2p-1}\mathcal{A}_f e^{-ar})' + \frac{2p-n}{n-1}\varphi^{2p}\mathcal{A}_f e^{-ar} + \frac{4p-n-1}{n-1}\varphi^{2p-1}m_H\mathcal{A}_f e^{-ar} \leq (2p-1)\mathrm{Ric}^H_{f\,-} \cdot \varphi^{2p-2}\mathcal{A}_f e^{-ar}. \tag{5}$$

Since $p > n/2$ and the assumption $r \leq \frac{\pi}{2\sqrt{H}}$, we have $m_H \geq 0$ and

$$\frac{4p-n-1}{n-1}\varphi^{2p-1}m_H\mathcal{A}_f e^{-ar} \geq 0.$$

Then we drop this term and have that

$$(\varphi^{2p-1}\mathcal{A}_f e^{-ar})' + \frac{2p-n}{n-1}\varphi^{2p}\mathcal{A}_f e^{-ar} \le (2p-1)\mathrm{Ric}^H_{f\,\_} \cdot \varphi^{2p-2}\mathcal{A}_f e^{-ar}.$$

We integrate the above inequality from 0 to r. Since

$$\varphi(0) = (m - m_H - \partial_r f - a)_+\big|_{r=0} = 0,$$

which comes from the theorem assumption: $a + \partial_r f \ge 0$, then

$$\varphi^{2p-1}\mathcal{A}_f e^{-ar} + \frac{2p-n}{n-1}\int_0^r \varphi^{2p}\mathcal{A}_f e^{-at}\mathrm{d}t \le (2p-1)\int_0^r \mathrm{Ric}^H_{f\,\_} \cdot \varphi^{2p-2}\mathcal{A}_f e^{-at}\mathrm{d}t.$$

This implies

$$\varphi^{2p-1}\mathcal{A}_f e^{-ar} \le (2p-1)\int_0^r \mathrm{Ric}^H_{f\,\_} \cdot \varphi^{2p-2}\mathcal{A}_f e^{-at}\mathrm{d}t. \tag{6}$$

and

$$\frac{2p-n}{n-1}\int_0^r \varphi^{2p}\mathcal{A}_f e^{-at}\mathrm{d}t \le (2p-1)\int_0^r \mathrm{Ric}^H_{f\,\_} \cdot \varphi^{2p-2}\mathcal{A}_f e^{-at}\mathrm{d}t. \tag{7}$$

By Holder inequality, we also have

$$\begin{aligned}\int_0^r \mathrm{Ric}^H_{f\,\_} \cdot \varphi^{2p-2}\mathcal{A}_f e^{-at}\mathrm{d}t \\ \le \left[\int_0^r \varphi^{2p}\mathcal{A}_f e^{-at}\mathrm{d}t\right]^{1-\frac{1}{p}} \cdot \left[\int_0^r \left(\mathrm{Ric}^H_{f\,\_}\right)^p \mathcal{A}_f e^{-at}\mathrm{d}t\right]^{\frac{1}{p}}.\end{aligned} \tag{8}$$

Combining (8) and (7), we immediately get (1). Then applying (1) and (8) to (6) yields (2).

If $H > 0$ and $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, then $m_H < 0$ in (5). It means that we cannot throw away the third term of (5) as before. To deal with this obstacle, multiplying by the integrating factor $\sin^{4p-n-1}(\sqrt{H}r)$ in (5) and integrating from 0 to r, we get

$$\begin{aligned}\sin^{4p-n-1}(\sqrt{H}r)\,\varphi^{2p-1}\mathcal{A}_f e^{-ar} + \frac{2p-n}{n-1}\int_0^r \varphi^{2p}\sin^{4p-n-1}(\sqrt{H}t)\mathcal{A}_f e^{-at}\mathrm{d}t \\ \le (2p-1)\int_0^r \mathrm{Ric}^H_{f\,\_} \cdot \varphi^{2p-2}\sin^{4p-n-1}(\sqrt{H}t)\mathcal{A}_f e^{-at}\mathrm{d}t.\end{aligned} \tag{9}$$

Similar to the above discussion, using the Holder inequality, we have

$$\begin{aligned}&\int_0^r \mathrm{Ric}^H_{f\,-}\cdot\varphi^{2p-2}\sin^{4p-n-1}(\sqrt{H}t)\mathcal{A}_f e^{-at}\mathrm{d}t\\&\le\left[\int_0^r\varphi^{2p}\sin^{4p-n-1}(\sqrt{H}t)\mathcal{A}_f e^{-at}\mathrm{d}t\right]^{1-\frac{1}{p}}\left[\int_0^r\sin^{4p-n-1}(\sqrt{H}t)\big(\mathrm{Ric}^H_{f\,-}\big)^p\mathcal{A}_f e^{-at}\mathrm{d}t\right]^{\frac{1}{p}}.\end{aligned}\tag{10}$$

Notice that two terms in the left-hand side of (9) are both positive. Then substituting (10) into (9), we get

$$\left\|\sin^{\frac{4p-n-1}{2p}}(\sqrt{H}t)\cdot\varphi\right\|_{2p\ f,a}(r)\le\left[\frac{(n-1)(2p-1)}{(2p-n)}\left\|\sin^{\frac{4p-n-1}{p}}(\sqrt{H}t)\cdot\mathrm{Ric}^H_{f\,-}\right\|_{p\ f,a}(r)\right]^{\frac{1}{2}},\tag{11}$$

which implies (3). Then putting (11) and (10) to (9) immediately proves (4) by only using an easy fact: $\sin^{\frac{4p-n-1}{p}}(\sqrt{H}t)\le 1$. □

3 Volume Comparison Estimate I

In Sect. 2, we have proved a weighted mean curvature comparison estimate when certain weighted integral of Bakry–Émery Ricci tensor bounds and ∇f has a lower bound, and one naturally hopes a corresponding volume comparison estimate under the same curvature assumptions. In this section, we will give these desired volume comparison estimates.

For an n-dimensional smooth metric measure space $(M^n, g, e^{-f}\mathrm{d}v_g)$, let $\mathcal{A}_f(t,\theta)$ denote the volume element of the weighted volume form $e^{-f}\mathrm{d}v_g=\mathcal{A}_f(t,\theta)\mathrm{d}t\wedge\mathrm{d}\theta_{n-1}$ in polar coordinate. That is,

$$\mathcal{A}_f(t,\theta)=e^{-f}\mathcal{A}(t,\theta),$$

where $\mathcal{A}(t,\theta)$ is the standard volume element of the metric g. We also let

$$A_f(x,r)=\int_{S^{n-1}}\mathcal{A}_f(r,\theta)\mathrm{d}\theta_{n-1},$$

which denotes the weighted volume of the geodesic sphere $S(x,r)=\{y\in M|\,d(x,y)=r\}$, and let $A_H(r)$ be the volume of the geodesic sphere in the model space M^n_H. We modify M^n_H to the weighted model space

$$M^n_{H,a}:=(M^n_H, g_H, e^{-h}\mathrm{d}v_{g_H}, O),$$

where (M^n_H, g_H) is the n-dimensional simply connected space with constant sectional curvature H, $O\in M^n_H$, and $h(x)=-a\cdot d(x,O)$. Let $\mathcal{A}^a_H$ be the h-volume element in $M^n_{H,a}$. Then

$$\mathcal{A}^a_H(r)=e^{ar}\mathcal{A}_H(r),$$

where $\mathcal{A}_H$ is the Riemannian volume element in M^n_H. We also have that

$$A_H(r) = \int_{S^{n-1}} \mathcal{A}_H(r,\theta)\mathrm{d}\theta_{n-1};$$

the corresponding weighted volume of the geodesic sphere in the weighted model space $M^n_{H,a}$ is defined by

$$A^a_H(r) = \int_{S^{n-1}} \mathcal{A}^a_H(r,\theta)\mathrm{d}\theta_{n-1}.$$

Hence,

$$A^a_H(r) = e^{ar} A_H(r).$$

Moreover, the weighted (or f-)volume of the ball $B(x,r) = \{y \in M | d(x,y) \le r\}$ is defined by

$$V_f(x,r) = \int_0^r A_f(x,t)\mathrm{d}t.$$

We also let $V^a_H(r)$ be the h-volume of the ball $B(O,r) \subset M^n_H$:

$$V^a_H(r) = \int_0^r A^a_H(t)\mathrm{d}t.$$

Clearly, we have

$$V_H(r) \le V^a_H(r) \le e^{ar} V_H(r).$$

Now we prove a comparison estimate for the area of geodesic spheres using the pointwise mean curvature estimate in Sect. 2.

Theorem 3.1 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \ge -a$$

for some constant $a \ge 0$, along all minimal geodesic segments from $x \in M$. Let $H \in \mathbb{R}$ and $p > n/2$ be given, and when $H > 0$ assume that $R \le \frac{\pi}{2\sqrt{H}}$. For $0 < r \le R$, we have

$$\left(\frac{A_f(x,R)}{A^a_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A^a_H(r)}\right)^{\frac{1}{2p-1}} \le C(n,p,H,R)\left(\left\|\mathrm{Ric}^H_{f\,-}\right\|_{p\,f,a}(R)\right)^{\frac{p}{2p-1}}, \tag{12}$$

where $C(n,p,H,R) := \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \cdot \int_0^R A_H(t)^{-\frac{1}{2p-1}}\mathrm{d}t$.

Moreover, if $H > 0$ *and* $\frac{\pi}{2\sqrt{H}} < r \le R < \frac{\pi}{\sqrt{H}}$, *then we have*

$$\begin{aligned} &\left(\frac{A_f(x,R)}{A_H^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H^a(r)}\right)^{\frac{1}{2p-1}} \\ &\le \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(R)\right)^{\frac{p}{2p-1}} \int_r^R \frac{(\sqrt{H})^{\frac{n-1}{2p-1}}}{\sin^2(\sqrt{H}t)}\mathrm{d}t. \end{aligned} \tag{13}$$

Remark 3.2 When $\mathrm{Ric}_{f\,-}^H \equiv 0$, that is, $\mathrm{Ric}_f \ge (n-1)H$, we exactly get Wei–Wylie's comparison result for the area of geodesic spheres (see (4.8) in [22]).

Proof of Theorem 3.1 We apply

$$\mathcal{A}_f' = m_f \mathcal{A}_f \quad \text{and} \quad {\mathcal{A}_H^a}' = (m_H + a)\mathcal{A}_H^a$$

to compute that

$$\frac{d}{\mathrm{d}t}\left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^a(t)}\right) = (m_f - m_H - a)\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^a(t)}.$$

Hence,

$$\begin{aligned} \frac{d}{\mathrm{d}t}\left(\frac{A_f(x,t)}{A_H^a(t)}\right) &= \frac{1}{Vol(S^{n-1})}\int_{S^{n-1}} \frac{d}{\mathrm{d}t}\left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^a(t)}\right)\mathrm{d}\theta_{n-1} \\ &\le \frac{1}{A_H^a(t)}\int_{S^{n-1}} \varphi\cdot\mathcal{A}_f(t,\theta)\mathrm{d}\theta_{n-1}. \end{aligned}$$

Using Holder's inequality and (2), we have

$$\begin{aligned} &\int_{S^{n-1}} \varphi\cdot\mathcal{A}_f(t,\theta)\mathrm{d}\theta_{n-1} \\ &\quad \le \left(\int_{S^{n-1}} \varphi^{2p-1}\mathcal{A}_f(x,t)\mathrm{d}\theta_{n-1}\right)^{\frac{1}{2p-1}} \cdot A_f(x,t)^{1-\frac{1}{2p-1}} \\ &\quad \le C(n,p)\, e^{\frac{at}{2p-1}}\left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(t)\right)^{\frac{p}{2p-1}} \cdot A_f(x,t)^{1-\frac{1}{2p-1}}, \end{aligned}$$

where $C(n,p) = \left[(2p-1)^p\left(\frac{n-1}{2p-n}\right)^{p-1}\right]^{\frac{1}{2p-1}}$. Hence, we have

$$\begin{aligned} \frac{d}{\mathrm{d}t}\left(\frac{A_f(x,t)}{A_H^a(t)}\right) &\le C(n,p)\left(\frac{A_f(x,t)}{A_H^a(t)}\right)^{1-\frac{1}{2p-1}} \\ &\quad \times \left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(t)\right)^{\frac{p}{2p-1}} \cdot \left(\frac{e^{at}}{A_H^a(t)}\right)^{\frac{1}{2p-1}}. \end{aligned} \tag{14}$$

Separating of variables and integrating from r to R, we obtain

$$\left(\frac{A_f(x,R)}{A_H^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H^a(r)}\right)^{\frac{1}{2p-1}} \\ \leq \left[\frac{n-1}{(2p-1)(2p-n)}\right]^{\frac{p-1}{2p-1}} \left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(R)\right)^{\frac{p}{2p-1}} \cdot \int_r^R \left(\frac{1}{A_H(t)}\right)^{\frac{1}{2p-1}} \mathrm{d}t.$$

Since the integral

$$\int_r^R \left(\frac{1}{A_H(t)}\right)^{\frac{1}{2p-1}} \mathrm{d}t \leq \int_0^R \left(\frac{1}{A_H(t)}\right)^{\frac{1}{2p-1}} \mathrm{d}t$$

converges when $p > n/2$, the conclusion (12) then follows.

For the case $H > 0$ and $\frac{\pi}{2\sqrt{H}} < r \leq R < \frac{\pi}{\sqrt{H}}$, we have

$$A_H^a(t) = e^{at}\left(\frac{\sin(\sqrt{H}t)}{\sqrt{H}}\right)^{n-1}.$$

Then we use this function and (2) instead of (4) to get (13) by following the above similar argument. □

Using (14), we can prove Theorem 1.3, similar to the argument of Petersen and Wei [18].

Proof of Theorem 1.3 Using

$$\frac{V_f(x,r)}{V_H^a(r)} = \frac{\int_0^r A_f(x,t)\mathrm{d}t}{\int_0^r A_H^a(t)\mathrm{d}t},$$

we compute that

$$\frac{d}{dr}\left(\frac{V_f(x,r)}{V_H^a(r)}\right) = \frac{A_f(x,r)\int_0^r A_H^a(t)\mathrm{d}t - A_H^a(r)\int_0^r A_f(x,t)\mathrm{d}t}{(V_H^a(r))^2}. \tag{15}$$

On the other hand, integrating (14) from t to r $(t \leq r)$ gives

$$\frac{A_f(x,r)}{A_H^a(r)} - \frac{A_f(x,t)}{A_H^a(t)} \\ \leq C(n,p)\int_t^r \frac{\left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(s)\right)^{\frac{p}{2p-1}}}{A_H(s)^{\frac{1}{2p-1}} \cdot A_H^a(s)^{1-\frac{1}{2p-1}}} \cdot A_f(x,s)^{1-\frac{1}{2p-1}}\mathrm{d}s$$

$$\le C(n,p)\frac{\left(\|\mathrm{Ric}_{f\,-}^{H}\|_{p\,f,a}(r)\right)^{\frac{p}{2p-1}}}{A_H(t)^{\frac{1}{2p-1}}\cdot A_H^a(t)^{1-\frac{1}{2p-1}}}\cdot\int_t^r A_f(x,s)^{1-\frac{1}{2p-1}}\mathrm{d}s$$
$$\le C(n,p)\frac{\left(\|\mathrm{Ric}_{f\,-}^{H}\|_{p\,f,a}(r)\right)^{\frac{p}{2p-1}}}{A_H(t)^{\frac{1}{2p-1}}\cdot A_H^a(t)^{1-\frac{1}{2p-1}}}\cdot(r-t)^{\frac{1}{2p-1}}\,V_f(x,r)^{1-\frac{1}{2p-1}}.$$

This implies that

$$\begin{aligned}&A_f(x,r)A_H^a(t)-A_H^a(r)A_f(x,t)\\&\quad\le C(n,p)\left(\|\mathrm{Ric}_{f\,-}^{H}\|_{p\,f,a}(r)\right)^{\frac{p}{2p-1}}\cdot A_H^a(r)\cdot e^{\frac{ar}{2p-1}}\cdot r^{\frac{1}{2p-1}}\,V_f(x,r)^{1-\frac{1}{2p-1}}.\end{aligned}$$

Plugging this into (15) gives

$$\begin{aligned}&\frac{d}{dr}\left(\frac{V_f(x,r)}{V_H^a(r)}\right)\\&\le C(n,p)\left(\|\mathrm{Ric}_{f\,-}^{H}\|_{p\,f,a}(r)\right)^{\frac{p}{2p-1}}\cdot A_H^a(r)\cdot e^{\frac{ar}{2p-1}}\cdot r^{\frac{2p}{2p-1}}\cdot\frac{V_f(x,r)^{1-\frac{1}{2p-1}}}{(V_H^a(r))^2}\\&=C(n,p)\left(\|\mathrm{Ric}_{f\,-}^{H}\|_{p\,f,a}(r)\right)^{\frac{p}{2p-1}}\cdot A_H(r)\left(\frac{r\,e^{ar}}{V_H^a(r)}\right)^{\frac{2p}{2p-1}}\left(\frac{V_f(x,r)}{V_H^a(r)}\right)^{1-\frac{1}{2p-1}}.\end{aligned}$$

Separating of variables and integrating from r to R ($r\le R$), we immediately get

$$\begin{aligned}&\left(\frac{V_f(x,R)}{V_H^a(R)}\right)^{\frac{1}{2p-1}}-\left(\frac{V_f(x,r)}{V_H^a(r)}\right)^{\frac{1}{2p-1}}\\&\le\left[\frac{n-1}{(2p-1)(2p-n)}\right]^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_{f\,-}^{H}\|_{p\,f,a}(R)\right)^{\frac{p}{2p-1}}\int_r^R A_H(t)\left(\frac{t\,e^{at}}{V_H^a(t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t.\end{aligned}$$

Since the integral

$$\begin{aligned}\int_r^R A_H(t)\left(\frac{t\,e^{at}}{V_H^a(t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t&\le\int_r^R A_H(t)\left(\frac{t\,e^{at}}{V_H(t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t\\&\le\int_0^R A_H(t)\left(\frac{t\,e^{at}}{V_H(t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t\end{aligned}$$

converges when $p>n/2$, the conclusion follows. □

As the classical case, the volume comparison estimate implies the volume doubling estimate, which is often useful in various geometric inequalities.

Corollary 3.3 (Volume doubling estimate) *Let* $(M, g, e^{-f}\mathrm{d}v)$ *be an* n*-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$*, along all minimal geodesic segments from* $x \in M$*. Given* $\alpha > 1$ *and* $p > n/2$*, there is an* $\epsilon = \epsilon(n, p, aR, |H|R^2, \alpha)$ *such that if* $R^2 \cdot \bar{k}(p, H, a, R) < \epsilon$*, then for all* $x \in M$ *and* $0 < r_1 < r_2 \leq R$ *(assume* $R \leq \frac{\pi}{2\sqrt{H}}$ *when* $H > 0$*), we have*

$$\frac{V_f(x, r_2)}{V_f(x, r_1)} \leq \alpha \frac{V_H^a(r_2)}{V_H^a(r_1)}.$$

Remark 3.4 We remark that $R^2 \cdot \bar{k}(p, H, a, R)$ is the scale invariant curvature quantity. Hence one can simply scale the metric so that one only need to work under the assumption that $\bar{k}(p, H, a, 1)$ is small.

Proof of Corollary 3.3 By Theorem 1.3, we get

$$\left(\frac{V_f(x, r_1)}{V_f(x, r_2)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{V_H^a(r_1)}{V_H^a(r_2)}\right)^{\frac{1}{2p-1}} (1 - \sigma), \tag{16}$$

where $\sigma := C(n, p, H, a, r_2)\, V_H^a(r_2)^{\frac{1}{2p-1}} \cdot \bar{k}^{\frac{p}{2p-1}}(p, H, a, r_2)$. Now we will estimate the quantity $(1 - \sigma)$. We claim that $\sigma(r)$ has some monotonicity in r (though it is not really monotonic). Indeed, since $C(n, p, H, a, r)$ is increasing in r, that is,

$$\sigma(r_2) V_H^a(r_2)^{-\frac{1}{2p-1}} \cdot \bar{k}^{-\frac{p}{2p-1}}(p, H, a, r_2) \leq \sigma(R) V_H^a(R)^{-\frac{1}{2p-1}} \cdot \bar{k}^{-\frac{p}{2p-1}}(p, H, a, R).$$

By the definition of $\bar{k}$, the above inequality implies

$$\sigma(r_2) V_H^a(r_2)^{-\frac{1}{2p-1}} \cdot V_f(x, r_2)^{\frac{1}{2p-1}} \leq \sigma(R) V_H^a(R)^{-\frac{1}{2p-1}} \cdot V_f(x, R)^{\frac{1}{2p-1}}.$$

Namely,

$$\sigma(r_2) \leq \sigma(R) \left(\frac{V_f(x, R)}{V_f(x, r_2)}\right)^{\frac{1}{2p-1}} \cdot \left(\frac{V_H^a(R)}{V_H^a(r_2)}\right)^{-\frac{1}{2p-1}}. \tag{17}$$

On the other hand, by Theorem 1.3 again, we have

$$\left(\frac{V_f(x, r_2)}{V_H^a(r_2)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{V_f(x, R)}{V_H^a(R)}\right)^{\frac{1}{2p-1}} \left[1 - C(n, p, H, a, R) V_H^a(R)^{\frac{1}{2p-1}} \cdot \bar{k}(R)^{\frac{p}{2p-1}}\right],$$

where $\bar{k}(R) = \bar{k}(p, H, a, R)$ and

$$C(n, p, H, a, R) := \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R A_H(t) \left(\frac{t\, e^{at}}{V_H^a(t)}\right)^{\frac{2p}{2p-1}} \mathrm{d}t.$$

We also have

$$C(n, p, H, a, R) V_H^a(R)^{\frac{1}{2p-1}} \le (e^{aR})^{\frac{2p+1}{2p-1}} R^{\frac{2p}{2p-1}} C(n, p, |H|R).$$

Hence

$$\left(\frac{V_f(x, r_2)}{V_f(x, R)}\right)^{\frac{1}{2p-1}} \ge \left(\frac{V_H^a(r_2)}{V_H^a(R)}\right)^{\frac{1}{2p-1}} \left[1 - C(n, p, |H|R) \cdot (e^{aR})^{\frac{2p+1}{2p-1}} \left(R^2 \bar{k}(R)\right)^{\frac{p}{2p-1}}\right].$$

When $R^2\bar{k}(R) \le \epsilon$ is small enough, which depends only on n, p, aR, and $|H|R$, the above inequality becomes

$$\left(\frac{V_f(x, r_2)}{V_f(x, R)}\right)^{\frac{1}{2p-1}} \ge \frac{1}{3} \left(\frac{V_H^a(r_2)}{V_H^a(R)}\right)^{\frac{1}{2p-1}}.$$

Substituting this into (17) yields

$$\sigma(r_2) \le 3\sigma(R).$$

Combining this with (16) and letting $\sigma(R)$ arbitrary small (as long as $R^2\bar{k}(R) \le \epsilon$ is small enough), the result follows. □

In the rest of this section, we will study the relative volume comparison estimate for annular regions and prove Theorem 1.5 in the introduction. The proof idea seems to be easy, using the twice procedures of proving Theorem 1.3.

Proof of Theorem 1.5 On one hand, using

$$\frac{V_f(x, r, R)}{V_H^a(r, R)} = \frac{\int_r^R A_f(x, t) \mathrm{d}t}{\int_r^R A_H^a(t) \mathrm{d}t},$$

we have

$$\frac{d}{dR}\left(\frac{V_f(x, r, R)}{V_H^a(r, R)}\right) = \frac{A_f(x, R) \int_r^R A_H^a(t) \mathrm{d}t - A_H^a(R) \int_r^R A_f(x, t) \mathrm{d}t}{(V_H^a(r, R))^2}. \tag{18}$$

Integrating (14) from t to R ($t \le R$) as before yields

$$\begin{aligned}
&\frac{A_f(x,R)}{A_H^a(R)} - \frac{A_f(x,t)}{A_H^a(t)}\\
&\le C(n,p)\frac{\left(\left\|\mathrm{Ric}_{f\,-}^H\right\|_{p\ f,a}(R)\right)^{\frac{p}{2p-1}}}{A_H(t)^{\frac{1}{2p-1}}\cdot A_H^a(t)^{1-\frac{1}{2p-1}}}\cdot\int_t^R A_f(x,s)^{1-\frac{1}{2p-1}}\mathrm{d}s\\
&\le C(n,p)\frac{\left(\left\|\mathrm{Ric}_{f\,-}^H\right\|_{p\ f,a}(R)\right)^{\frac{p}{2p-1}}}{A_H(t)^{\frac{1}{2p-1}}\cdot A_H^a(t)^{1-\frac{1}{2p-1}}}\cdot(R-t)^{\frac{1}{2p-1}}\left(V_f(x,t,R)\right)^{1-\frac{1}{2p-1}},
\end{aligned}$$

which gives that

$$\begin{aligned}
&A_f(x,R)A_H^a(t) - A_H^a(R)A_f(x,t)\\
&\le C(n,p)\left(\left\|\mathrm{Ric}_{f\,-}^H\right\|_{p\ f,a}(R)\right)^{\frac{p}{2p-1}} A_H^a(R)e^{\frac{at}{2p-1}}R^{\frac{1}{2p-1}}\left(V_f(x,t,R)\right)^{1-\frac{1}{2p-1}}.
\end{aligned}\tag{19}$$

Substituting this into (18),

$$\begin{aligned}
\frac{d}{dR}\left(\frac{V_f(x,r,R)}{V_H^a(r,R)}\right) &\le C(n,p)\left(\left\|\mathrm{Ric}_{f\,-}^H\right\|_{p\ f,a}(R)\right)^{\frac{p}{2p-1}}\cdot A_H(R)\\
&\quad\times\left(\frac{R\,e^{aR}}{V_H^a(r,R)}\right)^{\frac{2p}{2p-1}}\left(\frac{V_f(x,r,R)}{V_H^a(r,R)}\right)^{1-\frac{1}{2p-1}},
\end{aligned}$$

where we used the fact: $\int_r^R V_f(x,t,R)\mathrm{d}t \le V_f(x,r,R)$.

Separating of variables, integrating with respect to the variable R from R_1 to R_2 ($R_1 \le R_2$), and changing the variable r to r_2 ($r_2 \le R_1$), we get

$$\begin{aligned}
&\left(\frac{V_f(x,r_2,R_2)}{V_H^a(r_2,R_2)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r_2,R_1)}{V_H^a(r_2,R_1)}\right)^{\frac{1}{2p-1}}\\
&\le\left[\frac{n-1}{(2p-1)(2p-n)}\right]^{\frac{p-1}{2p-1}}\left(\left\|\mathrm{Ric}_{f\,-}^H\right\|_{p\ f,a}(R_2)\right)^{\frac{p}{2p-1}}\int_{R_1}^{R_2}A_H(t)\left(\frac{t\,e^{at}}{V_H^a(r_2,t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t.
\end{aligned}$$

On the other hand, similar to the above argument, we also have

$$\frac{d}{dr}\left(\frac{V_f(x,r,R)}{V_H^a(r,R)}\right) = \frac{A_H^a(r)\int_r^R A_f(x,t)\mathrm{d}t - A_f(x,r)\int_r^R A_H^a(t)\mathrm{d}t}{(V_H^a(r,R))^2}.\tag{20}$$

By (19), we also get that

$$
\begin{aligned}
&A_f(x,t)A_H^a(r) - A_H^a(t)A_f(x,r) \\
&\le C(n,p)\left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(t)\right)^{\frac{p}{2p-1}} A_H^a(t)\cdot e^{\frac{ar}{2p-1}}\cdot t^{\frac{1}{2p-1}}\,(V_f(x,r,t))^{1-\frac{1}{2p-1}}
\end{aligned}
$$

for $r \le t$. Substituting this into (20), and letting $R = R_1$, we have

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{V_f(x,r,R_1)}{V_H^a(r,R_1)}\right) &\le C(n,p)\left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(R_1)\right)^{\frac{p}{2p-1}}\cdot A_H(R_1) \\
&\quad\times\left(\frac{R_1\, e^{aR_1}}{V_H^a(r,R_1)}\right)^{\frac{2p}{2p-1}}\left(\frac{V_f(x,r,R_1)}{V_H^a(r,R_1)}\right)^{1-\frac{1}{2p-1}}.
\end{aligned}
$$

Separating of variables and integrating from r_1 to r_2 ($r_1 \le r_2$) with respect to the variable r, we immediately get

$$
\begin{aligned}
&\left(\frac{V_f(x,r_2,R_1)}{V_H^a(r_2,R_1)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r_1,R_1)}{V_H^a(r_1,R_1)}\right)^{\frac{1}{2p-1}} \\
&\le\left[\frac{n-1}{(2p-1)(2p-n)}\right]^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(R_1)\right)^{\frac{p}{2p-1}} A_H(R_1)\int_{r_1}^{r_2}\left(\frac{R_1\, e^{aR_1}}{V_H^a(t,R_1)}\right)^{\frac{2p}{2p-1}} \mathrm{d}t.
\end{aligned}
$$

Combining the above two aspects,

$$
\begin{aligned}
&\left(\frac{V_f(x,r_2,R_2)}{V_H^a(r_2,R_2)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r_1,R_1)}{V_H^a(r_1,R_1)}\right)^{\frac{1}{2p-1}} \\
&\le\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_{f\,-}^H\|_{p\ f,a}(R_2)\right)^{\frac{p}{2p-1}} \\
&\quad\times\left[\int_{R_1}^{R_2} A_H(t)\left(\frac{t\, e^{at}}{V_H^a(r_2,t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t + \int_{r_1}^{r_2} A_H(R_1)\left(\frac{R_1\, e^{aR_1}}{V_H^a(t,R_1)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t\right]
\end{aligned}
$$

for $0 \le r_1 \le r_2 \le R_1 \le R_2$. Hence the result follows. □

In particular, if f is constant and $a = 0$, we get volume comparison estimates for the annuluses on Riemannian manifolds with integral bounds for the Ricci curvature.

Corollary 3.5 *Let (M, g) be an n-dimensional complete Riemannian manifold. Let $H \in \mathbb{R}$ and $p > n/2$. For $0 \le r_1 \le r_2 \le R_1 \le R_2$ (assume $R_2 \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$), we have*

$$\begin{aligned}&\left(\frac{V(x,r_2,R_2)}{V_H(r_2,R_2)}\right)^{\frac{1}{2p-1}}-\left(\frac{V(x,r_1,R_1)}{V_H(r_1,R_1)}\right)^{\frac{1}{2p-1}}\\&\le\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\left\|\mathrm{Ric}_-^H\right\|_p(R_2)\right)^{\frac{p}{2p-1}}\\&\quad\times\left[\int_{R_1}^{R_2}A_H(t)\left(\frac{t}{V_H(r_2,t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t+\int_{r_1}^{r_2}A_H(R_1)\left(\frac{R_1}{V_H(t,R_1)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t\right].\end{aligned}$$

Remark 3.6 (1) If $r_1 = r_2 = 0$, we immediately get the Petersen–Wei's relative Bishop–Gromov volume comparison estimate in the integral sense [18].
(2) If $\mathrm{Ric}_-^H \equiv 0$, i.e., $\mathrm{Ric} \ge (n-1)H$, then we have a special case of the relative volume comparison estimate for annuluses on manifolds (see [25]).

4 Mean Curvature and Volume Comparison Estimate II

In this section, we shall prove a very general mean curvature comparison estimate on smooth metric measure spaces $(M, g, e^{-f}\mathrm{d}v)$ when only the integral Bakry–Émery Ricci tensor bounds (without any assumption on f), which might be useful in other applications.

In this case, we consider the following error form

$$\psi := (m_f - m_H)_+.$$

Using this, we have

Theorem 4.1 (Mean curvature comparison estimate II) *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Let $H \in \mathbb{R}$, and $r \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$. For any $p > \frac{n}{2}$ when $n \ge 3$ ($p > \frac{5}{4}$ when $n = 2$), we have*

$$\left(\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi(t)^{2p}\mathcal{A}_f\mathrm{d}t\right)^{\frac{1}{p}} \le \frac{2p-1}{2p-n}\Big(\mathcal{M}(r)+\mathcal{N}(r)\Big) \tag{21}$$

and

$$\mathrm{sn}_H^2(r)\psi(r)^{2p-1}e^{\frac{4p-2}{n-1}f(t)}\mathcal{A}_f \le \frac{(2p-1)^p}{(n-1)(2p-n)^{p-1}}\Big(\mathcal{M}(r)+\mathcal{N}(r)\Big)^p \tag{22}$$

along that minimal geodesic segment from x, where

$$\mathcal{M}(r) := \left(\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^{2p}\mathcal{A}_f\mathrm{d}t\right)^{\frac{1}{p}}$$

and

$$\mathcal{N}(r) := (n-1)\left(\int_0^r \mathrm{sn}_H^2(t) e^{\frac{4p-2}{n-1}f(t)} \left(\mathrm{Ric}_{f\,-}^H\right)^p \mathcal{A}_f \mathrm{d}t\right)^{\frac{1}{p}},$$

and where $m_H(r) = (n-1)\frac{\mathrm{sn}'_H(r)}{\mathrm{sn}_H(r)}$, *and* $\mathrm{sn}_H(r)$ *is the unique function satisfying*

$$\mathrm{sn}''_H(r) + H\mathrm{sn}_H(r) = 0, \quad \mathrm{sn}_H(0) = 0, \quad \mathrm{sn}'_H(0) = 1.$$

Moreover, if $H > 0$ *and* $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, *then we have*

$$\left(\int_0^r \sin^{4p-n-1}(\sqrt{H}t)\, e^{\frac{4p-2}{n-1}f(t)} \psi^{2p} \mathcal{A}_f \mathrm{d}t\right)^{\frac{1}{p}} \le \frac{2p-1}{2p-n}\left(\widetilde{\mathcal{M}}(r) + \widetilde{\mathcal{N}}(r)\right) \tag{23}$$

and

$$\sin^{4p-n-1}(\sqrt{H}r)\, e^{\frac{4p-2}{n-1}f(r)} \psi^{2p-1} \mathcal{A}_f \le (2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1} \left(\widetilde{\mathcal{M}}(r) + \widetilde{\mathcal{N}}(r)\right)^p \tag{24}$$

along that minimal geodesic segment from x, where

$$\widetilde{\mathcal{M}}(r) := \left(\int_0^r \sin^{4p-n-1}(\sqrt{H}t)\, e^{\frac{4p-2}{n-1}f(t)} m_H^{2p} \mathcal{A}_f \mathrm{d}t\right)^{\frac{1}{p}}$$

and

$$\widetilde{\mathcal{N}}(r) := (n-1)\left(\int_0^r \sin^{4p-n-1}(\sqrt{H}t)\, e^{\frac{4p-2}{n-1}f(t)} \left(\mathrm{Ric}_{f\,-}^H\right)^p \mathcal{A}_f \mathrm{d}t\right)^{\frac{1}{p}}.$$

It is unlucky that our theorem doesn't recover the classical case when the Ricci tensor has pointwise lower bound and f is constant. The main reason may be that we do not nicely deal with the "bad" term in the proof (see (25) below). It is interesting to know whether one has an improved estimate, which solves this problem.

Proof of Theorem 4.1 The proof's trick is partly inspired by the work of Wei–Wylie [22] and Petersen–Wei [18]. Recall that,

$$\begin{aligned}(m_f - m_H)' &\le -\frac{1}{n-1}[(m_f - m_H + \partial_r f)(m_f + m_H + \partial_r f)] + \mathrm{Ric}_{f\,-}^H \\ &= -\frac{1}{n-1}\Big[(m_f - m_H)^2 + 2(m_H + \partial_r f)(m_f - m_H) \\ &\qquad + \partial_r f(2m_H + \partial_r f)\Big] + \mathrm{Ric}_{f\,-}^H.\end{aligned}$$

Let $\psi := (m_f - m_H)_+$. Then

$$\psi' + \frac{\psi^2}{n-1} + \frac{2(m_H + \partial_r f)}{n-1}\psi \leq -\frac{\partial_r f}{n-1}(2m_H + \partial_r f) + \mathrm{Ric}^H_{f\,-}.$$

When $\partial_r f = 0$ and $\mathrm{Ric}^H_{f\,-} = 0$, we have $\psi = 0$, and get the classical mean curvature comparison. In general, notice that

$$-\frac{\partial_r f}{n-1}(2m_H + \partial_r f) = -\frac{(\partial_r f + m_H)^2}{n-1} + \frac{m_H^2}{n-1} \leq \frac{m_H^2}{n-1}. \tag{25}$$

Therefore,

$$\psi' + \frac{\psi^2}{n-1} + \frac{2(m_H + \partial_r f)}{n-1}\psi \leq \frac{m_H^2}{n-1} + \mathrm{Ric}^H_{f\,-}.$$

Multiplying this inequality by $(2p-1)\psi^{2p-2}\mathcal{A}_f$, we have

$$\begin{aligned}(2p-1)\psi^{2p-2}\psi'\mathcal{A}_f &+ \frac{2p-1}{n-1}\psi^{2p}\mathcal{A}_f + \frac{4p-2}{n-1}(m_H + \partial_r f)\psi^{2p-1}\mathcal{A}_f \\ &\leq \frac{2p-1}{n-1}m_H^2\psi^{2p-2}\mathcal{A}_f + (2p-1)\mathrm{Ric}^H_{f\,-}\cdot\psi^{2p-2}\mathcal{A}_f.\end{aligned} \tag{26}$$

Notice that

$$\begin{aligned}(\psi^{2p-1}\mathcal{A}_f)' &= (2p-1)\psi^{2p-2}\psi'\,\mathcal{A}_f + \psi^{2p-1}\mathcal{A}_f' \\ &= (2p-1)\psi^{2p-2}\psi'\,\mathcal{A}_f + \psi^{2p-1}m_f\mathcal{A}_f.\end{aligned}$$

So, (26) can be rewritten as

$$\begin{aligned}(\psi^{2p-1}\mathcal{A}_f)' - \psi^{2p-1}m_f\mathcal{A}_f &+ \frac{2p-1}{n-1}\psi^{2p}\mathcal{A}_f + \frac{4p-2}{n-1}(m_H + \partial_r f)\psi^{2p-1}\mathcal{A}_f \\ &\leq \frac{2p-1}{n-1}m_H^2\psi^{2p-2}\mathcal{A}_f + (2p-1)\mathrm{Ric}^H_{f\,-}\cdot\psi^{2p-2}\mathcal{A}_f.\end{aligned}$$

Rearranging some terms using $\psi := (m_f - m_H)_+$, we have

$$\begin{aligned}(\psi^{2p-1}\mathcal{A}_f)' + \frac{2p-n}{n-1}\psi^{2p}\mathcal{A}_f &+ \left(\frac{4p-n-1}{n-1}m_H + \frac{4p-2}{n-1}\partial_r f\right)\psi^{2p-1}\mathcal{A}_f \\ &\leq \frac{2p-1}{n-1}m_H^2\psi^{2p-2}\mathcal{A}_f + (2p-1)\mathrm{Ric}^H_{f\,-}\cdot\psi^{2p-2}\mathcal{A}_f.\end{aligned}$$

Multiplying this by the integrating factor $\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}$, we obtain

$$\begin{aligned}
&\left[\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\psi^{2p-1}\mathcal{A}_f\right]' + \frac{2p-n}{n-1}\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\psi^{2p}\mathcal{A}_f \\
&\qquad + \frac{4p-n-3}{n-1}\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}m_H\psi^{2p-1}\mathcal{A}_f \\
&\le \frac{2p-1}{n-1}\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}m_H^2\psi^{2p-2}\mathcal{A}_f \\
&\qquad + (2p-1)\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{2p-2}\mathcal{A}_f,
\end{aligned} \tag{27}$$

where we used $m_H(r)=(n-1)\frac{\mathrm{sn}_H'(r)}{\mathrm{sn}_H(r)}$. Since $p>n/2$ when $n\ge 3$, and $p>5/4$ when $n=2$, then

$$\frac{4p-n-3}{n-1}\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}m_H\psi^{2p-1}\mathcal{A}_f\ge 0.$$

Hence we can throw away this term from the above inequality, and get

$$\begin{aligned}
&\left[\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\psi^{2p-1}\mathcal{A}_f\right]' + \frac{2p-n}{n-1}\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\psi^{2p}\mathcal{A}_f \\
&\quad\le \frac{2p-1}{n-1}\left[\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}m_H^2\psi^{2p-2}\mathcal{A}_f + (n-1)\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{n-1}\mathcal{A}_f\right].
\end{aligned}$$

Since

$$\mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p-1}\mathcal{A}_f\Big|_{t=0}=0,$$

integrating the above inequality from 0 to r yields

$$\begin{aligned}
&\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\psi^{2p-1}\mathcal{A}_f + \frac{2p-n}{n-1}\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f\mathrm{d}t \\
&\le \frac{2p-1}{n-1}\bigg[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^2\psi^{2p-2}\mathcal{A}_f\mathrm{d}t \\
&\qquad + (n-1)\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\bigg].
\end{aligned}$$

Since $p>n/2$, the first two terms of the above inequality are non-negative. Hence,

$$\begin{aligned}
&\mathrm{sn}_H^2(r)e^{\frac{4p-2}{n-1}f(r)}\psi^{2p-1}\mathcal{A}_f \\
&\le \frac{2p-1}{n-1}\bigg[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^2\psi^{2p-2}\mathcal{A}_f\mathrm{d}t \\
&\qquad + (n-1)\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\bigg].
\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\frac{2p-n}{n-1}\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f\mathrm{d}t\\&\le\frac{2p-1}{n-1}\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^2\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\\&\qquad+(n-1)\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\Big].\end{aligned}\tag{29}$$

By Holder inequality, we also have

$$\begin{aligned}&\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^2\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\\&\le\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f\mathrm{d}t\Big]^{1-\frac{1}{p}}\cdot\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^{2p}\mathcal{A}_f\mathrm{d}t\Big]^{\frac{1}{p}}\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\\&\le\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f\mathrm{d}t\Big]^{1-\frac{1}{p}}\cdot\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\big(\mathrm{Ric}_{f\,-}^H\big)^p\mathcal{A}_f\mathrm{d}t\Big]^{\frac{1}{p}}.\end{aligned}\tag{31}$$

Finally, combining (30), (31), and (29), we obtain

$$\begin{aligned}\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f\mathrm{d}t\Big]^{\frac{1}{p}}&\le\frac{2p-1}{2p-n}\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}m_H^{2p}\mathcal{A}_f\mathrm{d}t\Big]^{\frac{1}{p}}\\&\quad+\frac{(n-1)(2p-1)}{2p-n}\Big[\int_0^r \mathrm{sn}_H^2(t)e^{\frac{4p-2}{n-1}f(t)}\big(\mathrm{Ric}_{f\,-}^H\big)^p\mathcal{A}_f\mathrm{d}t\Big]^{\frac{1}{p}},\end{aligned}\tag{32}$$

which implies (21). Combining (21), (30), (31), and (28) yields (22).

When $H>0$ and $\frac{\pi}{2\sqrt{H}}<r<\frac{\pi}{\sqrt{H}}$, we see that $m_H<0$ in (27). Similar to those discussion as before, multiplying by the integrating factor $\sin^{4p-n-3}(\sqrt{H}r)$ in (27) and integrating from 0 to r, we get

$$\begin{aligned}&\sin^{4p-n-1}(\sqrt{H}r)\cdot e^{\frac{4p-2}{n-1}f(r)}\psi^{2p-1}\mathcal{A}_f\\&\qquad+\frac{2p-n}{n-1}\int_0^r\sin^{4p-n-1}(\sqrt{H}t)\cdot e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f\,\mathrm{d}t\\&\le\frac{2p-1}{n-1}\Big[\int_0^r\sin^{4p-n-1}(\sqrt{H}t)\cdot e^{\frac{4p-2}{n-1}f(t)}m_H^2\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\\&\qquad+(n-1)\int_0^r\sin^{4p-n-1}(\sqrt{H}t)\cdot e^{\frac{4p-2}{n-1}f(t)}\mathrm{Ric}_{f\,-}^H\cdot\psi^{2p-2}\mathcal{A}_f\mathrm{d}t\Big].\end{aligned}\tag{33}$$

Using Holder inequality as before we get

$$\begin{aligned}
&\left[\int_0^r \sin^{4p-n-1}(\sqrt{H}t)e^{\frac{4p-2}{n-1}f(t)}\psi^{2p}\mathcal{A}_f \mathrm{d}t\right]^{\frac{1}{p}} \\
&\le \frac{2p-1}{2p-n}\left[\int_0^r \sin^{4p-n-1}(\sqrt{H}t)e^{\frac{4p-2}{n-1}f(t)}m_H^{2p}\mathcal{A}_f \mathrm{d}t\right]^{\frac{1}{p}} \\
&\quad + \frac{(n-1)(2p-1)}{2p-n}\left[\int_0^r \sin^{4p-n-1}(\sqrt{H}t)e^{\frac{4p-2}{n-1}f(t)}\left(\mathrm{Ric}_{f\,-}^H\right)^p\mathcal{A}_f \mathrm{d}t\right]^{\frac{1}{p}},
\end{aligned}$$

which is (23). Finally we substitute (23) into (33) gives (24) using the Holder inequality as before. □

In the following, we will apply mean curvature comparison estimate II to derive another weighted volume comparison estimate in the integral sense. At first, the weighted mean curvature comparison estimate II implies a tedious volume comparison estimate of geodesic spheres.

Theorem 4.2 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Let $H \in \mathbb{R}$ and $p > \frac{n}{2}$ when $n \ge 3$ ($p > \frac{5}{4}$ when $n = 2$) be given, and when $H > 0$ assume that $R \le \frac{\pi}{2\sqrt{H}}$. For $0 < r \le R$, we have*

$$\begin{aligned}
&\left(\frac{A_f(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}} \\
&\le C(n,p)\int_r^R \Big(\mathcal{M}(t)+\mathcal{N}(t)\Big)^{\frac{p}{2p-1}} \mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, e^{-\frac{2f(t)}{n-1}}\, A_H^{-\frac{1}{2p-1}}(t)\,\mathrm{d}t,
\end{aligned} \tag{34}$$

where

$$C(n,p) := \left(\frac{2p-n}{n-1}\right)^{\frac{1}{2p-1}}\left(\frac{2p-1}{2p-n}\right)^{\frac{p}{2p-1}},$$

$$\mathcal{M}(t) := \left(\int_0^t \mathrm{sn}_H^2(s)\, e^{\frac{4p-2}{n-1}f(s)}m_H^{2p}\mathcal{A}_f \mathrm{d}s\right)^{\frac{1}{p}},$$

and

$$\mathcal{N}(t) := (n-1)\left(\int_0^t \mathrm{sn}_H^2(s)\, e^{\frac{4p-2}{n-1}f(s)}\left(\mathrm{Ric}_{f\,-}^H\right)^p\mathcal{A}_f \mathrm{d}s\right)^{\frac{1}{p}}.$$

In particular, if further assume $|f| \le k$ for some constant $k \ge 0$; and $\frac{n}{2} < p < \frac{n}{2}+1$ when $n \ge 3$ (when $n = 2$, we assume $\frac{5}{4} < p < 2$). For $0 < r \le R$, we have

$$\begin{aligned}
&\left(\frac{A_f(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}} \\
&\le C(n,p)\, e^{\frac{4k}{n-1}}\Big(\mathcal{P}(R)+\mathcal{Q}(R)\Big)^{\frac{p}{2p-1}}\int_r^R \mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, A_H^{-\frac{1}{2p-1}}(t)\mathrm{d}t,
\end{aligned} \tag{35}$$

where

$$\mathcal{P}(R) := \left(\int_0^R \mathrm{sn}_H^2(t) m_H^{2p} \mathcal{A}_f \mathrm{d}t \right)^{\frac{1}{p}}$$

and

$$\mathcal{Q}(R) := (n-1) \Big(\int_0^R \mathrm{sn}_H^2(t) \big(\mathrm{Ric}_{f\,-}^H\big)^p \mathcal{A}_f \mathrm{d}t \Big)^{\frac{1}{p}}.$$

Remark 4.3 We remark that $\mathcal{P}(R)$ converges when $\frac{n}{2} < p < \frac{n}{2} + 1$, $n \geq 3$ (when $\frac{5}{4} < p < 2$, $n = 2$). However, for such p, if $r \to 0$, the integral

$$\int_r^R \mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, A_H^{-\frac{1}{2p-1}}(t) \mathrm{d}t$$

blows up.

Proof of Theorem 4.2 We apply $\mathcal{A}_f' = m_f \mathcal{A}_f$ and $\mathcal{A}_H' = m_H \mathcal{A}_H$ to compute that

$$\frac{d}{\mathrm{d}t} \left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H(t)} \right) = (m_f - m_H) \frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H(t)}.$$

Hence,

$$\begin{aligned} \frac{d}{\mathrm{d}t} \left(\frac{A_f(x,t)}{A_H(t)} \right) &= \frac{1}{Vol(S^{n-1})} \int_{S^{n-1}} \frac{d}{\mathrm{d}t} \left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H(t)} \right) \mathrm{d}\theta_{n-1} \\ &\leq \frac{1}{A_H(t)} \int_{S^{n-1}} \psi \cdot \mathcal{A}_f(t,\theta) \mathrm{d}\theta_{n-1}. \end{aligned}$$

Using Holder's inequality and (22), we have

$$\begin{aligned} &\int_{S^{n-1}} \psi \cdot \mathcal{A}_f(t,\theta) \mathrm{d}\theta_{n-1} \\ &\quad \leq \left(\int_{S^{n-1}} \psi^{2p-1} \mathcal{A}_f(x,t) \mathrm{d}\theta_{n-1} \right)^{\frac{1}{2p-1}} A_f(x,t)^{1-\frac{1}{2p-1}} \\ &\quad \leq C(n,p)\, \mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, e^{-\frac{2f(t)}{n-1}} \cdot \Big(\mathcal{M}(t) + \mathcal{N}(t) \Big)^{\frac{p}{2p-1}} A_f(x,t)^{1-\frac{1}{2p-1}}, \end{aligned}$$

where

$$C(n,p) := \left(\frac{2p-n}{n-1} \right)^{\frac{1}{2p-1}} \left(\frac{2p-1}{2p-n} \right)^{\frac{p}{2p-1}},$$

$$\mathcal{M}(t) := \left(\int_0^t \mathrm{sn}_H^2(s)\, e^{\frac{4p-2}{n-1} f(s)} m_H^{2p} \mathcal{A}_f \mathrm{d}s \right)^{\frac{1}{p}},$$

and

$$\mathcal{N}(t) := (n-1)\Big(\int_0^t \mathrm{sn}_H^2(s)\, e^{\frac{4p-2}{n-1}f(s)}\big(\mathrm{Ric}_{f\,-}^H\big)^p \mathcal{A}_f \mathrm{d}s\Big)^{\frac{1}{p}}.$$

Hence,

$$\begin{aligned}\frac{d}{\mathrm{d}t}\left(\frac{A_f(x,t)}{A_H(t)}\right) \le C(n,p)\,\mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, e^{-\frac{2f(t)}{n-1}}\cdot\left(\frac{A_f(x,t)}{A_H(t)}\right)^{1-\frac{1}{2p-1}}\\ \times\Big(\mathcal{M}(t)+\mathcal{N}(t)\Big)^{\frac{p}{2p-1}}\left(\frac{1}{A_H(t)}\right)^{\frac{1}{2p-1}}.\end{aligned} \tag{36}$$

Separating of variables and integrating from r to R, we obtain

$$\begin{aligned}&\left(\frac{A_f(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}}\\ &\le \frac{C(n,p)}{2p-1}\int_r^R \Big(\mathcal{M}(t)+\mathcal{N}(t)\Big)^{\frac{p}{2p-1}}\,\mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, e^{-\frac{2f(t)}{n-1}}\, A_H^{-\frac{1}{2p-1}}(t)\mathrm{d}t.\end{aligned}$$

Therefore we prove the first part of conclusions.

If we further assume $|f| \le k$ for some constant $k \ge 0$, then

$$\begin{aligned}&\left(\frac{A_f(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}}\\ &\le \frac{C(n,p)}{2p-1}\, e^{\frac{4k}{n-1}}\Big(\mathcal{P}(R)+\mathcal{Q}(R)\Big)^{\frac{p}{2p-1}}\int_r^R \mathrm{sn}_H^{-\frac{2}{2p-1}}(t)\, A_H^{-\frac{1}{2p-1}}(t)\mathrm{d}t,\end{aligned}$$

where

$$\mathcal{P}(R) := \Big(\int_0^R \mathrm{sn}_H^2(t) m_H^{2p}\mathcal{A}_f \mathrm{d}t\Big)^{\frac{1}{p}}$$

and

$$\mathcal{Q}(R) := (n-1)\Big(\int_0^R \mathrm{sn}_H^2(t)\big(\mathrm{Ric}_{f\,-}^H\big)^p \mathcal{A}_f \mathrm{d}t\Big)^{\frac{1}{p}}.$$

Notice that $\mathcal{P}(R)$ converges when $\frac{n}{2} < p < \frac{n}{2}+1$ if $n \ge 3$ (if $n = 2$, we assume $\frac{5}{4} < p < 2$). Hence the result follows. □

Similar to the first case of discussions (the case $\partial_r f \ge -a$), we can apply (36) to obtain the following volume comparison estimate when f is bounded.

Theorem 4.4 (Relative volume comparison estimate II) *Let* $(M, g, e^{-f}\mathrm{d}v)$ *be an* n*-dimensional smooth metric measure space. Assume that*

$$|f(x)| \leq k$$

for some constant $k \geq 0$. *Let* $H \in \mathbb{R}$ *and* $\frac{n}{2} < p < \frac{n}{2} + 1$ *when* $n \geq 3$ (*when* $n = 2$, *we assume* $\frac{5}{4} < p < 2$) *be given, and when* $H > 0$ *assume that* $R \leq \frac{\pi}{2\sqrt{H}}$. *For* $0 < r \leq R$, *we have*

$$\begin{aligned} &\left(\frac{V_f(x,R)}{V_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r)}{V_H(r)}\right)^{\frac{1}{2p-1}} \\ &\quad \leq C(n,p)\, e^{\frac{4k}{n-1}} \Big(\mathcal{P}(R) + \mathcal{Q}(R)\Big)^{\frac{p}{2p-1}} \int_r^R A_H(t) \left(\frac{t^{1-\frac{1}{p}}}{V_H(t)}\right)^{\frac{2p}{2p-1}} \mathrm{d}t. \end{aligned} \tag{37}$$

Here,

$$C(n,p) := \frac{4p-2}{p-1}\left(\frac{(n-1)^{-\frac{1}{p-1}}}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}},$$

$$\mathcal{P}(R) := \left(\int_0^R \mathrm{sn}_H^2(t) m_H^{2p} \mathcal{A}_f \mathrm{d}t\right)^{\frac{1}{p}}$$

and

$$\mathcal{Q}(R) := (n-1)\bigg(\int_0^R \mathrm{sn}_H^2(t)\big(\mathrm{Ric}_{f\ -}^H\big)^p \mathcal{A}_f \mathrm{d}t\bigg)^{\frac{1}{p}}.$$

Remark 4.5 We remark that $\mathcal{P}(R)$ converges when $\frac{n}{2} < p < \frac{n}{2} + 1$, $n \geq 3$ (when $\frac{5}{4} < p < 2$, $n = 2$). However, for such p, if $r \to 0$, then the integral

$$\int_r^R A_H(t) \left(\frac{t^{1-\frac{1}{p}}}{V_H(t)}\right)^{\frac{2p}{2p-1}} \mathrm{d}t$$

blows up.

Proof of Theorem 6.2 We use the formula

$$\frac{V_f(x,r)}{V_H(r)} = \frac{\int_0^r A_f(x,t)\mathrm{d}t}{\int_0^r A_H(t)\mathrm{d}t},$$

to compute that

$$\frac{d}{dr}\left(\frac{V_f(x,r)}{V_H(r)}\right) = \frac{A_f(x,r)\int_0^r A_H(t)\mathrm{d}t - A_H(r)\int_0^r A_f(x,t)\mathrm{d}t}{(V_H(r))^2}. \tag{38}$$

On the other hand, integrating (36) from t to r, and using the Holder inequality, we get

$$\begin{aligned}
&\frac{A_f(x,r)}{A_H(r)} - \frac{A_f(x,t)}{A_H(t)} \\
&\quad \le C(n,p,k) \int_t^r \frac{\big(\mathcal{P}(s)+\mathcal{Q}(s)\big)^{\frac{p}{2p-1}}}{\mathrm{sn}_H^{\frac{2}{2p-1}}(s)\cdot A_H(s)^{\frac{1}{2p-1}}} \cdot \left(\frac{A_f(x,s)}{A_H(s)}\right)^{1-\frac{1}{2p-1}} \mathrm{d}s \\
&\quad \le C(n,p,k) \frac{\big(\mathcal{P}(r)+\mathcal{Q}(r)\big)^{\frac{p}{2p-1}}}{A_H(t)} \cdot \int_t^r \mathrm{sn}_H^{-\frac{2}{2p-1}}(s)\cdot A_f(x,s)^{1-\frac{1}{2p-1}} \mathrm{d}s \\
&\quad \le C(n,p,k) \frac{\big(\mathcal{P}(r)+\mathcal{Q}(r)\big)^{\frac{p}{2p-1}}}{A_H(t)} \cdot \left(\int_t^r \mathrm{sn}_H^{-2}(s)\mathrm{d}s\right)^{\frac{1}{2p-1}} \cdot V_f(x,r)^{1-\frac{1}{2p-1}} \\
&\quad \le C(n,p,k) \frac{\big(\mathcal{P}(r)+\mathcal{Q}(r)\big)^{\frac{p}{2p-1}}}{A_H(t)} \cdot \left(\frac{4}{t}\right)^{\frac{1}{2p-1}} \cdot V_f(x,r)^{1-\frac{1}{2p-1}},
\end{aligned}$$

where

$$C(n,p,k) := \left(\frac{2p-n}{n-1}\right)^{\frac{1}{2p-1}} \left(\frac{2p-1}{2p-n}\right)^{\frac{p}{2p-1}} e^{\frac{4k}{n-1}}.$$

This implies that

$$\begin{aligned}
&A_f(x,r)A_H(t) - A_H(r)A_f(x,t) \\
&\qquad \le 4C(n,p,k)\big(\mathcal{P}(r)+\mathcal{Q}(r)\big)^{\frac{p}{2p-1}} \cdot A_H(r)\cdot t^{\frac{-1}{2p-1}}\, V_f(x,r)^{1-\frac{1}{2p-1}}.
\end{aligned}$$

Plugging this into (38) gives

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{V_f(x,r)}{V_H(r)}\right) \\
&\quad \le \frac{4(2p-1)}{2p-2} C(n,p,k)\big(\mathcal{P}(r)+\mathcal{Q}(r)\big)^{\frac{p}{2p-1}} A_H(r)\cdot r^{\frac{2p-2}{2p-1}} \cdot \frac{V_f(x,r)^{1-\frac{1}{2p-1}}}{(V_H(r))^2} \\
&\quad = \frac{4(2p-1)}{2p-2} C(n,p,k)\big(\mathcal{P}(r)+\mathcal{Q}(r)\big)^{\frac{p}{2p-1}} A_H(r) \left(\frac{r^{1-\frac{1}{p}}}{V_H(r)}\right)^{\frac{2p}{2p-1}} \left(\frac{V_f(x,r)}{V_H(r)}\right)^{1-\frac{1}{2p-1}}.
\end{aligned}$$

Separating of variables and integrating from r to R,

$$\begin{aligned}
&\left(\frac{V_f(x,R)}{V_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r)}{V_H(r)}\right)^{\frac{1}{2p-1}} \\
&\qquad \le \frac{2C(n,p,k)}{p-1} \big(\mathcal{P}(R)+\mathcal{Q}(R)\big)^{\frac{p}{2p-1}} \int_r^R A_H(t) \left(\frac{t^{1-\frac{1}{p}}}{V_H(t)}\right)^{\frac{2p}{2p-1}} \mathrm{d}t.
\end{aligned}$$

Then the conclusion follows. □

5 Applications of Comparison Estimates

In this section, we mainly apply mean curvature comparison estimate I and volume comparison estimate I to prove the global diameter estimate, eigenvalue upper estimate and the volume growth estimate when the normalized new L^p-norm of Bakry–Émery Ricci tensor below $(n-1)H$ is small.

We first prove Theorem 1.6 in the introduction.

Proof of Theorem 1.6 Let p_1, p_2 are two points in M, and x_0 be a middle point between p_1 and p_2. We also let $e(x)$ be the excess function for the points p_1 and p_2, i.e.,

$$e(x) := d(p_1, x) + d(p_2, x) - d(p_1, p_2).$$

By the triangle inequality, we have

$$e(x) \geq 0 \quad \text{and} \quad e(x) \leq 2r$$

on a ball $B(x_0, r)$, where $r > 0$. In the following, we will prove our result by contradiction. That is, if there exist two points p_1, p_2 in M, such that $d(p_1, p_2) > D$ for any sufficient large D, then we can show that excess function e is negative on $B(x_0, r)$, which is a contradiction. The detail discussion is as follows.

By the mean curvature estimate (1), using a suitably large comparison sphere we may choose any large D enough so that if $d(p_1, p_2) > D$, then

$$\Delta_f e \leq -K + \psi_1$$

on $B(x_0, r)$, where K is a large positive constant to be determined, and ψ_1 is an error term controlled by $C_1(n, p, a, H, r) \cdot \bar{k}(p, H, a, r)$.

Following the nice construction of Lemma 1.4 in Colding's paper in [7], let $\Omega_j \subseteq B(x_0, r)$ be a sequence of smooth star-shaped domains which converges to $B(x_0, r) - \mathrm{Cut}(x_0)$. Also let u_i be a sequence of smooth functions such that

$$|u_i - e| < i^{-1}, \quad |\nabla u_i| \leq 2 + i^{-1}, \quad \text{and} \quad \Delta_f u_i \leq \Delta_f e + i^{-1}$$

on $B(x_0, r)$. Set $h := d^2(x_0, \cdot) - r^2$, and then h is a negative smooth function on Ω_j. So by Green's formula with respect to the weighted measure $e^{-f}\mathrm{d}v$, we have

$$\int_{\Omega_j} (\Delta_f u_i)h - \int_{\Omega_j} u_i(\Delta_f h) = \int_{\partial\Omega_j} h(\nu u_i) - \int_{\partial\Omega_j} u_i(\nu h),$$

where ν is the outward unit normal direction to Ω_j. We notice that

$$\Delta_f u_i \leq -K + \psi_1 + i^{-1}$$

and

$$\begin{aligned} u_i(\Delta_f h) &\le (e+i^{-1})(2d\,\Delta_f d+2) \\ &\le (e+i^{-1})(2n+2ad+\psi_2) \\ &\le 3r(2n+2ar+\psi_2), \end{aligned}$$

where ψ_2 is another error term still controlled by $C_2(n,p,a,H,r)\cdot\bar{k}(p,H,a,r)$. Therefore, we have

$$\int_{\Omega_j}(-K+\psi_1+i^{-1})h-3r\int_{\Omega_j}(2n+2ar+\psi_2)\le\int_{\partial\Omega_j}h(2+i^{-1})-\int_{\partial\Omega_j}u_i(\nu h).$$

Since $u_i\to e$ when $i\to\infty$, by the dominated convergence theorem, the above inequality implies

$$\int_{\Omega_j}(-K+\psi_1)h-3r\int_{\Omega_j}(2n+2ar+\psi_2)\le 2\int_{\partial\Omega_j}h-\int_{\partial\Omega_j}e(\nu h). \tag{39}$$

We also notice that

$$\begin{aligned} \int_{B(x_0,r)}(-K+\psi_1)h &\ge \int_{B(x_0,\frac{r}{2})}-Kh-\int_{B(x_0,r)}r^2\psi_1 \\ &\ge \int_{B(x_0,\frac{r}{2})}\frac{3}{4}r^2K-\int_{B(x_0,r)}r^2\psi_1 \\ &= \frac{3}{4}r^2K\cdot V_f\left(x_0,\frac{r}{2}\right)-\int_{B(x_0,r)}r^2\psi_1 \end{aligned}$$

and hence,

$$\begin{aligned} &\int_{B(x_0,r)}(-K+\psi_1)h-3r\int_{B(x_0,r)}(2n+2ar+\psi_2) \\ &\ge \frac{3}{4}r^2K\cdot V_f\left(x_0,\frac{r}{2}\right)-6(nr+ar^2)V_f(x_0,r)-\int_{B(x_0,r)}(r^2\psi_1+3r\psi_2). \end{aligned}$$

By relative volume comparison estimate, if $\bar{k}(p,r,H,a)$ is small enough, we then have

$$V_f\left(x_0,\frac{r}{2}\right)\ge 2^{-1}e^{-ar}\left(\frac{\sin(\frac{r}{2})}{\sin r}\right)^n\cdot V_f(x_0,r).$$

Moreover, if $\bar{k}(p,H,a,r)$ is small enough, we also have

$$\int_{B(x_0,r)}(r^2\psi_1+3r\psi_2)\le(nr+r^2)V_f(x_0,r).$$

Thus, if we choose

$$K > \frac{8}{3}(7nr^{-1} + 6a + 1)e^{ar}\left(\frac{\sin r}{\sin(\frac{r}{2})}\right)^n,$$

then

$$\int_{B(x_0,r)}(-K + \psi_1)h - 3r\int_{B(x_0,r)}(2n + 2ar + \psi_2) > 0.$$

Combining this and (39) immediately yields

$$2\int_{\partial\Omega_j} h - \int_{\partial\Omega_j} e(\nu h) > 0$$

as $j \to \infty$. However, the first integral of the above inequality goes to zero as $j \to \infty$; while in the second integral of the above inequality: $\nu h \geq 0$ on $\partial\Omega_j$ for all j, as Ω_j is star-shaped. This forces that the excess function e must be negative on $B(x_0, r)$, which is a contradiction to the fact: $e \geq 0$. Hence $d(p_1, p_2) \leq D$ for some D. □

Next we apply the similar argument of Petersen–Sprouse [17] to prove Theorem 1.7 in the introduction.

Proof of Theorem 1.7 The proof is easy only by some direct computation. Recall that $B(\bar{x}_0, R)$ is a metric ball in the weighted model space $M^n_{H,a}$, where $R \leq \frac{\pi}{2\sqrt{H}}$. Let $\lambda_1^D(n, H, a, R)$ be the first eigenvalue of the h-Laplacian Δ_h with the Dirichlet condition in $M^n_{H,a}$, where $h(x) := -a\cdot d(\bar{x}_0, x)$, and $u(x) = \phi(r)$ be the corresponding eigenfunction, which satisfies

$$\phi'' + (m_H + a)\phi' + \lambda_1^D(n, H, a, R)\phi = 0, \quad \phi(0) = 1, \quad \phi(R) = 0.$$

It is easy to see that $0 \leq \phi \leq 1$, since $\phi' < 0$ on $[0, R]$. Now we consider the Rayleigh quotient of the function $u(x) = \phi(d(x_0, x))$. In the course of the proof, we will use the relative volume comparison estimate when volume normalization of some integral Bakry–Émery Ricci tensor is sufficient small. Now, a direct computation yields that

$$\begin{aligned}
\int_{B(x_0,R)} |\nabla u|^2 e^{-f}\mathrm{d}v &= \int_{S^{n-1}}\int_0^R (\phi')^2\mathcal{A}_f(t,\theta)\,\mathrm{d}t\mathrm{d}\theta_{n-1}\\
&= \int_{S^{n-1}}\left(\phi\phi'\mathcal{A}_f\big|_0^R - \int_0^R \phi(\phi'\mathcal{A}_f)'\,\mathrm{d}t\right)\mathrm{d}\theta_{n-1}\\
&= -\int_{S^{n-1}}\int_0^R \phi(\phi'' + m_f\phi')\mathcal{A}_f\,\mathrm{d}t\mathrm{d}\theta_{n-1}\\
&= -\int_{S^{n-1}}\int_0^R \phi(\phi'' + (m_H + a)\phi')\mathcal{A}_f\,\mathrm{d}t\mathrm{d}\theta_{n-1}
\end{aligned}$$

$$
\begin{aligned}
&-\int_{S^{n-1}}\int_0^R (m_f - m_H - a)\phi\phi'\mathcal{A}_f \,\mathrm{d}t\mathrm{d}\theta_{n-1} \\
&\le \lambda_1^D(n, H, a, R)\int_{S^{n-1}}\int_0^R \phi^2\mathcal{A}_f \,\mathrm{d}t\mathrm{d}\theta_{n-1} \\
&\quad + \int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+|\phi'|\mathcal{A}_f \,\mathrm{d}t\mathrm{d}\theta_{n-1}.
\end{aligned}
$$

Hence the Rayleigh quotient satisfies

$$
\begin{aligned}
Q &= \frac{\int_{B(x_0,R)} |\nabla u|^2 e^{-f}\mathrm{d}v}{\int_{B(x_0,R)} u^2 e^{-f}\mathrm{d}v} \\
&\le \lambda_1^D(n, H, a, R) + \frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+ |\phi'|\mathcal{A}_f \,\mathrm{d}t\mathrm{d}\theta_{n-1}}{\int_{S^{n-1}}\int_0^R \phi^2\mathcal{A}_f \,\mathrm{d}t\mathrm{d}\theta_{n-1}}.
\end{aligned}
$$

Now choose the first value $r = r(n, H, a, R)$ such that $\phi(r) = 1/2$. Then the last error term can be estimated:

$$
\begin{aligned}
&\frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+ |\phi'|\mathcal{A}_f}{\int_{S^{n-1}}\int_0^R \phi^2\mathcal{A}_f} \\
&\le \frac{\left(\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^2 \mathcal{A}_f\right)^{\frac{1}{2}} \cdot \left(\int_{S^{n-1}}\int_0^R |\phi'|^2\mathcal{A}_f\right)^{\frac{1}{2}}}{\frac{1}{2}V_f^{\frac{1}{2}}(x_0, r)\cdot\left(\int_{S^{n-1}}\int_0^R \phi^2\mathcal{A}_f\right)^{\frac{1}{2}}} \\
&\le 2\left(\frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^2 \mathcal{A}_f}{V_f(x_0, r)}\right)^{\frac{1}{2}}\sqrt{Q}.
\end{aligned}
$$

On the other hand, if $\bar{k}(p, H, a, R)$ is very small, then we have the following volume doubling estimate (see Corollary 3.3):

$$
\frac{V_f(x_0, R)}{V_f(x_0, r)} \le 4\frac{V_H^a(R)}{V_H^a(r)}.
$$

Putting this into the above error estimate, we have

$$
\begin{aligned}
&\frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+ |\phi'|\mathcal{A}_f}{\int_{S^{n-1}}\int_0^R \phi^2\mathcal{A}_f} \\
&\le 4\left(\frac{V_H^a(R)}{V_H^a(r)}\right)^{1/2}\left(\frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^2 \mathcal{A}_f}{V_f(x_0, R)}\right)^{\frac{1}{2}}\sqrt{Q}.
\end{aligned}
$$

By the Holder inequality, we observe that

$$\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^2\,\mathcal{A}_f \le \left(\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^{2p}\,\mathcal{A}_f e^{-at}\right)^{\frac{1}{p}} \times \left(\int_{S^{n-1}}\int_0^R \mathcal{A}_f e^{at}\right)^{1-\frac{1}{p}}$$
$$\le e^{a(1-\frac{1}{p})R}\cdot V_f(x_0,R)^{1-\frac{1}{p}}\left(\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^{2p}\,\mathcal{A}_f e^{-at}\right)^{\frac{1}{p}}.$$

Using this, we further have

$$\frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+\,|\phi'|\mathcal{A}_f}{\int_{S^{n-1}}\int_0^R \phi^2\mathcal{A}_f}$$
$$\le 4e^{\frac{(p-1)a}{2p}R}\left(\frac{V_H^a(R)}{V_H^a(r)}\right)^{\frac{1}{2}}\left(\frac{\int_{S^{n-1}}\int_0^R (m_f - m_H - a)_+^{2p}\,\mathcal{A}_f e^{-at}}{V_f(x_0,R)}\right)^{\frac{1}{2p}}\sqrt{Q}$$
$$\le C(n,p,H,a,R)(\bar{k}(p,H,a,R))^{\frac{1}{2}}\sqrt{Q}.$$

Therefore,

$$Q \le \lambda_1^D(n,H,a,R) + C(n,p,H,a,R)(\bar{k}(p,H,a,R))^{\frac{1}{2p}}\sqrt{Q},$$

which implies the desired result. □

Finally, we use Theorem 1.5 to prove Theorem 1.8. The proof method is similar to the classical case.

Proof of Theorem 1.8 Since $a = 0$ and $H = 0$, Theorem 1.5 in fact can be simply written as

$$\left(\frac{V_f(x,r_2,R_2)}{V_0(r_2,R_2)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r_1,R_1)}{V_0(r_1,R_1)}\right)^{\frac{1}{2p-1}} \le \mathcal{C}\cdot\left(\left\|\mathrm{Ric}^0_{f\,-}\right\|^p_{p\ f,0}(R_2)\right)^{\frac{1}{2p-1}},$$

for any $0 \le r_1 \le r_2 \le R_1 \le R_2$, where

$$\mathcal{C} := C(n,p)\left[\int_{r_1}^{r_2} R_1^{n-1}\left(\frac{R_1}{(R_1-t)^n}\right)^{\frac{2p}{2p-1}}\mathrm{d}t + \int_{R_1}^{R_2} t^{n-1}\left(\frac{t}{(t-r_2)^n}\right)^{\frac{2p}{2p-1}}\mathrm{d}t\right]$$
$$\le 2C(n,p)\,R_2^n\left(\frac{R_2}{(R_1-r_2)^n}\right)^{\frac{2p}{2p-1}}.$$

Let $x \in M$ be a point with $d(x_0, x) = R \geq 2$. Letting $r_1 = 0, r_2 = R - 1, R_1 = R$ and $R_2 = R + 1$ in the above inequality, then

$$\frac{V_f(x, R+1) - V_f(x, R-1)}{(R+1)^n - (R-1)^n} \leq \left[\left(\frac{V_f(x, R)}{R^n} \right)^{\frac{1}{2p-1}} + 2C(n, p)(R+1)^{n+\frac{2p}{2p-1}} \left(\|\mathrm{Ric}_{f\,-}^0\|_{p\ f,0}^p (R+1) \right)^{\frac{1}{2p-1}} \right]^{2p-1}.$$

Using the inequality $(a+b)^m \leq 2^{m-1}(a^m + b^m)$ for all $a > 0$ and $b > 0$ with $m = 2p - 1$, we have the following inequality

$$\frac{V_f(x, R+1) - V_f(x, R-1)}{(R+1)^n - (R-1)^n} \leq \tilde{C}(n, p)\frac{V_f(x, R)}{R^n} + \tilde{C}(n, p)(R+1)^{n(2p-1)+2p} \|\mathrm{Ric}_{f\,-}^0\|_{p\ f,0}^p (R+1)$$

for some constant $\tilde{C}(n, p)$. Multiplying this inequality by $\frac{(R+1)^n - (R-1)^n}{V_f(x, R+1)}$, by the definition of $\bar{k}$, we hence have

$$\frac{V_f(x, R+1) - V_f(x, R-1)}{V_f(x, R+1)} \leq \frac{D(n, p)}{R} + D(n, p)(R+1)^{n(2p-1)+2p} \cdot \bar{k}^p(p, 0, 0, R+1).$$

for some constant $D(n, p)$. Now we choose $\epsilon = \epsilon(n, p, R)$ small enough with $\bar{k}(p, 0, 0, R+1) < \epsilon$, such that

$$\frac{V_f(x, R+1) - V_f(x, R-1)}{V_f(x, R+1)} \leq \frac{2D(n, p)}{R}.$$

Since $B(x_0, 1) \subset B(x, R+1) \setminus B(x, R-1)$ and $B(x, R+1) \subset B(x_0, 2R+1)$, hence we have

$$V_f(x_0, 2R+1) \geq V_f(x, R+1) \geq \frac{V_f(x_0, 1)}{2D(n, p)} R$$

for the $R \geq 2$. □

Appendix: Comparison Estimates for Integral Bounds of *m*-Bakry–Emery Ricci Tensor

In this section, we will state f-mean curvature comparison estimates and relative f-volume comparison estimates when only the weighted integral bounds of the m-Bakry–Émery Ricci tensor. Since the proof is almost the same as the manifold case, we omit these proofs here.

Recall that another natural generalization of the Ricci tensor associated to smooth metric measure space $(M, g, e^{-f} dv_g)$ is called m-Bakry–Émery Ricci tensor, which is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric}_f - \frac{1}{m} df \otimes df$$

for some number $m > 0$. This curvature tensor is also introduced by Bakry and Émery [3]. Here m is finite, and we have the Bochner formula for the m-Bakry–Émery Ricci tensor

$$\begin{aligned} \frac{1}{2}\Delta_f |\nabla u|^2 &= |\mathrm{Hess} u|^2 + \langle \nabla \Delta_f u, \nabla u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u) \\ &\geq \frac{(\Delta_f u)^2}{m+n} + \langle \nabla \Delta_f u, \nabla u\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u) \end{aligned} \tag{40}$$

for some $u \in C^\infty(M)$, which is regarded as the Bochner formula of the Ricci curvature of an $(n+m)$-dimensional manifold. Hence many classical geometrical and topological results for manifolds with Ricci tensor bounded below can be easily extended to smooth metric measure spaces with m-Bakry–Émery Ricci tensor bounded below (without any assumption on f), see for example [4,5,14,21] for details.

Let $(M, g, e^{-f} dv_g)$ be an n-dimensional smooth metric measure space. For each $x \in M$, $m > 0$ and let $\lambda(x)$ denote the smallest eigenvalue for the tensor $\mathrm{Ric}_f^m : T_x M \to T_x M$. We define

$$\mathrm{Ric}_f^{m\,H}{}_- := \big((n+m-1)H - \lambda(x)\big)_+,$$

where $H \in \mathbb{R}$, which measures the amount of m-Bakry–Émery Ricci tensor below $(n+m-1)H$. We also introduce a L_f^p-norm of function ϕ, with respect to the weighted measure $e^{-f} dv_g$:

$$\|\phi\|_{p\,f}(r) := \sup_{x\in M}\left(\int_{B_x(r)} |\phi|^p \cdot e^{-f} dv_g\right)^{\frac{1}{p}}.$$

Clearly, $\|\mathrm{Ric}_f^{m\,H}{}_-\|_{p\,f}(r) = 0$ iff $\mathrm{Ric}_f^m \geq (n+m-1)H$. Notice that when f is constant, all above notations are just as the usual quantities on manifolds.

Let $r(y) = d(y, x)$ be the distance function from x to y, and define

$$\varphi := (\Delta_f r - m_H^{n+m})_+,$$

the error from weighted mean curvature comparison in [5]. Here m_H^{n+m} denotes the mean curvature of the geodesic sphere in M_H^{n+m}, the $n+m$-dimensional simply connected space with constant sectional curvature H. The weighted Laplacian comparison states that if $\mathrm{Ric}_f^m \geq (n+m-1)H$, then $\Delta_f r \leq m_H^{m+n}$ (see for example [5,21]). Using (40), following the discussion in Sect. 2, we can similarly generalize Petersen–Wei's and Aubry's comparison results to the case of smooth metric measure spaces with only the m-Bakry–Émery Ricci tensor integral bounds.

Theorem 6.1 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. For any $p > \frac{n+m}{2}$, $m > 0$, $H \in \mathbb{R}$ (assume $r \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$), then*

$$\|\varphi\|_{2p\,f}(r) \le \left[\frac{(n+m-1)(2p-1)}{2p-n-m}\left\|\mathrm{Ric}_f^{m\,H}{}_{-}\right\|_{p\,f}(r)\right]^{\frac{1}{2}}$$

and

$$\varphi^{2p-1}\mathcal{A}_f \le (2p-1)^p\left(\frac{n+m-1}{2p-n-m}\right)^{p-1}\cdot\int_0^r\left(\mathrm{Ric}_f^{m\,H}{}_{-}\right)^p\mathcal{A}_f\mathrm{d}t$$

along that minimal geodesic segment from x.

Moreover, if $H > 0$ and $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, then

$$\left\|\sin^{\frac{4p-n-m-1}{2p}}(\sqrt{H}t)\cdot\varphi\right\|_{2p\,f}(r) \le \left[\frac{(n+m-1)(2p-1)}{2p-n-m}\left\|\mathrm{Ric}_f^{m\,H}{}_{-}\right\|_{p\,f}(r)\right]^{\frac{1}{2}}$$

and

$$\sin^{4p-n-m-1}(\sqrt{H}r)\varphi^{2p-1}\mathcal{A}_f \le (2p-1)^p\left(\frac{n+m-1}{2p-n-m}\right)^{p-1}\cdot\int_0^r\left(\mathrm{Ric}_f^{m\,H}{}_{-}\right)^p\mathcal{A}_f\mathrm{d}t$$

along that minimal geodesic segment from x.

Using Theorem 6.1, we have the corresponding volume comparison estimate when only the weighted integral bounds of Ric_f^m.

Theorem 6.2 *Let $(M, g, e^{-f}\mathrm{d}v)$ be an n-dimensional smooth metric measure space. Let $H \in \mathbb{R}$ and $p > \frac{n+m}{2}$, $m > 0$. For $0 < r \le R$ (assume $R \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$),*

$$\left(\frac{V_f(x,R)}{V_H^{n+m}(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r)}{V_H^{n+m}(r)}\right)^{\frac{1}{2p-1}} \le C(n,m,p,H,R)\left(\left\|\mathrm{Ric}_f^H{}_{-}\right\|_{p\,f}^p(R)\right)^{\frac{1}{2p-1}}.$$

Here,

$$C(n,m,p,H,R) := \left(\frac{n+m-1}{(2p-1)(2p-n-m)}\right)^{\frac{p-1}{2p-1}}\int_0^R A_H^{n+m}(t)\left(\frac{t}{V_H^{n+m}(t)}\right)^{\frac{2p}{2p-1}}\mathrm{d}t,$$

where $V_H^{n+m}(t) = \int_0^t A_H^{n+m}(s)\mathrm{d}s$, $A_H^{n+m}(t) = \int_{S^{n-1}}\mathcal{A}_H^{n+m}(t,\theta)\mathrm{d}\theta_{n-1}$, and $\mathcal{A}_H^{n+m}$ is the volume element in the model space M_H^{n+m}.

Similar to the manifolds case, comparison estimates for the weighted integral bounds of Ric_f^m have many applications, which will be not treated here. Recall that

(40) allows one to extend many classical results for manifolds of pointwise Ricci tensor condition to smooth metric measure spaces of pointwise Ric_f^m condition, such as [4,5,14,21]. In a similar fashion, because Theorems 6.1 and 6.2 for n-dimensional smooth metric measure spaces are essentially the same as the usual $(n+m)$-manifolds case. We believe that many geometrical and topological results for the integral Ricci tensor, such as Myers' type theorems [1,17], finiteness fundamental group theorems [1,13], the first Betti number estimate [13], Gromov's bounds on the volume entropy [2], compactness theorems [18], heat kernel estimates [9], isoperimetric inequalities [11,17], Colding's volume convergence and Cheeger–Colding splitting theorems [19], local Sobolev constant estimates [10], etc. are all possibly extended to the case where the weighted integral of Ric_f^m bounds.

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