J Geom Anal (2018) 28:3553–3602
https://doi.org/10.1007/s12220-017-9969-y

Sewing Riemannian Manifolds with Positive Scalar Curvature

J. Basilio[1] · **J. Dodziuk**[2] · **C. Sormani**[3]

Received: 26 June 2017 / Published online: 11 December 2017

Abstract We explore to what extent one may hope to preserve geometric properties of three-dimensional manifolds with lower scalar curvature bounds under Gromov–Hausdorff and Intrinsic Flat limits. We introduce a new construction, called sewing, of three-dimensional manifolds that preserves positive scalar curvature. We then use sewing to produce sequences of such manifolds which converge to spaces that fail to have nonnegative scalar curvature in a standard generalized sense. Since the notion of nonnegative scalar curvature is not strong enough to persist alone, we propose that one pair a lower scalar curvature bound with a lower bound on the area of a closed minimal surface when taking sequences as this will exclude the possibility of sewing of manifolds.

Keywords Scalar curvature · Gromov-Hausdorff · Intrinsic flat

Mathematics Subject Classification 53C23

J. Basilio was partially supported as a doctoral student by NSF DMS 1006059. C. Sormani was partially supported by NSF DMS 1006059.

✉ C. Sormani
sormanic@gmail.com

J. Basilio
jorge.math.basilio@gmail.com

J. Dodziuk
jdodziuk@gmail.com

1 CUNY Graduate Center and Sarah Lawrence College, Bronxville, NY, USA

2 CUNY Graduate Center and Queens College, New York, NY, USA

3 CUNY Graduate Center and Lehman College, Bronx, NY, USA

1 Introduction

In this paper, we study three-dimensional manifolds with positive scalar curvature. The scalar curvature of a Riemannian manifold is the average of the Ricci curvatures which in turn is the average of the sectional curvatures. It can be determined more simply by taking the following limit:

$$\mathrm{Scal}(p) = \lim_{r\to 0} 30\frac{\mathrm{Vol}_{\mathbb{E}^3}(B(0,r)) - \mathrm{Vol}_{M^3}(B(p,r))}{r^2\,\mathrm{Vol}_{\mathbb{E}^3}(B(0,r))}, \tag{1}$$

where $\mathrm{Vol}_{\mathbb{E}^3}(B(0,r)) = (4/3)\pi r^3$ and $\mathrm{Vol}_{M^3}(B(p,r))$ is the Hausdorff measure of the ball about p of radius r in our manifold, M^3.

In [16], Gromov asks the following pair of deliberately vague questions which we paraphrase here: *Given a class of Riemannian manifolds*, $\mathcal{B}$, *what is the weakest notion of convergence such that a sequence of manifolds*, $M_j \in \mathcal{B}$, *subconverges to a limit* $M_\infty \in \mathcal{B}$ *where now we will expand* $\mathcal{B}$ *to include singular metric spaces? What is this generalized class of singular metrics spaces that should be included in* $\mathcal{B}$? Gromov points out that when $\mathcal{B}$ is the class of Riemannian manifolds with nonnegative sectional curvature then the "best known" answer to this question is Gromov–Hausdorff convergence and the singular limit spaces are then Alexandrov spaces with nonnegative Alexandrov curvature. When $\mathcal{B}$ is the class of Riemannian manifolds with nonnegative Ricci curvature, one uses Gromov–Hausdorff and metric measure convergence to obtain limits which are metric measure spaces with generalized nonnegative Ricci curvature as in work of Cheeger–Colding [8]. Work towards defining classes of singular metric measure spaces with generalized notions of nonnegative Ricci has been completed by Ambrosio-Gigli-Savare, Lott-Villani, Sturm and others [1,21,28].

Gromov then writes that *"the most tantalizing relation* $\mathcal{B}$ *is expressed with the scalar curvature by* $\mathrm{Scal} \geq k$" [16]. Bamler [4] and Gromov [15] have proven that under C^0 convergence to smooth Riemannian limits $\mathrm{Scal} \geq 0$ is preserved. In order to find the weakest notion of convergence which preserves $\mathrm{Scal} \geq 0$ in some sense, Gromov has suggested that one might investigate intrinsic flat convergence [16]. The intrinsic flat distance was first defined in work of the third author with Wenger [31], who also proved that for noncollapsing sequences of manifolds with nonnegative Ricci curvature, intrinsic flat limits agree with Gromov–Hausdorff and metric measure limits [30]. Intrinsic flat convergence is a weaker notion of convergence in the sense that there are sequences of manifolds with no Gromov–Hausdorff limit that have intrinsic flat limits, including Ilmanen's example of a sequence of three spheres with positive scalar curvature [31]. The third author has investigated intrinsic flat limits of manifolds with nonnegative scalar curvature under additional conditions with Lee, Huang, LeFloch and Stavrov [17,19,20,27]. These papers support Gromov's suggestion in the sense that the limits obtained in these papers have generalized nonnegative scalar curvature.

Here we construct a sequence of Riemannian manifolds, M_j^3, with positive scalar curvature that converges in the intrinsic flat, metric measure and Gromov–Hausdorff sense to a singular limit space, Y, which fails to satisfy (1) [Example 6.1]. In fact, the limit space is a sphere with a pulled thread:

$$Y = \mathbb{S}^3/\sim \text{ where } a \sim b \text{ iff } a, b \in C, \tag{2}$$

where C is one geodesic in $\mathbb{S}^3$ (see Sect. 4). The scalar curvature about the point $p_0 = [C(t)]$ formed from the pulled thread is computed in Lemma 6.3 to be

$$\lim_{r\to 0} \frac{\mathrm{Vol}_{\mathbb{E}^3}(B(0,r)) - \mathrm{Vol}_{M^3}(B(p,r))}{r^2\,\mathrm{Vol}_{\mathbb{E}^3}(B(0,r))} = -\infty. \tag{3}$$

In this sense, the limit space does not have generalized nonnegative scalar curvature.

We construct our sequence using a new method we call sewing developed in Propositions 3.1–3.3. Before we can sew the manifolds, the first two authors construct short tunnels between points in the manifolds building on prior work of Gromov–Lawson [12] and Schoen–Yau [32]. The details of this construction are in the Appendix. In a subsequent paper [7], we will extend this sewing technique to also provide examples whose limit spaces fail to satisfy the Scalar Torus Rigidity Theorem [12,32] and the Positive Mass Rigidity Theorem [33]. These examples, all constructed using the sewing techniques developed in this paper, demonstrate that Gromov–Hausdorff and Intrinsic Flat limit spaces of noncollapsing sequences of manifolds with positive scalar curvature may fail to satisfy key properties of nonnegative scalar curvature.

In light of these counter examples and the aforementioned positive results towards Gromov's conjecture, the third author has suggested in [26] to adapt the class $\mathcal{B}$. There it is proposed that the initial class of smooth Riemannian manifolds in $\mathcal{B}$ should have nonnegative scalar curvature, a uniform lower bound on volume (as assumed implicitly by Gromov), and also a uniform lower bound on the minimal area of a closed minimal surface in the manifold, MinA(M). The sequences of M_j^3 we construct using our new sewing methods have positive scalar curvature and a uniform lower bound on volume, but MinA$(M_j) \to 0$. Intuitive reasons as to why a uniform lower bound on MinA(M_j) is a natural condition are described in [26] along with a collection of related conjectures and open problems. Here we will simply propose the following possible revision of Gromov's vague conjecture:

Conjecture 1.1 *Suppose a sequence of Riemannian manifolds, M_j^3, have*

$$\mathrm{Scal}_j \ge 0,\ \mathrm{Vol}(M_j) \ge V_0 > 0,\ \textit{and } \mathit{MinA}(M_j) \ge A_0 > 0, \tag{4}$$

and the sequence converges in the intrinsic flat sense, $M_j \xrightarrow{\mathcal{F}} M_\infty$.

Then at every point $p \in M_\infty$ we have

$$\lim_{r\to 0} \frac{\mathrm{Vol}_{\mathbb{E}^3}(B(0,r)) - \mathrm{Vol}_Y(B(p,r))}{r^2\,\mathrm{Vol}_{\mathbb{E}^3}(B(0,r))} \ge 0. \tag{5}$$

This paper is part of the work towards Jorge Basilio's doctoral dissertation at the CUNY Graduate Center conducted under the advisement of Professors Józef Dodziuk and Christina Sormani. We would like to thank Jeff Jauregui, Marcus Khuri, Sajjad

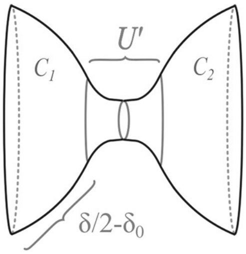

Fig. 1 The tunnel

Lakzian, Dan Lee, Raquel Perales, Conrad Plaut, and Catherine Searle for their interest in this work.

2 Background

In this section, we first briefly review Gromov–Lawson and Schoen–Yau's work. We then review Gromov–Hausdorff, Metric Measure, and Intrinsic Flat Convergence covering the key definitions as well as theorems applied in this paper to prove our example converges with respect to all three notions of convergence.

2.1 Gluing Gromov–Lawson and Schoen–Yau Tunnels

Using different techniques, Gromov–Lawson and Schoen–Yau described how to construct tunnels diffeomorphic to $\mathbb{S}^2 \times [0, 1]$ with metric tensors of positive scalar curvature that can be glued smoothly into three-dimensional spheres of constant sectional curvature [12,32]. See Fig. 1. These tunnels are the first crucial piece for our construction.

Here we need to explicitly estimate the volume and diameter of these tunnels. So the first and second authors prove the following lemma in the appendix.

Lemma 2.1 *Let* $0 < \delta/2 < 1$. *Given a complete Riemannian manifold,* M^3, *that contains two balls* $B(p_i, \delta/2) \subset M^3$, $i = 1, 2$, *with constant positive sectional curvature* $K \in (0, 1]$ *on the balls, and given any* $\epsilon > 0$, *there exists a* $\delta_0 > 0$ *sufficiently small so that we may create a new complete Riemannian manifold,* N^3, *in which we remove two balls and glue in a cylindrical region,* U, *between them:*

$$N^3 = M^3 \setminus (B(p_1, \delta/2) \cup B(p_2, \delta/2)) \sqcup U, \tag{6}$$

where $U = U(\delta_0)$ *has a metric of positive scalar curvature (See Fig. 1) with*

$$\mathrm{Diam}(U) \le h = h(\delta), \tag{7}$$

where

$$h(\delta) = O(\delta), \tag{8}$$

hence,

$$\lim_{\delta \to 0} h(\delta) = 0 \text{ uniformly for } K \in (0, 1]. \tag{9}$$

The collars $C_i = B(p_i, \delta/2) \setminus B(p_i, \delta_0)$ identified with subsets of N^3 have the original metric of constant curvature and the tunnel $U' = U \setminus (C_1 \cup C_2)$ has arbitrarily small diameter $O(\delta_0)$ and volume $O(\delta_0^3)$. Therefore with appropriate choice of δ_0, we have

$$(1-\epsilon)2\,\mathrm{Vol}(B(p,\delta/2)) \le \mathrm{Vol}(U) \le (1+\epsilon)2\,\mathrm{Vol}(B(p,\delta/2)) \tag{10}$$

and

$$(1-\epsilon)\,\mathrm{Vol}(M) \le \mathrm{Vol}(N) \le (1+\epsilon)\,\mathrm{Vol}(M). \tag{11}$$

We note that if M^3 has positive scalar curvature then so does N^3 and that, after inserting the tunnel, $\partial B(p_1, \delta/2)$ and $\partial B(p_2, \delta/2)$ are arbitrarily close together because of (9). Note that we have restricted to three dimensions here and required constant sectional curvature on the balls for simplicity. The first two authors will generalize these conditions in future work. This lemma suffices for proving all the examples in this paper.

2.2 Review GH Convergence

Gromov introduced the Gromov–Hausdorff distance in [14].

First recall that $\varphi : X \to Y$ is distance preserving iff

$$d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X. \tag{12}$$

This is referred to as a metric isometric embedding in [19] and is distinct from a Riemannian isometric embedding.

Definition 2.2 (Gromov) The Gromov–Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as

$$d_{GH}(X, Y) := \inf\, d_H^Z(\varphi(X), \psi(Y)), \tag{13}$$

where Z is a complete metric space, and $\varphi : X \to Z$ and $\psi : Y \to Z$ are distance preserving maps and where the Hausdorff distance in Z is defined as

$$d_H^Z(A, B) = \inf\{\epsilon > 0 : A \subset T_\epsilon(B) \text{ and } B \subset T_\epsilon(A)\}. \tag{14}$$

Gromov proved that this is indeed a distance on compact metric spaces: $d_{GH}(X, Y) = 0$ iff there is an isometry between X and Y. When studying metric spaces which are only precompact, one may take their metric completions before studying the Gromov–Hausdorff distance between them.

We write

$$X_j \xrightarrow{\text{GH}} X_\infty \quad \text{iff} \quad d_{GH}(X_j, X_\infty) \to 0. \tag{15}$$

Gromov proved that if $X_j \xrightarrow{\text{GH}} X_\infty$ then there is a common compact metric space Z and distance preserving maps $\varphi_j : X_j \to Z$ such that

$$d_H^Z(\varphi_j(X_j), \varphi_\infty(X_\infty)) \to 0. \tag{16}$$

We say $p_j \in X_j$ converges to $p_\infty \in X_\infty$ if there is such a set of maps such that $\varphi_j(p_j)$ converges to $\varphi_\infty(p_\infty)$ as points in Z. These limits are not uniquely defined but they are useful and every point in the limit space is a limit of such a sequence in this sense.

Theorem 2.3 (Gromov) *Suppose* $\epsilon_j \to 0$. *If a sequence of metric spaces* (X_j, d_j) *have* ϵ_j *almost isometries*

$$F_j : X_j \to X_\infty \tag{17}$$

such that

$$|d_\infty(F_j(p), F_j(q)) - d_j(p,q)| \le \epsilon_j \qquad \forall p, q \in X_j \tag{18}$$

and

$$X_\infty \subset T_{\epsilon_j}(F_j(X_j)) \tag{19}$$

then

$$X_j \xrightarrow{GH} X_\infty. \tag{20}$$

Note that $p_j \in X_j$ converges to $p_\infty \in X_\infty$ if $F_j(p_j) \to p_\infty \in X_\infty$.

Gromov's Compactness Theorem states that a sequence of manifolds with nonnegative Ricci (or Sectional) Curvature, and a uniform upper bound on diameter, has a subsequence which converges in the Gromov–Hausdorff sense to a geodesic metric space [14]. If a sequence of manifolds has nonnegative sectional curvature, then they satisfy the Toponogov Triangle Comparison Theorem. Taking the limits of the points in the triangles, one sees that the Gromov–Hausdorff limit of the sequence also satisfies the triangle comparison. Thus the limit spaces are Alexandrov spaces with nonnegative Alexandrov curvature (cf. [5]).

2.3 Review of Metric Measure Convergence

Fukaya introduced the notion of metric measure convergence of metric measure spaces (X_j, d_j, μ_j) in [10]. He assumed the sequence converged in the Gromov–Hausdorff sense as in (16) and then required that the push forwards of the measures converge as well,

$$\varphi_{j*}\mu_j \to \varphi_{\infty*}\mu_\infty \text{ weakly as measures in } Z. \tag{21}$$

Cheeger–Colding proved metric measure convergence of noncollapsing sequences of manifolds with Ricci uniformly bounded below in [8] where the measure on the limit is the Hausdorff measure. They proved metric measure convergence by constructing almost isometries and showing the Hausdorff measures of balls about converging points converge:

$$\text{If } p_j \to p_\infty \text{ then } \mathcal{H}^m(B(p_j, r)) \to \mathcal{H}^m(B(p_\infty, r)). \tag{22}$$

They also studied collapsing sequences obtaining metric measure convergence to other measures on the limit space. Cheeger and Colding applied this metric measure convergence to prove that limits of manifolds with nonnegative Ricci curvature have generalized nonnegative Ricci curvature. In particular they prove the limits satisfy the

Bishop–Gromov Volume Comparison Theorem and the Cheeger–Gromoll Splitting Theorem.

Sturm, Lott, and Villani then developed the CD(k,n) notion of generalized Ricci curvature on metric measure spaces in [21,28]. In [29], Sturm extended the study of metric measure convergence beyond the consideration of sequences of manifolds which already converge in the Gromov–Hausdorff sense, using the Wasserstein distance. This is also explored in Villani's text [34]. CD(k,n) spaces converge in this sense to CD(k,n) spaces. RCD(k,n) spaces developed by Ambrosio-Gigli-Savare are also preserved under this convergence [1]. RCD(k,n) spaces are CD(k,n) spaces which also require that the tangent cones almost everywhere are Hilbertian. There has been significant work studying both of these classes of spaces proving they satisfy many of the properties of Riemannian manifolds with lower bounds on their Ricci curvature.

2.4 Review of Integral Current Spaces

The Intrinsic Flat Distance is defined and studied in [31] by applying sophisticated ideas of Ambrosio–Kirchheim [2] extending earlier work of Federer–Fleming [9]. Limits of Riemannian manifolds under intrinsic flat convergence are integral current spaces, a notion introduced by the third author and Stefan Wenger in [31].

Recall that Federer–Fleming first defined the notion of an integral current as an extension of the notion of a submanifold of Euclidean space [9]. That is a submanifold $\psi : M^m \to \mathbb{E}^N$ can be viewed as a current $T = \psi_\#[M]$ acting on m-forms as follows:

$$T(\omega) = \psi_\#[M](\omega) = [M](\psi^*\omega) = \int_M \psi^*\omega. \tag{23}$$

If $\omega = f\, d\pi_1 \wedge \cdots \wedge d\pi_m$ then

$$T(\omega) = \psi_\#[M](\omega) = \int_M f \circ \psi \, d(\pi_1 \circ \psi) \wedge \cdots \wedge d(\pi_m \circ \psi). \tag{24}$$

They define boundaries of currents as $\partial T(\omega) = T(d\omega)$ so that then the boundary of a submanifold with boundary is exactly what it should be. They define integer rectifiable currents more generally as countable sums of images under Lipschitz maps of Borel sets. The integral currents are integer rectifiable currents whose boundaries are integer rectifiable.

Ambrosio–Kirchheim extended the notion of integral currents to arbitrary complete metric space [2]. As there are no forms on metric spaces, they use deGeorgi's tuples of Lipschitz functions,

$$T(f, \pi_1, \ldots, \pi_m) = \psi_\#[M](f, \pi_1, \ldots, \pi_m) = \int_M f \circ \psi \, d(\pi_1 \circ \psi) \wedge \cdots \wedge d(\pi_m \circ \psi). \tag{25}$$

This integral is well defined because Lipschitz functions are differentiable almost everywhere. They define boundary as follows:

$$\partial T(f, \pi_1, \ldots, \pi_m) = T(1, f, \pi_1, \ldots, \pi_m) \tag{26}$$

which matches with

$$d(f\, d\pi_1 \wedge \cdots \wedge d\pi_m) = 1\, df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m. \tag{27}$$

They also define integer rectifiable currents more generally as countable sums of images under Lipschitz maps of Borel sets. The integral currents are integer rectifiable currents whose boundaries are integer rectifiable.

The notion of an integral current space was introduced in [31].

Definition 2.4 An m-dimensional integral current space, (X, d, T), is a metric space, (X, d) with an integral current structure $T \in \mathbf{I}_m\left(\bar{X}\right)$ where $\bar{X}$ is the metric completion of X and set(T) $=$ X. Given an integral current space $M = (X, d, T)$ we will use set (M) or X_M to denote X, $d_M = d$ and $[\![M]\!] = T$. Note that set $(\partial \mathrm{T}) \subset \bar{\mathrm{X}}$. The boundary of (X, d, T) is then the integral current space:

$$\partial\left(X, d_X, T\right) := \left(\operatorname{set}(\partial \mathrm{T}), d_{\bar{\mathrm{X}}}, \partial \mathrm{T}\right). \tag{28}$$

If $\partial T = 0$ then we say (X, d, T) is an integral current without boundary.

A compact-oriented Riemannian manifold with boundary, M^m, is an integral current space, where $X = M^m$, d is the standard metric on M and T is integration over M. In this case $\mathbf{M}(M) = \mathrm{Vol}(M)$ and ∂M is the boundary manifold. When M has no boundary, $\partial M = 0$.

Ambrosio–Kirchheim defined the mass $\mathbf{M}(T)$ and the mass measure $||T||$ of a current in [2]. We apply the same notions to define a mass for an integral current space. Applying their theorems we have

$$\mathbf{M}(M) = \mathbf{M}(T) = \int_X \theta_T(x)\lambda(x)d\mathcal{H}^m(x), \tag{29}$$

where $\lambda(x)$ is the area factor and θ_T is the weight. In particular, $\lambda(x) = 1$ when the tangent cone at x is Euclidean which is true on a Riemannian manifold where the weight is also 1. This is true almost everywhere in the examples in this paper as well. The mass measure, $||T||$, is a measure on X and satisfies

$$||T||(A) = \int_A \theta_T(x)\lambda(x)d\mathcal{H}^m(x). \tag{30}$$

2.5 Review of the Intrinsic Flat Distance

The Intrinsic Flat distance was defined in work of the third author and Stefan Wenger [31] as a new distance between Riemannian manifolds based upon the Federer–Fleming flat distance [9] and the Gromov–Hausdorff distance [14].

Recall that the Federer–Fleming flat distance between m-dimensional integral currents $S, T \in \mathbf{I}_m(Z)$ is given by

$$d_F^Z(S, T) := \inf\{\mathbf{M}(U) + \mathbf{M}(V) : S - T = U + \partial V\}, \tag{31}$$

where $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$.

In [31], the third author and Wenger imitate Gromov's definition of the Gromov–Hausdorff distance (which he called the intrinsic Hausdorff distance) by replaced the Hausdorff distance by the Flat distance:

Definition 2.5 (Sormani and Wenger [31]) For $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2) \in \mathcal{M}^m$ let the intrinsic flat distance be defined:

$$d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2), \tag{32}$$

where the infimum is taken over all complete metric spaces (Z, d) and distance preserving maps $\varphi_1 : (\bar{X}_1, d_1) \to (Z, d)$ and $\varphi_2 : (\bar{X}_2, d_2) \to (Z, d)$ and the flat norm d_F^Z is taken in Z. Here $\bar{X}_i$ denotes the metric completion of X_i and d_i is the extension of d_i on $\bar{X}_i$, while $\varphi_\# T$ denotes the push forward of T.

They then prove that this distance is 0 iff the spaces are isometric with a current preserving isometry. They say

$$M_j \xrightarrow{\mathcal{F}} M_\infty \text{ iff } d_{\mathcal{F}}(M_j, M_\infty) \to 0. \tag{33}$$

And prove that this happens iff there is a complete metric space Z and distance preserving maps $\varphi_j : M_j \to Z$ such that

$$d_F^Z(\varphi_{j\#}T_j, \varphi_{\infty\#}T_\infty) \to 0. \tag{34}$$

Note that in contrast to Gromov's embedding theorem as stated in (16), the Z here is only complete and not compact.

There is a special integral current space called the zero space,

$$\mathbf{0} = (\emptyset, 0, 0). \tag{35}$$

Following the definition above, $M_j \xrightarrow{\mathcal{F}} \mathbf{0}$ iff $d_{\mathcal{F}}(M_j, \mathbf{0}) \to 0$ which implies there is a complete metric space Z and distance preserving maps $\varphi_j : M_j \to Z$ such that

$$d_F^Z(\varphi_{j\#}T_j, 0) \to 0 \tag{36}$$

Note that in this case the manifolds disappear and points have no limits.

Combining Gromov's Embedding Theorem with Ambrosio–Kirchheim's Compactness Theorem one has:

Theorem 2.6 (Sormani and Wenger [31]) *Given a sequence of m-dimensional integral current spaces $M_j = \left(X_j, d_j, T_j\right)$ such that X_j are equibounded and equicompact and with uniform upper bounds on mass and boundary mass. A subsequence converges in the Gromov–Hausdorff sense $\left(X_{j_i}, d_{j_i}\right) \xrightarrow{GH} (Y, d_Y)$ and in the intrinsic flat sense $\left(X_{j_i}, d_{j_i}, T_{j_i}\right) \xrightarrow{\mathcal{F}} (X, d, T)$, where either (X, d, T) is an m-dimensional integral current space with $X \subset Y$ or it is the $\mathbf{0}$ current space.*

Note that in [30], the third author and Wenger prove if the M_j have nonnegative Ricci curvature then in fact the intrinsic flat and Gromov–Hausdorff limits agree. Matveev and Portegies have extended this to more general lower bounds on Ricci curvature in [22]. With only lower bounds on scalar curvature the limits need not agree as seen in the Appendix of [31]. There are also sequences of manifolds with nonnegative scalar curvature that have no Gromov–Hausdorff limit but do converge in the intrinsic flat sense (cf. Ilmanen's example presented in [31] and also [18]).

In [35], Wenger proved that any sequence of Riemannian manifolds with a uniform upper bound on diameter, volume, and boundary volume has a subsequence which converges in the intrinsic flat sense to an integral current space (cf. [31]). It is possible that the limit space is just the **0** space which happens for example when the volumes of the manifolds converge to 0.

Note that when $M_j \xrightarrow{\mathcal{F}} M_\infty$ the masses are lower semicontinuous:

$$\liminf_{j\to\infty} \mathbf{M}(M_j) \geq \mathbf{M}(M_\infty), \tag{37}$$

where the mass of an integral current space is just the mass of the integral current structure. The mass is just the volume when M is a Riemannian manifold and can be computed using (29) otherwise. As there is not equality here, intrinsic flat convergence does not imply metric measure convergence.

In [23], Portegies has proven that when a sequence converges in the intrinsic flat sense and in addition $\mathbf{M}(M_j)$ is assumed to converge to $\mathbf{M}(M_\infty)$, then the spaces do converge in the metric measure sense, where the measures are taken to be the mass measures.

2.6 Useful Lemmas and Theorems Concerning Intrinsic Flat Convergence

The following lemmas, definitions, and theorems appear in work of the third author [25], although a few (labeled only as c.f. [25]) were used within proofs in older work of the third author with Wenger [30]. All are proven rigorously in [25].

Lemma 2.7 (c.f. Sormani [25]) *A ball in an integral current space, $M = (X, d, T)$, with the current restricted from the current structure of the Riemannian manifold is an integral current space itself,*

$$S\,(p, r) = (\mathrm{set}(T \llcorner B(p,r)), d, T \llcorner B\,(p,r)) \tag{38}$$

for almost every $r > 0$. Furthermore,

$$B(p,r) \subset \mathrm{set}(S(p,r)) \subset \bar{B}(p,r) \subset X. \tag{39}$$

Lemma 2.8 (c.f. Sormani [25]) *When M is a Riemannian manifold with boundary*

$$S(p,r) = \left(\bar{B}(p,r), d, T \llcorner B(p,r)\right) \tag{40}$$

is an integral current space for all $r > 0$.

Definition 2.9 (c.f. Sormani [25]) If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are a converging sequence that converge to $x_\infty \in \bar{X}_\infty$ if there exists a complete metric space Z and distance preserving maps $\varphi_i : X_i \to Z$ such that

$$\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty \text{ and } \varphi_i(x_i) \to \varphi_\infty(x_\infty). \tag{41}$$

If we say collection of points, $\{p_{1,i}, p_{2,i}, \ldots, p_{k,i}\}$, converges to a corresponding collection of points, $\{p_{1,\infty}, p_{2,\infty}, \ldots, p_{k,\infty}\}$, if $\varphi_i(p_{j,i}) \to \varphi_\infty(p_{j,\infty})$ for $j = 1, \ldots, k$.

Definition 2.10 (c.f. Sormani [25]) If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are Cauchy if there exists a complete metric space Z and distance preserving maps $\varphi_i : M_i \to Z$ such that

$$\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty \text{ and } \varphi_i(x_i) \to z_\infty \in Z. \tag{42}$$

We say the sequence is disappearing if $z_\infty \notin \varphi_\infty(X_\infty)$. We say the sequence has no limit in $\bar{X}_\infty$ if $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$.

Lemma 2.11 (c.f. Sormani [25]) *If a sequence of integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to an integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then every point x in the limit space X is the limit of points $x_i \in M_i$. In fact, there exists a sequence of maps $F_i : X \to X_i$ such that $x_i = F_i(x)$ converges to x and*

$$\lim_{i\to\infty} d_i(F_i(x), F_i(y)) = d(x, y). \tag{43}$$

Lemma 2.12 (c.f. Sormani [25]) *If $M_j \xrightarrow{\mathcal{F}} M_\infty$ and $p_j \to p_\infty \in \bar{X}_\infty$, then for almost every $r_\infty > 0$ there exists a subsequence of M_j also denoted M_j such that*

$$S(p_j, r_\infty) = \left(\bar{B}\left(p_j, r_\infty\right), d_j, T_j \llcorner B\left(p_j, r_\infty\right)\right) \tag{44}$$

are integral current spaces for $j \in \{1, 2, \ldots, \infty\}$ and we have

$$S(p_j, r_\infty) \xrightarrow{\mathcal{F}} S(p_\infty, r_\infty). \tag{45}$$

If p_j are Cauchy with no limit in $\bar{X}_\infty$ then there exists $\delta > 0$ such that for almost every $r \in (0, \delta)$ such that $S(p_j, r)$ are integral current spaces for $j \in \{1, 2, \ldots\}$ and we have

$$S(p_j, r) \xrightarrow{\mathcal{F}} 0. \tag{46}$$

If $M_j \xrightarrow{\mathcal{F}} \mathbf{0}$ then for almost every r and for all sequences p_j we have (46).

Theorem 2.13 (c.f. Sormani [25]) *Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces and*

$$M_i \xrightarrow{\mathcal{F}} M_\infty, \tag{47}$$

and suppose we have Lipschitz maps into a compact metric space Z,

$$F_i : X_i \to Z \textit{ with } \mathrm{Lip}(F_i) \le K, \tag{48}$$

then a subsequence converges to a Lipschitz map

$$F_\infty : X_\infty \to Z \textit{ with } \mathrm{Lip}(F_\infty) \le K. \tag{49}$$

More specifically, there exists distance preserving maps of the subsequence, $\varphi_i : X_i \to Z$, such that

$$d_F^Z(\varphi_{i\#} T_i, \varphi_\infty T_\infty) \to 0 \tag{50}$$

and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$ (i.e., $d_Z(\varphi_i(p_i), \varphi_\infty(p)) \to 0$), we have

$$\lim_{i \to \infty} F_i(p_i) = F_\infty(p_\infty). \tag{51}$$

Theorem 2.14 (c.f. Sormani [25]) *Suppose $M_i^m = (X_i, d_i, T_i)$ are integral current spaces which converge in the intrinsic flat sense to a nonzero integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$. Suppose there exists $r_0 > 0$ and a sequence $p_i \in M_i$ such that for almost every $r \in (0, r_0)$ we have integral current spaces, $S(p_i, r)$, for all $i \in \mathbb{N}$ and*

$$\liminf_{i \to \infty} d_{\mathcal{F}}(S(p_i, r), \mathbf{0}) = h_0 > 0. \tag{52}$$

Then there exists a subsequence, also denoted M_i, such that p_i converges to $p_\infty \in \bar{X}_\infty$.

Theorem 2.15 (c.f. Sormani [25]) *Let $M_i = (X_i, d_i, T_i)$ and $M_i' = (X_i', d_i', T_i)$ be integral current spaces with*

$$\mathbf{M}(M_i) \le V_0 \textit{ and } \mathbf{M}(\partial M_i) \le A_0 \tag{53}$$

such that

$$M_i \xrightarrow{\mathcal{F}} M_\infty \textit{ and } M_i' \xrightarrow{\mathcal{F}} M_\infty'. \tag{54}$$

Fix $\delta > 0$. Let $F_i : M_i \to M_i'$ be continuous maps which are isometries on balls of radius δ:

$$\forall x \in X_i, \;\; F_i : \bar{B}(x, \delta) \to \bar{B}(F_i(x), r) \textit{ is an isometry} \tag{55}$$

Then, when $M_\infty \neq \mathbf{0}$, we have $M'_\infty \neq \mathbf{0}$ and there is a subsequence, also denoted F_i, which converges to a (surjective) local current preserving isometry.

$$F_\infty : \bar{X}_\infty \to \bar{X}'_\infty \text{ satisfying (55).} \tag{56}$$

More specifically, there exists distance preserving maps of the subsequence $\varphi_i : X_i \to Z$, $\varphi'_i : X'_i \to Z'$, such that

$$d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0 \text{ and } d_F^{Z'}(\varphi'_{i\#}T'_i, \varphi'_\infty T'_\infty) \to 0 \tag{57}$$

and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$:

$$\lim_{i\to\infty} \varphi_i(p_i) = \varphi_\infty(p) \in Z \tag{58}$$

we have

$$\lim_{i\to\infty} \varphi'_i(F_i(p_i)) = \varphi'_\infty(F_\infty(p_\infty)) \in Z'. \tag{59}$$

When $M_\infty = \mathbf{0}$ and F_i are surjective, we have $M'_\infty = \mathbf{0}$.

3 Sewing Riemannian Manifolds with Positive Scalar Curvature

The main technique we will introduce in this paper is the construction of three-dimensional manifolds with positive scalar curvature through a process we call "sewing" which involved gluing a sequence of tunnels along a curve. We apply Lemma 2.1 which constructs Gromov–Lawson Schoen–Yau tunnels. The lemma is proven in the Appendix.

3.1 Gluing Tunnels Between Spheres

We begin by gluing tunnels between arbitrary collections of pairs of spheres as in Fig. 2.

Proposition 3.1 *Given a complete Riemannian manifold, M^3, and $A_0 \subset M^3$ a compact subset with an even number of points $p_i \in A_0$, $i = 1, \ldots, n$, with pairwise disjoint contractible balls $B(p_i, \delta)$ which have constant positive sectional curvature K, for some $\delta > 0$, define $A_\delta = T_\delta(A_0)$ and*

$$A'_\delta = A_\delta \setminus \left(\bigcup_{i=1}^{n} B(p_i, \delta/2) \right) \sqcup \bigcup_{i=1}^{n/2} U_i, \tag{60}$$

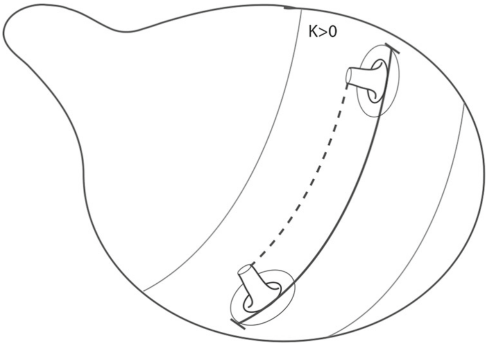

Fig. 2 Gluing two spheres with a tunnel

where U_i *are the tunnels as in Lemma 2.1 connecting* $\partial B(p_{2j+1}, \delta/2)$ *to* $\partial B(p_{2j+2}, \delta/2)$ *for* $j = 0, 1, \ldots, n/2 - 1$*. Then given any* $\epsilon > 0$*, shrinking* δ *further, if necessary, we may create a new complete Riemannian manifold,* N^3*,*

$$N^3 = (M^3 \setminus A_\delta) \sqcup A'_\delta \tag{61}$$

satisfying

$$(1 - \epsilon)\,\mathrm{Vol}(A_\delta) \le \mathrm{Vol}(A'_\delta) \le \mathrm{Vol}(A_\delta)(1 + \epsilon) \tag{62}$$

and

$$(1 - \epsilon)\,\mathrm{Vol}(M^3) \le \mathrm{Vol}(N^3) \le \mathrm{Vol}(M^3)(1 + \epsilon). \tag{63}$$

If, in addition, M^3 *has nonnegative or positive scalar curvature, then so does* N^3*. In fact,*

$$\inf_{x \in M^3} \mathrm{Scal}_x \ge \min\left\{0, \inf_{x \in N^3} \mathrm{Scal}_x\right\} \tag{64}$$

If $\partial M^3 \neq \emptyset$*, the balls avoid the boundary and* ∂M^3 *is isometric to* ∂N^3*.*

Definition 3.2 We say that we have glued the manifold to itself with a tunnel between the collection of pairs of sphere $\partial B(p_i, \delta)$ to $\partial B(p_{i+1}, \delta)$ for $i = 1$ to $n - 1$. See Fig. 2.

Proof For simplicity of notation, set $A = A_\delta$ and $A' = A'_\delta$.

By induction on n and Lemma 2.1, we see that N^3 can be given a metric of positive scalar curvature whenever M^3 has positive scalar curvature.

Using the fact that the balls are pairwise disjoint and of the same volume, and (10) from Lemma 2.1, we have the volume of A' can be estimated:

$$\begin{aligned}
\mathrm{Vol}(A') &= \mathrm{Vol}(A) - \sum_{i=1}^{n} \mathrm{Vol}(B(p_i, \delta/2)) + \sum_{i=1}^{n/2} \mathrm{Vol}(U_i) \\
&= \mathrm{Vol}(A) + \frac{n}{2} \cdot (\mathrm{Vol}(U_i) - 2\,\mathrm{Vol}(B(p_i, \delta/2))) \\
&\leq \mathrm{Vol}(A) + \frac{n}{2} \cdot (2\,\mathrm{Vol}(B(p_i, \delta/2)) \cdot \epsilon) \\
&= \mathrm{Vol}(A) + \epsilon \cdot (n\,\mathrm{Vol}(B(p_i, \delta/2))) \qquad \text{(by (10))} \\
&\leq \mathrm{Vol}(A) + \epsilon\,\mathrm{Vol}(A)
\end{aligned}$$

which yields the right-hand side of (62).

Similarly,

$$\begin{aligned}
\mathrm{Vol}(A') &= \mathrm{Vol}(A) - \sum_{i=1}^{n} \mathrm{Vol}(B(p_i, \delta/2)) + \sum_{i=1}^{n/2} \mathrm{Vol}(U_i) \\
&= \mathrm{Vol}(A) + \frac{n}{2} \cdot (\mathrm{Vol}(U_i) - 2\,\mathrm{Vol}(B(p_i, \delta/2))) \\
&\geq \mathrm{Vol}(A) + \frac{n}{2} \cdot (-2\,\mathrm{Vol}(B(p_i, \delta/2)) \cdot \epsilon) \\
&= \mathrm{Vol}(A) - \epsilon \cdot (n\,\mathrm{Vol}(B(p_i, \delta/2))) \qquad \text{(by (10))} \\
&\geq \mathrm{Vol}(A) - \epsilon\,\mathrm{Vol}(A)
\end{aligned}$$

which yields the left-hand side of (62).

To estimate the volume of N we will use the volume estimates for A'. Using (10) from Lemma 2.1 again, we have

$$\begin{aligned}
\mathrm{Vol}(N) &= \mathrm{Vol}(M) - \mathrm{Vol}(A) + \mathrm{Vol}(A') \\
&\leq \mathrm{Vol}(M) - \mathrm{Vol}(A) + (1+\epsilon)\,\mathrm{Vol}(A) \\
&= \mathrm{Vol}(M) + \epsilon\,\mathrm{Vol}(A) \qquad \text{(by (11))} \\
&\leq \mathrm{Vol}(M) + \epsilon\,\mathrm{Vol}(M),
\end{aligned}$$

which yields the right-hand side of (63).

Similarly,

$$\begin{aligned}
\mathrm{Vol}(N) &= \mathrm{Vol}(M) - \mathrm{Vol}(A) + \mathrm{Vol}(A') \\
&\geq \mathrm{Vol}(M) - \mathrm{Vol}(A) + (1-\epsilon)\,\mathrm{Vol}(A) \\
&= \mathrm{Vol}(M) - \epsilon\,\mathrm{Vol}(A) \qquad \text{(by (11))} \\
&\geq \mathrm{Vol}(M) - \epsilon\,\mathrm{Vol}(A),
\end{aligned}$$

which yields the left-hand side of (63).

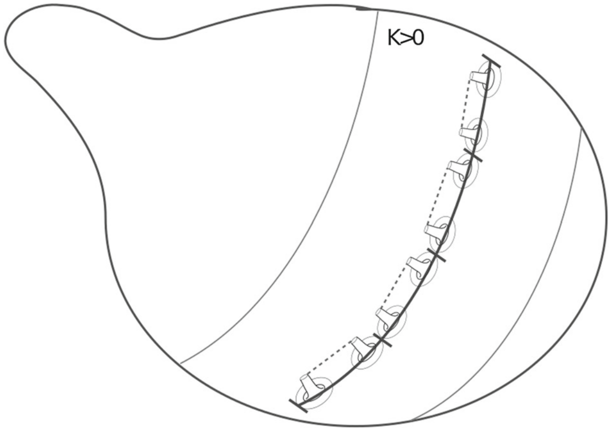

Fig. 3 Sewing a manifold through eight balls along a curve

Finally, observe that (64) follows since Lemma 2.1 shows that the tunnels U_i have positive scalar curvature. □

3.2 Sewing Along a Curve

We now describe our process we call sewing along a curve, where a sequence of balls is taken to be located along curve much like holes created when stitching a thread. We glue a sequence of tunnels to the boundaries of these balls as in Fig. 3. We say that we have sewn the manifold along the curve C through the given balls. By gluing tunnels in this precise way, we are able to shrink the diameter of the edited tubular neighborhood around the curve because travel along the curve can be conducted efficiently through the tunnels.

Proposition 3.3 *Given a complete Riemannian manifold, M^3, and $A_0 \subset M^3$ Riemannian isometric to an embedded curve, $C : [0, 1] \to \mathbb{S}^3_K$ possibly with $C(0) = C(1)$ and parametrized proportional to arclength, in a standard sphere of constant sectional curvature K, define $A_a = T_a(A_0)$ as in Proposition 3.1 and assume that A_a is Riemannian isometric to $T_a(C) \subset \mathbb{S}^3_K$. Then, given any $\epsilon > 0$ there exists n sufficiently large and $\delta = \delta(\epsilon, n, C, K) > 0$ sufficiently small as in (66) so that we can "sew along the curve" to create a new complete Riemannian manifold N^3,*

$$N^3 = (M^3 \setminus A_\delta) \sqcup A'_\delta, \tag{65}$$

exactly as in Proposition 3.1, for

$$\delta = \delta(\epsilon, n, C, K) \text{ such that } \delta < a, \ \lim_{n\to\infty} n \cdot h(\delta) = 0, \text{ and } \lim_{n\to\infty} n \cdot \delta = 0, \tag{66}$$

where h is defined in Lemma 2.1 and the disjoint balls $B(p_i, \delta)$ are to be centered at

$$p_{2j+1} = C\left(\frac{j}{n} + \frac{\delta}{L(C)}\right) \qquad p_{2j+2} = C\left(\frac{j+1}{n} - \frac{\delta}{L(C)}\right) \qquad j = 0, 1, \ldots, n-1 \tag{67}$$

and

$$A'_\delta = A_\delta \Big\backslash \left(\bigcup_{i=1}^{2n} B(p_i, \delta/2)\right) \sqcup \bigcup_{j=0}^{n-1} U_{2j+1}. \tag{68}$$

Thus, the tunnels U_{2j+1} connect $\partial B(p_{2j+1}, \delta)$ to $\partial B(p_{2j+2}, \delta)$ for $j = 0, 1, \ldots, n-1$.
Furthermore,

$$(1-\epsilon)\operatorname{Vol}(A_\delta) \le \operatorname{Vol}(A'_\delta) \le \operatorname{Vol}(A_\delta)(1+\epsilon) \tag{69}$$

and

$$(1-\epsilon)\operatorname{Vol}(M^3) \le \operatorname{Vol}(N^3) \le \operatorname{Vol}(M^3)(1+\epsilon) \tag{70}$$

and

$$\operatorname{Diam}(A'_\delta) \le H(\delta) = L(C)/n + (n+1)\,h(\delta) + (5n+2)\,\delta. \tag{71}$$

Since

$$\lim_{\delta\to 0} H(\delta) = 0 \text{ uniformly for } K \in (0, 1], \tag{72}$$

we say we have sewn the curve, A_0, arbitrarily short.

If, in addition, M^3 has nonnegative or positive scalar curvature, then so does N^3. In fact,

$$\inf_{x\in M^3} \operatorname{Scal}_x \ge \min\left\{0, \inf_{x\in N^3} \operatorname{Scal}_x\right\} \tag{73}$$

If $\partial M^3 \neq \emptyset$, the balls avoid the boundary and ∂M^3 is isometric to ∂N^3.

Proof By the fact that C is embedded, for n sufficiently large, the balls in the statement are disjoint even when $C(0) = C(1)$ so we may apply Proposition 3.1 to get (69) and (70).

For simplicity of notation, let $A = A_\delta$ and $A' = A'_\delta$.

We now verify the diameter estimate of A', (71). To do this, we define sets $C_i \subset A'$ which correspond to the sets $\partial B(p_i, \delta/2) \subset A$ which are unchanged because they are the boundaries of the edited regions:

$$C_i \cup C_{i+1} = \partial U_i, \tag{74}$$

whenever i is an odd value. Let

$$U = \bigcup_{j=0}^{n-1} U_{2j+1}. \tag{75}$$

Let x and y be arbitrary points in A'. We claim that there exists $j, k \in \{1, \ldots, 2n\}$ such that

$$d_{A'}(x, C_j) < \delta + L(C)/(2n) + h(\delta) \text{ and } d_{A'}(y, C_k) < \delta + L(C)/(2n) + h(\delta). \quad (76)$$

By symmetry we need only prove this for x. Note that in case I where

$$x \in A' \setminus U = A \Big\backslash \bigcup_{i=1}^{2n} B(p_i, \delta/2) \quad (77)$$

we can view x as a point in A. Let $\gamma_1 \subset A$ be the shortest path from x to the closest point $c_x \in C[0, 1]$ so that $L(\gamma_1) < \delta$.

If

$$\gamma_1 \cap B(p_j, \delta/2) \neq \emptyset \quad (78)$$

then

$$d_{A' \setminus U}(x, C_j) < \delta \quad (79)$$

and we have that (76) holds. Otherwise, still in Case I, if (78) fails then we have

$$d_{A' \setminus U}(x, C_j) \leq d_{A' \setminus U}(x, c_x) + d(c_x, C_j) \qquad \text{(by the triangle inequality)} \quad (80)$$

$$< \delta + \frac{L(C)}{2n}, \quad (81)$$

where the last inequality follows from $d_{A' \setminus U}(x, c_x) \leq L(\gamma_1) < \delta$ and the fact that $c_x \in C([0, 1])$ is at most $L(C)/(2n)$ away from the boundary of the nearest tunnel.

Alternatively, we have case II where $x \in U$. In this case, there exists j such that $x \in U_{2j+1}$ and so

$$d_{A'}(x, C_{2j+1}) \leq \mathrm{Diam}(U_{2j+1}) \leq h(\delta). \quad (82)$$

Thus, we have the claim in (76).

We now proceed to prove (71) by estimating $d_{A'}(x, y)$ for $x, y \in A'$. If $j = k$ in (76), then $d_{A'}(x, y) \leq 2(\delta + L(C)/(2n) + h(\delta))$ and we are done. Otherwise, by (76) and the triangle inequality, we have

$$d_{A'}(x, y) \leq d_{A'}(x, C_j) + d_{A'}(y, C_k) + \sup\{d_{A'}(z, w) \mid z \in C_j, w \in C_k\} \quad (83)$$

$$\leq 2(\delta + L(C)/(2n) + h(\delta)) + \sup\{d_{A'}(z, w) \mid z \in C_j, w \in C_k\}. \quad (84)$$

Without loss of generality, we may assume that $j < k$ and that j is odd. Thus, $C_j \subset \partial U_j$. If k is also odd then by the triangle inequality

$$\begin{aligned} \sup\{d_{A'}(z, w) \mid z \in C_j, w \in C_k\} \leq\ & \mathrm{Diam}(U_j) + \mathrm{dist}(U_j, U_{j+2}) \\ & + \mathrm{Diam}(U_{j+2}) + \cdots + \mathrm{Diam}(U_{k-2}) \\ & + \mathrm{dist}(U_{k-2}, U_k) \end{aligned} \quad (85)$$

and, when k is even,

$$\begin{aligned}\sup\{d_{A'}(z,w) \mid z\in C_j, w\in C_k\} \le \operatorname{Diam}(U_j) + \operatorname{dist}(U_j, U_{j+2})\\ + \operatorname{Diam}(U_{j+2}) + \cdots + \operatorname{Diam}(U_{k-2})\\ + \operatorname{dist}(U_{k-2}, U_{k-1}) + \operatorname{Diam}(U_{k-1}). \end{aligned} \tag{86}$$

We know that $\operatorname{Diam}(U_j) = \cdots = \operatorname{Diam}(U_k) \le h(\delta)$ from (7) of Lemma 2.1, and that the distance between any two adjacent tunnels is the same, and that there are at most n tunnels. Thus, in either case (85) or (86) we have

$$\sup\{d_{A'}(z,w) \mid z\in C_j, w\in C_k\} \le n\, h(\delta) + n\cdot \operatorname{dist}(U_j, U_{j+2}). \tag{87}$$

and by construction the distance between adjacent tunnels is

$$\operatorname{dist}(U_j, U_{j+2}) \le \operatorname{Diam}(C_{j+1}) + \operatorname{dist}(C_{j+1}, C_{j+2}) + \operatorname{Diam}(C_{j+2}) \tag{88}$$

$$\le \pi(\delta/2) + \delta + \pi(\delta/2) < 5\delta \tag{89}$$

since the balls $B(p_i, \delta/2)$ have constant sectional curvature K.

Therefore, combining (84), (87), and (89) we conclude that

$$d_{A'}(x,y) \le 2(\delta + L(C)/(2n) + h(\delta)) + n\, h(\delta) + 5n\delta \tag{90}$$

which is the desired diameter estimate (71).

We observe that by our choice of δ satisfying (66) and the fact that $h(\delta) = O(\delta)$ from Lemma 2.1 we have that (72) holds.

Finally, observe that (73) follows since Lemma 2.1 shows that the tunnels U_i have positive scalar curvature. □

4 Pulled String Spaces

The following notion of a pulled string metric space captures the idea that if a metric space is a patch of cloth and a curve in the patch is sewn with a string, then one can pull the string tight, identifying the entire curve as a single point, thus creating a new metric space. This notion was first described to the third author by Burago when they were working ideas related to [6]. See Fig. 4.

Proposition 4.1 *The notion of a metric space with a pulled string is a metric space (Y, d_Y) constructed from a metric space (X, d_X) with a curve $C : [0,1] \to X$, so that*

$$Y = X \setminus C[0,1] \sqcup \{p_0\}, \qquad p_0 = C(0), \tag{91}$$

where for $x_i \in Y$ we have

$$d_Y(x, p_0) = \min\{d_X(x, C(t)) : t\in[0,1]\} \tag{92}$$

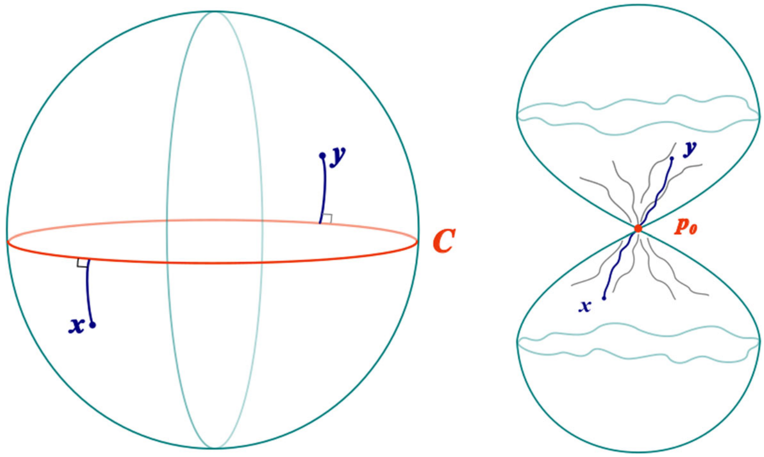

Fig. 4 A two sphere with the equator pulled to a point

and for $x_i \in X \setminus C[0,1]$ *we have*

$$d_Y(x_1, x_2) = \min\left\{ d_X(x_1, x_2), \min\{d_X(x_1, C(t_1)) + d_X(x_2, C(t_2)) : t_i \in [0,1]\} \right\}. \tag{93}$$

If (X, d, T) *is a Riemannian manifold then* $(Y, d, \psi_\# T)$ *is an integral current space whose mass measure is the Hausdorff measure on* Y *and*

$$\mathcal{H}^m_Y(Y) = \mathcal{H}^m_X(X) - \mathcal{H}^m_X(K). \tag{94}$$

If (X, d_X, T) *is an integral current space then* $(Y, d_Y, \psi_\# T)$ *is also an integral current space where* $\psi : X \to Y$ *such that* $\psi(x) = x$ *for all* $x \in X \setminus C[0,1]$ *and* $\psi(C(t)) = p_0$ *for all* $t \in [0,1]$. *So that*

$$\mathbf{M}(\psi_\# T) = \mathbf{M}(T) \tag{95}$$

We will in fact prove this proposition as a consequence of two lemmas about spaces with arbitrary compact subsets pulled to a point. Lemma 4.2 proves such a space is a metric space and Lemma 4.3 proves (94) and (95).

4.1 Pulled String Spaces Are Metric Spaces

Lemma 4.2 *Given a metric space* (X, d_X) *and a compact set* $K \subset X$, *we may define a new metric space* (Y, d_Y) *by pulling the set* K *to a point* $p_0 \in K$ *by setting*

$$Y := X \setminus K \sqcup \{p_0\}, \qquad p_0 \in K \text{ fixed}, \tag{96}$$

and, for $x \in Y$, *we have*

$$d_Y(x, p_0) = \min\{d_X(x, y) : y \in K\} \tag{97}$$

and, for $x_i \in Y \setminus \{p_0\}$, we have

$$d_Y(x_1, x_2) = \min\{d_X(x_1, x_2), \min\{d_X(x_1, y_1) + d_X(x_2, y_2) : y_i \in K\}\}. \tag{98}$$

Proof We first prove that (Y, d_Y) is a metric space. By definition, it is easy to see that d_Y is nonnegative and symmetric. To prove that d_Y satisfies the axiom of positivity, assume $x_1 = x_2$. Then either $x_i = p_0$, and $d_Y(x_1, x_2) = 0$ by Definitions (96)–(97), or $x_i \neq p_0$ and $d_X(x_1, x_2) = 0$ so by (98) we have $d_Y(x_1, x_2) \leq d_X(x_1, x_2) = 0$. Conversely, if $d_Y(x_1, x_2) = 0$ then either $d_X(x_1, x_2) = 0$ or

$$0 = \min\{d_X(x_1, y_1) + d_X(x_2, y_2) \mid y_i \in K\}. \tag{99}$$

In the first case, $x_1 = x_2$ since d_X is a metric, so assume otherwise. Then $d_X(x_1, x_2) \neq 0$ and (99) holds. Being that (99) is a sum of nonnegative numbers, it follows that $d_X(x_1, y_1) = 0$ and $d_X(x_2, y_2) = 0$ for some $y_i \in K$. Hence, $x_i = y_i$ which is impossible by the definition of Y unless $x_1 = x_2 = p_0$ which yields a contradiction. This proves that d_Y satisfies positivity.

Next, let us note that by virtue of (97) and (98), we always have

$$d_Y(x_1, x_2) \leq d_X(x_1, x_2), \qquad \forall x_1, x_2 \in Y \tag{100}$$

and

$$\text{if } d_Y(x_1, x_2) \neq d_X(x_1, x_2) \implies d_Y(x_1, x_2) = d_X(x_1, y_1) + d_X(x_2, y_2). \tag{101}$$

for some $y_i \in K$.

We now verify the triangle inequality: for any $x_1, x_2, x_3 \in Y$, we need to prove

$$d_Y(x_1, x_2) \leq d_Y(x_1, x_3) + d_Y(x_3, x_2). \tag{102}$$

It will be convenient to define $y_i \in K$ such that

$$d_X(x_i, y_i) = \min\{d_X(x_i, y) \mid y \in K\} \text{ for } i = 1, 2, 3. \tag{103}$$

Assume in Case I that $d_Y(x_1, x_2) \neq d_X(x_1, x_2)$. Then by (101) and (103),

$$d_Y(x_1, x_2) = d_X(x_1, y_1) + d_X(x_2, y_2). \tag{104}$$

We have three possibilities: (i) $d_Y(x_1, x_3) \neq d_X(x_1, x_3)$ and $d_Y(x_2, x_3) \neq d_X(x_2, x_3)$; (ii) $d_Y(x_1, x_3) = d_X(x_1, x_3)$ and $d_Y(x_2, x_3) = d_X(x_2, x_3)$; and (iii) (without loss of generality) $d_Y(x_1, x_3) \neq d_X(x_1, x_3)$ and $d_Y(x_2, x_3) = d_Y(x_2, x_3)$.

In Case I (i), we have

$$\begin{aligned} d_Y(x_1, x_2) &= d_X(x_1, y_1) + d_X(x_2, y_2) \qquad (\text{by}(104)) \\ &\leq d_X(x_1, y_1) + d_X(x_3, y_3) + d_X(x_2, y_2) + d_X(x_3, y_3) \\ &= d_Y(x_1, x_3) + d_Y(x_2, x_3). \qquad (\text{by assumption (i), (101), and(103)}) \end{aligned}$$

In Case I (ii), we have

$$\begin{aligned} d_Y(x_1, x_2) &\le d_X(x_1, x_2) \qquad \text{(by(100))} \\ &\le d_X(x_1, x_3) + d_X(x_2, x_3) \\ &= d_Y(x_1, x_3) + d_Y(x_2, x_3). \qquad \text{(by assumption (ii))} \end{aligned}$$

In Case I (iii), we have

$$\begin{aligned} d_X(x_2, y_2) &= \min\{d_X(x_2, K) \mid y \in K\} \qquad \text{(by(103))} \\ &\le d_X(x_2, y_3) \\ &\le d_X(x_2, x_3) + d_X(x_3, y_3) \qquad (105) \\ &\le d_Y(x_2, x_3) + d_X(x_3, y_3) \qquad \text{(by assumption (iii))} \qquad (106) \end{aligned}$$

so that

$$\begin{aligned} d_Y(x_1, x_2) &= d_X(x_1, y_1) + d_X(x_2, y_2) \qquad \text{(by(104))} \\ &\le d_X(x_1, y_1) + d_Y(x_2, x_3) + d_X(x_3, y_3) \qquad \text{(by(106))} \\ &= d_Y(x_1, x_3) + d_Y(x_2, x_3). \qquad \text{(by assumption (iii))} \end{aligned}$$

This proves the triangle inequality, (102), in Case I. Next, we assume, in Case II, that $d_Y(x_1, x_2) = d_X(x_1, x_2)$.

Again, we have three possibilities: (i) $d_Y(x_1, x_3) \neq d_X(x_1, x_3)$ and $d_Y(x_2, x_3) \neq d_X(x_2, x_3)$; (ii) $d_Y(x_1, x_3) = d_X(x_1, x_3)$ and $d_Y(x_2, x_3) = d_X(x_2, x_3)$; and (iii) (without loss of generality) $d_Y(x_1, x_3) \neq d_X(x_1, x_3)$ and $d_Y(x_2, x_3) = d_Y(x_2, x_3)$.

In Case II (i), we have

$$\begin{aligned} d_Y(x_1, x_2) &= d_X(x_1, x_2) \\ &\le d_X(x_1, y_1) + d_X(x_2, y_2) \qquad \text{(by (104))} \\ &\le d_X(x_1, y_1) + d_X(x_3, y_3) + d_X(x_2, y_2) + d_X(x_3, y_3) \\ &= d_Y(x_1, x_3) + d_Y(x_2, x_3). \qquad \text{(by assumption (i), (101), and (103))} \end{aligned}$$

In Case II (ii), (102) follows immediately from the triangle inequality for d_X.

Finally, in Case II (iii),

$$\begin{aligned} d_Y(x_1, x_2) &= d_X(x_1, x_2) \\ &\le d_X(x_1, y_1) + d_X(x_2, y_3) \qquad \text{(by (104))} \\ &\le d_X(x_1, y_1) + d_X(x_2, x_3) + d_X(x_3, y_3) \\ &= d_Y(x_1, x_3) + d_Y(x_2, x_3), \qquad \text{(by assumption (iii), (101), and (103))} \end{aligned}$$

which completes the proof. □

4.2 Hausdorff Measures and Masses of Pulled String Spaces

Lemma 4.3 *If (X, d_X, T) is an integral current space with a compact subset $K \subset X$ then $(Y, d_Y, \psi_\# T)$ is also an integral current space where (Y, d_Y) is defined as in Lemma 4.2 and where $\psi : X \to Y$ such that $\psi(x) = x$ for all $x \in X \setminus K$ and $\psi(q) = p_0$ for all $q \in K$. In addition*

$$\mathbf{M}(\psi_\# T) = \mathbf{M}(T) - ||T||(K) \tag{107}$$

If (X, d_X, T) is a Riemannian manifold then $(Y, d_Y, \psi_\# T)$ is an integral current space whose mass measure is the Hausdorff measure on Y and

$$\mathcal{H}^m_Y(Y) = \mathcal{H}^m_X(X) - \mathcal{H}^m_X(K). \tag{108}$$

Proof We must show that $(Y, d_Y, \psi_\# T)$ is an integral current space. We first observe that ψ as defined in the statement of the proposition is a 1-Lipschitz function: for $x, y \in X \setminus K$, there is no ambiguity so we may view them as elements of $Y \setminus \{p_0\}$ and $d_Y(\psi(x), \psi(y)) = d_Y(x, y) \le d_X(x, y)$ by definition of d_Y. Otherwise, we may assume, without loss of generality, that $x \in K$ and $y \notin K$. In this case, $d_Y(\psi(x), \psi(y)) = d_Y(p_0, \psi(y)) = d_Y(p_0, y) = \min\{d_X(z, y) : z \in K\} \le d_X(x, y)$, as $x \in K$. Thus, $\psi_\# T$ is an integral current on Y since ψ is a 1-Lipschitz function and the well-known inequality

$$\|\psi_\# T\| \le \mathrm{Lip}(\psi)^m \|T\| \tag{109}$$

implies that $\psi_\# T$ has finite mass because T does. To show that $(Y, d_Y, \psi_\# T)$ is an integral current space there remains to show that it is completely settled, or $\psi_\# T$ has positive density at p_0.

Let $f : Y \to \mathbb{R}$ be a bounded Lipschitz map and $\pi_j : Y \to \mathbb{R}$ be Lipschitz maps. Then

$$\begin{aligned}
(\psi_\# T)(f, \pi_1, \ldots, \pi_m) &= T(f \circ \psi, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&= T(f \cdot 1_{X\setminus K} + f(p_0) \cdot 1_K, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&= T(f \cdot 1_{X\setminus K}, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&\quad + f(p_0) T(1_K, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&= T(f \cdot 1_{X\setminus K}, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) + 0
\end{aligned}$$

by locality since $\pi_i \circ \psi$ are constant on $\{1_K \neq 0\}$ (see [2]) so

$$\begin{aligned}
(\psi_\# T)(f, \pi_1, \ldots, \pi_m) &= T(f \cdot 1_{X\setminus K}, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&= (T \llcorner 1_{X\setminus K})(f, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&= (T \llcorner 1_{X\setminus K})(f \circ \psi, \pi_1 \circ \psi, \ldots, \pi_m \circ \psi) \\
&\qquad \text{because } \psi(x) = x \text{ on } X \setminus K, \\
&= \psi_\#(T \llcorner 1_{X\setminus K})(f, \pi_1, \ldots, \pi_m).
\end{aligned}$$

So, using the characterization of mass from [2], (2.6) of Proposition 2.7,

$$\begin{aligned}\mathbf{M}(\psi_\# T) &= \mathbf{M}(\psi_\#(T \llcorner 1_{X\setminus K})) \\ &= \mathbf{M}(T \llcorner 1_{X\setminus K})\end{aligned}$$

because $\psi(x) = x$ on $X \setminus K$, so since $\mathbf{M}(\cdot) = \|\cdot\|$,

$$\begin{aligned}(\psi_\# T)(f, \pi_1, \dots, \pi_m) &= \|T \llcorner 1_{X\setminus K}\|(X) \\ &= \sup\left\{\sum_{j=1}^{\infty} |(T \llcorner 1_{X\setminus K})(1_{A_j}, \pi_1^j, \dots, \pi_m^j)|\right\},\end{aligned}$$

where the supremum is taken over all Borel partitions $\{A_j\}$ of X such that $X = \cup_j A_j$ and all Lipschitz functions $\pi_i^j \in \mathrm{Lip}(X)$ with $\mathrm{Lip}(\pi_i^j) \le 1$, then continuing

$$\begin{aligned}(\psi_\# T)(f, \pi_1, \dots, \pi_m) &= \sup\left\{\sum_{j=1}^{\infty} |T(1_{X\setminus K} \cdot 1_{A_j}, \pi_1^j, \dots, \pi_m^j)|\right\} \\ &= \sup\left\{\sum_{j=1}^{\infty} |T(1_{\tilde{A}_j}, \tilde{\pi}_1^j, \dots, \tilde{\pi}_m^j)|\right\},\end{aligned}$$

where the second supremum is taken over all Borel partitions $\{\tilde{A}_j\}$ of $X \setminus K$ such that $X \setminus K = \cup_j \tilde{A}_j$ and all Lipschitz functions $\tilde{\pi}_i^j \in \mathrm{Lip}(X \setminus K)$ with $\mathrm{Lip}(\tilde{\pi}_i^j) \le 1$. So, by the characterization of mass we have

$$\begin{aligned}(\psi_\# T)(f, \pi_1, \dots, \pi_m) &= \sup\left\{\sum_{j=1}^{\infty} |T(1_{\tilde{A}_j}, \tilde{\pi}_1^j, \dots, \tilde{\pi}_m^j)|\right\} \\ &= ||T||(X \setminus K) \\ &= ||T||(X) - ||T|(K) \\ &= \mathbf{M}(T) - ||T||(K),\end{aligned}$$

which proves (107).

Finally, assume that the m-dimensional integral current space (X, d_X, T) is a Riemannian manifold. We show that the mass measure of $(Y, d_Y, \psi_\# T)$ is the Hausdorff measure on (Y, d_Y).

We claim that

$$\mathcal{H}_Y^m \llcorner (Y \setminus \{p_0\}) = \mathcal{H}_X^m \llcorner (X \setminus K). \tag{110}$$

First, observe that since ψ is 1-Lipschitz,

$$\mathcal{H}_Y^m(\psi(X \setminus K)) \le (\mathrm{Lip}(\psi))^m \mathcal{H}_X^m(X \setminus K),$$

by Proposition 3.1.4 on page 37 from [3], hence

$$\mathcal{H}^m_Y(Y \setminus \{p_0\}) \leq \mathcal{H}^m_X(X \setminus K).$$

Thus, there remains to show the opposite inequality in (110).

Define sets

$$C_j = \{y \in Y \mid d_Y(y, p_0) \geq 1/j\}$$

for each $j \in \mathbb{N}$. Then the C_j are closed sets, $C_j \subset C_{j+1}$ and $Y \setminus \{p_0\} = \cup_{j\in\mathbb{N}} C_j$. So we may use Theorem 1.1.18 from [3]:

$$\mathcal{H}^m_Y(Y \setminus \{p_0\}) = \mathcal{H}^m_Y(\cup_{j\in\mathbb{N}} C_j) = \lim_{j\to\infty} \mathcal{H}^m_Y(C_j). \tag{111}$$

Consider, for each $j \in N$,

$$D_j = \psi^{-1}(C_j) = \{x \in X \mid d_X(x, K) \geq 1/j\}$$

which are closed in X, $D_j \subset D_{j+1}$, and $X \setminus K = \cup_{j\in\mathbb{N}} D_j$. Using Theorem 1.1.8 from [3] again:

$$\mathcal{H}^m_X(X \setminus K) = \mathcal{H}^m_X(\cup_{j\in\mathbb{N}} D_j) = \lim_{j\to\infty} \mathcal{H}^m_X(D_j). \tag{112}$$

Next, we claim that

$$\mathcal{H}^m_X(D_j) \leq \mathcal{H}^m_Y(C_j), \qquad j \in \mathbb{N}. \tag{113}$$

Fix j. Fix $\delta < \frac{1}{2j}$. Let $\{E_l\}_{l\in\mathbb{N}}$ be a countable cover of C_j with $\mathrm{Diam}(E_l) < \delta$, for all l. Then

$$\mathrm{dist}(E_l, p_0) > \frac{1}{2j}, \qquad l \in \mathbb{N}. \tag{114}$$

To see this, assume otherwise. Then since $\mathrm{dist}_Y(p_0, E_l) < \frac{1}{2j}$ and the definition of distance (as an infimum), there is $e \in E_l$ such that $d_Y(p_0, e) < \frac{1}{2j}$. Now, we also know that $E_l \cap C_j \neq \emptyset$. So, there is $c \in C_j \cap E_l$. So, $d_Y(e, c) \leq \mathrm{Diam}_Y(E_l) < \delta < \frac{1}{2j}$. Also, by the triangle inequality, $d_Y(p_0, c) \leq d_Y(p_0, e) + d_Y(e, c) < 1/j$. But this contradicts that $c \in C_j$ as by definition of C_j, $d_Y(p_0, c) > 1/j$.

Next, we show that

$$\mathrm{Diam}_Y(E_l) = \mathrm{Diam}_X(\psi^{-1}(E_l)), \tag{115}$$

i.e., ψ^{-1} is an isometry when restricted to $\{E_l\}$. In fact, we prove

$$d_X(\psi^{-1}(a), \psi^{-1}(b)) = d_Y(a, b), \qquad \forall\, a, b \in E_l,\ j \in \mathbb{N}.$$

Let $a, b \in E_l$. Then since $\mathrm{Diam}(E_l) < \delta < \frac{1}{2j}$ we have $d_Y(a,b) \le \mathrm{Diam}_Y(E_l) < \delta < \frac{1}{2j}$, so

$$d_Y(a,b) < \frac{1}{2j}. \tag{116}$$

By definition of the distance d_Y, since $\psi^{-1}(a) = a$ and $\psi^{-1}(b) = b$,

$$d_Y(a,b) = \min\left\{ d_X(a,b),\ \min\{\, d_X(a,k_1) + d_X(b,k_2) \mid k_i \in K\}\right\}.$$

If $d_Y(a,b) = d_X(a,b)$, we're done. If not, then there exists $k_1, k_2 \in K$ so that

$$d_Y(a,b) = d_X(a,k_1) + d_X(b,k_2). \tag{117}$$

By (114),

$$d_Y(a,p_0) \ge \frac{1}{2j} \qquad \text{and} \qquad d_Y(b,p_0) \ge \frac{1}{2j}$$

which implies

$$\mathrm{dist}_X(a,K) \ge \frac{1}{2j} \qquad \text{and} \qquad \mathrm{dist}_X(b,K) \ge \frac{1}{2j}.$$

But then

$$\begin{aligned}\frac{1}{j} &\le \mathrm{dist}_X(a,K) + \mathrm{dist}_X(b,K)\\ &\le d_X(a,k_1) + d_X(b,k_2)\\ &= d_X(a,b) \qquad \text{(by (117))}\\ &< \frac{1}{j}, \qquad \text{(by (116))}\end{aligned}$$

which is a contradiction.

Next, observe that $\{\psi^{-1}(E_l)\}_{l\in\mathbb{N}}$ is necessarily a cover of D_j so

$$\begin{aligned}\mathcal{H}^m_X(D_j) &\le \sum_{l=1}^{\infty} \omega_m \left(\frac{\mathrm{Diam}_X(\psi^{-1}(E_l))}{2}\right)^m\\ &= \sum_{l=1}^{\infty} \omega_m \left(\frac{\mathrm{Diam}_Y(E_l)}{2}\right)^m. \qquad \text{(by (115))}\end{aligned}$$

Taking the infimum over all covers of C_j with diameters less than δ gives

$$\mathcal{H}^m_X(D_j) \le \mathcal{H}^m_{Y,\delta}(C_j)$$

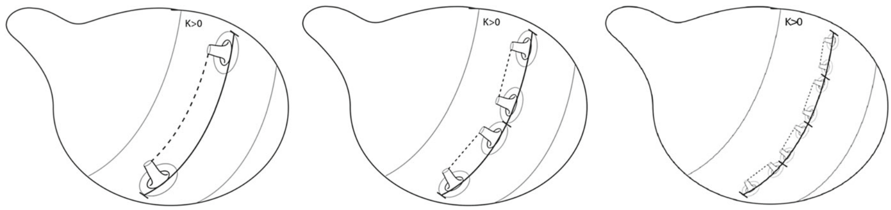

Fig. 5 A sequence of increasingly tightly sewn manifolds

then taking the limit as $\delta \to 0$ shows

$$\mathcal{H}^m_X(D_j) \le \mathcal{H}^m_Y(C_j)$$

which proves the claim (113).

To finish, we take the limit in (113) as $j \to \infty$ and use (111) and (112) to complete the proof. □

5 Sewn Manifolds Converging to Pulled Strings

In this section, we consider sequences of sewn manifolds being sewn increasingly tightly and prove they converge in the Gromov–Hausdorff and Intrinsic Flat sense to metric spaces with pulled strings.

To be more precise, we consider the following sequences of *increasingly tightly sewn* manifolds:

Definition 5.1 Given a single Riemannian manifold, M^3, with a curve, $A_0 = C([0,1]) \subset M$, with a tubular neighborhood $A = T_a(A_0)$ which is Riemannian isometric to a tubular neighborhood of a compact set $V \subset \mathbb{S}^3_K$, in a standard sphere of constant sectional curvature K, satisfying the hypothesis of Proposition 3.3. We can construct its sequence of increasingly tightly sewn manifolds, N^3_j, by applying Proposition 3.3 taking $\epsilon = \epsilon_j \to 0$, $n = n_j \to \infty$, and $\delta = \delta_j \to 0$ to create each sewn manifold, $N^3 = N^3_j$ and the edited regions $A'_\delta = A'_{\delta_j}$ which we simply denote by A'_j. This is depicted in Fig. 5. Since these sequences N^3_j are created using Proposition 3.3, they have positive scalar curvature whenever M^3 has positive scalar curvature, and $\partial N^3_j = \partial M^3$ whenever M^3 has a nonempty boundary.

In this section, we prove Lemmas 5.5, 5.6, and 5.7 which immediately imply the following theorem:

Theorem 5.2 *The sequence N^3_j as in Definition 5.1 converges in the Gromov–Hausdorff sense*

$$N^3_j \xrightarrow{GH} N_\infty, \tag{118}$$

the metric measure sense

$$N^3_j \xrightarrow{mGH} N_\infty, \tag{119}$$

and the intrinsic flat sense

$$N_j^3 \xrightarrow{\mathcal{F}} N_\infty, \tag{120}$$

where N_∞ is the metric space created by pulling the string, $A_0 = C([0,1]) \subset M$, to a point as in Proposition 4.1.

In fact, our lemmas concern more general sequences of manifolds which are constructed from a given manifold M and *scrunch* a given compact set $K \subset M$ down to a point as follows:

Definition 5.3 Given a single Riemannian manifold, M^3, with a compact set, $A_0 \subset M$. A sequence of manifolds,

$$N_j^3 = (M^3 \setminus A_{\delta_j}) \sqcup A'_{\delta_j} \tag{121}$$

is said to scrunch A_0 down to a point if $A_\delta = T_\delta(A_0)$ and A'_δ satisfies:

$$(1-\epsilon)\,\mathrm{Vol}(A_\delta) \le \mathrm{Vol}(A'_\delta) \le \mathrm{Vol}(A_\delta)(1+\epsilon) \tag{122}$$

and

$$(1-\epsilon)\,\mathrm{Vol}(M^3) \le \mathrm{Vol}(N^3) \le \mathrm{Vol}(M^3)(1+\epsilon) \tag{123}$$

and

$$\mathrm{Diam}(A'_\delta) \le H \tag{124}$$

where $\epsilon = \epsilon_j \to 0$ and where $H = H_j \to 0$ and $2\delta_j < H_j$.

Note that by Proposition 3.3, a sequence of increasingly tightly sewn manifolds sewn along a curve $C([0,1])$ as in Definition 5.1 is a sequence of manifolds which scrunches $A_0 = C([0,1])$ down to a point as in Definition 5.3. So we will prove lemmas about sequences of manifolds which scrunch a compact set and then apply them to prove Theorem 5.2 in the final subsection of this section.

5.1 Constructing Surjective Maps to the Limit Spaces

Before we prove convergence of the scrunched sequence of manifolds to the pulled thread space, we construct surjective maps from the sequence to the proposed limit space.

Lemma 5.4 *Given M^3 a compact Riemannian manifold (possibly with boundary) and a smooth embedded compact zero to three-dimensional submanifold $A_0 \subset M^3$ (possibly with boundary), and N_j as in Definition 5.3. Then for j sufficiently large there exist surjective Lipschitz maps*

$$F_j : N_j^3 \to N_\infty \text{ with } \mathrm{Lip}(F_j) \le 4, \tag{125}$$

where N_∞ is the metric space created by taking M^3 and pulling A_0 to a point p_0 as in Lemmas 4.2–4.3.

Note that when A_0 is the image of a curve, N_∞, is a pulled thread space as in Proposition 4.1.

Proof First observe that by the construction in Definition 5.3 there are maps

$$P_j : M^3 \to N_\infty \tag{126}$$

which are Riemannian isometries on regions which avoid A_0 and map A_0 to p_0. These define Riemannian isometries

$$P_j : N_j^3 \setminus A'_j \tilde{=} M^3 \setminus T_{\delta_j}(A_0) \to N_\infty^3 \setminus T_{\delta_j}(p_0). \tag{127}$$

In addition, sufficiently small balls lying in these regions are isometric to convex balls in M^3.

Observe also that for $\delta > 0$ sufficiently small, the exponential map:

$$exp : \{(p, v) : \;\; p \in A_0, \;\; v \in V_p \; |v| < 2\delta\} \to T_{2\delta}(A_0) \tag{128}$$

is invertible where

$$V_p = \{v \in T_p M : \;\; d_M(exp_p(tv), p) = d_M(exp_p(tv), A_0)\}. \tag{129}$$

Taking $\delta = \delta_{A_0} > 0$ even smaller (depending on the submanifold A_0), we can guarantee that $\forall v_i \in V_p, |v_i| < 2\delta_{A_0}, t_i \in (0, 1)$ we have

$$d_M(exp_{p_1}(t_1 v_1), exp_{p_2}(t_2 v_2)) \le 2 d_M(exp_{p_1}(v_1), exp_{p_2}(v_2)) + 2|t_1 - t_2|. \tag{130}$$

This is not true unless A_0 is a smooth embedded compact submanifold with either no boundary or a smooth boundary.

Define $F_j : N_j^3 \to N_\infty$ as follows:

$$F_j(x) = P_j(x) \qquad \forall x \in N_j^3 \setminus T_{\delta_j}(A'_j) \tag{131}$$

and

$$F_j(x) = p_0 \qquad \forall x \in A'_j. \tag{132}$$

Between these two regions, we take

$$F_j(x) = f_j(P_j(x)) \qquad \forall x \in T_{\delta_j}(A'_j) \setminus A'_j, \tag{133}$$

where $f_j : N_\infty \to N_\infty$ is a surjective map:

$$f_j : Ann_{p_0}(\delta_j, 2\delta_j) \to B_{2\delta_j}(p_0) \setminus \{p_0\} \tag{134}$$

which takes a point q to

$$f_j(q) = \gamma_q\left((d_{N_\infty}(p_0, q) - \delta_j)/\delta_j\right), \tag{135}$$

where γ_q is the unique minimal geodesic from $\gamma_q(0) = p_0$ to $\gamma_q(1) = q$. Here we are assuming $\delta_j < \delta_{A_0}$. So

$$d_{N_\infty}(p_0, P_j(x)) = d_{M^3}(A_0, x) \tag{136}$$

and

$$\gamma_q(t) = P_j(exp_{q'}(tv')) \text{ where } P_j(exp_{q'}(v')) = q. \tag{137}$$

In particular for $x \in \partial T_{\delta_j}(A'_j)$,

$$f_j(P_j(x)) = \gamma_{P_j(x)}((2\delta_j - \delta_j)/\delta_j) = \gamma_{P_j(x)}(1) = P_j(x) \tag{138}$$

and for $x \in \partial A'_j$,

$$f_j(P_j(x)) = \gamma_{P_j(x)}((\delta_j - \delta_j)/\delta_j) = \gamma_{P_j(x)}(0) = p_0 \tag{139}$$

so that F_j is continuous.

We claim

$$\mathrm{Lip}(F_j) = 0 \text{ on } A'_j \tag{140}$$

$$\mathrm{Lip}(F_j) \le 4 \text{ on } T_{\delta_j}(A'_j) \setminus A'_j \tag{141}$$

$$\mathrm{Lip}(F_j) = 1 \text{ on } N_j \setminus T_{\delta_j}(A'_j). \tag{142}$$

Only the middle part is difficult. By the definition of d_{N_∞}, we have the following two possibilities

$$\text{Case I:} \quad d_{N_\infty}(q_1, q_2) = d_M(P_j^{-1}(q_1), P_j^{-1}(q_2)) \tag{143}$$

$$\text{Case II:} \quad d_{N_\infty}(q_1, q_2) = d_M(P_j^{-1}(q_1), A_0) + d_M(P_j^{-1}(q_2), A_0). \tag{144}$$

In Case II, we see that the minimal geodesic from q_1 to q_2 passes through p_0. Since $f_j(q_1)$ and $f_j(q_2)$ lie on this geodesic, we have

$$d_{N_\infty}(f_j(q_1), f_j(q_2)) \le d_{N_\infty}(q_1, q_2). \tag{145}$$

In Case I, we apply (130) with

$$t_i = (d_M(P_j^{-1}(q_i), A_0) - \delta_j)/\delta_j \tag{146}$$

because $t_i \in (0, 1)$ due to (141) so that by the reverse triangle inequality

$$|t_1 - t_2| = |d_M(P_j^{-1}(q_1), A_0) - d_M(P_j^{-1}(q_2), A_0)|/\delta_j \tag{147}$$

$$\le d_M(P_j^{-1}(q_1), q_2)/\delta_j \tag{148}$$

$$\le d_{N_\infty}(q_1, q_2) \tag{149}$$

to see that

$$
\begin{aligned}
d_{N_\infty}(f_j(q_1), f_j(q_2)) &\le d_M(P_j^{-1}(f_j(q_1)), P_j^{-1}(f_j(q_2))) && (150)\\
&\le 2d_M(P_j^{-1}(q_1), P_j^{-1}(q_2)) + 2|t_1 - t_2| \text{ by (130)}, && (151)\\
&\le 2d_{N_\infty}(q_1, q_2) + 2|t_1 - t_2| \text{ by Case I hypothesis}, && (152)\\
&\le 4d_{N_\infty}(q_1, q_2). && (153)
\end{aligned}
$$

This gives our claim.

We claim $\mathrm{Lip}(F_j) \le 4$ everywhere. Given $x_1, x_2 \in N_j^3$, we have a minimizing geodesic $\eta : [0, 1] \to N_j$ such that $\eta(0) = x_1$ and $\eta(1) = x_2$. Then

$$d_{N_\infty}(F_j(x_1), F_j(x_2)) \le L(F_j \circ \eta). \tag{154}$$

Since $|(F_j \circ \eta)'(t)| \le 2|\eta'(t)|$ by our localized Lipschitz estimates and because the function F_j is continuous, we are done. □

5.2 Constructing Almost Isometries

See Sect. 2.2 for a review of the Gromov–Hausdorff distance.

Lemma 5.5 *Given N_j^3 as in Definition 5.3, the maps $F_j : N_j^3 \to N_\infty$ defined in (131)–(133) in the proof of Lemma 5.4 are H_j-almost isometries with $\lim_{j\to\infty} H_j = 0$. Thus*

$$N_j \xrightarrow{GH} N_\infty. \tag{155}$$

Proof Before we begin the proof recall that

$$\mathrm{Diam}(A'_j) \le H_j \to 0 \tag{156}$$

in (124) of Definition 5.3.

By Theorem 2.3 of Gromov, to prove (155) it suffices to show that F_j are H_j-almost isometries. To see this, examine $x, y \in N_j$ and join them by a minimizing curve $\sigma : [0, 1] \to N_j$.

If $\sigma[0, 1] \subset N_j \setminus A'_j$, then by (131) we have

$$L(\sigma) = L(F_j \circ \sigma) \tag{157}$$

and so

$$d_{N_j}(x, y) \ge d_{N_\infty}(F_j(x), F_j(y)). \tag{158}$$

Otherwise we have

$$d_{N_j}(x,y) \geq d_{N_j}(x,A'_j) + d_{N_j}(y,A'_j) \qquad T_{\delta_j}(A'_j) \text{ to } A'_j \tag{159}$$
$$= d_{N_\infty}(F_j(x), B_{\delta_j}(p_0)) + d_{N_\infty}(F_j(y), B_{\delta_j}(p_0)) \tag{160}$$
$$= d_{N_\infty}(F_j(x), p_0) - \delta_j + d_{N_\infty}(F_j(y), p_0) - \delta_j \tag{161}$$
$$\geq d_{N_\infty}(F_j(x), F_j(y)) - 2\delta_j. \tag{162}$$

Next we join $F_j(x)$ to $F_j(y)$ by a minimizing curve γ. If $\gamma[0,1] \subset N_\infty \setminus B_{\delta_j}(p_0)$ then there is a curve η such that $\gamma = F_j \circ \eta$ with $\eta[0,1] \subset N_j \setminus A'_j$ and so by (131)

$$d_{N_j}(x,y) \leq L(\eta) = L(\gamma) = d_{N_\infty}(F_j(x), F_j(y)). \tag{163}$$

Otherwise we have

$$d_{N_j}(x,y) \leq d_{N_j}(x,A'_j) + \text{Diam}(A'_j) + d_{N_j}(y,A'_j) \tag{164}$$
$$\leq d_{N_j}(x,A'_j) + H_j + d_{N_j}(y,A'_j) \tag{165}$$
$$= d_{N_\infty}(F_j(x), B_{\delta_j}(p_0)) + d_{N_\infty}(F_j(y), B_{\delta_j}(p_0)) + H_j \tag{166}$$
$$\leq L(\gamma) + H_j = d_{N_\infty}(F_j(x), F_j(y)) + H_j. \tag{167}$$

Hence, F_j is an H_j isometry since $2\delta_j < H_j$. □

5.3 Metric Measure Convergence

Recall metric measure convergence as reviewed in Sect. 2.3.

Lemma 5.6 *Given $N_j^3 \to N_\infty$ as in Lemma 5.4 endowed with the Hausdorff measures, then we have metric measure convergence if A_0 has $\mathcal{H}^3$-measure 0.*

Proof Recall the maps $F_j : N_j^3 \to N_\infty$ defined in (131)–(133) in the proof of Lemma 5.4. We need only show that for almost every $p \in N_\infty$ and for almost every $r < r_p$ sufficiently small we have

$$\mathcal{H}^3(B(p,r)) = \lim_{j\to\infty} \mathcal{H}^3(B(p_j,r)), \tag{168}$$

where $F_j(p_j) = p$ and that for any sequence $p_{0j} \to p_0$ we have r_0 sufficiently small that for all $r < r_0$

$$\mathcal{H}^3(B(p_0,r)) = \lim_{j\to\infty} \mathcal{H}^3(B(p_{0j},r)). \tag{169}$$

In fact, take any $p \neq p_0$ in N_∞ and choose

$$r < r_p < d_{N^3_\infty}(p,p_0)/2. \tag{170}$$

Then for j large enough that $\delta_j < r_p$ we have

$$B(p,r) \cap B(p_0,\delta_j) = \emptyset. \tag{171}$$

Thus

$$B(p_j, r) \cap A'_j = \emptyset. \tag{172}$$

Thus by (131), F_j is an isometry from $B(p_j, r) \subset N_j^3$ onto $B(p, r) \subset N_\infty$ and so we have

$$\mathcal{H}^3(B(p, r)) = \mathcal{H}^3(B(p_j, r)) \qquad \forall r < r_p. \tag{173}$$

Next we examine p_0. Observe that by (108)

$$\mathcal{H}^3_{N_\infty}(B(p_0, r)) = \mathcal{H}^3_M(T_r(A_0)) - \mathcal{H}^3_M(A_0) = \mathrm{Vol}_M(T_r(A_0) \setminus A_0). \tag{174}$$

For any $p_{0,j} \to p_0$, we have by (125)

$$r_j = d_{N_j}(p_{0,j}, A'_j) \le 4d_{N_\infty}(F_j(p_{0,j}), p_0) \to 0 \tag{175}$$

Thus

$$B(p_{0,j}, r) \subset T_{r+r_j}(A'_j). \tag{176}$$

So

$$\begin{aligned} \mathrm{Vol}_{N_j}(B(p_{0,j}, r)) &\le \mathrm{Vol}_{N_j}(T_{r+r_j}(A'_j)) & (177)\\ &\le \mathrm{Vol}_{N_j}(T_{r+r_j}(A'_j) \setminus A'_j) + \mathrm{Vol}_{N_j}(A'_j) & (178)\\ &= \mathrm{Vol}_M\left(T_{r+r_j+\delta_j}(A_0) \setminus T_{\delta_j}(A_0)\right) + \mathrm{Vol}_{N_j}(A'_j). & (179) \end{aligned}$$

Thus

$$\begin{aligned} \limsup_{j\to\infty} \mathrm{Vol}_{N_j}(B(p_{0,j}, r)) &\le \mathrm{Vol}_M\left(T_r(A_0) \setminus A_0\right) + \limsup_{j\to\infty} \mathrm{Vol}_{N_j}(A'_j) & (180)\\ &= \mathcal{H}^3(B(p_0, r)) & (181) \end{aligned}$$

since we claim that

$$\lim_{j\to\infty} \mathrm{Vol}_{N_j}(A'_j) = 0. \tag{182}$$

This follows because $\epsilon_j \to 0$ and (122) implies

$$(1 - \epsilon_j)\,\mathrm{Vol}_M(A_{\delta_j}) \le \mathrm{Vol}_{N_j}(A'_j) \le (1 + \epsilon_j)\,\mathrm{Vol}_M(A_{\delta_j}). \tag{183}$$

The assumption that $\mathcal{H}^3(A_0) = 0$ then implies (182) after taking the limit.

Similarly, we have for j sufficiently large

$$T_{r-H_j-r_j}(A'_j) \subset B(p_{0,j}, r). \tag{184}$$

So

$$\begin{aligned} \mathrm{Vol}_{N_j}(B(p_{0,j}, r)) &\ge \mathrm{Vol}_{N_j}(T_{r-H_j-r_j}(A'_j)) & (185)\\ &= \mathrm{Vol}_{N_j}(T_{r-H_j-r_j}(A'_j) \setminus A'_j) + \mathrm{Vol}_{N_j}(A'_j) & (186)\\ &= \mathrm{Vol}_M\left(T_{r-H_j-r_j+\delta_j}(A_0) \setminus T_{\delta_j}(A_0)\right) + \mathrm{Vol}_{N_j}(A'_j). & (187) \end{aligned}$$

Thus

$$\liminf_{j\to\infty} \mathrm{Vol}_{N_j}(B(p_{0,j}, r)) \geq \mathrm{Vol}_M\left(T_r(A_0)\setminus A_0\right) + \liminf_{j\to\infty} \mathrm{Vol}_{N_j}(A'_j) \tag{188}$$

$$= \mathcal{H}^3(B(p_0, r)), \quad \text{by (182)} \tag{189}$$

which completes the proof. □

5.4 Intrinsic Flat Convergence

For a review of intrinsic flat convergence see Sect. 2.5.

Lemma 5.7 *Let $N_j^3 \xrightarrow{GH} N_\infty$ be exactly as in Lemmas 5.4 and 5.5 where we assume M is compact and we have a compact set, $A_0 \subset M \setminus \partial M$. Then there exists an integral current space N such that $\bar{N}$ is isometric to N_∞ and*

$$N_j \xrightarrow{\mathcal{F}} N. \tag{190}$$

and when A_0 has Hausdorff measure 0

$$\mathbf{M}(N_j) \to \mathbf{M}(N) = \mathcal{H}^3(N). \tag{191}$$

When $A_0 = C([0, 1])$ then $N = N_\infty$.

Proof By (123), we have uniformly bounded volume

$$\mathrm{Vol}(N_j^3) \leq 2\,\mathrm{Vol}(M^3). \tag{192}$$

Since $\partial N_j^3 = \partial M^3$, we have uniformly bounded boundary volume

$$\mathrm{Vol}(\partial N_j^3) = \mathrm{Vol}(\partial M^3). \tag{193}$$

Combining this with Lemma 5.5 and Theorem 2.6, there exists an integral current space N possibly $N = \mathbf{0}$ such that a subsequence

$$N_j \xrightarrow{\mathcal{F}} N. \tag{194}$$

We claim that $N \neq \mathbf{0}$. If not, then by the final line in Lemma 2.12, for any sequence $p_j \in N_j$ and almost every r, $S(p_j, r) \xrightarrow{\mathcal{F}} \mathbf{0}$. However, taking p_j and r such that

$$B(p_j, r) \subset N_j^3 \setminus A'_j \tag{195}$$

we know there is some $p \in M^3$ with $B(p, r) \subset N_\infty \setminus \{p_0\}$ that $d_{\mathcal{F}}(S(p_j, r), S(p, r)) = 0$ for $p \in M^3$, so $S(p_j, r) \xrightarrow{\mathcal{F}} S(p, r) \neq \mathbf{0}$ which is a contradiction.

By Theorem 2.13, we know that after possibly taking a subsequence we obtain a limit map

$$F_\infty : N \to N_\infty. \tag{196}$$

We claim that F_∞ is distance preserving. Let $p, q \in N$. By Theorem 2.11, we have $p_j, q_j \in N_j$ converging to p, q in the sense of Definition 2.9, i.e.,

$$d_{N_j}(p_j, q_j) \to d_N(p, q). \tag{197}$$

Since the F_j are ϵ_j-almost isometries and $\epsilon_j \to 0$, we have

$$d_{N_\infty}(F_j(p_j), F_j(q_j)) \to d_N(p, q). \tag{198}$$

By the definition of F_∞ we have $F_j(p_j) \to F_\infty(p)$ and $F_j(q_j) \to F_\infty(q)$. Thus

$$d_{N_\infty}(F_\infty(p), F_\infty(q)) = d_N(p, q). \tag{199}$$

We claim that F_∞ maps onto at least $N_\infty \setminus \{p_0\}$. Let $x \in N_\infty \setminus \{p_0\}$. Since F_j are surjective, there exists $x_j \in N_j$ such that $F_j(x_j) = x$. Since $x \neq p_0$, we may define

$$r = \min\{d_{N_\infty}(x, p_0)/3, \mathrm{ConvexRad}_M(x)\}, \tag{200}$$

where $\mathrm{ConvexRad}_M(x)$ is the convexity radius about x viewed as a point in M. Then there exists j sufficiently large such that $\delta_j < r$ so that

$$B(x_j, r) \subset N_j \setminus T_{\delta_j}(A'_j). \tag{201}$$

Furthermore, these balls are isometric to the convex ball $B(x, r) \subset M^3$.

So

$$d_{\mathcal{F}}(S(x_j, r), \mathbf{0}) = d_{\mathcal{F}}(S(x, r), \mathbf{0}) > 0. \tag{202}$$

Thus by Theorem 2.14 with $h_0 = d_{\mathcal{F}}(S(x, r), \mathbf{0})$, and $N_j \xrightarrow{\mathcal{F}} N$, a subsequence of the x_j converges to $x_\infty \in N$. By the definition of F_∞, we have $F_j(x_j) \to F_\infty(x_\infty) \in N_\infty$. But since $F_j(x_j) = x$ it follows that $F_\infty(x_\infty) = x$, hence F_∞ maps onto $N_\infty \setminus p_0$.

Taking the metric completions of N and $N_\infty \setminus \{p_0\}$, we have an isometry

$$F_\infty : \bar{N} \to N_\infty. \tag{203}$$

Since N_j are Riemannian manifolds,

$$\mathbf{M}([\![N_j]\!]) = \mathrm{Vol}(N_j) = \mathcal{H}^3(N_j). \tag{204}$$

By the lower semicontinuity of mass and the metric measure convergence of N_j to N we know that

$$\mathbf{M}([\![N_\infty]\!]) \le \liminf_{j\to\infty} \mathbf{M}([\![N_j]\!]) = \mathcal{H}^3(N). \tag{205}$$

On the other hand by (29)

$$\mathbf{M}([\![N_\infty]\!]) \geq \mathcal{H}^3(N) \tag{206}$$

because almost every tangent cone is Euclidean and it has integer weight everywhere. Thus we have (191). In fact, equality in these inequalities implies that N has weight one everywhere.

Recall that the set of an integral current space only includes points of positive density. Since

$$\liminf_{r\to 0} \frac{\mathrm{Vol}_{N_\infty}(B(p_0, r))}{r^3} = \liminf_{r\to 0} \frac{\mathrm{Vol}_M(T_r(A_0) \setminus A_0)}{r^3} \tag{207}$$

Thus N is isometric to N_∞ when this liminf is positive and N is isometric to $N_\infty \setminus \{p_0\}$ when this liminf is 0. When $A_0 = C([0, 1])$ is a curve in a 3-dimensional Riemannian manifold we have

$$\liminf_{r\to 0} \frac{\mathrm{Vol}_M(T_r(A_0) \setminus A_0)}{r^3} = \liminf_{r\to 0} \frac{\pi r^2 L(C)}{r^3} = +\infty > 0. \tag{208}$$

Thus N is isometric to N_∞.

Thus N does not depend on the subsequence in (194) and in fact the original sequence (given a consistent orientation) converges in the intrinsic flat sense to N. □

5.5 The Proof of Theorem 5.2

Proof In Proposition 3.3, we show that given any $\epsilon_j \to 0$ we can find $n_j \to \infty$ and $\delta_j \to 0$ so fast that $\delta_j n_j \to 0$ and we have $h(\delta_j)n_j \to 0$ as well such that the sewn manifolds:

$$N_j^3 = (M^3 \setminus A_{\delta_j}) \sqcup A'_{\delta_j}, \tag{209}$$

satisfy:

$$(1-\epsilon)\,\mathrm{Vol}(A_\delta) \leq \mathrm{Vol}(A'_\delta) \leq \mathrm{Vol}(A_\delta)(1+\epsilon) \tag{210}$$

and

$$(1-\epsilon)\,\mathrm{Vol}(M^3) \leq \mathrm{Vol}(N^3) \leq \mathrm{Vol}(M^3)(1+\epsilon) \tag{211}$$

and

$$\mathrm{Diam}(A'_\delta) \leq H(\delta) = L(C)/n + (n+1)\,h(\delta) + (5n+2)\,\delta, \tag{212}$$

where

$$\lim_{\delta\to 0} H(\delta) = 0 \text{ uniformly for } K \in (0, 1]. \tag{213}$$

Thus we have a sequence N_j which is scrunching a set $A_0 = C([0, 1])$ to a point as in Definition 5.3.

Lemma 5.5 implies that

$$N_j \xrightarrow{\mathrm{GH}} N_\infty, \tag{214}$$

where N_∞ is the pulled string space. Lemma 5.6 implies we have metric measure to N_∞ convergence because $A_0 = C([0, 1])$ has $\mathcal{H}^3$-measure 0.

Lemma 5.7 implies that

$$N_j \xrightarrow{\mathcal{F}} N_\infty \tag{215}$$

and

$$\mathbf{M}(N_j) \to \mathbf{M}(N_\infty) = \mathcal{H}^3(N), \tag{216}$$

completing the proof of Theorem 5.2. □

6 Sewing a Sphere to Obtain our Limit Space

Here we construct the specific example of a sequence of manifolds with positive scalar curvature that converges to a limit space which fails to have generalized nonnegative scalar curvature as discussed in the introduction. More specifically:

Example 6.1 We define a sequence N_j^3 of manifolds with positive scalar curvature constructed from the standard $\mathbb{S}^3$ sewn along a closed geodesic $C : [0, 1] \to \mathbb{S}^3$ with $\delta = \delta_j \to 0$ as in Proposition 3.3. Then by Theorem 5.2 we have

$$N_j^3 \xrightarrow{\text{mGH}} N_\infty \text{ and } N_j^3 \xrightarrow{\mathcal{F}} N_\infty, \tag{217}$$

where N_∞ is the metric space created by taking the standard sphere and pulling the geodesic to a point as in Proposition 4.1. By Lemma 6.3 below we see that at the pulled point $p_0 \in N_\infty$, we have (3). Thus we have produces a sequence of three-dimensional manifolds with positive scalar curvature converging to a limit space which fails to satisfy generalized scalar curvature defined using limits of volumes of balls as in (1).

Remark 6.2 Note that with $\delta_j \to 0$, the neck in the center of the tunnels has a rotationally symmetric minimal surface whose area is $\le 4\pi\delta_j^2$ which converges to 0. So this sequence, and in fact any sewn sequence created as in Definition 5.1, has $\mathrm{MinA}(N_j) \to 0$.

Lemma 6.3 *At the pulled point $p_0 \in N_\infty$ of Example 6.1 we have*

$$\lim_{r\to 0}\left(\frac{\mathrm{Vol}_{\mathbb{E}^3}(B(0,r)) - \mathrm{Vol}_{N_\infty}(B(p_0,r))}{r^2\,\mathrm{Vol}_{\mathbb{E}^3}(B(0,r))}\right) = -\infty. \tag{218}$$

Proof First, observe that

$$\mathrm{Vol}_{N_\infty}(B(p_0,r)) = \mathcal{H}^3_{N_\infty}(B(p_0,r)) \tag{219}$$

$$= \mathcal{H}^3_{N_\infty}(B(p_0,r) \setminus \{p_0\}) \tag{220}$$

$$= \mathcal{H}^3_{\mathbb{S}^3}\left(T_r(C([0,1]))\right). \tag{221}$$

Since $C([0, 1])$ is a closed geodesic of length 2π in a three-dimensional sphere, we have

$$\lim_{r\to 0} \frac{\mathcal{H}^3_{\mathbb{S}^3}\left(T_r(C([0,1]))\right)}{2\pi(\pi r^2)} = 1. \tag{222}$$

Thus

$$\lim_{r\to 0}\frac{\mathrm{Vol}_{\mathbb{E}^3}(B(0,r))-\mathrm{Vol}_{N_\infty}(B(p_0,r))}{r^2\,\mathrm{Vol}_{\mathbb{E}^3}(B(0,r))}=\lim_{r\to 0}\frac{(4/3)\pi r^3-2\pi(\pi r^2)}{(4/3)\pi r^5}=-\infty \tag{223}$$

as claimed. □

Acknowledgements J. Basilio was partially supported as a doctoral student by NSF DMS 1006059. C. Sormani was partially supported by NSF DMS 1006059.

Appendix: Short Tunnels with Positive Scalar Curvature by Jorge Basilio and Józef Dodziuk

There is a deep connection between the geometry of Riemannian manifolds M^n with positive scalar curvature and surgery theory. The subject began with the surprising discovery by Gromov and Lawson [12] (for $n \geq 3$) and Schoen and Yau [32] that a manifold obtained via a surgery of codimension 3 from a manifold M^n with a metric of positive scalar curvature may also be given a metric with positive scalar curvature. The key to the tunnel construction of [12] is defining a curve γ which begins along the vertical axis then bends upwards as it moves to the right and ends with a horizontal line segment, cf. Fig. 6 below. The tunnel then is the surface of revolution determined by γ. We note that the "bending argument" has attracted some attention (See [24]).

As the goals of the surgery theory were topological in nature, Gromov and Lawson did not estimate with diameters or volumes of these tunnels. Indeed, the tunnels they constructed may be thin but long (See [11]). To build sewn manifolds, we need tunnels with diameters shrinking to zero as the size of the original balls decreases to zero (see (7), (8) (9)). Therefore, we prove Lemma 2.1 to obtain a refinement of the Gromov and Lawson construction showing the existence of tiny (in sense of (10)) and arbitrarily short tunnels with a metric of positive scalar curvature.

Proof of Lemma 2.1 To aid the reader, we provide a summary of our proof and introduce additional notation.

Outline of Proof of Lemma 2.1

To aid the reader, we provide a summary of our proof and introduce additional notation.

Step 1: Setup and notation

Let $\epsilon > 0$ be given. We shall specify $0 < \delta_0 < \delta/2$ below.

Given that $B_1 = B(p_1, \delta/2) \subset M^3$ has constant sectional curvature $K > 0$, we may choose coordinates so that it is realized as a hypersurface of revolution. This is also true for $B(p_1, \delta_0) \subset B_1$ for $0 < \delta_0 < \delta/2$ centered at the same p_1. Thus, $B(p_1, \delta_0)$ is a hypersurface of revolution U'_{γ_0} with the induced metric in $\mathbb{R}^4$ determined by revolving a segment of the circle γ_0 in the (x_0, x_1)-plane about the x_0-axis. We set things up so that the vertical x_1-axis corresponds to boundary points of $B(p_1, \delta_0)$. We then proceed as Gromov and Lawson to deform γ_0 away from vertical axis bending it upwards as

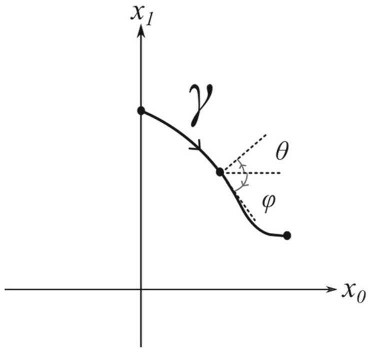

Fig. 6 The curve γ

we move to the right and ending with an arbitrarily short horizontal line segment. We call this curve γ, cf. Fig. 6. The curve γ begins exactly as γ_0 so that we may attach the corresponding hypersurface onto the larger $B(p_1, \delta/2)$ in a natural way. We do exactly the same for $B_2 \subset M^3$ and identify the two hypersurfaces along their common boundary, i.e., the "tiny neck," forming $2U'_\gamma = U'_\gamma \sqcup U'_\gamma$. We then define the tunnel $U = U_\delta$ by

$$U = U_\delta = ((B(p_1, \delta/2) \setminus B(p_1, \delta_0)) \sqcup (2U'_{\delta_0,\gamma}) \sqcup ((B(p_2, \delta/2) \setminus B(p_2, \delta_0)), \quad (224)$$

where $0 < \delta_0 < \delta/2$ and $U'_\gamma = U'_{\delta_0,\gamma}$ is a modified Gromov–Lawson tunnel, see Fig. 1.

The boundary of $2U'_\gamma$ is isometric to a collar of $B(p_1, \delta_0) \sqcup B(p_2, \delta_0)$, so we may smoothly attach it to form (224).

Step 2: Construction of the curve γ, Part 1: C^1

In this step, we construct a C^1, and piecewise C^∞, curve γ. The construction is based on the bending argument of Gromov and Lawson and uses the fundamental theorem of plane curves, i.e., the fact that a smooth curve parametrized by arclength is uniquely determined by its curvature, the initial point, and the initial tangent vector. Care must be taken to ensure that the induced metric on U'_γ maintains positive scalar curvature and that the length of γ is controlled to yield diameter and volume estimates of Lemma 2.1. This step is quite technical and forms the heart of the proof.

Step 3: Construction of the curve γ, Part 2: from C^1 to C^∞

In this step, we show how to modify the curve constructed in Step 2 to obtain a smooth curve $\bar{\gamma}$ while maintaining all the required features. The modification is elementary and, once it is completed, we rename $\bar{\gamma}$ back to γ.

Step 4: Diameter estimates (7), (9) and volume estimates (10), (11)

This is very straightforward since the previous steps give an estimate of the length of the tunnel.

We remark here that the choice of δ_0 is used only to insure that the tunnel U' (see Fig. 1) has sufficiently small volume.

Step 1 of the Proof

We now set up our notation further, describe U explicitly in terms of a special curve γ, and state the important curvature formulas needed in later steps. The construction of γ is done in the next two sub-sections (Steps 2 and 3).

As mentioned in Sect. 1, because we assume that B_1 and B_2 have constant sectional curvature K, we may work directly in Euclidean space $\mathbb{R}^4$ with coordinates (x_0, x_1, x_2, x_3) and its standard metric. Let $\gamma(s)$ be a curve in the (x_0, x_1)-plane, parametrized by arc-length, written as $\gamma(s) = (x_0(s), x_1(s))$. This curve specifies a hypersurface in $\mathbb{R}^4$ (by rotating γ about the x_0-axis),

$$U' = U'_\gamma = \{\, (x_0, x_1, x_2, x_3 \in \mathbb{R}^4 \mid x_0 = x_0(s),\ x_1^2 + x_2^2 + x_3^2 = x_1(s)^2 \,\}, \quad (225)$$

which we endow with the induced metric. Our curve γ will always lie in the first quadrant of (x_0, x_1)-plane and will be parametrized so that $x_0(s)$ will be increasing. We denote by $\theta(s)$ the angle between the horizontal direction and the upward normal vector, and by $\varphi(s)$ the angle between the horizontal direction and the tangent vector to γ.

We remark that the two angle functions are related by

$$\theta(s) = \varphi(s) + \frac{\pi}{2}, \quad (226)$$

See Fig. 6. In particular, $\varphi \in (-\pi/2, 0]$.

Denote by $k(s)$ the geodesic curvature of γ. It is a signed quantity so that γ bends away from the horizontal axis if $k(s) > 0$ and towards the x_0-axis when $k(s) < 0$. If $\gamma(s_0) = (c, d)$ and $\varphi_0 = \varphi(s_0)$ then (cf. Theorem 6.7, [13]) the function $k(s)$ determines γ by the formulae

$$\varphi(s) = \varphi_0 + \int_{s_0}^{s} k(u)\, \mathrm{d}u \quad (227)$$

and

$$\gamma(s) = \left(c + \int_{s_0}^{s} \cos(\varphi(u))\, \mathrm{d}u,\ d + \int_{s_0}^{s} \sin(\varphi(u))\, \mathrm{d}u \right). \quad (228)$$

Our aim is to define a function $k(s)$ so that the resulting threefold of revolution U' has positive scalar curvature. The formula on page 226 of [12] for $n = 3$ gives a relation between the two curvatures. Namely

$$\mathrm{Scal}_{U'}(s) = \frac{2\, \sin\theta(s)}{x_1(s)} \left[\frac{\sin\theta(s)}{x_1(s)} - 2k(s) \right], \quad (229)$$

where $\mathrm{Scal}_{U'}(s)$ is the scalar curvature of the induced metric on U' and k is the geodesic curvature of γ. In particular, the formula holds if γ is the intersection of the 3-sphere around the origin with the (x_0, x_1)-plane in which case k is a negative constant.

We begin defining our curve $\gamma(s)$ so that $\gamma(0)$ corresponds to a point on $\partial B(p_1, \delta_0)$ and $\gamma(s)$, for small values of $s \in [0, s_0]$, parametrizes the intersection of $B(p_1, \delta_0)$ with the (x_0, x_1)-plane. In particular, for small s, $k(s) \equiv -\sqrt{K}$. We choose $s_0 = \delta_0/2$ and then extend (in Step 2, Sect. 1) the function $k(s)$ to a suitable step function on a longer interval $[0, L]$ so that the resulting curve $\gamma(s)$ has the following properties.

(I) The graph of γ lies strictly in the first quadrant, beginning at $p_I = \gamma(0) = (0, \cos(-\pi/2+\delta_0)/\sqrt{K})$ and ending at $p_F = \gamma(L)$ with $x_0(L) > 0, x_1(L) > 0$, where L is the length of the curve. Moreover, a point of γ moves to the right when s increases.

(II) Let $\theta(s)$ be the angle between the upward pointing normal to γ and the x_0-axis. The curve γ ends at p_F with $\theta(L) = \pi/2$ and has $\theta = \pi/2$ (so that it is a horizontal line segment) for an arbitrarily small interval $(L', L]$ (where $L' < L$).

(III) The curve γ has constant curvature $-\sqrt{K}$ near 0 so that the boundary of U has a neighborhood that is isometric to a collar of $B_1 \cup B_2$.

(IV) The curvature function $k(s)$ satisfies

$$k(s) < \frac{\sin(\theta(s))}{2x_1(s)} \qquad s \in [0, L], \tag{230}$$

so that the expression on the right-hand side of (229) is positive for all $s \in [0, L]$. We remark here that in certain stages of the construction $k(s)$ will have discontinuities so that $\mathrm{Scal}_{U'}(s)$ is not defined but this will cause no difficulties.

(V) The length of γ, L, is $O(\delta_0)$.

Due to properties (I) and (II) of γ above, we may smoothly attach two copies of U' along their common boundary at $s = L$ to define $2U' = U'_\gamma \sqcup U'_\gamma$ and then, using property (III), attach $2U'$ to form U as in (224).

In the next step, we construct a piecewise C^1 curve γ in the (x_0, x_1)-plane which satisfies properties (I) through (V). Then, in Step 3, we modify the construction once more to produce a smooth curve, $\bar{\gamma}$, with these same properties.

Step 2 of the Proof: Construction of γ, Part 1: C^1

As above, let $s_0 = \delta_0/2$ and let $q_0 = (a_0, b_0)$ be the coordinates of the point $\gamma(s_0)$ that is already defined. By choosing δ_0 sufficiently small, we can assume that the tangent vector to γ at $s = s_0$ is nearly vertical and is pointing downward at $s = s_0$. We also have $k(s) \equiv -\sqrt{K}$ on $[0, s_0]$.

We will use a finite induction to define a sequence of extensions of γ over intervals $[s_i, s_{i+1}]$, with $s_i < s_{i+1}$ for a finite number of steps $0 \le i \le n$, where $n = n(\delta_0)$ is the number of steps required such that properties (I), (III), (IV), and (V) all hold at each extension. We denote by (a_i, b_i) the coordinates of the point $\gamma(s_i)$ for $0 \le i \le n$.

Let us first choose the curvature function $k(s)$ of $\gamma(s)$ on the first extended interval $[s_0, s_1]$. Observe that equation (230) limits the amount of positive curvature allowed for $k(s)$. In fact, we choose $k(s)$ to be the constant $k_1 > 0$ over the interval $[s_0, s_1]$ based only the initial data at s_0

$$k_1 = \frac{\sin(\theta(s_0))}{4b_0} > 0, \tag{231}$$

where $\theta(s_0) = \frac{\pi}{2} + \varphi(s_0) = \delta_0 - \sqrt{K}s_0 > 0$ and $b_0 = x_1(s_0)$. Note that constant positive curvature means that $\gamma(s)$ moves along the arc of a circle of curvature $1/\sqrt{k_1}$ bending away from the origin.

We verify that property (IV) holds with our choice of k_1 in (231). From (227), we see that $\varphi(s)$ is an increasing function with range in the interval $(-\pi/2, 0)$, hence $\theta(s)$ is also increasing by (226). Moreover, from (227) and (228), we see that the x_1-coordinate function is decreasing on the interval (s_0, s_1) since $x_1'(s) = \sin(\varphi(s)) < 0$. Thus, the expression on the right-hand side of (230), $\sin(\theta(s))/(2x_1(s))$, is an increasing function on (s_0, s_1) so that

$$\frac{\sin(\theta(s_0))}{2x_1(s_0)} \le \frac{\sin(\theta(s))}{2x_1(s)} \qquad s \in [s_0, s_1]. \tag{232}$$

Since $k(s) \equiv k_1$ is constant it follows that the property (IV) holds for $s \in [s_0, s_1]$.

Next, we choose the length of the extension $\Delta s_1 = s_1 - s_0$, so that properties (I) and (V) hold. This is achieved by setting

$$\Delta s_1 = \frac{b_0}{2} > 0 \tag{233}$$

Observe that $x_0(s)$ is increasing since $x_0'(s) = \cos(\varphi(s)) > 0$ as $\varphi \in (-\pi/2, 0)$.

Clearly we have

$$b_0 < \delta_0 \tag{234}$$

since b_0 is the vertical distance of $\gamma(s_0)$ to the x_0-axis which is less than the distance along the sphere.

Of course, we do not achieve a final angle of $\pi/2$ of the normal at s_1 and gain only a small but definite increase in the angle. The change in angle of the normal with the x_0-axis is

$$\Delta\theta_1 = \theta(s_1) - \theta(s_0) = \int_{s_0}^{s_1} k(s)\,\mathrm{d}s = k_1 \cdot \Delta s_1 = \frac{\sin(\theta(s_0))}{8} > 0$$

by (231) and (233).

With γ extended over the first interval $[s_0, s_1]$, we now inductively define further extensions. Assume that Δs_j, s_j and k_j have been chosen for $j = 1, 2, \ldots, (i-1)$, and γ extended on the intervals $[s_j, s_{j+1}]$, we then define

$$\Delta s_i = \frac{b_{i-1}}{2}, \qquad s_i = s_{i-1} + \Delta s_i \qquad \text{and} \qquad k_i = \frac{\sin(\theta(s_{i-1}))}{4b_{i-1}}, \tag{235}$$

where $\gamma(s_i) = (a_i, b_i)$. In what follows we will also write θ_j and φ_j for $\theta(s_j)$ and $\varphi(s_j)$, respectively. We remark that $b_{i+1} < b_i$ by (228) since the angle φ is negative and that $k_{i+1} > k_i$ since the ratio $\frac{\sin(\theta(s))}{x_1(s)}$ is increasing. Observe that properties (I), (IV), and (V) of γ hold on $[s_{i-1}, s_i]$ for all i by our choices in (235) by arguments analogous to those given for the first extension of γ on $[s_0, s_1]$.

We observe that we gain a definite amount of angle θ with each extension since, by (235), for each $j \in \{1, 2, \ldots, i\}$,

$$\begin{aligned} \Delta\theta_j = \theta(s_j) - \theta(s_{j-1}) = \int_{s_{j-1}}^{s_i} k(s)\,\mathrm{d}s = k_j \cdot \Delta s_j = \frac{\sin(\theta(s_{j-1}))}{8} \\ \geq \frac{\sin(\theta(s_0))}{8}, \end{aligned} \tag{236}$$

because $\theta(s_{j-1}) \geq \theta(s_0)$ and the values of θ are in the range $(0, \pi/2)$ so that the sine is an increasing function. We stop the construction when $\theta(s)$ reaches the value $\pi/2$. Thus the total change in the angle θ over the interval $[0, s_i]$ is bounded from below by

$$\Delta\theta = \sum_{j=1}^{i} \Delta\theta_j \geq i \cdot \frac{\sin(\theta_0)}{8}. \tag{237}$$

To prove property (V), that the length of γ is on the order of δ_0, we need the sequence of b_i's to be summable and will want to compare it to the geometric progression. The difficulty here is that, since our curve is bending more and more upwards, the ratios b_i/b_{i-1} increase. For this reason, we stop our induction when θ reaches the value of $\pi/4$. It will turn out that once this value is reached, we can complete the construction of $k(s)$ by a single extension albeit with Δs not given by (235).

Thus, define $n = n(\delta_0)$ to be the first positive integer with

$$\frac{\pi}{4} \leq \theta_n \tag{238}$$

which exists by (237). Moreover, if $\theta_n > \pi/4$ we re-define s_n to be the exact value in (s_{n-1}, ∞) such that $\theta(s_n) = \pi/4$. Thus, for the modified value of s_n

$$\theta_n = \theta(s_n) = \frac{\pi}{4}. \tag{239}$$

The following Lemma gives the desired comparison.

Lemma 7.1 *There exists a universal constant $C \in (0, 1)$, independent of δ_0 and K, such that for all $i \leq n$*

$$b_i \leq C \cdot b_{i-1},$$

where $n = n(\delta_0)$ is as above.

The Lemma, to be proven shortly below, implies that the length of the curve γ on the entire interval $[0, s_n]$ is no larger than a constant (independent of δ_0) times δ_0. Namely,

$$L(\gamma([0, s_n])) = s_n = \sum_{j=1}^{n} \Delta s_j. \tag{240}$$

Thus, from (235) and Lemma (7.1), we have

$$\sum_{j=1}^{n} \Delta s_j = \sum_{j=1}^{n} \frac{b_{j-1}}{2} \leq \frac{b_0}{2} \left(\sum_{j=1}^{n-1} C^j \right) \leq C_1 \delta_0 \tag{241}$$

by the lemma and (234). So, $L(\gamma([0, s_n])) \leq C_1 b_0$ with $C_1 = \frac{1}{2-2C}$ which is independent of δ_0 since C is. This proves that $L(\gamma([0, s_n])) = O(\delta_0)$.

Proof of Lemma 7.1 Let $1 \leq i \leq n$. We compute explicitly using (227), (228), and (235),

$$\varphi(s_i) = \varphi(s_{i-1}) + k_i \cdot \Delta s_i = \varphi(s_{i-1}) + \frac{\sin(\theta_{i-1})}{8} \tag{242}$$

and

$$\begin{aligned} b_i &= x_1(s_i) \\ &= b_{i-1} + \int_{s_{i-1}}^{s_i} \sin(\varphi(s_{i-1}) + k_i(u - s_{i-1}))\, \mathrm{d}u \\ &= b_{i-1} - \frac{1}{k_i} (\cos(\varphi(s_i)) - \cos(\varphi(s_{i-1}))) \\ &= b_{i-1} - \frac{4b_{i-1}}{\sin(\theta(s_{i-1}))} \left(\cos\left(\varphi(s_{i-1}) + \frac{\sin(\theta_{i-1})}{8} \right) - \cos(\varphi(s_{i-1})) \right). \end{aligned}$$

Thus,

$$\frac{b_i}{b_{i-1}} = 1 - \frac{4}{\sin(\theta(s_{i-1}))} \left(\cos\left(\varphi(s_{i-1}) + \frac{\sin(\theta_{i-1})}{8} \right) - \cos(\varphi(s_{i-1})) \right).$$

Therefore, by the Mean Value Theorem, there exists $\mu_i \in (\varphi(s_{i-1}), \varphi(s_{i-1}) + \sin(\theta(s_{i-1}))/8)$ such that

$$\frac{b_i}{b_{i-1}} = 1 - \frac{4}{\sin(\theta(s_{i-1}))}(-\sin(\mu_i)) \cdot \frac{\sin(\theta(s_{i-1}))}{8} = 1 + \frac{\sin(\mu_i)}{2}.$$

To complete the proof of the claim, we seek a constant $0 < C < 1$, independent of δ_0, such that

$$1 + \frac{\sin(\mu_i)}{2} < C < 1. \tag{243}$$

Recall that the angle function φ takes negative values throughout.

We claim that the choice

$$C = 1 + \frac{1}{4}\sin\left(-\frac{\pi}{4} + \frac{\cos(-\frac{\pi}{4})}{8}\right) \approx 0.8395 \tag{244}$$

will satisfy our requirement.

This follows from the fact that the sine is an increasing function on the interval $(\varphi(s_{i-1}), \varphi(s_{i-1}) + \sin(\theta(s_{i-1}))/8)$ and the fact that both the angles φ_i and θ_i are increasing, so

$$\begin{aligned} 1 + \frac{\sin(\mu_i)}{2} &\le 1 + \frac{1}{2}\sin\left(\varphi(s_{i-1}) + \frac{\sin(\theta(s_{i-1}))}{8}\right) \\ &\le 1 + \frac{1}{2}\sin\left(\varphi(s_n) + \frac{\cos(\varphi(s_n))}{8}\right). \end{aligned}$$

By our choice of s_n, $\theta(s_n) = \pi/4$ from (239) and $\varphi(s_n) = -\pi/4$ so that

$$\begin{aligned} 1 + \frac{\sin(\mu_i)}{2} &\le 1 + \frac{1}{2}\sin\left(-\frac{\pi}{4} + \frac{\cos\left(-\frac{\pi}{4}\right)}{8}\right) \\ &< 1 + \frac{1}{4}\sin\left(-\frac{\pi}{4} + \frac{\cos\left(-\frac{\pi}{4}\right)}{8}\right) \\ &= C < 1. \end{aligned}$$

This finishes the proof of the Lemma. □

At this stage of the construction, γ has angle $\theta = \pi/4$ at the endpoint s_n. We make one additional extension of our step function.

We now define $s_{n+1} > s_n$ and $k_{n+1} > 0$ as follows.

By (227) $\varphi(s)$ in $[s_n, s_{n+1}]$ will be given by

$$\varphi(s) = \varphi_n + \int_{s_n}^{s} k(u)\,\mathrm{d}u = \varphi_n + k_{n+1}(s - s_n). \tag{245}$$

Let s_{n+1} be determined by k_{n+1} as the first value such that $\varphi(s_{n+1}) = 0$ (equivalently $\theta(s_{n+1}) = \pi/2$). Then

$$0 = \varphi(s_{n+1}) = \varphi_n + k_{n+1}(s_{n+1} - s_n) \tag{246}$$

so that

$$s_{n+1} = s_n - \frac{\varphi_n}{k_{n+1}}. \tag{247}$$

We require in addition that $b(s_{n+1}) > 0$ (that is, γ remains above the x_0-axis). Using (247) and (228), we obtain

$$b(s_{n+1}) = b_n + \int_{s_n}^{s_{n+1}} \sin(\varphi(s))\,\mathrm{d}s = b_n - \frac{\cos(\varphi(s_{n+1})) - \cos(\varphi(s_n))}{k_{n+1}} \\ = b_n - \frac{1 - \cos(\varphi(s_n))}{k_{n+1}} \tag{248}$$

so that $b(s_{n+1}) > 0$ is equivalent to

$$b_n - \frac{1 - \cos(\varphi(s_n))}{k_{n+1}} > 0$$

or

$$k_{n+1} \cdot b_n > 1 - \cos(\varphi(s_n)). \tag{249}$$

On the other hand, k_{n+1} has to be bounded from above in order to guarantee (230). Therefore, we require that

$$k_{n+1} < \frac{\sin(\theta(s_n))}{2b_n},$$

or

$$k_{n+1} \cdot b_n < \frac{\sin(\theta(s_n))}{2}. \tag{250}$$

Combining (249) and (250) gives conditions for k_{n+1}

$$1 - \cos(\varphi(s_n)) < k_{n+1} \cdot b_n < \frac{\sin(\theta(s_n))}{2}. \tag{251}$$

Since $\sin(\theta(s)) = \cos(\varphi(s))$, (251) is equivalent to

$$1 - \cos(\varphi(s_n)) < k_{n+1} \cdot b_n < \frac{\cos(\varphi(s_n))}{2}. \tag{252}$$

Now, recall that s_n was chosen in (239) so that $\varphi(s_n) = -\pi/4$ so

$$1 - \cos(\varphi(s_n)) = \frac{2 - \sqrt{2}}{2} < \frac{\cos(\varphi(s_n))}{2} = \frac{\sqrt{2}}{4}.$$

Now, choose arbitrarily any α, satisfying

$$\frac{2 - \sqrt{2}}{2} < \alpha < \frac{\sqrt{2}}{4} \tag{253}$$

and define k_{n+1} by

$$k_{n+1} = \alpha / b_n. \tag{254}$$

With this choice (252), and therefore, (249) and (250) hold.

To ensure property (II), we choose $L > s_{n+1}$ so that $L - s_{n+1}$ is arbitrarily small. We extend γ to the interval $[s_{n+1}, L]$, where γ is a straight horizontal line on $[s_{n+1}, L]$ by choosing $k(s) = 0$ there. To check that the length of the curve we constructed is $O(\gamma_0)$ we observe that

$$s_{n+1} = s_n - \varphi_n / k_{n+1} = s_n + \frac{\pi}{4\alpha} b_n \le s_n + \frac{\pi}{4\alpha} b_0 = O(\delta_0) \tag{255}$$

by (234), (241), and (255).

We note that the choice of L is arbitrary. It will be made explicit in the next step when we construct the curve $\bar{\gamma}$, the C^∞ version of γ.

This completes the construction of the continuously differentiable curve γ defined on the interval $[0, L]$ satisfying properties (I) through (V).

Step 3 of the Proof: Construction of γ, Part 2: From C^1 to C^∞

In this step, barred quantities will refer to the C^∞ curve $\bar{\gamma}(s)$ to be constructed in this step and all the other quantities related to the construction (for example, $\bar{\theta}$, $\bar{\varphi}$, $\bar{k}(s)$, etc.). Unbarred quantities will refer to the C^1 curve constructed in the previous step.

The general plan is to replace $k(s)$ as chosen in Step 2 with a smooth version $\bar{k}(s)$ as depicted in Fig. 7, which will then define $\bar{\gamma}$ by the formulae (227) and (228). Set $k_0 = -K^{1/2}$ and modify $k(s)$ on $[s_i, s_{i+1}]$ for $i = 0, 1, 2, \ldots, n$ so that the graph of $\bar{k}(s)$ will connect to the constant function equal to k_i smoothly at s_i, will rise steeply to the value k_{i+1} in a very short interval $[s_i, s_i + \alpha]$ and will connect smoothly with constant function equal to k_{i+1} in $[s_i + \alpha, s_{i+1}]$. For each $i = 0, 1, 2, \ldots n$, $\bar{k}|[s_i, s_{i+1}]$ can be constructed as follows. Choose and fix a C^∞ function $g(s)$ which is identically 0 for $s < 0$, identically 1 for $s > 1$, and strictly increasing on $[0, 1]$. Then $\bar{k}|[s_i, s_{i+1}]$ is constructed by appropriate rescaling and translations of the graph of $g(s)$ in both vertical and horizontal directions. The values of k_i and k_{i+1} determine the transformations along the vertical axis but rescaling of the independent variable remains a free parameter α to be set sufficiently small later. We will use the same value of α for every $i = 1, 2, \ldots n$.

Since

$$\Delta\bar{\theta} = \int_0^{s_{n+1}} \bar{k}\,\mathrm{d}s \le \int_0^{s_{n+1}} k\,\mathrm{d}s = \Delta\theta,$$

we loose a small amount of "bend" so that $\bar{\theta}(s_{n+1}) < \frac{\pi}{2}$ by a very small amount controlled by α. We compensate for this by one final extension of $\bar{k}$ to an interval $[s_{n+1}, L]$ with $L = s_{n+1} + 2\beta$. We choose $\bar{k}$ so that it connects smoothly with k_{n+1} at s_{n+1}, drops smoothly to zero over $[s_{n+1}, s_{n+1} + \beta]$ and continues identically zero on $[s_{n+1} + \beta, s_{n+1} + 2\beta]$. β and $\bar{k}$ are chosen so that

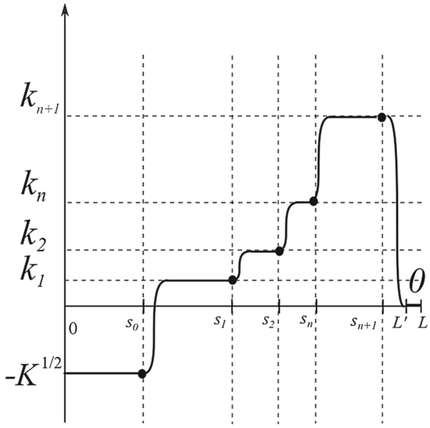

Fig. 7 Graph of the smooth curvature $\bar{k}(s)$ with "full bend"

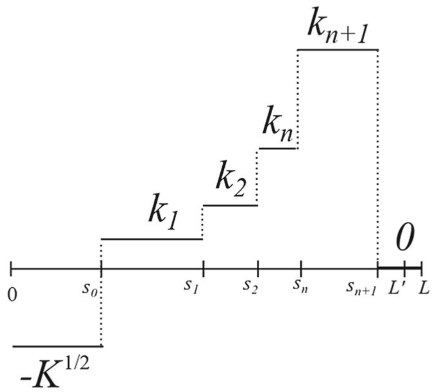

Fig. 8 Graph of the curvature, $k(s)$, with "full bend" as a step function

$$\int_{s_{n+1}}^{s_{n+1}+\beta} \bar{k}(s)\,\mathrm{d}s = \frac{\pi}{2} - \bar{\theta}(s_{n+1}).$$

This ensures that $\bar{\theta} = \frac{\pi}{2}$ in the interval $[s_{n+1}+\beta, s_{n+1}+2\beta]$. This final extension is constructed as the preceding ones except that we have to use the reflection $s \mapsto -s$ before rescaling and translating the original function g. We note that $\beta = O(\alpha)$ is determined by the choice of α and the requirement that $\bar{\theta}(L) = \frac{\pi}{2}$. We also observe that as α tends to zero, the functions $\bar{\varphi}$, $\bar{\theta}$, $\bar{x_0}$, and $\bar{x_1}$ will converge uniformly on $[0, L]$ to φ, θ, x_0, and x_1, respectively, as follows from (227) and (228).

We now check that the properties (I) through (V) on page (I) hold for the curve $\bar{\gamma}$ for sufficiently small choice of α. Only (IV) and (V) need a verification. (V) follows since $L = s_{n+1}+2\beta = O(\delta_0)+O(\alpha)$. To prove (IV) we use the uniform convergence on $[0, s_{n+1}]$ as α approaches 0 of $\frac{\sin\bar{\theta}(s)}{2\bar{x}_1(s)}$ to $\frac{\sin\theta(s)}{2x_1(s)}$. More precisely, on $[s_i, s_{i+1}]$,

$$\frac{\sin\bar{\theta}(s)}{2\bar{x}_1(s)} - \bar{k}(s) = \left(\frac{\sin\bar{\theta}(s)}{2\bar{x}_1(s)} - k_{i+1}\right) + \left(k_{i+1} - \bar{k}(s)\right).$$

For sufficiently small α, the first term on the right becomes positive by the property (IV) for the curve γ while the second term is nonnegative by construction (cf. Fig. 8). Finally, in the last interval $[s_{n+1}, L]$ the ratio $\frac{\sin\bar{\theta}(s)}{2\bar{x}_1(s)}$ is nondecreasing so that

$$\frac{\sin\bar{\theta}(s)}{2\bar{x}_1(s)} \geq \frac{\sin\bar{\theta}(s_{n+1})}{2\bar{x}_1(s_{n+1})} > k_{n+1}$$

since the last inequality was verified for $s = s_{n+1}$ already. Property (IV) follows since $k_{n+1} > \bar{k}(s)$ in $[s_{n+1}, L]$. This finishes the construction of $\bar{\gamma}$.

Step 4 of the Proof: Diameter and Volume Estimates of Lemma 2.1

Given the definition of U in (224), the diameter of U is estimated by

$$\mathrm{Diam}(U) \leq \pi\delta + \delta + 2L = O(\delta) + O(\delta_0) = O(\delta).$$

To estimate the volume of U', note that the intersection of U' with the hyperplane $x_0 = x_0(s) = c$ for $0 < s < L$ is a sphere of two dimensions and of radius $x_1(s) < \delta_0$. It follows by Fubini's theorem that $\mathrm{Vol}(U') = O(\delta_0^3)$. To prove (10) recall that U is obtained from the union of two disjoint balls of radius δ by removing balls of radius δ_0 and attaching U' along the common boundary (cf. Fig. 1). Since the volumes of the removed balls and of the added tunnel are $O(\delta_0^3)$, the estimate (10) follows by choosing δ_0 sufficiently small depending on ϵ. The estimate (11) is proved in the same way. The proof of Lemma 2.1 is now complete. □

References

1. Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. **163**(7), 1405–1490 (2014)
2. Ambrosio, L., Kirchheim, B.: Currents in metric spaces. Acta Math. **185**(1), 1–80 (2000)
3. Ambrosio, L., Tilli, P.: Topics on Analysis in Metric Spaces, Volume 25 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford (2004)
4. Bamler, R.: A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature. Math. Res. Lett. **23**(2), 325–337 (2016)
5. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry, Volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence (2001)
6. Burago, D., Ivanov, S.: Area spaces: first steps, with appendix by nigel higson. Geom. Funct. Anal. **19**(3), 662–677 (2009)
7. Basilio, J., Sormani, C.: Sequences of three dimensional manifolds with positive scalar curvature. Preprint to appear (2017)
8. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. I. J. Differ. Geom. **46**(3), 406–480 (1997)
9. Federer, H., Fleming, W.H.: Normal and integral currents. Ann. Math. **2**(72), 458–520 (1960)
10. Fukaya, K.: Collapsing of Riemannian manifolds and eigenvalues of Laplace operator. Invent. Math. **87**(3), 517–547 (1987)

11. Gromov, M., Blaine Lawson, H.: The classification of simply connected manifolds of positive scalar curvature. Ann. Math. **111**(3), 423–434 (1980)
12. Gromov, M., Blaine Lawson Jr., H.: Spin and scalar curvature in the presence of a fundamental group. I. Ann. Math. **111**(2), 209–230 (1980)
13. Gray, A.: Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd edn. CRC Press, Boca Raton (1998)
14. Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces, volume 152 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates
15. Gromov, M.: Dirac and Plateau billiards in domains with corners. Century Eur. J. Math. **12**(8), 1109–1156 (2014)
16. Gromov, M.: Plateau-Stein manifolds. Century Eur. J. Math. **12**(7), 923–951 (2014)
17. Huang, L.-H., Lee, D., Sormani, C.: Stability of the positive mass theorem for graphical hypersurfaces of Euclidean space. Journal fur die Riene und Angewandte Mathematik (Crelle's Journal) **727**, 269–299 (2017)
18. Lakzian, S., Sormani, C.: Smooth convergence away from singular sets. Commun. Anal. Geom. **21**(1), 39–104 (2013)
19. Lee, D.A., Sormani, C.: Stability of the positive mass theorem for rotationally symmetric Riemannian manifolds. Journal fur die Riene und Angewandte Mathematik (Crelle's Journal), 686 (2014)
20. LeFloch, P.G., Sormani, C.: The nonlinear stability of rotationally symmetric spaces with low regularity. J. Funct. Anal. **268**(7), 2005–2065 (2015)
21. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. Math. **169**(3), 903–991 (2009)
22. Matveev, R., Portegies, J.: Intrinsic flat and Gromov-Hausdorff convergence of manifolds with Ricci curvature bounded below. J. Geom. Anal. **27**(3), 1855–1873 (2017)
23. Portegies, J.W.: Semicontinuity of eigenvalues under intrinsic flat convergence. Calc. Var. Partial Differ. Equ. **54**(2), 1725–1766 (2015)
24. Rosenberg, J., Stolz, S.: Metrics of positive scalar curvature and connections with surgery. Surveys on Surgery Theory, Number 149 in Annals of Mathematics Studies 2. Princeton University Press, Princeton (2001)
25. Sormani, C.: Intrinsic flat Arzela-Ascoli theorems. Commun. Anal. Geom. 27(1), (2019) (on arxiv since 2014)
26. Sormani, C.: Scalar curvature and intrinsic flat convergence. In: Gigli, N. (ed) Measure Theory in Non-Smooth Spaces, Chap. 9, pp. 288–338. De Gruyter Press (2017)
27. Sormani, C., Stavrov, I.: Geometrostatic manifolds of small ADM mass (2017). arXiv: 1707.03008
28. Sturm, K.-T.: A curvature-dimension condition for metric measure spaces. C. R. Math. Acad. Sci. Paris **342**, 197?200 (2006)
29. Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. **196**(1), 65–131 (2006)
30. Sormani, C., Wenger, S.: Weak convergence and cancellation, appendix by Raanan Schul and Stefan Wenger. Calc. Var. Partial Differ. Equ., 38(1–2) (2010)
31. Sormani, C., Wenger, S.: Intrinsic flat convergence of manifolds and other integral current spaces. J. Differ. Geom., 87 (2011)
32. Schoen, R., Yau, S.T.: On the structure of manifolds with positive scalar curvature. Manuscripta Math. **28**(1–3), 159–183 (1979)
33. Schoen, R., Yau, S.T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. **65**(1), 45–76 (1979)
34. Villani, C.: Optimal transport, volume 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (2009). Old and new
35. Wenger, S.: Compactness for manifolds and integral currents with bounded diameter and volume. Calc. Var. Partial Differ. Equ. **40**(3–4), 423–448 (2011)