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Curvature flow with driving force on fixed boundary points

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Abstract In this paper, we consider the curvature flow with driving force on fixed boundary points in the plane. We give a general local existence and uniqueness result of this problem with C^2 initial curve. For a special family of initial curves, we classify the solutions into three categories. Moreover, in each category, the asymptotic behavior is given.

Keywords Curvature flow · Driving force · Fixed boundary points

Mathematics Subject Classification 35A01 · 35A02 · 35K55 · 53C44

1 Introduction

In this paper, we consider the curvature flow with driving force on fixed boundary points given by

$$V = -\kappa + A, \text{ on } \Gamma(t), \ 0 < t < T; \tag{1.1}$$

$$\Gamma(0) = \Gamma_0; \tag{1.2}$$

$$\partial\Gamma(t) = \{P, Q\}, \ 0 \le t < T. \tag{1.3}$$

Here V denotes the upward normal velocity(the definition of "upward" is given by Remark 2.2). The sign κ is chosen such that the problem is parabolic. A is a positive constant. P, Q are two different fixed points in $\mathbb{R}^2$

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Let $s \in [0, L(t)]$ be the arc length parameter on $\Gamma(t)$ and $F \in C^{2,1}([0, L(t)] \times [0, T) \to \mathbb{R}^2)$ such that $\Gamma(t) = \{F(s,t) \in \mathbb{R}^2 \mid 0 \le s \le L(t)\}$. Equations (1.1) and (1.2) can be written as

$$\frac{\partial}{\partial t} F(s,t) = \kappa N - AN, \ 0 < s < L(t), \tag{1.4}$$

$$F(s,0) = F_0(s), \ 0 \le s \le L_0. \tag{1.5}$$

Here $\Gamma_0 = \{F_0(s) \in \mathbb{R}^2 \mid 0 \le s \le L_0\}$; N denotes the unit downward normal vector(the definition of "downward" is given by Remark 2.2) and $L(t)$ denotes the length of $\Gamma(t)$. And the notation $\frac{\partial}{\partial t} F(s,t)$ denotes the partial derivative with respect to t by fixing s. Noting the assumption that the sign of κ is chosen such that the problem (1.1) is parabolic, combining Frenet formulas, there holds

$$\kappa N = \frac{\partial^2}{\partial s^2} F(s,t).$$

The boundary condition 1.3 can be written as follows:

$$F(0,t) = P, \ F(L(t),t) = Q. \tag{1.6}$$

Main results Here we give our main theorems. In the following paper, p is denoted as the arc length parameter on Γ_0.

Theorem 1.1 *Assume that $F_0 \in C^2([0, L_0] \to \mathbb{R}^2)$ is embedding. Then there exist $T > 0$ and a unique embedding map $\tilde{F} \in C^{2,1}([0, L_0] \times [0, T_0) \to \mathbb{R}^2)$ such that the following results hold:*

(1) $\frac{\partial}{\partial t} \tilde{F}(p,t) = \kappa N - AN$, $0 < p < L_0$, $0 < t < T$;
(2) $\tilde{F}(p,0) = F_0(p)$, $0 \le p \le L_0$;
(3) $\tilde{F}(0,t) = P$, $\tilde{F}(L_0,t) = Q$, $0 \le t < T$. Moreover, the flow

$$\Gamma(t) = \{\tilde{F}(p,t) \mid p \in [0, L_0]\}, \ 0 \le t < T$$

satisfies (1.1), (1.2), (1.3).

Assume that $P = (-a, 0)$, $Q = (a, 0)$, where $0 < a \le 1/A$. Before giving the three categories result, we introduce two equilibrium solutions of (1.1) with boundary condition (1.6). Denote

$$\Gamma_* = \{(x,y) \in \mathbb{R}^2 \mid y = \sqrt{1/A^2 - x^2} - \sqrt{1/A^2 - a^2}, -a \le x \le a\}$$

and

$$\Gamma^* = \partial B_{\frac{1}{A}}\big((0, \sqrt{1/A^2 - a^2})\big) \setminus \{(x,-y) \in \mathbb{R}^2 \mid (x,y) \in \Gamma_*\}.$$

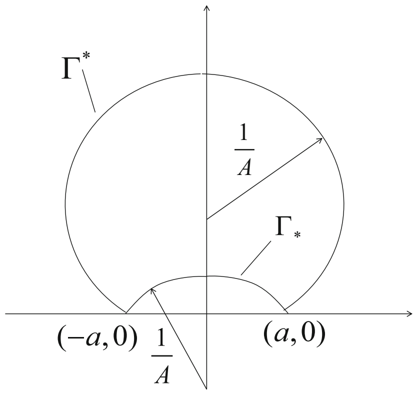

Fig. 1 Equilibrium solutions of (1.1)

Here $B_r\big((x, y)\big)$ denotes the ball centered at (x, y) with radii r (Fig. 1).

Obviously, on Γ_* and Γ^*, there hold $\kappa = A$ and the fixed boundary condition. Here we give the three categories theorem. In the following theorem, we consider a family of initial curves given by

$$\Gamma_\sigma = \{(x, y) \in \mathbb{R}^2 \mid y = \sigma\varphi(x), -a \le x \le a\}.$$

Here φ is even, $\varphi \in C^2\big([-a, a]\big)$, $\varphi(-a) = \varphi(a) = 0$, and $\varphi''(x) \le 0$, $-a < x < a$. And assume that for all $\sigma \in \mathbb{R}$, Γ_σ intersects Γ^* at most fourth(including the boundary points). Denote $\Gamma_\sigma(t)$ being the solution with $\Gamma_\sigma(0) = \Gamma_\sigma$.

Theorem 1.2 *There exists $\sigma^* > 0$ such that*

(1) For $\sigma > \sigma^*$, there exists $T^*_\sigma < T_\sigma$ such that $\Gamma_\sigma(t) \succ \Gamma^*$, $T^*_\sigma < t < T_\sigma$;
(2) For $\sigma = \sigma^*$, $T_\sigma = \infty$ and $\Gamma_\sigma(t) \to \Gamma^*$ in C^1, as $t \to \infty$;
(3) For $\sigma < \sigma^*$, $T_\sigma = \infty$ and $\Gamma_\sigma(t) \to \Gamma_*$ in C^1, as $t \to \infty$.

Here T_σ denotes the maximal existence time of $\Gamma_\sigma(t)$.

The notation "$\succ$" can be seen as an order. The precise definition is given in Sect. 2. We will interpret the sense of C^1 convergence in Definition 2.9.

Main method. Theorem 1.1 can be easily proven by transport map. The transport map is first used by [1] to consider the curvature flow under the non-graph condition. For the three categories result, we use the intersection number principle to classify the type of the solutions in Lemma 4.3. Since Γ_σ intersects Γ^* at most fourth, the intersection number between $\Gamma_\sigma(t)$ and Γ^* can only be two or four. In Lemma 4.3, one of the following three conditions can hold:

(1) The curve $\Gamma_\sigma(t)$ intersects Γ^* twice and $\Gamma_\sigma(t) \succ \Gamma^*$ eventually;
(2) The curve $\Gamma_\sigma(t)$ intersects Γ^* fourth for every $t > 0$.
(3) The curve $\Gamma_\sigma(t)$ intersects Γ^* twice and $\Gamma^* \succ \Gamma_\sigma(t)$ eventually.

Considering future, under the condition (2) above, $\Gamma_\sigma(t) \to \Gamma^*$ in C^1, as $t \to \infty$; under the condition (3) above, $\Gamma_\sigma(t) \to \Gamma_*$ in C^1, as $t \to \infty$. In this paper, we prove the asymptotic behavior by using Lyapunov function introduced in Sect. 5.

A short review for mean curvature flow. For the classical mean curvature flow: $A = 0$ in (1.1), there are many results. Concerning this problem, Huisken [9] shows that any solution that starts out as a convex, smooth, compact surface remains so until it shrinks to a "round point" and its asymptotic shape is a sphere just before it disappears. He proves this result for hypersurfaces of $\mathbb{R}^{n+1}$ with $n \geq 2$, but Gage and Hamilton [4] show that it still holds when $n = 1$, the curves in the plane. Gage and Hamilton also show that embedded curve remains embedded, i.e. the curve will not intersect itself. Grayson [8] proves the remarkable fact that such family must become convex eventually. Thus, any embedded curve in the plane will shrink to "round point" under curve shortening flow.

For fixed boundary point problem, Forcadel et al. [3] consider a family of half lines evolved by (1.1), and one boundary point is fixed at the origin. Precisely, the family of curves is given by polar coordinates,

$$\begin{cases} x = \rho \cos \theta(\rho, t), \\ y = \rho \sin \theta(\rho, t), \end{cases}$$

for $0 \leq \rho < \infty$. Therefore, $\theta(\rho, t)$ satisfies

$$\rho\theta_t = A\sqrt{1 + \rho^2\theta_\rho^2} + \theta_\rho \left(\frac{2 + \rho^2\theta_\rho^2}{1 + \rho^2\theta_\rho^2} \right) + \frac{\rho\theta_{\rho\rho}}{1 + \rho^2\theta_\rho^2}, \; t > 0, \rho > 0. \tag{1.7}$$

Obviously, this problem is singular near $\rho = 0$. They consider the solution of (1.7) in viscosity sense. Since near the fixed boundary point, curvature flow has singularity by using polar coordinates, there are some papers considering this problem by digging a hole. For example, Giga et al. [5] consider anisotropic curvature flow equation with driving force in the ring domain $r < \rho < R$. At the boundary, the family of the curves is imposed being perpendicular to the boundary, as seen in Fig. 2.

Motivation of this research. Ohtsuka et al. first prove the existence and uniqueness of spiral crystal growth for (1.1) by level set method in [6] and [12]. Moreover, they also consider this problem by digging a hole near the fixed points. Recently, [13]

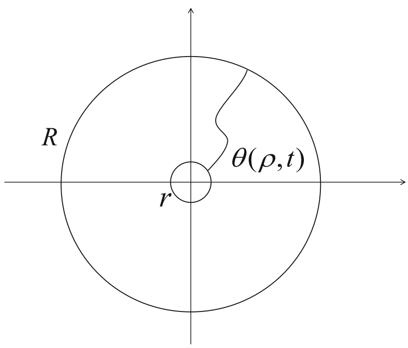

Fig. 2 Research in [5]

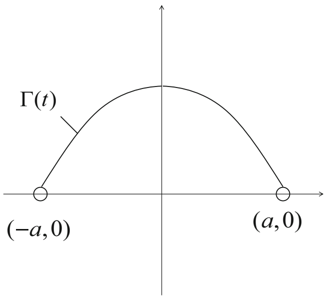

Fig. 3 Evolution of level set #1

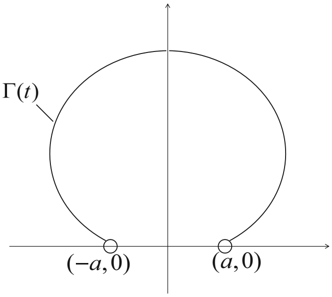

Fig. 4 Evolution of level set #2

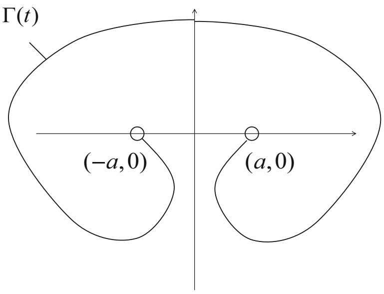

Fig. 5 Evolution of level set #3

simulates the level set of the solution given in [12] by numerical method. In their paper, for $a > 1/A$, the level set evolves as shown in Figs. 3, 4 and 5.

Although, in our paper, we only consider the problem under the condition $a \leq 1/A$, the simulated results in [13] give the hit about this research. We are devoted to considering them in an analytic way.

The rest of this paper is organized as follows. In Sect. 2, we give some preliminary knowledge including the definition of semi-order, comparison principle, and intersection number principle. In Sect. 3, we give the existence and uniqueness result for the fixed boundary points problem. Moreover, in Lemma 3.5, we give a sufficient condition for the solution $\Gamma_\sigma(t)$ remaining regular. In Sect. 4, we give the asymptotic behavior of the solution $\Gamma_\sigma(t)$ when σ is large or small. Lemma 4.3 gives an important result for classifying $\Gamma_\sigma(t)$ by intersection number. In Sect. 5, we prove the asymptotic behavior for the condition (3) in Lemma 4.3 by Lyapunov function. In Sect. 6, we give the proof of Theorem 1.2.

2 Preliminary

Semi-order We want to define a semi-order for curves with the same fixed boundary points.

Definition 2.1 For any points $P, Q \in \mathbb{R}^2$ and $P \neq Q$, assume that maps $F_i(s) \in C([0, l_i] \to \mathbb{R}^2)$ are injection and F_i are differentiable at 0 and l_i.The curves γ_i are given by $\gamma_i = \{F_i(s) \mid 0 \le s \le l_i, F_i(0) = P, F_i(l_i) = Q\}$, where l_i is the length of γ_i, $i = 1, 2$. It is easy to see that γ_i have the same boundary points P, Q, $i = 1, 2$. We say $\gamma_1 \succ \gamma_2$, if

(1) There exists connect, bounded and open domain Ω such that $\partial\Omega = \gamma_1 \cup \gamma_2$;
(2) $\frac{\mathrm{d}}{\mathrm{d}s} F_1(0) \cdot \frac{\mathrm{d}}{\mathrm{d}s} F_2(0) \neq 1$ and $\frac{\mathrm{d}}{\mathrm{d}s} F_1(l_1) \cdot \frac{\mathrm{d}}{\mathrm{d}s} F_2(l_2) \neq 1$;
(3) The domain Ω is located in the right hand side of γ_1, when someone walks along γ_1 from P to Q.

Here "$\cdot$" denotes the inner product in $\mathbb{R}^2$. We say $\gamma_1 \succeq \gamma_2$, if there exist two sequences of curves $\{\gamma_{in}\}_{n\ge1}$, $i = 1, 2$ such that

(1) $\lim\limits_{n\to\infty} d_H(\gamma_{in}, \gamma_i) \to 0, i = 1, 2;$
(2) $\gamma_{1n} \succ \gamma_{2n}, n \ge 1.$

Here $d_H(A, B)$ denotes the Hausdorff distance for set $A, B \subset \mathbb{R}^2$.

Let $F(s) \in C^2([0, l] \to \mathbb{R}^2)$ be embedding and $\gamma = \{F(s) \mid s \in [0, l], F(0) = P, F(l) = Q\}$. Using the definition of semi-order, we can define a shuttle neighborhood of γ. Considering the assumption of γ, we can extend γ by γ^* such that γ^* is C^1 curve and divides $\mathbb{R}^2$ into two connect parts denoted by Ω_1 and Ω_2. Moreover, Ω_1 is located in the left hand side when someone walks along γ^* from P to Q (Fig. 6).

Remark 2.2 We say the normal vector of γ is *upward*(*downward*), if the normal vector points to the domain $\Omega_1(\Omega_2)$.

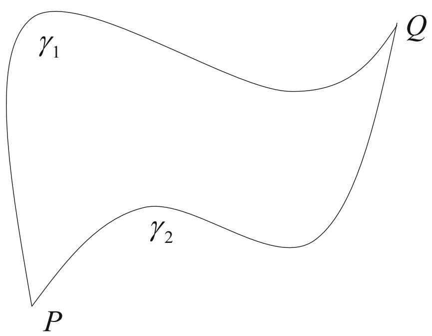

Fig. 6 Definition 2.1

Definition 2.3 (*Shuttle neighborhood*) We say V is a *shuttle neighborhood* of γ, if there exist two embedded curves γ_1 and γ_2 such that

(1) $\gamma_i \subset \Omega_i, i = 1, 2$;
(2) $\gamma_1 \succ \gamma \succ \gamma_2$;
(3) $\partial V = \gamma_1 \cup \gamma_2$.

Comparison principle and intersection number principle Here we introduce the comparison principle and intersection number principle. The intersection number principle can help us classify the solutions.

For giving comparison principle, we must define sub,super-solution of (1.4).

Definition 2.4 We say a continuous family of continuous curves $\{\gamma(t)\}$ is a sub(super)-solution of (1.4) and (1.6), if

(1) $\gamma(t)$ are continuous curves and have the same boundary points P, Q;
(2) Let $\{S(t)\}$ be a smooth flow with boundary points P, Q. For some point P^* and some time $t_0 > 0$ satisfying $P^* \in \gamma(t_0)$ with $P^* \neq P, \ Q$. If near the point P^* and time t_0, $\{S(t)\}$ only intersects $\{\gamma(t)\}$ at P^* and time t_0 from above(below). Let $V_{S(t)}$ denote the upward normal velocity of $S(t)$ and $\kappa_{S(t)}(P)$ denote the curvature at $P \in S(t)$. Then

$$V_{S(t_0)}(P^*) \leq (\geq) - \kappa_{S(t_0)}(P^*) + A.$$

Theorem 2.5 *(Comparison principle) For two families of curves $\{\gamma_1(t)\}_{0\leq t\leq T}$ and $\{\gamma_2(t)\}_{0\leq t\leq T}$, assume $\{\gamma_1(t)\}_{0\leq t\leq T}$ is a super-solution of (1.4) and (1.6), $\{\gamma_2(t)\}_{0\leq t\leq T}$ is a subsolution of (1.4) and (1.6). If $\gamma_1(0) \succeq \gamma_2(0)$, then $\gamma_1(t) \succeq \gamma_2(t)$, $0 \leq t \leq T$. Moreover, If $\gamma_1(0) \succeq \gamma_2(0)$ and $\gamma_1(0) \neq \gamma_2(0)$, then $\gamma_1(t) \succ \gamma_2(t)$. $0 \leq t \leq T$.*

We can prove this theorem by contradiction. Using local coordinate representation, by maximum principle and Hopf lemma, the conclusion can be got easily. Here we omit the detail.

In this paper, besides intersection number $Z[\cdot, \cdot]$, we introduce a related notion $SGN[\cdot, \cdot]$(first used by [2]), which turns out to be exceedingly useful in classifying the types of the solutions.

Definition 2.6 For two curves γ_1 and γ_2 satisfying the same conditions in Definition 2.1, we define

(1) $Z[\gamma_1, \gamma_2]$ is the number of the intersections between curves γ_1 and γ_2. Noting that γ_1 and γ_2 have the same boundary points, then $Z[\gamma_1, \gamma_2] \geq 2$;
(2) $SGN[\gamma_1, \gamma_2]$ is defined when $Z[\gamma_1, \gamma_2] < \infty$. Denoting $n+1 := Z[\gamma_1, \gamma_2] < \infty$, let $P = P_0, P_1, \cdots, P_{n-1}, P_n = Q$ be the intersections. Here we assume $P_{i+1}P_0 > P_iP_0$ and $\widetilde{P_{i+1}P_0} > \widetilde{P_iP_0}$, $i = 1, \cdots, n$, where P_iP_j denotes the arc length of γ_1 between P_i and P_j; $\widetilde{P_iP_j}$ denotes the arc length of γ_2 between P_i and P_j. If $\gamma_1 \mid_{P_iP_{i-1}} \succeq \gamma_2 \mid_{\widetilde{P_iP_{i-1}}}$, we say the sign between P_i and P_{i-1} is "+"; Respectively, $\gamma_2 \mid_{\widetilde{P_iP_{i-1}}} \succeq \gamma_1 \mid_{P_iP_{i-1}}$, we say the sign between P_i and P_{i-1} is "−", $i = 1, \cdots, n$. Where $\gamma_1 \mid_{P_iP_{i-1}}$ and $\gamma_2 \mid_{\widetilde{P_iP_{i-1}}}$ denote the restriction between P_{i-1} and P_i.

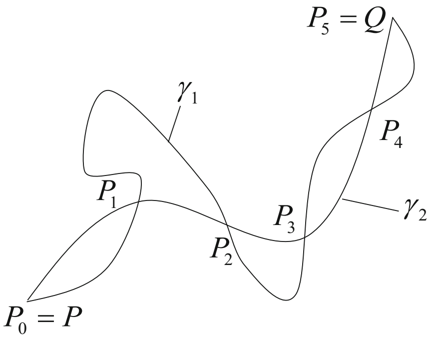

Fig. 7 Example for $SNG[\cdot,\cdot]$

$SGN[\gamma_1, \gamma_2]$ called *ordered word set* consists the sign between P_i and P_{i-1}, $i = 1, \cdots, n$.

For explaining Definition 2.6, we give an example. Considering Fig. 7, $Z[\gamma_1, \gamma_2] = 6$ and

$$SGN[\gamma_1, \gamma_2] = [- \; + \; - \; + \; -].$$

Let A and B be two ordered word sets, we write $A \triangleright B$, if B is a sub ordered word set of A. For example,

$$[+\,-] \triangleright B \text{ for } B = [+\,-],\ [+],\ [-],\ \text{but not } [+\,-] \triangleright [-\,+].$$

Remark 2.7 For the curve shortening flow ($A = 0$), we can deduce that for all $t_1 < t_2$,

$$Z[\gamma_1(t_2), \gamma_2(t_2)] \le Z[\gamma_1(t_1), \gamma_2(t_1)],\ SGN[\gamma(t_2), \gamma(t_2)] \triangleleft SGN[\gamma(t_1), \gamma(t_1)].$$

However, for the curve shortening flow with driving force ($A > 0$), even if $\gamma_1(t)$ and $\gamma_2(t)$ satisfy (1.4) and (1.6), we can not guarantee that for all $t_1 < t_2$,

$$Z[\gamma_1(t_2), \gamma_2(t_2)] \le Z[\gamma_1(t_1), \gamma_2(t_1)],\ SGN[\gamma(t_2), \gamma(t_2)] \triangleleft SGN[\gamma(t_1), \gamma(t_1)].$$

For giving the intersection number principle, we need assume $\gamma_1(t)$ and $\gamma_2(t)$ are homeomorphism to a curve.

Theorem 2.8 *(Intersection number principle) Let M be C^1 embedded curve with boundary points P, Q, V be a shuttle neighborhood of M and flow $\phi : M \times (-\delta, \delta) \to V$ satisfy that for every $z \in V$, there exist unique $z^* \in M$ and unique $\alpha_0 \in (-\delta, \delta)$ such that*

$$\phi(z^*, \alpha_0) = z.$$

Assume that two families of curves $\{\gamma_i(t)\}_{0 \le t \le T} \subset V$ satisfy (1.4) and (1.6) and there exist

$$u_i : M \times [0, T] \to \mathbb{R}$$

such that

$$\gamma_i(t) = \{\phi(z, u(z, t)) \mid z \in M\},\ 0 \le t \le T$$

for $i = 1, 2$. *Then one of the following conditions holds*

(1)

$$\gamma_1(t) \equiv \gamma_2(t)$$

for all $0 \le t \le T$*;*

(2) $Z[\gamma_1(t), \gamma_2(t)] < \infty$ *for all* $0 < t \le T$. *Moreover,*

$$Z[\gamma_1(t_2), \gamma_2(t_2)] \le Z[\gamma_1(t_1), \gamma_2(t_1)],\ SGN[\gamma(t_2), \gamma(t_2)] \lhd SGN[\gamma(t_1), \gamma(t_1)],$$

for all $0 < t_1 < t_2 \le T$.

Proof If $\gamma_1(t) \neq \gamma_2(t)$, by the basic parabolic theory, the intersections are discrete. Therefore, $Z[\gamma_1(t), \gamma_2(t)] < \infty$ for all $0 < t \le T$. It is necessary to prove that for any t_0, there exists ϵ_0 such that

$$Z[\gamma_1(t_2), \gamma_2(t_2)] \le Z[\gamma_1(t_1), \gamma_2(t_1)],\ SGN[\gamma(t_2), \gamma(t_2)] \lhd SGN[\gamma(t_1), \gamma(t_1)],$$

for all $t_0 - \epsilon_0 \le t_1 < t_2 \le t_0 + \epsilon_0$.

We can use the local coordinate to represent the two curves near the intersections and time t_0. Using the classical intersection number principle, for example, considering [2], we can prove this results easily. We omit the detail safely. □

Definition 2.9 For a C^1 curve γ and a sequence of C^1 curves γ_n with boundary points P, Q, we say $\gamma_n \to \gamma$ in C^1, if

(1) There exist a C^1 curve M with boundary points P, Q and maps

$$\varphi,\ \varphi_n : M \to \mathbb{R}^2$$

such that

$$\gamma = \{\varphi(z) \mid z \in M\},\ \gamma_n = \{\varphi_n(z) \mid z \in M\}.$$

(2)

$$\|\varphi_n - \varphi\|_{C^1(M \to \mathbb{R}^2)} \to 0,$$

as $n \to \infty$.

3 Time local existence and uniqueness of solution

In this section, we introduce the the transport map first used by [1] and prove Theorem 1.1.

Lemma 3.1 *For Γ_0 satisfying the assumption in Theorem 1.1, there exist a shuttle neighborhood V of Γ_0 and a vector field $X \in C^1(\overline{V} \to \mathbb{R}^2)$ such that*

$$X(z) \cdot N(z) < 0, \ z \in \Gamma_0$$

and in V, there holds

$$|X| \geq \delta > 0, \textit{for some } \delta > 0,$$

where N denotes the unit downward normal vector of Γ_0.

Proof We extend Γ_0 by Γ_0^* such that Γ_0^* is a C^2 curve and divide $\mathbb{R}^2$ into two connect parts Ω_1 and Ω_2. Assume $\Omega_1(\Omega_2)$ locates in the left(right) side of Γ_0^*("left side" and "right side" are defined as in Sect. 2).

Let $d(x)$ be the signed distance function defined as follows:

$$d(x) = d(x, \Omega_2) - d(x, \Omega_1), \ x \in \mathbb{R}^2.$$

Since Γ_0^* is C^2, as we know, there exists a tubular neighborhood U of Γ_0^* such that d is C^2 in U. Moreover, there exists a projection map P such that for all $z \in U$, there exists a unique point $z^* \in \Gamma_0^*$ such that

$$Pz = z^*$$

and $\nabla d(z) = \nabla d(z^*) = -N(z^*)$. We choose two curves $\Gamma_1, \Gamma_2 \subset U$ and $\Gamma_i \subset \Omega_i$, $i = 1, 2$, such that $\Gamma_1 \succ \Gamma_0 \succ \Gamma_2$. Let V be the domain satisfying $\Gamma_0 \subset V$, $\partial V = \Gamma_1 \cup \Gamma_2$, and $X(z) = \nabla d(z)$. Obviously

$$|X|(z) = 1, \ z \in V$$

and

$$X(z) \cdot N(z) = -1, \ z \in \Gamma_0.$$

□

Transport map Let $\phi : \Gamma_0 \times (-\delta, \delta) \to V$ be the map generated by vector field X, precisely,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\alpha}\phi(z, \alpha) = X(\phi), \ z \in \Gamma_0, \\ \phi(z, 0) = z, \ z \in \Gamma_0. \end{cases}$$

Recalling $\Gamma_0 = \{F_0(p) \mid 0 \leq s \leq L_0\}$ and $F_0 \in C^2([0, L_0] \to \mathbb{R}^2)$, let

$$\psi(p, \alpha) = \phi(F_0(p), \alpha).$$

Considering the assumption of F_0 and X, ψ_p, ψ_α, ψ_{pp}, $\psi_{p\alpha}$, $\psi_{\alpha\alpha}$ are all continuous vectors for $0 \leq p \leq L_0$, $-\delta < \alpha < \delta$.

If $\Gamma(t) \subset V$ is C^1 close to Γ_0 and satisfies (1.4), (1.6), $0 < t < T$ with initial data $\Gamma(0) = \Gamma_0$, then there exists a function $u(\cdot, t) : [0, L_0] \to \mathbb{R}$ such that

$$\Gamma(t) = \{\psi(p, u(p, t)) \mid 0 \leq p \leq L_0\}.$$

Moreover, u satisfies

$$\begin{cases} u_t = \dfrac{1}{|\psi_p + \psi_\alpha u_p|^2} u_{pp} + \dfrac{\det(\psi_p + \psi_\alpha u_p, \psi_{pp} + 2u_p\psi_{p\alpha} + \psi_{\alpha\alpha}u_p^2)}{\det(\psi_p, \psi_\alpha)|\psi_p + \psi_\alpha u_p|^2} \\ \qquad + A\dfrac{|\psi_{p\alpha} + \psi_\alpha u_p|}{\det(\psi_p, \psi_\alpha)}, \\ 0 < p < L_0, \ 0 < t < T, \\ u(0, t) = u(L_0, t) = 0, \ 0 \leq t < T, \\ u(p, 0) = 0, \ 0 \leq p \leq L_0, \end{cases} \tag{3.1}$$

where $\det(\cdot, \cdot)$ denotes the determinant. Indeed, the upward normal velocity and the curvature are given by

$$V = \frac{\det(\psi_p, \psi_\alpha)u_t}{|\psi_p + \psi_\alpha u_p|}$$

and

$$\kappa = \frac{\det(\psi_p, \psi_\alpha)}{|\psi_p + \psi_\alpha u_p|^3} u_{pp} + \frac{\det(\psi_p + \psi_\alpha u_p, \psi_{pp} + 2u_p\psi_{p\alpha} + \psi_{\alpha\alpha}u_p^2)}{|\psi_p + \psi_\alpha u_p|^3}.$$

For the computation, we can see, for example, [11].

Following Proposition 3.2 implies Theorem 1.1.

Proposition 3.2 *There exist $T_0 > 0$ and a unique $u \in C([0, L_0] \times [0, T_0)) \cap C^{2+\alpha, 1+\alpha/2}([0, L_0] \times (0, T_0))$ such that u satisfies (3.1) for $T = T_0$.*

Proof Since $\psi_p(p, 0) \cdot \psi_\alpha(p, 0) = 0$, $0 \leq p \leq L_0$, then

$$|\det(\psi_p, \psi_\alpha)|(p, 0) = 1, \ 0 \leq p \leq L_0.$$

There exist $\delta_1 > 0$ and α_0 such that for all $-\alpha_0 < \alpha < \alpha_0$ and $0 \leq p \leq L_0$,

$$|\det(\psi_p, \psi_\alpha)|(p, \alpha) > \delta_1.$$

By the quasilinear parabolic theory in [10], we can deduce that there exist T_0 and $u \in C([0, L_0] \times [0, T_0)) \cap C^{2+\alpha, 1+\alpha/2}([0, L_0] \times (0, T_0))$ such that u satisfies (3.1) and $|u| \leq \alpha_0$, $0 \leq t < T_0$. For the uniqueness, since ψ_p, ψ_α, ψ_{pp}, $\psi_{p\alpha}$, $\psi_{\alpha\alpha}$ are all continuous vectors for $0 \leq p \leq L_0$, $-\alpha_0 < \alpha < \alpha_0$, the uniqueness can be obtained easily. □

Proof of Theorem 1.1 Let T_0 and u be given by Proposition 3.2. Obviously, $\tilde{F}(p,t) = \psi(p, u(p,t))$ is the unique solution.

Let $\Gamma(t) = \{\tilde{F}(p,t) \mid 0 \leq p \leq L_0\}$, $0 \leq t < T_0$ and s be the arc length parameter on $\Gamma(t)$. Then $F(s,t) = \tilde{F}(p,t)$ satisfies (1.4), (1.5), (1.6). □

Remark 3.3 The assumption for initial curve can be weakened. In this paper, we assume $F_0 \in C^2([0, L_0] \to \mathbb{R}^2)$. Indeed, the initial curve can be assumed to be Lipschitz continuous. Recently, [11] has considered the curve-shortening flow with Lipschitz initial curve, under the Neumann boundary condition. Since the purpose of this paper is to get the three categories of solutions, we do not introduce this part in detail.

Lemma 3.4 *For $\Gamma(t)$ satisfying (1.4), (1.6), for $0 < t < T$, then the curvature $\kappa(s,t)$ satisfies*

$$\begin{cases} \kappa_t = \kappa_{ss} - \kappa\kappa_s^2 + \kappa^2(\kappa - A), \ 0 < s < L(t), \ 0 < t < T \\ \kappa(0,t) = A, \ \kappa(L(t),t) = A, 0 < t < T, \end{cases} \tag{3.2}$$

where κ_t denotes the derivative with respect to t by fixing s.

For the proof of the first equation, the calculation can be seen in [7]. Since at the boundary points, $\Gamma(t)$ does not move, the boundary condition is obvious.

Lemma 3.5 *For $\sigma > 0$, $\Gamma_\sigma(t)$ given in Theorem 1.2, let $F_\sigma(s,t)$ satisfy*

$$\Gamma_\sigma(t) = \{F_\sigma(s,t) \mid 0 \leq s \leq L_\sigma(t)\},$$

where $L_\sigma(t)$ is the length of $\Gamma_\sigma(t)$. If $\frac{\partial}{\partial s}F_\sigma(0,t) \cdot (0,1) > 0$, for all $0 \leq t \leq t_0$, then $t_0 < T_\sigma$.

This lemma gives a sufficient condition under which $\Gamma_\sigma(t)$ does not become singular. The assumption $\frac{\partial}{\partial s}F_\sigma(0,t) \cdot (0,1) > 0$ means that the y-component of the tangential vector $\frac{\partial}{\partial s}F_\sigma(0,t)$ is positive.

Proof Considering the choice of $\Gamma_\sigma(0)$, then $\kappa_\sigma(s,0) \geq 0$, $0 \leq s \leq L_\sigma(0)$, for $\sigma > 0$. Combining Lemma 3.4 and maximum principle, $\kappa_\sigma(s,t) > 0$, $0 < s < L(t)$, $0 < t < T_\sigma$.

If $T_\sigma = \infty$, the result is trivial. We assume $T_\sigma < \infty$.

We prove the result by contradiction, assuming $t_0 \geq T_\sigma$. We claim that every half-line given by

$$y = kx, \ y \geq 0, \ \text{or } x = 0, \ y \geq 0$$

intersects $\Gamma_\sigma(t)$ only once, $0 < t < T_\sigma$.

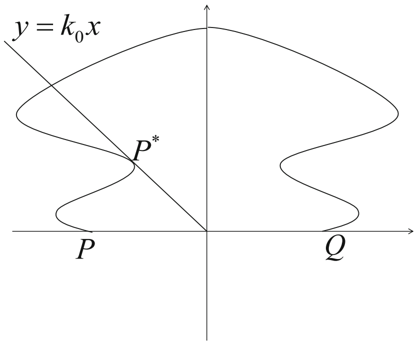

Fig. 8 Proof of claim

First, for all $0 < t < T_\sigma$, we prove $x = 0$, $y \geq 0$ intersects $\Gamma_\sigma(t)$ only once. If not, suppose that there exists $t_1 < T_\sigma$ such that $x = 0$, $y \geq 0$ intersects $\Gamma_\sigma(t_1)$ more than once. Since $\Gamma_\sigma(t)$ is symmetric about y-axis, it is easy to see that $\Gamma_\sigma(t)$ becomes singular at t_1. This contradicts to $t_1 < T_\sigma$.

Next, by contradiction, assume that there exist $t_2 < T_\sigma$ and $k < 0$ such that $y = kx,\ y \geq 0$ intersects $\Gamma_\sigma(t_2)$ more than once. Combining our assumption $\frac{\partial}{\partial s}F_\sigma(0, t) \cdot (0, 1) > 0$, we can choose k_0 satisfying $k_0 < k < 0$ such that half-line $y = k_0 x,\ y \geq 0$ intersects $\Gamma_\sigma(t_2)$ tangentially at some point P^*, and near P^*, $\Gamma_\sigma(t_2)$ is located under the half-line. It is easy to deduce that the curvature at P^*, $\kappa_\sigma(P^*, t_2) \leq 0$. This contradicts to that the curvatures on $\Gamma_\sigma(t)$ are all positive, $0 < t < T_\sigma$. Here we complete the proof of claim (Fig. 8).

Regarding $\frac{\partial}{\partial s}F_\sigma(0, t) \cdot (0, 1) > 0$ and the claim above, $\Gamma_\sigma(t) \subset \{(x, y) \mid y \geq 0\}$, $t < T_\sigma$. The claim implies that we can express $\Gamma_\sigma(t)$ by polar coordinate. For $(x, y) \in \Gamma_\sigma(t)$, let

$$\begin{cases} x = \rho_\sigma(\theta, t)\cos\theta, \\ y = \rho_\sigma(\theta, t)\sin\theta, \end{cases}$$

for $0 \leq \theta \leq \pi$, $0 \leq t < T_\sigma$. Consequently, ρ_σ satisfies

$$\begin{cases} \rho_t = \dfrac{\rho_{\theta\theta}}{\rho^2 + \rho_\theta^2} - \dfrac{2\rho_\theta^2 + \rho^2}{\rho(\rho_\theta^2 + \rho^2)} + \dfrac{1}{\rho}A\sqrt{\rho_\theta^2 + \rho^2},\ 0 < \theta < \pi,\ 0 < t < T_\sigma, \\ \rho(0, t) = a,\ \rho(\pi, t) = a,\ 0 \leq t < T_\sigma, \end{cases} \tag{3.3}$$

recalling $P = (-a, 0)$, $Q = (a, 0)$.

Since for $\sigma > 0$, $\Gamma_\sigma(0) \succ \Lambda_0 = \{(x, y) \mid y = 0,\ -a \leq x \leq a\}$. It is easy to see that Λ_0 is a subsolution of (1.4) and (1.6). By comparison principle, $\Gamma_\sigma(t) \succ \Lambda_0$ for $0 < t < T_\sigma$. Let

$$\Lambda_* = \{(x, y) \mid y = \sqrt{R^2 - x^2} - \sqrt{R^2 - a^2}\}.$$

For all $t_3 > 0$, we can choose sufficiently large $R > 1/A$ such that

$$\Gamma_\sigma(t_3) \succ \Lambda_*.$$

It is easy to check that Λ_* is a subsolution. Then

$$\Gamma_\sigma(t) \succ \Lambda_*,\ t_3 < t < T_\sigma.$$

This implies that there exists $\rho_1 > 0$ such that $\rho_\sigma(\theta, t) \geq \rho_1$ for $t_3 < t < T_\sigma$, $0 \leq \theta \leq \pi$. On the other hand, since $T_\sigma < \infty$, there exists $\rho_2 > 0$ such that $\rho_\sigma(\theta, t) \leq \rho_2$ for $0 < t < T_\sigma$, $0 \leq \theta \leq \pi$. Therefore, the quasilinear theory in [10] shows that for $\epsilon > 0$, there exists C_ϵ such that

$$\|\rho_\sigma(\cdot, t)\|_{C^2[0,\pi]} \leq C_\epsilon,\ t_3 + \epsilon < t < T_\sigma.$$

Therefore, the curvature of $\Gamma_\sigma(t)$ is uniformly bounded for t close to T_σ. This implies that the solution $\Gamma_\sigma(t)$ can be extended over time T_σ. This contradicts to that T_σ is the maximal existence time. □

Lemma 3.6 *[Continuous dependence on the initial curve] Assume ρ and ρ_n are the solutions of (3.3) for $0 \leq \theta \leq \pi$, $0 < t < T$. If ρ is bounded from below for some positive constant and*

$$\lim_{n\to\infty} \|\rho_n(\cdot, 0) - \rho(\cdot, 0)\|_{C^1[0,\pi]} = 0,$$

then for all $0 < t < T$,

$$\lim_{n\to\infty} \|\rho_n(\cdot, t) - \rho(\cdot, t)\|_{C^2[0,\pi]} = 0.$$

By our assumption that ρ is bounded from below for some positive constant, this lemma can be proven by Schauder estimate in [10] directly.

4 Behavior for σ sufficient small or large

Proposition 4.1 *There exists $\sigma_1 > 0$ such that, for all $\sigma > \sigma_1$, there exists some time $T_\sigma^* < T_\sigma$ such that $\Gamma_\sigma(t) \succ \Gamma^*$, for $T_\sigma^* < t < T_\sigma$*

For proving this proposition, we introduce the Grim reaper for the curve shortening flow. Grim reaper is given by

$$G(x,t) = C - \frac{t}{b} + b \ln \cos \frac{x}{b},\ -\frac{b\pi}{2} < x < \frac{b\pi}{2},\ t > 0,$$

where $b > 0$ and $C \in \mathbb{R}$. It is easy to see that $G(x, t)$ satisfies

$$G_t = \frac{G_{xx}}{1 + G_x^2}.$$

The Grim reaper $G(x,t)$ is a traveling wave moving downward with speed $1/b$.

Lemma 4.2 *If $b < 2a/\pi$, the curve*

$$\gamma_G(t) = \left\{(x,y) \mid y = \max\{G(x,t), 0\},\ |x| < \frac{b\pi}{2}\right\} \cup \left\{(x,y) \mid y = 0,\ \frac{b\pi}{2} \le |x| \le a\right\}$$

is a subsolution of (1.4) and (1.6) in the sense of Definition 2.4.

Proof When $0 < t < bC$, let $x(t) > 0$ be a point such that $G(x(t), t) = 0$.

For $|x| < x(t)$, $\gamma_G = \{(x,y) \mid y = G(x,t)\}$. Therefore,

$$G_t \le \frac{G_{xx}}{1+G_x^2} + A\sqrt{1+G_x^2},\ |x| < x(t).$$

For $x(t) < |x| < a$, $\gamma_G = \{(x,y) \mid y = 0\}$. Obviously, $y = 0$ is a subsolution of

$$u_t \le \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ x(t) < |x| < a.$$

At the point $x = x(t)(x = -x(t))$, it is impossible that for smooth flow $S(t)$, near $x = x(t)(x = -x(t))$, $S(t)$ touches $\gamma_G(t)$ at $x = x(t)(x = -x(t))$ only once from above.

Therefore, $\gamma_G(t)$ is a subsolution of (1.4) and (1.6), for $0 < t < bC$.

When $t \ge bC$, $\gamma_G = \Lambda_0 = \{(x,y) \mid y = 0,\ |x| \le a\}$(given in the proof of Lemma 3.5). Obviously, γ_G is a subsolution of (1.4) and (1.6), for $t \ge bC$. □

Following lemma gives the result for the classification of the solution $\Gamma_\sigma(t)$.

Lemma 4.3 *For $\Gamma_\sigma(t)$ given by Theorem 1.2, for $\sigma > 0$, $\Gamma_\sigma(t)$ satisfies one of the following four conditions:*

(1) $SGN(\Gamma_\sigma(t), \Gamma^*) = [-]$ for $t < T_\sigma$. Moreover, $T_\sigma = \infty$;
(2) There exists t_σ^* such that $SGN(\Gamma_\sigma(t), \Gamma^*) = [-\ +\ -]$ for $t < t_\sigma^*$ and $SGN(\Gamma_\sigma(t), \Gamma^*) = [-]$ for $t_\sigma^* < t < T_\sigma$. Moreover, $T_\sigma = \infty$;
(3) $SGN(\Gamma_\sigma(t), \Gamma^*) = [-\ +\ -]$ for $t < T_\sigma$. Moreover, $T_\sigma = \infty$;
(4) There exists $T_\sigma^* < T_\sigma$ such that $SGN(\Gamma_\sigma(t), \Gamma^*) = [-\ +\ -]$ for $t < T_\sigma^*$ and $SGN(\Gamma_\sigma(t), \Gamma^*) = [+]$, $T_\sigma^* < t < T_\sigma$.

Proof Considering the assumption in Theorem 1.2, there exists $\sigma_0 > 0$ such that

(a) $0 < \sigma \le \sigma_0$, $\Gamma^* \succeq \Gamma_\sigma$;
(b) $\sigma > \sigma_0$, Γ_σ intersects Γ^* fourth.

Step 1 For $0 < \sigma \le \sigma_0$, by comparison principle, $\Gamma^* \succ \Gamma_\sigma(t)$, for $0 < t < T_\sigma$. Noting $\sigma > 0$, $\Gamma_\sigma(t) \succ \Lambda_0$, for $0 < t < T_\sigma$. Therefore, $\frac{\partial}{\partial s} F_\sigma(0,t) \cdot (0,1) > 0$, $0 < t < T_\sigma$. By contradiction and the same method in the proof of Lemma 3.5, we can prove $T_\sigma = \infty$. Therefore, for $0 < \sigma \le \sigma_0$, condition (1) holds.

Step 2 For $\sigma > \sigma_0$, considering the choice of Γ_σ, $SGN(\Gamma_\sigma, \Gamma^*) = [- \ + \ -]$. Let τ_0 depending on σ satisfy

$$\tau_0 = \sup\{\tau \mid \frac{\partial}{\partial s} F_\sigma(0,t) \cdot (0,1) > 0, 0 < t < \tau\}.$$

Since $\frac{\partial}{\partial s} F_\sigma(0,0) \cdot (0,1) > 0$, we can deduce $\tau_0 > 0$. Therefore, Lemma 3.5 implies $T_\sigma > \tau_0$. Moreover, $\Gamma_\sigma(t)$ can be represented by polar coordinate, $0 < t < \tau_0$. This means that $\Gamma_\sigma(t)$ satisfies the assumption of Theorem 2.8 for $0 < t < \tau_0$. Then

$$SGN(\Gamma_\sigma(t), \Gamma^*) \lhd [- \ + \ -], \ 0 < t < \tau_0.$$

Considering the symmetry of $\Gamma_\sigma(t)$, then for $t < \tau_0$, one of the following three conditions holds

(i) $SGN(\Gamma_\sigma(t), \Gamma^*) = [+]$;
(ii) $SGN(\Gamma_\sigma(t), \Gamma^*) = [-]$;
(iii) $SGN(\Gamma_\sigma(t), \Gamma^*) = [- \ -]$.
(iv) $SGN(\Gamma_\sigma(t), \Gamma^*) = [- \ + \ -]$.

Step 3 If for some $t_\sigma^* < \tau_0$ (ii) or (iii) holds, then $\Gamma^* \succeq \Gamma_\sigma(t_\sigma^*)$. Then by comparison principle, $\Gamma^* \succ \Gamma(t) \succ \Lambda_0$, $t_\sigma^* < t < T_\sigma$. Therefore, by the same argument in Step 1, condition (2) holds.

If for some $T_\sigma^* < \tau_0$ (i) holds, this means that $\Gamma_\sigma(T_\sigma^*) \succ \Gamma^*$. By comparison principle, $\Gamma_\sigma(t) \succ \Gamma^*$, $T_\sigma^* < t < T_\sigma$. Therefore, condition (4) holds.

If for every $t < \tau_0$, there holds $SGN(\Gamma_\sigma(t), \Gamma^*) = [- + -]$. Combining $\Gamma_\sigma(t) \succ \Lambda_0$, $t < \tau_0$, there exists $\delta > 0$ such that

$$\frac{\partial}{\partial s} F_\sigma(0,t) \cdot (0,1) > \delta, \ t < \tau_0.$$

If $\tau_0 < \infty$, by the definition of τ_0, $\frac{\partial}{\partial s} F_\sigma(0,\tau_0) \cdot (0,1) = 0$. This yields a contradiction. Therefore, $\tau_0 = \infty$. Consequently, $T_\sigma = \infty$. Condition (3) holds.

We complete the proof. □

In the following, we consider two circles.

$$\partial B_1 = \left\{(x,y) \mid (x-R)^2 + \left(y - (1+R/a)\sqrt{1/A^2 - a^2}\right)^2 = R^2\right\}$$

and

$$\partial B_2 = \left\{(x,y) \mid (x-R)^2 + \left(y - (1+R/a)\sqrt{1/A^2 - a^2}\right)^2 = \frac{1}{A^2}(1+R/a)^2\right\},$$

where $R > 1/A$. We denote

$$(0, K) = \partial B_2 \cap \{x = 0 \mid y > 0\}.$$

It is easy to check that ∂B_2 intersects Γ^* tangentially at $(-a, 0)$. Let $R(t)$ be the solution of

$$R'(t) = A - 1/R(t) \tag{4.1}$$

with $R(0) = R$. Since $R(0) > 1/A$, $R(t)$ is increasing and $\lim_{t\to\infty} R(t) = \infty$. Noting $a \le 1/A$ and $(1 + R/a)/A > R$, there exists t^* such that $R(t^*) = (1 + R/a)/A$.

Lemma 4.4 *Let point* $(0, y_\sigma(t)) = \Gamma_\sigma(t) \cap \{(x, y) \mid x = 0\}$, $t < T_\sigma$. *There exists* σ_1*(indeed* σ_1 *in this lemma is the one we want to choose in Proposition 4.1) such that for all* $\sigma > \sigma_1$ *there holds that if* $t^* < T_\sigma$,

$$y_\sigma(t) > K,\ \ t < t^*.$$

We use the Grim reaper to prove this lemma.

Proof Considering that Grim reaper given by Lemma 4.2

$$G(x, t) = C - \frac{t}{b} + b \ln \cos \frac{x}{b}$$

is a traveling wave with uniform speed $1/b$, then choose C large enough such that $G(0, t) = C - t/b > C - t^*/b > K, t < t^*$.

We can choose σ_1 such that for all $\sigma > \sigma_1$, $\Gamma_\sigma \succ \gamma_G(0)$.

If $t^* < T_\sigma$, Lemma 4.2 implies that $\Gamma_\sigma(t) \succ \gamma_G(t)$, for $t < t^*$. This means that $y_\sigma(t) > K, t < t^*$. □

Proof of Proposition 4.1 Choose σ_1 as in Lemma 4.4.

Step 1. For $\sigma > \sigma_1$, if $T_\sigma \le t^*$(t^* is given in Lemma 4.4), this means that $T_\sigma < \infty$. By Lemma 4.3, only the condition (4) in Lemma 4.3 can hold. Consequently, the result is true.

Step 2. For $\sigma > \sigma_1$, if $t^* < T_\sigma$, by Lemma 4.4, $y_\sigma(t) > K, t < t^*$. Here we prove

$$\Gamma_\sigma(t) \succ \Gamma^*,\ \ t^* < t < T_\sigma.$$

Let

$$\partial B(t) = \left\{(x, y) \mid (x - R)^2 + \left(y - (1 + R/a)\sqrt{1/A^2 - a^2}\right)^2 = R(t)^2\right\},$$

where $R(t)$ is given by (4.1). It is easy to see that $\partial B(t)$ evolves by $V = -\kappa + A$.

Let $\Sigma(t) = \partial B(t) \cap \{(x, y) \mid x \le 0, y \ge 0\}$. There exists δ satisfying $0 < \delta < t^*$ such that

$$R(\delta) = \sqrt{R^2 + (1 + R/a)^2(1/A^2 - a^2)}.$$

Obviously, $\partial B(\delta)$ passes through the origin $(0, 0)$. As seen in the Figs. 9 and 10 and noting the choice of t^*, the boundary of $\Sigma(t)$ does not intersect $\Gamma_\sigma(t)$, $t < t^*$. By

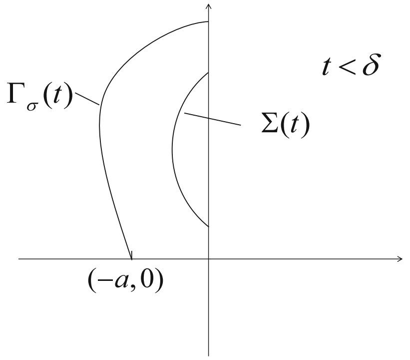

Fig. 9 Proof of Proposition 4.1

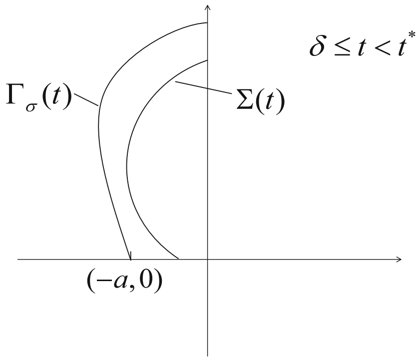

Fig. 10 Proof of Proposition 4.1

maximum principle, $\Sigma(t)$ cannot intersect $\Gamma_\sigma(t)$ interior. Therefore, $\Gamma_\sigma(t)$ does not intersect $\Sigma(t)$, $t < t^*$. Here we omit the detail.

As seen in the Fig. 11, $\Sigma(t^*)$ intersects Γ^* tangentially at $(-a, 0)$. Since $\Gamma_\sigma(t)$ does not intersect $\Sigma(t)$ for $t < t^*$, then $\Gamma(t^*) \succ \Gamma^*$. Therefore, $\Gamma(t) \succ \Gamma^*$ for $t^* < t < T_\sigma$. Let $T_\sigma^* = t^*$, we complete the proof. □

Proposition 4.5 *There exists $\sigma_2 > 0$ such that for all $\sigma < \sigma_2$, $T_\sigma = \infty$ and $\Gamma_\sigma(t) \to \Gamma_*$ in C^1, as $t \to \infty$.*

Proof **Step 1.** (Upper bound)

There exists σ_2 such that for all $\sigma < \sigma_2$, $\Gamma_* \succ \Gamma_\sigma$. Since Γ_σ is represented by the graph of $\sigma\varphi$, then $\Gamma_\sigma(t)$ can time locally be represented by the graph of some function $u_\sigma(x, t)$. Let T_σ^g be the maximal time such that

$$\Gamma_\sigma(t) = \{(x, y) \mid y = u_\sigma(x, t)\},\ 0 \le t < T_\sigma^g.$$

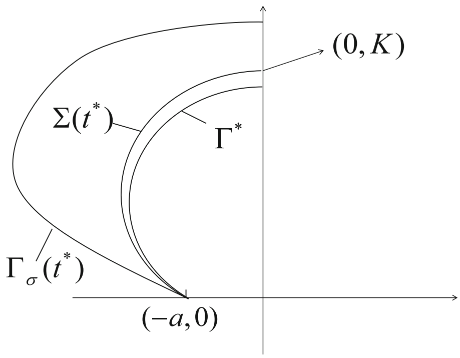

Fig. 11 Proof of Proposition 4.1

Therefore, $u_\sigma(x,t)$ satisfies

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \ -a < x < a, \ 0 < t < T_\sigma^g, \\ u(-a,t) = u(a,t) = 0, \ 0 < t < T_\sigma^g, \\ u(x,0) = \sigma\varphi(x), \ -a \le x \le a. \end{cases} \tag{4.2}$$

Since for all $\sigma < \sigma_2$ there holds $\sigma\varphi(x) \le \sqrt{1/A^2 - x^2} - \sqrt{1/A^2 - a^2}, -a \le x \le a$, by comparison principle, we have

$$u_\sigma(x,t) < \sqrt{1/A^2 - x^2} - \sqrt{1/A^2 - a^2}, \ -a < x < a, \ 0 < t < T_\sigma^g. \tag{4.3}$$

Step 2. Lower bound and derivative estimate.
If $0 \le \sigma < \sigma_2$, by comparison principle,

$$u_\sigma(x,t) > 0, \ -a < x < a, \ 0 < t < T_\sigma^g. \tag{4.4}$$

Combining (4.3) and (4.4),

$$-\frac{a}{\sqrt{1/A^2 - a^2}} < \frac{\partial}{\partial x}u_\sigma(a,t) < 0 \text{ and } 0 < \frac{\partial}{\partial x}u_\sigma(-a,t) < \frac{a}{\sqrt{1/A^2 - a^2}}, \ 0 < t < T_\sigma^g. \tag{4.5}$$

Differentiating the first equation in (4.2) by x and combining boundary condition (4.5), by maximum principle, we have

$$\left|\frac{\partial}{\partial x}u_\sigma(x,t)\right| < \frac{a}{\sqrt{1/A^2 - a^2}}, \ -a \le x \le a, \ 0 < t < T_\sigma^g. \tag{4.6}$$

If $\sigma < 0$, let $k > 0$ satisfy $k := \sigma\varphi'(a)$. We denote function

$$\underline{u}(x) = \max\{-k(x+a), k(x-a)\},\ -a \le x \le a.$$

Obviously, $\underline{u}(x) \le \sigma\varphi$, $-a \le x \le a$ and $\underline{u}$ is a subsolution of (4.2) in viscosity sense.

Therefore, by maximum principle, $u_\sigma(x,t) > \underline{u}(x)$, $-a < x < a$, $0 < t < T_\sigma^g$. Combining (4.3), we have

$$\left|\frac{\partial}{\partial x}u_\sigma(x,t)\right| < \max\left\{k, \frac{a}{\sqrt{1/A^2 - a^2}}\right\},\ -a \le x \le a,\ 0 < t < T_\sigma^g. \quad (4.7)$$

Consequently, (4.6) and (4.7) imply that there exists C_σ such that

$$\left|\frac{\partial}{\partial x}u_\sigma(x,t)\right| \le C_\sigma,\ -a \le x \le a,\ 0 < t < T_\sigma^g. \quad (4.8)$$

Step 3. We prove the convergence in this step.

By [10], for $\epsilon > 0$, $(u_\sigma)_{xx}(x,t)$ is bounded for all $-a \le x \le a$, $\epsilon \le t < T_\sigma^g$. This means that $T_\sigma^g = \infty$. Therefore, by [10] again, $u_\sigma(x,t)$, $(u_\sigma)_t(x,t)$, $(u_\sigma)_{tt}(x,t)$, $(u_\sigma)_x(x,t)$, $(u_\sigma)_{xx}(x,t)$ and $(u_\sigma)_{xxx}(x,t)$ are all bounded for some constant $D_\sigma > 0$, $-a \le x \le a$, $\epsilon \le t < \infty$. By Ascoli–Arzela Theorem, for any sequence $t_n \to \infty$, there exist a subsequence t_{n_j} and function $v(x,t)$ (v may depend on the choice of the subsequence. In Step 5, we will prove v is independent of the choice of the subsequence) such that

$$u_\sigma(\cdot, \cdot + t_{n_j}) \to v,\ \text{in } C^{2,1}([-a,a] \times [\epsilon, \infty)),$$

as $j \to \infty$.

Step 4. In this step, we introduce the Lyapunov function for (4.2).

Let

$$J[u] = \int_{-a}^{a} \sqrt{1 + u_x^2}dx.$$

If u is a solution of (4.2), we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J[u] &= \int_{-a}^{a} \frac{u_x u_{xt}}{\sqrt{1+u_x^2}}\mathrm{d}x = -\int_{-a}^{a} \frac{u_t u_{xx}}{(1+u_x^2)^{3/2}}\mathrm{d}x = -\int_{-a}^{a} \frac{(u_t)^2}{\sqrt{1+u_x^2}}\mathrm{d}x \\ &\quad + A\int_{-a}^{a} u_t \mathrm{d}x \\ &= -\int_{-a}^{a} \frac{(u_t)^2}{\sqrt{1+u_x^2}}\mathrm{d}x + A\frac{\mathrm{d}}{\mathrm{d}t}\int_{-a}^{a} u\mathrm{d}x.\end{aligned}$$

Therefore, there hold

$$J[u(\cdot,t)] \leq J[u(\cdot,\epsilon)] + A\int_{-a}^{a} u(x,t)\mathrm{d}x - A\int_{-a}^{a} u(x,\epsilon)\mathrm{d}x$$

and

$$\begin{aligned}\int_{\epsilon}^{\infty}\int_{-a}^{a}\frac{(u_t)^2}{\sqrt{1+u_x^2}}\mathrm{d}x\mathrm{d}t = A\lim_{t\to\infty}\int_{-a}^{a} u(x,t)\mathrm{d}x \\ -A\int_{-a}^{a} u(x,\epsilon)\mathrm{d}x + J[u(\cdot,\epsilon)] \\ -\lim_{t\to\infty} J[u(\cdot,t)].\end{aligned}$$

Step 5. Using the Lyapunov function, we complete the proof.

For u_σ given above, $u_\sigma(x,t)$ is uniformly bounded for $-a \leq x \leq a$, $0 < t < \infty$. Then, the integral

$$\left|\int_{-a}^{a} u_\sigma(x,t)\mathrm{d}x\right|$$

is bounded for $0 < t < \infty$. Consequently, $J[u_\sigma(\cdot,t)]$ is bounded for $0 < t < \infty$. Therefore, the integral

$$\int_{\epsilon}^{\infty}\int_{-a}^{a}\frac{(u_{\sigma t})^2}{\sqrt{1+u_{\sigma x}^2}}\mathrm{d}x\mathrm{d}t$$

is bounded. Then for all $s_0 > 0$,

$$\begin{aligned}&\int_{s_0}^{s_0+1}\int_{-a}^{a}\frac{(u_{\sigma t})^2}{\sqrt{1+u_{\sigma x}^2}}(x,t+t_{n_j})\mathrm{d}x\mathrm{d}t \\ &= \int_{s_0+t_{n_j}}^{s_0+1+t_{n_j}}\int_{-a}^{a}\frac{(u_{\sigma t})^2}{\sqrt{1+u_{\sigma x}^2}}(x,t)\mathrm{d}x\mathrm{d}t \to 0\end{aligned}$$

as $j \to \infty$. Considering $u_\sigma(\cdot,\cdot+t_{n_j}) \to v$, in $C^{2,1}([-a,a]\times[\epsilon,\infty))$, as $j \to \infty$, we have

$$\int_{s_0}^{s_0+1}\int_{-a}^{a}\frac{(v_t)^2}{\sqrt{1+v_x^2}}(x,t)\mathrm{d}x\mathrm{d}t = 0.$$

Then $v_t(x,t) = 0$, for all $-a \leq x \leq a$, $s_0 \leq t \leq s_0+1$. Considering that the choice of s_0 is arbitrary,

$$v_t(x,t) = 0,$$

for all $-a \le x \le a$, $0 < t < \infty$. Therefore, v is independent on t and is a stationary solution of (4.2). Then

$$v = \sqrt{1/A^2 - x^2} - \sqrt{1/A^2 - a^2},\ -a \le x \le a.$$

Here we get that for any sequence $t_n \to \infty$, there exists a subsequence t_{n_j} such that

$$u_\sigma(\cdot, \cdot + t_{n_j}) \to v,\ \text{in } C^{2,1}([-a, a] \times [\epsilon, \infty)),$$

as $j \to \infty$. Consequently,

$$u_\sigma(\cdot, t) \to v,\ \text{in } C^2([-a, a]),$$

as $t \to \infty$.

The proof of this proposition is completed. □

5 Asymptotic behavior for the condition (3) in Lemma 4.3

Considering Lemma 3.5 and the proof of Lemma 4.3, under the condition (3) in Lemma 4.3, we can assume there exists ρ_σ such that

$$\Gamma_\sigma(t) = \{(\rho_\sigma(\theta, t)\cos\theta, \rho_\sigma(\theta, t)\sin\theta) \mid 0 \le \theta \le \pi\},\ 0 \le t < \infty.$$

Moreover ρ_σ satisfies (3.3) for $T_\sigma = \infty$.

Lemma 5.1 *Let $L_\sigma(t)$ be the length of $\Gamma_\sigma(t)$ and $S_\sigma(t)$ be the area of the domain surrounded by $\Gamma_\sigma(t)$ and $y = 0$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\sigma(t) = -\int_0^{L_\sigma(t)} (\kappa - A)^2 \mathrm{d}s + A \frac{\mathrm{d}}{\mathrm{d}t} S_\sigma(t). \tag{5.1}$$

Remark 5.2 (1) Noting that under the condition (3) in Lemma 4.3, $\Gamma_\sigma(t)$ located in $\{y \ge 0\}$, the definition of $S_\sigma(t)$ is well defined.

(2) The result of this lemma is a general condition for the Lyapunov function in the proof of Proposition 4.5.

Proof Considering the calculation in [14],

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\sigma(t) = \int_0^{L_\sigma(t)} (A\kappa - \kappa^2) \mathrm{d}s.$$

Recall N being the unit downward normal vector. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_\sigma(t) &= -\int_0^{L_\sigma(t)}(\kappa - A)^2\mathrm{d}s + \int_0^{L_\sigma(t)}(-A\kappa + A^2)\mathrm{d}s \\ &= -\int_0^{L_\sigma(t)}(\kappa - A)^2\mathrm{d}s \\ &\quad +A\int_0^{L_\sigma(t)}\frac{\partial}{\partial t}F(s,t)\cdot(-N)\mathrm{d}s,\end{aligned}$$

where F is the point on the curve $\Gamma_\sigma(t)$ and for convenience, we omit the subscript of $F_\sigma(s,t)$. Let

$$\gamma_\sigma(t) = \Gamma_\sigma(t) \cup \{(x,y) \mid y = 0, -a \le x \le a\}.$$

By Green's formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}S_\sigma(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_\sigma(t)}F(s,t)\cdot(-N)\mathrm{d}s,$$

where F is the point on the curve $\gamma_\sigma(t)$ and N is the unit inner normal vector. Since the curve

$$\{(x,y) \mid y = 0, -a \le x \le a\}$$

is independent on t,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_\sigma(t)}F(s,t)\cdot(-N)\mathrm{d}s &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_\sigma(t)}F(s,t)\cdot(-N)\mathrm{d}s \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{L_\sigma(t)}F(s,t)\cdot(-N)\mathrm{d}s.\end{aligned}$$

Computing as in [14],

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{L_\sigma(t)}F(s,t)\cdot(-N)\mathrm{d}s &= \frac{1}{2}\int_0^{L_\sigma(t)}\frac{\partial}{\partial t}F(s,t)\cdot(-N)\mathrm{d}s \\ &\quad +\frac{1}{2}\int_0^{L_\sigma(t)}F(s,t)\cdot\left(-\frac{\partial}{\partial t}N\right)\mathrm{d}s \\ &\quad +\frac{1}{2}\int_0^{L_\sigma(t)}F(s,t)\cdot(-N)(A\kappa - \kappa^2)\mathrm{d}s.\end{aligned}$$

Considering the calculation in [14],

$$\frac{\partial}{\partial t}N = -\frac{\partial \kappa}{\partial s}T,$$

where $T = F_s$ is the unit tangential vector, then

$$\frac{1}{2}\int_0^{L_\sigma(t)} F(s,t)\cdot\left(-\frac{\partial}{\partial t}N\right)\mathrm{d}s = \frac{1}{2}\int_0^{L_\sigma(t)} F(s,t)\cdot\left(\frac{\partial \kappa}{\partial s}T\right)\mathrm{d}s := I.$$

Considering the symmetry of $\Gamma_\sigma(t)$ and $\kappa(0,t) = \kappa(L_\sigma(t),t) = A$, at the boundary,

$$\begin{aligned} F(0,t)\cdot T(0,t)\kappa(0,t) &= F(L_\sigma(t),t)\cdot T(L_\sigma(t),t)\kappa(L_\sigma(t),t) \\ &= AF(0,t)\cdot T(0,t). \end{aligned}$$

Integrating I by parts, there holds

$$\begin{aligned} I = -\frac{1}{2}\int_0^{L_\sigma(t)} F_s\cdot T\kappa\mathrm{d}s - \frac{1}{2}\int_0^{L_\sigma(t)} F\cdot T_s\kappa\mathrm{d}s = -\frac{1}{2}\int_0^{L_\sigma(t)}\kappa\mathrm{d}s \\ -\frac{1}{2}\int_0^{L_\sigma(t)} F\cdot F_{ss}\kappa\mathrm{d}s = -\frac{1}{2}\int_0^{L_\sigma(t)}\kappa\mathrm{d}s - \frac{1}{2}\int_0^{L_\sigma(t)} F\cdot N\kappa^2\mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_0^{L_\sigma(t)} F(s,t)\cdot(-N)\mathrm{d}s &= \frac{1}{2}\int_0^{L_\sigma(t)}\frac{\mathrm{d}}{\mathrm{d}t}F(s,t)\cdot(-N)\mathrm{d}s \\ &\quad -\frac{1}{2}\int_0^{L_\sigma(t)}\kappa\mathrm{d}s \\ &\quad -\frac{1}{2}\int_0^{L_\sigma(t)} AF(s,t)\cdot(\kappa N)\mathrm{d}s \\ &= \frac{1}{2}\int_0^{L_\sigma(t)}\frac{\partial}{\partial t}F(s,t)\cdot(-N)\mathrm{d}s \\ &\quad -\frac{1}{2}\int_0^{L_\sigma(t)}\kappa\mathrm{d}s \\ &\quad -\frac{1}{2}\int_0^{L_\sigma(t)} AF(s,t)\cdot(T_s)\mathrm{d}s \\ &= \frac{1}{2}\int_0^{L_\sigma(t)}\frac{\partial}{\partial t}F(s,t)\cdot(-N)\mathrm{d}s \\ &\quad +\frac{1}{2}\int_0^{L_\sigma(t)}(A-\kappa)\mathrm{d}s \\ &= \int_0^{L_\sigma(t)}\frac{\partial}{\partial t}F(s,t)\cdot(-N)\mathrm{d}s. \end{aligned}$$

In the last second equality, we use integral by parts. Therefore,

$$\int_0^{L_\sigma(t)}\frac{\partial}{\partial t}F(s,t)\cdot(-N)\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}S_\sigma(t).$$

Consequently, (5.1) holds. □

Lemma 5.3 *Under the condition (3) in Lemma 4.3,* $SGN(\Gamma_\sigma(t), \Gamma^*) = [- \; + \; -]$ *for* $t < \infty$*, there exist* $\rho_2 > \rho_1 > 0$ *such that*

$$\rho_1 < \rho_\sigma(\theta, t) < \rho_2,$$

for $0 \le \theta \le \pi$*,* $0 < t < \infty$*.*

Proof First, we prove $\rho_\sigma < \rho_2$. We prove this by contradiction, assuming ρ_σ is not bounded from above.

If $\rho_\sigma(\pi/2, t)$ is bounded for all t, we can easily prove that there exist some $0 < \theta_0 < \pi/2$ and t_0 such that $\kappa_\sigma(\theta_0, t_0) \le 0$. This contradicts to that $\kappa_\sigma(\theta, t) > 0$, for all $0 < \theta < \pi, t < \infty$.

Therefore, $\rho_\sigma(\pi/2, t)$ is not bounded. Assume for some t_0, $\rho_\sigma(\pi/2, t_0)$ is large enough. We can use the Grim reaper argument as in Proposition 4.1 to prove that $\Gamma_\sigma(t) \succ \Gamma^*$ in finite time. This contradicts to that $SGN(\Gamma_\sigma(t), \Gamma^*) = [- \; + \; -]$ for $t < \infty$. Therefore, there exists $\rho_2 > 0$ such that $\rho_\sigma(\theta, t) < \rho_2$, for all $0 < \theta < \pi$, $t < \infty$.

On the other hand, we note that $\Gamma_\sigma \succ \Lambda_0 = \{(x, y) \mid y = 0, \; -a \le x \le a\}$, $0 < t < \infty$. Then there exists $\rho_1 > 0$ such that $\rho_\sigma(\theta, t) > \rho_1, 0 < \theta < \pi, t < \infty$.

We complete the proof. □

Here we give the asymptotic behavior under the condition (3) in Lemma 4.3.

Proposition 5.4 *Under the condition (3) in Lemma 4.3,* $SGN(\Gamma_\sigma(t), \Gamma^*) = [- + -]$ *for* $t < \infty$*,*

$$\Gamma_\sigma(t) \to \Gamma^* \text{ in } C^1$$

as $t \to \infty$*.*

Proof Since $\rho_1 < \rho_\sigma < \rho_2$ and ρ_σ satisfies (3.3), then there exists $\epsilon > 0$ such that $\rho_{\sigma t}, \rho_{\sigma tt}, \rho_{\sigma\theta}, \rho_{\sigma\theta\theta}$ and $\rho_{\sigma\theta\theta\theta}$ are bounded for $0 \le \theta \le \pi, \epsilon \le t < \infty$. Therefore, for any $t_n \to \infty$, there exist a subsequence t_{n_j} and a function $r(\theta, t)$ such that

$$\rho_\sigma(\cdot, \cdot + t_{n_j}) \to r \text{ in } C^{2,1}([0, \pi] \times [\epsilon, \infty)),$$

as $j \to \infty$. Let

$$\gamma_r(t) = \{(x, y) \mid x = r(\theta, t)\cos\theta, \; y = r(\theta, t)\sin\theta, \; 0 \le \theta \le \pi\},$$

and the curvature $\kappa_\sigma(\cdot, t)$ be the curvature on $\Gamma_\sigma(t)$, $\kappa_r(\cdot, t)$ be the curvature on $\gamma_r(t)$. Therefore, $\kappa_\sigma(\cdot, \cdot + t_{n_j}) \to \kappa_r$ in $C([0, \pi] \times [\epsilon, \infty))$. Obviously, the length $L_\sigma(t)$ and the area $S_\sigma(t)$ are bounded. Using the same argument in Proposition 4.5 and the Lyapunov function in Lemma 5.1, we can deduce that $\kappa_r \equiv A$. Consequently, $r_t(\theta, t) = 0$ for all $0 \le \theta \le \pi$ and $t > 0$. Considering the curvature of γ_r is a positive constant, γ_r can only be a part of circle with radius $1/A$. Considering $r(0,t) = -a$ and $r(\pi, t) = a$, then $\gamma_r = \Gamma^*$ or $\gamma_r = \Gamma_*$. But for all $t > 0$,

$$\Gamma_\sigma(t + t_{n_j}) \to \gamma_r \text{ in } C^1 \text{(indeed, the convergence can be proved in} C^2\text{)},$$

as $j \to \infty$. If $\gamma_r = \Gamma_*$, then for t large enough, $\Gamma^* \succ \Gamma_\sigma(t)$. This yields a contradiction. Therefore, $\gamma_r = \Gamma^*$. Consequently,

$$\Gamma_\sigma(t) \to \Gamma^* \text{ in } C^1 \text{(indeed, the convergence can be proved in} C^2\text{)},$$

as $t \to \infty$.

Here we complete the proof. □

6 Proof of Theorem 1.2

Lemma 6.1 *The set*

$$B_* = \{\sigma \in \mathbb{R} \mid \Gamma_\sigma(t) \to \Gamma_* \textit{ in } C^1, \textit{ as } t \to \infty\}$$

is open and connect.

Proof Proposition 4.5 implies that $(-\infty, 0] \subset B_* \neq \emptyset$. In following argument, let $\sigma_1 > 0$ and $\sigma_1 \in B_*$.

(1) (Proof of the property connect) We claim that, for all $\sigma < \sigma_1$, $\Gamma_\sigma(t) \to \Gamma_*$ in C^1, as $t \to \infty$.

Since $\Gamma_{\sigma_1}(t) \to \Gamma_*$ in C^1, as $t \to \infty$, then there holds $\Gamma^* \succ \Gamma_{\sigma_1}(t)$ for t large enough. By comparison principle, $\Gamma_{\sigma_1}(t) \succ \Gamma_\sigma(t)$, $t < T_\sigma$. These imply that the condition (4) in Lemma 4.3 can not hold. Therefore, $T_\sigma = \infty$. By the same argument in Proposition 4.5, we can prove

$$\Gamma_\sigma(t) \to \Gamma_* \text{ in } C^1,$$

as $t \to \infty$. Here we prove that B_* is connect.

(2) (Proof of the property open) We are going to prove B_* is open. We only need prove that there exists $\epsilon_0 > 0$, $(\sigma_1, \sigma_1 + \epsilon_0) \subset B_*$. We divide this proof into two steps.

Step 1. Let

$$\tau_0(\sigma) = \max\{\tau \mid \frac{\partial}{\partial s} F_\sigma(0, t) \cdot (0, 1) > 0, 0 < t < \tau\}, \text{ for } \sigma > 0.$$

By comparison principle, we can prove that $\tau_0(\sigma)$ is non-increasing with respect to σ. let $\tau^* = \sup\{\tau_0(\sigma) \mid \sigma > \sigma_1\} = \lim_{\sigma \downarrow \sigma^1} \tau_0(\sigma)$. We claim that $\tau^* = \infty$.

For all $t < \tau^*$, there exists δ_0, for $\sigma \in (\sigma_1, \sigma_1 + \delta_0)$, $\tau_0(\sigma) > t$. Therefore, for $\sigma \in (\sigma_1, \sigma_1 + \delta_0)$, $\Gamma_\sigma(t)$ can be represented by polar coordinate and not become singular. By Lemma 3.6,

$$\Gamma_\sigma(t) \to \Gamma_{\sigma_1}(t), \text{ in } C^2$$

as $\sigma \to \sigma_1$. Considering that the condition (1) or (2) in Lemma 4.3 hold for $\Gamma_{\sigma_1}(t)$, we can prove that there exists $\delta > 0$ such that

$$\frac{\partial}{\partial s} F_{\sigma_1}(0,t) \cdot (0,1) > \delta, \ t < \infty.$$

Consequently,

$$\lim_{\sigma \downarrow \sigma^1} \frac{\partial}{\partial s} F_\sigma(0,t) \cdot (0,1) \geq \delta, \ t < \tau^*.$$

Therefore, $\tau^* = \infty$.

Step 2. We complete the proof.

We choose two curves γ_1 and γ_2 such that $\Gamma^* \succ \gamma_1 \succ \Gamma_* \succ \gamma_2$ and the domain V be the shuttle neighborhood of Γ_* satisfying $\partial V = \gamma_1 \cup \gamma_2$.

Since

$$\Gamma_{\sigma_1}(t) \to \Gamma_* \text{ in } C^1,$$

as $t \to \infty$, for t_0 large enough $\Gamma_{\sigma_1}(t_0) \subset V$. Considering the result in Step 1, for σ close to σ_1, $\Gamma_\sigma(t_0)$ can be represented by polar coordinate and not become singular. By Lemma 3.6,

$$\Gamma_\sigma(t_0) \to \Gamma_{\sigma_1}(t_0), \text{ in } C^1,$$

as $\sigma \to \sigma_1$. Then there exists ϵ_0 for $\sigma \in (\sigma_1, \sigma_1 + \epsilon_0)$, $\Gamma_\sigma(t_0) \subset V$. Using the Lyapunov function given by Lemma 5.1 and the same argument in Proposition 5.4, for all $\sigma \in (\sigma_1, \sigma_1 + \epsilon_0)$,

$$\Gamma_\sigma(t) \to \Gamma_* \text{ in } C^1,$$

as $t \to \infty$.

We complete the proof. □

Lemma 6.2 *The set*

$$B^* = \{\sigma \in \mathbb{R} \mid \textit{there exists } T_\sigma^* > 0 \textit{ such that } \Gamma_\sigma(t) \succ \Gamma^*, \ T_\sigma^* < t < T_\sigma\}$$

is open and connect.

Proof Propositions 4.1 and 4.5 show that $B^* \subset (0,\infty)$ is not empty. In the following argument, let $\sigma_2 > 0$ and $\sigma_2 \in B^*$. Then there exists $T_{\sigma_2}^* > 0$ such that

$$\Gamma_{\sigma_2}(t) \succ \Gamma^*, \ T_{\sigma_2}^* < t < T_{\sigma_2}.$$

(1) (Proof the property connect) We claim $(\sigma_2, \infty) \subset B^*$.

For $\sigma > \sigma_2$, if $T_\sigma < \infty$, then only the condition (4) in Lemma 4.3 can hold. The result is true for $T_\sigma < \infty$.

In the following argument, we assume $T_\sigma = \infty$, then by comparison principle,

$$\Gamma_\sigma(t) \succ \Gamma_{\sigma_2}(t) \succ \Gamma^*,\ T^*_{\sigma_2} < t < T_{\sigma_2}.$$

Here we complete the proof that B^* is connect.

(2) (Proof of the property open) We prove B^* is open. We only need to prove that there exists $\epsilon_0 > 0$ such that $(\sigma_2 - \epsilon_0, \sigma_2) \subset B^*$.

We can choose t_0 such that $\Gamma_{\sigma_2}(t_0) \succ \Gamma^*$ and

$$\frac{\partial}{\partial s} F_{\sigma_2}(0, t) \cdot (0, 1) > 0,\ 0 < t \leq t_0.$$

By Lemma 3.5 and comparison principle, it is easy to see that for all $0 < \sigma < \sigma_2$, $T_\sigma > t_0$. For σ close to σ_2, $\Gamma_\sigma(t)$ can be represented by polar coordinate for $0 < t \leq t_0$. By Lemma 3.6,

$$\Gamma_\sigma(t_0) \to \Gamma_{\sigma_2}(t_0),\ \text{in } C^1$$

as $\sigma \to \sigma_2$. Therefore, there exists $\epsilon_0 > 0$ such that for all $\sigma \in (\sigma_2 - \epsilon_0, \sigma_2)$, $\Gamma_\sigma(t_0) \succ \Gamma^*$. By the comparison principle, we can get the result easily.

We complete the proof. □

Corollary 6.3 *There exist* $0 < \sigma_* \leq \sigma^*$ *such that*

$$B^* = (\sigma^*, \infty)$$

and

$$B_* = (-\infty, \sigma_*).$$

Proof Let $\sigma^* = \inf B^*$ and $\sigma_* = \inf B_*$. Obviously, $\sigma_* \leq \sigma^*$.

Lemmas 6.1 and 6.2 imply that

$$B^* = (\sigma^*, \infty)$$

and

$$B_* = (-\infty, \sigma_*).$$

Proposition 4.5 shows that $\sigma_* > 0$. The proof is completed. □

Proposition 6.4 *If* $\Gamma_{\sigma_0}(t) \to \Gamma^*$ *in* C^1, *for some* σ_0, *as* $t \to \infty$, *then* $(\sigma_0, \infty) \subset B^*$.

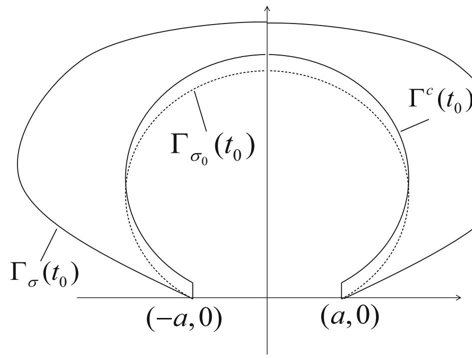

Fig. 12 Construction of $\Gamma_c(t)$

Proof For $\sigma > \sigma_0$, if $T_\sigma < \infty$, then the claim holds. In the following proof, we only consider $T_\sigma = \infty$. By comparison principle, $\Gamma_\sigma(t) \succ \Gamma_{\sigma_0}(t)$, $t > 0$.

Since

$$\Gamma_{\sigma_0}(t) \to \Gamma^* \text{ in } C^1,$$

there exist t_0 and $\delta > 0$ such that for all $t \geq t_0$, there hold

$$\frac{\partial}{\partial s} F_{\sigma_0}(0, t) \cdot (0, 1) \geq \delta \tag{6.1}$$

and

$$\frac{\partial}{\partial s} F_{\sigma_0}(0, t) \cdot (1, 0) \leq -\delta. \tag{6.2}$$

Since $\Gamma_\sigma(t_0) \succ \Gamma_{\sigma_0}(t_0)$, then we can choose a small positive constant c such that

$$\Gamma_\sigma(t_0) \succ \Gamma^c(t_0),$$

where

$$\begin{aligned}\Gamma^c(t) = \{(x, y + c) \mid (x, y) \in \Gamma_{\sigma_0}(t)\} \cup \{(-a, y) \mid 0 \leq y \leq c\} \\ \cup \{(a, y) \mid 0 \leq y \leq c\},\ t > 0.\end{aligned}$$

We claim that $\Gamma^c(t)$ is a subsolution for $t \geq t_0$. Indeed, the part $\Gamma^c(t) \cap \{y > c\}$ is a translation of $\Gamma_{\sigma_0}(t)$. $\Gamma^c(t) \cap \{y > c\}$ satisfies (1.4). Since the part $\Gamma^c(t) \cap \{y < c\}$ consists of two straight lines, then the part is a subsolution of (1.4). Next at the points $\{(-a, c), (a, c)\} = \Gamma^c(t) \cap \{y = c\}$, considering (6.1), (6.2) for any smooth flow $S(t)$ can not touch $(-a, c)$ or (a, c) above only once. Therefore, $\Gamma^c(t)$ is a subsolution of (1.4) and (1.6) in the sense of Definition 2.4 (Fig. 12).

By $\Gamma_\sigma(t_0) \succ \Gamma^c(t_0)$ and $\Gamma^c(t)$ being subsolution for $t > t_0$,

$$\Gamma_\sigma(t) \succ \Gamma^c(t), \ t > t_0. \tag{6.3}$$

Note that $\Gamma^c(t) \to \Gamma^{*c}$ in C, as $t \to \infty$, where

$$\begin{aligned}\Gamma^{*c} = \{(x, y+c) \mid (x, y) \in \Gamma^*\} \cup \{(-a, y) \mid 0 \le y \le c\} \\ \cup\{(a, y) \mid 0 \le y \le c\}.\end{aligned}$$

If $\Gamma_\sigma(t)$ satisfies the condition (3) in Lemma 4.3, by Proposition 5.4, $\Gamma_\sigma(t) \to \Gamma^*$. Combining (6.3), we get $\Gamma^* \succeq \Gamma^{*c}$. It is impossible.

If $\Gamma_\sigma(t)$ satisfies the condition (1) or (2) in Lemma 4.3, by the same argument in Proposition 5.4, we can prove the derivative of Γ_σ are all bounded. Using Ascoli-Arzela Theorem and Lyapunov function as in Proposition 5.4, we can get $\Gamma_\sigma(t) \to \Gamma_*$, as $t \to \infty$. Combining (6.3), $\Gamma_* \succeq \Gamma^{*c}$. But they are also impossible. Therefore, only the condition (4) in Lemma 4.3 holds.

The proof is completed. □

Proof of Theorem 1.2 Let σ_* and σ^* be given by Corollary 6.3.

Considering the definition of σ_*, $\sigma_* \notin B^*$ and $\sigma_* \notin B_*$. Therefore, $\Gamma_{\sigma_*}(t)$ only satisfies the condition (3) in Lemma 4.3. The result in Sect. 5 shows that $\Gamma_{\sigma_*}(t) \to \Gamma^*$, as $t \to \infty$.

By Proposition 6.4, $(\sigma_*, \infty) = B^*$. Consequently, $\sigma_* = \sigma^*$.

The proof of Theorem 1.2 is completed. □

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Compliance with ethical standards

Conflict of interest We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

References

1. Angenent, S.B.: Parabolic equaitons for curves on surfaces-partII. Ann. Math. **113**, 171–215 (1991)
2. Ducrot, A., Giletti, T., Matano, H.: Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations. Trans. Am. Math. Soc. **366**, 5541–5566 (2014)
3. Forcadel, N., Imbert, C., Monneau, R.: Uniqueness and existence of spirals moving by forced mean curvature motion. Interfaces Free Bound. **14**, 365–400 (2012)
4. Gage, M., Hamilton, R.S.: The heat equation shrinking convex plane curves. J. Diff. Geom. **23**, 69–96 (1986)
5. Giga, Y., Ishimura, N., Kohsaka, Y.: Spiral solutions for a weakly anisotropic curvature flow equation. Adv. Math. Sci. Appl. **12**, 393–408 (2002)
6. Goto, S., Nakagawa, M., Ohtsuka, T.: Uniqueness and existence of generalized motion for spiral crystal growth. Indiana Univ. Math. J. **57**, 2571–2599 (2008)

7. Guo, J.S., Matano, H., Shimojo, M., Wu, C.H.: On a free boundary problem for the curvature flow with driving force. Arch. Ration. Mech. Anal. **Volume 219**(Issue 3), 1207–1272 (2016)
8. Grayson, M.: The heat equation shrinks emmbedded plane curves to round points. J. Diff. Geom. **26**, 285–314 (1987)
9. Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Diff. Geom. **31**, 285–299 (1990)
10. Ladyzhenskaya, O.A., Solonnikov, V., Ural'ceva, N.: Linear and quasilinear equations of parabolic type (Translations of Mathematical Monographs). AMS, Providence (1968)
11. Mori, R.: A free boundary problem for a curve-shortening flow with Lipschitz initial data (to appear)
12. Ohtsuka, T.: A level set method for spiral crystal growth. Adv. Math. Sci. Appl. **13**, 225–248 (2003)
13. Ohtsuka, T., Tsai, Y.-H.R., Giga, Y.: A level set approac reflecting sheet structure with single auxiliary function for evolving spirals on crystal surface. J. Sci. Comput. **62**, 831–874 (2015)
14. Zhang, L.J.: Asymptotic behavior for curvature flow with driving force when curvature blowing up. To appear in Adv. Math. Sci. Appl