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Geometry and Topology of the Space of Plurisubharmonic Functions

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Abstract Let Ω be a strongly pseudoconvex domain. We introduce the Mabuchi space of strongly plurisubharmonic functions in Ω. We study the metric properties of this space using Mabuchi geodesics and establish regularity properties of the latter, especially in the ball. As an application, we study the existence of local Kähler–Einstein metrics.

Keywords Geodesics · Mabuchi space · Monge–Ampère equation · Pseudoconvex domain

Mathematics Subject Classification 53C55 · 32W20 · 32U05

Introduction

Let Y be a compact Kähler manifold and $\alpha_Y \in H^{1,1}(Y, \mathbb{R})$ a Kähler class. The space $\mathcal{H}_{\alpha_Y}$ of Kähler metrics ω_Y in α_Y can be seen as an infinite dimensional riemannian manifold whose tangent spaces $T_{\omega_Y}\mathcal{H}_{\alpha_Y}$ can all be identified with $\mathcal{C}^\infty(Y, \mathbb{R})$. Mabuchi has introduced in [27] an L^2-metric on $\mathcal{H}_{\alpha_Y}$, by setting

$$\langle f, g\rangle_{\omega_Y} := \int_Y f\, g\, \frac{{\omega_Y}^n}{V_{\alpha_Y}},$$

where $n = \dim_{\mathbb{C}} Y$ and $V_{\alpha_Y} = \int_Y {\omega_Y}^n = \alpha_Y^n$ denotes the volume of α_Y. Mabuchi studied the corresponding geometry of $\mathcal{H}_{\alpha_Y}$, showing in particular that it can formally

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be seen as a locally symmetric space of non-positive curvature. The (geometry) metric study of the space $(\mathcal{H}_{\alpha_Y}, \langle , \rangle_{\omega_Y})$ has been motivated a lot of interesting works in the last decades, see notably [7,10–15,17,20,26].

The purpose of this article is to extend some of these studies to the case when Y is a smooth strongly pseudoconvex bounded domain of $\mathbb{C}^n$. We note here that this problem of extension to the local case has recently been considered by Rashkovskii [28] and Hosono [24]. The geodesics for plurisubharmonic functions in the Cegrell class $\mathcal{F}_1$ on a bounded hyperconvex domain were first studied by Rashkovskii. He has shown geodesics for plurisubharmonic functions in the Cegrell class $\mathcal{F}_1$ on a bounded hyperconvex domain. Hosono has described the behaviour of the weak geodesics between the toric psh functions with poles at the origin.

Our first interest is the geometry of the space of plurisubharmonic functions. We equipped the space of plurisubharmonic functions with a Levi-Civita connection D and describe the tensor curvature and the sectional curvature as in the paper of Mabuchi [27]. Our first main result is to the establish that the space of plurisubharmonic functions is a locally symmetric space:

Theorem A *The Mabuchi space $\mathcal{H}$ equipped with the Levi-Civita connection D is a locally symmetric space.*

Following the work of Donaldson [20] and Semmes [29] in the compact setting, we reinterpret the geodesics as a solution to a homogeneous complex Monge–Ampère equation. Weak geodesics are introduced as an envelope of functions:

$$\Phi(z,\zeta) = \sup\{u(z,\zeta)/u \in \mathcal{F}(\Omega \times A, \Psi)\}.$$

Our second main result is to establish regularity properties of geodesics in the ball by adapting the celebrated result of Bedford-Taylor [1]:

Theorem B *By taking $\mathbb{B}$ as the unit ball in $\mathbb{C}^n$. Let φ_0 and φ_1 be the geodesic end points which are $C^{1,1}$. Then the Perron–Bremermann envelope*

$$\Phi(z,\zeta) = \sup\{u(z,\zeta)/u \in \mathcal{F}(\Omega \times A, \Psi)\},$$

admits second-order partial derivatives almost everywhere with respect to variable $z \in \mathbb{B}$ which is locally bounded uniformly with respect to $\zeta \in A$, i.e for any compact subset $K \subset \mathbb{B}$ there exists C that depends on K, φ_0 and φ_1 such that

$$\|D_z^2 \Phi\|_{L^\infty(K\times A)} \leq C.$$

The existence of local Kähler–Einstein metrics was studied by Guedj et al. [22] in bounded smooth strongly pseudoconvex domains which are circled. This is equivalent to the resolution of the following Dirichlet problem

$$(MA)_1 \quad \begin{cases} (dd^c\varphi)^n = \frac{e^{-\varphi}\mu}{\int_\Omega e^{-\varphi} d\mu}, & \text{in } \Omega \\ \varphi = 0, & \text{on } \Omega \end{cases}.$$

They treated also the following family of Dirichlet problems

$$(MA)_t \quad \begin{cases} (dd^c\varphi_t)^n = \frac{e^{-t\varphi_t}\mu}{\int_\Omega e^{-t\varphi_t}d\mu}, & \text{in } \Omega \\ \varphi_t = 0, & \text{on } \Omega \end{cases},$$

showing that there is a solution for $t < (2n)^{1+1/n}(1+1/n)^{(1+1/n)}$. We apply our study of the geodesics problem and an idea of [16,18] to prove that the existence of a solution to $(MA)_t$ is equivalent to the coercivity of the Ding functional:

Theorem C *Let $\Omega \subset \mathbb{C}^n$ be a smooth strongly pseudoconvex circled domain. If there exists $\varepsilon(t)$, $M(t) > 0$ such that,*

$$\mathcal{F}_t(\psi) \leq \varepsilon(t)E(\psi) + M(t) \quad \forall \psi \in \mathcal{H},$$

then $(MA)_t$ admits a S^1-invariant smooth strictly plurisubharmonic function solution.

Conversely if $(MA)_t$ admits such a solution φ_t and Ω is strictly φ_t-convex, then there exists $\varepsilon(t)$, $M(t) > 0$ such that,

$$\mathcal{F}_t(\psi) \leq \varepsilon(t)E(\psi) + M(t) \quad \forall \psi \in \mathcal{H}.$$

The organization of the paper is as follows:

- Sect. 1 is devoted to preliminary results and definition of the space $\mathcal{H}$ and its geometry.
- In Sect. 2, we show that geodesics are continuous (sometimes even Lipschitz) up to the boundary of $\Omega \times A$.
- In Sect. 3, we prove the Theorem B.
- we prove finally the Theorem C in Sect. 4.

1 Mabuchi Geometry in Pseudoconvex Domains

In this section, we study the geometry of the space of plurisubharmonic functions in a strongly pseudoconvex domain, based upon works of Mabuchi [27], Semmes [29] and Donaldson [20], as it was clarified through lecture notes of Guedj [21] and Kolev [25].

1.1 Preliminaries

In this section, we recall some analytic tools that will be used in the sequel. Let $\Omega \Subset \mathbb{C}^n$ be a smooth pseudoconvex bounded domain. Recall that a bounded domain $\Omega \Subset \mathbb{C}^n$ is strictly pseudoconvex if there exists a smooth function ρ defined in neighbourhood Ω' of $\bar{\Omega}$ such that $\Omega = \{z \in \Omega' \ /\rho(z) < 0\}$ with $dd^c\rho > 0$, where

$$d := \partial + \bar{\partial}\ ,\ d^c := \frac{i}{2\pi}(\partial - \bar{\partial}).$$

Definition 1.1 We let $PSH(\Omega)$ denote the set of plurisubharmonic functions in Ω. In particular a function $\varphi \in PSH(\Omega)$ is L^1_{loc}, upper semi-continuous and such that

$$dd^c\varphi \geq 0,$$

in the weak sense of positive currents.

The following cone of "test functions" has been introduced by Cegrell [8]:

Definition 1.2 [8] We let $\mathcal{E}_0(\Omega)$ denote the convex cone of all bounded plurisubharmonic functions φ defined in Ω such that $\lim_{z\to\zeta}\varphi(\zeta) = 0$, for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < +\infty$.

Definition 1.3 [8] The class $\mathcal{E}^p(\Omega)$ is a set of functions u for which there exists a sequence of functions $u_j \in \mathcal{E}_0(\Omega)$decreasing towards u in all of Ω, and so that $\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty$.

We need the following maximum principle:

Proposition 1.4 [1] *Let u,v be locally bounded plurisubharmonic functions in Ω such that* $\liminf_{z\to\partial\Omega}(u-v) \geq 0$. *Then*

$$(dd^c u)^n \leq (dd^c v)^n \Longrightarrow v \leq u \; in \; \Omega.$$

1.2 The Mabuchi Space of Plurisubharmonic Functions

We begin this section by defining the Mabuchi space of plurisubharmonic functions in Ω.

Definition 1.5 The Mabuchi space of plurisubharmonic functions in Ω is

$$\mathcal{H} := \{\varphi \in C^\infty(\bar{\Omega}, \mathbb{R}) / dd^c\varphi > 0 \; in \; \bar{\Omega} \;\; \varphi = 0 \; on \; \partial\Omega\}.$$

We now consider the tangent space of $\mathcal{H}$ in every $C^\infty(\bar{\Omega}, \mathbb{R})$.

Definition 1.6 The tangent space of $\mathcal{H}$ at point φ denoted by $T_\varphi\mathcal{H}$ that is the linearisation of $\mathcal{H}$ defined by

$$T_\varphi\mathcal{H} = \{\gamma'(0) / \;\; \varphi : [-\varepsilon, \varepsilon] \to \mathcal{H} \;\; and \;\; \gamma(0) = \varphi\}.$$

The tangent space of $\mathcal{H}$ at φ can be identified with

$$T_\varphi\mathcal{H} \cong \{\xi \in C^\infty(\bar{\Omega}, \mathbb{R}) \; / \;\; \xi = 0 \;\; on \; \partial\Omega\}.$$

Indeed, let $\xi \in \{\xi \in C^\infty(\bar{\Omega}, \mathbb{R}) \; / \;\; \xi = 0 \; on \; \partial\Omega\}$, we put $\gamma(s) := \varphi + s\xi$ for s close enough to 0 we have $\gamma_s \in \mathcal{H}$, and

$$\gamma(0) = \varphi \;\; and \;\; \gamma'(0) = \xi$$

this implies that $\xi \in T_\varphi \mathcal{H}$ hence

$$\{\xi \in C^\infty(\bar{\Omega}, \mathbb{R}) \ / \ \ \xi|_{\partial\Omega} = 0\} \subset T_\varphi \mathcal{H}.$$

Conversely, let $\gamma \in \mathcal{H}$ which gives $\gamma_s|_{\partial\Omega} = 0$ for every s. In particular $\dot{\gamma}(0)|_{\partial\Omega} = 0$, therefore

$$\xi = \dot{\gamma}(0) \in \{\xi \in C^\infty(\bar{\Omega}, \mathbb{R}) \ / \ \ \xi = 0 \ on \ \partial\Omega\}.$$

Definition 1.7 [27] The Mabuchi metric is the L^2 Riemannian metric and is defined by

$$<< \psi_1, \psi_2 >>_\varphi := \int_\Omega \psi_1 \psi_2 (dd^c\varphi)^n,$$

where $\varphi \in \mathcal{H}$, $\psi_1, \psi_2 \in T_\varphi \mathcal{H}$.

1.3 Mabuchi Geodesics

Geodesics between two points φ_0, φ_1 in $\mathcal{H}$ are defined as the extremals of the Energy functional

$$\varphi \longmapsto H(\varphi) := \frac{1}{2} \int_0^1 \int_\Omega (\dot{\varphi}_t)^2 (dd^c \varphi_t)^n,$$

where $\varphi = \varphi_t$ is a path in $\mathcal{H}$ joining φ_0 to φ_1. The geodesic equation is obtained by computing the Euler–Lagrange equation of the functional H.

Theorem 1.8 *The geodesics equation is*

$$\ddot{\varphi}(t) - |\nabla \ \dot{\varphi}(t)|^2_{\varphi(t)} = 0 \tag{1}$$

where ∇ is the gradient relative to the metric $\omega_\varphi = dd^c\varphi$.

Proof We need to compute the Euler–Lagrange equation of the Energy functional. Let $(\phi_{s,t})$ be a variation of φ with fixed end points,

$$\phi_{0,t} = \varphi_t, \phi_{s,0} = \varphi_0, \phi_{s,1} = \varphi_1 \ and \ \phi_{s,t} = 0 \ on \ \partial\Omega$$

Set $\psi_t := \frac{\partial \phi}{\partial s}|_{s=0}$ and observe that $\psi_0 \equiv \psi_1 \equiv 0$ and $\psi_t = 0$ on $\partial\Omega$. Thus

$$\phi_{s,t} = \varphi_t + s\psi_t + \circ(s) \ and \ \frac{\partial \phi_{s,t}}{\partial t} = \dot{\varphi} + s\dot{\psi}_t + \circ(s)$$

and

$$(dd^c\phi_{s,t})^n = (dd^c(\varphi_t + s\psi_t))^n = (dd^c\varphi_t)^n + s.ndd^c\psi_t \wedge (dd^c\varphi_t)^{n-1}.$$

A direct computation yields

$$\begin{aligned} H(\phi_{s,t}) &= \frac{1}{2}\int_0^1\int_\Omega (\dot{\phi_{s,t}})^2(dd^c\phi_{s,t})^n \mathrm{d}t \\ &= H(\varphi_t) + s\int_0^1\int_\Omega \dot{\varphi}_t\dot{\psi}(dd^c\varphi_t)^n \mathrm{d}t \\ &\quad + \frac{ns}{2}\int_0^1\int_\Omega \dot{\varphi_t}^2 dd^c\psi_t \wedge (dd^c\varphi_t)^{n-1}\mathrm{d}t. \end{aligned}$$

Using the integration by part, and the fact that $\psi_0 \equiv \psi_1 \equiv 0$ yields

$$\begin{aligned} \int_0^1\int_\Omega \dot{\varphi}_t\dot{\psi}(dd^c\varphi_t)^n \mathrm{d}t = -\int_0^1\int_\Omega \psi_t\{&\ddot{\varphi}_t(dd^c\varphi_t)^n \\ &+ n\dot{\varphi}_t dd^c\dot{\varphi}_t \wedge (dd^c\varphi_t)^{n-1}\}\mathrm{d}t. \end{aligned}$$

also we have by Stokes and the fact that $\dot{\varphi}_t = 0$ on $\partial\Omega$ the following equality

$$\begin{aligned} \int_0^1\int_\Omega (\dot{\varphi})^2 dd^c\psi_t \wedge (dd^c\varphi_t)^{n-1}\mathrm{d}t = 2\int_0^1\int_\Omega \psi_t(&d\dot{\varphi} \wedge d^c\dot{\varphi}_t \\ &+ \dot{\varphi}_t dd^c\dot{\varphi}_t \wedge (dd^c\varphi_t)^{n-1})\mathrm{d}t \end{aligned}$$

hence

$$\begin{aligned} H(\varphi_{s,t}) = H(\varphi_t) + s\int_0^1\int_\Omega \psi_t\Big\{&-\ddot{\varphi}_t(dd^c\varphi_t)^n \\ &+ nd\dot{\varphi}_t \wedge d^c\dot{\varphi} \wedge (dd^c\varphi_t)^{n-1}\Big\}\mathrm{d}t + \circ(s) \end{aligned}$$

which implies

$$\begin{aligned} 0 &= d_{\varphi_t}H.\psi_t \\ &= \lim_{s\to 0}\frac{H(\varphi_{s,t}) - H(\varphi_t)}{s} \\ &= \int_0^1\int_\Omega \psi_t\left\{-\ddot{\varphi}_t(dd^c\varphi_t)^n + nd\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge (dd^c\varphi_t)^{n-1}\right\}\mathrm{d}t. \end{aligned}$$

Therefore (φ_t) is critical point of H if and only if

$$\ddot{\varphi}_t(dd^c\varphi_t)^n = nd\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge (dd^c\varphi_t)^{n-1}.$$

□

1.4 Levi-Civita Connection

As for Riemanniann manifolds of finite dimension, one can find the local expression of the Levi-Civita connection by polarizing the geodesics equation.

Definition 1.9 We define the covariant derivative of the vector field ψ_t along the path φ_t in $\mathcal{H}$ by the formula

$$D\psi := \frac{d\psi}{\mathrm{d}t} - < \nabla\psi, \nabla\, \dot{\varphi} >_{\varphi} .$$

Theorem 1.10 *D is the Levi-Civita connection.*

Proof To show that D is a Levi-Civita connection, we must show that the connection D is metric-compatible and torsion-free.

(i) Metric-compatibility: Let ψ_1, ψ_2 be two vector fields

$$\begin{aligned}\frac{d}{\mathrm{d}t} << \psi_1, \psi_2 >>_{\varphi} &= \frac{d}{\mathrm{d}t}\int_{\Omega} \psi_1\psi_2(dd^c\varphi)^n \\ &= \int_{\Omega} (\dot{\psi_1}\psi_2 + \psi_1\dot{\psi_2})(dd^c\varphi)^n \\ &\quad + n\psi_1\psi_2 dd^c\dot{\varphi}\wedge(dd^c\varphi)^{n-1} \\ &= \int_{\Omega} (\dot{\psi_1}\psi_2 + \psi_1\dot{\psi_2} - < \nabla\, (\psi_1\psi_2), \nabla\, \dot{\varphi} >_{\varphi})(dd^c\varphi)^n \\ &= \int_{\Omega} ((\dot{\psi_1} - < \nabla\psi_1, \nabla\dot{\varphi} >_{\varphi})\psi_2(dd^c\varphi)^n \\ &\quad + \int_{\Omega} \psi_1(\dot{\psi_2} - < \nabla\, \psi_2, \nabla\, \dot{\varphi} >_{\varphi})(dd^c\varphi)^n \\ &= << D\psi_1, \psi_2 >>_{\varphi} + << \psi_1, D\psi_2 >>_{\varphi} .\end{aligned}$$

(The passage from the second line to the third line is a result of the equation

$$\begin{aligned}d(\psi_1\psi_2 d^c\dot{\varphi}\wedge(dd^c\varphi)^{n-1}) = d(\psi_1\psi_2)\wedge d^c\dot{\varphi}\wedge(dd^c\varphi)^{n-1} \\ + \psi_1\psi_2 dd^c\dot{\varphi}\wedge(dd^c\varphi)^{n-1}\end{aligned}$$

and Stokes theorem).

(ii) D is torsion-free, because

$$D_s\frac{d\varphi}{\mathrm{d}t} = D_t\frac{d\varphi}{\mathrm{d}s}.$$

Thus D is a Levi-Civita connection. □

1.5 Curvature Tensor

We define the curvature tensor and the sectional curvature and we give their expressions. We finish by proving that the space of plurisubharmonic functions is locally symmetric. We start by giving some definitions and conventions.

Definition 1.11 Let ψ and θ be two functions in the tangent space of $\mathcal{H}$ at φ. The Poisson bracket of ψ and θ compared to the form $\omega_\varphi = dd^c\varphi$ is

$$\{\psi, \theta\} = \{\psi, \theta\}_\varphi := i \sum_{\alpha,\beta=1} \varphi^{\alpha\bar{\beta}} \left(\frac{\partial\psi}{\partial\bar{z}_\beta} \frac{\partial\theta}{\partial z_\alpha} - \frac{\partial\psi}{\partial z_\alpha} \frac{\partial\theta}{\partial\bar{z}_\beta} \right),$$

where $(\varphi^{\alpha\bar{\beta}})$ is the inverse matrix of $(\varphi_{\alpha\bar{\beta}})$.

Lemma 1.12 *Let ψ, θ and η be three functions belonging to the tangent space of $\mathcal{H}$ at φ. The Poisson bracket satisfies the following properties :*

(i) $\{\psi, \theta\} = -\{\theta, \psi\}$.
(ii) $\{\psi, \theta\} = \omega_\varphi(X_\psi, X_\theta)$.
(iii) $\{\psi, \theta + \eta\} = \{\psi, \theta\} + \{\psi, \eta\}$.
(iv) $[X_\psi, X_\theta] = X_\psi(X_\theta) - X_\theta(X_\psi) = X_{\{\psi,\theta\}}$.
(v) $\int_\Omega \{\psi, \theta\}\eta(dd^c\varphi)^n = \int_\Omega \psi\{\theta, \eta\}(dd^c\varphi)^n$.
(vi) $D\{\psi, \theta\} = \{D\psi, \theta\} + \{\psi, D\theta\}$.

Where $X_\psi := i\nabla\psi$ and $[,]$ is the Lie bracket.

Let ψ be a function in a tangent space, the Hessian of ψ is defined by Hess $\psi = \nabla^\varphi d\psi$, where ∇^φ is the Levi-Civita connection, respectively, to the form $\omega_\varphi = dd^c\varphi$. We recall in the next lemma some properties of the Hessian well known in the literature.

Lemma 1.13 *Let X and Y be two vector fields. Then the Hessian satisfies the following properties:*

(i) $\text{Hess}\,\psi(X, Y) = < \nabla^\varphi_X \nabla^\varphi \psi, Y >_\varphi$.
(ii) $\text{Hess}\,\psi(X, Y) = X(Y(\psi)) - \nabla^\varphi_X Y(\psi)$.
(iii) $dd^c\psi(X, iY) = \text{Hess}\,\psi(X, Y) + \text{Hess}\,\psi(iX, iY)$.

Where ∇^φ and $<,>_\varphi$ are the Levi-Civita connection and the metric, respectively, associated to the form $\omega_\varphi = dd^c\varphi$.

In the sequel of this section, we consider a 2-parameter family $\varphi(t, s) \in \mathcal{H}$ and a vector field $\psi(t, s) \in T_\varphi\mathcal{H}$ defined along φ. We denote by

$$\varphi_t = \frac{d\varphi}{\mathrm{d}t}, \quad \varphi_s = \frac{d\varphi}{\mathrm{d}s}.$$

Definition 1.14 The curvature tensor of the Mabuchi metric in $\mathcal{H}$ is defined by

$$R_\varphi(\varphi_t, \varphi_s)\psi := D_t D_s \psi - D_s D_t \psi,$$

where $\varphi(s,t) \in \mathcal{H}$ is 2-parameter family and vector field $\psi(s,t) \in T_\varphi\mathcal{H}$.

The sectional curvature is defined by

$$K_\varphi(\varphi_t, \varphi_s) :=<< R_\varphi(\varphi_t, \varphi_s)\varphi_t, \varphi_s >>_\varphi .$$

Theorem 1.15 *The curvature tensor of the Mabuchi metric in $\mathcal{H}$ can be expressed as*

$$R_\varphi(\varphi_t, \varphi_s)\psi = -\{\{\varphi_t, \varphi_s\}, \psi\}.$$

The sectional curvature is the following

$$K_\varphi(\varphi_t, \varphi_s) = -||\{\varphi_t, \varphi_s\}||_\varphi^2 \leq 0,$$

where $\{,\}_\varphi$ is the Poisson bracket associate to the form $\omega_\varphi = dd^c\varphi$.

Proof To compute the curvature tensor of D, we compute the first term in the definition of the curvature tensor . Indeed, let ψ be the vector field, its derivative along the path φ_s is

$$D_s\psi = \psi_s - < \nabla\psi, \nabla\varphi_s >_\varphi = \psi_s + \Gamma_\varphi(\psi, \varphi_s),$$

where

$$\Gamma_\varphi(\psi, \varphi_s) = - < \nabla\psi, \nabla\varphi_s >_\varphi,$$

we derive the $D_s\psi$ along the path φ_t as follows:

$$\begin{aligned} D_t D_s\psi &= D_t(\psi_s + \Gamma_\varphi(\psi, \varphi_s) \\ &= \frac{d}{\mathrm{d}t}(\psi_s + \Gamma_\varphi(\psi, \varphi_s)) + \Gamma_\varphi(\psi_s + \Gamma_\varphi(\psi, \varphi_s), \varphi_t)) \\ &= \psi_{st} + \frac{d}{\mathrm{d}t}(\Gamma_\varphi(\psi, \varphi_s)) + \Gamma_\varphi(\psi_s, \varphi_t) + \Gamma_\varphi(\Gamma_\varphi(\psi, \varphi_s), \varphi_t). \end{aligned}$$

We express the second term in RHS of the last equation:

$$\begin{aligned} \frac{d}{\mathrm{d}t}\Gamma_\varphi(\psi, \varphi_s) &= \frac{d}{\mathrm{d}t}(- < \nabla\psi, \nabla\varphi_s >_\varphi) \\ &= -\frac{d}{\mathrm{d}t}\varphi^{\alpha\bar{\beta}}\varphi_{s\alpha}\psi_{\bar{\beta}} \\ &= -\varphi^{\alpha\bar{\beta}}\varphi_{s\alpha t}\psi_{\bar{\beta}} - \varphi^{\alpha\bar{\beta}}\varphi_{s\alpha}\psi_{\bar{\beta}t} + \varphi^{\alpha\bar{m}}\varphi^{n\bar{\beta}}\varphi_{n\bar{m}t}\varphi_{s\alpha}\psi_{\bar{\beta}} \\ &= \Gamma_\varphi(\psi, \varphi_{ts}) + \Gamma_\varphi(\psi_t, \varphi_s) + dd^c\varphi_t(\nabla\varphi_s, i\nabla\psi). \end{aligned}$$

By applying the three properties of Lemma 1.13 by taking $X = \nabla\varphi_s$ and $Y = \nabla\psi$, we express the last term in the last equation as follows:

$$dd^c\varphi_t(\nabla\varphi_s, i\nabla\psi) = \mathrm{Hess}(\varphi_t)(\nabla\varphi_s, \nabla\psi) + \mathrm{Hess}(\varphi_t)(i\nabla\varphi_s, i\nabla\psi),$$

which gives

$$\frac{d}{\mathrm{d}t}\Gamma_\varphi(\varphi_t,\psi) = \Gamma_\varphi(\psi,\varphi_{ts}) + \Gamma_\varphi(\psi_s,\varphi_t) \\ +\mathrm{Hess}(\varphi_t)(\nabla\,\varphi_s,\nabla\psi) + \mathrm{Hess}(\varphi_t)(i\nabla\,\varphi_s, i\nabla\,\psi).$$

We develop the fourth term in the RHS in the last equation by applying the second properties of Lemma 1.13, by taking $X=\nabla\varphi_s$ and $Y=\nabla\psi$:

$$\begin{aligned}\mathrm{Hess}(\varphi_t)(\nabla\,\varphi_s,\nabla\,\psi) &= \nabla\,\varphi_s(\nabla\,\psi(\varphi_t)) - (\nabla^\varphi_{\nabla\,\varphi_s}\nabla\,\psi)(\varphi_t)\\ &= \nabla\,\varphi_s(<\nabla\,\varphi_t,\nabla\,\psi>_\varphi) - <\nabla\,\varphi_t,\nabla^\varphi_{\nabla\,\varphi_s}\nabla\,\psi>_\varphi\\ &= \Gamma_\varphi(\Gamma_\varphi(\varphi_t,\psi),\varphi_s) - \mathrm{Hess}(\psi)(\nabla\,\varphi_s,\nabla\,\varphi_t)\end{aligned}$$

We have also by applying the first properties of Lemma 1.13:

$$\begin{aligned}\mathrm{Hess}(\varphi_t)(i\nabla\,\varphi_s, i\nabla\,\psi) &= <\nabla^\varphi_{i\nabla\,\varphi_s}\nabla\,\varphi_t, i\nabla\,\psi>_\varphi\\ &= <\nabla^\varphi_{i\nabla\,\varphi_s}(i\nabla\,\varphi_t), i(i\nabla\,\psi)>_\varphi\\ &= \omega_\varphi(\nabla^\varphi_{X_{\varphi_s}}X_{\varphi_t}, X_\psi),\end{aligned}$$

where $X_h = i\nabla\,h$. Then we have

$$\frac{d}{\mathrm{d}t}\Gamma_\varphi(\varphi_s,\psi) = \Gamma_\varphi(\psi,\varphi_{ts}) + \Gamma_\varphi(\psi_t,\varphi_s) + \Gamma_\varphi(\Gamma_\varphi(\varphi_s,\psi),\varphi_t) \\ -\mathrm{Hess}(\psi)(\nabla\,\varphi_t,\nabla\,\varphi_s) + \omega_\varphi(\nabla^\varphi_{X_{\varphi_t}}X_{\varphi_s}, X_\psi).$$

By the previous equations, we get the expression of $D_tD_s\psi$ as follows:

$$\begin{aligned}D_tD_s\psi &= \psi_{st} + \Gamma_\varphi(\psi,\varphi_{ts}) + \Gamma_\varphi(\psi_t,\varphi_s)\\ &\quad +\Gamma_\varphi(\Gamma_\varphi(\varphi_s,\psi),\varphi_t) - \mathrm{Hess}(\psi)(\nabla\,\varphi_t,\nabla\,\varphi_s)\\ &\quad +\omega_\varphi(\nabla^\varphi_{X_{\varphi_t}}X_{\varphi_s}, X_\psi) + \Gamma_\varphi(\psi_s,\varphi_t) + \Gamma_\varphi(\Gamma_\varphi(\psi,\varphi_s),\varphi_t).\end{aligned}$$

We get the expression of $D_sD_t\psi$ by reversing the roles of t and s as follows:

$$\begin{aligned}D_sD_t\psi &= \psi_{st} + \Gamma_\varphi(\psi,\varphi_{st}) + \Gamma_\varphi(\psi_s,\varphi_t)\\ &\quad +\Gamma_\varphi(\Gamma_\varphi(\varphi_t,\psi),\varphi_s) - \mathrm{Hess}(\psi)(\nabla\,\varphi_s,\nabla\,\varphi_t)\\ &\quad +\omega_\varphi(\nabla^\varphi_{X_{\varphi_s}}X_{\varphi_t}, X_\psi) + \Gamma_\varphi(\psi_t,\varphi_s) + \Gamma_\varphi(\Gamma_\varphi(\psi,\varphi_t),\varphi_s).\end{aligned}$$

Therefore we get

$$\begin{aligned}R_\varphi(\varphi_t,\varphi_s)\psi &= D_tD_s\psi - D_sD_t\psi\\ &= \omega_\varphi(\nabla^\varphi_{X_{\varphi_t}}X_{\varphi_s}, X_\psi) - \omega_\varphi(\nabla^\varphi_{X_{\varphi_s}}X_{\varphi_t}, X_\psi)\\ &= \omega_\varphi([X_{\varphi_t}, X_{\varphi_s}], X_\psi)\end{aligned}$$

$$= \omega_\varphi(\{\varphi_t, \varphi_s\}, X_\psi)$$
$$= -\{\{\varphi_t, \varphi_s\}, \psi\}.$$

In the line three, we use the fact that the Levi-Civita connection is torsion free. In the line four, we use the fourth property in Lemma 1.12, in the last line, we use the second property in Lemma 1.12. We calculate the sectional curvature as follows:

$$\begin{aligned} K_\varphi(\varphi_t, \varphi_s) &= << R_\varphi(\varphi_t, \varphi_s)\varphi_t, \varphi_s >>_\varphi \\ &= \int_\Omega R_\varphi(\varphi_t, \varphi_s)\varphi_t\varphi_s(dd^c\varphi)^n \\ &= -\int_\Omega \{\{\varphi_t, \varphi_s\}, \varphi_t\}\varphi_s(dd^c\varphi)^n \\ &= -\int_\omega \{\varphi_t, \varphi_s\}\{\varphi_t, \varphi_s\}(dd^c\varphi)^n \\ &= -||\{\varphi_t, \varphi_s\}||_\varphi^2. \end{aligned}$$

We use in line three the expression of the curvature tensor and in the line four, we use the fifth property in Lemma 1.12. □

Definition 1.16 We say a connection D in $\mathcal{H}$ is locally symmetric if its curvature tensor is parallel i.e $DR = 0$.

Theorem 1.17 *The Mabuchi space $\mathcal{H}$ provided by the Levi-Civita connection D is a locally symmetric space.*

Proof Let $\varphi(t, s, r)$ be 3-parameter family in $\mathcal{H}$.

$$\begin{aligned} D_r(R_\varphi(\varphi_t, \varphi_s)\psi) &= D_r(-\{\{\varphi_t, \varphi_s\}, \psi\} \\ &= -\{D_r\{\varphi_t, \varphi_s\}, \psi\} - \{\{\varphi_t, \varphi_s\}, D_r\psi\} \\ &= -\{\{D_r\varphi_t, \varphi_s\} + \{\varphi_t, D_r\varphi_s\}, \psi\} - \{\{\varphi_t, \varphi_s\}, D_r\psi\} \\ &= -\{\{\{D_r\varphi_t, \varphi_s\}\} - \{\varphi_t, D_r\varphi_s\}, \psi\} - \{\{\varphi_t, \varphi_s\}, D_r\psi\} \\ &= R_\varphi(D_r\varphi_t, \varphi_s)\psi + R_\varphi(\varphi_t, D_r\varphi_s)\psi + R_\varphi(\varphi_t, D_r\varphi_s)(D_r\psi). \end{aligned}$$

We use the expression of the curvature tensor and the sixth property in the Lemma 1.12 of the Poisson bracket. Therefore

$$\begin{aligned} (D_rR_\varphi)(\varphi_t, \varphi_s)\psi = D_r(R_\varphi(\varphi_t, \varphi_s)\psi) - R_\varphi(D_r\varphi_t, \varphi_s)\psi \\ -R_\varphi(\varphi_t, D_r\varphi_s)\psi - R_\varphi(\varphi_t, \varphi_s)(D_r\psi) = 0, \end{aligned}$$

hence $\mathcal{H}$ is locally symmetric. □

2 The Dirichlet Problem

We now study the regularity of geodesics using pluripotential theory, the used tools are developed by Bedford and Taylor [1,2].

2.1 Semmes Trick

We are interested in the boundary value problem for the geodesic equation: given φ_0,φ_1 two distinct points in $\mathcal{H}$, can one find a path $(\varphi(t))_{0\leq t\leq 1}$ in $\mathcal{H}$ which is a solution of(1) with end points $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$? For each path $(\varphi_t)_{t\in[0,1]}$ in $\mathcal{H}$, we set

$$\Phi(z, \zeta) = \varphi_t(z) \ , z \in \Omega \ \ and \ \ \zeta = e^{t+is} \in A = \{\zeta \in \mathbb{C}/1 < |\zeta| < e\}.$$

We show in this section that the geodesic equation in $\mathcal{H}$ is equivalent to the Monge–Ampère equation on $\Omega \times A$ as in Semmes [29].

Lemma 2.1 *The Monge–Ampère measure of the function Φ in $\Omega \times A$ is*

$$\begin{aligned}(dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} &= (dd^c_z\Phi(z,\zeta))^{n+1} + (n+1)(dd^c_z\Phi(z,\zeta))^n \wedge R \\ &\quad + \frac{n(n+1)}{2}(dd^c_z\Phi(z,\zeta))^{n-1} \wedge R^2,\end{aligned}$$

where

$$R = R(z,\zeta) = d_z d^c_\zeta \Phi + d_\zeta d^c_z \Phi + d_\zeta d^c_\zeta \Phi.$$

Proof We write $d_{z,\zeta}\Phi = d_z\Phi + d_\zeta\Phi$ and $d^c_{z,\zeta}\Phi = d^c_z\Phi + d^c_\zeta\Phi$, and we give also the expression of $dd^c_{z,\zeta}\Phi(z,\zeta)$ in $\Omega \times A$. Indeed,

$$\begin{aligned}dd^c_{x,z}\Phi &= (d_z + d_\zeta)(d^c_z\Phi + d^c_\zeta\Phi) \\ &= d_z d^c_z\Phi + d_z d^c_\zeta\Phi + d_\zeta d^c_z\Phi + d_\zeta d^c_\zeta\Phi \\ &= d_z d^c_z\Phi + R(z,\zeta)\end{aligned}$$

with $R = d_z d^c_\zeta\Phi + d_\zeta d^c_z\Phi + d_\zeta d^c_\zeta\Phi$ such that $R^3 = 0$. Then we can find the expression of $(dd^c_{x,z}\Phi)^{n+1}$ in $\Omega \times A$. Indeed,

$$\begin{aligned}(dd^c_{z,\zeta}\Phi)^{n+1} &= (dd^c_z\Phi + R)^{n+1} \\ &= \sum_{j=0}^{n+1} C^j_{n+1}(dd^c_z\Phi)^j \wedge (R)^{n+1-j} \\ &= (dd^c_z\Phi)^{n+1} + (n+1)(dd^c_z\Phi)^n \wedge R \\ &\quad + \frac{n(n+1)}{2}(dd^c_z\Phi)^{n-1} \wedge R^2.\end{aligned}$$

On the second line, we use the Leibniz formula and the fact that $R^3 = R \wedge R \wedge R = 0$ on the third line. □

Theorem 2.2 $(\varphi_t)_{0\leq t\leq 1}$ *is a geodesic if and only if* $(dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} = 0$.

Proof From the previous Lemma, we have

$$(dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} = (dd^c_z\Phi(z,\zeta))^{n+1} + (n+1)(dd^c_z\Phi(z,\zeta))^n \wedge R$$
$$+\frac{n(n+1)}{2}(dd^c_z\Phi(z,\zeta))^{n-1} \wedge R^2.$$

The first term in RHS of the last equation equal to 0 like a result of the bi-degree. We have

$$d_\zeta\Phi = \partial_\zeta\Phi + \bar{\partial}_\zeta\Phi = \frac{\partial\Phi}{\partial\zeta}d\zeta + \frac{\partial\Phi}{\partial\bar{\zeta}}d\bar{\zeta} = \dot{\varphi}_t(z)(d\zeta + d\bar{\zeta}),$$

and

$$d^c_\zeta\Phi = \frac{i}{2}\left(\bar{\partial}\Phi - \partial\Phi\right) = \frac{i}{2}\left(\frac{\partial\Phi}{\partial\bar{\zeta}}d\bar{\zeta} - \frac{\partial\Phi}{\partial\zeta}d\zeta\right)$$
$$= \frac{i}{2}\dot{\varphi}_t(z)(d\zeta - d\bar{\zeta}),$$

also we have $d_\zeta d^c_\zeta\Phi = i\ddot{\varphi}_t(z)d\zeta \wedge d\bar{\zeta}$, which gives

$$R = i\ddot{\varphi}_t(z)d\zeta \wedge d\bar{\zeta} + \frac{i}{2}d_z\dot{\varphi}_t \wedge d\bar{\zeta} - \frac{i}{2}d_z\dot{\varphi}_t \wedge d\zeta$$
$$+d^c_z\dot{\varphi}_t \wedge d\zeta + d^c_z\dot{\varphi}_t \wedge d\bar{\zeta},$$

and

$$R^2 = 2id_z\dot{\varphi}_t \wedge d^c_z\dot{\varphi}_t \wedge d\zeta \wedge d\bar{\zeta}.$$

Now we can explain the second term also. Indeed,

$$(dd^c_z\Phi)^n \wedge R = (dd^c_z\varphi_t(z))^n \wedge (i\ddot{\varphi}_t(z)d\zeta \wedge d\bar{\zeta} + \frac{i}{2}d_z\dot{\varphi}_t \wedge d\bar{\zeta}$$
$$-\frac{i}{2}d_z\dot{\varphi}_t \wedge d\zeta + d^c_z\dot{\varphi}_t \wedge d\zeta + d^c_z\dot{\varphi}_t \wedge d\bar{\zeta})$$
$$= i\ddot{\varphi}_t(dd^c_z\varphi_t)^n \wedge d\zeta \wedge d\bar{\zeta}.$$

For the third term, we have

$$(dd^c_z\Phi)^{n-1} \wedge R^2 = (dd^c\varphi_t(z))^{n-1} \wedge R \wedge R$$
$$= (dd^c\varphi_t(z))^{n-1} \wedge 2id_z\dot{\varphi}_t \wedge d^c_z\dot{\varphi}_t \wedge d\zeta \wedge d\bar{\zeta}$$
$$= -2id_z\dot{\varphi}_t \wedge d^c_z\dot{\varphi} \wedge (dd^c\varphi_t(z))^{n-1} \wedge d\zeta \wedge d\bar{\zeta}.$$

By the previous equations we have,

$$\begin{aligned}(dd^c_{z,\zeta}\Phi)^{n+1} &= (n+1)(dd^c_z\Phi(z,\zeta))^n \wedge R \\ &\quad + \frac{n(n+1)}{2}(dd^c_z\Phi(z,\zeta))^{n-1} \wedge R^2 \\ &= i(n+1)(\ddot{\varphi}_t(dd^c_z\varphi_t)^n - nd_z\dot{\varphi}_t \wedge d^c_z\dot{\varphi}_t \\ &\quad \wedge (dd^c\varphi_t(z))^{n-1} \wedge d\zeta \wedge d\bar{\zeta} \\ &= i(n+1)\left(\ddot{\varphi}_t - \frac{nd_z\dot{\varphi}_t \wedge d^c_z\dot{\varphi}_t \wedge (dd^c\varphi_t(z))^{n-1}}{(dd^c_z\varphi_t)^n}\right) \\ &\quad (dd^c_z\varphi_t)^n \wedge d\zeta \wedge d\bar{\zeta}.\end{aligned}$$

From the fact that $nd(\dot{\varphi}_t) \wedge d^c(\dot{\varphi}_t) \wedge (dd^c\varphi_t)^{n-1} = \ddot{\varphi}_t(dd^c\varphi_t)^n$, we infer that φ_t is a geodesic between φ_0 and φ_1 if and only if

$$(dd^c_{z,\zeta}\Phi(z,\zeta)^{n+1} = 0.$$

□

By the previous theorem, we deduce that the geodesics problem in Mabuchi space is equivalent to the following Dirichlet problem:

$$\begin{cases} (dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} = 0 & \Omega \times A \\ \Phi(z,\zeta) = \varphi_0(z) & \Omega \times \{|\zeta| = 1\} \\ \Phi(z,\zeta) = \varphi_1(z) & \Omega \times \{|\zeta| = e\} \quad (3) \\ \Phi(z,\zeta) = 0 & \partial\Omega \times A \end{cases}$$

2.2 Continuous Envelopes

In the sequel of this paper, we assume that φ_0 and φ_1 are only $C^{1,1}$.

Definition 2.3 The Perron–Bremermann envelope is defined by

$$\Phi(z,\zeta) = \sup\{u(z,\zeta) \in \mathcal{F}(\Psi, \Omega \times A)\}$$

with

$$\mathcal{F}(\Psi, \Omega \times A) = \{u \in PSH(\Omega \times A) \ / \ u^* \leq \Psi \text{ on } \partial(\Omega \times A)\},$$

where $\Psi|_{\partial\Omega\times\bar{A}} = 0 and \Psi_{\Omega\times\partial A} = \begin{cases} \varphi_0(z), \ \Omega \times \{|\zeta| = 1\} \\ \varphi_1(z), \ \Omega \times \{|\zeta| = e\}. \end{cases}$.

Theorem 2.4 *If* $\Psi \in C^0(\partial(\Omega \times A))$. *Then the Perron–Bremermann envelope* Φ *satisfies the following conditions:*

(i) $\Phi \in PSH(\Omega \times A) \cap C^0(\bar{\Omega} \times \bar{A})$.
(ii) $\Phi|_{\partial(\Omega\times A)} = \Psi$.
(iii) $(dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} = 0$ *in* $\Omega \times A$.

Proof Let ρ be a strictly plurisubharmonic defining function of $\Omega = \{\rho < 0\}$. Observe that the family $\mathcal{F}(\Psi, \Omega \times A)$ is not empty .

(i) We start by proving the plurisubharmonicity of Φ in $\Omega \times A$. We can write the Dirichlet problem on the following way:

$$\begin{cases} (dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} = 0 & \Omega \times A \\ \Phi(z,\zeta) = \Psi(z,\zeta) & \partial(\Omega \times A), \end{cases}$$

with $\Psi(z,\zeta) = \frac{1}{e^2-1}(\varphi_1(z)(|\zeta|^2 - 1) - \varphi_0(z)(|\zeta|^2 - e^2))$. Let $h \in Har(\Omega \times A) \cap C^0(\bar{\Omega} \times \bar{A})$ be a harmonic function in $\Omega \times A$, continuous up to the boundary of $\Omega \times A$, the solution of the following Dirichlet problem

$$\begin{cases} \Delta_{z,\zeta} h(z,\zeta) = 0, & \Omega \times A \\ h = \Psi, & \partial(\Omega \times A), \end{cases}$$

has a solution , since $\Omega \times A$ is a regular domain.
For all $v \in \mathcal{F}(\Psi, \Omega \times A)$, we have $v^* \leq \Psi$ on $\partial(\Omega \times A)$, which implies

$$(v - h)^* \leq 0 \; on \; \partial(\Omega \times A),$$

furthermore we have

$$\Delta_{z,\zeta}(v - h)(z,\zeta) = \Delta_{z,\zeta} v(z,\zeta) \geq 0 \;\; in \;\; \Omega \times A.$$

Then by the maximum principle

$$v(z,\zeta) \leq h(z,\zeta) \;\; in \;\; \Omega \times A,$$

the last inequality holds for every function in $\mathcal{F}(\Psi, \Omega \times A)$, hence it holds for upper envelope of subsolution

$$\Phi(z,\zeta) \leq h(z,\zeta) \;\; in \;\; \Omega \times A.$$

It also holds for its upper semi-continuous regularization on the boundary $(\Omega \times A)$ and we get

$$(\Phi(z,\zeta))^* \leq \Psi(z,\zeta) \;\; on \;\; \partial(\Omega \times A),$$

consequently

$$\Phi^* \in \mathcal{F}(\Psi, \Omega \times A).$$

Since the function Φ^* is plurisubharmonic in $\Omega \times A$ and

$$\Phi(z,\zeta) \leq \Phi(z,\zeta))^* \;\; in \;\; \Omega \times A,$$

we infer that

$$(\Phi(z,\zeta))^* = \Phi(z,\zeta) \ \ in \ \ \Omega \times A.$$

Hence, Φ is plurisubharmonic function in $\Omega \times A$.
Since Φ is plurisubharmonic in $\Omega \times A$, implies that Φ is upper semi-continuous. We now prove that the lower upper semi-continuous. Indeed, fix $\epsilon > 0$ and since $\partial(\Omega \times A) = (\partial\Omega \times \bar{A}) \cup (\bar{\Omega} \times \partial A)$ is compact and the function Ψ is continuous on $\partial(\Omega \times A)$, we can choose $\beta > 0$ so small that

$$\begin{aligned}(z,\zeta) \in \Omega \times A, \forall(\xi,\eta) \in \partial(\Omega \times A)\|(z,\zeta) - (\xi,\eta)\| \\ \le \beta \Rightarrow |\Phi(z,\zeta) - \Psi(\xi,\eta)| \le \epsilon.\end{aligned}$$

Fix $a = (a_1, a_2) \in \mathbb{C}^n \times \mathbb{C}$ with $\|a\| \le \beta$. So, we have the following inequality

$$\begin{aligned}\Phi(\xi + a_1, \eta + a_2) \le \Psi(\xi,\eta) + \varepsilon \ \ if \ \ (\xi,\eta) \\ \in (\Omega \times A \setminus \{a\}) \cup \partial(\Omega \times A)\end{aligned}$$

and

$$\begin{aligned}\Phi^*(z + a_1, \zeta + a_2) \le \Psi(z + \alpha, \zeta + a_2) + \varepsilon \le \Phi(z,\zeta) \\ +\varepsilon \ \ if \ \Omega \times A \cap \partial((\Omega \times A) \setminus \{a\}).\end{aligned}$$

It follows that the function

$$W(z,\zeta) = \begin{cases} \max(\Phi(z,\zeta), \Phi(z + a_1, \zeta + a_2) - 2\varepsilon & (z,\zeta) \in (\Omega \times A) \setminus (\Omega \times A) \setminus \{a\}; \\ \Phi(z,\zeta), & (z,\zeta) \in (\Omega \times A) \cap (\Omega \times A) \setminus \{a\}. \end{cases}$$

is plurisubharmonic in $\Omega \times A$ because

(1) if $(z,\zeta) \in (\Omega \times A) \cap (\Omega \times A) \setminus \{a\}$ it coincides with Φ which is plurisubharmonic.

(2) if $(z,\zeta) \in (\Omega \times A) \setminus (\Omega \times A) \setminus \{a\}$, it is the maximum of two plurisubharmonic functions .

(3) by the two previous inequalities, we infer that the function W coincides on the boundary, furthermore

$$W \le \Psi \ \ on \ \ \partial(\Omega \times A),$$

which implies $W \in \mathcal{F}(\Omega \times A, \Psi)$, finally we get

$$\begin{aligned}\Phi(z + a_1, \zeta + a_2) - 2\varepsilon \le \Phi(z,\zeta) \ for \ (z,\zeta) \in \Omega \\ \times A \ and \ a \in \mathbb{C}^{n+1}, ||a|| \le \beta.\end{aligned}$$

Thus Φ is lower semi-continuous, therefore it is continuous.

(ii) We are going to prove that

$$\lim_{\Omega\times A\ni(z,\zeta)\to(\xi_0,\eta_0)\in\partial(\Omega\times A)} \Phi(z,\zeta) = \Psi(\xi_0,\eta_0).$$

Firstly, since $\Phi \in \mathcal{F}(\Psi, \Omega \times A)$ we have

$$\limsup_{(z,\zeta)\to(\xi_0,\eta_0)} \Phi(z,\zeta) \leq \Psi(\xi_0,\eta_0)\ \forall(\xi_0,\eta_0) \in \partial(\Omega \times A).$$

To prove the reverse of inequality, we construct a plurisubharmonic barrier function at each point $(\xi_0, \eta_0) = \gamma_0 \in \partial(\Omega \times A)$. Since ρ is strictly plurisubharmonic function, we can choose B large enough so that the function

$$b(z,\xi) := B\rho(z) - |z-\xi_0|^2 - |\zeta - \eta_0|^2$$

is plurisubharmonic in $\Omega \times A$ and continuous up to the boundary such that $b(\xi_0, \eta_0) \leq 0$ with $b < 0$ for all $(z,\zeta) \in \bar{\Omega} \times \bar{A} \setminus \gamma_0$.
Fix $\epsilon > 0$ and take $\eta > 0$ such that $\Psi(\gamma_0) - \epsilon \leq \Psi(\gamma)\ \forall\gamma \in \partial(\Omega \times A)$ and $|\gamma - \gamma_0| \leq \eta$. We choose $C > 1$ big enough so that

$$Cb + \Psi(\gamma_0) - \varepsilon \leq \Psi \ \ on \ \ \partial(\Omega \times A).$$

This implies that the function $V(z,\zeta) = Cb(z,\zeta) + \Psi(\gamma_0) - \varepsilon \in PSH(\Omega \times A)$ satisfies

$$V \leq \Psi \ \ on \ \ \partial(\Omega \times A).$$

Thus we have $V \in \mathcal{F}(\Psi, \Omega \times A)$ which implies $V(z,\zeta) \leq \Phi(z,\zeta)$ in $\Omega \times A$. We get

$$\Psi(\xi_0,\eta_0) - \varepsilon \leq \liminf_{(z,\zeta)\to(\xi_0;\eta_0)} \Phi(z,\zeta),$$

therefore

$$\lim_{(z,\zeta)\to(\xi_0;\eta_0)} \Phi(z,\zeta) = \Psi(\xi_0,\eta_0)\ \forall(\xi_0,\eta_0) \in \partial(\Omega \times A).$$

(iii) The Perron–Bremermann envelope

$$\Phi(z,\zeta) = \sup\{u(z,\zeta) \in \mathcal{F}(\Omega \times A, \Psi)\}$$

is plurisubharmonic continuous up the boundary of $\Omega \times A$ and $\Phi|_{\partial(\Omega\times A)} = \Psi$. By a Lemma due to Choquet, this envelope can be realized by a countable family

$$\Phi(z,\zeta) = \sup\{u(z,\zeta) \in \mathcal{F}(\Omega \times A, \Psi)\} = \sup_j\{u_j(z,\zeta) \in \mathcal{F}(\Omega \times A, \Psi)\}.$$

We put

$$\Phi_j(z,\zeta) = \max(u_1(z,\zeta), u_2(z,\zeta), \dots, u_j(z,\zeta)) \nearrow \Phi(z,\zeta),$$

the function Φ_j is increasing and satisfies

$$(\Phi(z,\zeta))^* = (\sup_j\{\Phi_j(z,\zeta)\})^*.$$

Let $\mathbb{B} \subset\subset \Omega \times A$ be any ball, we consider the following Dirichlet problem

$$\begin{cases} (dd^c(u_j(z,\zeta))^{n+1} = 0, & \mathbb{B}; \\ u_j = \Phi_j, & \partial\mathbb{B}. \end{cases}$$

since

$$(dd^c_{z,\zeta} u_j(z,\zeta))^{n+1} \leq (dd^c_{z,\zeta}\Phi_j(z,\zeta))^{n+1} \quad \text{in} \quad \mathbb{B},$$

and

$$u_j = \Phi_j \quad \text{on} \quad \partial\mathbb{B},$$

we have

$$\Phi_j(z,\zeta) \leq u_j(z,\zeta) \;\; in \;\; \mathbb{B}.$$

We consider the following function

$$\Theta(z,\zeta) = \begin{cases} u_j(z,\zeta), & (z,\zeta) \in \mathbb{B}; \\ \Phi_j(z,\zeta), & (z,\zeta) \in \bar{\Omega} \times \bar{A} \setminus \mathbb{B}. \end{cases}$$

The function Θ_j belongs to $\mathcal{F}(\Omega \times A, \Psi)\}$. This implies

$$\Theta_j(z,\zeta) \leq \Phi_j(z,\zeta) \; in \; \Omega \times A,$$

furthermore

$$\Theta_j = \Phi_j = \Psi \; on \; \partial(\Omega \times A),$$

then

$$u_j(z,\zeta) = \Phi_j(z,\zeta) \; in \; \mathbb{B},$$

therefore

$$(dd^c_{z,\zeta}(\Phi_j(z,\zeta)))^{n+1} = (dd^c_{z,\zeta}(u_j(z,\zeta)))^{n+1} = 0 \; in \; \mathbb{B},$$

since $\mathbb{B}$ is arbitrary, we give

$$(dd^c_{z,\zeta}\Phi_j(z,\zeta))^{n+1} = 0 \; in \; \Omega \times A.$$

By the continuity property of Monge–Ampère operators of Bedford and Taylor along monotone sequences, we have

$$(dd^c_{z,\zeta}(\Phi_j(z,\zeta))^{n+1} \longrightarrow (dd^c_{z,\zeta}(\Phi(z,\zeta))^{n+1} = 0,$$

i.e

$$(dd^c_{z,\zeta}(\Phi(z,\zeta))^{n+1} = 0 \;\; in \;\; \Omega \times A.$$

□

2.3 Lipschitz Regularity

In this subsection, we give the geodesic regularity Lipschitz in time and in space. We begin by Lipschitz regularity with respect to the time variable. We use a barrier argument as noted by Berndtsson [6].

Proposition 2.5 *The Perron–Bremermann envelope* $\Phi(z,\zeta) = \sup\{u(z,\zeta)/u \in \mathcal{F}(\Omega \times A, \Psi\}$ *is a Lipschitz function with respect to the variable* $t = \log|\zeta|$.

Proof The proof follows from a classical balayage technique. Indeed, we consider the following function

$$\chi(z,\zeta) = \max(\varphi_0(z) - A\log|\zeta|, \varphi_1(z) + A(\log|\zeta| - 1)),$$

where $A > 0$ is a big constant. Furthermore,

$$\begin{aligned}
\chi(z,\zeta)|_{\Omega\times\{|\zeta|=1\}} &= \max(\varphi_0(z), \varphi_1(z) - A) = \varphi_0(z)\\
\chi(z,\zeta)|_{\Omega\times\{|\zeta|=e\}} &= \max(\varphi_0(z) - A, \varphi_1(z)) = \varphi_1(z)\\
\chi(z,\zeta)|_{\partial\Omega\times A} &= \max(-A\log|\zeta|, A(\log|\zeta| - 1) \le 0,
\end{aligned}$$

the last line follows from the fact that $\varphi_0 = \varphi_1 = 0$ on $\partial\Omega$ and $1 < |\zeta| < e$. Then χ it belongs to $\mathcal{F}(\Omega \times A, \Psi)$ and

$$\chi(z,\zeta) \le \Phi(z,\zeta) \; in \; \Omega \times A.$$

Since $\Phi(z,\zeta) = \varphi(z, \log|\zeta|)$ and $\chi(z,\zeta) = \chi(z, \log|\zeta|)$, which implies

$$\frac{\varphi(z,\log|\zeta|)-\varphi(z,1)}{\log|\zeta|} \geq \frac{\chi(z,\zeta)-\varphi(z,1)}{\log|\zeta|} = \frac{\chi(z,\zeta)-\chi(z,1)}{\log|\zeta|}$$

$$\lim_{|\zeta|\to 1} \frac{\chi(z,\zeta)-\chi(z,1)}{\log|\zeta|} = \lim_{|\zeta|\to 1} \frac{\varphi_0(z) - A\log(|\zeta|) - \varphi_0(z)}{\log|\zeta|} = -A$$

which gives $\dot{\varphi}(z,0) \geq -A$, similarly for other case $\dot{\varphi}(z,1) \leq A$. Since the function φ_t is convex along t (by subharmonicity in ζ), we infer that for almost everywhere z,t,

$$-A \leq \dot{\varphi}(z,0) \leq \dot{\varphi}(z,t) \leq \dot{\varphi}(z,1) \leq A.$$

Then φ_t is uniformly Lipschitz at $t = \log|\zeta|$. □

We prove the regularity Lipschitz in space by adapting the method of Bedford and Taylor [1] (see also [23]).

Theorem 2.6 *The Perron–Bremermann envelope* $\Phi(z,\zeta) = \sup\{u(z,\zeta)/u \in \mathcal{F}(\Omega\times A, \Psi\}$ *is a Lipschitz function up to the boundary with respect to space variable* z.

Proof Let ρ be a smooth defining of Ω which is strictly psh in a neighbourhood Ω' of Ω, and also α be a smooth defining of A which is strictly psh in a neighbourhood A' of A. We construct $C^{1,1}$ an extension of function defined on the boundary of $\Omega \times A$ by

$$\Psi(z,\zeta) = \begin{cases} \varphi_0(z) & \Omega\times\{|\zeta|=1\} \\ \varphi_1(z) & \Omega\times\{|\zeta|=e\} \\ 0 & \partial\Omega\times A. \end{cases}$$

Let χ be a smooth function with compact support defined in [0, 1] by $\chi(t) = 1$ near of 0 and by $\chi(t) = 0$ near of 1. We put

$$\tilde{\chi}(\zeta) = \chi(\log|\zeta|),$$

is a smooth function in $\bar{A}$. We have $\tilde{\chi}(\zeta) = 1$ near of $|\zeta| = 1$ and $\tilde{\chi}(\zeta) = 0$ near of $|\zeta| = e$.

We consider the following function:

$$F(z,\zeta) = \tilde{\chi}(\zeta)\tilde{\varphi}_0(z,\zeta) + (1-\tilde{\chi}(\zeta))\tilde{\varphi}_1(z,\zeta) + B\alpha(\zeta),$$

where $\tilde{\varphi}_0(z,\zeta) = \varphi_0(z)$, $\tilde{\varphi}_1(z,\zeta) = \varphi_1(z)$. The function F satisfies

$$F|_{\Omega\times\partial A} = \begin{cases} \varphi_0(z), & \Omega\times\{|\zeta|=1\} \\ \varphi_1(z), & \Omega\times\{|\zeta|=e\} \\ 0, & \partial\Omega\times A. \end{cases}$$

The function F is plurisubharmonic extension of the function Ψ defined on $\Omega\times\partial A$ to $\Omega\times A$. We can also extend the function Ψ defined in $\partial\Omega\times A$ by putting

$$F(z,\zeta) = D\rho(z),$$

where D is a big constant.

On two cases the function F satisfies the following properties

$$F \le \Phi \; on \; \partial(\Omega \times A) \quad \text{and} \quad (dd^c_{z,\zeta} F)^{n+1} \ge (dd^c_{z,\zeta} \Phi)^{n+1} \; in \; \Omega \times A.$$

By the maximum principle, we get

$$F(z,\zeta) \le \Phi(z,\zeta) \quad \text{in} \quad \Omega \times A.$$

Applying the same process to the boundary data $-\Psi$ we choose $C^{1,1}$ function defined in $\Omega \times A$ such that $G = -\Psi$ on $\partial(\Omega \times A)$, the maximum Principle implies

$$\Phi(z,\zeta) \le -G(z,\zeta) \quad \text{in} \quad \Omega \times A$$

by the two previous inequalities we have

$$F(z,\zeta) \le \Phi(z,\zeta) \le -G(z,\zeta) \quad \text{in} \quad \Omega \times A.$$

Since $F(.,\zeta) \le \Phi(,\zeta)$ in Ω, the envelope $\Phi(,\zeta)$ can be extended, respectively, to variable z as a plurisubharmonic function in Ω' by setting $\Phi(,\zeta) = F(,\zeta)$ in $\Omega' \setminus \Omega$ with ζ fixed in A. Fix $\delta > 0$ so small that $z + h \in \Omega$ whenever $z \in \bar{\Omega}$ and $||h|| < \delta$, this set noted in sequel by Ω_h. Fix $h \in \mathbb{C}^n$ such that $||h|| < \delta$. Recall that F and G are Lipschitz in each variable, thus

$$|F(z+h,\zeta) - F(z,\zeta)| \le C||h|| \; and \; |G(z+h,\zeta) - G(z,\zeta)| \le C||h||,$$

for any $z \in \bar{\Omega}$ and $\zeta \in \bar{A}$.

Observe that the function $v(z,\zeta) := \Phi(z+h,\zeta) - C||h||$ is well-defined psh in the open set $\Omega \times A$. If $z \in \partial\Omega \cap \Omega_h$ and $\zeta \in \bar{A}$, then

$$v(z,\zeta) = \Phi(z+h,\zeta) - C||h|| \le -G(z+h,\zeta) - C||h|| \le -G(z,\zeta) = \Psi(z,\zeta).$$

If $z \in \Omega \cap \partial\Omega_h$ and $\zeta \in \bar{A}$, then

$$v(z,\zeta) = \Phi(z+h,\zeta) - C||h|| \le F(z+h,\zeta) - C||h|| \le F(z,\zeta) \le \Phi(z,\zeta).$$

This shows that the function w defined by

$$w(z,\zeta) := \begin{cases} \max(v(z,\zeta), \Phi(z,\zeta)) & \text{if} \quad (z,\zeta) \in \Omega \cap \Omega_h \times A \\ \Phi(z,\zeta) & \text{if} \quad (z,\zeta) \in \Omega \setminus \Omega_h \times A \end{cases}$$

is plurisubharmonic in $\Omega \times A$. Since $w \le \Psi$ on $\partial(\Omega \times A)$ we get $w \le \Phi$ in $\Omega \times A$, hence $v \le \Phi$ in $\Omega \times A$. We have shown that

$$\Phi(z+h,\zeta) - \Phi(z,\zeta) \le C||h||$$

whenever $z \in \Omega \cap \Omega_h$, $||h|| \leq \delta$ and $\zeta \in A$. Replacing h by $-h$ shows that

$$|\Phi(z+h, \zeta) - \Phi(z, \zeta)| \leq C||h||,$$

which proves that $\Phi(, \zeta)$ is Lipschitz in every $z \in \bar{\Omega}$. □

3 Case of the Unit Ball

In this section, we shall show how to use the proof of Bedford and Taylor [1], which is simplified by Demailly [19] in the unit ball for giving the regularity in space variable for our geodesics problem. We need some preparation to prove this regularity. The open subset giving by

$$\mathbb{B} := \{z \in \mathbb{C}^n \ / \ |z_1|^2 + |z_2|^2, \cdots, +|z_n|^2 < 1\},$$

is called the unit ball. First, we shall define the Mobius transformation of the unit ball. Let $a \in \mathbb{B} \setminus \{0\} \subset \mathbb{C}^n$. Denote by P_a the orthogonal projection onto the subspace of in $\mathbb{C}^n$ generated by the vector a by,

$$P_a(z) := \frac{< z, a > a}{||a||^2}.$$

The Mobius transformation associated with a is the mapping

$$T_a(z) := \frac{P_a(z) - a + \sqrt{(1 - ||a||^2)}(z - P_a(z))}{1 - < z, a >}.$$

With $< z, a >= \sum_{i=1}^n z_i \bar{a}_i$ denote the hermitian scalar product to z and a. For every $a \in B$, the Mobius transformation has the following properties

(i) $T_a(0) = -a$ and $T_0(a) = 0$.
(ii) an elementary computation yields

$$T_a(z) = z - a + < z, a > a + O(||a||^2) = z - h + O(||a||^2), \tag{2}$$

with $h = h(a, z) := a - < z, a > z$ and $O(||a||^2)$ is uniformly with respect of the variable $z \in \bar{\mathbb{B}}$.

We need in the sequel the following useful Lemma to giving the regularity in unit ball.

Lemma 3.1 *Let u be a plurisubharmonic function in domain $\Omega \subset\subset \mathbb{C}^n$, assume that there exists $B, \delta > 0$ such that*

$$u(z+h) + u(z-h) - 2u(z) \leq B||h||^2, \ \ \forall 0 < ||h|| < \delta,$$

and for all $z \in \Omega$ and $dist(z, \Omega) > \delta$. Then u is $C^{1,1}$-smooth and its second derivative, which exists almost everywhere, satisfies

$$||D^2 u||_{L^\infty(\Omega)} \leq B.$$

Proof Let $u_\varepsilon = u * \chi_\varepsilon$ denote the standard regularization of u defined in $\Omega_\varepsilon = \{z \in \Omega \,/ dist(\partial\Omega, z) > \varepsilon\}$ for $0 < \varepsilon << 1$. Fix $\delta > 0$ small enough and $0 < \varepsilon < \frac{\delta}{2}$. Then for $0 < ||h|| < \frac{\delta}{2}$ we have

$$u_\varepsilon(z+h) + u_\varepsilon(z-h) - 2u_\varepsilon(z) \leq B||h||^2. \tag{3}$$

It follows from Taylor's formula that if $z \in \Omega_\epsilon$

$$\frac{d^2}{\mathrm{d}t^2} u_\varepsilon(z+th)|_{t=0} := \lim_{t \to 0^+} \frac{u_\varepsilon(z-th) + u_\varepsilon(z+th) - 2u_\varepsilon(z)}{t^2},$$

therefore by having $D^2 u_\varepsilon(z).h^2 \leq B||h||^2$ for all $z \in \Omega_\varepsilon$ and $h \in \mathbb{C}^n$. Now for $z \in \Omega_\varepsilon$,

$$D^2 u_\varepsilon(z).h^2 = \sum_{i,j=1}^{n} \left(\frac{\partial^2 u_\varepsilon}{\partial z_i \partial z_j} h_i h_j + 2\frac{\partial^2 u_\varepsilon}{\partial z_i \partial \bar{z}_j} h_i \bar{h}_j + \frac{\partial^2 u_\varepsilon}{\partial \bar{z}_i \partial \bar{z}_j} \bar{h}_i \bar{h}_j \right).$$

Recall that u_ϵ is plurisubharmonic in Ω_ε hence

$$D^2 u_\varepsilon(z).h^2 + D^2 u_\varepsilon(z).(ih)^2 = 4 \sum_{i,j=1}^{n} \frac{\partial^2 u_\varepsilon}{\partial z_i \partial \bar{z}_j} h_i \bar{h}_j \geq 0.$$

The above upper-bound also yields a lower-bound of $D^2 u_\varepsilon$

$$D^2 u_\varepsilon.h^2 \geq -D^2 u_\varepsilon.(ih)^2 \geq -B||h||^2.$$

For any $z \in \Omega$ and $h \in \mathbb{C}^n$. This implies that

$$||D^2 u_\varepsilon||_{L^\infty(\Omega)} \leq B$$

Thus, we have shown that Du_ε is uniformly Lipschitz in Ω_ε. We infer that Du is Lipschitz in Ω and $Du_\varepsilon \longrightarrow Du$ uniformly in compact subsets of Ω. Since the dual of L^1 is L^∞, it follows from the Alaoglu–Banach theorem that, up to extracting a subsequence, there exists a bounded function V such that $D^2 u_\varepsilon \longrightarrow V$ weakly in L^∞. Now $D^2 u_\varepsilon \longrightarrow D^2 u$ in the sense of distributions hence $V = D^2 u$. Therefore, u is $C^{1,1}$ in Ω and its second-order derivative exists almost everywhere with $||D^2 u||_{L^\infty(\Omega)} \leq B$. □

Theorem 3.2 *By taking $\mathbb{B}$ is the unit ball in $\mathbb{C}^n$. Let φ_0 and φ_1 be the geodesic end points which are $C^{1,1}$. Then the Perron–Bremermann envelope*

$$\Phi(z,\zeta) = \sup\{u(z,\zeta)/u \in \mathcal{F}(\Omega \times A, \Psi)\},$$

admits second-order partial derivatives almost everywhere with respect to the variable $z \in \mathbb{B}$ which locally uniformly bounded with respect to the variable $\zeta \in A$, i.e for any compact subset $K \subset \mathbb{B}$ there exists C which depends on K, φ_0 and φ_1 such that

$$\|D_z^2\Phi\|_{L^\infty(K\times A)} \leq C.$$

Proof To prove the theorem, we weed to prove the following inequality

$$\Phi(z+h,\zeta) + \Phi(z-h,\zeta) - 2\Phi(z,\zeta) \leq A||h||^2,$$

for any $||h|| << 1$, $z \in \mathbb{B}$ and $\zeta \in A$.

The idea is to study the boundary behaviour of the plurisubharmonic function $(z,\zeta) \longmapsto \frac{1}{2}(\Phi(z+h,\zeta)+\Phi(z-h,\zeta)$ in order to compare it with the function Φ in $\mathbb{B}\times A$. This does not make sense since the translations do not preserve the boundary. We are instead going to move point z by automorphisms of the unit ball: the group of holomorphic automorphisms of the latter acts transitively on it and this is the main reason why we prove this result for the unit ball rather than for a general strictly pseudoconvex domain (which has generically few such automorphisms).

By the fact that Φ is Lipschitz with respect to the variable z (Theorem 2.6) and expansion (2), we have

$$|\Phi(T_a(z),\zeta) - \Phi(z-h,\zeta)| \leq C||T_a(z) - (z-h)|| \leq C||a||^2,$$

and

$$|\Phi(T_{-a}(z),\zeta) - \Phi(z+h,\zeta)| \leq C||T_{-a}(z) - (z+h)|| \leq C||a||^2,$$

which implies

$$\Phi(z+h,\zeta) + \Phi(z-h,\zeta) \leq \Phi(T_a(z),\zeta) + \Phi(T_{-a}(z),\zeta) + 2C||a||^2.$$

We consider the following functions:

$$F(z,\zeta) := \Phi(T_a(z),\zeta) + \Phi(T_{-a}(z),\zeta) + 2C||a||^2,$$

and $G(z,\zeta) = 2\Phi(z,\zeta) + D||a||^2$, we observe that the functions F and G are well defined in $\mathbb{B}\times A$ and plurisubharmonic s in $\mathbb{B}\times A$. We need to show that

$$F(z,\zeta) \leq G(z,\zeta) \quad \text{in} \quad \mathbb{B}\times A.$$

To show the last inequality, we apply the maximum principle, then we need to prove

$$F(z,\zeta) \le G(z,\zeta) \quad \text{on} \quad \partial(\mathbb{B}\times A)$$

and

$$(dd^c_{z,\zeta}F(z,\zeta))^{n+1} \ge (dd^c_{z,\zeta}G(z,\zeta))^{n+1} \quad \text{in} \quad \mathbb{B}\times A.$$

The last inequality follows from the fact that F is a plurisubharmonic and $(dd^c_{z,\zeta}\Phi)^{n+1} = 0$ in $\mathbb{B}\times A$ by (Theorem 2.4).

We need to compare F and G in the boundary of $\mathbb{B}\times A$. Indeed, since $\partial(\mathbb{B}\times A) = (\partial\mathbb{B}\times \bar{A}) \cup (\bar{\mathbb{B}}\times\partial A)$, then we compare in two parts, we begin first by the part that $\partial\mathbb{B}\times\bar{A}$, in this part we get

$$F|_{\partial B\times\bar{A}} = 2C||a||^2 \ \ \textit{and} \ \ G|_{\partial\mathbb{B}\times\bar{A}} = D||a|^2.$$

To show that $F|_{\partial\mathbb{B}\times\bar{A}} \le G_{\partial\mathbb{B}\times\bar{A}}$, we take just

$$D = 2C.$$

For the second part $\bar{\mathbb{B}}\times\partial A$, we only compare $\mathbb{B}\times\bar{A}$, because $\partial\mathbb{B}\times\bar{A}$ belongs to the previous part, since $\partial A = \{|\zeta| = 1\} \cup \{|\zeta| = e\}$, we begin this part by comparing the case $\mathbb{B}\times\{|\zeta| = 1\}$, we have

$$F|_{\mathbb{B}\times\{|\zeta|=1\}} = \varphi_0(T_a(z)) + \varphi_0(T_{-a}(z)) + 2C||a||^2,$$

and

$$G|_{\mathbb{B}\times\{|\zeta|=1\}} = 2\varphi_0(z) + D||a||^2.$$

We apply Taylor expansion and we get

$$\varphi_0(T_a(z)) = \varphi_0(z - h + O(|a|^2) = \varphi_0(z) - d\varphi(z).h + O(|a|^2),$$

and

$$\varphi_0(T_{-a}(z)) = \varphi_0(z + h + O(|a|^2) = \varphi_0(z) + d\varphi(z).h + O(|a|^2),$$

which implies

$$\varphi_0(T_a(z)) + \varphi_0(T_{-a}(z)) \le 2\varphi_0(z) + 2C_0|a|^2,$$

where C_0 depends only on the φ_0 then

$$F(z,\zeta) \le 2\varphi_0(z) + 2C_1|a|^2 + 2C|a|^2.$$

If we take $D = 2(C_0 + C)$, we get $F(z,\zeta) \le G(z,\zeta)$ on $\mathbb{B} \times \{|\zeta| = 1\}$. By the same methods, we get that $F(z,\zeta) \le G(z,\zeta)$ on $\mathbb{B} \times \{|\zeta| = 1\}$ for $D = 2(C_1 + C)$, where C_1 depends only on the φ_1 which concludes the second part.

Through the part one and two we infer that,

$$F(z,\zeta) \le G(z,\zeta) \quad \text{on} \quad \partial(\mathbb{B} \times A).$$

From the maximum Principle we get,

$$F(z,\zeta) \le G(z,\zeta) \quad \text{in} \quad \mathbb{B} \times A,$$

Which implies

$$\begin{aligned}\Phi(z+h,\zeta) + \Phi(z-h,\zeta) - 2\Phi(z,\zeta) &\le \Phi(T_a(z),\zeta) + \Phi(T_{-a}(z),\zeta) \\ &\quad +2C||a||^2 - 2\Phi(z,\zeta) \\ &\le \Phi(T_a(z),\zeta) + \Phi(T_{-a}(z),\zeta) \\ &\quad +2C||a||^2 - 2\Phi(z,\zeta) \\ &\le D||a||^2.\end{aligned}$$

Observe that the mapping $a \longmapsto h(a,z) = a- < z, a > z$ is a local diffeomorphism in neighbourhood of the origin as long as $||z|| < 1$, which depends on $z \in \mathbb{B}$ smoothly and its inverse $h \longmapsto a(h,z)$ which is linear with a norm less than or equal to $\frac{1}{1-||z||^2}$ since

$$||h|| \ge ||a|| - ||a||||z||^2 = ||a||(1 - ||z||^2),$$

which gives

$$\Phi(z+h,\zeta) + \Phi(z-h,\zeta) - 2\Phi(z,\zeta) \le \frac{D||h||^2}{(1-||z||^2)^2}.$$

Fix a set $K \subset \mathbb{B}$ compact, there exists $\delta > 0$ such that $\forall z \in K$ and $\forall 0 < ||h|| < \delta$ we have

$$\Phi(z+h,\zeta) + \Phi(z-h,\zeta) - 2\Phi(z,\zeta) \le \frac{D||h||^2}{dist(K,\partial\mathbb{B})^2},$$

by the previous Lemma we get

$$||D_z^2\Phi||_{L^\infty(K\times A)} \le D,$$

where $C = \frac{D}{dist(K,\partial\mathbb{B})^2}$. □

4 Moser–Trudinger Inequalities

In this section, we assume that Ω is a strictly pseudoconvex circled domain. We consider the following Monge–Ampère equation

$$(dd^c\varphi_t)^n = \frac{e^{-t\varphi_t}\mu}{\int_\Omega e^{-t\varphi_t}d\mu} \tag{4}$$

with φ_t smooth and plurisubharmonic, $\varphi_{t\,|\partial\Omega} = 0$ and μ is just the Lebesgue normalized so that $\mu(\Omega) = 1$. It is known that this equation admits a solution if $t < (2n)^{1+1/n}$ $(1+1/n)^{(1+1/n)}$ [3,9,22]

- We can solve this equation if t is not too large ($t = 1$ is treated in [22] and even $t < (2n)^{1+1/n}(1+1/n)^{(1+1/n)}$).
- One cannot solve the equation if t is too large, cf [22, Section 6.2] and [3].
- The above equation was also studied by Cegrell [9].

We denote by

$$E(\varphi) := \frac{1}{n+1}\int_\Omega \varphi(dd^c\varphi)^n,$$

the Monge–Ampère energy functional of a plurisubharmonic function φ, which is defined as the primitive of Monge–Ampère operator. The expression

$$\mathcal{F}_t(\varphi) := E(\varphi) + \frac{1}{t}\log\left[\int_\Omega e^{-t\varphi}d\mu\right],$$

defines the Ding functional.

Definition 4.1 We say the functional $\mathcal{F}_t$ is coercive, if there exist $\varepsilon > 0$ and $B > 0$ such that

$$\mathcal{F}_t(\varphi) \le \varepsilon E(\varphi) + B \quad \forall\varphi \in \mathcal{H},$$

Definition 4.2 Set $\Phi_s(z) = \Phi(z, e^s)$. The continuous family $(\Phi_s)_{0\le s\le 1}$ is called the geodesic joining φ_0 and φ_1.

We show that E is linear along of geodesics, this result was proven in [22, Lemma 22]. It was also proven by Rashkovskii [28] in the Cegrell class. For convenience of the reader, we reproduce the proof here.

Lemma 4.3 *Let $(\Phi_s)_{0\le s\le 1}$ be a continuous geodesic. Then $s \longmapsto E(\Phi_s)$ is affine.*

Proof by the Proof of Theorem 2.2 we have

$$(dd^c_{z,\zeta}\Phi(z,\zeta))^{n+1} = (n+1)(dd^c_z\Phi(z,\zeta))^n \wedge R$$
$$+\frac{n(n+1)}{2}(dd^c_z\Phi(z,\zeta))^{n-1} \wedge R^2$$

$$= (n+1)\Big(d_\zeta d^c_\zeta \Phi \wedge (d_z d^c_z \Phi)^n - n d_z d^c_\zeta \Phi \wedge d_\zeta d^c_z \Phi \wedge (d_z d^c_z \Phi)^{n-1}\Big).$$

We have by definition of E

$$E(\Phi(.,\zeta) = \frac{1}{n+1}\int_\Omega \Phi(z,\zeta)(d_z d^c_z \Phi(z,\zeta))^n.$$

Which implies

$$d^c_\zeta E(\Phi) = \frac{1}{n+1}\int_\Omega d^c_\zeta \Phi \wedge (d_z d^c_z \Phi)^n$$

$$\begin{aligned} d_\zeta d^c_\zeta E(\Phi) &= \frac{1}{n+1}\Big(\int_\Omega d_\zeta d^c_\zeta \Phi \wedge (d_z d^c_z \Phi)^{n-1} \\ &\quad + n\int_\Omega d^c_\zeta \Phi \wedge d_\zeta d_z d^c_z \Phi \wedge (d_z d^c_z \Phi)^n\Big) \\ &= \frac{1}{n+1}\Big(\int_\Omega d_\zeta d^c_\zeta \Phi \wedge (d_z d^c_z \Phi)^{n-1} \\ &\quad - n\int_\Omega d_z d^c_\zeta \Phi \wedge d_\zeta d^c_z \Phi \wedge (d_z d^c_z \Phi)^{n-1}\Big) \\ &= \frac{1}{(n+1)^2}\int_\Omega (dd^c_{z,\zeta}\Phi)^{n+1}, \end{aligned}$$

where the second equality follows from Stokes theorem and the fact that $d_\zeta \Phi = 0$ on $\partial\Omega$.

Thus, it follows from Theorem 2.4 that $\zeta \in A \longmapsto E(\Phi(.,\zeta) \in \mathbb{R}$ is harmonic in ζ. Since Φ is invariant by rotation with respect to the variable ζ, hence it is affine in $t = \log|\zeta|$. □

We recall here [22, Proposition 23].

Proposition 4.4 *Assume that Ω is circled, let φ_t be an S^1-invariant solution of $(MA)_t$. Then*

$$\mathcal{F}_t(\varphi_t) = \sup_{\psi \in I(\Omega)} \mathcal{F}_t(\psi),$$

where $I(\Omega)$ denotes all S^1-invariant plurisubharmonic functions ψ in Ω which are continuous up to the boundary, with zero boundary value.

Proof Let $(\Phi)_{0\leq s\leq 1}$ be a geodesic joining $\Phi_0 := \varphi_t$ to $\Phi_1 = \psi$. It follows from work of Berndtsson [5] that

$$s \longmapsto -\frac{1}{t}\log\left(\int_\Omega e^{-t\Phi_s} d\mu\right)$$

is convex, since $s \longmapsto E(\Phi_s)$ is affine from Lemma 4.3. Then $s \longmapsto \mathcal{F}(\Phi_s)$ is concave.

Therefore, it is sufficient to show that the derivative of $\mathcal{F}_t(\Phi_s)$ at $s = 0$ is non-negative to conclude $\mathcal{F}_t(\varphi_t) = \mathcal{F}(\Phi_0) \geq \mathcal{F}_t(\Phi_s)$ for all s, in particular at $s = 1$ where it yields $\mathcal{F}_t(\varphi_t) \geq \mathcal{F}_t(\psi)$. When $\longmapsto \Phi_s$ is smooth, a direct computation yields, for $s = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}_t(\Phi_s) = \int_\Omega \dot{\Phi}_s\left[(dd^c\Phi_s)^n - \frac{e^{-t\Phi_s}\mu}{\int_\Omega e^{-t\Phi_s}d\mu}\right] = 0$$

For the general case, one can argue as in the proof of [4, Theorem 6.6]. □

Lemma 4.5 *The Functional $\mathcal{F}_t$ is upper semi-continuous in $\mathcal{E}^1_C(\Omega) = \{\psi \in \mathcal{E}^1(\Omega)/\psi = 0$ on $\partial\Omega$ and $E(\psi) \geq -C\}$.*

Proof Recall $\mathcal{F}_t(\psi) = E(\psi) + \frac{1}{t}\log\left(\int_\Omega e^{-t\psi}d\mu\right)$. The first term is upper semi-continuous in $\mathcal{E}^1(\Omega)$. For the second term, we apply Skoda uniform integrability Theorem [30].

Assume without loss of generality that $t = 1$. We need to check that $\psi \in \mathcal{E}^1_C(\Omega) \longmapsto \int_\Omega e^{-\psi}d\mu$ is upper semi-continuous.

Let ψ_j be a sequence in $\mathcal{E}^1_C(\Omega)$ converging to ψ, these functions have zero Lelong number. The following extension:
$g_j = \psi_j + \psi$ to $\Omega \subset K \subset \Omega'$ as $\tilde{g}_j = g_j$ in Ω, $\tilde{g}_j = 0$ in $\Omega' \setminus \Omega$. We apply Skoda's uniform integrability estimates:

$$\int_\Omega e^{-2(\psi+\psi_j)}d\mu \leq \int_K e^{-2(\psi+\psi_j)}d\mu \leq C.$$

$$|\int_\Omega e^{-\psi_j}d\mu - \int_\Omega e^{-\psi}d\mu| \leq \int_\Omega |\psi - \psi_j|e^{-(\psi_j+\psi)}d\mu \leq C||\psi_j - \psi||_{L^2(\mu)},$$

as follows from the Cauchy–Schwarz inequality and the elementary inequality

$$|e^a - e^b| \leq |a-b|e^{a+b},\ \ for\ all\ a, b \geq 0.$$

The conclusion follows since (ψ_j) converges to ψ in $L^2(\mu)$. □

We recall that the Dirichlet problem $(MA)_t$ has a solution for $t = 1$ by [22], moreover we have uniqueness if Ω is strictly φ-convex(Ω is strictly convex dor the metric $dd^c\varphi$). We recall here the main result of [22].

Theorem 4.6 *Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain which is circled. Let φ be a smooth S^1-invariant strictly plurisubharmonic solution of the complex Monge–Ampère problem $(MA)_1$. If Ω is strictly φ-convex, then φ is the unique S^1-invariant solution of $(MA)_1$.*

Inspired by Dinezza-Guedj [16, Theorem 5.5], we now prove the following theorem.

Theorem 4.7 *Let $\Omega \subset \mathbb{C}^n$ be a smooth strongly pseudoconvex circled domain. If there exists $\varepsilon(t), M(t) > 0$ such that,*

$$\mathcal{F}_t(\psi) \leq \varepsilon(t)E(\psi) + M(t) \quad \forall \psi \in \mathcal{H},$$

then $(MA)_t$ admits a S^1-invariant smooth strictly plurisubharmonic function solution.

Conversely if $(MA)_t$ admits such a solution φ_t and Ω is strictly φ_t-convex, then there exists $\varepsilon(t), M(t) > 0$ such that,

$$\mathcal{F}_t(\psi) \leq \varepsilon(t)E(\psi) + M(t) \quad \forall \psi \in \mathcal{H}.$$

Proof If we assume the following inequality holds,

$$\mathcal{F}_t(\psi) \leq \varepsilon(t)E(\psi) + M(t)$$

then the same method of [22]applies , if only we change φ by $t\varphi$.

Conversely, as φ_t is a solution of $(MA)_t$ then from the (Proposition 4.4) we have

$$\mathcal{F}_t(\varphi_t) := \sup\{\mathcal{F}_t(\psi)/\psi \in \mathcal{H} \cap I(\Omega)\} \tag{5}$$

assume for contradiction that there is no $\varepsilon > 0$ such that

$$\mathcal{F}_t(\psi) \leq \varepsilon E(\psi) + M$$

for all $\psi \in \mathcal{H}$. Put $\varepsilon_j = \frac{1}{j}$ and $M = \mathcal{F}_t(\varphi_t) + 1$. Then we can find a sequence $(\varphi_j) \subset \mathcal{H}$ such that

$$\mathcal{F}_t(\varphi_j) > \frac{E(\varphi_j)}{j} + \mathcal{F}_t(\varphi_t) + 1.$$

We discuss here two cases, the first case if $E(\varphi_j)$ does not blow up to $-\infty$, we reach a contradiction, by letting j go to $+\infty$. Indeed, we can assume that $E(\varphi_j)$ is bounded and φ_j converges to some $\psi \in \mathcal{E}^1(\Omega)$ which is S^1-invariant. Since $\mathcal{F}_t$ is upper semi-continuous by Lemma 4.5, we infer $\mathcal{F}_t(\psi) \geq \mathcal{F}_t(\varphi_t) + 1 > \mathcal{F}_t(\varphi_t)$ contradiction because φ_t is the solution of $(MA)_t$.

The second case if $E(\varphi_j) \to -\infty$. It follows that $d_j = -E(\varphi_j) \to +\infty$. We let $(\phi_{s,j})_{0 \leq s \leq d_j}$ denote the weak geodesic joining φ_t to φ_j and set $\psi_j := \phi_{1,j}$. We know that $s \longmapsto E(\phi_{s,j})$ is affine along of the Mabuchi geodesic. Thus $E(\phi_{s,j}) = a_j s + b_j$, where a_j and b_j are real numbers. For $s = 0$ we have

$$E(\phi_{0,j}) = b_j = E(\varphi_t),$$

and for $s = d_j$ we have

$$E(\varphi_j) = E(\phi_{d_j,j}) = a_j d_j + E(\varphi_t)$$

therefore $a_j = \frac{E(\varphi_j)-E(\varphi_t)}{d_j}$. Then

$$E(\phi_{s,j}) = \frac{E(\varphi_j) - E(\varphi_t)}{d_j} s + E(\varphi_t). \tag{6}$$

Since $s \longmapsto E(\phi_{s,j})$ is affine along of the Mabuchi geodesic and by Berndtsson [5] convexity result, we infer that the map $s \longmapsto \mathcal{F}_t(\phi_{s,j})$ is concave, which implies with (5) that

$$0 \geq \mathcal{F}_t(\phi_{1,j}) - \mathcal{F}_t(\phi_{0,j}) \geq \frac{\mathcal{F}_t(\phi_{d_j,j}) - \mathcal{F}_t(\phi_{0,j})}{d_j} > -\frac{1}{j} + \frac{1}{d_j}.$$

Thus $\mathcal{F}_t(\psi_j) \longrightarrow \mathcal{F}_t(\varphi_t)$. This shows that (ψ_j) is a maximizing sequence for $\mathcal{F}_t$. If we take $t = 1$ on Eq. (6), we get

$$E(\psi_j) = \frac{E(\varphi_j) - E(\varphi_t)}{d_j} + E(\varphi_t) = -1 - \frac{E(\varphi_t)}{d_j} + E(\varphi_t) \geq -1 + E(\varphi_t). \tag{7}$$

Passing to subsequence, we can assume that ψ_j converges to $\psi \in \mathcal{E}^1(\Omega)$ which is S^1-invariant. Since $\mathcal{F}_t$ is upper semi-continuous and ψ_j is a maximizing sequence for $\mathcal{F}_t$ then we have $\mathcal{F}_t(\psi) = \mathcal{F}_t(\varphi_t)$ and so $\psi = \varphi_t$ thanks to the uniqueness. Letting j to infinity in (7) we get

$$E(\psi) = -1 + E(\varphi_t),$$

this yields a contradiction. □

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