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The Two-Dimensional Inverse Conductivity Problem

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Abstract
In this article, we introduce a process to reconstruct a Riemann surface with boundary equipped with a linked conductivity tensor from its boundary and the Dirichlet–Neumann operator associated with this conductivity. When initial data come from a two- dimensional real Riemannian surface equipped with a conductivity tensor, this process recovers its conductivity structure.

Keywords Riemann surface · Dirichlet-to-Neumann problem · Green function · Conductivity · Shock wave · Embedding

Mathematics Subject Classification Primary 32c25, 32d15, 32v15, 35r30, 58j32 · Secondary 35r30

This paper is organized as follows. Section 1 gives a short non-exhaustive history of the subject and Sect. 2 contains some of our main results. Section 3 is meant to fix definitions and notation about conductivity structures but also to state some results which, if not new, are not completely explicit in literature. Nodal manifolds are inevitably involved in the reconstruction methods proposed here. Section 4.1 contains what we need about them. Sections 4.2 and 5 are devoted to the proofs of Theorems 5 and 3. Section 6 is about the effective reconstruction of a bordered Riemann surface from its Dirichlet–Neumann operator. This is a key case for the inverse conductivity problem. Our method is based on a new a priori analysis of decompositions of two variables holomorphic function as a sum of shock waves functions, that is holomorphic solutions of $\frac{\partial h}{\partial y} = h\frac{\partial h}{\partial x}$. Section 7 enables to link the key number p of these sought shock waves to the Euler characteristic of a computable complex curve of $\mathbb{C}^2$.

In January 2016, my friend Gennadi Henkin, with whom I had worked for more than fifteen years, passed away. This paper, which is on a subject he brought, is dedicated to him. The numerous citations from the articles he authored show the depth of his mathematical thought.

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1 Introduction

We define a (two-dimensional) conductivity structure as a couple (M, σ) where M is a connected real surface with boundary[1] equipped with a conductivity $\sigma : T^*\overline{M} \to T^*\overline{M}$, that is a tensor such that

$$T_p^*\overline{M} \times T_p^*\overline{M} \ni (a, b) \mapsto \frac{a \wedge \sigma_p(b)}{\mu_p}$$

is a positive symmetric bilinear form, μ being a fixed volume form for $\overline{M}$. In the sequel, we get rid of brackets for the action of σ on a differential form ω by writing $\sigma\omega$ for $\sigma(\omega)$, that is the form $\overline{M} \ni p \mapsto \sigma_p(\omega_p)$. The above definition of a conductivity is perhaps unusual but is nothing than an intrinsic reformulation[2] of the one given by [29]. In this paper, conductivities are assumed to be at least of class C^3 though it is not mandatory for all statements.

For any continuous function $u : bM \to \mathbb{R}$, we denote $E_\sigma u$ the unique solution of the following Dirichlet problem :

$$\mathrm{d}\sigma\mathrm{d}U = 0 \ \& \ U|_{bM} = u. \tag{1}$$

Some authors prefer to consider a Riemannian metric g on M and solutions of the Dirichlet problem $\left(\Delta_g U = 0 \ \& \ U|_{bM} = u\right)$, where Δ_g is the Laplace–Beltrami operator. Writing in coordinates the equations $\mathrm{d}\sigma\mathrm{d}U = 0$ and $\Delta_g U = 0$, one sees that these two formulations are equivalent only when $\det\sigma_p = 1$ for all $p \in \overline{M}$.

The positive function $s_\sigma = \sqrt{\det\sigma}$ plays a special role in our subject. We call it the coefficient of σ. In Sect. 3, we establish that σ can be uniquely factorized in the form $\sigma = s_\sigma c_\sigma$ where c_σ is a conductivity of coefficient 1 and also the conjugation operator acting on $T^*\overline{M}$ of a complex structure $\mathcal{C}_\sigma$ uniquely associated with σ. Thus, the condition that $\det\sigma$ is constant means that (M, σ) is nothing more than the Riemann surface $(M, \mathcal{C}_\sigma)$.

The inverse conductivity problem we consider belongs to Electrical Impedance Tomographic problems ; in physics, U should be considered as an electrical potential, $\sigma(\mathrm{d}U)$ as the electrical current generated by U and $\mathrm{d}\sigma\mathrm{d}U = 0$ as the Maxwell divergence equation when there is no time dependence. The EIT problem is generally

[1] We think of a surface with boundary M as a dense open subset of an oriented two-dimensional real manifold with boundary $\overline{M}$ whose all connected components are bounded by pure one-dimensional real manifolds ; so the topological boundary bM of M is $\overline{M}\backslash M$; in the sequel ∂M is bM equipped with the natural orientation induced by M. A Riemann surface with boundary is a connected complex manifold of dimension 1 which is also a real surface with boundary.

[2] If we fix a point p in $\overline{M}$, some coordinates (x, y) around p and we set as in [29] $(\xi, \eta) = (\mathrm{d}y, -\mathrm{d}x)$ then $\sigma(\mathrm{d}x) = r\xi + t\eta$ and $\sigma(\mathrm{d}y) = u\xi + s\eta$, for $a = a_x\mathrm{d}x + a_y\mathrm{d}y$ and $b = b_x\mathrm{d}x + b_y\mathrm{d}y$ in $T_p\overline{M}$, $\sigma_p(b) = \left(b_x r + b_y u\right)\xi + \left(b_x t + b_y s\right)\eta$ and

$$\begin{aligned} a \wedge \sigma_p(b) &= \left(a_x\mathrm{d}x + a_y\mathrm{d}y\right) \wedge \left[\left(b_x r + b_y u\right)\mathrm{d}y - \left(b_x t + b_y s\right)\mathrm{d}x\right] \\ &= \left(ra_x b_x + ua_x b_y + ta_y b_x + sb_x b_y\right)\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

thought as the reconstruction of (M,σ) from ∂M, the boundary bM of M orientated by M, $T^*_{bM}\overline{M} = \underset{p\in bM}{\cup} T^*_p\overline{M}$, $\sigma\Big|_{T^*_{bM}\overline{M}}$ and the Dirichlet–Neumann operator associated with σ. This formulation is somehow ambiguous because it does not tell if M has to be determined as an abstract manifold, an embedded manifold, or even more precisely as a particular submanifold of some standard space. Success depends of the chosen position. Before going into what can be recovered and how it can be, we have to clarify what is a Dirichlet–Neumann operator.

To do so, one can use a metric (see Sect. 3) but we prefer to use the «differential» Dirichlet–Neumann operator N^σ_d whose action on a sufficiently smooth function $u : bM \to \mathbb{R}$ is defined by

$$N^\sigma_d u = [\sigma d\,(E_\sigma u)]|_{bM}\,. \tag{2}$$

Hence, in physics, $N^\sigma_d u$ is the measurement along bM of the current generated by the electrical potential $E_\sigma u$.

When M is a domain in $\mathbb{R}^2$, the conductivity is often thought as the matrix $\left(\sigma_{jk}\right) = Mat^{(\mathrm{d}x_2,-\mathrm{d}x_1)}_{(\mathrm{d}x_1,\mathrm{d}x_2)}(\sigma)$ which represents at each point p the linear map σ_p from $T^*_p\overline{M}$ with $(\mathrm{d}x_1,\mathrm{d}x_2)$ as domain basis to $T^*_p\overline{M}$ with $(\mathrm{d}x_2,-\mathrm{d}x_1)$ as range basis, (x_1,x_2) being the standard coordinates of $\mathbb{R}^2$; (1) turns to be

$$\sum_{j,k=1,2}\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial U}{\partial x_k}\right) = 0 \;\;\&\;\; U|_{bM} = u, \tag{3}$$

and the conditions constraining σ as a conductivity translate into the fact that $\left(\sigma_{jk}\right)$ is symmetric and positive.

The task, understood as the reconstruction of $\left(\sigma_{jk}\right)$ from $\left(\partial M, N^\sigma_d\right)$, has no natural solution because it is known from a remark of Tartar cited by [26], that when $\varphi \in C^1\left(\overline{M},\overline{M}\right)$ is a diffeomorphism matching identity on bM and Φ is the Jacobian matrix of φ, $(\sigma'_{jk}) = \frac{1}{\det\Phi}{}^t\Phi\left(\sigma_{jk}\right)\Phi$ defines a conductivity σ' such that $N^{\sigma'}_d = N^\sigma_d$. However, Lemma 8 of Sect. 3 shows that φ is a biholomorphism between the Riemann surfaces $(M,\mathcal{C}_\sigma)$ and $(M,\mathcal{C}_{\sigma'})$, where $\mathcal{C}_\sigma$ (resp. $\mathcal{C}_{\sigma'}$) is the complex structure where $\sigma = sc$ (resp. $\sigma' = s'c'$), s (resp. s') being a positive function on $\overline{M}$ and c (resp. c') the conjugation operator on $T^*\overline{M}$ associated with $\mathcal{C}_\sigma$ (resp. $\mathcal{C}_{\sigma'}$). Though they have the same underlying set, it is more accurate to see $(M,\mathcal{C}_\sigma)$ and $(M,\mathcal{C}_{\sigma'})$ as two different embeddings of the same abstract Riemann surface.

This example leads to consider the two-dimensional inverse conductivity problem as the reconstruction of M, an abstract Riemann surface with boundary, and of a function $s : \overline{M} \to \mathbb{R}^*_+$ from the knowledge of bM, $s\,|_{bM}$, the action on $T^*_{bM}\overline{M}$ of the conjugation operator c of M, and the Dirichlet–Neumann operator

$$N^{sc}_d : \mathcal{F}(bM) \ni u \mapsto d^c E_{sc} u\big|_{bM}\,,$$

where $\mathcal{F}(M)$ is any reasonable functions space like $C^0(bM)$, $C^\infty(bM)$ or $H^{1/2}(bM)$, $d^c = i\left(\overline{\partial} - \partial\right)$, $\partial = d - \overline{\partial}$, and $\overline{\partial}$ is the Cauchy–Riemann operator of M. In particular, even if data come from a Riemannian manifold (M, g) equipped with a conductivity tensor σ, we think our inverse problem as the reconstruction of the Riemann surface (M, C_σ) and of the coefficient of σ. Note that this formulation does not mention the auxiliary volume form μ because as explained in Sect. 3, the knowledge of the complex structure of M along bM enables to bypass it.

When (M, σ) is a two-dimensional conductivity structure embedded in a real or complex affine space, M can also be endowed the complex structure $\mathcal{C}$ induced by restriction of the ambient space metric. If c denotes the conjugation operator of $\mathcal{C}$ acting on $T^*\overline{M}$, σ is said to be isotropic (relatively to c or $\mathcal{C}$) if there is a function $s : \overline{M} \to \mathbb{R}_+^*$ such that $\sigma = sc$. In other words, to assume that σ is isotropic (relatively to the ambient metric) means to suppose the complex structure $\mathcal{C}_\sigma$ associated with σ is already known. In such circumstances, the inverse problem we talk about is to recover the positive function $s_\sigma = \sigma/c = \sqrt{\det \sigma}$.

At this point, one may ask what can happen if the starting point is a known Riemann surface X embedded in $\mathbb{R}^3$ whose complex structure $\mathcal{C}$ is inherited from the standard euclidean structure of $\mathbb{R}^3$ and σ is any conductivity on X. When σ is isotropic relatively to $\mathcal{C}$, $\mathcal{C}_\sigma = \mathcal{C}$ and the reconstruction task is done by the Henkin–Novikov theorem 1. For a non-isotropic conductivity, should an atlas of the abstract Riemann surface $(X, \mathcal{C}_\sigma)$ be recovered from N_d^σ, any constructive metric embedding X' of it in $\mathbb{R}^3$ could be considered also as recovered from N_d^σ. Of course, X and X' will be homeomorphic but $(X, \mathcal{C})$ and X' will be different Riemann surfaces. Moreover, in practical cases, only the boundary of X may be known. So it is not necessarily relevant to consider that X is already embedded in some standard space to which $\mathcal{C}_\sigma$ would be unrelated. Besides, in the main theorem of [24] quoted by Theorem 2, (M, σ) is given as embedded in $\mathbb{R}^3$ but is considered for the proof as embedded in $\mathbb{C}^3$ with an anisotropic conductivity while in [22], M is thought as embedded in $\mathbb{CP}_3$.

For a bounded domain M of $\mathbb{R}^2$ equipped with an isotropic conductivity σ, it is known that σ is completely determined by its Dirichlet–Neumann operator. This uniqueness is established for a real analytic conductivity by Kohn and Vogelius in [25]. For a smooth isotropic conductivity, an effective reconstruction process has been given by Novikov in [31] and for a conductivity with a positive lower bound and of class $W^{2,p}$, $p > 1$, by Nachman in [30]. Another proof of this result has been written by Gutarts in [12] for a smooth conductivity. When M is a connected Riemann surface whose genus is known, Henkin and Novikov in [22, Theorem 1.2] generalize and correct the reconstruction results of an isotropic conductivity of [18]. The necessarily technical aspect of the main result of [22, Theorem 1.2] limits us to give here only a sketch of it.

Theorem 1 (Henkin–Novikov, 2011) *Let M be a Riemann surface of genus g equipped with an isotropic conductivity $\sigma = sc$ where $s \in C^3\left(M, \mathbb{R}_+^*\right)$ and c is the conjugation operator of M acting on 1-forms. Then s can be recovered from the Dirichlet–Neumann operator N_d^σ by solving g Fredholm equations associated with g generic data of N_d^σ and then by solving g explicit systems which, in the case where M is a domain of $\left\{z \in \mathbb{C}^2;\ P(z) = 0\right\}$, $P \in \mathbb{C}_N[X]$, are linear systems of $N(N-1)$ equations with $N(N-1)$ unknowns.*

When the conductivity is not isotropic, authors have focused on the injectivity up to diffeomorphism of $\sigma \mapsto N_d^\sigma$, that is on the reverse of Tartar's remark. This injectivity is proved by Nachman [30] for a bounded domain of class C^3 in $\mathbb{R}^2$ and a conductivity of class C^3 after Sylvester [34] proved it with additional hypothesis. In [5], it is established for a conductivity of class L^∞ but for a simply connected domain of $\mathbb{R}^2$.

In the special case where the conductivity coefficient is constant, the question is to know if two conformal structures on M are identical when they share the same Dirichlet–Neumann operator. A positive answer is claimed by Lassas and Uhlmann in [27] when M is connected and Belishev confirmed it in [6] by showing that M can be seen as the spectra of the algebra of restrictions to bM of holomorphic on M extending continuously to $\overline{M}$.

In [27] and [6], the complete knowledge of the Dirichlet–Neumann operator is necessary to get the uniqueness of the conformal structure. In [17], it is said that it is determined by the action of the Dirichlet–Neumann operator on only three generic functions but the proof provided for this result is correct only if one strengthens a little the generic conditions required for these functions as it is done in [19]. This uniqueness can also be obtained by increasing the number of generic functions as in [21]. Theorem 3 gives a proof with the hypothesis of [17] and at the end of this section, we propose a new reconstruction of the Riemann surface $(M, \mathcal{C}_\sigma)$.

In [23] for a domain of $\mathbb{R}^2$ and in [24, Theorem 1.1] for the general case of a real two- dimensional connected manifold M, Henkin and Santacesaria made a major breakthrough in the theory by proving that the Dirichlet–Neumann operator determines the complex structure $\mathcal{C}_\sigma$ of (M, σ) as a nodal Riemann surface nodal with boundary embedded in $\mathbb{C}^2$. We refer to Sect. 4.1 for definitions and notation about nodal surfaces.

Theorem 2 (Henkin–Santacesaria, 2012) *Let (M, σ) be a conductivity structure, σ being of class C^3. Then, there exists in $\mathbb{C}^2$ a nodal Riemann surface with boundary $\mathcal{M}$ and a C^3-normalization $F : \overline{M} \to \overline{\mathcal{M}}$ such that $F_*\sigma = tc_{\mathcal{M}}$, where $t \in C^3(\mathcal{M}, \mathbb{R}_+^*)$ and $c_{\mathcal{M}}$ is the conjugation operator of the complex structure induced by $\mathbb{C}^2$ on $\mathcal{M}$. If in addition $F : \overline{M} \to \overline{\mathcal{M}'}$ is another C^3-normalization of the same kind, $\mathcal{M}$ and $\mathcal{M}'$ are roughly isomorphic in the sense of* [19]. *Lastly, the boundary value of F and in particular $b\mathcal{M}$ are determined by bM, $\sigma\Big|_{T^*_{bM}\overline{M}}$ and the Dirichlet–Neumann operator N_d^σ of (M, σ).*

Note that, thanks to Lemma 8, F is holomorphic in the sense that for any subset V of M such that $F(V)$ is a branch of $\mathcal{M}$, F is analytic from $(V, \mathcal{C}_\sigma)$ to $\mathbb{C}^2$. Besides, this theorem's proof implies that the singularities of $\mathcal{M}$ are the points of $F(\overline{M})$ with many preimages by F. So, when $\mathcal{M}$ has no singularity, F is a diffeomorphism from $\overline{M}$ onto $\overline{\mathcal{M}}$ satisfying the hypothesis of Lemma 8, which makes it an isomorphism of Riemann surfaces with boundary from $(M, \mathcal{C}_\sigma)$ onto $\mathcal{M}$.

In [24], it is said that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic without providing a precise meaning for it. Let us succinctly prove it involves at least rough isomorphism as defined in Sect. 4.1. Suppose that $F : \overline{M} \to \overline{\mathcal{M}}$ and $G : \overline{M} \to \overline{\mathcal{M}'}$ are C^3-normalizations of the above kind. Set $F_{\text{reg}} = F\Big|_{F^{-1}(\text{Reg}\,\mathcal{M})}^{\text{Reg}\,\mathcal{M}}$, $G_{\text{reg}} = G\Big|_{G^{-1}(\text{Reg}\,\mathcal{M}')}^{\text{Reg}\,\mathcal{M}'}$ and denotes by H_{reg} the map from $\text{Reg}\,\mathcal{M}' \cap G\left(F^{-1}(\text{Reg}\,\mathcal{M})\right)$ to $\text{Reg}\,\mathcal{M} \cap F\left(\text{Reg}\,\mathcal{M}'\right)$ defined by $H_{\text{reg}}(z) = F_{\text{reg}}\left(G_{\text{reg}}^{-1}(z)\right)$. Because F and G are normalizations, H_{reg} extends holomorphically along any branch of $\mathcal{M}'$ as a (multivalued) map H from $\mathcal{M}'$ to $\mathcal{M}$. By construction, $H\left(\mathcal{M}'\right)$ and $\mathcal{M}$ are complex curves which are different at most at a finite number of points. Hence, they are equal and in particular, Sing $\mathcal{M}$ and Sing $\mathcal{M}'$ have the same cardinal. It follows that $\mathcal{M}$ and $\mathcal{M}'$ are roughly isomorphic. The analysis of Theorem 2 is carried on in the next section.

2 Main Results

The nodal Riemann surfaces $\mathcal{M}$ and $\mathcal{M}'$ involved in Theorem 2 are actually isomorphic in the strong sense of this article. Indeed, by lifting to M, $\mathcal{M}$ and $\mathcal{M}'$ induce complex structures on M which coincide on bM and share the same Dirichlet–Neumann operator. Then, Theorem 3 enables to tell that these lifted Riemann surfaces with boundary are isomorphic and hence, that $\mathcal{M}$ and $\mathcal{M}'$ are so as nodal Riemann surfaces with boundary. The proof of Theorem 3 is given in Sect. 3. When $n = 2$, it completes the proof of Theorem 1 of [17] whose arguments really had to be corrected. By the way, as said before, Theorem 3 also proves the isomorphism claim of [24, Theorem 1.1].

In the statement below, $[w_0 : \cdots : w_n]$ denotes the standard homogeneous coordinates of $\mathbb{CP}_n$. If $\omega_0, \ldots, \omega_n$ are $(1,0)$-forms of $\mathbb{CP}_n$ without common zero and are pairwise proportional, we denote by $[\omega_0 : \cdots : \omega_n]$ or $[\omega]$ the map defined on each $\left\{\omega_j \neq 0\right\}$ by $[\omega] = \left[\frac{\omega_0}{\omega_j} : \cdots : \frac{\omega_n}{\omega_j}\right]$. Note that the hypothesis required for $(u_0, \ldots, u_n)$ in the theorem below is generically verified within n-uples of smooth functions on the boundary (see [17,19]).

Theorem 3 (Henkin–Michel, 2007) *Let M and M', two smooth Riemann surfaces bordered by the same real curve γ. Set $\partial = d - \overline{\partial}$ (resp. $\partial' = d - \overline{\partial'}$), $\overline{\partial}$ (resp. $\overline{\partial'}$) being the Cauchy–Riemann operator of M (resp. M'). If $u \in C^\infty(\gamma)$, denote $\widetilde{u}$ (resp. $\widehat{u}$) the harmonic extension of u to M (resp. M') and set $\theta u = \left(\partial\widetilde{u}\right)\big|_\gamma$ (resp. $\theta' u = \left(\partial'\widehat{u}\right)\big|_\gamma$); θ (resp. θ') is also the operator θ_c^σ defined by (9) when σ is the conjugation operator of M (resp. M') acting on 1-forms.*

Select $u = (u_0, \ldots, u_n) \in C^\infty(\gamma)^{n+1}$ where $n \in \mathbb{N}^*$, suppose that for all $j \in \{0, \ldots, n\}$, $\theta u_j = \theta' u_j$, the map $[\theta u] = [\theta u_0 : \cdots : \theta u_n] = \left[\theta' u\right]$ is well defined, realizes an embedding of γ in $\{w \in \mathbb{CP}_n;\ w_0 \neq 0\}$ and suppose in addition that $[\partial\widetilde{u}]$ (resp. $\left[\partial'\widehat{u}\right]$) is well defined on M (resp. M') and extends meromorphically $[\theta u]$ (resp. $\left[\theta' u\right]$) to M (resp. M'). Under these conditions, there exists an isomorphism of Riemann surfaces with boundary from $\overline{M}$ onto $\overline{M'}$ whose restriction to γ is identity.

Hence, the regular part of the nodal Riemann surface $\mathcal{M}$ produced by the Henkin–Santacesaria theorem is a model for the complex structure of $\left(M \backslash F^{-1}(\text{Sing}\,\mathcal{M}), \sigma\right)$. This model is effectively computable. Indeed, $\mathcal{M}$ is a complex curve of $\mathbb{C}^2 \backslash b\mathcal{M}$ which in the sense of currents satisfies $d[\mathcal{M}] = F_*[\partial M]$ where $[\mathcal{M}]$ denotes the integration current on $\mathcal{M}$ and $[\partial M]$ the one of bM oriented by M. In this situation, one knows, essentially since the works of Harvey and Lawson [13,14], that $\mathcal{M}$ is computable thanks to Cauchy type formulas (see e.g., [17, Theorem 2] or [24, Proposition 1]). More specifically, because $\mathcal{M}$ lies in $\mathbb{C}^2$, these formulas directly give the symmetric functions of the functions whose graphs describes the intersections of $\mathcal{M}$ with a chosen family of complex lines.

Meanwhile, as only the boundary values of F are known, there is an ambiguity on how to unfold the possible nodes of $\mathcal{M}$. To really know the complex structure $\mathcal{C}_\sigma$ of M, one has to know an atlas of it or a true embedding of it in some classical space. When the coefficient of σ is constant, it is the same thing as recovering $(M, \mathcal{C}_\sigma)$. This particular case is studied in [21, Theorem 4] and with the remark made at page 327, we readily have the result below for which we refer to [21] for the precise meaning of *generic*. Note also that though [21] is formally only about Riemann surfaces, the only part of the theorem which is not explicit in [21] is the isotropy statement but it is a plain consequence of the fact that Θ is a biholomorphism from $(M, \mathcal{C}_\sigma)$ to S.

The theorem below introduces operators which play a crucial role in this paper. When (M, σ) is a conductivity structure, we set $\partial^\sigma = d - \overline{\partial}^\sigma$ and $d^\sigma = i\left(\overline{\partial}^\sigma - \partial^\sigma\right)$ where $\overline{\partial}^\sigma$ is the Cauchy–Riemann operator of Riemann surface $(M, \mathcal{C}_\sigma)$. The operator θ_c^σ acts on $u \in C^\infty(bM)$ by $\theta_c^\sigma u = (\partial^\sigma \widetilde{u})\,|_{bM}$, $\widetilde{u}$ being the $\mathcal{C}_\sigma$-harmonic extension to M of u. The theorem does not mention the regularity of σ because what matters is that $(M, \mathcal{C}_\sigma)$ is a smooth manifold with boundary so that Stokes formula holds.

Theorem 4 (Henkin–Michel, 2015) *Let (M, σ) be a conductivity structure. Then for generic $u = (u_0, \dots, u_3)$ in $C^\infty(bM, \mathbb{R})^4$, the map $\left[\theta_c^\sigma u\right] = \left[\theta_c^\sigma u_0 : \cdots : \theta_c^\sigma u_3\right]$ is the boundary value of a map Θ which embeds $(M, \mathcal{C}_\sigma)$ in $\mathbb{CP}_3$ as a Riemann surface S with boundary. Moreover, $\Theta = [\partial^\sigma \widetilde{u}]$ where $\widetilde{u}$ is the $\mathcal{C}_\sigma$-harmonic extension of u to M, and $\Theta_* \sigma$ is a conductivity isotropic relatively to the complex structure of S.*

One should be careful here because the operator θ_c^σ cannot be thought as directly available from N_d^σ. Even if σ is the identity on the fibers of $T^*\overline{M}$ along bM, what is immediately available from N_d^σ are the boundary values of the derivatives of solutions of Dirichlet problems $\mathrm{d}\sigma\mathrm{d}U = 0$ and $U\,|_{bM} = u$ while what is required to apply Theorem 4 are the boundary values of the derivatives of solutions of Dirichlet problems $dd^\sigma U = 0$ and $U\,|_{bM} = u$. Unless the coefficient of σ is constant, one cannot expect these boundary values to be the same. To cope with this difficulty, we have Theorem 5 which is a new result.

Before stating it, we explain some notation but complete details and proofs are written in Sect. 4.2. We say that the conductivity structure $(\widetilde{M}, \widetilde{\sigma})$ extends plainly (M, σ) if $M \subset\subset \widetilde{M}$, $\widetilde{\sigma}$ is of the same class as σ, $\widetilde{\sigma}\,|_M = \sigma$, and $\widetilde{\sigma}\,\Big|_p = Id_{T_p^*\overline{\widetilde{M}}}$ for all $p \in b\widetilde{M}$. Let then F, $\mathcal{M}$, and $\widetilde{\mathcal{M}}$ be as below. The nodal Green function g we use for the possibly singular curve $\mathcal{M} = F(M)$ is defined in Corollary 12 of Sect. 4.2 but for a rough picture, the reader can think it as a kernel with the usual logarithmic singularities on the diagonal but with no boundary vanishing condition. Then the double-layer potential $D_g u$ of $u \in C^0(b\mathcal{M})$ is defined for any regular point q of $\widetilde{\mathcal{M}} \backslash b\mathcal{M}$ by $\left(D_g u\right)(q) = \int_{\partial\mathcal{M}} u d^c g_q$ where $g_q = g(q, .)$. When u is sufficiently smooth, the functions $D_g^+ u = \left(D_g u\right)|_{\mathcal{M}}$ and $D_g^- u = \left(D_g u\right)\Big|_{\widetilde{\mathcal{M}}\backslash\mathcal{M}}$ extend up to the boundary into (nodal) C^1-functions whose restrictions to $b\mathcal{M}$ are denoted as $A_g^+ u$ and $A_g^- u$. The conditional Green operator $B_g = Id + N_g^\#$ is defined for any $u \in C^\infty(b\mathcal{M})$ and $p \in b\mathcal{M}$ by $\left(N_g^\# u\right)(p) = 2PV\left(\int_{\partial\mathcal{M}} u(q)\,\frac{\partial g}{\partial \nu_p}(p, q)\,\tau_q^*\right)$ where PV means principal value and (ν, τ) is a frame for $T_{b\mathcal{M}}\overline{\mathcal{M}}$, direct and orthonormal with respect to the ambient Hermitian metric of $\mathbb{C}^2$, τ being tangent to $b\mathcal{M}$.

Theorem 5 *Let (M, σ) be a conductivity structure, σ being of class C^3. Select, which is always possible, a conductivity structure $(\widetilde{M}, \widetilde{\sigma})$ extending plainly (M, σ). We denote $F : \widetilde{M} \to \widetilde{\mathcal{M}} \subset \mathbb{C}^2$ the normalization obtained by applying Theorem 2 to $(\widetilde{M}, \widetilde{\sigma})$ and we set $f = F \Big|_{bM}^{F(bM)}$. g, $D_g^{\pm}$, $A_g^{\pm}$ and B_g, and τ are defined as above.*

Then, $Id + A_g^-$ is an endomorphism of $C^\infty(b\mathcal{M})$, its kernel and the kernel of B_g are finite dimensional subspaces of $C^\infty(b\mathcal{M})$, and for any $u \in C^\infty(bM, \mathbb{R})$ such that $\int_{\partial\mathcal{M}} (f_* u)\, w\tau^* = 0$ when $w \in \ker B_g$, the equation $f_* u = w + A_g^- w$ can be solved in $C^\infty(b\mathcal{M}, \mathbb{R})$ and for any solution w, $\theta_c^\sigma u = \left(F^* \partial D_g^+ w\right)|_{bM}$.

The main difficulty in the proof of Theorem 5 comes from the fact that harmonic Dirichlet problems in a nodal curve have unique solutions only if data are specified for nodal points (see [19, Proposition 2]). By the way, should $\mathcal{M}$ have no singularity, there would be nothing to do since $\mathcal{M}$ would be already an embedding of $(M, \mathcal{C}_\sigma)$ in $\mathbb{C}^2$.

Since the boundary values of F are computable from N_d^σ and since the Green function we use is so from $\mathcal{M}$ and $\mathcal{M}$ is computable from N_d^σ, Theorem 5 gives a tool to compute from N_d^σ as many $\theta_c^\sigma u$ as needed to apply Theorem 4 and so, to get the boundary values of an embedding Θ of the Riemann surface $(M, \mathcal{C}_\sigma)$ onto a Riemann surface S of $\mathbb{CP}_3$ for which $\Theta_* \sigma$ is isotropic.

If S itself is computed, the Henkin–Novikov Theorem 1 enables the reconstruction of the conductivity coefficient s of $\Theta_* \sigma$. Finally, denoting c the conjugation operator of S, (S, sc) is an explicit solution of the problem posed if it is understood as producing a conductivity structure, abstract or embedded in a standard space, whose oriented boundary and Dirichlet–Neumann operator are those specified.

It remains to explain how to recover the above Riemann surface S, or, which is the same, the conductivity structure (S, c). As S is a complex submanifold of $\mathbb{CP}_3$, the problem is no longer to recover c but to recover S as a set. Without loss of generality, S is supposed to be a relatively compact domain in an open Riemann surface $\widetilde{S}$ of $\mathbb{CP}_3$. For a generic choice of the 4-uple (u_0, u_1, u_2, u_3) of functions used in Theorem 4, we can also assume that the projections $\pi_2 : (w_0 : w_1 : w_2 : w_3) \mapsto (w_0 : w_1 : w_2)$ and $\pi_3 : (w_0 : w_1 : w_2 : w_3) \mapsto (w_0 : w_1 : w_3)$ immerse $\widetilde{S}$ in $\mathbb{CP}_2$ on nodal curves $\widetilde{S}_2$ and $\widetilde{S}_3$ such that $\pi_3^{-1}\left(\mathrm{Sing}\, \widetilde{S}_3\right) \cap \pi_2^{-1}\left(\mathrm{Sing}\, \widetilde{S}_2\right) \cap \widetilde{S} = \varnothing$. Therefore, to obtain an atlas of S, it is sufficient to get one for $Q_j = \pi_j(S)$, $j = 2, 3$, that is for a nodal Riemann surface with boundary which is a relatively compact domain Q in an open nodal Riemann surface $\widetilde{Q}$ of $\mathbb{CP}_2$ and whose oriented boundary ∂Q is known. This reconstruction problem is studied in [17, Theorem 2] but the suggested algorithm is not truly effective since the polynomials P_m arising from a non-empty intersection of Q with $\{w_0 = 0\}$ cannot be computed as easily as claimed.

In this paper, we provide a new approach to this problem with an effective method of computing these polynomials. How this can be done is described below but details and technical notation are postponed as most as possible to Sect. 6. Theorem 39 which specifies a linear system to solve to find some crucial auxiliary polynomials and Proposition 41 which enables to extract from them functions with geometric meaning are new and part of our main results. They are written in Sects. 6.4 and 6.5.

What we have at our hand is an oriented real-curve ∂Q which is known to be the boundary of a complex curve Q of $\mathbb{CP}_2$; without loss of generality, we assume that $\{w_0 = 0\} \cap bQ = \varnothing$. In such a situation, it is classical to use the Cauchy–Fantapié indicators of Q. Denoting U the open subset of $\mathbb{C}^2$ whose elements are points $z = (x, y)$ of $\mathbb{C}^2$ such that bQ does not meet $L_z = \{w \in \mathbb{CP}_2;\ xw_0 + yw_1 + w_2 = 0\}$, these are the functions G_k, $k \in \mathbb{N}$, defined on U by

$$G_k(z) = \frac{1}{2\pi i}\int_{\partial Q} \Omega_z^k,\ \ \Omega_z^k = \left(\frac{w_1}{w_0}\right)^k \frac{1}{x + y\frac{w_1}{w_0} + \frac{w_2}{w_0}} d\left(x + y\frac{w_1}{w_0} + \frac{w_2}{w_0}\right). \tag{4}$$

By Proposition 21, which is a result of Dolbeault and Henkin, we know that for all $k \in \mathbb{N}$, there exists $P_k \in \mathbb{C}(Y)_k[X]$ such that $G_k - P_k$ is the k-nth Newton symmetric function $N_{h,k}$ of locally defined shock waves functions $h_1, \dots, h_p$ which determine the intersections of Q with the lines L_z. The polynomials P_k are generated by points in $Q^\infty = Q \cap \{w_0 = 0\}$. In the favorable but unlikely case $Q^\infty = \varnothing$, all P_k are 0, Q is contained in the affine space $\{w_0 \neq 0\}$, and well-known techniques enable to compute these functions h_j.

When the number q^∞ of points in Q^∞ is 1 or 2, Agaltsov and Henkin [1] give an explicit procedure to recover Q and they claim that it should be efficient for any value of q^∞. Meanwhile, they provide no proof of it and it is not clear to us how to cope with the algebraic systems involved.

The new method we propose below focuses on the number p of the involved shock waves functions and works for any value of p or q^∞. For $q^\infty \in \{1, 2\}$, it is difficult to compare the Agaltsov–Henkin procedure to ours because fixing p or q^∞ to small values are really different hypothesis; from Corollary 24, $p = q^\infty + \delta$ where $\delta \in \mathbb{Z}$ is computed from G_1. Our reconstruction process goes in five steps.

1. If G_1 is algebraic in y and affine in x, Q is contained, according to Lemma 40, in a connected algebraic curve K such that $K \cap L_z = Q \cap L_z$ for $z \in Z$ where as specified by 24, $Z \subset U$ is a domain of the form $\underset{|y|>\lesssim}{\cup} D(0, \alpha|y|) \times \{y\}$. In this situation, we choose other coordinates in order that at least one of the lines L_z, $z \in Z$, meets Q and $K \backslash Q$. Thus, we assume that $\frac{\partial^2 G_1}{\partial x^2} \neq 0$ on Z for the remaining of the process.
2. We assume that for some $d \in \mathbb{N}^*$, we have found in $\mathbb{C}[X]^d$ a solution $\mu = (\mu_1, \dots, \mu_d)$ for the differential linear system S_d such that $B_\mu(0, y) \underset{y \to 0^*}{\rightarrow} 1$ and $\Delta_\mu \neq 0$, these three conditions being specified in Theorem 39. Note that S_d is actually a linear system on the coefficients of μ. According to Theorem 39, $G_1 = -s_1 + 1 \otimes \frac{A}{B} + X \otimes \frac{B'}{B}$ with $A, B \in \mathbb{C}[Y]$, $\deg A < \deg B = r = d - \delta$, $B(0) = 1$, and $s_k = \frac{e^H}{1 \otimes B}\left(\sum_{k \leqslant j \leqslant d} \mathcal{F}^{j-k}\left(\mu_j \otimes 1\right)\right)$, $1 \leqslant k \leqslant d$, where H is a function defined on $Z^+ = Z \backslash (\mathbb{C} \times \mathbb{R}_-)$ and $\mathcal{F}$ is an operator, both being specified in Definition 30 and computable from G_1.

3. According to Corollary 33, outside an analytic subset of Z, the s_k are the symmetric functions of shock waves functions $g_1, \dots, g_d$. Applying to the family (g_j) the reduction described in the beginning of Sect. 6.5 and applying Proposition 41, we conclude that $d \geqslant p$ where p is the number p of the locally defined shock waves functions h_j we are looking for, $r \geqslant q^\infty$ and that if $(\widetilde{g}_j)_{1 \leqslant j \leqslant p}$ is the set of functions obtained from (g_j) by reduction, $\{\widetilde{g}_1, \dots, \widetilde{g}_p\} = \{h_1, \dots, h_p\}$ and $P_1 = 1 \otimes \frac{A}{B} + X \otimes \frac{B'}{B}$. Consequently, $(P_k)_{k \in \mathbb{N}^*}$ is the algebraic extension of $(G_k - N_{\widetilde{g},k})_{k \in \mathbb{N}^*}$ where the $N_{\widetilde{g},k}$ are the Newton symmetric functions of the $\widetilde{g}_j$.
4. We know from Proposition 21, that there exists a locally constant function π with values in $\mathbb{N}$ such that for z_* in Z but outside some analytic subset of Z, there exists a neighborhood U_{z_*} of z_* in Z and mutually distinct shock waves $h_1^{z_*}, \dots, h_{\pi(z_*)}^{z_*}$ such that Q contains $Q_{z_*} = \underset{1 \leqslant k \leqslant \pi(z_*)}{\cup} \left\{ \left(1 : h_j^{z_*}(z) : -x - y h_j^{z_*}(z)\right);\ z \in U_{z_*} \right\}$ and $\left(G_k \left|_{U_{z_*}}\right.\right)_{k \in \mathbb{N}^*} = \left(N_{h^{z_*},k} + P_k \left|_{U_{z_*}}\right.\right)_{k \in \mathbb{N}^*}$ where the $N_{h^{z_*},k}$ are the Newton symmetric functions of the $h_j^{z_*}$. Thanks to Newton's formulas (27) and what precede, we can hence compute the symmetric functions $S_{h^{z_*},k}$ of the $h_j^{z_*}$. Moreover, $\pi(z_*) = G_0 \left|_{U_{z_*}}\right. - q^\infty$ is known. We can hence individually compute the functions $h_j^{z_*}$, $1 \leqslant j \leqslant \pi(z_*)$ from $\left(S_{h^{z_*},k}\right)_{1 \leqslant k \leqslant \pi(z_*)}$.
5. Thanks to Lemma 20, $Q \cap \{w_0 \neq 0\}$ and hence Q are known.

From a practical point of view, it would be very convenient to know a priori p since it would enable to write directly a relevant system S_d. Inequality (5) of Theorem 6 delivers an upper bound $p_{\max}$ for this number p. Note that data needed to think (5) as effective, mainly $\mathcal{M}$, $(D\partial^\sigma \widetilde{u_0}) |_{bM}$, and $\theta_c^\sigma u_0 = \partial^\sigma \widetilde{u_0} |_{bM}$ are, as explained in the proof which is given at the end of Sect. 7, computable from available boundary data. It would be useful to have a formula delivering $\mathcal{X}\left(\overline{\mathcal{M}}\right)$ in terms of Dirichlet–Neumann boundary data but such a formula is not known and $\overline{\mathcal{M}}$ has to be computed in order get its Euler characteristic.

Theorem 39 implies that S_d has a non-trivial solution for some d between 1 and $p_{\max}$. In addition, with results of Sect. 6.5, we know that from any non- trivial solution of some S_d, we can extract the sought shock waves. Hence, in the second step of the above process, we have at most $p_{\max}$ linear systems S_d to solve and this process may be considered as effective for any value of p or q^∞.

In Theorem 6, the generic hypothesis that $Q \in \{Q_1, Q_2\}$ is assumed to satisfy is that Q is a well-defined nodal open-bordered Riemann surface of $\mathbb{CP}_2$ whose boundary is a smooth real curve such that $bQ \subset \{w_0 w_1 w_2 \neq 0\}$, $(0:0:1)$ and $(0:1:0)$ are not in $Q^\infty = Q \cap \{w_0 = 0\}$ which is supposed to be transversal and contained in Reg Q. The number p_j is, according to Proposition 21 when $Q \in \{Q_1, Q_2\}$, the number of shock waves functions $h_{j,1}, \dots, h_{j,p_j}$ such the function G_k defined by (4) can be written on the set Z defined by (24) in the form $\left(h_{j,1}\right)^k + \cdots + \left(h_{j,p_j}\right)^k + P_{j,k}$ where $P_{j,k} \in \mathbb{C}(Y)_k[X]$. The complex differential operator ∂^σ of $(M, \mathcal{C}_\sigma)$ is defined as before.

Theorem 6 *Let (M, σ) be a conductivity structure. We equip the bundle $\Lambda^{1,0} T^* \overline{M}$ of $(1,0)$-forms of $(M, \mathcal{C}_\sigma)$ with an Hermitian metric and a Chern connection D as in The-*

orem 44. Denote by $\mathcal{M}$ the nodal Riemann surface designed by Theorem 2 and denote $\chi\left(\overline{\mathcal{M}}\right)$ the Euler characteristic of $\overline{\mathcal{M}}$. Assume that $u = (u_0, u_1, u_2, u_3) \in C^\infty(bM)^4$ satisfies the following generic hypothesis : the $\mathcal{C}_\sigma$-harmonic extension $\widetilde{u}$ of u is such that $[\partial^\sigma \widetilde{u}]$ is an embedding of $\overline{M}$ in $\mathbb{CP}_3$ and $Q_j = \left[\partial^\sigma \widetilde{u_0} : \partial^\sigma \widetilde{u_1} : \partial^\sigma \widetilde{u_j}\right](M)$, $j = 2, 3$, satisfies the generic hypothesis stated above. Let $p = \max(p_2, p_3)$ and $\delta = \max(\delta_2, \delta_3)$ where $\delta_j = \frac{1}{2\pi i}\int_{\partial Q_j} \frac{d(w_1/w_0)}{w_1/w_0}$ is the number δ defined in Lemma 23 and p_j is the number of shock waves functions involved in Proposition 21 when z_* is in the set Z defined by (24). Then

$$p \leqslant \delta + \frac{1}{2\pi i}\int_{\partial M} \frac{D\partial^\sigma \widetilde{u_0}}{\partial^\sigma \widetilde{u_0}} - \chi\left(\overline{\mathcal{M}}\right). \tag{5}$$

3 Conductivity Structures and Metrics

Requirements on σ to be a conductivity indicate a metric is involved. It is noticed in [17] that once a volume form μ is chosen for $\overline{M}$, one can design a natural metric $g_{\mu,\sigma}$ on $\overline{M}$ by setting for all $t, t' \in T\overline{M}$

$$g_{\mu,\sigma}\left(t, t'\right) = \frac{\sigma^{-1}(t \lrcorner \mu) \wedge \left(t' \lrcorner \mu\right)}{\mu}.$$

Its conformal class or complex structure $\mathcal{C}_\sigma$ does not depend on μ and σ factorizes (see [17]) through $\mathcal{C}_\sigma$ in the sense that there exists a function $s_\sigma : \overline{M} \to \mathbb{R}_+^*$ with the same regularity as σ, called conductivity coefficient in this article, such that when (x_1, x_2) is a couple of local isothermal coordinates for $\mathcal{C}_\sigma$,

$$Mat_{\mathrm{d}x}^{(\mathrm{d}x_2, -\mathrm{d}x_1)}(\sigma_p) = s_\sigma(p)\, I_2 \tag{6}$$

for all p in the open subset of $\overline{M}$ where (x_1, x_2) is defined, I_2 being the 2×2 identity matrix, and $\mathrm{d}x = (\mathrm{d}x_1, \mathrm{d}x_2)$. Denote by $\det \sigma$ the map which to a point p of $\overline{M}$ associates the determinant of the linear map σ_p ; (6) implies $s_\sigma = \sqrt{\det \sigma}$. If c_σ is the conductivity defined by

$$\sigma = s_\sigma \cdot c_\sigma = \sqrt{\det \sigma} \cdot c_\sigma, \tag{7}$$

$\mathcal{C}_\sigma$ is also the conformal class associated with c_σ ; when (x_1, x_2) is a couple of local isothermal coordinates for $\mathcal{C}_\sigma$,

$$Mat_{\mathrm{d}x}^{\mathrm{d}x}(c_\sigma) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overset{def}{=} J.$$

In other words, c_σ is also the conjugation operator acting on 1-forms of M. Moreover, if $d^\sigma = c_\sigma d$, $\overline{\partial}^\sigma = \frac{1}{2}(d - id^\sigma)$ is the Cauchy–Riemann operator associated with $\mathcal{C}_\sigma$ and

$$\mathrm{d}\sigma\mathrm{d}U = ds_\sigma d^\sigma U$$

for all functions $U \in C^2\left(\overline{M}\right)$. Note that by definition, $\partial^\sigma = \partial^{c_\sigma}$, $\overline{\partial}^\sigma = \overline{\partial}^{c_\sigma}$, and $d^\sigma = d^{c_\sigma}$; these operators are associated with the complex structure $\mathcal{C}_\sigma$.

Let us suppose that $\mathcal{C}$ is a complex structure on $\overline{M}$, that is an atlas for $\overline{M}$ which makes M a Riemann surface with boundary. If x_1 and x_2 are the real and imaginary part of a same holomorphic coordinate for M, Jacobian matrices relatives to (x_1, x_2) of holomorphic maps commute with J. This means that one can define a tensor $c : T\overline{M} \to T\overline{M}$ by the fact that in such coordinates, $Mat_{\mathrm{d}x}^{\mathrm{d}x}(c) = J$. By construction, c is a conductivity whose coefficient is 1, $c \circ d = i\left(\overline{\partial} - \partial\right) \overset{def}{=} d^c$ and c is the conjugation operator of $\mathcal{C}$ and also the Hodge star operator acting on 1-forms when M is equipped the metric dual of the one given on each $T_p^*\overline{M}$ by $\langle a, b\rangle\, \mu = a \wedge *b = \frac{1}{\sqrt{\det \sigma}} a \wedge \sigma\,(b)$.

So, decomposition (7) shows a complex structure naturally associated with σ. It is unique in the sense that if c' is the conjugation operator of T^*M associated with a complex structure $\mathcal{C}'$ and if $s' \in \left(\mathbb{R}_+^*\right)^{M'}$, the identity $\sigma = s'.c'$ forces, because $\det c_\sigma = 1 = \det c'$, first $s_\sigma = s'$, and then $c_\sigma = c'$.

Formula (6) shows that for all $p \in M$, σ_p commute with the orthogonal automorphisms of $\left(T_pM, \left(g_{\mu,\sigma}\right)_p\right)$. When M is a submanifold embedded in $\mathbb{R}^3$, in particular if M is a domain of $\mathbb{R}^2$, and when $g_{\mu,\sigma}$ is induced by the standard metric of $\mathbb{R}^3$, this means that σ is isotropic in the usual sense (see [29] and [34] for example). The proposition below sums up what precedes.

Proposition 7 *Let M be a real two-dimensional surface with boundary. A complex structure $\mathcal{C}$ on $\overline{M}$ defines a conductivity tensor with coefficient equal to* 1. *Reciprocally, for all conductivity σ on $\overline{M}$, there exists a unique complex structure $\mathcal{C}_\sigma$ such that $\sigma = \sqrt{\det \sigma}\, c_\sigma$ where c_σ is the conjugation operator associated with $\mathcal{C}_\sigma$.*

Hence, it is natural to say that a complex-valued function f defined on an open set U of M is σ-holomorphic if $\overline{\partial}^\sigma f = 0$, or equivalently, when for all charts $z : V \to \mathbb{C}$ of the holomorphic atlas of $(M, \mathcal{C}_\sigma)$, $f \circ z^{-1}$ is holomorphic on $z^{-1}(U)$ in the usual sense.

If $\left(M', \sigma'\right)$ is an another conductivity structure, a map f from an open subset U of M to M' is said $\left(\sigma, \sigma'\right)$-analytic if for all holomorphic charts $z' : V' \to \mathbb{C}$ of $\left(M', \mathcal{C}_{\sigma'}\right)$, $z' \circ f$ is σ-holomorphic on $f^{-1}\left(V'\right) \cap U$, that is if $z' \circ f \circ z^{-1}$ is holomorphic on $z^{-1}\left(f^{-1}\left(V'\right) \cap U\right)$ in the usual sense for all holomorphic charts $z : V \to \mathbb{C}$ of $(M, \mathcal{C}_\sigma)$. This also can be characterized by the following lemma.

Lemma 8 *Let (M, σ) and $\left(M', \sigma'\right)$ be two conductivity structures, U an open subset of M and $f : U \to M'$ a differentiable map. Then f is $\left(\sigma, \sigma'\right)$-analytic if and only if $\left({}^tDf\right) \circ c_{\sigma'} = c_\sigma \circ \left({}^tDf\right)$. When f realizes a diffeomorphism φ from U to $f(U)$, φ is $\left(\sigma, \sigma'\right)$-analytic if and only if $\varphi_* c_\sigma = c_{\sigma'}$ and in particular if $\varphi_*\sigma = \sigma'$.*

Proof Consider holomorphic charts $z : V \to \mathbb{C}$ and $z' : V' \to \mathbb{C}$ of $(M, \mathcal{C}_\sigma)$ and $\left(M', \mathcal{C}_{\sigma'}\right)$. Set $F = Mat_{(\mathrm{d}x,\mathrm{d}y)}^{(\mathrm{d}x',\mathrm{d}y')}(Df)$ where $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$ and $\left(x', y'\right) = \left(\operatorname{Re} z', \operatorname{Im} z'\right)$. Then

$$Mat_{(\mathrm{d}x',\mathrm{d}y')}^{(\mathrm{d}x,\mathrm{d}y)}\left(\left({}^{t}Df\right)\circ c_{\sigma'}\right)=Mat_{(\mathrm{d}x',\mathrm{d}y')}^{(\mathrm{d}x,\mathrm{d}y)}\left({}^{t}Df\right)Mat_{(\mathrm{d}x',\mathrm{d}y')}^{(\mathrm{d}x',\mathrm{d}y')}\left(c_{\sigma'}\right)={}^{t}FJ$$
$$Mat_{(\mathrm{d}x',\mathrm{d}y')}^{(\mathrm{d}x,\mathrm{d}y)}\left(c_{\sigma}\circ\left({}^{t}Df\right)\right)=Mat_{(\mathrm{d}x,\mathrm{d}y)}^{(\mathrm{d}x,\mathrm{d}y)}\left(c_{\sigma}\right)Mat_{(\mathrm{d}x',\mathrm{d}y')}^{(\mathrm{d}x,\mathrm{d}y)}\left({}^{t}Df\right)=J^{t}F$$

So, the equality $\left({}^{t}Df\right)\circ c_{\sigma'}=c_{\sigma}\circ\left({}^{t}Df\right)$ holds if and only if $JF=FJ$. Translating this on matrix coefficients, this is equivalent to the fact that Re f and Im f satisfy the Cauchy–Riemann equations, that is $\frac{\partial f}{\partial\overline{z}}=0$.

Suppose now that $\varphi=f\Big|_{U}^{f(U)}$ is a diffeomorphism. Since by definition, $\varphi_{*}c_{\sigma}=\left({}^{t}Df\right)_{\psi}^{-1}\circ(c_{\sigma})_{\psi}\circ{}^{t}(D\varphi)_{\psi}$ where $\varphi=\psi^{-1}$, the preceding point gives that φ is $\left(\sigma,\sigma'\right)$-analytic if and only if $\varphi_{*}c_{\sigma}=c_{\sigma'}$. Besides, $\varphi_{*}c_{\sigma}=(\det\sigma)_{\psi}.\varphi_{*}c_{\sigma}=\det\left(\sigma_{\psi}\right).\varphi_{*}c_{\sigma}$. So, $\sigma'=\varphi_{*}\sigma=\left({}^{t}Df\right)_{\psi}^{-1}\circ(c_{\sigma})_{\psi}\circ{}^{t}(D\varphi)$ forces $\det c_{\sigma'}=\det\left(\sigma_{\psi}\right)$ and $\varphi_{*}c_{\sigma}=c_{\sigma'}$. □

This lemma enables to justify our comment in the introduction about Tartar's remark. The conductivity σ' is defined by $Mat_{(\mathrm{d}x_1,\mathrm{d}x_2)}^{(\mathrm{d}x_2,-\mathrm{d}x_1)}\left(\sigma'\right)=\frac{1}{\det\Phi}{}^{t}\Phi\left(\sigma_{jk}\right)\Phi$ where Φ is the Jacobian matrix of φ. But $Mat_{(\mathrm{d}x_1,\mathrm{d}x_2)}^{(\mathrm{d}x_1,\mathrm{d}x_2)}(\sigma)=JMat_{(\mathrm{d}x_1,\mathrm{d}x_2)}^{(\mathrm{d}x_2,-\mathrm{d}x_1)}(\sigma)$ and the same holds for σ'. Since $\frac{-1}{\det\Phi}J^{t}\Phi J=\Phi^{-1}$ and $J^{2}=-I_{2}$, we get $Mat_{(\mathrm{d}x_1,\mathrm{d}x_2)}^{(\mathrm{d}x_1,\mathrm{d}x_2)}\left(\sigma'\right)=\Phi^{-1}Mat_{(\mathrm{d}x_1,\mathrm{d}x_2)}^{(\mathrm{d}x_1,\mathrm{d}x_2)}(\sigma)\,\Phi$ which means $\sigma'=\varphi_{*}\sigma$. Hence, φ is a biholomorphic map between $(M,\mathcal{C}_{\sigma})$ and $\left(M',\mathcal{C}_{\sigma'}\right)$.

We now turn our attention to the Dirichlet–Neumann operator itself. Assume again that M is also equipped with an arbitrary Riemannian metric g ; this in particular the case when M is a real surface in $\mathbb{R}^3$ with a non-isotropic conductivity. Denote by ν and τ vector fields defined along bM such that for all $p\in bM$, $\left(\nu_p,\tau_p\right)$ is a direct g-orthonormal basis for $T_p\overline{M}$ and $\tau_p\in T_pbM$. The «normal» Dirichlet–Neumann operator N_{ν}^{σ} is then defined for any sufficiently smooth function $u:bM\to\mathbb{R}$ by

$$N_{\nu}^{\sigma}u=\left.\frac{\partial E_{\sigma}u}{\partial\nu}\right|_{bM} \tag{8}$$

where $E_{\sigma}u$ is the unique solution of (1). So, when $u:bM\to\mathbb{R}$ is sufficiently smooth

$$\mathrm{d}E_{\sigma}u=(E_{\sigma}u\cdot\nu)\,\nu^{*}+(E_{\sigma}u\cdot\tau)\,\tau^{*}=\left(N_{\nu}^{\sigma}u\right)\nu^{*}+(\mathrm{d}u\cdot\tau)\,\tau^{*}.$$

This formula shows that data from N_{ν}^{σ} which depends of a choice of metric, can be replaced by data from the «differential» Dirichlet–Neumann operator $N_{d}^{\sigma}=\sigma dE_{\sigma}$ defined by (2).

In the particular case where $\det\sigma=1$, $\sigma=c_{\sigma}$ and it is noticed in [17] that $\partial^{c_{\sigma}}E_{c_{\sigma}}u\big|_{bM}=\left(L_{\nu}^{c_{\sigma}}u\right)(\nu^{*}+i\tau^{*})$ where $\partial^{c_{\sigma}}=d-\overline{\partial}^{\sigma}$ and $\overline{\partial}^{c_{\sigma}}$ is the Cauchy–Riemann operator of $(M,\mathcal{C}_{\sigma})$ and where $L_{\nu}^{c_{\sigma}}u=\frac{1}{2}\left(N_{\nu}^{c_{\sigma}}u-i\frac{\partial u}{\partial\tau}\right)$. So, one can consider in this case the «complex» Dirichlet–Neumann operator θ_{c}^{σ} defined on sufficiently smooth functions $u:bM\to\mathbb{R}$ by

$$\theta_{c}^{\sigma}u=\partial^{c_{\sigma}}E_{c_{\sigma}}u\big|_{bM}=\left(L_{\nu}^{c_{\sigma}}u\right)\left(\nu^{*}+i\tau^{*}\right) \tag{9}$$

For a general $\det\sigma$, we still let $\theta_c^\sigma = \theta_c^{c_\sigma}$. This means that for $u \in C^\infty(bM)$, $\theta_c^\sigma u$ is still defined by (9) even if σ and c_σ are no longer equal. Hence, θ_c^σ and N_d^σ correspond to Dirichlet problems associated with different operators, namely, $dc_\sigma d$ for the first and $\mathrm{d}\sigma d = \mathrm{d}s_\sigma c_\sigma d$ for the second.

To end this section, we explain how to get rid of the auxiliary volume form μ. As in the inverse problem studied here, $T^*_{bM}\overline{M}$ and $\sigma\Big|_{T^*_{bM}\overline{M}}$ are supposed to be known, the conjugation operator c_σ associated with the complex structure $\mathcal{C}_\sigma$ of (M,σ) is known when it acts on $T^*_{bM}\overline{M}$. Having chosen a smooth generating section τ^* of T^*bM, we set $\nu_s^* = -(c_\sigma)_s\,\tau_s^*$ for any $s \in bM$. By definition of conductivity, $bM \ni s \mapsto \tau_s^* \wedge \nu_s^*$ is then a smooth section of the volume forms bundle of $\overline{M}$ and can be extended to a smooth volume form μ on $\overline{M}$. Though this extension is not unique, any tensor which would be a conductivity for one of these extensions would be so for any.

4 Recovering the Complex Dirichlet–Neumann Operator

Nodal Riemann surfaces are discussed in [19] and the reader can refer to it. Meanwhile, for sake of simplicity [19] does not consider the case where nodes are allowed in the boundary. Since the nodal Riemann surface we have to consider is produced as the solution of a boundary problem for a real smooth curve and since as pointed out in [14, Sect. 3.2] such complex curves may present this type of singularity, we give some basics in Sect. 4.1. Then, we prove the existence of nodal Green functions for such surfaces. At the end of this section, is written the proof of Theorem 5 which enables the recovering of the complex Dirichlet–Neumann operator θ_c^σ. This result is new wether or not nodes at the boundary are present. Besides, existence of such nodes should be considered as exceptional.

4.1 Nodal Riemann Surfaces and Harmonic Distributions

In this article a nodal Riemann surface with boundary Q is a set of the form $\left(\overline{S}/\mathcal{R}\right)\backslash\pi(bS)$ where S is a Riemann surface with boundary, $\mathcal{R}$ a nodal relation which means that $\mathcal{R}$ is an equivalence relation on $\overline{S}$ identifying a finite number of points of $\overline{S}$ but such that two distinct points of bS are in two different classes and π is the natural projection of $\overline{S}$ on $\overline{S}/\mathcal{R}$. In particular, $\pi_{bS} = \pi\,\Big|_{bS}^{bS}$ is a bijection.

We equip $\overline{S}/\mathcal{R}$ with the quotient topology so that Q is an open subset, $\overline{Q} = \overline{S}/\mathcal{R}$ and $bQ = \pi(bS)$. One denotes by $\operatorname{Reg} Q$ the set of points of Q having only one preimage by π and we set $\operatorname{Sing} Q = Q\backslash\operatorname{Reg} Q$; $\operatorname{Reg}\overline{Q}$ and $\operatorname{Sing}\overline{Q}$ are defined similarly.

If $q \in \overline{Q}$ (resp. $q \in bQ$), an inner (resp. boundary) branch of $\overline{Q}$ at q is any subset B of Q (resp. $\overline{Q}$) for which there exists an open connected subset V of S (resp. $\overline{S}$) and $s \in V \cap \pi^{-1}(q)$ such that $\overline{V}\backslash\{s\} \subset \pi^{-1}\left(\operatorname{Reg}\overline{Q}\right)$, π realizes a bijection from V to B and, if $q \in bQ$, $V \cap bS$ is a neighborhood of s in bS. A set of inner branches at a point q of $\overline{Q}$ is complete if their union with the possible boundary branch of $\overline{Q}$ at q is a neighborhood of q in $\overline{Q}$.

Q carries a natural (nodal) complex structure which is characterized by the fact that for any inner branch B of $\overline{Q}$, there exists an open connected subset V of S such that π

is a biholomorphism from V to B. Likewise, one gives a natural meaning to notions of nodal conductivities (for which considerations of the preceding section apply) and to nodal function or maps between nodal Riemann surfaces, holomorphic or of class C^k, $0 \leqslant k \leqslant \infty$. With such definitions, $\pi : \overline{S} \to \overline{Q}$ becomes a normalization of $\overline{Q}$.

As pointed out in [19, Proposition 2], isomorphisms between nodal Riemann surfaces are a little bit trickier since nodes can be mixed. Let us consider another nodal Riemann surface with boundary Q' which is the quotient of a Riemann surface with boundary S' and denote π' the natural projection of $\overline{S'}$ to $\overline{Q'}$. Take a nodal map $\varphi : \overline{Q} \longrightarrow \overline{Q'}$; so, φ is univalued on $\operatorname{Reg}\overline{Q}$ and multivalued on $\operatorname{Sing}\overline{Q}$. We say that φ is an isomorphism of nodal Riemann surfaces with boundary if the following conditions are satisfied :

(i) φ is an homeomorphism from $\varphi^{-1}\left(\operatorname{Reg}\overline{Q'}\right) \cap \operatorname{Reg}\overline{Q}$ onto $\varphi\left(\operatorname{Reg}\overline{Q}\right) \cap \operatorname{Reg}\overline{Q'}$.
(ii) For all inner (resp. boundary) branches B' of $\overline{Q'}$, there exists an inner (resp. boundary) branch B of $\overline{Q}$ such that $\varphi\left(B \cap \operatorname{Reg}\overline{Q}\right) = B' \cap \operatorname{Reg}\overline{Q'}$ and the continuous extension $\varphi\big|_B^{B'}$ of the map $B \cap \operatorname{Reg}\overline{Q} \to B'$, $q \mapsto \varphi(q)$, is an isomorphism of Riemann surfaces (resp. with boundary).
(iii) For all $q \in \overline{Q}$, the branches of $\overline{Q'}$ at $\varphi(q)$ are the images by φ of the branches of $\overline{Q}$ at q.

If φ satisfies only (i) and (ii), we says as in [19, Proposition 2] that φ is a rough isomorphism.

Distributions and currents are defined on nodal Riemann surfaces as usual by duality and of course, harmonic distributions are by definition those in the kernel of dd^c. According to [19, Proposition 2] whose proof applies without change to the case $\left(\operatorname{Sing}\overline{Q}\right) \cap bQ \neq \varnothing$, a distribution u on a open set W of $\overline{Q}$ is harmonic if and only if it is harmonic in the usual sense on $W \cap \operatorname{Reg} Q$, continuous on $W \cap \operatorname{Reg}\overline{Q}$ as well as in all boundary branches of $\overline{Q}$ contained in W, and if for any singular point q of $\overline{Q}$ the two conditions below are satisfied :

1. for all inner branches B of $\overline{Q}$ at q sufficiently small so it admits a holomorphic coordinate z centered at q, there exists $c_B \in \mathbb{C}$ such that $u\big|_{Q_{q,j}\setminus\{q\}} - 2c_B \ln|z|$ extends to B as a usual harmonic function.
2. $\sum_{B\in\mathcal{B}} c_B = 0$ where $\mathcal{B}$ is a complete set of inner branches of $\overline{Q}$ at q.

This implies that a same continuous function u on bQ extends to Q in many harmonic distributions ; the Dirichlet problem for u is well posed only if for the extension U, one specifies for all $q \in \operatorname{Sing}\overline{Q}$ and all inner branches B of $\overline{Q}$ at q, the residue c_B of $\partial U\,|_B$ at q. In particular, $\widehat{u}$ denoting the harmonic extension of $u \circ \pi_{bS}^{-1}$ to S, $\pi_* \widehat{u}$ is the only harmonic distribution which is continuous along any branch of $\overline{Q}$ and coincides with u on bQ ; we call it the simple harmonic extension of u.

For a nodal Riemann surface Q, we define the complex Dirichlet–Neumann operator as the operator $\theta_c^Q = \theta_c^{c_Q}$ where c_Q is the conjugation operator associated with the complex structure of Q and where in (9) simple harmonic extensions are used.

4.2 Recovering of θ_c^σ, Proof of Theorem 5

4.2.1 Green Functions in the Smooth Case

This section is about classical facts on Green functions for a smooth open-bordered Riemann surface S which are generalized to the nodal case in Sect. 4.2.2.

A Green function for S is a function g defined on $\overline{S} \times \overline{S}$ without its diagonal $\Delta_{\overline{S}}$ such that for all $q \in S$, $g_q = g(q, .)$ is harmonic on $S \backslash \{q\}$, continuous on $\overline{S} \backslash \{q\}$ and has an isolated logarithmic singularity at q, which means that given a holomorphic coordinate z of S defined near q and centered at q, $g_q - \frac{1}{2\pi} \ln |z|$ extends harmonically around q. g is said principal if it is symmetric, real valued and its partial functions g_q vanishes on bS. The Perron method shows that such a function exists and the maximum principle implies it is unique.

The problem we want to address is the computation from g of the operator θ_c^S which to $u \in C^\infty(bS)$ associates $(\partial \widetilde{u})|_{bS}$ where $\widetilde{u}$ is the harmonic extension of u to S. Without loss of generality, we assume that S is a relatively compact domain in an open Riemann surface $\widetilde{S}$ for which g is a Green function. We also assume that g is symmetric and real valued.

First, one builds the operator T_g which to $u \in C^0(bS)$ associates the harmonic function $T_g u$ defined on $\widetilde{S} \backslash bS$ by

$$T_g u : \widetilde{S} \backslash bS \ni q \mapsto \frac{2}{i} \int_{\partial S} u \partial g_q \tag{10}$$

and which splits in $T_g^{\pm} u = (T_g u)|_{S^\pm}$ where $S^+ = S$ and $S^- = \widetilde{S} \backslash \overline{S}$. Let us choose an Hermitian metric for $\widetilde{S}$ and for $T\widetilde{S}$ near bS, a direct orthonormal frame (ν, τ) such that $\tau|_{bS} \in T_{bS} S$. When f is differentiable function near bS, we can write

$$\partial f = \frac{1}{2} \left(\frac{\partial f}{\partial \nu} - i \frac{\partial f}{\partial \tau} \right) (\nu^* + i \tau^*) . \tag{11}$$

Since the pull back of ν^* by the natural injection of bS into $\widetilde{S}$ is 0, we get that for any $u \in C^1(bS)$ and $q \in \widetilde{S} \backslash bS$,

$$(T_g u)(q) = \int_{\partial S} u \frac{\partial g_q}{\partial \nu} \tau^* + i \int_{\partial S} u' g_q \tau^* \stackrel{def}{=} D_g u + i S_g u' \tag{12}$$

where $u' = \frac{\partial u}{\partial \tau}$ and where $D_g u$ and $S_g u'$ are the so called double-layer and single-layer potentials of u and u'. Since $d^c = i(\overline{\partial} - \partial)$, we also get from (11) that for any $u \in C^0(bS)$ and $q \in \widetilde{S} \backslash bS$,

$$(D_g u)(q) = \int_{\partial S} u d^c g_q \tag{13}$$

Like T_g, D_g and S_g split in sided operators $D_g^{\pm}$ and $S_g^{\pm}$. Then it is well known that for any $u \in C^2(bS)$, $D_g^{\pm}u = \left(D_g u\right)|_{S^{\pm}}$ and $S_g^{\pm}u = \left(S_g u\right)|_{S^{\pm}}$ extend to $\overline{S^{\pm}}$ as C^1-functions, that S_g is continuous on $\widetilde{S}$ and that if $u \in C^2(bS)$, the boundary values $A_g^{\pm}u = \left(D_g^{\pm}u\right)|_{bS}$ satisfy

$$A_g^+ u - A_g^- u = u \ \ \& \ \ A_g^+ u + A_g^- u = N_g u \tag{14}$$

where $N_g u$ is defined for $p \in bS$ by

$$\left(N_g u\right)(p) = 2PV\left(\int_{\partial S} u d^c g_q\right),$$

PV standing for principal value. According to (12), when $u \in C^2(bS)$, $T_g^{\pm}u$ also extend to $\overline{S^{\pm}}$ as C^1-functions which verify

$$A_{g,c}^+ u - A_{g,c}^- u = u \ \ \& \ \ A_{g,c}^+ u + A_{g,c}^- u = N_{g,c} u$$

where $A_{g,c}^{\pm}u = \left(T_g^{\pm}u\right)|_{bS} = A_g^{\pm}u - iS_g u'$ and where $N_{g,c}u$ is defined for $p \in bS$ by

$$\left(N_{g,c}u\right)(p) = 2PV\left(\frac{2}{i}\int_{\partial S} u\partial g_q\right)$$

This goes back to the works of Sohotksy in 1873 or, later, of Plemelj and can be found in many books. The reader can refer for example to [35, Chapter 7, §§11] where these operators and formulas are proven to make sense for u in the distributional sense in Sobolev spaces. A direct proof for $T_{g,c}$ and C^2-functions can be found as a particular case in [28] which addresses similar problems in Stein manifolds.

We also use the operator $N_g^{\#}$ defined on any Sobolev space $H^s(bS)$ by density of $C^\infty(bS)$ and by, when $u \in C^\infty(bS)$,

$$\forall p \in bS, \ \left(N_g^{\#}u\right)(p) = 2PV\left(\int_{\partial S} u(q)\frac{\partial g}{\partial \nu_p}(p,q)\,\tau_q^*\right)$$

From [35, Proposition 11.3], we know that in the distributional sense

$$\forall p \in bS, \ \left(N_g^{\#}u\right)(p) = u(p) + 2\lim_{\varepsilon\to 0^+}\frac{\partial S^- u}{\partial \nu}\left(p - \varepsilon\nu_p\right) \tag{15}$$

Assume that for some $u \in C^\infty(bS)$ and we have found a solution $w \in C^\infty(bS)$ to the equation

$$u = w + A_g^- w, \tag{16}$$

that is, u belongs to the range of $Id + A_G^-$. Then $D_g^+ w$ is a smooth function on $\overline{S}$ such that $\left(D_g^+ w\right)|_{bS} = A_g^+ w = w + A_g^- w = u$, which entails that $D_g^+ w$ is the harmonic extension $\widetilde{u}$ of u to S and that $\theta_c^S u = \left(\partial D_g^+ w\right)|_{bS}$ can be computed, which is our goal. Thus, the question which arises is the characterization of the range of $Id + A_G^-$.

As g is symmetric and real, we know (see e.g., [35, chapter 7, §§11]) that for any real s, $Id + A_g^-$ is a Fredholm operator from $H^s(bS)$ to itself and has index 0. This implies that the obstruction to solve (16) in $H^s(bS)$ for data in $H^s(bS)$ is only finite dimensional and that $Id + A_g^-$ is an isomorphism if it is injective or surjective. Consider the standard identification $H^{-s}(bS)$ of the dual of $H^s(bS)$ by defining the duality pairing $\langle ., . \rangle$ by density of $C^\infty(bS)^2$ in $H^s(bS) \times H^{-s}(bS)$ and by

$$\langle u, w \rangle = \int_{\partial S} u w \tau^*$$

when $u, w \in C^\infty(bS)$. Then we can define the adjoint L^* of any operator L of $H^s(bS)$ and get the identity $\overline{\operatorname{Im} L} = (\ker L^*)^\perp$. Since $Id + A_g^-$ has a closed range as a Fredholm operator, we get $\operatorname{Im}\left(Id + A_g^-\right) = \left(\ker\left(Id + A_g^-\right)^*\right)^\perp$. From (14), it comes $Id + A_g^- = \frac{1}{2}\left(Id + N_g\right)$ and $N_g = I + 2A_g^-$. For $w \in C^\infty(bS)$, we obtain that for any $p \in bS$,

$$\left(N_g w\right)(p) = w(p) + 2\lim_{\varepsilon \to 0}\left(D_g^- w\right)\left(p - \varepsilon \nu_p\right)$$

in the distributional sense. With (15) and the Fubini theorem, we deduce that for $u, w \in C^\infty(bS)$

$$\begin{aligned}
\left\langle u, N_g w\right\rangle &= \langle u, w \rangle + 2\lim_{\varepsilon \to 0^+}\int_{\partial S} u(p)\left(D_g^- w\right)\left(p - \varepsilon \nu_p\right)\tau_p^* \\
&= \langle u, w \rangle + 2\lim_{\varepsilon \to 0^+}\int_{\partial S} u(p)\left(\int_{\partial S} w(q)\frac{\partial g}{\partial \nu_q}\left(p - \varepsilon \nu_p, q\right)\tau_p^*\right)\tau_q^* \\
&= \langle u, w \rangle + 2\lim_{\varepsilon \to 0^+}\int_{\partial S} w(q)\left(\int_{\partial S} u(p)\frac{\partial g}{\partial \nu_q}\left(p - \varepsilon \nu_p, q\right)\tau_q^*\right)\tau_p^* \\
&= \left\langle w, N_g^\# u\right\rangle.
\end{aligned}$$

This proves that $\left(N_g\right)^* = N_g^\#$, which entails $\ker\left(Id + A_g^-\right)^* = \ker\left(Id + N_g^\#\right)$. We summarize the above discussion within the following lemma.

Lemma 9

(1) *Let* $B_g = Id + N_g^\#$. *Then* $\ker B_g \subset C^\infty(bS)$ *and a function* $u \in H^s(bS)$ *is in the range of* $Id + A_g^-$ *if and only if* $\langle u, w \rangle = 0$ *for any* $w \in \ker B_g$.

(2) *Let* $u \in C^{\infty}(bS, \mathbb{R})$ *be orthogonal to* $\ker B_g$ *and* $w \in H^s(bS)$ *such that* $u = w + A_g^- w$. *Then* $w \in C^{\infty}(bS, \mathbb{R})$ *and* $\theta_c^S u = \left(\partial D_g^+ w\right)|_{bS}$.

(3) *When G is the principal Green function for of S,* $T_G^+ = D_G^+$ *and* $Id + A_G^-$ *is an automorphism of* $H^s(bS)$.

Proof (1) and (2) have been already proved except for $w \in C^{\infty}(bS)$ and $\ker B_g \subset C^{\infty}(bS)$. Both are consequences of the fact that $N_g^{\#}$ and A_g^- are a pseudo-differential operators of order -1 (see [35]). For a smooth real valued u and its harmonic extension $\widetilde{u}$ to S, Stokes Formula applied on S without an arbitrary small conformal disk Δ_ε around $q \in S$ gives

$$\begin{aligned}
\left(T_G^+ u\right)(q) &= \frac{2}{i}\int_{\partial S}\left(\widetilde{u}\partial G_q + G_q\overline{\partial}\widetilde{u}\right)\\
&= \frac{2}{i}\int_{\partial\Delta_\varepsilon}\left(\widetilde{u}\partial G_q + G_q\overline{\partial}\widetilde{u}\right) + \frac{2}{i}\int_{S\setminus\Delta_\varepsilon}\left(\overline{\partial}\widetilde{u}\wedge\partial G_q + \partial G_q\wedge\overline{\partial}\widetilde{u}\right)\\
&= \frac{2}{i}\int_{\partial\Delta_\varepsilon}\widetilde{u}\partial G_q + O(\varepsilon\ln\varepsilon) + 0 \underset{\varepsilon\to 0}{\to} \widetilde{u}(q)
\end{aligned}$$

As G and u are real valued,

$$\overline{T_G^+ u} = -\frac{2}{i}\int_{\partial S}\left(\widetilde{u}\overline{\partial}G_q + G_q\partial\widetilde{u}\right) = -\frac{2}{i}\int_{\partial S}\left[d\left(\widetilde{u}G_q\right) - \widetilde{u}\partial G_q\right] = T_G^+ u$$

This yields $D_G^+ u = T_G^+ u = \widetilde{u}$. Thus, $A_G^+ = Id + A_G^-$ is surjective and, because its index is 0, an isomorphism of $H^s(bS)$ as claimed in (3). □

Remark It is also known that $Id + A_g^-$ is an isomorphism of $H^s(bS)$ when $S \subset \mathbb{C}$ is bounded and has a connected complement (see e.g., [35]). In the general case, it is not difficult to prove that functions in $\ker\left(Id + A_g^-\right)$ are boundary values of holomorphic function on $\widetilde{S}\setminus\overline{S}$ smooth up to the boundary and that the Dirichlet–Neumann operator $\mathcal{N} : C^{\infty}(bS) \ni u \mapsto \frac{\partial\widetilde{u}}{\partial\nu}|_{bS}$ realizes an isomorphism from $\ker B_g$ to $\ker\left(Id + A_g^-\right)$.

Thus, to have at hand the principal Green function of S enables to bypass the resolution of (16). Unhappily, the standard method introduced by Fredholm in 1900 to build principal Green functions consists precisely in finding for each $q \in S$ a function w_q such that $g_q = w_q + A_g^- w_q$ and then to set $G_q = g_q - D_g^+ w_q$. Happily, in our problem it is not necessarily relevant to compute G because we only have to compute sufficiently many $\theta_c^S u$.

As mentioned in the next session, all of these considerations readily apply to the nodal setting.

4.2.2 Green Functions in the Nodal Case

Definition 10 Let $\mathcal{Z}$ be an open complex curve, possibly singular, of an open subset of $\mathbb{C}^2$. A Green function for $\mathcal{Z}$ is a function g defined on $(\mathrm{Reg}\,\mathcal{Z} \times \mathrm{Reg}\,\mathcal{Z})\setminus\Delta_{\mathrm{Reg}\,\mathcal{Z}}$

such that for all $q_* \in \operatorname{Reg} \mathcal{Z}$, $g_{q_*} = g(q_*, .)$ extends to $\mathcal{Z}$ as a current and $i\partial\overline{\partial} g_{q_*}$ is the Dirac current δ_{q_*} supported by $\{q_*\}$ - this implies in particular that ∂g_{q_*} is a weakly holomorphic $(1,0)$-form on $\mathcal{Z} \backslash \{q_*\}$ in the sense of [32].

When $\mathcal{Z}$ is an open nodal Riemann surface, quotient of Σ, an open Riemann surface, by an equivalence relation and when π is the canonical projection of Σ onto $\mathcal{Z}$, a simple Green function for $\mathcal{Z}$ is a is symmetric function g defined on $(\operatorname{Reg} \mathcal{Z} \times \operatorname{Reg} \mathcal{Z}) \backslash \Delta_{\operatorname{Reg} \mathcal{Z}}$ for which there exists a real valued Green function $\widetilde{g}$ for Σ such that $g = \pi_* \widetilde{g}$ in the following sense : for any branch $\mathcal{B}$ of $\mathcal{Z}$ at q_*, image by π of an open subset V of Σ such that $V \backslash \{s_*\} \subset \pi^{-1}(\operatorname{Reg} \mathcal{Z})$ where $s_* \in \pi^{-1}(q_*)$, $g_q|_{\mathcal{B}} = \pi_*\left(\widetilde{g}_{s_*}|_V\right)$ in a neighborhood of q_* in $\mathcal{B}$.

A principal Green function for a nodal bordered Riemann surface $\mathcal{Z}$ is a symmetric real valued simple Green function g such that if $\mathcal{B}$ is any boundary branch of $\mathcal{Z}$, $g\left|_{\overline{\mathcal{B}}}\right.$ extends continuously to $\overline{\mathcal{B}}$ with the value 0 on $\overline{\mathcal{B}} \cap b\mathcal{Z}$.

Let us now detail the explicit formula of [20, Proposition 17] establishing the existence of Green functions for a 1-parameter family of complex curves whose possible singularities are arbitrary. Consider a complex curve $\mathcal{Y}$ in an open subset of $\mathbb{C}^2$, Ω a Stein neighborhood of $\mathcal{Y}$ in $\mathbb{C}^2$, Φ a holomorphic function on Ω such that $\mathcal{Y} = \{\Phi = 0\}$ and $d\Phi\left|_{\mathcal{Y}}\right. \neq 0$ then a strictly pseudoconvex domain Ω_0 of $\mathbb{C}^2$ verifying

$$\mathcal{Y}_0 = \mathcal{Y} \cap \Omega_0 \subset \Omega,$$

and lastly a symmetric function $\Psi \in \mathcal{O}\left(\Omega \times \Omega, \mathbb{C}^2\right)$ such that for all $\left(z, z'\right) \in \mathbb{C}^2$,

$$\Phi\left(z'\right) - \Phi(z) = \left\langle \Psi\left(z', z\right), z' - z \right\rangle$$

where $\langle v, w \rangle = v_1 w_1 + v_2 w_2$ when $v, w \in \mathbb{C}^2$. We define on $\operatorname{Reg} \mathcal{Y}$ a $(1,0)$-form ω by setting

$$\omega = \frac{-\mathrm{d}z_1}{\partial\Phi/\partial z_2} \text{ on } \mathcal{Y}^1 = \mathcal{Y} \cap \{\partial\Phi/\partial z_2 \neq 0\}$$
$$\omega = \frac{+\mathrm{d}z_2}{\partial\Phi/\partial z_1} \text{ on } \mathcal{Y}^2 = \mathcal{Y} \cap \{\partial\Phi/\partial z_1 \neq 0\}$$

and we consider

$$k\left(z', z\right) = \det\left[\frac{\overline{z'} - \overline{z}}{|z' - z|^2}, \Psi\left(z', z\right)\right].$$

When $q_* \in \operatorname{Reg} \mathcal{Y}_0$, [20, Proposition 17] tells that the formula

$$g_c(q_*, q) = g_{c,q_*}(q) = \frac{1}{4\pi^2} \int_{q' \in \mathcal{Y}_0} \overline{k\left(q', q\right)} k\left(q_*, q'\right)\ i\omega\left(q'\right) \wedge \overline{\omega}\left(q'\right). \quad (17)$$

defines a Green function for $\mathcal{Y}_0$. In addition, the proof of [20, Proposition 17] gives that if $q_* \in \operatorname{Reg}\mathcal{Y}_0$

$$\partial g_{c,q_*} = \widetilde{k}_{q_*}\omega$$

where $\widetilde{k}_{q_*} = \frac{1}{2\pi}k\,(.,q_*)$. The proposition below gives a useful complement.

Proposition 11 *Suppose $\mathcal{Y}_0$ has only nodal singularities. Then, the function*

$$\begin{aligned} g\,(q_*,q) = \operatorname{Re} g_c\,(q_*,q) = \frac{1}{4\pi^2}\int_{q'\in\mathcal{Y}_0}\frac{1}{2}\Big(\overline{k\,(q',q)}k\left(q_*,q'\right) \\ +k\left(q',q\right)\overline{k\,(q_*,q')}\Big)\,i\omega\left(q'\right)\wedge\overline{\omega}\left(q'\right) \end{aligned} \tag{18}$$

is a simple Green function for $\mathcal{Y}_0$.

Proof Let us begin by proving that q_* being fixed in $\operatorname{Reg}\mathcal{Y}_0$, g_{c,q_*} extends as a usual harmonic function along the branches of $\mathcal{Y}_0\backslash\{q_*\}$. As g_{c,q_*} is a harmonic distribution on $\mathcal{Y}_0\backslash\{q_*\}$, we already know that $g_{c,q_*}\big|_{(\operatorname{Reg}\mathcal{Y}_0)\backslash\{q_*\}}$ is a usual harmonic function and according to [19, Proposition 2], that for any branch $\mathcal{B}$ of $\mathcal{Y}_0$ at q, $g_{c,q_*}|_{\mathcal{B}}$ has at most an isolated logarithmic singularity at q. Equivalently, this means that $\partial g_{c,q_*}$ has at most a simple pole at q. Fix q in $\operatorname{Sing}\mathcal{Y}_0$ and $\mathcal{B}$ a branch of $\mathcal{Y}_0$ at q. Decreasing $\mathcal{B}$ and with a possible change of coordinates, we get the case where $q=0$ and Φ is in a neighborhood of 0 of the form

$$\Phi\,(z) = (z_2-\varphi\,(z_1))\,\Theta\,(z) \tag{19}$$

with φ holomorphic in a sufficiently small disk $V=D\,(0,r)$ and $\Theta\,|_{\mathcal{B}}$ vanishing only at 0. In particular, there exists a function holomorphic θ on V such that $\theta\,(0)\neq 0$ and $\Theta\,(z_1,\varphi\,(z_1)) = z_1^{\nu-1}\theta\,(z_1)$ when $z_1\in V$, ν being the number of branches of $\mathcal{Y}_0$ at q. On $\mathcal{B}\backslash\{q\}$, we get hence $\omega = \frac{\mathrm{d}z_1}{\theta(z_1)z_1^{\nu-1}}$. Consider then a $(0,1)$-form χ compactly supported in $\mathcal{B}$; so $\chi = \xi\mathrm{d}\overline{z_1}$ with $\xi\in\mathcal{D}\,(V)$. Hence, by definition,

$$\left\langle\partial g_{c,q_*},\chi\right\rangle = \lim_{\varepsilon\downarrow 0^+}\int_{z_1\in V\backslash D(0,\varepsilon)}\frac{\widehat{k}_{q_*}\,(z_1)\,\xi\,(z_1)}{\theta\,(z_1)\,z_1^{\nu-1}}i\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}$$

where $\widehat{k}_{q_*}\,(z_1) = \widetilde{k}_{q_*}\,(z_1,\varphi\,(z_1))$. Let us write

$$\begin{aligned} \frac{\widehat{k}_{q_*}\,(z_1)\,\xi\,(z_1)}{\theta\,(z_1)} &= \sum_{\alpha+\beta<\nu-1} c_{\alpha,\beta}z_1^{\alpha}\overline{z_1}^{\beta} \\ &\quad+\int_0^1\frac{(1-t)^{\nu-2}}{(\nu-2)!}\,D^{\nu-1}\left(\widehat{k}_{q_*}\xi/\theta\right)\Big|_{tz_1}\cdot z_1^{\nu-2}\mathrm{d}t\;i\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1} \end{aligned}$$

where $D^p f|_w\cdot z^p$ is understood has the value taken by the total differential of order p of f at w on the vector $(z,\ldots,z)$. Since $\int_0^{2\pi}e^{i\theta(\alpha-\beta-\nu+1)}\mathrm{d}\theta = 0$ when $\alpha+\beta<\nu-1$, we get

$$\left\langle \partial g_{c,q_*}, \chi \right\rangle = \int_{z_1 \in V} \int_0^1 \frac{(1-t)^{\nu-2}}{(\nu-2)!} \, D^{\nu-1} \left(\widehat{k}_{q_*} \xi / \theta \right) \Big|_{tz_1} \cdot 1^{\nu-1} \mathrm{d}t \; i \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \tag{20}$$

Moreover, there exists $c \in \mathbb{C}$ and $h \in \mathcal{O}(V)$ such that the expression of $\partial g_{c,q_*} |_{\mathcal{B}}$ is $\frac{c}{z_1} \mathrm{d}z_1 + h \mathrm{d}z_1$ in the coordinate z. Hence

$$\left\langle \partial g_{c,q_*}, \chi \right\rangle = \lim_{\varepsilon \downarrow 0^+} \int_{z_1 \in V \setminus D(0,\varepsilon)} \left(\frac{c}{z_1} + h(z_1) \right) \xi(z_1) \, i \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1}$$

Let us write $\xi(z_1) = \xi(0) + \xi_{1,0} z_1 + \xi_{0,1} \overline{z_1} + \int_0^1 (1-t) \; D^2 \xi \big|_{tz_1} \cdot z_1^2 \mathrm{d}t$. Then comes

$$\left\langle \partial g_{c,q_*}, \chi \right\rangle = \pi r^2 \xi_{1,0} c + \int_{z_1 \in V} h(z_1) \, \xi(z_1) \, i \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \tag{21}$$

As (20) shows no derivation of the Dirac measure at 0, comparison with (21) forces $c = 0$. Hence $g_{c,q_*} |_{\mathcal{B}}$ and $g_{q_*} |_{\mathcal{B}}$ are usual harmonic functions.

Next, we check that $i \partial \overline{\partial} g_{c,q_*}$ is the Dirac current at q_*. Since g_{c,q_*} has no singularity in any branch of $\mathcal{Y}_0 \backslash \{q_0\}$, we get thanks to the nodal version of Stokes formula that for that any test function χ on $\mathcal{Y}_0$, $\left\langle i \partial \overline{\partial} g_{c,q_*}, \chi \right\rangle = \left\langle i \partial g_{c,q_*}, \overline{\partial} f \right\rangle$ is the limit when $\varepsilon \to 0^+$ of $\frac{1}{i} \int_{\partial \Delta_\varepsilon} \chi \partial g_{c,q_*}$ where Δ_ε is a conformal disk of radius ε centered at q_*. Using the same notation as above with $\nu = 1$ and q replaced by q_* which we can assume to be 0, we find that

$$\left\langle i \partial \overline{\partial} g_{c,q_*}, \chi \right\rangle = \frac{\Psi_2(0,0) - \varphi'(0) \, \Psi_1(0,0)}{\theta(0) \left(1 + |\varphi'(0)|^2 \right)} \chi(q_*)$$

From (19) we get by differentiation that $\Psi_2(0,0) = \theta(0)$ and $\Psi_1(0,0) = -\overline{\varphi'(0)}$. Hence, $\left\langle i \partial \overline{\partial} g_{c,q_*}, \chi \right\rangle = \chi(q_*)$ which means $i \partial \overline{\partial} g_{c,q_*} = \delta_{q_*}$. Since δ_{q_*} is real valued on real valued test functions, this entails $i \partial \overline{\partial} g_{q_*} = \delta_{q_*}$.

Fix now q_s in Sing $\mathcal{Y}_0$. Consider a branch $\mathcal{B}$ of $\mathcal{Y}_0$ at q_s sufficiently small so we have for it a holomorphic coordinate z centered at q_s. Since g is symmetric from (17), what precedes implies that when $q_* \in \mathcal{B} \backslash \{q_s\}$, $q \mapsto g_{q_*}(q) - \frac{1}{2\pi} \ln |z(q) - z(q_*)| = g_q(q_*) - \frac{1}{2\pi} \ln |z(q) - z(q_*)|$ is a usual harmonic function on $\mathcal{B}$. Hence, when $q_* \in \mathcal{B} \backslash \{q_s\}$ tends to q_s, $g_{q_*} - \frac{1}{2\pi} \ln |z - z(q_*)|$ converges uniformly on $\mathcal{B}$ to a harmonic function of the form $g^{\mathcal{B}}_{\mathcal{B},q_s} - \frac{1}{2\pi} \ln |z|$ where $g^{\mathcal{B}}_{\mathcal{B},q_s}$ is harmonic on $\mathcal{B} \backslash \{q_s\}$. For the same reason, if $\mathcal{B}'$ is another branch of $\mathcal{Y}_0$ at q_s or a branch of $\mathcal{Y}_0$ relatively compact in $\mathcal{Y}_0 \backslash \{q_s\}$, g_{q_*} converges uniformly on $\mathcal{B}'$ to a harmonic function $g^{\mathcal{B}'}_{\mathcal{B},q_s}$ when $q_* \in \mathcal{B} \backslash \{q_s\}$ tends to q_s. When $\mathcal{B}'$ describes the set of branches of $\mathcal{Y}_0$, these functions $g^{\mathcal{B}'}_{\mathcal{B},q_s}$ match into a function $g_{\mathcal{B},q_s}$ which is harmonic on $\mathcal{B}' \backslash b\mathcal{Y}_0$ for all branches $\mathcal{B}'$ of $\mathcal{Y}_0 \backslash \mathcal{B}$, whose restriction to $\mathcal{B}$ has a logarithmic singularity at q_s and such that g_{q_*} tends to $g_{\mathcal{B},q_s}$ in the sense of currents when $q_* \in \mathcal{B} \backslash \{q_s\}$ tends to q_s. Proceeding so for all singulars point of $\mathcal{Y}_0$, we find that g is a simple Green function for $\mathcal{Y}_0$. □

We now apply what precedes to the situation of Theorem 5. We recall that $F:\widetilde{M}\to\mathbb{C}^2$ is the map obtained by applying Theorem 2 to a plain extension $(\widetilde{M},\widetilde{\sigma})$ of (M,σ). We set $\mathcal{Y}=F(\widetilde{M})$ and we fix a Stein neighborhood Ω of $\mathcal{Y}$ in $\mathbb{C}^2$, that is a neighborhood of $\mathcal{Y}$ which is a Stein manifold. As $\mathcal{M}=F(M)$ is relatively compact in $\mathcal{Y}$, we can pick up in $\mathbb{C}^2$ a strictly pseudoconvex domain Ω_0 verifying $\mathcal{M}\subset\subset\mathcal{Y}_0=\mathcal{Y}\cap\Omega_0\subset\Omega$. We use then Proposition 11 and get a Green function for $\mathcal{M}$. The corollary below tells it comes from a Green function for M.

Corollary 12 *Hypothesis and notation remains as in Theorem 5 and g is the function defined by (18). Then, $g_M=F^*g\left|_{\overline{M}\times\overline{M}\backslash\Delta_M}\right.$ is a Green function for $(M,\mathcal{C}_\sigma)$.*

Proof Since $F:M\to\mathcal{M}$ is a $(c_\sigma,c_{\mathcal{M}})$-analytic normalization, $h=F^*g$ is well defined on $\overline{M}_{\rm reg}\times\overline{M}_{\rm reg}\backslash\Delta_{\overline{M}_{\rm reg}}$ where $\overline{M}_{\rm reg}=F^{-1}\left(\operatorname{Reg}\overline{Q}\right)$, symmetric and for all $x\in\overline{M}$, $h_x=h(.,x)$ is harmonic on $\overline{M}_{\rm reg}\backslash bM\cup\{x\}$, continuous on $\overline{M}_{\rm reg}\backslash\{x\}$ and $i\partial^\sigma\overline{\partial^\sigma}h$ is the Dirac current δ_x of M at x. When $p\in F^{-1}\left(\operatorname{Sing}\overline{\mathcal{M}}\right)\cap M$ and V is a connected open neighborhood of p in M, $B=F(V)$ is an inner branch of $\overline{\mathcal{M}}$ at $q=F(p)$ and we can set $g_{M,p}=F^*g_{p,B}$. Proposition 11 implies that g_M so built is a Green function for M. □

Thus, we can apply the methods of Sect. 4.2.1 to g_M and then push forward their results to $\mathcal{M}$. Meanwhile, as in our problem $\mathcal{M}$ and θ_c^σ have to be computed before M can be, it is more relevant to apply directly these methods to $\mathcal{M}$ and g. As $b\mathcal{M}$ is smooth, Sobolev spaces on $b\mathcal{M}$ are defined as usual and the discussion of Sect. 4.2.1 can be readily followed. So the operators T_g, D_g, $A_g^\pm$, N_g etc., are defined as above (with $\mathcal{M}$ instead of S) and Lemma 9 holds. We are now ready to prove Theorem 5.

Proof of Theorem 5 Consider $u\in C^\infty(bM)$ and $\widetilde{u}$ its $\mathcal{C}_\sigma$-harmonic extension to M. As $d=\overline{\partial}^\sigma+\partial^\sigma$ and $d^\sigma=i\left(\overline{\partial}^\sigma-\partial^\sigma\right)$, we get $2i\partial^\sigma\overline{\partial}^\sigma=dd^\sigma$ and $\widetilde{u}$ is the unique solution in $C^\infty\left(\overline{M}\right)$ of

$$i\partial^\sigma\overline{\partial}^\sigma U=0\ \ \&\ \ U\left|_{bM}\right.=u.$$

and $\theta_c^\sigma u$ is the restriction to bM of the $\mathcal{C}_\sigma$-holomorphic $(1,0)$-form $\partial^\sigma\widetilde{u}$. By definition, when B is a branch of $\mathcal{M}$, there is a (unique) open subset V of M such that the map $F_B=F\left|_V^B\right.$ is a $(c_\sigma,c_{\mathcal{M}})$-biholomorphism. Since $\widetilde{u}$ is smooth, we deduce that $F_*\widetilde{u}$ is smooth along any branch B of $\mathcal{M}$ and satisfies $\left(i\partial\overline{\partial}\left(F_B\right)_*\widetilde{u}\right)|_B=\left(F_B\right)_*i\partial^\sigma\overline{\partial}^\sigma\widetilde{u}=0$. Hence, $\widetilde{u}\circ\left(F_{\operatorname{Reg}\mathcal{M}}\right)^{-1}$ harmonically extends along branches of $\mathcal{M}$ and define on $\mathcal{M}$ a distribution W which is the unique continuous solution along branches of $\mathcal{M}$ for the problem

$$i\partial\overline{\partial}W=0\ \ \&\ \ W\left|_{b\mathcal{M}}\right.=f_*u \tag{22}$$

This yields $F^*W=\widetilde{u}$ which means that $\widetilde{u}\left|_V\right.=\left(F\left|_V^{F(V)}\right.\right)^*W$ whenever $V\subset M$ is such that $F(V)$ is a branch of $\mathcal{M}$. Lemma 8 yields that $F:M\to\mathcal{M}$ is a holomorphic map from (M,c_σ) to $(\mathcal{M},c_{\mathcal{M}})$. Since the complex differential operators of these (nodal) Riemann surfaces are ∂^{σ} and ∂, we get $\partial^{\sigma}\widetilde{u} = \partial^{\sigma} F^{*}W = F^{*}\partial W$ and W is the simple harmonic extension $\widehat{f_{*}u}$ of $f_{*}u$ to $\mathcal{M}$. So, we get $\theta_c^{\sigma} u = (F^{*}\partial \widehat{f_{*}u})\,|_{bM}$.

The kernel of B_g (in its nodal issue) is a finite dimensional subspace of $C^{\infty}(b\mathcal{M})$ and when $u \in C^{\infty}(bM, \mathbb{R})$ is such that $f_{*}u$ is orthogonal to it, any solution w of the equation $f_{*}u = w + A_g^{-} w$ is in $C^{\infty}(b\mathcal{M}, \mathbb{R})$ and delivers $\widehat{f_{*}u}$ under the form $T_g^{+} w$. Hence, $\theta_c^{\sigma} u = \left(F^{*}\partial T_g^{+} w\right)|_{bM}$. □

Remark The above proof contains the fact that for any $u \in C^{\infty}(bM)$, $\widetilde{u} = F^{*}\widehat{f_{*}u}$ and $\theta_c^{\sigma} u = F^{*}\theta_c^{\mathcal{M}} f_{*}u$ where $\widetilde{u}$ is the $\mathcal{C}_{\sigma}$-harmonic extension of u to M and $\widehat{f_{*}u}$ is the simple harmonic extension of $f_{*}u$ to $\mathcal{M}$.

5 Proof of the Uniqueness Theorem 3

In this section, we prove Theorem 3 and as mentioned in Sect. 2, we complete so the proof of [17, Theorem 1] and also the isomorphism claim of [24, Theorem 1.1]. One of the steps of the proof of Theorem 3 uses Lemmas 11 to 14 of [21] which were initially written by the author of these lines to give a complete proof of Theorem 3.

We note (U_{ℓ}) and $\left(U'_{\ell}\right)$ the harmonic extensions of u to M and M' respectively. By hypothesis $F = [\partial U] : \overline{M} \longrightarrow \mathbb{CP}_n$ and $F' = \left[\partial U'\right] : \overline{M'} \longrightarrow \mathbb{CP}_n$ are well defined, coincide on γ and $f = F\left|_{\gamma} = F'\right|_{\gamma}$ embeds γ in $\{w_0 \neq 0\}$ where $w_0, \dots, w_n$ are the standard homogeneous coordinates of $\mathbb{CP}_n$. We equip $\delta = f(\gamma)$ with the orientation of γ brought by f. The regularity hypothesis made on M and M' implies that F and F' are of class C^1. We set

$$Y = F(M)\backslash\delta,\ \ \Gamma = F^{-1}(\delta),$$
$$\widetilde{M} = M\backslash\Gamma,\ \ \widetilde{F} = F\Big|_{M\backslash\Gamma}^{\mathbb{CP}_n\backslash\delta},$$
$$\overline{M}_r = \{dF \neq 0\}\ \ \&\ \ M_s = \{dF = 0\}$$

Since f is an embedding of γ in $\{w_0 \neq 0\}$ which is isomorphic to $\mathbb{C}^n$, there exists an open neighborhood G of γ in $\overline{M}$ such that $F_G = F\,|_G$ is an embedding of G in $\mathbb{C}^2$; the orientation of δ is hence also induced by the natural one of G. When A is a topological space, we note $CC(A)$ the set of the connected components of A. If $A \subset \overline{M}$ and $B \subset F(A)$, we denote $\nu(F, A, B)$ the degree of $F\big|_A^B$ if it exists. We agree for M' similarly notation to those for M. $\mathcal{D}_{p,q}(U)$ stands for the space of (p,q)-forms of class C^{∞} compactly supported in an open subset U of a complex manifold. $\mathcal{H}^d(E)$ denotes the Hausdorff d-dimensional measure of a set E when this is meaningful.

Lemma 13 *$\Gamma\backslash\gamma$ is a compact of M and Y is a complex curve of $\mathbb{CP}_n\backslash\delta$.*

Proof Since F_G is embeds G in $\mathbb{C}^2$, $\Gamma \cap G = \gamma$ and $\Gamma\backslash\gamma = \Gamma \cap \left(\overline{M}\backslash G\right)$ is a compact of M. In particular, $\widetilde{M} = M\backslash\Gamma$ is an open surface Riemann. By construction, $\widetilde{F}$ is proper because if L is a compact of $\mathbb{CP}_n\backslash\delta$, $\widetilde{F}^{-1}(L)$ is a compact of $\overline{M}$ which does not meet Γ and hence is a compact of $\widetilde{M}$. By a theorem of Remmert, unnecessary in the very simple case $n = 1$, $Y = \widetilde{F}(\widetilde{M})$ is an analytic subset of $\mathbb{CP}_n\backslash\delta$. □

Lemma 14 *$F_*[M]$ is a normal positive current supported by $\overline{Y}$ and $dF_*[M]=[\delta]$.*

Proof If χ is a compactly supported smooth form of $\mathbb{CP}_n$,

$$\langle F_*[M],\chi\rangle=\int_M F^*\chi.$$

$F_*[M]$ is thus a current of bidegree (1, 1) supported by $\overline{F(M)}$, that is $\overline{Y}$. It is positive because if $\chi\in\mathcal{D}_{1,1}(\mathbb{CP}_n)$ is positive, $(F^*\chi)|_M$ is a positive (1, 1)-form of M since F is holomorphic and hence $\langle F_*[M],\chi\rangle\geqslant 0$. Let $\xi\in C^\infty(\mathbb{CP}_n)$ be such that $\chi=\xi\omega_{FS}$ where and $\omega_{FS}=\frac{i}{2\pi}\partial\overline{\partial}\ln|w|^2$ is the (1, 1)-form defining the Fubini–Study metric. We get then

$$|\langle F_*[M],\chi\rangle|\leqslant\int_M|\xi|\,F^*\omega_{FS}\leqslant\|\xi\|_\infty\int_M F^*\omega_{FS}$$

As $\|\chi\|=\sup\limits_{p\in\mathbb{CP}_n}\|\chi_p\|$ and

$$\begin{aligned}\|\chi_p\|&=\max_{s,t\in T_p\mathbb{CP}_n,\ \|s\|_{FS}=\|t\|_{FS}=1}\left|\chi_p.(s,t)\right|\\&=|\xi(p)|\max_{s,t\in T_p\mathbb{CP}_n,\ \|s\|_{FS}=\|t\|_{FS}=1}\left|(\omega_{FS})_p.(s,t)\right|=|\xi(p)|,\end{aligned}$$

we get that the mass of $F_*[M]$ is finite and at most $\int_M F^*\omega_{FS}$. If $\chi\in\mathcal{D}(\mathbb{CP}_n)$,

$$\langle dF_*[M],\chi\rangle=\langle F_*[M],d\chi\rangle=\int_M F^*d\chi=\int_M dF^*\chi=\int_\gamma F^*\chi=\left\langle F_*\left[\gamma\right],\chi\right\rangle$$

In other words, $dF_*[M]=F_*\left[\gamma\right]=[\delta]$. In particular, the mass of $dF_*[M]$ is finite ; $F_*[M]$ is a normal current supported by $\overline{Y}$. □

Lemma 15 *$F_*[M]\left|_{\mathbb{CP}_n\backslash\delta}\right.$ is a positive holomorphic chain of $\mathbb{CP}_n\backslash\delta$ supported by Y.*

Proof Given that $T=F_*[M]$ is supported by $\overline{Y}$ and that $Y=\overline{Y}\backslash\delta$, $S=T\left|_{\mathbb{CP}_n\backslash\delta}\right.$ is a normal, and hence locally rectifiable, current of $\mathbb{CP}_n\backslash\delta$, without boundary and supported by Y. According to the Structure Theorem 2.1 of [14], there exists hence $\left(n_j\right)_{1\leqslant j\leqslant N}\in\mathbb{Z}^{\mathbb{N}}$ such that $S=\sum_{1\leqslant j\leqslant N}n_j\left[Y_j\right]$ where $\left(Y_j\right)$ is the family of irreducible components of Y. S being moreover a positive current according to Lemma 14, the n_j are natural integers. □

Lemma 16 *$F_*[M]=F_*\left[M'\right]$ and $Y'=Y$.*

Proof According to Lemma 14, the current $T=F_*[M]-F'_*\left[M'\right]$ is a boundaryless normal current of bidegree (1, 1) supported by $\overline{Y}\cup\overline{Y'}$. It is hence of the form $\sum_{1\leqslant j\leqslant N}n_j\left[Z_j\right]$ where $\left(n_j\right)\in(\mathbb{Z}^*)^{\mathbb{N}}$ and the Z_j are irreducible compact complex curves of $\mathbb{CP}_n$ lying in $\overline{Y}\cup\overline{Y'}$. Let Z one of these curves. $Z\cap\delta\neq\varnothing$ because otherwise $F^{-1}(Z)$ is a compact complex curve lying in M or M', which is excluded. One of the connected components of δ, says β, is hence contained in Z; we equip β of the orientation induce by δ. β being smooth, there exists in Z a Riemann (smooth) surface B such that $B\backslash\beta$ is included in $(\mathbb{CP}_n\backslash\delta) \cap \operatorname{Reg}\overline{Y} \cap \operatorname{Reg}\overline{Y'}$ and has only two connected components, B^- and B^+.

By construction, B^- is an open connected Riemann surface included in the complex curve $Y \cup Y'$ and hence, at least one of the two numbers $\mathcal{H}^2\left(B^- \cap Y\right)$ or $\mathcal{H}^2\left(B^- \cap Y'\right)$ is positive, says $\mathcal{H}^2\left(B^- \cap Y\right) > 0$. As B^- is connected, this implies[3] that $B^- \subset Y$. Given that β is a subset of the boundaries of Y and B, we infer that after decreasing B if necessary, $Y \cap B \subset Z$ and hence $Y \cap B \subset B^- \cup B^+$.

Suppose that $\mathcal{H}^2\left(B^+ \cap Y\right) = 0$. Then, as $B \subset \operatorname{Reg}\overline{Y}$, $B^+ \cap Y = \varnothing$, $Y \cap B = B^-$ and, by force, $B^+ \subset Y'$. Suppose in addition that $\mathcal{H}^2\left(B^- \cap Y'\right) = 0$, then, decreasing B if necessary, we get as before $Y' \cap B = B^+$ and so $d[Y] = -d\left[Y'\right]$ near β. This does not match the fact that $F_*[M]$ and $F'_*\left[M'\right]$ are two positive holomorphic chains of $\mathbb{CP}_n\backslash\delta$ supported respectively by Y and Y'. So, $\mathcal{H}^2\left(B^- \cap Y'\right) > 0$ and hence, $B^- \subset Y'$. Hence $B \subset Y'$ and $Z \subset Y'$, which is again a contradiction. Going back to our first assumption, we get that $\mathcal{H}^2\left(B^+ \cap Y\right) > 0$ and hence $B \subset Y$, still an impossibility. The lemma is proven. □

Lemma 17 *When $y \in \overline{Y}$, $\overline{M_y} = F^{-1}(\{y\})$ is a finite set and $\nu : \overline{Y} : y \mapsto \operatorname{Card}\overline{M}_y$ is bounded.*

Proof Suppose that $F^{-1}(\{y\})$ is infinite for some $y \in \overline{Y}$. If $F^{-1}(\{y\})$ has an accumulation point in M, $F = y$ on a connected component of M and hence on a non-empty open subset of γ. In the contrary case , $F^{-1}(\{y\})$ has an accumulation point in γ and dF vanishes at this point. In both cases, this contradicts that $F\big|_\gamma$ is an embedding.

Suppose that ν is unbounded. There exists then $(y_m) \in \overline{Y}^{\mathbb{N}}$ such that $(\nu_m) = (\nu(y_m))$ admits $+\infty$ as limit and (y_m) converges to $y_* \in \overline{Y}$. Since $\overline{M}$ is compact, there exists in $\overline{M}^{\mathbb{N}}$ a convergent sequence with limit $x_*^0 \in F^{-1}(\{y_*\})$ and a strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$ such that $y_{\varphi(m)} = F(x_m)$ for all $m \in \mathbb{N}$. If $dF\Big|_{x_*^0} \neq 0$, there exists an open neighborhood U_0 of x_*^0 in $\overline{M}$ such that $V_0 = F(U_0)$ is a Riemann surface (with boundary if $x_*^0 \in \gamma$) and $F\Big|_{U_0}^{V_0}$ is a biholomorphism (of Riemann surfaces with boundary if $x_*^0 \in \gamma$) ; we set $m_*^0 = 1$ in this case. If $dF\Big|_{x_*^0} = 0$, $x_0^* \notin \gamma$ and we can choose in a neighborhood of y_* in $\mathbb{CP}_n$, holomorphic coordinates $(\zeta_1, \ldots, \zeta_n)$ such that the vanishing order m_* of $(d(\zeta_1 \circ F), \ldots, d(\zeta_n \circ F))$ at x_*^0 is also the one of $d(\zeta_1 \circ F)$ at x_*^0. In this case, there exists an open neighborhood U_0 of x_*^0 in M such that if $y \in V_0 = F(U_0)$, $\zeta_1(F(y))$ has exactly m_*^0 preimages by $\zeta_1 \circ F$ in U_0, mutually distinct if $y \neq y_*$; if $y \in V_0 = F(U_0)$, y has at least one preimage by F in U_0 and at most m_*^0.

Suppose that we have got $k+1$ mutually distinct points $x_*^0, \ldots, x_*^k$ in $F^{-1}(y_*)$ and open neighborhoods $U_0, \ldots, U_k$ of these points in $\overline{M}$ such that for all $j \in \{1, \ldots, k\}$,

[3] Since $B^- \cap \delta = \varnothing$, $B^- = \left(B^- \cap Y\right) \cup \left(B^- \backslash \overline{Y}\right)$. $B^- \cap Y$ is an open subset B^- because by construction, $B^- \subset \operatorname{Reg}\overline{Y} \cap \operatorname{Reg}\overline{Y'}$. It is non-empty by hypothesis. Hence $B^- = B^- \cap Y \subset Y$.

$1 \leqslant \operatorname{Card} F^{-1}(y_*) \cap U_j \leqslant m_*^j$ and $U_j \subset \overline{M} \backslash V_{j-1}$ where $V_{j-1} = \underset{1 \leqslant \ell \leqslant j-1}{\cup} U_\ell$. Then $\operatorname{Card} F^{-1}(y_*) \cap V_k \leqslant \sum_{0 \leqslant j \leqslant k} m_*^j$ and since $\overline{M} \backslash V_{k+1}$ is compact, we can find a strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $m \in \mathbb{N}$, $F^{-1}\left(y_{\varphi(m)}\right) \cap \left(\overline{M} \backslash V_{k+1}\right)$ contains at least a point x_m^{k+1} which tends, when m goes to infinity, toward a point $x_*^{k+1} \in F^{-1}(\{y_*\})$. As before, we can then find an integer m_*^{k+1} and a neighborhood U_{k+1} of x_*^{k+1} in $\overline{M}$ such that $1 \leqslant \operatorname{Card} F^{-1}(y_*) \cap U_k \leqslant m_*^k$.

The values of the sequence $\left(x_*^k\right)_{k \in \mathbb{N}}$ so built are mutually distinct points of $\overline{M}_y$, which is impossible. ν is hence bounded. □

Lemma 18 *Consider* $h \in \mathcal{O}(M) \cap C^0\left(\overline{M}\right)$. *Then* $F_* h$ *is holomorphic and bounded on* $\operatorname{Reg} Y$. *In addition,* $F'^* F_* h = (F_* h) \circ F' \in \mathcal{O}\left(M'\right) \cap C^0\left(\overline{M'}\right)$

Proof By definition $F_* h$ is the function defined on Y by $(F_* h)(y) = \sum_{x \in F^{-1}(y)} h(x)$. Let $y_* \in (\operatorname{Reg} Y) \backslash F(\{dF = 0\})$. Set $F^{-1}(y_*) = \{x_{*1}, \ldots x_{*k}\}$ where $k = \nu(y)$. There exists a neighborhood B of y in $\operatorname{Reg} Y$ such that for all $j \in \{1, \ldots, k\}$, there exists a neighborhood A_j of x_{*j} in M for which $F_j = F\Big|_{A_j}^{B}$ is a biholomorphism. Suppose that $(y_\nu) \in B^{\mathbb{N}}$ converges to y_* and $\operatorname{Card} F^{-1}\{y_n\} \geqslant k$ for all n. Then, for each $n \in \mathbb{N}$ there exists $a_n \in M \backslash \left\{F_1^{-1}(y_n), \ldots, F_k^{-1}(y_n)\right\}$ such that $F(a_n) = y_n$. Possibly after extracting a subsequence, (a_n) converges to a point a of $\overline{M}$ which satisfies $F(a) = y_*$. Given that $y \in Y = F(M) \backslash F(bM)$, $a \notin bM$ and there exists $j \in \{1, \ldots, k\}$ such that $a = x_{*j}$. For n big enough, a_n and $F_j^{-1}(y_n)$ are then two distinct points of A_j sharing the same image y_n by F. This is absurd. Hence, $F_* h = \sum_{1 \leqslant j \leqslant k} h \circ F_j^{-1}$ is holomorphic in a neighborhood of y. Furthermore, $|F_* h| \leqslant k \|h\|_\infty$ and $k = \nu(y)$. $F_* h$ is thus bounded according to Lemma 17. Given that $(\operatorname{Reg} Y) \cap F(\{dF = 0\})$ is finite, $F_* h$ extends holomorphically to $\operatorname{Reg} Y$. This implies that $F'^* F_* h = (F_* h) \circ F'$ is holomorphic and is bounded on $M' \backslash F'^{-1}(\operatorname{Sing} Y)$. As $F'^{-1}(\operatorname{Sing} Y)$ is a finite set, $F'^* F_* h$ extends holomorphically to M'. □

Lemma 19 *If* $\omega' \in C^{1,0}\left(\overline{M'}\right) \cap \Omega^{1,0}\left(M'\right)$, *there exists* $\omega \in C^{1,0}\left(\overline{M}\right) \cap \Omega^{1,0}(M)$ *such that* $\omega\big|_\gamma = \omega'\big|_\gamma$.

Proof We have to check that $\omega'\big|_\gamma$ verifies the moment condition when γ is seen as the boundary of M. So, let $h \in \mathcal{O}(M) \cap C^0\left(\overline{M}\right)$. According to Lemma 18, $g = F'^* F_* h \in \mathcal{O}\left(M'\right) \cap C^0\left(\overline{M'}\right)$. Since $f_*[\gamma] = [\delta]$,

$$\int_\gamma h\omega' = \int_\gamma F^* F_*\left(h\omega'\right) = \int_\delta F_*\left(h\omega'\right)$$
$$= \int_\gamma \left(F'^* F_*\right)\left(h\omega'\right) = \int_{M'} d\left(F'^* F_*\right)\left(h\omega'\right) = 0.$$

because $F'^* F_* h \in \mathcal{O}\left(M'\right) \cap C^0\left(\overline{M'}\right)$ and $\omega' \in \Omega^{1,0}\left(M'\right)$. □

Proof of Theorem 3 Since by hypothesis $\left[(\partial U_\ell)_{0\leqslant\ell\leqslant n}\right]$ is a well defined map from $\overline{M}$ to $\mathbb{CP}_n$, we can use the Adjonction Lemma 12 of [21] which, though written for the particular case $n=2$, applies without any change for arbitrary n in $\mathbb{N}^*$: there exists harmonic functions $U_{n+1},\ldots,U_N$ on M and continuous on $\overline{M}$ such that $\left[(\partial U_\ell)_{0\leqslant\ell\leqslant N}\right]$ is an embedding of M in $\mathbb{CP}_N$. Similarly, there exists harmonic functions $U'_{N+1},\ldots,U'_{N'}$ on M' and continuous on $\overline{M'}$ such that $\left[\left(\partial U'_\ell\right)_{\ell\in\{0,\ldots,n,N+1,\ldots,N'\}}\right]$ is an embedding of M' in $\mathbb{CP}_{n+N'-N}$. When $\ell\in\left\{N+1,\ldots,N+N'\right\}$, Lemma 19 gives that $\left(\partial U'_\ell\right)\big|_{\gamma'}$ extends to M as a $(1,0)$-form holomorphic Σ_ℓ. Also, when $\ell\in\{n+1,\ldots,N\}$, $(\partial U_\ell)\big|_\gamma$ extends to M' as a $(1,0)$-form holomorphic Σ'_ℓ. Consider then

$$\Sigma=\left(\partial U_0,\ldots,\partial U_n,\partial U_{n+1}\ldots,\partial U_N,\Sigma_{N+1},\ldots,\Sigma_{N+N'}\right)\overset{def}{=}(\Sigma_\ell)_{0\leqslant\ell\leqslant L}$$
$$\Sigma'=\left(\partial U'_0,\ldots,\partial U'_n,\Sigma'_{n+1},\ldots,\Sigma'_{N'},\partial U'_{N+1},\ldots,\partial U'_{N+N'}\right)\overset{def}{=}\left(\Sigma'_\ell\right)_{0\leqslant\ell\leqslant L}$$

By construction Σ and Σ' coincide on γ. Note $(w_\ell)_{0\leqslant\ell\leqslant L}$ the natural coordinates of $\mathbb{C}^{L+1}$. When $0\leqslant\ell_*\leqslant n$, $[\Sigma]\big|_{\{\partial U_\ell\neq0\}}$ can be written $\left(\partial U_\ell/\partial U_{\ell_*}\right)_{\ell\neq\ell^*}$ in the natural coordinates of $\mathbb{C}^L$ identified to $\left\{w_{\ell_*}\neq0\right\}$. Note p_{ℓ_*} the natural projection of $\mathbb{C}^L$ on $\mathbb{C}^N$, $(z_\ell)_{\ell\neq\ell_*}\mapsto(z_\ell)_{0\leqslant\ell\leqslant N,\,\ell\neq\ell_*}$. The map $\left(\partial U_\ell/\partial U_{\ell_*}\right)_{0\leqslant\ell\leqslant N,\,\ell\neq\ell_*}$ is by construction an embedding of $\{\partial U_\ell\neq0\}$ in $\mathbb{C}^N$. $[\Sigma]$ is moreover injective because $\overline{M}=\underset{0\leqslant\ell\leqslant n}{\cup}\{\partial U_\ell\neq0\}$ and because a relation of the form $[\Sigma](x)=[\Sigma](y)$ impose $y\in\underset{(\partial U_\ell)_x\neq0}{\cap}\{\partial U_\ell\neq0\}$. $[\Sigma]$ is thus an embedding of $\overline{M}$ in $\mathbb{CP}_L$. Also, $\left[\Sigma'\right]$ is an embedding of M' in $\mathbb{CP}_L$. Noting that the proof of Lemma 14 does not use that F is a canonical map, that is of the form $[\partial U]$, or noting that Lemma 8 of [21] shows that Σ and Σ' are necessarily of this kind, we conclude that $\Sigma(M)=\Sigma'\left(M'\right)$ then that M and M' are isomorphic through a map whose restriction to γ is the identity. □

6 Reconstruction of a Riemann Surface

As explained in Sect. 2, one of the steps in the reconstruction of a general conductivity structure is the particular case of the reconstruction of a Riemann surface from its Dirichlet–Neumann operator which itself comes down to the reconstruction from its oriented boundary ∂Q of a relatively compact domain Q of an open nodal Riemann surface $\widetilde{Q}$ of $\mathbb{CP}_2$.

This last job is done in this section with the help of the Cauchy–Fantapié indicators of Q defined by Formula (4). Theorem 39 and Proposition 41 which are the main result of this Sect. 6.5 are novelties about characterization and uniqueness of decomposition in sums of shock waves of these indicators.

For the reader's convenience, we list here some of the notation used in this section. U, L_z and G_k are defined with (4); Q^∞, q^∞, b^q, E^∞, U_{reg}, Z, Z_{reg}, Z^+, Z^+_{reg}, ρ, $\widetilde{\rho}$ are defined at the beginning of Sect. 6.1; $N_{h,k}$ and $S_{h,k}$: (25); $\mathbb{C}[X,Y]$ and $\mathbb{C}_k[X,Y]$: Proposition 21; N^Q_k and S^Q_k : end of Sect. 6.1; P_k : (29); B^∞ and $p_{k,\nu}$: (31); δ, $G_{k,m}$

and $\widetilde{G}_{k,m}$: Lemma 23; $(\partial Q)_0$: beginning of Sect. 6.2; e_m, κ_m, κ_m^r, L : (38); $S_{k,r}$ and $\mathcal{P}$: Definition 28; H, $\mathcal{E}$, Π, $\mathcal{F}$: Definition 30; $\mathcal{F}_k$: Corollary 33.

6.1 Decomposition of Cauchy–Fantapié Indicators

This section specifies background notation for Section 6 and recall a result of Dolbeault and Henkin which gives a decomposition of the Cauchy–Fantapié related to intersections of the lines L_z with the nodal Riemann surface Q to be reconstructed.

Without loss of generality, we suppose that $bQ \subset \{w_0 w_1 w_2 \neq 0\}$. From now, we also assume the generic hypothesis and so little restrictive, that

$$(0:0:1)\,,(0:1:0) \notin Q^\infty = Q \pitchfork \{w_0 = 0\} \subset \operatorname{Reg} Q$$

where $\pitchfork$ denotes a transverse intersection. In this situation, $u_0 = \frac{w_0}{w_2}$ can be taken as a coordinate for Q in a neighborhood of points of Q^∞ and there exists for each $q \in Q^\infty$ a function g^q holomorphic near 0 in $\mathbb{C}$ such that in a neighborhood of q in $\mathbb{CP}_2$, Q coincide with $\{(u_0 : u_1 : 1)\,;\ u_1 = g^q(u_0)\}$. We note then $(\Sigma g_\nu^q u_0^\nu)$ the Taylor expansion of g^q at 0. So, for $q \in Q^\infty$,

$$q = \left(0 : g_0^q : 1\right) \overset{def}{=} \left(0 : b^q : 1\right).$$

We also set

$$E^\infty = \mathbb{C} \times \left\{-1/b^q;\ q \in Q^\infty\right\}.$$

In this section, U is the open subset of $\mathbb{C}^2$ where the G_k are defined. For any subset X of U, we denote X_{reg} the subset of $\mathbb{C}^2$ made by points $z = (x, y)$ of X such that Q and $L_z = \{w \in \mathbb{CP}_2;\ x w_0 + y w_1 + w_2 = 0\}$ meet transversely at each point of $Q \cap L_z$; we set $X_{\text{sing}} = X \backslash X_{\text{reg}}$ so that U_{sing} is an analytic subset of U.

Though U may be complicated, it contains a convenient open subset. Let us define

$$\rho = \max\left(\max_{w\in bQ} |w_2/w_1|\,, 5\frac{\max\limits_{w\in bQ} |w_2/w_0|}{\min\limits_{w\in bQ} |w_1/w_0|}\right)\,,\quad \widetilde{\rho} = \max\left\{\rho,\ \left|1/b^q\right|;\ q \in Q^\infty\right\} \tag{23}$$

and pick a real α such that $0 < \alpha < \frac{1}{4}\min\limits_{w\in bQ} |w_1/w_0|$. Then the sets defined below are contained in U and play a crucial role :

$$\begin{aligned} Z &= \left\{(x,y) \in \mathbb{C}^2;\ \rho < |y|\ \&\ |x| < \alpha\,|y|\right\}\ \ \&\ \ Z^+ = Z \backslash (\mathbb{C} \times \mathbb{R}_-) \\ \widetilde{Z} &= \left\{(x,y) \in \mathbb{C}^2;\ \widetilde{\rho} < |y|\ \&\ |x| < \alpha\,|y|\right\}\ \ \&\ \ \widetilde{Z}^+ = \widetilde{Z} \backslash (\mathbb{C} \times \mathbb{R}_-) \end{aligned} \tag{24}$$

Remark The hypothesis $(0:1:0), (0:0:1) \notin Q$ (which ensures $Q^\infty \subset \{w_1 w_2 \neq 0\}$) and $Q^\infty \subset \operatorname{Reg} Q$ simplifies some statements and calculus but are not all mandatory. We indicate for some formulas a version for the case $Q^\infty \cap \operatorname{Sing} Q \neq \varnothing$.

The lemma below ensures that the reconstruction process initiated by Proposition 21 ends to a complete knowledge of Q ; thorough this paper $\mathbb{D}$, is the unit open disk of $\mathbb{C}$.

Lemma 20 *For all $w_* \in Q \cap \{w_0 \neq 0\}$ and all $R \in \mathbb{R}_+^*$, there exists $z \in U_{\mathrm{reg}} \cap \left(\mathbb{C} \times \mathbb{C} \backslash R\overline{\mathbb{D}}\right)$ such that $w_* \in L_z$.*

Proof Let $R \in [\widetilde{\rho}, +\infty[$ and $w_* \in Q$ such that $w_{*0} \neq 0$. Set $\zeta_* = \left(\frac{w_{*1}}{w_{*0}}, \frac{w_{*2}}{w_{*0}}\right)$. The points $z = (x, y)$ of $\mathbb{C}^2$ such that $w_* \in L_z$ form the line $L_{w_*}^*$ of equation $x + y\zeta_{*1} + \zeta_{*2} = 0$. If $L_{w_*}^*(R) = L_{w_*}^* \cap \left(\mathbb{C} \times \mathbb{C} \backslash R\overline{\mathbb{D}}\right)$ does not meet U, for all $y \in \mathbb{C} \backslash R\overline{\mathbb{D}}$, there exists in bQ an element $w = (1:\zeta_1:\zeta_2)$ which is also in $L_{(-y\zeta_{*1}-\zeta_{*2}, y)}$ so that $y = -\frac{\zeta_{*2}-\zeta_2}{\zeta_{*1}-\zeta_1}$. Given that bQ is a real curve, $\mathbb{C} \backslash R\overline{\mathbb{D}}$ cannot be contained in the image of bQ by $\zeta \mapsto -\frac{\zeta_{*2}-\zeta_2}{\zeta_{*1}-\zeta_1}$. Hence, $L_{w_*}^*(R) \cap U$ is a non-empty open subset of $L_{w_*}^*$.

Cover $Q \cap \{w_0 \neq 0\}$ by a locally finite family $\mathcal{B}$ of branches of Q. For each $B \in \mathcal{B}$, we pick a function f holomorphic in an open subset V_B of $\mathbb{C}^2$ such that

$$B = \{(1:\zeta_1:\zeta_2)\,;\; (\zeta_1, \zeta_2) \in V_B \;\&\; f_B(\zeta_1, \zeta_2) = 0\}$$

and df_B does not vanish in B. Denote $E(R)$ the set of points $z \in L_{w_*}^*(R)$ such that L_z and Q are tangential at some point of $L_z \cap Q$. A point $z = (x, y) \in \mathbb{C}^2$ belongs to $E(R)$ when $|y| > R$ and there exists $B \in \mathcal{B}$ and $\zeta \in V_B$ verifying the conditions

$$\begin{gathered} f_B(\zeta) = 0, \quad x + y\zeta_{*1} + \zeta_{*2} = 0, \quad x + y\zeta_1 + \zeta_2 = 0, \\ \frac{\partial f_B}{\partial \zeta_2}(\zeta) \neq 0, \quad y = \frac{\partial f_B/\partial \zeta_1}{\partial f_B/\partial \zeta_2}(\zeta), \quad x = -\frac{\partial f_B/\partial \zeta_1}{\partial f_B/\partial \zeta_2}(\zeta)\,\zeta_{*1} - \zeta_{*2} \end{gathered}$$

When $\zeta \neq \zeta_*$, this forces $\zeta_{*1} \neq \zeta_1$ and $-\frac{\partial f_B/\partial \zeta_1}{\partial f_B/\partial \zeta_2}(\zeta) = \frac{\zeta_{*2}-\zeta_2}{\zeta_{*1}-\zeta_1}$. The points ζ satisfying this equation form an analytic subset C_B of B. For this reason, C_B is either discrete, or equal to B.

Suppose that $C_B = B$ for an element B of $\mathcal{B}$. Then $\partial f_B/\partial \zeta_2$ does not vanish in V_B and we can find locally a holomorphic function φ such that $f_B(\zeta) = 0$ if and only if $\zeta_2 = \varphi(\zeta_1)$. The function φ verifies then $\varphi'(\zeta_1) + \frac{1}{\zeta_{*1}-\zeta_1}\varphi(\zeta_1) = \frac{\zeta_{*2}}{\zeta_{*1}-\zeta_1}$, that is $\left(\frac{1}{\zeta_{*1}-\zeta_1}\varphi(\zeta_1)\right)' = \left(\frac{\zeta_{*2}}{\zeta_{*1}-\zeta_1}\right)'$. Hence $\varphi(\zeta_1) = (\zeta_1 - \zeta_{*1})\,c + \zeta_{*2}$ where c is a constant. In this case, B is an open subset of the line defined by the equation $\zeta_2 = (\zeta_1 - \zeta_{*1})\,c + \zeta_{*2}$. Since Q is connected and has only nodal singularities, this implies that Q itself lies in this line. It suffices then to pick any y sufficiently large to get that $L_{(-y\zeta_{*1}-\zeta_{*2}, y)}$ meets Q only not tangentially. When C_B is a discrete subset of B, the set $E(R, B)$ of elements z in $L_{w_*}^*(R)$ such that L_z are B are tangential at some point of $L_z \cap B$ is contained, because of the above relations, in a discrete set. Since $\mathcal{B}$ is locally finite, the study of these two cases shows that $L^*_{w_*}(R)$ meets $U_{\text{reg}} \cap \left(\mathbb{C} \times \mathbb{C} \backslash R\overline{\mathbb{D}}\right)$. □

The starting point of all this section is Proposition 21 about the Cauchy–Fantapié indicators of Q defined by (4). This result can be extracted as a particular case from Theorem II and Lemma 4.2.2 obtained by Dolbeault and Henkin in [10]; their proof applies without change when some knots of $\overline{Q}$ are in Q^∞. In this statement and after, we use the following notation when $h_1, \ldots, h_p$ are complex-valued functions and $k \in \mathbb{N}$,

$$N_{h,k} = \sum_{1 \leqslant j \leqslant p} h_j^k \;\;\&\;\; S_{h,k} = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant k} h_{j_1} \cdots h_{j_k}. \tag{25}$$

The Newton identities state that for all $k \in \mathbb{N}^*$,

$$N_{h,k} = (-1)^{k-1} k S_{h,k} + \sum_{1 \leqslant j \leqslant k-1} (-1)^{k-j-1} S_{h,j} N_{h,k-j} \tag{26}$$

$$S_{h,k} = \frac{(-1)^{k-1}}{k} N_{h,k} + \frac{1}{k} \sum_{1 \leqslant j \leqslant k-1} (-1)^{j-1} S_{h,j} N_{h,k-j} \tag{27}$$

We denote $\mathbb{C}[X, Y)$ the set of elements of $\mathbb{C}(X, Y)$ which are polynomials in X. $\mathbb{C}_k[X, Y) = \mathbb{C}(Y)_k[X]$ denotes the ring of polynomials in X of degree at most k whose coefficients are algebraic fractions in Y. A shock wave is by definition a holomorphic function h on an open subset of $\mathbb{C}^2$ such that in the standard coordinates system (x, y)

$$\frac{\partial h}{\partial y} = h \frac{\partial h}{\partial x} \tag{28}$$

Proposition 21 (Dolbeault–Henkin, 1997) *Let $z_* \in U_{\text{reg}} \backslash E^\infty$ and $p = \text{Card}\left(L_{z_*} \cap Q\right)$. If U_* is a sufficiently small neighborhood of z_* in U_{reg}, there exists shock waves $h_1, \ldots, h_p$ on U_* whose images are mutually disjoint such that for all $z \in U_*$,*

$$L_z \cap Q = \left\{\left(1 : h_j(z) : -x - y h_j(z)\right);\; 1 \leqslant j \leqslant p\right\}.$$

Moreover, for all $k \in \mathbb{N}$, there exists $P_k \in \mathbb{C}_k[X, Y)$ such that for all $z \in U_*$

$$G_k(z) = N_{h,k}(z) + P_k(z). \tag{29}$$

In addition, η denoting the natural injection of Q in $\mathbb{CP}_2$, $P_k = \sum_{q \in Q^\infty} \text{Res}\left(\eta^* \Omega_z^k, q\right)$ and

$$\frac{\partial P_k}{\partial Y} = \frac{k}{k+1} \frac{\partial P_{k+1}}{\partial X}.$$

In practical terms, the difficulty to extract from the Eqs. (29) the symmetric functions of the h_j comes from the polynomials P_k. [1] contains a method when $q^\infty \in \{1, 2\}$. For the one proposed in this paper, the first step is to get precision on (P_k).

Lemma 22 *$P_0 = -q^\infty$ where $q^\infty = \text{Card}\, Q^\infty$ and setting $P_k = \sum_{0 \leqslant \nu \leqslant k} X^\nu \otimes p_{k,\nu}$ when $k \in \mathbb{N}^*$,*

$$p_{k,k} = \frac{1}{(k-1)!} p_{1,1}^{(k-1)} \;\; \& \;\; p_{k,\nu} = \frac{k}{\nu!\,(k-\nu)} p_{k-\nu,0}^{(\nu)}, \quad \nu \in \{0, \ldots, k-1\} \tag{30}$$

Moreover, if we set

$$B^\infty = \prod_{q \in Q^\infty} \left(1 + Yb^q\right) \tag{31}$$

Then

$$p_{1,1} = \sum_{q \in Q^\infty} \frac{b^q}{1 + Yb^q} = \frac{B^{\infty\prime}}{B^\infty} \tag{32}$$

$$p_{1,0} = -\sum_{q \in Q^\infty} \frac{g_1^q}{1 + Yb^q} \overset{def}{=} \frac{A^\infty}{B^\infty} \tag{33}$$

$$p_{k,0} = \sum_{j=1}^{k} \sum_{q \in Q^\infty} \frac{p_{k,0,j}^q}{(1 + Yb^q)^j} = \sum_{j=1}^{k} \frac{p_{k,0,j}}{(B^\infty)^j}, \; k \in \mathbb{N} \tag{34}$$

where the $p_{k,0,j}^q$ are universal polynomials in the coefficients of the jet of order $k-j+1$ of Q at q and $p_{k,0,j} = \sum_q p_{k,0,j}^q \prod_{q' \neq q} \left(1 + Yb^{q'}\right)^j$. In particular, P_k does not depend on z_* and is entirely determined by the $k\,(q^\infty + 1)$ numbers b^q, $p_{k,0,j}^q$, $(q, j) \in Q^\infty \times \{1, \ldots, k\}$.

Furthermore, P_k admits a Laurent series expansion of the form $\sum_{m \leqslant -1} P_{k,m} \otimes Y^m$ where $P_{k,m} \in \mathbb{C}_{k-1}[X]$ when $-1 \geqslant m > -k$ and $P_{k,m} \in \mathbb{C}_k[X]$ when $-k \geqslant m$.

Remark In the case where $Q^\infty \cap \text{Sing}\, Q \neq \varnothing$, Formula (31) becomes $B^\infty = \prod_{q \in Q^\infty} (1 + Yb^q)^{\nu(q)}$ where $\nu(q)$ denotes the number of branches of Q at q, (32) stay unchanged and in (33), g_1^q has to replaced by $\sum_B g_1^{B,q}$ where the sum is done on a complete set of inner branches of $\overline{Q}$ at q and $g_1^{B,q} = \left(g^B\right)'(0)$, g^B denoting the holomorphic function such that in a neighborhood of 0, an equation of the branch B is $u_1 = g^B(u_0)$.

Proof Suppose that (30) is verified for a positive integer k. Then

$$\begin{aligned} P_{k+1} &= P_{k+1}(0,Y) + \frac{k+1}{k}\left(\sum_{0\leqslant m\leqslant k-1} p'_{k,m}\frac{X^{m+1}}{m+1} + p'_{k,k}\frac{X^{k+1}}{k+1}\right) \\ &= p_{k+1,0} + \sum_{0\leqslant m\leqslant k-1}\frac{k+1}{(m+1)!\,(k-m)}p^{(m+1)}_{k-m,0}X^{m+1} + \frac{1}{k!}p^{(k)}_{1,1}X^{k+1} \\ &= p_{k+1,0} + \sum_{1\leqslant m\leqslant k}\frac{k+1}{(m+1)!\,(k+1-m)}p^{(m)}_{k+1-m,0}X^{m} + \frac{1}{k!}p^{(k)}_{1,1} \end{aligned}$$

which proves (30) with a recurrence.

Let now $k\in\mathbb{N}$ and $z=(x,y)\in U\backslash E^{\infty}$. In the affine coordinates $(u_0,u_1)=\left(\frac{w_0}{w_2},\frac{w_1}{w_2}\right)$ of $\mathbb{CP}_2$, Ω_z^k has the form

$$\Omega_z^k = \left(\frac{u_1}{u_0}\right)^k \frac{d\frac{xu_0+yu_1+1}{u_0}}{\frac{xu_0+yu_1+1}{u_0}} = \left(\frac{u_1}{u_0}\right)^k\left(\frac{x\mathrm{d}u_0+y\mathrm{d}u_1}{xu_0+yu_1+1}-\frac{\mathrm{d}u_0}{u_0}\right).$$

We fix a point q in Q^{∞} and in order to simplify the scripts, we write g instead of g^q (an so, g_ν stands for g_ν^q) and u in place of u_0. In a neighborhood of q in Q, the form $\eta^*\Omega_z^k$ written in the coordinate u is

$$\eta^*\Omega_z^k = \left(\frac{(x+yg')\,g^k}{u^k\,(1+xu+yg)} - \frac{g^k}{u^{k+1}}\right)\mathrm{d}u.$$

Denoting by $\langle f,u^\nu\rangle$ the coefficient of u^ν in the Taylor expansion at 0 of a function f holomorphic in a neighborhood of 0, one gets

$$P_k^q(z) \overset{def}{=} \mathrm{Res}\left(\eta^*\Omega_z^k,q\right) = \mathrm{Res}\left(\frac{(x+yg')\,g^k}{(1+xu+yg)\,u^k},0\right) - \left\langle g^k,u^k\right\rangle.$$

In particular $P_0^q(z)=-1$ and hence $P_0=-\mathrm{Card}\,Q^{\infty}$. Suppose now $k\geqslant 1$. Then

$$\frac{yg'g^k}{1+xu+yg} = \left(1-\frac{1+xu}{1+xu+yg}\right)g'g^{k-1}$$

and if $g'g^{k-1}=\sum_{n\in\mathbb{N}}\alpha_n u^n$, $\frac{1}{k}\left(g^k-g_0^k\right)=\sum_{n\in\mathbb{N}^*}\frac{\alpha_{n-1}}{n}u^n$, which gives

$$\mathrm{Res}\left(\frac{g^k}{u^{k+1}}\mathrm{d}u,0\right) = k\frac{\alpha_{k-1}}{k} = \mathrm{Res}\left(\frac{g'g^{k-1}}{u^k}\mathrm{d}u,0\right)$$

This entails,

$$P_k^q(z) = \text{Res}\left(\frac{xg^k}{(1+xu+yg)\,u^k},0\right) - \text{Res}\left(\frac{1+xu}{(1+xu+yg)\,u^k}g'g^{k-1},0\right)$$
$$= \text{Res}\left(\frac{x\left(g-ug'\right)-g'}{(1+xu+yg)\,u^k}g^{k-1},0\right).$$

Since $g-g_0=O(u)$ and $(x,y)\notin E^\infty$, $1+yg_0\neq 0$ and it comes furthermore that for u small enough

$$\frac{1}{1+xu+yg} = \frac{(1+yg_0)^{-1}}{1+\frac{xu+y(g-g_0)}{1+yg_0}} = \sum_{n\in\mathbb{N}^*}\frac{(-1)^{n-1}}{(1+yg_0)^n}\left[xu+y\,(g-g_0)\right]^{n-1}.$$

Burt for all $n\in\mathbb{N}^*$

$$\left[x\left(g-ug'\right)-g'\right]g^{k-1}\left[xu+y\,(g-g_0)\right]^{n-1}$$
$$=\sum_{m=0}^{n-1}C_{n-1}^m g^{k-1}\begin{pmatrix}x^{m+1}y^{n-1-m}\left(g-ug'\right)(g-g_0)^{n-1-m}\,u^m\\ -g'x^my^{n-1-m}\,(g-g_0)^{n-1-m}\,u^m\end{pmatrix}$$
$$=\begin{matrix}\sum_{m=1}^{n}C_{n-1}^{m-1}x^my^{n-m}g^{k-1}\left(g-ug'\right)(g-g_0)^{n-m}\,u^{m-1}\\ -\sum_{m=0}^{n-1}C_{n-1}^{m}x^my^{n-1-m}g'g^{k-1}\,(g-g_0)^{n-1-m}\,u^m\end{matrix}$$
$$=\begin{matrix}-y^{n-1}g'g^{k-1}\,(g-g_0)^{n-1}+x^ng^{k-1}\left(g-ug'\right)u^{n-1}\\ +\sum_{m=1}^{n-1}x^my^{n-1-m}\begin{pmatrix}yC_{n-1}^{m-1}g^{k-1}\left(g-ug'\right)(g-g_0)\\ -C_{n-1}^mg'g^{k-1}u\end{pmatrix}(g-g_0)^{n-1-m}\,u^{m-1}\end{matrix}$$

So,

$$P_k^q(z) = -\sum_{n=1}^{k}\frac{(-1)^n}{(1+yg_0)^n}$$
$$\begin{pmatrix}-y^{n-1}\left\langle g'g^{k-1}\,(g-g_0)^{n-1},u^{k-1}\right\rangle + x^n\left\langle g^{k-1}\left(g-ug'\right),u^{k-n}\right\rangle\\ +\sum_{m=1}^{n-1}x^my^{n-1-m}\left\langle\begin{pmatrix}yC_{n-1}^{m-1}g^{k-1}\left(g-ug'\right)(g-g_0)\\ -C_{n-1}^mg'g^{k-1}u\end{pmatrix}(g-g_0)^{n-1-m},u^{k-m}\right\rangle\end{pmatrix}$$

Hence $P_k^q(z)=\sum_{m=0}^k p_{k,m}^q(y)\,x^m$ with

$$p_{k,0}^q = \sum_{n=1}^k\frac{(-1)^nY^{n-1}}{(1+Yg_0)^n}\left\langle g'g^{k-1}\,(g-g_0)^{n-1},u^{k-1}\right\rangle,\quad p_{k,k}^q=\frac{(-1)^{k+1}g_0^k}{(1+Yg_0)^k}$$

and for $1 \leqslant m \leqslant k-1$,

$$p^q_{k,m} = -\sum_{n=m+1}^{k} \frac{(-1)^n\, Y^{n-1-m}}{(1+Yg_0)^n}$$
$$\left\langle \left(Y C^{m-1}_{n-1} g^{k-1}\left(g - ug'\right)(g-g_0) + C^m_{n-1} g' g^{k-1} u \right)(g-g_0)^{n-1-m}, u^{k-m} \right\rangle$$

In particular,

$$p^q_{1,0} = \frac{-g_1}{1+Yg_0} \;\;\&\;\; p^q_{1,1} = \frac{g_0}{1+Yg_0}.$$

Furthermore, for all $n \in \mathbb{N}$,

$$\begin{aligned} \frac{(-1)^n\, Y^{n-1}}{(1+Yg_0)^n} &= \frac{(-1)^n}{g_0^{n-1}(1+Yg_0)}\left(1 - \frac{1}{1+Yg_0}\right)^{n-1} \\ &= (-1)^n \sum_{j=1}^{n} \frac{(-1)^{j-1}\, C^{j-1}_{n-1} g_0^{-(n-1)}}{(1+Yg_0)^j}, \end{aligned}$$

Hence

$$\begin{aligned} p^q_{k,0} &= \sum_{j=1}^{k} \frac{(-1)^{j-1}}{(1+Yg_0)^j} \sum_{n=j}^{k} \frac{(-1)^n\, C^{j-1}_{n-1}}{g_0^{n-1}} \left\langle g' g^{k-1}(g-g_0)^{n-1}, u^{k-1} \right\rangle \\ &= \frac{-g_1^k}{(1+Yg_0)^k} + \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{(1+Yg_0)^j} \sum_{n=j}^{k} \frac{(-1)^n\, C^{j-1}_{n-1}}{g_0^{n-1}} \left\langle g' g^{k-1}(g-g_0)^{n-1}, u^{k-1} \right\rangle \end{aligned}$$

Note that $\left\langle g' g^{k-1}(g-g_0)^{k-1}, u^{k-1} \right\rangle = g_1 g_0^{k-1} g_1^{k-1} = g_1^k g_0^{k-1}$ and

$$\begin{aligned} &\left\langle g' g^{k-1}(g-g_0)^{k-2}, u^{k-1} \right\rangle \\ &\quad = \left(g_1 + 2g_2 u + O\left(u^2\right) \right) \\ &\qquad \left(g_0 + g_1 u + O\left(u^2\right) \right)^{k-1} \left(g_1 u + g_2 u^2 + O\left(u^3\right) \right)^{k-2} \\ &\quad = (g_1 + 2g_2 u)\left(g_0^{k-1} + (k-1)\, g_0^{k-2} g_1 u \right) \\ &\qquad \left(g_1^{k-2} u^{k-2} + (k-2)\, g_1^{k-3} g_2 u^{k-1} \right) + O\left(u^k\right) \\ &\quad = \left(g_1 g_0^{k-1} + \left(2g_2 g_0^{k-1} + (k-1)\, g_0^{k-2} g_1^2 \right) u \right) \\ &\qquad \left(g_1^{k-2} u^{k-2} + (k-2)\, g_1^{k-3} g_2 u^{k-1} \right) + O\left(u^k\right) \\ &\quad = g_1^{k-1} g_0^{k-1} u^{k-2} \end{aligned}$$

$$
\begin{aligned}
&+\left[g_1 g_0^{k-1}(k-2)\, g_1^{k-3} g_2+\left[2 g_2 g_0^{k-1}+(k-1)\, g_0^{k-2} g_1^2\right] g_1^{k-2}\right] u^{k-1} \\
&+O\left(u^k\right) \\
=\; & g_1^{k-1} g_0^{k-1} u^{k-2}+\left[k g_0^{k-1} g_1^{k-2} g_2+(k-1)\, g_0^{k-2} g_1^k\right] u^{k-1} \\
&+O\left(u^k\right)
\end{aligned}
$$

which gives

$$
\frac{(-1)^k\, C_{k-1}^{k-1}}{g_0^{k-1}}\left\langle g' g^k\,(g-g_0)^{n-1}, u^{k-1}\right\rangle=(-1)^k\, g_1^k
$$

and

$$
\begin{aligned}
&\sum_{n=k-1}^{k} \frac{(-1)^n\, C_{n-1}^{k-2}}{g_0^{n-1}}\left\langle g' g^k\,(g-g_0)^{n-1}, u^{k-1}\right\rangle \\
&=\frac{(-1)^k\,(k-1)}{g_0^{k-1}}\left\langle g' g^k\,(g-g_0)^{k-1}, u^{k-1}\right\rangle+\frac{(-1)^{k-1}}{g_0^{k-2}}\left\langle g' g^k\,(g-g_0)^{k-2}, u^{k-1}\right\rangle \\
&=(-1)^k\,(k-1)\, g_1^k+\frac{(-1)^{k-1}}{g_0^{k-2}}\left[k g_0^{k-1} g_1^{k-2} g_2+(k-1)\, g_0^{k-2} g_1^k\right] \\
&=(-1)^k\left((k-1)\, g_1^k-\left[k g_0 g_1^{k-2} g_2+(k-1)\, g_1^k\right]\right)=-k\,(-1)^k\, g_0 g_1^{k-2} g_2
\end{aligned}
$$

So,

$$
p_{k,0}^q=\frac{-g_1^k}{(1+Y g_0)^k}+\frac{-k g_0 g_1^{k-2} g_2}{(1+Y g_0)^{k-1}}+\sum_{j=1}^{k-2} \frac{p_{k,0,j}^q}{(1+Y g_0)^j}
$$

with

$$
p_{k,0,j}^q=(-1)^{j-1} \sum_{n=j}^{k} \frac{(-1)^n\, C_{n-1}^{j-1}}{g_0^{n-1}}\left\langle g' g^{k-1}\,(g-g_0)^{n-1}, u^{k-1}\right\rangle
$$

Summing on the elements q of Q^{∞} the above equalities, we get the relations claimed in the statement.

Writing the Laurent series at infinity of $p_{k,\nu}$, $0 \leqslant \nu \leqslant k$, in the form $\sum_{m\leqslant -1}\left\langle p_{k,\nu} Y^m\right\rangle Y^m$, we get $P_k=\sum_{m\leqslant -1} P_{k,m}\otimes Y^m$ and $P_{k,m}=\sum_{0\leqslant\nu\leqslant k}\left\langle p_{k,\nu}, Y^m\right\rangle X^{\nu}$ for any m. Since (30) implies $\left\langle p_{k,k}, Y^m\right\rangle=0$ when $m>-k$, we obtain that $P_{k,m}=\sum_{0\leqslant j<k}\left\langle p_{k,j}^m, Y^m\right\rangle X^j \in \mathbb{C}_{k-1}[X]$ for $-1\geqslant m>-k$ and that $P_{k,m}=\sum_{0\leqslant j\leqslant k}\left\langle p_{k,j}^m, Y^m\right\rangle X^j \in \mathbb{C}_k[X]$ for $-k\geqslant m$. □

6.2 Expansion of Indicators

The form of fractions P_k given by Lemma 22 suggests to study the functions G_k on the domain Z defined by (24). In this section and after, $(\partial Q)_0$ stands for the real orientated curve of $\mathbb{C}^2$ which is the image of ∂Q by the coordinates map $w \mapsto \left(\frac{w_1}{w_0}, \frac{w_2}{w_0}\right)$.

Lemma 23 *We note δ the integer $\frac{1}{2\pi i}\int_{\partial Q} \frac{d(w_1/w_0)}{w_1/w_0}$. G_0 is constant on Z and for all $k \in \mathbb{N}^*$, G_k admits on Z a Laurent expansion of the form*

$$G_k(x, y) = \sum_{n\in\mathbb{N}^*} \frac{G_{k,-n}(x)}{y^n} = (-1)^k \delta \frac{x^k}{y^k} + \sum_{n\in\mathbb{N}^*} \frac{\widetilde{G}_{k,-n}(x)}{y^n} \tag{35}$$

with normal convergence on Z and where for all $n \in \mathbb{N}^*$, $\widetilde{G}_{k,-n} = \sum_{0\leqslant\nu<n} G^{\nu}_{k,-n} X^{\nu}$ is a polynomial of degree at most $n-1$. In particular, $G_{k,0} = \delta_{k,0}\delta$, $G_{k,-n} = \delta_{k,n}(-1)^n \delta X^n + \widetilde{G}_{k,-n} \in \mathbb{C}_{n-1+\delta_{k,n}}[X]$ and

$$G_1(x, y) = \frac{G^0_{1,-1} - \delta x}{y} + \sum_{n\geqslant 2} \frac{G_{1,-n}(x)}{y^n} \tag{36}$$

with $G^0_{1,-1} = \frac{-1}{2\pi i}\int_{\partial Q} \frac{w_2}{w_1} d\frac{w_1}{w_0}$.

Proof Fix k in $\mathbb{N}^*$. Let $(x, y) \in Z$. Then for all $(z_1, z_2) \in (\partial Q)_0$, $\left|\frac{x+z_2}{yz_1}\right| < \frac{1}{2}$ since by definition of ρ, $|x + z_2| \leqslant \alpha |y| + \max_{(\zeta_1,\zeta_2)\in(\partial Q)_0} |\zeta_2| < \frac{1}{2}|y| \min_{(\zeta_1,\zeta_2)\in(\partial Q)_0} |\zeta_1| \leqslant \frac{1}{2}|yz_1|$. Hence

$$\begin{aligned}
G_k(x, y) &= \frac{1}{2\pi i}\int_{(\partial Q)_0} z_1^{k-1} \mathrm{d}z_1 + \frac{1}{2\pi i}\int_{(\partial Q)_0} z_1^{k-1} \frac{z_1 \mathrm{d}z_2 - (x+z_2)\,\mathrm{d}z_1}{x + yz_1 + z_2} \\
&= 0 + \frac{1}{2\pi i}\int_{(\partial Q)_0} \frac{z_1^{k-2}}{y} \frac{z_1 \mathrm{d}z_2 - (x+z_2)\,\mathrm{d}z_1}{1 + \frac{x+z_2}{yz_1}} \\
&= \frac{1}{2\pi i}\int_{(\partial Q)_0} \sum_{\nu\in\mathbb{N}} \frac{(-1)^{\nu} z_1^{k-2-\nu}}{y^{\nu+1}} (x+z_2)^{\nu} \left(z_1 \mathrm{d}z_2 - (x+z_2)\,\mathrm{d}z_1\right) \\
&= \sum_{n\in\mathbb{N}^*} \frac{G_{k,-n}(x)}{y^n}
\end{aligned}$$

with normal convergence on Z and for any $n \in \mathbb{N}^*$

$$G_{k,-n}(x) = \frac{(-1)^{n-1}}{2\pi i}\int_{(\partial Q)_0} z_1^{k-n-1} (x+z_2)^{n-1} \left(z_1 \mathrm{d}z_2 - (x+z_2)\,\mathrm{d}z_1\right).$$

Hence, $G_{k,-n}$ is a polynomial of degree at most n. Let us write it $\sum_{0\leqslant\nu\leqslant n} G^{\nu}_{k,-n}X^n$. The coefficient $G^n_{k,-n}$ of X^n in $G_{k,-n}$ is given by the formula

$$G^n_{k,-n} = \frac{(-1)^n}{2\pi i}\int_{(\partial Q)_0} z_1^{k-n-1}\mathrm{d}z_1 = \delta_{k,n}(-1)^n\,\delta.$$

With $\widetilde{G}_{k,-n} = \sum_{0\leqslant\nu<n} G^{\nu}_{k,-n}X^{\nu}$, we get

$$G_k(x,y) = \sum_{n\in\mathbb{N}^*}\frac{\delta_{k,n}(-1)^n\,\delta x^n + \widetilde{G}_{k,-n}(x)}{y^n} = (-1)^k\,\delta\frac{x^k}{y^k} + \sum_{n\in\mathbb{N}^*}\frac{\widetilde{G}_{k,-n}(x)}{y^n}$$

Besides,

$$G_{1,-1}(x) = \frac{1}{2\pi i}\int_{(\partial Q)_0} z_1^{-1}\left(z_1\mathrm{d}z_2 - (x+z_2)\,\mathrm{d}z_1\right) = G^0_{1,-1} + xG^1_{1,-1}$$

with $G^0_{1,-1} = 0 + \frac{-1}{2\pi i}\int_{w\in(\partial Q)_0}\frac{w_2}{w_1}d\frac{w_1}{w_0}$ and $G^1_{1,-1} = \frac{-1}{2\pi i}\int_{(\partial Q)_0} z_1^{-1}\mathrm{d}z_1 = -\delta$.

By definition, G_0 is the function $U \ni (x,y) \mapsto \frac{1}{2\pi i}\int_{\partial Q}\frac{d[(xw_0+yw_1+w_2)/w_0)]}{xw_0+yw_1+w_2/w_0}$. Hence, it is continuous and integer valued. So it is constant on Z and equal to its limit value when $x=0$ and $y\to\infty$, that is δ. Thus, $G_{0,-n}=0$ for all $n\in\mathbb{N}^*$. □

Corollary 24 *The number p of functions $h_1,\ldots,h_p$ involved in Proposition 21 is the same for all points of $Z_{\mathrm{reg}}\backslash E^{\infty}$: $p=\delta+q^{\infty}$ where $q^{\infty} = \mathrm{Card}\ Q^{\infty}$.*

Proof Denote temporarily $p(z)$ the number of functions $h_1,\ldots,h_{p(z)}$ involved in Proposition 21 when $z\in U_{\mathrm{reg}}$. Since $P_0=-q^{\infty}$, we know that $G_0(z) = p(z)-q^{\infty}$ and so that p is an integer valued function continuous on the connected set $Z_{\mathrm{reg}}\backslash E^{\infty}$. It is thus constant and since $G_0(x,y) = \delta + \sum_{m\in\mathbb{N}^*}\frac{G_{0,m}(x)}{y^m}$ when $(x,y)\in Z_{\mathrm{reg}}$, we conclude that $\delta = p-q^{\infty}$. □

Remark In the case where $Q^{\infty}\cap\mathrm{Sing}\,Q\neq\varnothing$, $q^{\infty} = \sum_{q\in Q^{\infty}}\nu(q)$. Corollary 45 of Sect. 7 gives a formula linking q^{∞} and the genus of Q via the Dirichlet–Neumann operator.

Corollary 25 *Notation and hypothesis remains as stated in Proposition 21. For all $k\in\mathbb{N}^*$, $N_{h,k}$ extends to $Z\backslash E^{\infty}$ as a holomorphic function N^Q_k which does not depend of z_* and which expands in Laurent series on $\widetilde{Z}$ in the form $N^Q_k(x,y) = \sum_{n\in\mathbb{N}^*}\frac{N^Q_{k,n}(x)}{y^n}$ where the $N^Q_{k,n}$ are polynomials of degree at most n. Moreover, for all $z\in Z_{\mathrm{reg}}$, there exists shock waves $h^z_1,\ldots,h^z_p$ whose images are mutually distinct and such for z' sufficiently close to z, $\left(N^Q_k(z')\right)_{k\in\mathbb{N}} = \left(N_{h^z,k}(z')\right)_{k\in\mathbb{N}}$ and $L_{z'}\cap Q = \left\{\left(1:h_j(z'):-x-yh_j(z')\right);\ 1\leqslant j\leqslant p\right\}$.*

Proof Let $k\in\mathbb{N}$. We know that $N_{h,k} = G_k - P_k$ on U_* and thanks to Lemma 22 that P_k is an algebraic fraction which does not depend on z_* and which is defined on $Z\backslash E^{\infty}$. Hence, $N_k^Q = G_k - P_k$ extends $N_{h,k}$ as a holomorphic function on Z. Applying Proposition 21 and Corollary 24 with an arbitrary point z of $Z_{\mathrm{reg}}\backslash E^{\infty}$, we obtain shock waves $h_1^z, \ldots, h_p^z$ with the claimed properties. Furthermore, Lemma 22 also gives that

$$P_k = \sum_{0\leqslant \nu \leqslant k} p_{k,\nu} X^{\nu} = \frac{1}{(k-1)!} p_{1,1}^{(k-1)} X^k + \sum_{0\leqslant \nu w<k} \frac{k}{\nu!\,(k-\nu)} p_{k-\nu,0}^{(\nu)} X^{\nu}$$

with $p_{1,1} = \sum_{q\in Q^{\infty}} \frac{b^q}{1+Yb^q}$ and $p_{\nu,0} = \sum_{j=1}^{\nu} \sum\limits_{q\in Q^{\infty}} \frac{p_{\nu,0,j}^q}{(1+Yb^q)^j}$. For $|y| > \widetilde{\rho}$, one get

$$p_{1,1}(y) = \sum_{n\in\mathbb{N}^*} \frac{(-1)^{n-1}}{y} \sum_{q\in Q^{\infty}} \left(b^q\right)^{n-1} = \sum_{n\in\mathbb{N}^*} \frac{(-1)^{n-1} S_{b,n-1}}{y^n}$$

$$p_{\nu,0}(y) = \sum_{j=1}^{\nu} \sum_{n\in\mathbb{N}^*} \frac{(j-1)!\,(-1)^{n+j-1}}{y^{n+j-1}} \sum_{q\in Q^{\infty}} b^q p_{\nu,0,j}^q = \sum_{m\in\mathbb{N}^*} \frac{p_{\nu,0}^{\infty,m}}{y^m}$$

with $p_{\nu,0}^{\infty,m} = (-1)^m \sum_{(n,j)\in\mathbb{N}^*\times\{1,\ldots,\nu\},\ n+j=m+1} (j-1)! \sum_{q\in Q^{\infty}} \left(b^q\right)^{-n} p_{\nu,0,j}^q$. It suffices then to combine these formulas with Lemma 23 in order to get the announced statements. □

Corollary 26 *Notation and hypothesis remain as stated in Proposition 21. Denote by S_k^Q, $k \in \mathbb{N}^*$, the functions obtained from (26) and $\left(N_k^Q\right)_{k\in\mathbb{N}^*}$ which is defined in Corollary (25); locally the S_k^Q are the symmetric functions of the functions $h_1, \ldots, h_p$ of Proposition 21. Then for all $k \in \mathbb{N}^*$, S_k^Q expands in Laurent series on $\widetilde{Z}$.*

6.3 A Genesis of Multiple Shock Wave

Let $A, B \in \mathbb{C}[Y]$ with $\deg A < r = \deg B$, $B(0) = 1$. Define $P \in \mathbb{C}[X,Y)$ and N by

$$P(X,Y) = \frac{A(Y)}{B(Y)} + \frac{B'(Y)}{B(Y)} X \ \ \&\ \ N = G_1 - P.$$

In this section, we look for a characterization of when N is a multiple shock wave, that is a sum of shock waves. Theorem 4 of [17] gives a characterization of such sums but in this article, we use one which is more adapted to the present situation. This two characterizations correspond more or less to emphasize one of the variables x or y and rely on the following lemma whose proof is omitted since it follows easily from [17, Lemma 16] and the proof of [17, Proposition 17]

Lemma 27 (Henkin–Michel, 2007)

Let D be a domain of $\mathbb{C}^2$, $N \in \mathcal{O}(D)$ and $d \in \mathbb{N}^*$. There exists mutually distinct local shock waves $h_1, \ldots, h_d$ such that $N = h_1 + \cdots + h_d$ if and only if there exists $s_1, \ldots, s_d \in \mathcal{O}(D)$ such that $s_1 = -N$ and

$$-s_d\frac{\partial N}{\partial x}+\frac{\partial s_d}{\partial y}=0,\quad -s_k\frac{\partial N}{\partial x}+\frac{\partial s_k}{\partial y}=\frac{\partial s_{k+1}}{\partial x},\quad 1\leqslant k\leqslant d-1, \tag{37}$$

and if the discriminant of the polynomial $\Sigma = T^d + s_1T^{d-1}+\cdots+s_d \in \mathcal{O}(D)[T]$ *is not identically zero on* D. *In this case, we say that* N *is a* d*-shock waves.*

In order to define integro-differential operators adapted to the resolution of the system (37), we introduce notation linked to Laurent series and their primivitization. For $m\in\mathbb{Z}$, we set

$$\begin{aligned} &e_m:\mathbb{C}^*\ni y\mapsto(-1)^{|m|-1}(|m|-1)!\,y^m \text{ if } m\leqslant -1\\ &e_m:\mathbb{C}^*\ni y\mapsto \frac{1}{m!}y^m \text{ if } m\geqslant 0 \end{aligned}\tag{38}$$

and we denote by $\kappa_m=\frac{e_m}{e_1^m}$ the real number such that $e_m(y)=\kappa_m y^m$ for any $y\in\mathbb{C}^*$. We also make use of the notation $\kappa_m^r=\frac{\kappa_r\kappa_{m-r}}{\kappa_m}$ when $0\leqslant r\leqslant m$. The main reason of this normalization is that for any $m\in\mathbb{Z}\backslash\{-1\}$, e_{m+1} is a primitive of e_m. Note that $\kappa_1=\kappa_{-1}=1$. We denote by L the principal determination of the logarithm on $\mathbb{C}\backslash\mathbb{R}_-$.

Definition 28 For $(k,r)\in\mathbb{Z}\times\mathbb{N}$, we denote by $S_{k,r}$ the set of holomorphic functions F on Z^+ such that there exists a family $\left(c_{m,s}\right)_{m\leqslant k,\ 0\leqslant s\leqslant r}$ of entire functions such that for each $s\in\{0,\ldots,r\}$, the series $(\sum_{m\leqslant k}c_{m,s}\otimes e_m)$ is normally convergent on subsets of Z whose first projection is bounded and such that $F=\sum_{m\leqslant k,\ 0\leqslant s\leqslant r}c_{m,s}\otimes e_mL^s$ on Z^+.

We define an operator $\mathcal{P}$ on $S_{*,*}=\underset{(k,r)\in\mathbb{Z}\times\mathbb{N}}{\cup}S_{k,r}$ by setting $\mathcal{P}F=\sum_{m\leqslant k,\ 0\leqslant s\leqslant r}c_{m,s}\otimes\mathcal{P}(e_mL^s)$ when $F=\sum_{m\leqslant k,\ 0\leqslant s\leqslant r}c_{m,s}\otimes e_mL^s\in\mathcal{S}_{k,r}$, the action of $\mathcal{P}$ on e_mL^s being defined by

$$\begin{aligned} &\mathcal{P}(e_m)=e_{m+1} \text{ if } m\neq-1, \mathcal{P}e_{-1}=L\\ &\mathcal{P}(e_mL^s)=(-1)^0A_s^0a_m^0e_{m+1}L^s+\cdots+(-1)^sA_s^sa_m^se_{m+1}L^0 \text{ if } m\neq-1,\\ &\mathcal{P}(e_{-1}L^s)=\tfrac{1}{s+1}L^{s+1}=\tfrac{1}{s+1}e_0L^{s+1} \end{aligned}$$

where $a_m=-m$ if $m\leqslant-2$ and $a_m=\frac{1}{m+1}$ if $m\geqslant0$.

Lemma 29 *For any* $F=\sum_{m\leqslant k,\ 0\leqslant s\leqslant r}c_{m,s}\otimes e_mL^s\in\mathcal{S}_{k,r}$, $\mathcal{P}F\in c_{k,r}\otimes e_{k+1}L^r+\frac{c_{-1,r}}{r+1}\otimes L^{r+1}+S_{k,r}$ *and* $\mathcal{P}F$ *is a partial primitive of* F *in the sense that* $\frac{\partial}{\partial y}\mathcal{P}F=F$.

Proof We only need to check that for a given $(m,s)\in\mathbb{Z}\times\mathbb{N}$, $\left[\mathcal{P}(e_mL^s)\right]'=e_mL^s$. The cases $m=-1$ or $(s=0\ \&\ m\neq-1)$ are quite evident. Assume $s\neq0$ and $m\neq-1$. Then

$$\int_{[1;y]}\left(e_mL^s\right)(\tau)\,d\tau=\left[e_{m+1}L^s\right]_1^y-\int_{[1;y]}e_{m+1}(\tau)\frac{s}{\tau}L^{s-1}(\tau)\,d\tau$$

If $m \leqslant -2$, $e_{m+1}(\tau)\frac{1}{\tau} = (-1)^{|m|}|m|!\tau^m = -me_m$ and if $m \geqslant 0$, $e_{m+1}(\tau)\frac{1}{\tau} = \frac{1}{(m+1)!}\tau^m = \frac{1}{m+1}e_m$. Thus

$$\begin{aligned}\int_{[1;y]}\left(e_m L^s\right)(\tau)\,d\tau &= e_{m+1}L^s - sa_m\int_{[1;y]}\left(e_m L^{s-1}\right)(\tau)\,d\tau\\ &= A_s^0 a_m^0 e_{m+1}L^s + \cdots + (-1)^{s-1}A_s^{s-1}a_m^{s-1}e_{m+1}L^1\\ &\quad + (-1)^s A_s^s a_m^s\int_{[1;y]} e_m(\tau)\,d\tau\\ &= A_s^0 a_m^0 e_{m+1}L^s + \cdots + (-1)^s A_s^s a_m^s e_{m+1}L^0 = \mathcal{P}\left(e_m L^s\right)\end{aligned}$$

and $\mathcal{P}(e_m L^s)$ is indeed a primitive of $e_m L^s$. □

Definition 30 Let H be the function defined on Z^+ by

$$H = \mathcal{P}\frac{\partial G_1}{\partial x} = -\delta\otimes L + \sum_{m\leqslant -1}\frac{G'_{1,m-1}}{\kappa_{m-1}}\otimes e_m = -\delta\otimes L + \widetilde{H}$$

We then define operators $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ on $S_{*,*}$ in the following way

$$\mathcal{D} = e^{-H}\frac{\partial}{\partial x}e^H = \frac{\partial}{\partial x} + \frac{\partial H}{\partial x},\quad \mathcal{E} = \mathcal{P}\circ\mathcal{D}\ \ \&\ \ \mathcal{F} = \Pi\mathcal{E} \tag{39}$$

where Π is the operator which to $F = \sum_{m\leqslant k,\ 0\leqslant s\leqslant r} c_{m,s}\otimes e_m L^s \in \mathcal{S}_{k,r}$ associates $\sum_{m\leqslant k} c_{m,0}\otimes e_m$.

The lemma below collects some basic facts about the crucial function H.

Lemma 31 *$\widetilde{H} = I + J$ where for any $(x, y)\in Z$,*

$$I(x,y) = \frac{1}{2\pi i}\int_{(\partial Q)_0}\frac{z_1\mathrm{d}z_2 - (x+z_2)\,\mathrm{d}z_1}{x + yz_1 + z_2}$$
$$J(x,y) = \frac{-1}{2\pi i}\int_{(\partial Q)_0} L\left(\frac{x+yz_1+z_2}{yz_1}\right)\mathrm{d}z_1$$

$H = -\delta\otimes L + \sum_{m\leqslant -1} H_m\otimes e_m$ with $H_m\in\mathbb{C}_{|m|-1}[X]$ for any $m\leqslant -1$ and

$$\frac{\partial H}{\partial y} = \frac{\partial G_1}{\partial x}. \tag{40}$$

e^H extends holomorphically to Z and

$$e^H = \left(1\otimes e_1^{-\delta}\right)e^{\widetilde{H}} \tag{41}$$

so that $\mathcal{D}$ is in fact defined on $\mathcal{O}(Z)$. Furthermore, δ is given for all $x \in \mathbb{C}$ by the formula

$$\delta = \lim_{|y| \to +\infty} \frac{\ln \left| e^{-H(x,y)} \right|}{\ln |y|}. \tag{42}$$

Proof Formula (40) is the main purpose of setting $H = \mathcal{P}\frac{\partial G_1}{\partial x}$, (41) just takes in account that $\delta \in \mathbb{Z}$ and (42) follows from (41). For any $m \leqslant -1$, $H_m = -\frac{1}{\kappa-2} G'_{1,m-1} \in \mathbb{C}_{|m-1|-2}[X] = \mathbb{C}_{|m|-1}[X]$. To prove that $\widetilde{H} = I + J$, we note that for $(x, y) \in U$,

$$\begin{aligned}\frac{\partial G_1}{\partial x}(x,y) &= \frac{-1}{2\pi i} \int_{(\partial Q)_0} \frac{y z_1 \mathrm{d}z_1 + z_1 \mathrm{d}z_2}{(x + y z_1 + z_2)^2} \\ &= \frac{-1}{2\pi i} \int_{(\partial Q)_0} \frac{\mathrm{d}z_1}{x + y z_1 + z_2} + \frac{1}{2\pi i} \int_{(\partial Q)_0} \frac{(x + z_2)\,\mathrm{d}z_1 - z_1 \mathrm{d}z_2}{(x + y z_1 + z_2)^2} \\ &= \frac{-1}{2\pi i} \int_{(\partial Q)_0} \frac{\mathrm{d}z_1}{x + y z_1 + z_2} + \frac{\partial I}{\partial y}(x,y).\end{aligned}$$

When $(z_1, z_2) \in (\partial Q)_0$, $\frac{x+yz_1+z_2}{yz_1} \in \mathbb{R}^*_-$ only if $y \in]0; \frac{-x-z_2}{z_1}]$, which cannot happen since $\left|\frac{-x-z_2}{z_1}\right| \leqslant \frac{\alpha}{\min\limits_{(\zeta_1,\zeta_2)\in(\partial Q)_0}|\zeta_1|} |y| + \max\limits_{(\zeta_1,\zeta_2)\in(\partial Q)_0} |\zeta_2| \leqslant \frac{1}{2}|yz_1| < |yz_1|$. Hence J is well defined on Z and

$$\frac{\partial J}{\partial y} = \frac{-1}{2\pi i} \int_{(\partial Q)_0} \left(\frac{1}{x + y z_1 + z_2} - \frac{1}{y z_1} \right) \mathrm{d}z_1 = \frac{\delta}{y} + \frac{-1}{2\pi i} \int_{(\partial Q)_0} \frac{\mathrm{d}z_1}{x + y z_1 + z_2}$$

Thus, $\frac{\partial (I+J)}{\partial y} = \frac{\partial \widetilde{H}}{\partial y}$ and since both $H(x,.)$ and $(I+J)(x,.)$ have limit 0 at infinity when x is fixed, we get $I + J = \widetilde{H}$. □

The operator $\mathcal{F}$ enables to design a machinery adapted to the system (37).

Proposition 32 *Let $s_1, \ldots, s_d \in \mathcal{O}(Z \backslash E^\infty)$. Then $(s_1, \ldots, s_d)$ is a solution of (37) with $N = G_1 - P$ if and only if each $(1 \otimes B) s_j$ extends holomorphically to Z and there exists $\mu_1, \ldots, \mu_d \in \mathcal{O}(\mathbb{C})$ which satisfy the system below on Z^+,*

$$(1 \otimes B) s_k = \left[\mathcal{F}^0 (\mu_k \otimes e_0) + \cdots + \mathcal{F}^{d-k} (\mu_d \otimes e_0) \right] e^H, \ d \geqslant k \geqslant 1 \tag{43}$$

Proof Since $N = G_1 - \frac{A}{B} - Id \otimes \frac{B'}{B}$, we note that if $s \in \mathcal{O}(Z)$ and $\widetilde{B} = 1 \otimes B$

$$\begin{aligned}\widetilde{B}\left(-s\frac{\partial N}{\partial x} + \frac{\partial s}{\partial y}\right) &= s\left(-\widetilde{B}\frac{\partial G_1}{\partial x} + \widetilde{B}'\right) + \widetilde{B}\frac{\partial s}{\partial y} \\ &= -\left(\widetilde{B}s\right)\frac{\partial G_1}{\partial x} + \frac{\partial \widetilde{B}s}{\partial y} = e^H \frac{\partial e^{-H}\widetilde{B}s}{\partial y}\end{aligned}$$

As e^H extends holomorphically to Z, $(s_1, \ldots, s_d) \in \mathcal{O}(Z \backslash E^\infty)^d$ is a solution of (37) if and only if the equations

$$\frac{\partial e^{-H} \widetilde{B} s_d}{\partial y} = 0 \quad \& \quad \frac{\partial e^{-H} \widetilde{B} s_k}{\partial y} = e^{-H} \frac{\partial \widetilde{B} s_{k+1}}{\partial x}, \quad 1 \leqslant k \leqslant d-1 \tag{44}$$

are satisfied on $Z \backslash E^\infty$. The first one is equivalent to the existence of a function μ_d defined on $\mathbb{C}$ such that for all $(x, y) \in Z \backslash E^\infty$,

$$B(y)\, s_d(x, y) = \mu_d(x)\, e^{H(x,y)} \tag{45}$$

Such a function μ_d is actually holomorphic on $\mathbb{C}$ since for all $y \in \mathbb{C} \backslash \widetilde{\rho} \overline{\mathbb{D}}$, it would be given on $D(0, \alpha|y|)$ by the formula $\mu_d = s_d(., y) \frac{e^{H(.,y)}}{B(y)}$. Hence, (45) also implies that $\widetilde{B} s_d$ holomorphically extends to Z. Suppose that for $k \in \{1, \ldots, d-1\}$, $\mu_d, \ldots, \mu_k \in \mathcal{O}(\mathbb{C})$ satisfy on $Z \backslash E^\infty$

$$\widetilde{B} s_j = \left[\mathcal{F}^0 \left(\mu_j \otimes e_0 \right) + \cdots + \mathcal{F}^{d-j} \left(\mu_d \otimes e_0 \right) \right] e^H$$

when $d \geqslant j \geqslant k+1$ and that each of these $\widetilde{B} s_j$ extends holomorphically to Z. The equation $\frac{\partial}{\partial y} \left(\widetilde{B} s_k e^{-H} \right) = e^{-H} \frac{\partial}{\partial x} \left(\widetilde{B} s_{k+1} \right)$ is then equivalent to the existence of a function μ_k defined on $\mathbb{C}$ such that for all $(x, y) \in Z^+ \backslash E^\infty$,

$$B(y)\, s_k(x, y)\, e^{-H(x,y)} = \mu_k(x) + \mathcal{P} \left(e^{-H} \frac{\partial}{\partial x} \left(B s_{k+1} \right) \right)(x, y). \tag{46}$$

Since $\widetilde{B} s_{k+1}$ and e^{-H} extends holomorphically to Z, the only logarithmic term (46) may have comes from $\mathcal{P}$ applied to some elements of $\mathcal{O}(\mathbb{C}) \otimes e_{-1}$. As $\widetilde{B} s_k e^{-H}$ expands in usual Laurent series in Z, theses logarithmic terms have to compensate. Hence, it turns out that the right side of (46) expands in usual Laurent series in Z, which yields that $\widetilde{B} s_k$ holomorphically extends to Z and $\mu_k \in \mathcal{O}(\mathbb{C})$. We also get

$$\begin{aligned} \widetilde{B} s_k e^{-H} &= \Pi \left(\left(1 \otimes \widetilde{B} \right) s_k e^{-H} \right) = \mu_k \otimes e_0 + \Pi \mathcal{P} \left(e^{-H} \frac{\partial}{\partial x} \left(\widetilde{B} s_{k+1} \right) \right) \\ &= \mu_k \otimes e_0 + \Pi \sum_{k+1 \leqslant j \leqslant d} \mathcal{P} \left(e^{-H} \frac{\partial}{\partial x} \left(e^H \mathcal{F}^{j-k-1} \left(\mu_j \otimes e_0 \right) \right) \right) \\ &= \sum_{1 \leqslant j \leqslant d} \mathcal{F}^{j-k-1} \left(\mu_j \otimes e_0 \right). \end{aligned}$$

□

We derive from Proposition 32 a process to construct a priori some functions which may be multiple shock wave.

Corollary 33 *For $\mu_1, \dots, \mu_d \in \mathcal{O}(\mathbb{C})$, we define on Z holomorphic functions $s_k(\mu, B)$, $1 \leqslant k \leqslant d$, by*

$$s_k(\mu, B) = \frac{e^{\widetilde{H}}}{1 \otimes e_1^{\delta} B} \mathcal{F}_k(\mu) \ \ \& \ \ \mathcal{F}_k(\mu) = \sum_{j=k}^{d} \mathcal{F}^{j-k}\left(\mu_j \otimes e_0\right), \ \ 1 \leqslant k \leqslant d.$$

Let $\mathbb{C}_{\mathcal{B}}[Y] = \{B \in \mathbb{C}[Y];\ B(0) = 1\}$. Then the map $\mathcal{O}(\mathbb{C})^d \times \mathbb{C}_{\mathcal{B}}[Y] \ni (\mu, B) \mapsto (s_k(\mu, B))_{1 \leqslant k \leqslant d}$ is injective. Moreover, $-s_1(\mu, B)$ is a d-shock waves on Z if and only if

$$-s_1(\mu, B) = G_1 - P$$

and the discriminant $\Delta(\mu, B)$ of $S(\mu, B) = T^d + s_1(\mu, B) T^{d-1} + \cdots + s_d(\mu, B) \in \mathcal{O}(Z)[T]$ is not identically zero.

Proof Suppose that (μ, B) and (ν, C) are two elements of $\mathcal{O}(\mathbb{C})^d \times \mathbb{C}_{\mathcal{B}}[Y]$ such that $(s_k(\mu, B))_{1 \leqslant k \leqslant n} = (s_k(\nu, C))_{1 \leqslant k \leqslant d}$. Then on $Z \backslash E^{\infty}$, $\mu_d \otimes \frac{1}{B} = \nu_d \otimes \frac{1}{B}$. As $B, C \in \mathbb{C}_{\mathcal{B}}[Y]$, this implies $B = C$ and $\mu_d = \nu_d$. Suppose that $\mu_j = \nu_j$ when $d \geqslant j \geqslant k > 1$. The relation $s_{k-1}(\mu, B) = s_{k-1}(\nu, C)$ can be then written $\mathcal{F}_{k-1}(\mu) = \mathcal{F}_{k-1}(\nu)$ and this gives immediately $\mu_{k-1} = \nu_{k-1}$. Hence, $\mu = \nu$.

Since $e^H = \left(1 \otimes e_1^{-\delta}\right) e^{\widetilde{H}}$, Proposition 32 gives that $(s_k(\mu, B))_{1 \leqslant k \leqslant d}$ verifies system (37). When $-s_1(\mu, B) = G_1 - P$, $\Delta(\mu, B) \neq 0$ ensures that $-s_1(\mu, B)$ is the sum of d shock waves mutually distinct whose symmetric functions are the $(-1)^k s_k(\mu, B)$. □

The proposition below shows that the system (43) can be seen as a classical differential system with unknowns $\mu_1, \dots, \mu_d$.

Proposition 34 *We define holomorphic functions $\mathcal{F}_{k,k}, \dots, \mathcal{F}_{k,0}$ on Z for all $k \in \mathbb{N}$ by the following relations*

$$\mathcal{F}_{k,k} = 1 \otimes e_k, \ \ \mathcal{F}_{k+1,0} = \mathcal{F}^k \Pi \mathcal{P} \frac{\partial H}{\partial x}, \ \ \mathcal{F}_{k+1,j} = \Pi \mathcal{P} \mathcal{F}_{k,j-1} + \mathcal{F} \mathcal{F}_{k,j}, \ 1 \leqslant j \leqslant k$$

where $\mathcal{F}_{k,\nu} = 0$ if $\nu < 0$. Then for all $f \in \mathcal{O}(\mathbb{C})$,

$$\mathcal{F}^k(f \otimes e_0) = \sum_{0 \leqslant j \leqslant k} \left(f^{(j)} \otimes e_0\right) \mathcal{F}_{k,j}.$$

Proof By definition, for all $f \in \mathcal{O}(\mathbb{C})$, $\mathcal{D}(f \otimes e_0) = f' \otimes e_0 + (f \otimes e_0) \frac{\partial H}{\partial x}$ and hence $\mathcal{F}(f \otimes e_0) = \Pi \mathcal{P} \mathcal{D}(f \otimes e_0) = \left(f' \otimes e_0\right) \mathcal{F}_{1,1} + (f \otimes e_0) \mathcal{F}_{1,0}$ with $\mathcal{F}_{1,1} = 1 \otimes e_1$ and $\mathcal{F}_{1,0} = \Pi \mathcal{P} H$. Suppose lemma's result true for a given $k \in \mathbb{N}^*$. Then for $f \in \mathcal{O}(\mathbb{C})$

$$
\begin{aligned}
&\mathcal{F}^{k+1}\left(f \otimes e_0\right) \\
&\quad = \sum_{0\leqslant j\leqslant k} \Pi\mathcal{P}\frac{\partial}{\partial x}\left(f^{(j)}\otimes e_0\right)\mathcal{F}_{k,j} + \Pi\mathcal{P}\left(\frac{\partial H}{\partial x}\sum_{0\leqslant j\leqslant k}\left(f^{(j)}\otimes e_0\right)\mathcal{F}_{k,j}\right) \\
&\quad = \sum_{0\leqslant j\leqslant k} \Pi\mathcal{P}\left(\left(f^{(j+1)}\otimes e_0\right)\mathcal{F}_{k,j} + \left(f^{(j)}\otimes e_0\right)\frac{\partial \mathcal{F}_{k,j}}{\partial x}\right) \\
&\qquad + \sum_{0\leqslant j\leqslant k}\left(f^{(j)}\otimes e_0\right)\Pi\mathcal{P}\left(\mathcal{F}_{k,j}\frac{\partial H}{\partial x}\right) \\
&\quad = \sum_{0\leqslant j\leqslant k}\left(f^{(j+1)}\otimes e_0\right)\Pi\mathcal{P}\mathcal{F}_{k,j} + \sum_{0\leqslant j\leqslant k}\left(f^{(j)}\otimes e_0\right)\Pi\mathcal{P}\frac{\partial \mathcal{F}_{k,j}}{\partial x} \\
&\qquad + \sum_{0\leqslant j\leqslant k}\left(f^{(j)}\otimes e_0\right)\Pi\mathcal{P}\left(\mathcal{F}_{k,j}\frac{\partial H}{\partial x}\right)
\end{aligned}
$$

which gives the expected formula with

$$
\begin{aligned}
\mathcal{F}_{k+1,k+1} &= \Pi\mathcal{P}\mathcal{F}_{k,k} = \Pi\mathcal{P}\left(1\otimes e_k\right) = 1\otimes e_{k+1}, \\
\mathcal{F}_{k+1,j} &= \Pi\mathcal{P}\mathcal{F}_{k,j-1} + \Pi\mathcal{P}\left(\frac{\partial \mathcal{F}_{k,j}}{\partial x} + \mathcal{F}_{k,j}\frac{\partial H}{\partial x}\right) \\
&= \Pi\mathcal{P}\mathcal{F}_{k,j-1} + \mathcal{F}\mathcal{F}_{k,j},\ 1\leqslant j\leqslant k, \\
\mathcal{F}_{k+1,0} &= \Pi\mathcal{P}\left(\frac{\partial \mathcal{F}_{k,0}}{\partial x} + \mathcal{F}_{k,0}\frac{\partial H}{\partial x}\right) \\
&= \mathcal{F}\mathcal{F}_{k,0} = \mathcal{F}\mathcal{F}^{k-1}\Pi\mathcal{P}\frac{\partial H}{\partial x} = \mathcal{F}^{k}\Pi\mathcal{P}\frac{\partial H}{\partial x}.
\end{aligned}
$$

□

Going further in the analysis of (37), we are about to prove that the functions μ_j are polynomials. We start by two elementary lemmas.

Lemma 35 *Let $k\in\mathbb{N}$ and $F=\sum_{m\leqslant k} c_m\otimes e_m\in S_{k,r}$. Then $\mathcal{F}F\in c_k'\otimes e_{k+1}+S_{k,r}$ and $\langle\mathcal{F}F, e_0\rangle=0$.*

Proof Let k and F be as above. Since $\mathcal{F}F=\Pi\mathcal{P}\mathcal{D}F$ and $\left\langle\mathcal{P}\left(e_jL^s\right), e_0\right\rangle=0$ for any (j,s), we get $\langle\mathcal{F}F, e_0\rangle=0$. Furthermore,

$$
\mathcal{P}\frac{\partial F}{\partial x}=\sum_{m\leqslant k} c_m'\otimes\mathcal{P}e_m\in c_k'\otimes e_{k+1}+S_{k,r}.
$$

As H_{-1} is constant, $\frac{\partial H}{\partial x} = \sum_{m \leqslant -2} H'_m \otimes e_m$ and the expected relation follows from

$$\Pi\mathcal{P}\left(F\frac{\partial H}{\partial x}\right) = \Pi\mathcal{P}\sum_{j\leqslant k}\sum_{\nu\leqslant -2} c_j H'_\nu \otimes \frac{\kappa_j\kappa_\nu}{\kappa_{j+\nu}} e_{\nu+j}$$
$$= \Pi\mathcal{P}\sum_{\ell\leqslant k-2}\left(\sum_{\substack{\nu+j=\ell \\ \nu\leqslant -2\ \&\ j\leqslant k}} \kappa^j_{j+\nu}\, c_j H'_\nu\right)\otimes e_\ell$$
$$= \sum_{0\neq m\leqslant k-1}\left(\sum_{\substack{\nu+j=m-1 \\ \nu\leqslant -2\ \&\ j\leqslant k}} \kappa^j_{j+\nu} c_j H'_\nu\right)\otimes e_m \in S^{\rho,\delta}_{k,r}.$$

□

Lemma 36 *Denote by B_{q^∞} the leading coefficient of B. Then, there exists $(\lambda_m) \in \mathbb{C}[X]^{\mathbb{Z}_-}$ such that*

$$\frac{e^H}{1\otimes B} = \frac{1}{B_{q^\infty}}\sum_{m\leqslant 0}\lambda_m \otimes \frac{e_m}{e_p/\kappa_p} \tag{47}$$

with $\lambda_0 = 1$ and $\deg \lambda_m \leqslant |m| - 1$ *for all $m \in \mathbb{Z}^*_-$.*

Proof For a suitable family $\left(B_{-1,m}\right) \in \mathbb{C}^{\mathbb{Z}_-}$, $\frac{1}{B} = \frac{\kappa_{q^\infty}}{B_{q^\infty} e_{q^\infty}}\sum_{m\leqslant 0} B_{-1,m} e_m$ with $B_{-1,0} = 1$. Since $H = -\delta L + \sum_{m\leqslant -1} H_m \otimes e_m$,

$$e^{-H} = \frac{\kappa_\delta}{e_\delta}\left[1 + \sum_{n\in\mathbb{N}^*}\frac{1}{n!}\left(-\sum_{\nu\leqslant -1} H_\nu \otimes e_\nu\right)^n\right] = \frac{\kappa_\delta}{e_\delta}\sum_{m\leqslant 0} h_m \otimes e_m$$

with $h_0 = 1$ and for $m \in \mathbb{N}^*$, $h_m = \sum_{1\leqslant n\leqslant |m|}\frac{(-1)^n}{n!}\sum_{\nu\in(\mathbb{Z}^*_-)^n;\ \nu_1+\cdots+\nu_n=m} H_{\nu_1}\cdots H_{\nu_n} \in \mathbb{C}_{|m|-1}[X]$ because if $\nu \in \left(\mathbb{Z}^*_-\right)^n$ and $\nu_1 + \cdots + \nu_n = m$, $\deg H_{\nu_1}\cdots H_{\nu_n} \leqslant \sum_{1\leqslant j\leqslant n}\left(\left|\nu_j\right| - 1\right) = |m| - n \leqslant |m| - 1$. As $p = \delta + q^\infty$, $\frac{\kappa_\delta\kappa_{q^\infty}}{e_\delta e_{q^\infty}} = \frac{\kappa_p}{e_p}$ and we get (47) with $\lambda_0 = 1$ and for all $m \in \mathbb{Z}^*_-$, $\lambda_m = \sum_{r+s=m,\ 0\geqslant r,s} h_r B_{-1,s}$ which is a polynomial of degree at most $\max_{0\geqslant r\geqslant m} \deg h_r$, that is $|m| - 1$. □

Proposition 37 *Let $f \in \mathcal{O}(\mathbb{C})$ and $k \in \mathbb{N}^*$. Then,*

$$\mathcal{F}^k(f\otimes e_0) = f^{(k)}\otimes e_k + \sum_{m\leqslant k-2} P_{k,m}(f)\otimes e_m = \sum_{m\leqslant k} P_{k,m}(f)\otimes e_m$$

with $P_{k,k} = \frac{\partial^k}{\partial x_k}$, $P_{k,k-1} = P_{k,0} = 0$ *and for* $m \in \mathbb{Z}\cap]-\infty, k-1]$, $P_{k,m} = \sum_{(m+1)^+ \leqslant j \leqslant k-1} P^j_{k,m} \frac{\partial^j}{\partial x^j}$ *where for any* j, $P^j_{k,m} \in \mathbb{C}_{j-m-1}[X]$ *which means that* $P^j_{k,m} = 0$ *when* $j < m+1$.

Proof Note that if $\nu \in \mathbb{Z}^*_-$, $\deg H'_\nu = (|\nu|-1)-1 = |\nu|-2$. Set $F = f \otimes e_0$ and for $m \in \mathbb{Z}, \langle \mathcal{F}^k F, e_m \rangle = c_{k,m}$. By definition of $\mathcal{F}$, $\mathcal{F}^1 F = f' \otimes e_1 + \sum_{m \leqslant -1} H'_{m-1} f \otimes e_m$. As when $m \in \mathbb{Z}^*_-$, $P_{1,m} \overset{def}{=} P^0_{1,m} \overset{def}{=} H'_{m-1}$ has degree $|m|-1$, the claims are true for $k=1$.

Let $k \in \mathbb{N}\setminus\{0,1\}$ be such that $c_{k,m} = 0$ when $m \in \mathbb{Z}\cap[k, +\infty[$, $c_{k,k} = f^{(k)}$, $c_{k,k-1} = c_{k,0} = 0$ whereas for $m \in \mathbb{Z}\cap]-\infty, k-1]$, $c_{k,m} = P_{k,m}(f)$ with $P_{k,m} = \sum_{0\leqslant j\leqslant k-1} P^j_{k,m}\frac{\partial^j}{\partial x^j}$ and $P^j_{k,m} \in \mathbb{C}_{j-m-1}[X]$ for all j. Since $H'_{-1} = 0$, with $\kappa^r_m = \frac{\kappa_r \kappa_{m-r}}{\kappa_m}$, we get

$$\begin{aligned}
\mathcal{F}^{k+1}F = \Pi\mathcal{E}\mathcal{F}^k F &= \sum_{0\neq m\leqslant k+1} c'_{k,m-1}\otimes e_m + \Pi\mathcal{P}(\sum_{r\leqslant k} c_{k,r}\otimes e_r)\left(\sum_{s\leqslant -2} H'_s \otimes e_s\right)\\
&= \sum_{0\neq m\leqslant k+1} c'_{k,m-1}\otimes e_m + \Pi\mathcal{P}\sum_{m\leqslant k-2}\left(\sum_{m+2\leqslant r\leqslant k}\kappa^r_m c_{k,r} H'_{m-r}\right)\otimes e_m\\
&= c'_{k,k}\otimes e_{k+1} + c'_{k,k-1}\otimes e_k\\
&\quad + \sum_{0\neq m\leqslant k-1}\left(c'_{k,m-1} + \sum_{m+1\leqslant r\leqslant k}\kappa^r_{m-1} c_{k,r} H'_{m-1-r}\right)\otimes e_m
\end{aligned}$$

Thus $c_{k+1,k+1} = c'_{k,k} = f^{(k+1)}$, $c_{k+1,k} = c'_{k,k-1} = 0$ and $c_{k+1,m} = 0$ if $m \geqslant k+1$ where $m = 0$. For $m \in \mathbb{Z}^*\cap\big]-\infty, k]$, it comes

$$c_{k+1,m} = c'_{k,m-1} + \sum_{m+1\leqslant r\leqslant k}\kappa^r_{m-1} H'_{m-1-r} c_{k,r} \tag{48}$$

Let $m \in \mathbb{Z}^*\cap\big]-\infty, k-1]$. Formula (48) and the induction hypothesis give

$$\begin{aligned}
c_{k+1,m} &= \left(\sum_{(m+1)^+\leqslant j\leqslant k-1} P^j_{k,m-1} f^{(j)}\right)' + \sum_{m+1\leqslant r\leqslant k}\ \sum_{(m+1)^+\leqslant j\leqslant k-1}\kappa^r_{m-1} P^j_{k,r} H'_{m-1-r} f^{(j)}\\
&= \sum_{(m+1)^+\leqslant j\leqslant k-1}(P^j_{k,m-1} f^{(j)})'\\
&\quad + \sum_{(m+1)^+\leqslant j\leqslant k-1}\left[\sum_{m+1\leqslant r\leqslant k}\kappa^r_{m-1} P^j_{k,r} H'_{m-1-r}\right] f^{(j)} = P_{k+1,m}(f)
\end{aligned}$$

with $P_{k+1,m}=\sum_{(m+1)^+-1\leqslant j\leqslant k} P^j_{k+1,m}\frac{\partial}{\partial x_j}$ and

$$P^k_{k+1,m}=P^{k-1}_{k,m-1} \tag{49}$$

$$P^j_{k+1,m}=P^{j-1}_{k,m-1}+\left(P^j_{k,m-1}\right)'+\sum_{r=m+1}^{k}\kappa^r_{m-1}P^j_{k,r}H'_{m-1-r},\ \ (m+1)^+\leqslant j<k \tag{50}$$

$$P^{(m+1)^+-1}_{k+1,m}=\left(P^{(m+1)^+-1}_{k,m-1}\right)'+\sum_{m+1\leqslant r\leqslant k}\kappa^r_{m-1}P^{(m+1)^+-1}_{k,r}H'_{m-1-r} \tag{51}$$

Assume $1\leqslant m\leqslant k-1$. Then (51) becomes $P^m_{k+1,m}=\left(P^m_{k,m-1}\right)'+\sum_{m+1\leqslant r\leqslant k}\kappa^r_{m-1}P^m_{k,r}H'_{m-1-r}$. We know that $\deg P^m_{k,m-1}=m-(m-1)-1=0$ and that when $m+1\leqslant r\leqslant k$, $P^m_{k,r}=0$ since $m\leqslant r-1<r+1$. Hence $P^m_{k+1,m}=0$. When $m+1\leqslant j\leqslant k-1$, $\deg P^{j-1}_{k,m-1}\leqslant j-1-(m-1)-1=j-m-1$, $\deg\left(P^j_{k,m-1}\right)'\leqslant (j-(m-1)-1)-1=j-m-1$ and for $m+1\leqslant r\leqslant k$, $\deg P^j_{k,r}H'_{m-1-r}\leqslant (j-r-1)+(r+1-m)-2=j-m-2$. Thus, (50) gives that $\deg P^j_{k+1,m}\leqslant j-m-1$. Lastly, $\deg P^k_{k+1,m}=\deg P^{k-1}_{k,m-1}\leqslant k-1-(m-1)-1=k-m-1$.

Assume now $m\leqslant -1$. Degree computations for $P^k_{k+1,m}$ and $P^j_{k+1,m}$ when $1\leqslant j\leqslant k-1$ are still valid. Formula (51) becomes $P^0_{k+1,m}=\left(P^0_{k,m-1}\right)'+\sum_{m+1\leqslant r\leqslant k}\kappa^r_{m-1}P^0_{k,r}H'_{m-1-r}$ and gives $\deg P^0_{k+1,m}\leqslant 0-m-1$ because $\deg\left(P^0_{k,m-1}\right)'\leqslant (0-(m-1)-1)-1=-m-1$ and for $m+1\leqslant r\leqslant k$, $\deg P^0_{k,r}H'_{m-1-r}\leqslant (0-r-1)+(r+1-m)-2=-m-2$. The proof is complete. □

Proposition 38 *Assume that $(s_1,\ldots,s_d)\in\mathcal{O}(Z\backslash E^\infty)^d$ is a solution of (37) with $-s_1=G_1-P_1$ and let $(\mu_1,\ldots,\mu_d)\in\mathcal{O}(\mathbb{C})^d$ satisfies the system (43). Then $d=p$, μ_p is a polynomial of degree p and $\mu_p^{(p)}=p!B_{q^\infty}$ where $B_{q^\infty}=\prod_{q\in Q^\infty}b^q$ is the leading coefficient of B. Moreover, for all $j\in\{1,\ldots,p-1\}$, μ_j is a polynomial of degree at most $p-1$.*

Proof The proof relies on a downward induction starting on p and on the comparison of the Laurent series of s_1, series we have to compute, to the expansion of $-G_1+P_1$ which we know because of Lemmas 23 and 22 : $G_1=\sum_{m\leqslant -1}\frac{G_{1,m}}{\kappa_m}\otimes e_m$ and $P_1=\sum_{m\leqslant -1}\frac{P_{1,m}}{\kappa_m}\otimes e_m$ with $G_{1,-1}=G^0_{1,1}-\delta x$, $G_{1,m}\in\mathbb{C}_{|m|-1}[X]$ when $m\leqslant -2$, $P_{1,1}=q^\infty X+\langle p_{1,0},e_{-1}\rangle$ and $P_{1,m}\in\mathbb{C}_1[X]$ for all m. Thanks to Proposition 37 and to (47), we get

$$
\begin{aligned}
s_1 &= \frac{e^H}{1\otimes B}\sum_{1\leqslant j\leqslant d}\mathcal{F}^{j-1}\left(\mu_j\otimes e_0\right) = \frac{e^H}{1\otimes B}\sum_{1\leqslant j\leqslant d}\sum_{m\leqslant j-1}P_{j-1,m}\left(\mu_j\right)\otimes e_m \\
&= \frac{1}{B_{q^\infty}}\left(\sum_{m\leqslant 0}\lambda_m\otimes\frac{e_m}{e_p/\kappa_p}\right)\sum_{m\leqslant d-1}\left(\sum_{m^++1\leqslant j\leqslant p}P_{j-1,m}\left(\mu_j\right)\right)\otimes e_s \\
&= \frac{1}{B_{q^\infty}}\sum_{m\leqslant d-1}\sum_{m-d+1\leqslant r\leqslant 0}\left(\sum_{(m-r)^++1\leqslant j\leqslant d}\kappa_m^r\lambda_r P_{j-1,m-r}\left(\mu_j\right)\right)\otimes\frac{e_m}{e_p/\kappa_p} \\
&= \frac{1}{B_{q^\infty}}\sum_{m\leqslant d-1}\frac{\kappa_m}{\kappa_{m-p}}\widetilde{s}_{1,m}\otimes e_{m-p}
\end{aligned}
$$

with for $m\leqslant p-1$, $\widetilde{s}_{1,m}=\sum_{m-d+1\leqslant r\leqslant 0}\sum_{(m-r)^++1\leqslant j\leqslant d}\kappa_m^r\lambda_r P_{j-1,m-r}\left(\mu_j\right)$. In particular, when $0\leqslant m\leqslant d-1$,

$$
\widetilde{s}_{1,m}=\sum_{m+1\leqslant j\leqslant d}\sum_{m-j+1\leqslant r\leqslant 0}\kappa_m^r\lambda_r d_{j-1,m-r}\left(\mu_j\right)=\sum_{m+1\leqslant j\leqslant d}\widetilde{P}_{1,m}^j\left(\mu_j\right)
$$

where for $m+1\leqslant j\leqslant d$,

$$
\widetilde{P}_{1,m}^j=\sum_{m-j+1\leqslant r\leqslant 0}\kappa_m^r\lambda_r P_{j-1,m-r}=\kappa_m^0 P_{j-1,m}+\sum_{m-j+1\leqslant r\leqslant -1}\kappa_m^r\lambda_r P_{j-1,m-r}
$$

Thus, $\widetilde{P}_{1,m}^{m+1}\left(\mu_{m+1}\right)=\kappa_m^0 P_{m,m}\left(\mu_{m+1}\right)=\mu_{m+1}^{(m)}$ since $\kappa_m^0=1$. So

$$
\widetilde{s}_{1,m}=\mu_{m+1}^{(m)}+\sum_{m+2\leqslant j\leqslant d}\widetilde{P}_{1,m}^j\left(\mu_j\right) \tag{52}
$$

Moreover,

$$
\begin{aligned}
\widetilde{P}_{1,m}^j\left(\mu_j\right) &= \sum_{0\leqslant t\leqslant j-2}P_{j-1,m}^t\mu_j^{(t)}+\kappa_m^{m-j+1}\lambda_{m-j+1}P_{j-1,j-1}\left(\mu_j\right) \\
&\quad+\sum_{m-j+2\leqslant r\leqslant -1}\sum_{0\leqslant t\leqslant j-2}\kappa_m^r\lambda_r P_{j-1,m-r}^{(t)}\mu_j^{(t)} \\
&= \kappa_m^{m-j+1}\lambda_{m-j+1}\mu_j^{(j-1)} \\
&\quad+\sum_{0\leqslant t\leqslant j-2}\left(P_{j-1,m}^t+\sum_{m-j+2\leqslant r\leqslant -1}\kappa_m^r\lambda_r P_{j-1,m-r}^t\right)\mu_j^{(t)}
\end{aligned}
$$

Formula (52) implies $\widetilde{s}_{1,d-1}=\mu_d^{(p-1)}$ so that $s_1\in\frac{\kappa_{d-1}}{B_{q^\infty}}\mu_d^{(d-1)}e_{d-p-1}+S_{d-p-2,0}$. Yet, $s_1=-N_1=-G_1+P_1$, $G_1\in\left(G_{1,1}^0-\delta Id\right)\otimes e_{-1}+S_{-2,0}$ and $P_1\in$

$\left(q^\infty Id + \langle p_{1,0}, e_{-1}\rangle\right) \otimes e_{-1} + S_{-2,0}$. So, $d - p - 1$ has to be equal to -1, that is $d = p$, and

$$\mu_p^{(p-1)} = \frac{B_{q^\infty}}{\kappa_{p-1}} \left(pId - G_{1,1}^0 - \langle p_{1,0}, e_{-1}\rangle\right) = p! B_{q^\infty} Id \\ - (p-1)! B_{q^\infty} \left[G_{1,1}^0 - \langle p_{1,0}, e_{-1}\rangle\right]$$

In particular, $\mu_p \in \mathbb{C}_p[X]$ and $\mu_p^{(p)} = p! B_{q^\infty}$.

Assume now that $0 \leqslant m \leqslant p-2$ and that $\mu_p, \ldots, \mu_{m+2}$ are polynomials. Then for $m+2 \leqslant j \leqslant p$, $\widetilde{P}_{1,m}^j(\mu_j)$ is of the same kind and as

$$\begin{aligned} \deg \lambda_{m-j+1} \mu_j^{(j-1)} &\leqslant (j-m-1) - 1 + \deg \mu_j - j + 1 = \deg \mu_j - m - 1 \\ \deg P_{j-1,m}^t \mu_j^{(t)} &\leqslant (t-m-1) + \deg \mu_j - t = \deg \mu_j - m - 1 \\ \deg \lambda_r P_{j-1,m-r}^t \mu_j^{(t)} &\leqslant (|r|-1) + (t-m+r) + \deg \mu_j - t = \deg \mu_j - m - 1 \end{aligned}$$

we get

$$\deg \widetilde{P}_{1,m}^j(\mu_j) \leqslant \deg \mu_j - m - 1$$

Thus, $\widetilde{s}_{1,m}$ is polynomial and there exists a polynomial R_m such that

$$\deg \widetilde{s}_{1,m} = \mu_{m+1}^{(m)} + R_m \;\;\&\;\; \deg R_m \leqslant \max_{m+2 \leqslant j \leqslant p} \deg \mu_j - m - 1$$

Moreover,

$$-G_{1,m-p} + P_{1,m-p} = s_{1,m-p} = \frac{1}{B_{q^\infty}} \frac{\kappa_m}{\kappa_{m-p}} \widetilde{s}_{1,m},$$

$G_{1,m-p} \in \mathbb{C}_{|m|-1}[X]$ since $m - p \leqslant -2$ and $P_{1,m} \in \mathbb{C}_1[X]$. From

$$\mu_{m+1}^{(m)} = B_{q^\infty} \frac{\kappa_{m-p}}{\kappa_m} \left(-G_{1,m-p} + P_{1,m-p}\right) + R_m,$$

we first recover that the functions μ_j are all polynomials then, with $m = p-2$ that

$$\deg \mu_{p-1}^{(p-2)} \leqslant \max\left\{p-(p-2)-1, 1, \deg \mu_p - (p-2) - 1\right\} = 1$$

and hence that $\deg \mu_{p-1} \leqslant p-1$. Assuming $\deg \mu_j \leqslant p-1$ when $m+2 \leqslant j \leqslant p-1$, we obtain

$$\deg \mu_{m+1}^{(m)} \leqslant \max\{p-m-1, 1, p-m-1\} = p-m-1$$

and thus $\deg \mu_{m+1} \leqslant p-1$, which end this induction proof. □

6.4 A Linear System

According to Proposition 21, Lemma 22, and Corollary 25, there exists $A^{\infty}, B^{\infty} \in \mathbb{C}[Y]$ with $\deg A < \deg B^{\infty} = q^{\infty}$ and $B^{\infty}(0) = 1$ such that on $Z \backslash E^{\infty}$,

$$G_1 = N_1^Q + X \otimes \frac{B^{\infty\prime}}{B^{\infty}} + 1 \otimes \frac{A^{\infty}}{B^{\infty}}$$

where N_1^Q is locally the sum of the shock wave functions $h_1, \ldots, h_p$ involved in Proposition 21. According to Lemma 27, Corollary 25, Propositions 32, and 38, these local functions define on $Z \backslash E^{\infty}$ global symmetric functions $(-1)^k s_k^Q$, $1 \leqslant k \leqslant p$, which can be written in the form

$$s_k^Q = \frac{e^H}{1 \otimes B^{\infty}} \mathcal{F}_k \left(\mu^Q\right), \ p \geqslant k \geqslant 1,$$

where $\mu^Q = \left(\mu_1^Q, \ldots, \mu_p^Q\right) \in \mathbb{C}[X]^p$ is such that and $\deg \mu_j^Q < \deg \mu_p^Q = p$ when $1 \leqslant j \leqslant p$. In the above formula, $\mathcal{F}_k$ is defined for any $\mu \in \mathbb{C}[X]^d$ and arbitrary $(d, k) \in \mathbb{N}^* \times \mathbb{N}$ by

$$\mathcal{F}_0(\mu) = \mathcal{F}\mathcal{F}_1(\mu) \ \ \& \ \ \mathcal{F}_k(\mu) = \sum_{k \leqslant j \leqslant d} \mathcal{F}^{j-k}\left(\mu_j \otimes e_0\right), \ k \geqslant 1, \tag{53}$$

where $\mathcal{F}$ is the operator defined by (39).

In Theorem 39, the system S_d defined by the Eqs. (54) to (58) is a linear system whose nature is to have infinitely many solutions when the zero function is not the only one. The first part of Theorem 39 says in other words that, because bM is known to be the boundary of a Riemann surface, 0 is not the only solution of S_d at least when $d = q^{\infty} + \delta = p$. The second part of Theorem 39 is a kind of reverse. If we manage to find a non-zero solution to S_d where d is some positive integer, one gets a decomposition (62) of the kind we are looking for. Meanwhile, it is not clear that such a decomposition is really meaningful. The next section clarifies this point : the right decomposition can be deduced from (62) by tossing some parasite terms.

Theorem 39 *Assume that* $\frac{\partial^2 G_1}{\partial x^2} \neq 0$*, fix* d *in* $\mathbb{N}^*$*, set* $r = d - \delta$ *and consider* $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{C}[X]^d$ *such that for* $j \in \{1, \ldots, d-1\}$*,* $\deg \mu_j < \deg \mu_d = d$*.*

1. *Assume that* $d = p$ *and* $\mu = \mu^Q$*. Then* $r = q^{\infty}$ *and*

$$\frac{\partial}{\partial y}\left(\frac{1}{\partial^2 G_1/\partial x^2} \frac{\partial}{\partial y}\left[e^{-H} \frac{\partial}{\partial x} \mathcal{F}_0(\mu)\right]\right) = 0 \tag{54}$$

$$\frac{\partial}{\partial x} e^H \mathcal{F}_0(\mu) - \frac{\partial H/\partial x}{\partial^2 G_1/\partial x^2} e^H \frac{\partial}{\partial y}\left[e^{-H} \frac{\partial}{\partial x} \mathcal{F}_0(\mu)\right]$$
$$-e^H \frac{\partial}{\partial x}\left(\frac{1}{\partial^2 G_1/\partial x^2} \frac{\partial}{\partial y}\left[e^{-H} \frac{\partial}{\partial x} \mathcal{F}_0(\mu)\right]\right) = 0 \tag{55}$$

$$\frac{\partial^2 G_1}{\partial x^2}\frac{\partial^{r+1}}{\partial y^{r+1}}\left[e^H\mathcal{F}_0(\mu)\right]-\left(\frac{\partial^{r+1}e^H}{\partial y^{r+1}}\right)\frac{\partial}{\partial y}\left[e^{-H}\frac{\partial}{\partial x}\mathcal{F}_0(\mu)\right]=0 \quad (56)$$

$$\frac{\partial^2 G_1}{\partial x^2}\frac{\partial^r}{\partial y^r}\left[e^H\left(G_1\mathcal{F}_0(\mu)+\mathcal{F}_1(\mu)\right)\right]$$
$$-\frac{\partial^r e^H G_1}{\partial y^r}\frac{\partial}{\partial y}\left[e^{-H}\frac{\partial}{\partial x}\mathcal{F}_0(\mu)\right]=0 \quad (57)$$

$$\mathcal{E}\mathcal{F}_1(\mu)=\Pi\mathcal{E}\mathcal{F}_1(\mu)=\mathcal{F}\mathcal{F}_1(\mu)=\mathcal{F}_0(\mu) \quad (58)$$

and $B_\mu=e^H\left(\mathcal{F}_0(\mu)-\frac{1}{\partial^2 G_1/\partial x^2}\frac{\partial}{\partial y}e^{-H}\frac{\partial}{\partial x}\mathcal{F}_0(\mu)\right)$ *satisfies* $B_\mu(0,y)\underset{\mathbb{C}^*\ni y\to 0}{\to}1.$

2. *Assume that* μ *satisfies the differential linear system* S_d *defined by the equations* (54) *to* (58) *and that* $B_\mu(0,y)\underset{\mathbb{C}^*\ni y\to 0}{\to}1$. *Then there exists* $(c_0,A,B)\in\mathcal{O}(\mathbb{C})\times\mathbb{C}_{r-1}[Y]\times\mathbb{C}_r[Y]$ *with* $B(0)=1$ *and such that*

$$c_0\otimes 1=\frac{1}{\partial^2 G_1/\partial x^2}\frac{\partial}{\partial y}e^{-H}\frac{\partial}{\partial x}\mathcal{F}_0(\mu) \quad (59)$$

$$1\otimes B=\left(\mathcal{F}_0(\mu)-c_0\otimes 1\right)e^H \quad (60)$$

$$1\otimes A=(1\otimes B)\,G_1+e^H\mathcal{F}_1(\mu)-X\otimes B' \quad (61)$$

Moreover, taking in account that e^H *extends holomorphically to* Z, $s_1^\mu=\frac{e^H}{1\otimes B}\mathcal{F}_1(\mu)$ *define a holomorphic function on* $Z\backslash E^\infty$ *such that*

$$G_1=-s_1+X\otimes\frac{B'}{B}+1\otimes\frac{A}{B} \quad (62)$$

and which is a d*-shock waves outside the zero locus of the discriminant* Δ_μ *of* $T^d+\sum_{1\leqslant k\leqslant d}s_k T^{d-k}$ *where* $\left(s_k^\mu\right)=\left(\frac{e^H}{1\otimes B}\mathcal{F}_{d-k}(\mu)\right)_{d\geqslant k\geqslant 1}$.

Proof **1** Set $(A,B)=(A^\infty,B^\infty)$. According to the results quoted in the beginning of this section, we know that

$$1\otimes A=(1\otimes B)\,G_1+e^H\mathcal{F}_1(\mu)-X\otimes B' \quad (63)$$

In particular, the right member of (63) is independent of X. Since $\frac{\partial G_1}{\partial x}=\frac{\partial H}{\partial y}$, we get

$$0=e^{-H}\frac{\partial(1\otimes A)}{\partial x}=e^{-H}\left[(1\otimes B)\frac{\partial H}{\partial y}-\left(1\otimes B'\right)\right]+e^{-H}\frac{\partial}{\partial x}e^H\mathcal{F}_j(\mu)$$
$$=-\frac{\partial(1\otimes B)\,e^{-H}}{\partial y}+\mathcal{D}\mathcal{F}_1(\mu) \quad (64)$$

Hence $\frac{\partial(1\otimes B)e^{-H}}{\partial y} = \mathcal{DF}_1(\mu)$ and we get an entire function c_0 such that

$$\mathcal{PDF}_1(\mu) = \mathcal{P}\frac{\partial}{\partial y}(1\otimes B)\, e^{-H} = (1\otimes B)\, e^{-H} + c_0\otimes 1. \tag{65}$$

As e^{-H} has a usual Laurent series on $\widetilde{Z}$, $\mathcal{PDF}_1(\mu)$ cannot have any logarithmic term, which means that (58) is satisfied. Then, (65) implies that B is given by (60) though we do not know yet c_0. As B does not depend on x, we obtain

$$0 = \frac{\partial}{\partial x} e^{H}\mathcal{F}_0(\mu) - (c_0\otimes 1)\frac{\partial H}{\partial x} e^{H} - \left(c_0'\otimes 1\right) e^{H} \tag{66}$$

As $\frac{\partial H}{\partial y} = \frac{\partial G_1}{\partial x}$, this entails

$$0 = \frac{\partial}{\partial y} e^{-H}\frac{\partial}{\partial x}\mathcal{F}_0(\mu) - (c_0\otimes 1)\frac{\partial^2 G_1}{\partial x^2},$$

which implies that c_0 is actually defined by (59). With this value of c_0, (54) is the statement that c_0 does not depend on y and (66) become the compatibility equation (55). As the right member 60) have to be in $\mathbb{C}_r[Y]$, we also get

$$0 = \frac{\partial^{r+1}}{\partial y^{r+1}}\left[(\mathcal{F}_0(\mu) - c_0\otimes 1)\, e^{H}\right] = \frac{\partial^{r+1}}{\partial y^{r+1}}\left[\mathcal{F}_0(\mu)\, e^{H}\right] - (c_0\otimes 1)\frac{\partial^{r+1} e^{H}}{\partial y^{r+1}}$$

which become (56) when (59) is used for c_0. Moreover, as the right member of (63) have to be in $\mathbb{C}_{r-1}[Y]$, $\deg B < r$ and as (59) has been already proven, we also get

$$\begin{aligned}
0 &= \frac{\partial^r}{\partial y^r}\left[(1\otimes B)\, G_1 - X\otimes B' + e^{H}\mathcal{F}_1(\mu)\right] \\
&= \frac{\partial^r}{\partial y^r}\left[(1\otimes B)\, G_1 + e^{H}\mathcal{F}_1(\mu)\right] \\
&= \frac{\partial^r}{\partial y^r}\left[(\mathcal{F}_0(\mu) - c_0\otimes 1)\, e^{H} G_1 + e^{H}\mathcal{F}_1(\mu)\right] \\
&= \frac{\partial^r}{\partial y^r}\left[e^{H}\left(G_1\mathcal{F}_0(\mu) + \mathcal{F}_1(\mu)\right)\right] - (c_0\otimes 1)\frac{\partial^r}{\partial y^r}\left[e^{H} G_1\right]
\end{aligned}$$

which becomes (57) when (59) is used for c_0. Note that S_d is a differential linear system because of Proposition 34.

2 Conversely, assume that $\frac{\partial^2 G_1}{\partial x^2} \neq 0$ and that the system S_d is satisfied by μ. Then, thanks to (54), the right member of (59) depends only of its first variable so it defines a function c_0. As $\frac{\partial}{\partial x}\left[(\mathcal{F}_0(\mu) - c_0\otimes 1)\, e^{H}\right]$ is equal to the right member of (66), (55) means that $(\mathcal{F}_0(\mu) - c_0\otimes 1)\, e^{H}$ does not depend on x so that (60) defines correctly a function B. Since

$$\frac{\partial^{r+1}}{\partial y^{r+1}}\left[\left(\mathcal{F}_0(\mu)-c_0\otimes 1\right)e^H\right]=\frac{\partial^r}{\partial y^r}\left[\mathcal{F}_0(\mu)\,e^H\right]-(c_0\otimes 1)\frac{\partial^r e^H}{\partial y^r},$$

(56) tells that B is a polynomial of degree at most r. As $B_\mu = e^H\left(\mathcal{F}_0(\mu)-c_0\otimes 1\right) = 1\otimes B$, $B(0)=\lim_{y\to 0^*} B_\mu(0,y)=1$. Denote by $\mathcal{A}$ the right member of (61). Then

$$\begin{aligned}
e^{-H}\frac{\partial\mathcal{A}}{\partial x} &= e^{-H}\left[(1\otimes B)\frac{\partial H}{\partial y}-\left(1\otimes B'\right)\right]+e^{-H}\frac{\partial}{\partial x}\left[e^H\mathcal{F}_1(\mu)\right]\\
&=\mathcal{D}\mathcal{F}_1(\mu)-\frac{\partial\,(1\otimes B)\,e^{-H}}{\partial y}\\
&=\mathcal{D}\mathcal{F}_1(\mu)-\frac{\partial\left(\mathcal{F}_0(\mu)-c_0\otimes 1\right)}{\partial y}=\mathcal{D}\mathcal{F}_1(\mu)-\frac{\partial}{\partial y}\mathcal{F}_0(\mu)
\end{aligned}$$

so that $\frac{\partial\mathcal{A}}{\partial x}=0$ because of (58). Hence (61) defines correctly a function A, which because of (57), is a polynomial of degree at most $r-1$. The other claims of (2) are now consequences of Corollary 33. □

Remark If $c\in\mathbb{C}^*$, $(cA, cB)\in\mathbb{C}[Y]^2$ also verifies $G_1=-s_1+\frac{X\otimes B'+1\otimes A}{B}$. Hence, the condition $B_\mu(0,y)\underset{\mathbb{C}^*\ni y\to 0}{\to}1$ can be seen as a kind of normalization of B. However, the theorem does not address uniqueness.

For a given d, the system S_d can be explicitly written thanks to Proposition 34 which gives formulas for the coefficients of the operators $\mathcal{F}^k$ and $\mathcal{F}_0$. The case $d=0$ is impossible when $\partial^2 G_1/\partial x^2\neq 0$. The case $d=1$ corresponds to the case where the complex lines L_z, $z\in Z$, meets Q only one time. In this case, S_1 is an over determined system on the coefficients of only one affine function μ_1. It can easily be written but is already space consuming. For example, (54) which means that some function of the two variables x and y actually depends only on one of them, takes some space even for $d=p=1$. In this case $q^\infty=1-\delta$, $\delta\in\mathbb{Z}\cap]-\infty,1]$ and taking in account 53, Definition 30, writing $e^{-H}=\sum_{m\leqslant q^\infty-1}\widetilde{h}_m\otimes e_m$ and $\frac{1}{\partial^2 G_1/\partial x^2}=\sum_{m\leqslant 2} g_{1,m}\otimes e_m$, we get after some calculus that

$$\begin{aligned}
&\frac{1}{\partial^2 G_1/\partial x^2}\frac{\partial}{\partial y}\left[e^{-H}\frac{\partial}{\partial x}\mathcal{F}_0(\mu)\right]\\
&=\sum_{m\leqslant q^\infty-1}\sum_{t=m-2}^{q^\infty-3}\left(\sum_{s=t-q^\infty+2}^{-1} g_{1,m-t}\frac{\left(\mu_1 G''_{1,s-2}\right)'}{\kappa_{s-2}}\widetilde{h}_{t+1-s}\right)\otimes e_m
\end{aligned}$$

So the vanishing of the y-derivative of the left member of the above equation yields infinitely many linear equations on the two coefficients of μ_1. Certainty 0 is not the only solution comes only from the fact that we have assumed that p is equal to 1. For a general p, the number of μ_j increases but also their degree. Hence, Theorem 6 which gives an upper bound for p is of practical importance. In this article, we spare space by avoiding to write out completely explicitly S_d.

6.5 Uniqueness of Shock Wave Decompositions

Assume that $\frac{\partial^2 G_1}{\partial x^2} \neq 0$ and let $\mathcal{R} = \underset{d\in\mathbb{N}^*}{\cup} \mathcal{R}_d$ where $\mathcal{R}_d$ is the set of $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{C}[X]^d$ with $\deg \mu_j < d = \deg \mu_d$ for $j \in \{1, \dots, d\}$ such that μ is a solution of S_d, $B_\mu(0, y) \underset{y\to 0^*}{\to} 1$ and $\Delta_\mu \neq 0$ where B_μ and Δ_μ are defined in Theorem 39 . This theorem tells that $\mathcal{R}_p \neq \varnothing$ and that if $\mu \in \mathcal{R}_d$, μ produces by explicit formulas a decomposition of G_1 in the form $-s_1 + X \otimes \frac{B'}{B} + 1 \otimes \frac{A}{B}$ where $-s_1$ is a d-shock waves function in $Z \backslash \left(E^\infty \cup \{\Delta_\mu = 0\}\right)$ and where $A, B \in \mathbb{C}[Y]$ with $\deg A < \deg B = r - \delta$ and $B(0) = 1$. Thus, we know thanks to Proposition 33 that for $z_* \in Z$ outside a proper analytic subset S of Z and for a sufficiently small neighborhood U_* of z_*, there exists shock waves $g_1, \dots, g_d$ on U_* whose images are mutually distinct such that for all $z \in U_*$,

$$
\begin{aligned}
&-s_1(z) = N_{g,1}(z) \\
&N_{g,1}(z) + P(z) = G_1(z) = N_{h,1}(z) + P_1(z) = N_{Q,1}(z) + P_1
\end{aligned}
$$

where the functions h_j are the shock waves $h_j^{z_*}$ defined in Corollary 25, that is the shock waves generated by the collision of Q with the lines L_z, $z \in U_*$.

A priori, nothing guaranties that $\{g_1, \dots, g_d\} = \{h_1, \dots, h_p\}$ because for example, it may happen that there exists a finite non-empty subset J of $\{1, \dots, d\}$ such that $\sum_{j\in J} g_j$ extends as an element of the space $\mathbb{C}(Y)_1[X]$ of rational functions which are affine in X. In this case, $G_1 = N_{\widetilde{g},1} - \widetilde{P}$ with $\widetilde{P} = P - \sum_{j\in J} g_j \in \mathbb{C}(Y)_1[X]$ and $\{\widetilde{g}_1, \dots, \widetilde{g}_{\widetilde{d}}\}$ where $\widetilde{d} = d - \operatorname{Card} \widetilde{J} \in \{0, \dots, d-1\}$. Iterating this reduction, arrises the situation where

$$
\forall J \in \mathcal{P}(\{1, \dots, d\}) \setminus \{\varnothing\}, \ \sum_{j\in J} g_j \notin \mathbb{C}(Y)_1[X]. \tag{67}
$$

The case $d = 0$ happens at the end of these iterations only if at the beginning, $\sum_{1\leqslant j\leqslant d} g_j$ and so G_1, extends as an element of $\mathbb{C}(Y)_1[X]$. The lemma below studies this case.

Lemma 40 *We use notation of Corollary 25. G_1 extends as an element of $\mathbb{C}(Y)_1[X]$ if and only if Q is a domain in a compact connected curve K such that for all z_* in Z_{reg} and z in a sufficiently small neighborhood U_* of z_* in Z_{reg},*

$$
K \cap L_z = \left\{\left(1 : h_j^{z_*}(z) : -x - y h_j^{z_*}(z)\right);\ 1 \leqslant j \leqslant p\right\} = Q \cap L_z.
$$

Proof Suppose at first that K is a compact curve with the above properties. Fix z_* and U_* as in the statement. Since K is an algebraic curve, we know from Abel's work that $\sum_{1\leqslant j\leqslant p} h_j^{z_*} \in \mathbb{C}(Y)_1[X]$ (see e.g., [15]). It follows that $G_1 = N_{h^{z*},1} + P_1$ is, on U_* and so on Z, rational in y and affine in x.

Conversely, suppose that $G_1 \in \mathbb{C}(Y)_1[X]$. Then $N_{h^{z*},1} = G_1 - P_1$ is on U_* algebraic in y and affine at x. Since $\left\{\left(1 : h_j^{z_*}(z) : -x - y h_j^{z_*}(z)\right);\ 1 \leqslant j \leqslant p\right\} =$

$Q \cap L_z$ for all $z \in U_*$, a theorem of Wood [37] states the existence of a compact algebraic curve K of degree p containing Q. Since the degree of K is p, $K \cap L_z = \left\{\left(1 : h_j(z) : -x - yh_j(z)\right);\ 1 \leqslant j \leqslant \lambda\right\} = Q \cap L_z$ for all $z \in U$. □

In case G_1 is algebraic in y and affine in x, the algebraic curve K of Lemma 40 is known in a neighborhood of bQ. We can then pick generically homogeneous coordinates w in order at least one line L_z, $z \in U$, meets $K \backslash Q$. We are thus brought back to the general case since Lemma 40 ensures then that even after reduction, d is not zero.

With Proposition 41 which is proved thanks to results of Henkin [15] and of Collion [8], we know that when this reduction ends, the remaining shock waves functions are those we are looking for.

Proposition 41 *Notation remains as stated in this section and we suppose* (67) *verified. For the case where Q is contained in an algebraic curve, $\widehat{Q}$ denoting then the smallest one with this property, we suppose that* $(0 : 1 : 0) \notin \widehat{Q}$ *and at least one of the lines L_z, $z \in U$, meets Q and $\widehat{Q} \backslash Q$. That being so, $\{g_1, \dots, g_d\} = \left\{h_1, \dots, h_p\right\}$ and $P = P_1$.*

Proof After a possible renumbering, we assume that $g_\nu = h_\nu$, $1 \leqslant \nu \leqslant t \in \mathbb{N}$ and $\{g_{t+1}, \dots, g_d\} \cap \left\{h_{t+1}, \dots, h_p\right\} = \varnothing$.

1. Suppose that Q is not contained in an algebraic curve. Then $d \in \mathbb{N}^*$ because otherwise, $N_{h,1} \in \mathbb{C}(Y)_1[X]$ and G_1, which is the sum of $N_{h,1}$ and P_1, appears to be the restriction to U of an element of $\mathbb{C}(Y)_1[X]$. According to Lemma 40, this would contradict our hypothesis.
   Suppose $t < \min(p, d)$. Up to a change of the reference point z_* and a decrease of U_*, we suppose that the curves $H_\nu = \{(1 : h_\nu(z) : -x - yh_\nu(z));\ z \in U_*\}$, $t + 1 \leqslant \nu \leqslant p$ and $C_\nu = \{(1 : g_\nu(z) : -x - yg_\nu(z));\ z \in U_*\}$, $t + 1 \leqslant \nu \leqslant d$ are smooths and mutually disjoint. We then denote φ the differential form defined on the union C of these curves by $\varphi\big|_{H_\nu} = d\frac{w_1}{w_0}$ when $t + 1 \leqslant \nu \leqslant p$ and $\varphi\big|_{C_\nu} = -d\frac{w_1}{w_0}$ when $t + 1 \leqslant \nu \leqslant d$. We note AR the Abel–Radon transform of the current $\varphi \wedge [C]$. By definition (see [15], [8] or [16]),

$$AR = d\left(\sum_{t+1\leqslant\nu\leqslant p} h_\nu - \sum_{t+1\leqslant\nu\leqslant q} g_\nu\right).$$

   But hypothesis imply,

$$\sum_{t+1\leqslant\nu\leqslant p} h_\nu - \sum_{t+1\leqslant\nu\leqslant q} g_\nu = N_{h,1} - N_{g,1} = R - P_1.$$

   AR is hence algebraic in the sense of [8] so that Theorem 1.2 of [8] applies and gives in particular the existence of an algebraic curve Λ containing C. Since Q is not contained in Λ, the connectedness of Q entails that none of the curves H_ν is contained in Λ and thus that $\left\{h_1, \dots, h_p\right\} \subset \{g_1, \dots, g_d\}$. Hence, $\sum_{p<\nu\leqslant d} g_\nu$ is an algebraic function affine in x, which is impossible due to the reduction made on $\left(g_j\right)_{1\leqslant j\leqslant d}$. So, $t = \min(p, d)$.

If $t = d < p$, the relation $N_{g,1} + P = N_{h,1} + P_1$ reads also $h_{t+1} + \cdots + h_p = P_1 - P \in \mathbb{C}(Y)_1[X]$ and the theorem of Wood implies, since Q is connected, that Q is contained in an algebraic curve which is excluded by hypothesis. If $t = p < d$, $g_{t+1} + \cdots + g_d = N_{g,1} - N_{h,1} + P - P_1 \in \mathbb{C}(Y)_1[X]$ which is excluded by to the reduction made on the family (g_j).
Finally $t = p = d$, $\{h_1, \ldots, h_p\} = \{g_1, \ldots, g_d\}$ and $P_1 = R$.

2. Suppose now that Q is contained in an algebraic curve $\widehat{Q}$, minimal with respect to inclusion. By hypothesis $(0:1:0) \notin \widehat{Q}$, and $\widehat{Q} \backslash Q$ is bounded by $-\partial Q$. Up to a change of reference point z_* and a decrease of U_*, we can suppose that for all $z \in U_*$, L_z meets transversely $\widehat{Q}$. We note then $h_{p+1}, \ldots, h_{\widehat{p}}$ the shock waves on U_* such that for all $z \in U$,

$$(\widehat{Q} \backslash Q) \cap L_z = \{(1 : h_\nu(z) : -x - y h_\nu(z))\,;\ p+1 \leqslant \nu \leqslant \widehat{p}\}.$$

Since $\widehat{Q}$ is an algebraic curve, $N_{\widehat{Q},1} \overset{def}{<=} N_{h,1} + N_{h_{p+1},\ldots,h_{\widehat{p}}} \overset{def}{=} N_{h,1} + \widehat{N}_1$ is algebraic and affine in x. Hence

$$N_{g,1} + \widehat{N}_1 = N_{g,1} - N_{h,1} + N_{\widehat{Q},1} = P_1 - R + N_{\widehat{Q},1} \in \mathbb{C}(Y)_1[X]$$

The sum $N_{g,1} + \widehat{N}_1$ can be written $\sum_{1 \leqslant \lambda \leqslant s} c_\lambda f_\lambda$ where $f_1, \ldots f_s$ are the mutually distinct functions of the union of $\{g_\nu;\ 1 \leqslant \nu \leqslant q\}$ and $\{h_\nu;\ p+1 \leqslant \nu \leqslant \widehat{p}\}$ and where $c_\lambda = 2$ if f_λ is in the intersection of this two sets and 1 otherwise. As previously we can choose z_* and U_* in order that the functions f_λ has images mutually disjoint. We can then introduce the form ψ which on $F_\lambda = \{(1 : f_\lambda(z) : -x - y f_\lambda(z))\,;\ z \in U\}$ is $d\frac{w_1}{w_0}$ if $c_\lambda = 1$ and $2d\frac{w_1}{w_0}$ if $c_\lambda = 2$. The form $\sum_{1 \leqslant \lambda \leqslant s} c_\lambda d f_\lambda$ is the Abel–Radon transform $\psi \wedge [F]$ where $F = \cup F_\lambda$. This one being algebraic, the principal theorem of Henkin in [15] applies and gives in particular the existence of an algebraic curve $\widetilde{F}$ and an algebraic form Ψ such that for all λ, $\Psi\big|_{F_\lambda} = \psi$ and for all $z \in U_*$, $\widetilde{F} \cap L_z = \cup L_z \cap F_\lambda$. Given that $\widehat{Q} \cap \widetilde{F}$ contains $(\widehat{Q} \backslash Q) \underset{z \in U_*}{\cup} L_z$, $\widehat{Q} \subset \widetilde{F}$. If $\widetilde{F} \neq \widehat{Q}$, $\overline{\widehat{Q} \backslash \widetilde{F}}$ is an algebraic curve whose intersections with the L_z, $z \in U_*$, are parametrized with a sub-family of the g_j. This is impossible since because of hypothesis, $d \neq 0$ and no sub-family of (g_j) has a sum algebraic in y and affine in x. Thus, $\widehat{Q} = \widetilde{F}$ and when $z \in U_*$, $\widehat{Q} \cap L_z$ is the union of $(\widehat{Q} \backslash Q) \cap L_z$ and of $\{(1 : g_j(z) : -x - y g_\lambda(z))\,;\ 1 \leqslant j \leqslant d\}$. This entails $\{h_1, \ldots, h_p\} = \{g_1, \ldots, g_d\}$ and $P_1 = R$. □

7 Genus of a Riemann Surface with Boundary

Formula (71) of Theorem 44 links the genus $g(M)$ of M to data associated with the complex structure $\mathcal{C}_\sigma$ of (M, σ). It is probably well known to specialists but we did not find a reference for it. The link with the complex Dirichlet–Neumann operator θ_c^σ comes from Corollary 45. The formula so obtained is not yet effective because we do not know the Euler characteristic of $\overline{M}$. But as explained in Theorem 6 whose proof is given at the end of this section, Theorem 2 and Lemma 47 enable to deduce from Corollary 45 an effective bound for the key number p of unknown shock waves sought in the reconstruction process described in Sect. 2.

Let us recall that $g(M)$ is by definition the genus of the compact manifold obtained by gluing κ (pairwise disjoint) conformal disks along the κ connected components of bM. In [6], Belishev gives for a connected boundary the formula

$$2g(M) = \text{rg}\left(T + \left(N^\nu J\right)^2 T\right)$$

where T is the tangential derivation, N^ν is the Dirichlet–Neumann operator of $(M, \mathcal{C}_\sigma)$ in its metric issue, that is the one which to $u \in C^\infty(bM)$ associates the normal derivative along bM of the harmonic extension of u to M and J is the natural primitivization operator defined on the space of function u whose integral over ∂M is 0. However, a priori calculus of the rank of $T + (N^\nu J)^2 T$ is not easy and this formula is limited to connected boundaries. To bypass this difficulty, [7] and [33] propose to use Dirichlet–Neumann operators acting on forms. This gives simple formulas for $g(M)$ when the conductivity reduces to a complex structure but it is not clear that these operators have physics meaning.

To produce formulas whose ingredients are computable from N_d^σ, we use special volume forms for M and special metrics for the bundle $\Lambda^{1,0}T^*\overline{M}$ of the $(1,0)$-forms on $\overline{M}$.

Definition 42 Let M be a Riemann surface with boundary and ρ a defining function of bM, which means that $\rho \in C^\infty\left(\overline{M}, \mathbb{R}\right)$ is such that $\rho|_M < 0$, $\rho|_{bM} = 0$ and $(d\rho)_s \neq 0$ for any $s \in bM$. Under these conditions, any section ω of $\Lambda^{p,q}T^*\overline{M}$ of class C^k, $k \geqslant 1$, on an open subset U of $\overline{M}$ can be written in the form $\omega_0 + \rho\omega_1$ where ω_j, $j = 0, 1$, is a section of $\Lambda^{p,q}T^*\overline{M}$ on U of class C^{k-j}, the couple $(\omega_\rho^{(0)}, \omega_\rho^{(1)}) = (\omega_0|_{U\cap bM}, \omega_1|_{U\cap bM})$ being the same for all (ω_0, ω_1) such that $\omega = \omega_0 + \rho\omega_1$. The fact that $\omega_\rho^{(1)}$ vanishes does not depend of the choice of the chosen defining function ρ. ω is said tangent to bM when $\omega_\rho^{(1)} = 0$.

The existence of a decomposition $\omega = \omega_0 + \rho\omega_1$ follows from the fact that ρ can be chosen as part of a system of real coordinates for $\overline{M}$ near bM. Uniqueness of $(\omega_\rho^{(0)}, \omega_\rho^{(1)})$ proceed from the same reason and if ρ' is another defining function of bM, one can write $\rho' = \lambda\rho$ where λ is a never vanishing function, so that vanishing of $\omega_{\rho'}^{(1)} = \lambda|_M\,\omega_\rho^{(1)}$ and $\omega_\rho^{(1)}$ are simultaneous.

Note that when M is equipped with a Hermitian metric and ρ is the distance to bM, $\omega_\rho^{(1)} = \frac{\partial\omega}{\partial\rho}|_{bM}$ is nothing else that the derivative of ω with respect to the unitary vector directing the exterior normal to $\overline{M}$ at points of bM. The lemma below ensures the existence of volume forms satisfying the hypothesis of this section's main theorem.

Lemma 43 *Let (M, σ) be a conductivity structure. Then $\overline{M}$ admits a volume form of class C^2 tangential to its boundary and whose restriction to bM is computable from boundary data associated with (M, σ).*

Proof As it is pointed out at the end of Sect. 3, we can design from boundary data a smooth section μ_0 over bM of the bundle of volume forms of $\overline{M}$. Let $\widehat{M}$ be the double of M (see the proof of Theorem 44 for a detailed construction), V an arbitrary volume form of class C^2 on $\widehat{M}$ and $\rho \in C^\infty(\widehat{M}, \mathbb{R})$ such that $M = \{\rho < 0\}$, $bM = \{\rho = 0\}$ and $(d\rho)_s \neq 0$ for any $s \in bM$. Using the Whitney extension theorem (see [3,4, Proposition 2.2]), one can constructs a section $\widetilde{V}$ of $\Lambda^{1,1}T\widehat{M}$ of class C^2 such that $\widetilde{V}\,|_{bM} = \mu_0$ and $V_\rho^{(1)} = \frac{\partial \widetilde{V}}{\partial \rho}\,|_{bM} = 0$. By continuity, there exists a neighborhood Σ of bM in $\widehat{M}$ such that $\widetilde{V}\,|_\Sigma$ is a volume form. Choose $\chi \in C^\infty(M, [0, 1])$ equal to 1 in a neighborhood of bM in Σ and whose support is contained in Σ. $W = \chi\widetilde{V} + (1-\chi)V$ is a volume form W of class C^2 on $\widehat{M}$ such that $W_\rho^{(1)} = \frac{\partial W}{\partial \rho}\,|_{bM} = 0$. □

Let (M, σ) be a conductivity structure and μ a volume form for $\overline{M}$ as in Lemma 43. Denote $*$ and $\Lambda^{1,0}T^*\overline{M}$ the conjugation operator and the bundle of $(1,0)$-forms associated with $(M, \mathcal{C}_\sigma)$. For simplicity of notation, we set in this section $\partial = \partial^\sigma = d - \overline{\partial}$ where $\overline{\partial} = \overline{\partial}^\sigma$ is the Cauchy–Riemann operator of $(M, \mathcal{C}_\sigma)$. We equip $\Lambda^{1,0}T^*\overline{M}$ with the metric h^* defined for $s \in \overline{M}$ and $\alpha, \beta \in \Lambda^{1,0}T_s^*\overline{M}$ by

$$h_s^*(\alpha, \beta) = \frac{\alpha \wedge *_s\overline{\beta}}{\mu_s} \tag{68}$$

Denote by D the Chern connection of h. A definition can be found in [9], [11, p. 73] or [36] but we recall here some basics. Consider a fixed non- vanishing smooth section e of $\Lambda^{1,0}T^*\overline{M}$ over an open set W of $\overline{M}$, holomorphic in $W \cap M$, and let $|e|_{h^*} = \sqrt{h^*(e, e)}$ be the point wise norm of e with respect to h^*. Then,

$$\eta_e = \frac{\partial\,|e|_{h^*}^2}{|e|_{h^*}^2} = \partial \ln h^*(e, e) \tag{69}$$

is the connection form of D associated with the holomorphic frame e, the curvature $\Theta = d\eta_e = \overline{\partial}\eta_e$ of D does not depend of e and if $\omega = \lambda e$, $\lambda \in C^\infty(W)$, is any smooth section of $\Lambda^{1,0}T_s^*\overline{M}$ over W, $D\omega$ is the 1-form valued in $\Lambda^{1,0}T_W^*\overline{M}$ given by $D\omega = (d\lambda)\,e + \eta_e\omega$. If ω is also holomorphic in $W \cap M$, we get $\frac{D\omega}{\omega} = \frac{\partial\lambda}{\lambda} + \eta_e$. Note that in particular, $\eta_e = \frac{De}{e}$.

When $\sigma\Big|_{T_{bM}^*\overline{M}}$ is assumed to be known, so it is for $\frac{D\omega}{\omega}\,|_{bS}$ when ω is a $(1,0)$-form near bM. Indeed, thanks to Theorem 5, we know that with the nodal Riemann surface $\mathcal{M}$ designed by Theorem 2, we can find smooth non-vanishing sections of $\Lambda^{1,0}T_{bM}^*\overline{M}$ which extends holomorphically to M by computing $\theta_c^\sigma u$ for adequate $u \in C^\infty(bM)$. For such an u and its $\mathcal{C}_\sigma$-harmonic extension to M, $\partial\widetilde{u}$ is a holomorphic frame for $\Lambda^{1,0}T_W^*\overline{M}$ where $W = \{\partial\widetilde{u} \neq 0\}$ and (69) becomes

$$\frac{D\partial\widetilde{u}}{\partial\widetilde{u}} = \eta_{\partial\widetilde{u}} = \partial \ln h^*(\partial\widetilde{u}, \partial\widetilde{u}) = \partial \ln\left(\frac{\partial\widetilde{u} \wedge *\overline{\partial\widetilde{u}}}{\mu}\right) \tag{70}$$

Since the complex structure of $\overline{M}$ is known along bM and since $\partial\widetilde{u}$ is holomorphic, the Cauchy–Riemann equations enable to compute the normal derivative of $\partial\widetilde{u}$ from its tangential derivative. This means that in (70), derivatives coming from $\partial\widetilde{u}$ are computable on bM from available boundary data. As the volume form μ is tangential to bM, its normal derivative is zero on bM and its tangential derivative is known on bM. Hence $\frac{D\partial\widetilde{u}}{\partial\widetilde{u}}|_{bM}$, that is $\eta_{\partial\widetilde{u}}|_{bM}$, is computable from available boundary data, what we had to check.

Note that for the computation of a connection form along bM, it is not mandatory to use a holomorphic frame of the form $\partial\widetilde{u}$. Indeed, let $F : M \to \mathcal{M}$ be the normalization of the nodal complex curve $\mathcal{M}$ of $\mathbb{C}^2$ designed by Theorem 2 and let γ be an open subset of $b\mathcal{M}$. We can choose any non-vanishing smooth section φ of $\Lambda^{1,0}T^*_\gamma\overline{\mathcal{M}}$ which extends into a $(1,0)$-form $\widetilde{\varphi}$ smooth on $\mathcal{W}$ and holomorphic on $\mathcal{W}\backslash b\mathcal{M}$ where $\mathcal{W}$ is an open subset of $\overline{\mathcal{M}}$ containing γ and such that $\mathcal{W}\backslash b\mathcal{M} \subset \operatorname{Reg}\mathcal{M}$. Let $W = F^{-1}(\mathcal{W}) \cup f^{-1}(\gamma)$ where $f = F\Big|_{bM}^{b\mathcal{M}}$. Then $\left(F\Big|_{W\backslash bM}^{\mathcal{W}\backslash\mathcal{B}\mathcal{M}}\right)^*\widetilde{\varphi}$, which we abbreviate into $F^*\widetilde{\varphi}$, is a holomorphic $(1,0)$-form of $(M, \mathcal{C}_\sigma)$ which extends smoothly to W and whose restriction to $f^{-1}(\gamma)$ is $F^*\varphi$. The connection form $\eta_{F^*\widetilde{\varphi}} = \partial \ln h^*(F^*\widetilde{\varphi}, F^*\widetilde{\varphi})$ associated with $F^*\widetilde{\varphi}$ is computable on bM from available boundary data as before. Moreover, since F is holomorphic from $(M, \mathcal{C}_\sigma)$ to $\mathcal{M}$, we can also make computation on $\mathcal{M} \subset \mathbb{C}^2$ and then pull back the result to bM by F :

$$\eta_{F^*\widetilde{\varphi}} = F^*\partial \ln \frac{\partial\widetilde{\varphi} \wedge *\overline{\partial}\widetilde{\varphi}}{F_*\mu}$$

where here $\partial = d - \overline{\partial}$ and $\overline{\partial}$ is the Cauchy–Riemann operator of $\overline{\mathcal{M}}$ and $*$ its Hodge star operator.

We can now state Theorem 44. It is more about the Riemann surface $(M, \mathcal{C}_\sigma)$ than (M, σ).

Theorem 44 *Let (M, σ) be a conductivity structure and κ the number of connected components of bM. Choose a volume form μ as in Lemma 43, equip the bundle $\Lambda^{1,0}T^*\overline{M}$ of $(1,0)$-forms of $(M, \mathcal{C}_\sigma)$ with the metric h^* defined by (68) and denote by D its Chern connection. Then, when ω is a $\mathcal{C}_\sigma$-meromorphic $(1,0)$-form on $\overline{M}$, without pole or zero on bM and*

$$\frac{1}{2\pi i}\int_{\partial M} \frac{D\omega}{\omega} = N_z(\omega) - N_p(\omega) + 2 - 2g(M) - \kappa \tag{71}$$

where $N_z(\omega)$ and $N_p(\omega)$ are respectively the number of zeros and of poles of ω counted with their multiplicity or order.

Remark Suppose that μ' is a volume form for $\overline{M}$ with the same properties as μ. The function $\lambda : \overline{M} \to \mathbb{R}$ such that $\mu = e^{2\lambda}\mu'$ satisfies $D_\mu = D_{\mu'} - \partial\lambda$, which gives $\int_{\partial M} \frac{D_\mu\omega}{\omega} = \int_{\partial M} \frac{D_{\mu'}\omega}{\omega} - \int_{\partial M} j^*_{bM}\partial\lambda$. (71) indicates then $\int_{\partial M} j^*_{bM}\partial\lambda = 0$. To check this a priori, let us consider a defining function ρ of bM. From the relation $\frac{\partial\mu}{\partial\rho} = e^\lambda\frac{\partial\mu'}{\partial\rho} + \mu'\frac{\partial\lambda}{\partial\rho}$ which holds on bM, we get $\frac{\partial\lambda}{\partial\rho}|_{bM} = 0$. Equip M with a

Hermitian metric and consider a smooth section (ν,τ) of $\left(T_{bM}\overline{M}\right)^2$ such that for any $s\in bM$, (ν_s,τ_s) is an orthonormal direct basis of $T_s\overline{M}$. Then, for all $s\in bM$, $(\partial\lambda)_s=\frac{1}{2}\left((\nu\lambda)_s-i\,(\tau\lambda)_s\right)\left(\tau_s^*+i\nu_s^*\right)$ where $\left(\tau_s^*,\nu_s^*\right)$ is the dual basis of (ν_s,τ_s). When $s\in bM$, the fact that $\frac{\partial\lambda}{\partial\rho}(s)=0$ indicates that $(d\lambda)_s\in\mathbb{R}\tau_s^*$ and hence $(\nu\lambda)_s=0$, which gives $(\partial\lambda)_s=\frac{1}{2i}(\tau\lambda)_s\left(\tau_s^*-i\nu_s^*\right)$. Thus, $j_{bM}^*\partial\lambda=\frac{1}{2i}(\tau\lambda)\,\tau^*\,|_M=\frac{1}{2i}j_{bM}^*d\lambda$. So, $j_{bM}^*\partial\lambda$ is exact and its integral over ∂M is zero.

With Formula (74), we obtain Corollary 45 as a particular case of Theorem 44.

Corollary 45 *Hypothesis and notation remains as in Theorem 44. Let $u\in C^\infty(bM)$, $\widetilde{u}$ its $\mathcal{C}_\sigma$-harmonic extension to M and q the number $N_z\left(\partial^\sigma\widetilde{u}\right)$ of zeros of $\partial^\sigma\widetilde{u}$ counted with multiplicity where $\partial^\sigma=d-\overline{\partial}^\sigma$ and $\overline{\partial}^\sigma$ is the Cauchy–Riemann operator of $(M,\mathcal{C}_\sigma)$. We assume that $\partial^\sigma\widetilde{u}$ has no zero on bM. Then*

$$q=\frac{1}{2\pi i}\int_{\partial M}\frac{D\partial^\sigma\widetilde{u}}{\partial^\sigma\widetilde{u}}-\chi\left(\overline{M}\right). \tag{72}$$

Proof of Theorem 44 Let us begin by detailing a construction of the double $\widehat{M}$ of M which for example can be found in [2]. Let $\mathcal{U}$ be an atlas of M. We use the following notation : for $\nu\in\{-1,+1\}$ and $X\subset\overline{M}$, $X_\nu=X\times\{\nu\}$ and if $(s,\nu)\in M_1\cup M_{-1}$, $\pi(s,\nu)=s$; when $s\in bM$, the points of $\widehat{M}=\overline{M_1}\cup\overline{M_{-1}}$ of the form $(s,-1)$ and $(s,1)$ are identified and form the real curve γ. M_1 is equipped with the complex structure associated with the atlas $\mathcal{U}_1$ formed by the maps $\varphi_1:U_1\ni p\mapsto\varphi(\pi(p))$ where $\varphi:U\to\mathbb{C}$ is arbitrary $\mathcal{U}$. For M_{-1}, we use the atlas $\mathcal{U}_{-1}$ of the maps $\varphi_{-1}:U_{-1}\ni p\mapsto-\overline{\varphi(\pi(p))}$, $\varphi:U\to\mathbb{C}$ arbitrary in $\mathcal{U}$. One gets an atlas $\widehat{\mathcal{U}}=\mathcal{U}_1\cup\mathcal{U}_b\cup\mathcal{U}_{-1}$ giving to $\widehat{M}$ a complex structure by letting $\mathcal{U}_b$ be the set of maps φ_b defined as follows : consider a boundary chart for $\overline{M}$ that is $\varphi\in C^\infty(U,\mathbb{C})$ where U is an open subset of $\overline{M}$ such $b_UM=\overline{U}\cap bM$ is open in bM, $\varphi(U\backslash M)=\mathbb{D}^+=\mathbb{D}\cap\{\mathrm{Im}>0\}$ and $\varphi(b_UM)=]-1,1[$; φ_b is the map from $U_b=U_1\cup U_{-1}$ to $\mathbb{C}$ obtained by setting $\varphi_b(s,1)=\varphi(s)$ and $\varphi_b(s,-1)=\overline{\varphi(s)}$ for any $s\in U$.

We define volume forms μ_1 and μ_{-1} on $\overline{M}_1$ and $\overline{M}_{-1}$ by letting when $\varphi:U\to\mathbb{C}$ is a chart of $\overline{M}$,

$$\begin{aligned}(\varphi_{1*}\mu_1)_z&=(\varphi_*\mu)_z=\lambda_\varphi(z)\,i\mathrm{d}z\wedge\mathrm{d}\overline{z},\ z\in U\\(\varphi_{-1*}\mu_{-1})_w&=(\varphi_*\mu)_{-\overline{w}}=\lambda_\varphi(-\overline{w})\,i\mathrm{d}w\wedge\mathrm{d}\overline{w},\ \ -\overline{w}\in U\end{aligned}$$

This definition is obviously coherent for μ_1. Suppose $\psi:V\to\mathbb{C}$ is another chart of M and $\psi_*\mu=\lambda_\psi i\mathrm{d}z\wedge\mathrm{d}\overline{z}$. Denote $\Phi:\psi(U\cap V)\ni z\mapsto\varphi\left(\psi^{-1}(z)\right)$ the change of chart from ψ to φ. Hence, $\lambda_\psi=\left|\Phi'\right|^2\lambda_\varphi\circ\Phi$. The transition map from $\psi_{-1}:V_{-1}\to\mathbb{C}$ to $\varphi_{-1}:U_{-1}\to\mathbb{C}$ is then the map Φ_{-1} defined on $\psi_{-1}(V_{-1}\cap U_{-1})=-\overline{\psi}(U\cap V)$ by

$$\begin{aligned}\Phi_{-1}(w)&=\varphi_{-1}\left((\psi_{-1})^{-1}w\right)=\varphi_{-1}\left(\psi^{-1}(-\overline{w}),-1\right)\\&=-\overline{\varphi}\left(\psi^{-1}(-\overline{w})\right)=-\overline{\Phi(-\overline{w})}.\end{aligned}$$

Thus,

$$\begin{aligned}\Phi_{-1}^{*}\left(\lambda_{\varphi}\left(-\overline{z}\right)i\mathrm{d}z\wedge\mathrm{d}\overline{z}\right)&=\lambda_{\varphi}\left(\Phi\left(-\overline{w}\right)\right)i\left(-\frac{\partial\overline{\Phi\left(-\overline{w}\right)}}{\partial w}\mathrm{d}w\right)\wedge\left(\left(-\frac{\partial\Phi\left(-\overline{w}\right)}{\partial\overline{w}}\mathrm{d}\overline{w}\right)\right)\\&=\lambda_{\varphi}\left(\Phi\left(-\overline{w}\right)\right)\left|\Phi'\left(-\overline{w}\right)\right|^{2}i\mathrm{d}w\wedge\mathrm{d}\overline{w}\\&=\lambda_{\psi}\left(-\overline{w}\right)i\mathrm{d}w\wedge\mathrm{d}\overline{w},\end{aligned}$$

which proves the coherency of the definition of μ_{-1}.

The forms μ_1 and μ_{-1} continuously glue along γ in a volume form $\widehat{\mu}$ for $\widehat{M}$. Indeed, consider a boundary chart $\varphi : U \to \mathbb{C}$ and $\overline{M}$ and the chart $\varphi_b : \overline{U_b} \to \mathbb{C}$ defined as above. Set $\varphi_* \mu = \lambda_\varphi i\mathrm{d}z \wedge \mathrm{d}\overline{z}$. When $s \in U$, $\varphi_b\left(s,-1\right) = \overline{\varphi\left(s\right)}$ and $\varphi_{-1}\left(s,-1\right) = -\overline{\varphi\left(s\right)}$. Hence, the transition map from φ_b to φ_{-1} is $\overline{U} \to -\overline{U}$, $z \mapsto -z$. Thus,

$$\left(\left(\varphi_b\right)_* \mu_{-1}\right)_z = \lambda_\varphi\left(\overline{z}\right) i\mathrm{d}z \wedge \mathrm{d}\overline{z} = \left(\varphi_{1*}\mu_1\right)_{\overline{z}}$$

for all $z \in \mathbb{D}^- \cup \left]-1,1\right[$ where $\mathbb{D}^- = \mathbb{D} \cap \{\mathrm{Im} > 0\}$. Given that $\varphi\left(b_U M\right) = \left]-1,1\right[$, this shows that $\mu_{-1} = \mu_1$ at each point of $\gamma \cap U$. Develop in a neighborhood in $\mathbb{D}^+ \cup \left]-1,1\right[$ the function λ_φ under the form $\lambda_{\varphi,0}\left(x\right) + \lambda_{\varphi,1}\left(x\right) y + \lambda_{\varphi,2}\left(x\right) y^2 + o\left(y^2\right)$. As μ is tangential to bM by hypothesis, $0 = \lambda_{\varphi,1}$ on bM and it appears that $\widehat{\mu}$ is of class C^2.

One can now equip $\Lambda^{1,0} T_p^* \widehat{M}$, $p \in \widehat{M}$, with the metric $\widehat{h_p^*}$ defined by

$$\widehat{h_p^*}\left(\alpha,\beta\right) = \frac{\alpha \wedge *\overline{\beta}}{\widehat{\mu}_p}$$

for all $\alpha, \beta \in \Lambda^{1,0} T_p^* \widehat{M}$. The Chern connection D of $\widehat{h^*}$ is thus of class C^2. Consider a meromorphic $(1,0)$-form ω on $\overline{M}$ without pole nor zero on bM. As recalled previously, when e is a local holomorphic frame for $\Lambda^{1,0} T^* \widehat{M}$ and $\omega = \lambda e$, $\frac{D\omega}{\omega} = \frac{d\lambda}{\lambda} + \widehat{\eta}$ where $\widehat{\eta}$ is the connection form of D associated with e. Since λ has to be meromorphic with same zeros and poles as ω where the formula $\omega = \lambda e$ is valid and since $d\widehat{\eta}$ is the curvature $\widehat{\Theta}$ of D, the Stokes formula, applied to the domains obtained by removing from M_1 arbitrary small conformal disks around the zeros and poles of ω, gives

$$\frac{1}{2\pi i}\int_{\partial M}\frac{D\omega}{\omega} = \frac{1}{2\pi i}\int_{\partial M_1}\frac{D\omega}{\omega} = N_z\left(\omega\right) - N_p\left(\omega\right) - \frac{1}{2\pi}\int_{M_1} i\widehat{\Theta} \tag{73}$$

If one agrees that $\frac{1}{2\pi}\int_{M_1} i\widehat{\Theta} = \frac{1}{2\pi}\int_{M_{-1}} i\widehat{\Theta}$, (71) results from (73) and (74) because, since $\widehat{M}$ is compact and D of class C^2, we get then $\frac{1}{2\pi}\int_{M_1} i\widehat{\Theta} = \frac{1}{2}\frac{1}{2\pi}\int_{\widehat{M}} i\widehat{\Theta} = \frac{1}{2}c_1(\widehat{M}) = g(\widehat{M}) - 1$ where $c_1(\widehat{M})$ is the first Chern class of $\widehat{M}$. A proof of the last equality can be found for example in [35, Theorem 9.1 p. 284 of 1st ed.] or in [9, p. 319] where it is called Hurwitz's formula.

Denote j the natural symmetry of $\widehat{M}$ with respect to γ and c the conjugation of $\mathbb{C}$. When $\varphi : U \to \mathbb{C}$ is a chart of M, the expression of j in the charts φ_1 and φ_{-1} is $\varphi_{-1} \circ j \circ (\varphi_1)^{-1}$ that is $-c\left|{}_{U}^{\overline{U}}\right.$. Thus, j exchange the orientations of M_1 and M_{-1} which gives

$$\int_{M_1} \widehat{\Theta} = -\int_{M_{-1}} j^* \widehat{\Theta}.$$

When $\psi : V \to \mathbb{C}$ is a chart of $\widehat{M}$, the map $\widetilde{\psi} : j(V) \to \mathbb{C}$ defined by $\widetilde{\psi} = \overline{\psi} \circ j$ is also a chart of $\widehat{M}$. This enables (see [2] for example) starting with a section ω of $\Lambda T^* \widehat{M}$ on a subset X of $\widehat{M}$, to define a section $\widetilde{\omega}$ of $\Lambda T^* \widehat{M}$ on $j(X)$ by setting for any chart $\psi : V \to \mathbb{C}$ of $\widehat{M}$ such that $V \cap X \neq \varnothing$, $\left(\widetilde{\psi}_* \widetilde{\omega}\right)_w = \beta(\overline{w})\,\mathrm{d}w + \alpha(\overline{w})\,\mathrm{d}\overline{w}$ when $\psi_* \omega = \alpha \mathrm{d}z + \beta \mathrm{d}\overline{z}$ and $\overline{w} \in \psi(V \cap X)$. In particular, ω being a fixed section of $\Lambda^{1,0} T^* M$ without zero on $\overline{M}$, holomorphic on bM and of class C^∞ on $\overline{M}$, $\omega_1 = \pi^* \omega$ (resp. $\omega_{-1} = \widetilde{\omega_1}$) is a section of $\Lambda^{1,0} T^* \widehat{M}$ without zero on $\overline{X_1}$ (resp. $\overline{X_{-1}}$), holomorphic on X_1 (resp. X_{-1}) and of class C^∞ on (resp. $\overline{X_{-1}}$). Setting $f_\nu = \ln \widehat{h}(\omega_\nu)^2$, we then knows that

$$\widehat{\Theta}\,\Big|_{M_\nu} = d\partial f_\nu, \;\; \nu = \pm 1.$$

Fix a chart $\varphi : U \to \mathbb{C}$ and set $\varphi_* \omega = \alpha \mathrm{d}z$. Then $(\varphi_1)_* \omega_1 = \alpha \mathrm{d}z$ and $(\widetilde{\varphi_1})_* \omega_{-1} = \overline{\alpha(\overline{w})}\mathrm{d}w$. Since $*$ acts on $(0, 1)$-forms as multiplication by $\frac{i}{2}$, one gets

$$(\widetilde{\varphi_1})_* (\omega_{-1} \wedge *\overline{\omega_{-1}}) = \overline{\alpha(\overline{w})}\mathrm{d}w \wedge \frac{i}{2}\alpha(\overline{w})\,\mathrm{d}\overline{w} = |\alpha(\overline{w})|^2 \frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\overline{w}$$

Set $\mu = \lambda_\varphi \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. In the chart φ_{-1}, μ_{-1} writes as $\varphi_{-1*}\mu_{-1} = \lambda_\varphi(-\overline{z})\,\frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. $\widetilde{\varphi_1}$ is also a chart defined on $j(U_1) = U_{-1}$ and the transition map from $\widetilde{\varphi_1}$ to φ_{-1} is the map Φ which to $w \in \widetilde{\varphi_1}(U_{-1}) = \overline{U}$ associates the number $\Phi(w)$ defined by

$$\begin{aligned}\Phi(w) &= \widetilde{\varphi_1}\left((\varphi_{-1})^{-1}(w)\right) = (\overline{\varphi_1} \circ j)\left(\varphi^{-1}(-\overline{w}), -1\right) \\ &= \overline{\varphi_1\left(\varphi^{-1}(-\overline{w}), 1\right)} = \overline{\varphi\left(\varphi^{-1}(-\overline{w})\right)} = -w.\end{aligned}$$

Thus, for $w \in \mathbb{D}^- \cup [-1, 1]$,

$$\begin{aligned}((\widetilde{\varphi_1})_* \mu_{-1})_w &= \left((\widetilde{\varphi_1})^{-1}\right)^* \varphi_{-1}^* \varphi_{-1*}\mu_{-1} = \left(\varphi_{-1} \circ (\widetilde{\varphi_1})^{-1}\right)^* \varphi_{-1*}\mu_{-1} \\ &= \left(\Phi^{-1}\right)^* \varphi_{-1*}\mu_{-1} = \left(\Phi^{-1}\right)^* \left(\lambda_\varphi(-\overline{z})\,\frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}\right) \\ &= \lambda_\varphi(\overline{w})\,\frac{i}{2}\mathrm{d}w \wedge \mathrm{d}\overline{w} = (\varphi_{1*}\mu_1)_{\overline{w}}\end{aligned}$$

and hence

$$\left((\widetilde{\varphi}_1)_*\widehat{h}\,(\omega_{-1})\right)(w)=\frac{(\widetilde{\varphi}_1)_*\,(\omega_{-1}\wedge *\overline{\omega_{-1}})}{\varphi_{-1*}\mu_{-1}}\,(w)=\frac{|\alpha\,(\overline{w})|^2}{\lambda\,(\overline{w})}$$
$$=(\varphi_1)_*\left(\widehat{h}\,(\omega_1)\right)(\overline{w})$$

We infer $\widehat{h}\,(\omega_{-1})\circ\widetilde{\varphi}_1^{\;-1}=\widehat{h}\,(\omega_1)\circ(\varphi_1)^{-1}\circ c$ and so $(\widetilde{\varphi}_1)_*\,f_{-1}=(\varphi_1)_*\,f_1\circ c$ (which gives also $f_{-1}=f_1\circ j$). Derivating twice this relation and using $d\overline{\partial}=-d\partial$, one gets finally $j^*\widehat{\Theta}=-\widehat{\Theta}$ and hence $\int_{M_1}\widehat{\Theta}=\int_{M_{-1}}\widehat{\Theta}$, which ends the proof provided Lemma 46 is proved. □

Lemma 46 *Let M be a Riemann surface with boundary. Denote κ the number of connected components of bM and $\widehat{M}$ the double of M. The genus $g(\widehat{M})$ of $\widehat{M}$ and the Euler characteristic $\chi\left(\overline{M}\right)$ of $\overline{M}$ are linked to the genus $g\,(M)$ of M by the formulas*

$$g(\widehat{M})=2g\,(M)+\kappa-1\ \ \&\ \ \chi\left(\overline{M}\right)=2-2g\,(M)-\kappa. \tag{74}$$

Proof Consider a triangulation T of $\overline{M}$. When α is in the set $\mathcal{C}$ of connected components of $\gamma=bM$, we denote Σ_γ the set of vertices of elements of T which lie on γ and A_γ the one of the edges of elements of T which are contained in γ. We set $\Sigma^b=\underset{\gamma\in\mathcal{C}}{\cup}M_\gamma$ and $A^b=\underset{\gamma\in\mathcal{C}}{\cup}T_\gamma$. For each $\gamma\in\mathcal{C}$, $\left|\Sigma_\gamma\right|=\left|A_\gamma\right|$ and assuming, up to a change of triangulation, that the sets $\underset{t\in T,\ T\cap M_\gamma\neq\varnothing}{\cup}$ are pairwise disjoint when γ describes $\mathcal{C}$, one gets $\left|\Sigma^b\right|=\left|A^b\right|$. Lastly, denotes by $\sigma\,(T)$ the number of vertices of T, $a\,(T)$ the number of edges of T, $f\,(T)$ the number of faces of T and set $\widetilde{M}=\widehat{M}\backslash\overline{M}$. Denotes $\widetilde{T}$ the triangulation of $\widetilde{M}$ obtained by symmetrization of T, that is the one obtained by letting act on T the natural involution of $\widehat{M}$. $\widehat{T}=T\cup\widetilde{T}$ is then a triangulation of $\widehat{M}$. Par definition of the Euler characteristic, one gets then

$$\begin{aligned}\chi(\widehat{M})&=\sigma\left(\widehat{T}\right)-a\left(\widehat{T}\right)+f\left(\widehat{T}\right)\\&=\left[2\left(\sigma\,(T)-\Sigma^b\right)+\Sigma^b\right]-\left[2\left(a\,(T)-A^b\right)+A^b\right]+2f\,(T)\\&=\left[2\sigma\,(T)-\Sigma^b\right]-\left[2a\,(T)-A^b\right]+2f\,(T)\\&=2\sigma\,(T)-2a\,(T)+2f\,(T)=2\chi\left(\overline{M}\right).\end{aligned}$$

Thanks to the usual theory of compact Riemann surfaces, $\chi(\widehat{M})=2-2g(\widehat{M})$. Thus, $g(\widehat{M})=1-\chi\left(\overline{M}\right)$. Denotes M' the surface obtained by gluing κ conformal disks along connected components of γ. Then $\chi\left(M'\right)=\chi\left(\overline{M}\right)+\kappa$ and by definition, $g\,(M)=g\left(M'\right)$. Thus,

$$\chi\left(\overline{M}\right)=\chi\left(M'\right)-\kappa=2-2g\,(M)-\kappa$$

and

$$g(\widehat{M}) = 1 - (2 - 2g\,(M) - \kappa) = 2g\,(M) + \kappa - 1.$$

□

We need one last lemma before proving Theorem 6.

Lemma 47 *Let Q be a nodal Riemann surface with boundary which is a quotient of a Riemann surface with boundary S. For $q \in \operatorname{Sing} \overline{Q}$, denote by $\nu\,(q)$ the number of branches of $\overline{Q}$ at q. Then the Euler characteristics of $\overline{S}$ and $\overline{Q}$ are linked by the relation*

$$\chi\left(\overline{S}\right) = \chi\left(\overline{Q}\right) + \sum_{q \in \operatorname{Sing} \overline{Q}} (\nu\,(q) - 1)\,.$$

Proof Let π be the natural projection of S onto Q and consider a triangulation T of $\overline{S}$ such that any point of $X = \pi^{-1}\left(\operatorname{Sing} \overline{Q}\right)$ is a vertex of T. We can also assume that T is sufficiently refined so that a same triangle of T contains at most one point of X. Denote by V the set of vertices of T, E its sets of edges and F its set of faces. Then π and T induce a natural triangulation $\pi_* T$ of $\overline{Q}$ whose set $\pi_* V$ of its vertices is $\pi\,(V \backslash X) \cup \left(\operatorname{Sing} \overline{Q}\right)$. As any triangle of T contains at most one point of X, $\pi_* T$ and T have the same number of

$$\begin{aligned}|\pi_* V| = |\pi\,(V \backslash X)| + \left|\operatorname{Sing} \overline{Q}\right| = |V| - |X| \\ + \left|\operatorname{Sing} \overline{Q}\right| = |V| - \sum_{q \in \operatorname{Sing} \overline{Q}} (\nu\,(q) - 1)\end{aligned}$$

Lemma 46 gives that $\chi\left(\overline{S}\right) = 1 - g\,(S) - \kappa$. Thus,

$$\begin{aligned}\chi\left(\overline{S}\right) &= |V| - |E| + |F| \\ &= |\pi_* V| - |E| + |F| + \sum_{q \in \operatorname{Sing} \overline{Q}} (\nu\,(q) - 1) = \chi\left(\overline{Q}\right) + \sum_{q \in \operatorname{Sing} \overline{Q}} (\nu\,(q) - 1)\,.\end{aligned}$$

□

Proof of Theorem 6 Let $j \in \{1, 2\}$ and $q_j^\infty = \operatorname{Card} Q_j \cap \{w_0 = 0\}$. Then, $p_j = \delta_j + q^\infty \leqslant \delta_j + N_z\,(\partial \widetilde{u_0})$. Thus, Formula (72) gives

$$p \leqslant \delta + \frac{1}{2\pi i} \int_{\partial M} \frac{D \partial^\sigma \widetilde{u_0}}{\partial^\sigma \widetilde{u_0}} - \chi\left(\overline{M}\right)$$

As $\mathcal{M}$ is a nodal quotient of M by the nodal relation induced by F, we can apply Lemma 47. So, $\chi\left(\overline{M}\right) \geqslant \chi\left(\overline{\mathcal{M}}\right)$ and we get the sought inequality. As mentioned after Theorem 2, $\mathcal{M}$ is computable from boundary data and as explained above in this section with Formula (70), $\frac{D \partial^\sigma \widetilde{u_0}}{\partial^\sigma \widetilde{u_0}}\,|_{bM}$ is computable from available boundary data. The proof is complete. □

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