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Several Special Complex Structures and Their Deformation Properties

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Abstract We introduce a natural map from the space of pure-type complex differential forms on a complex manifold to the corresponding one on the infinitesimal deformations of this complex manifold. By use of this map, we generalize an extension formula in a recent work of K. Liu, X. Yang, and the first author. As direct corollaries, we prove several deformation invariance theorems for Hodge numbers. Moreover, we also study the Gauduchon cone and its relation with the balanced cone in the Kähler case, and show that the limit of the Gauduchon cone in the sense of D. Popovici for a generic fiber in a Kählerian family is contained in the closure of the Gauduchon cone for this fiber.

Keywords Deformations of complex structures · Deformations and infinitesimal methods · Formal methods · Deformations · Hermitian and Kählerian manifolds

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1 Introduction

We introduce an extension map from the space of complex differential forms on a complex manifold to the corresponding one on the infinitesimal deformations of the complex manifold and generalize an extension formula in [33] with more complete deformation significance. As direct corollaries, we prove several deformation invariance theorems for Hodge numbers in sufficiently general situations by a power series approach, which is analogously used to reprove the classical Kodaira–Spencer's local stability of Kähler structures in a recent paper [46]. We will also study the Gauduchon cone and its relation with the balanced one in the Kähler case, to explore the deformation properties on the Gauduchon cone of an **sGG** manifold introduced by D. Popovici [41]. We are much motivated by Popovici's remarkable work on [40, Conjecture 1.1], which confirms that if the central fiber X_0 of a holomorphic family of complex manifolds admits the deformation invariance of (0, 1)-type Hodge numbers or a so-called strongly Gauduchon metric and the generic fiber X_t $(t \neq 0)$ of this family is projective, then X_0 is Moishezon.

We will mostly follow the notations in [33]. All manifolds in this paper are assumed to be n-dimensional compact complex manifolds. A *Beltrami differential* is an element in $A^{0,1}(X, T_X^{1,0})$, where $T_X^{1,0}$ denotes the holomorphic tangent bundle of X. Then i_ϕ or $\phi\lrcorner$ denotes the contraction operator with $\phi \in A^{0,1}(X, T_X^{1,0})$ alternatively if there is no confusion. We also follow the convention

$$e^{\spadesuit} = \sum_{k=0}^{\infty} \frac{1}{k!} \spadesuit^k, \tag{1.1}$$

where $\spadesuit^k$ denotes k-time action of the operator $\spadesuit$. Since the dimension of X is finite, the summation in the above formulation is always finite.

Consider the smooth family $\pi : \mathcal{X} \to B$ of n-dimensional complex manifolds over a small domain B in $\mathbb{R}^k$ as in Definition 2.1, with the central fiber $X_0 := \pi^{-1}(0)$ and the general fibers $X_t := \pi^{-1}(t)$. Set $k = 1$ for simplicity. Denote by $\zeta := (\zeta_j^\alpha(z, t))_{\alpha=1}^n$ the holomorphic coordinates of X_t induced by the family with the holomorphic coordinates $z := (z^i)_{i=1}^n$ of X_0, under a coordinate covering $\{\mathcal{U}_j\}$ of $\mathcal{X}$, when t is assumed to be fixed. Suppose that this family induces the integrable Beltrami differential $\varphi(z, t)$, which is denoted by $\varphi(t)$ and φ interchangeably. These are reviewed at the beginning of Sect. 2. Then we have the following crucial calculation:

Lemma 1.1 (=Lemma 2.4)

$$\begin{pmatrix} \frac{\partial z}{\partial \zeta} & \frac{\partial z}{\partial \bar\zeta} \\ \frac{\partial \bar z}{\partial \zeta} & \frac{\partial \bar z}{\partial \bar\zeta} \end{pmatrix} = \begin{pmatrix} (\mathbb{1} - \varphi\overline{\varphi})^{-1} \left(\frac{\partial \zeta}{\partial z}\right)^{-1} & -\varphi \left(\mathbb{1} - \overline{\varphi}\varphi\right)^{-1} \left(\overline{\frac{\partial \zeta}{\partial z}}\right)^{-1} \\ -\left(\mathbb{1} - \overline{\varphi}\varphi\right)^{-1} \overline{\varphi} \left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \left(\overline{\mathbb{1} - \varphi\overline{\varphi}}\right)^{-1} \left(\overline{\frac{\partial \zeta}{\partial z}}\right)^{-1} \end{pmatrix},$$

where $\varphi\overline{\varphi}$, $\overline{\varphi}\varphi$ stand for the two matrices $(\varphi_{\bar k}^i \overline{\varphi_{\bar j}^k})_{\substack{1\le i\le n \\ 1\le j\le n}}$, $(\overline{\varphi_{\bar k}^i} \varphi_{\bar j}^k)_{\substack{1\le i\le n \\ 1\le j\le n}}$, respectively, and $\mathbb{1}$ is the identity matrix.

Using this calculation and its corollaries, we are able to reprove an important result (Proposition 2.7) in deformation theory of complex structures, which asserts that the holomorphic structure on X_t is determined by $\varphi(t)$. Actually, we obtain that for a differentiable function f defined on an open subset of X_0

$$\overline{\partial}_t f = e^{i_{\overline{\varphi}}} \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \lrcorner (\overline{\partial} - \varphi \lrcorner \partial) f \right),$$

where the differential operator d is decomposed as $d = \partial_t + \overline{\partial}_t$ with respect to the holomorphic structure on X_t and $e^{i_{\overline{\varphi}}}$ follows the notation (1.1).

Motivated by the new proof of Proposition 2.7, we introduce a map

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}} : A^{p,q}(X_0) \to A^{p,q}(X_t),$$

which plays an important role in this paper and is given in Definition 2.8. This map is a real linear isomorphism as t is arbitrarily small. Based on this, we achieve:

Proposition 1.2 (=Proposition 2.13) *For any* $\alpha \in A^{*,*}(X_0)$,

$$\bar{\partial}_t \left(e^{i_\varphi | i_{\bar{\varphi}}}(\alpha) \right) = 0$$

amounts to

$$\left([\partial, i_\varphi] + \bar{\partial} \right) (\mathbb{1} - \bar{\varphi}\varphi) \,\exists\, \alpha = 0,$$

where '$\exists$' is the simultaneous contraction introduced in Sect. 2.2.

This proposition provides a criterion for a specific $\overline{\partial}$-extension from $A^{p,q}(X_0)$ to $A^{p,q}(X_t)$ and generalizes [33, Theorem 3.4] (or Proposition 2.3) in deformation significance. As a direct application of Proposition 1.2, we consider the deformation invariance of Hodge numbers. Before stating the main theorems in Sect. 3, we recall several definitions of related cohomology groups and mappings.

Let X be a compact complex manifold of complex dimension n with the following commutative diagram

$$\begin{array}{ccccc} & & H^{p,q}_{\partial}(X) & & \\ & \overset{\iota^{p,q}_{BC,\partial}}{\nearrow} & & \overset{\iota^{p,q}_{\partial,A}}{\searrow} & \\ H^{p,q}_{BC}(X) & & \xrightarrow{\quad \iota^{p,q}_{BC,A} \quad} & & H^{p,q}_{A}(X) \\ & \underset{\iota^{p,q}_{BC,\bar{\partial}}}{\searrow} & & \underset{\iota^{p,q}_{\bar{\partial},A}}{\nearrow} & \\ & & H^{p,q}_{\bar{\partial}}(X) & & \end{array} .$$

Dolbeault cohomology groups $H^{\bullet,\bullet}_{\bar\partial}(X)$ of X are defined by

$$H^{\bullet,\bullet}_{\bar\partial}(X) := \frac{\ker \bar\partial}{\operatorname{im} \bar\partial},$$

with $H^{\bullet,\bullet}_{\partial}(X)$ similarly defined, while Bott–Chern and Aeppli cohomology groups are defined as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker \partial \cap \ker \bar\partial}{\operatorname{im} \partial\bar\partial} \quad \text{and} \quad H^{\bullet,\bullet}_{A}(X) := \frac{\ker \partial\bar\partial}{\operatorname{im} \partial + \operatorname{im} \bar\partial},$$

respectively. The dimensions of $H^{p,q}_{\bar\partial}(X)$, $H^{p,q}_{BC}(X)$, $H^{p,q}_{A}(X)$, and $H^{p,q}_{\partial}(X)$ over $\mathbb{C}$ are denoted by $h^{p,q}_{\bar\partial}(X)$, $h^{p,q}_{BC}(X)$, $h^{p,q}_{A}(X)$, and $h^{p,q}_{\partial}(X)$, respectively; the first three of which are usually called (p,q)-*Hodge numbers*, *Bott–Chern numbers*, and *Aeppli numbers*. From the very definition of these cohomology groups, the following equalities clearly hold

$$h^{p,q}_{BC} = h^{q,p}_{BC} = h^{n-q,n-p}_{A} = h^{n-p,n-q}_{A}, h^{n-p,n-q}_{\bar\partial} = h^{p,q}_{\bar\partial} = h^{q,p}_{\partial} = h^{n-q,n-p}_{\partial}.$$

Now let us describe our basic philosophy to consider the deformation invariance of Hodge numbers briefly. The Kodaira–Spencer's upper semi-continuity theorem ([28, Theorem 4]) tells us that the function

$$t \longmapsto h^{p,q}_{\bar\partial_t}(X_t) = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial_t}(X_t, \mathbb{C})$$

is always upper semi-continuous for $t \in B$ and thus, to approach the deformation invariance of $h^{p,q}_{\bar\partial_t}(X_t)$, we only need to obtain the lower semi-continuity. Here our main strategy is a modified iteration procedure, originally from [34] and developed in [33,52,53,63], which is to look for an injective extension map from $H^{p,q}_{\bar\partial}(X_0)$ to $H^{p,q}_{\bar\partial_t}(X_t)$. More precisely, for a nice uniquely chosen representative σ_0 of the initial Dolbeault cohomology class in $H^{p,q}_{\bar\partial}(X_0)$, we try to construct a convergent power series

$$\sigma_t = \sigma_0 + \sum_{j+k=1}^{\infty} t^k \bar{t}^j \sigma_{k\bar{j}} \in A^{p,q}(X_0),$$

with σ_t varying smoothly on t such that for each small t:

(1) $e^{i_\varphi | i_{\bar\varphi}}(\sigma_t) \in A^{p,q}(X_t)$ is $\bar\partial_t$-closed with respect to the holomorphic structure on X_t;
(2) The extension map $H^{p,q}_{\bar\partial}(X_0) \to H^{p,q}_{\bar\partial_t}(X_t) : [\sigma_0]_{\bar\partial} \mapsto [e^{i_\varphi | i_{\bar\varphi}}(\sigma_t)]_{\bar\partial_t}$ is injective.

One main theorem in Sect. 3 can be stated as

Theorem 1.3 (=Theorem 3.1) *If the injectivity of the mappings* $\iota^{p+1,q}_{BC,\partial}$, $\iota^{p,q+1}_{\bar\partial,A}$ *on the central fiber* X_0 *and the deformation invariance of the* $(p,q-1)$*-Hodge number* $h^{p,q-1}_{\bar\partial_t}(X_t)$ *holds, then* $h^{p,q}_{\bar\partial_t}(X_t)$ *are deformation invariant.*

Obviously, a classical result that a complex manifold satisfying the $\partial\bar\partial$-lemma admits the deformation invariance of all-type Hodge numbers follows by this theorem and induction. Three examples 3.2, 3.3, and 3.4 in the Kuranishi family of the Iwasawa manifold (cf. [3, Appendix]) are found that the deformation invariance of the (p,q)-Hodge number fails when one of the three conditions in Theorem 1.3 does not hold, while the other two do. It indicates that the three conditions above may not be omitted in order to state a theorem for the deformation invariance of all the (p,q)-Hodge numbers. We also refer the readers to [61] (based on [24]) for the negative counterpart of invariance of Hodge numbers.

The speciality of the types may lead to the weakening of the conditions in Theorem 1.3, such as $(p,0)$ and $(0,q)$:

Theorem 1.4 (=Theorems 3.6 + 3.7)

(1) *If the injectivity of the mappings* $\iota^{p+1,0}_{\bar\partial,A}$ *and* $\iota^{p,1}_{\bar\partial,A}$ *on* X_0 *holds, then* $h^{p,0}_{\bar\partial_t}(X_t)$ *are independent of* t*;*
(2) *If the surjectivity of the mapping* $\iota^{0,q}_{BC,\bar\partial}$ *on* X_0 *and the deformation invariance of* $h^{0,q-1}_{\bar\partial_t}(X_t)$ *holds, then* $h^{0,q}_{\bar\partial_t}(X_t)$ *are independent of* t.

As mentioned in Remark 3.8, for the case $q=1$ of Theorem 1.4.(2), the surjectivity of the mapping $\iota^{0,1}_{BC,\bar\partial}$ is equivalent to the **sGG** condition proposed by Popovici–Ugarte [41,45], from [45, Theorem 2.1 (iii)]. Hence, the **sGG** manifolds can be examples of Theorem 3.7, where the Frölicher spectral sequence does not necessarily degenerate at the E_1-level, by [45, Proposition 6.3]. Inspired by the deformation invariance of the (0, 1), (0, 2), and (0, 3)-Hodge numbers of the Iwasawa manifold $\mathbb{I}_3$ shown in [3, Appendix], we prove

Corollary 1.5 (=Corollary 3.9) *Let* $X=\Gamma\backslash G$ *be a complex parallelizable nilmanifold of complex dimension* n*, where* G *is a simply connected complex nilpotent Lie group and* Γ *is denoted by a discrete and co-compact subgroup of* G*. Then* X *is an* ***sGG*** *manifold. In addition, the* $(0,q)$*-Hodge numbers of* X *are deformation invariant for* $1\le q\le n$.

Inspired by Console–Fino–Poon [14, Sect. 6], we use the proof of Theorem 1.4.(1) to give in Example 3.11 a holomorphic family of nilmanifolds of complex dimension 5 with the central fiber endowed with an abelian complex structure, which admits the deformation invariance of the $(p,0)$-Hodge numbers for $1\le p\le 5$, but not the (1, 1)-Hodge number or (1, 1)-Bott–Chern number. This shows the function of Theorem 1.4.(1) possibly beyond Kodaira–Spencer's squeeze [28, Theorem 13] in this case.

Here is an interesting question:

Question 1.6 *What are the sufficient and necessary conditions for a class of compact complex manifolds to satisfy the deformation invariance for each prescribed-type Hodge number and all-type Hodge numbers?*

In Sect. 4, we will study various cones to explore the deformation properties of **sGG** manifolds. Here are several notations. The Kähler cone $\mathcal{K}_X$ and its closure $\overline{\mathcal{K}}_X$, the numerically effective cone (shortly nef cone), are important geometric objects on a compact Kähler manifold X, extensively studied such as in [9,15–17,22,41,45,58]. Fu and Xiao [22] study the relation between the balanced cone $\mathcal{B}_X$ and the Kähler cone $\mathcal{K}_X$. Meanwhile, Popovici [41], together with Ugarte [45], investigates geometric properties of the Gauduchon cone $\mathcal{G}_X$ and its related cones. The *Gauduchon cone* $\mathcal{G}_X$ is defined by

$$\mathcal{G}_X = \left\{ \left[\Omega\right]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R}) \;\middle|\; \Omega \text{ is a } \partial\bar{\partial}\text{-closed positive } (n-1,n-1)\text{-form} \right\}.$$

More detailed descriptions of real Bott–Chern groups $H_{\mathrm{BC}}^{p,p}(X,\mathbb{R})$, Aeppli groups $H_{\mathrm{A}}^{p,p}(X,\mathbb{R})$, and these cones will appear at the beginning of Sect. 4.

Inspired by all these, we hope to understand the relation of the balanced cone $\mathcal{B}_X$ and the Gauduchon cone $\mathcal{G}_X$ via the mapping $\mathscr{J} : H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R}) \to H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R})$ induced by the identity map. Another direct motivation of this part is the following conjecture:

Conjecture 1.7 ([44, Conjecture 6.1]) *Each compact complex manifold X satisfying the $\partial\bar{\partial}$-lemma admits a balanced metric.*

One possible approach is to prove $\mathscr{J}^{-1}(\mathcal{G}_X) = \mathcal{B}_X$, since the Gauduchon cone of a compact complex manifold is never empty and $\mathscr{J}$ is an isomorphism from the $\partial\bar{\partial}$-lemma. See the important argument in [44, Sect. 6] or [12, Sect. 2] relating a slightly different conjecture with the quantitative part of Transcendental Morse Inequalities Conjecture for differences of two nef classes as in [9, Conjecture 10.1.(ii)] and (more precisely) also their main Conjecture 1.10.

A weaker question comes up:

Question 1.8 *Does the mapping $\mathscr{J}$ map the balanced cone $\mathcal{B}_X$ bijectively onto the Gauduchon cone $\mathcal{G}_X$ on the Kähler manifold X?*

It is clear that $\mathscr{J}$ maps $\mathcal{B}_X$ injectively into $\mathcal{G}_X$ from the $\partial\bar{\partial}$-lemma of Kähler manifolds. The affirmation of this question is equivalent to the equality

$$\mathcal{E}_X = \mathscr{L}^{-1}(\mathcal{E}_{\partial\bar{\partial}}) \tag{1.2}$$

by Proposition 4.13. The *pseudo-effective cone* $\mathcal{E}_X$ is generated by Bott–Chern classes in $H_{\mathrm{BC}}^{1,1}(X,\mathbb{R})$ represented by d-closed positive $(1,1)$-currents and the convex cone $\mathcal{E}_{\partial\bar{\partial}} \subseteq H_{\mathrm{A}}^{1,1}(X,\mathbb{R})$ is generated by Aeppli classes represented by $\partial\bar{\partial}$-closed positive $(1,1)$-currents, with the natural isomorphism $\mathscr{L} : H_{\mathrm{BC}}^{1,1}(X,\mathbb{R}) \to H_{\mathrm{A}}^{1,1}(X,\mathbb{R})$ induced by the identity map. The pull-back cone $\mathscr{L}^{-1}(\mathcal{E}_{\partial\bar{\partial}})$ denotes the inverse image of the cone $\mathcal{E}_{\partial\bar{\partial}}$ under the isomorphism $\mathscr{L}$. The closed convex cone $\mathcal{M}_X \subseteq$

$H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R})$ is called the movable cone, originating from [9], and $(\mathcal{M}_X)^{\vee_c}$ denotes its dual cone (cf. Definitions 4.7 and 4.14).

Lemma 1.9 (See Lemma 4.15 and its remarks) *Let X be a compact Kähler manifold. There exist the following inclusions:*

$$\mathcal{E}_X \subseteq \mathscr{L}^{-1}(\mathcal{E}_{\partial\bar{\partial}}) \subseteq (\mathcal{M}_X)^{\vee_c}.$$

By the inclusions in this lemma, the equality (1.2) is actually a part of:

Conjecture 1.10 ([9, Conjecture 2.3]) *Let X be a compact Kähler manifold. Then the equality holds*

$$\mathcal{E}_X = (\mathcal{M}_X)^{\vee_c}.$$

An analogous conjecture of the balanced case is proposed as [22, Conjecture 5.4]. The following theorem provides some evidence for the assertion of Question 1.8.

Theorem 1.11 (= Theorem 4.17) *Let X be a compact Kähler manifold and* $[\alpha]_{\mathrm{BC}}$ *a nef class. Then* $[\alpha^{n-1}]_{\mathrm{A}} \in \mathcal{G}_X$ *implies that* $[\alpha^{n-1}]_{\mathrm{BC}} \in \mathcal{B}_X$*. Hence* $\bar{\mathbb{I}}(\overline{\mathcal{K}}_X)\bigcap \mathcal{B}_X$ *and* $\overline{\mathbb{K}}(\overline{\mathcal{K}}_X)\bigcap \mathcal{G}_X$ *can be identified by the mapping* $\mathbb{J}$.

The mappings $\bar{\mathbb{I}}$ and $\overline{\mathbb{K}}$ are contained in the pair of diagrams $(\mathrm{D}, \overline{\mathrm{D}})$ as in the beginning of Sect. 4.2. The proof relies on several important results on solving complex Monge–Ampère equations on the compact Kähler manifold X. One is the Yau's celebrated results of solutions of the complex Monge–Ampère equations for Kähler classes [62]. The other one is the Boucksom–Eyssidieux–Guedj–Zeriahi's work on the equations for the nef and big classes [10].

Popovici and Ugarte in [45, Theorem 5.7] prove that the following inclusion holds

$$\mathcal{G}_{X_0} \subseteq \lim_{t\to 0} \mathcal{G}_{X_t}$$

for the family $\pi : \mathcal{X} \to \Delta_\epsilon$ over a small complex disk with the central fiber an **sGG** manifold, where $\lim_{t\to 0} \mathcal{G}_{X_t}$ is defined by

$$\lim_{t\to 0} \mathcal{G}_{X_t} = \left\{[\Omega]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X_0,\mathbb{R}) \middle| \mathrm{P}_t \circ \mathrm{Q}_0([\Omega]_{\mathrm{A}}) \in \mathcal{G}_{X_t} \text{ for } t \text{ sufficiently small}\right\}.$$

The canonical mappings $\mathrm{P}_t : H_{\mathrm{DR}}^{2n-2}(X_t,\mathbb{R}) \to H_{\mathrm{A}}^{n-1,n-1}(X_t,\mathbb{R})$ are surjective for all t and the mapping $\mathrm{Q}_0 : H_{\mathrm{A}}^{n-1,n-1}(X_t,\mathbb{R}) \to H_{\mathrm{DR}}^{2n-2}(X_t,\mathbb{R})$, depending on a fixed Hermitian metric ω_0 on X_0, is injective, which satisfies $\mathrm{P}_0 \circ \mathrm{Q}_0 = \mathrm{id}_{H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R})}$. Here we give another inclusion from the other side as follows, where Demailly's regularization of closed positive currents (Theorem 4.21) plays an important role in the proof.

Theorem 1.12 (= Theorem 4.22) *Let $\pi : \mathcal{X} \to \Delta_\epsilon$ be a holomorphic family with the Kählerian central fiber X_0. Then we have*

$$\lim_{t \to \tau} \mathcal{G}_{X_t} \subseteq \mathcal{N}_{X_\tau} \quad \textit{for each} \quad \tau \in \Delta_\epsilon,$$

where $\mathcal{N}_{X_\tau}$ is the convex cone generated by Aeppli classes of $\partial_\tau \overline{\partial}_\tau$-closed positive $(n-1, n-1)$-currents on X_τ. Moreover, the following inclusion holds, for $\tau \in \Delta_\epsilon \setminus \bigcup S_\nu$,

$$\lim_{t \to \tau} \mathcal{G}_{X_t} \subseteq \overline{\mathcal{G}}_{X_\tau}.$$

Here $\bigcup S_\nu$ is a countable union of analytic subvarieties S_ν of Δ_ϵ. And Theorem 4.23 deals with the case of the fiber, satisfying the equality $\overline{\mathcal{K}}_X = \mathcal{E}_X$, in a Kähler family.

In [46], Wan and the authors will apply the extension methods developed here to a power series proof of Kodaira–Spencer's local stability theorem of Kähler metrics, which is motivated by:

Problem 1.13 (Remark 1 on [37, p. 180]) *A good problem would be to find an elementary proof* (*for example, using power series methods*). *Our proof uses non-trivial results from partial differential equations.*

2 An Extension Formula for Complex Differential Forms

Inspired by the classical Kodaira–Spencer–Kuranishi deformation theory of complex structures and the recent work [33], we will present an extension formula for complex differential forms. For a holomorphic family of compact complex manifolds, we adopt the definition [27, Definition 2.8]; while for the differentiable one, we follow:

Definition 2.1 ([27, Definition 4.1]) Let $\mathcal{X}$ be a differentiable manifold, B a domain of $\mathbb{R}^k$, and π a smooth map of $\mathcal{X}$ onto B. By a *differentiable family of n-dimensional compact complex manifolds* we mean the triple $\pi : \mathcal{X} \to B$ satisfying the following conditions:

(i) The rank of the Jacobian matrix of π is equal to k at every point of $\mathcal{X}$.
(ii) For each point $t \in B$, $\pi^{-1}(t)$ is a compact connected subset of $\mathcal{X}$.
(iii) $\pi^{-1}(t)$ is the underlying differentiable manifold of the n-dimensional compact complex manifold X_t associated to each $t \in B$.
(iv) There is a locally finite open covering $\{\mathcal{U}_j \mid j = 1, 2, \ldots\}$ of $\mathcal{X}$ and complex-valued smooth functions $\zeta_j^1(p), \ldots, \zeta_j^n(p)$, defined on $\mathcal{U}_j$ such that for each t,

$$\left\{ p \to \left(\zeta_j^1(p), \ldots, \zeta_j^n(p) \right) \mid \mathcal{U}_j \cap \pi^{-1}(t) \neq \emptyset \right\}$$

form a system of local holomorphic coordinates of X_t.

2.1 Extension Maps for Deformations

Let us introduce several new notations. For $\phi \in A^{0,s}(X, T_X^{1,0})$ on a complex manifold X, the contraction operator can be extended to

$$i_\phi : A^{p,q}(X) \to A^{p-1,q+s}(X).$$

For example, if $\phi = \eta \otimes Y$ with $\eta \in A^{0,q}(X)$ and $Y \in \Gamma(X, T_X^{1,0})$, then for any $\omega \in A^{p,q}(X)$,

$$(i_\phi)(\omega) = \eta \wedge (i_Y \omega).$$

Let $\varphi \in A^{0,p}(X, T_X^{1,0})$ and $\psi \in A^{0,q}(X, T_X^{1,0})$, locally written as

$$\varphi = \frac{1}{p!} \sum \varphi^i_{\bar{j}_1,\dots,\bar{j}_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p} \otimes \partial_i \text{ and } \psi = \frac{1}{q!} \sum \psi^i_{\bar{k}_1,\dots,\bar{k}_q} d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q} \otimes \partial_i.$$

Then we have

$$[\varphi, \psi] = \sum_{i,j=1}^{n} \left(\varphi^i \wedge \partial_i \psi^j - (-1)^{pq} \psi^i \wedge \partial_i \varphi^j \right) \otimes \partial_j,$$

where

$$\partial_i \varphi^j = \frac{1}{p!} \sum \partial_i \varphi^j_{\bar{j}_1,\dots,\bar{j}_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p}$$

and similarly for $\partial_i \psi^j$. In particular, if $\varphi, \psi \in A^{0,1}(X, T_X^{1,0})$,

$$[\varphi, \psi] = \sum_{i,j=1}^{n} \left(\varphi^i \wedge \partial_i \psi^j + \psi^i \wedge \partial_i \varphi^j \right) \otimes \partial_j.$$

For any $\phi \in A^{0,q}(X, T_X^{1,0})$, we can define $\mathcal{L}_\phi$ by

$$\mathcal{L}_\phi = (-1)^q d \circ i_\phi + i_\phi \circ d.$$

According to the types, we can decompose

$$\mathcal{L}_\phi = \mathcal{L}_\phi^{1,0} + \mathcal{L}_\phi^{0,1},$$

where

$$\mathcal{L}_\phi^{1,0} = (-1)^q \partial \circ i_\phi + i_\phi \circ \partial$$

and

$$\mathcal{L}_{\phi}^{0,1} = (-1)^q \overline{\partial} \circ i_\phi + i_\phi \circ \overline{\partial}.$$

Then one has the following commutator formula, which originated from [54,55] and whose various versions appeared in [4,13,19,31,34] and also [32,33] for vector bundle-valued forms.

Lemma 2.2 *For $\phi, \phi' \in A^{0,1}(X, T_X^{1,0})$ on a complex manifold X and $\sigma \in A^{*,*}(X)$,*

$$[\phi, \phi'] \lrcorner \sigma = -\partial(\phi' \lrcorner (\phi \lrcorner \sigma)) - \phi' \lrcorner (\phi \lrcorner \partial \sigma) + \phi \lrcorner \partial(\phi' \lrcorner \sigma) + \phi' \lrcorner \partial(\phi \lrcorner \sigma),$$

or equivalently,

$$i_{[\phi,\phi']} = \mathcal{L}_{\phi}^{1,0} \circ i_{\phi'} - i_{\phi'} \circ \mathcal{L}_{\phi}^{1,0}. \tag{2.1}$$

Let $\phi \in A^{0,1}(X, T_X^{1,0})$ and i_ϕ be the contraction operator. Define an operator

$$e^{i_\phi} = \sum_{k=0}^{\infty} \frac{1}{k!} i_\phi^k,$$

where $i_\phi^k = \underbrace{i_\phi \circ \cdots \circ i_\phi}_{k \text{ copies}}$. Since the dimension of X is finite, the summation in the above formulation is also finite.

Proposition 2.3 ([33, Theorem 3.4]). *Let $\phi \in A^{0,1}(X, T_X^{1,0})$. Then on the space $A^{*,*}(X)$,*

$$e^{-i_\phi} \circ d \circ e^{i_\phi} = d - \mathcal{L}_\phi - i_{\frac{1}{2}[\phi,\phi]} = d - \mathcal{L}_{\phi}^{1,0} + i_{\overline{\partial}\phi - \frac{1}{2}[\phi,\phi]}. \tag{2.2}$$

Or equivalently

$$e^{-i_\phi} \circ \overline{\partial} \circ e^{i_\phi} = \overline{\partial} - \mathcal{L}_{\phi}^{0,1} \tag{2.3}$$

and

$$e^{-i_\phi} \circ \partial \circ e^{i_\phi} = \partial - \mathcal{L}_{\phi}^{1,0} - i_{\frac{1}{2}[\phi,\phi]}.$$

Proof Note that (2.3) proved in [13, Lemma 8.2] will not be used in this new proof, but only the commutator formula (2.1) and

$$i_{[\phi,\phi]} \circ i_\phi = i_\phi \circ i_{[\phi,\phi]} \tag{2.4}$$

by a formula on [13, Page 361].

Let us first define a bracket

$$\left[d, i_\phi^k\right] = d \circ i_\phi^k - i_\phi^k \circ d.$$

Obviously, $[d, i_\phi] = -\mathcal{L}_\phi$ and (2.2) is equivalent to

$$[d, e^{i_\phi}] = e^{i_\phi} \circ [d, i_\phi] - e^{i_\phi} \circ i_{\frac{1}{2}[\phi,\phi]}. \tag{2.5}$$

We check the Leibniz rule for the bracket: for $k \geq 2$,

$$\left[d, i_\phi^k\right] = \sum_{j=1}^{k} i_\phi^{j-1} \circ [d, i_\phi] \circ i_\phi^{k-j}.$$

As for $k = 2$,

$$\left[d, i_\phi^2\right] = d \circ i_\phi^2 - i_\phi \circ d \circ i_\phi + i_\phi \circ d \circ i_\phi - i_\phi^2 \circ d = [d, i_\phi] \circ i_\phi + i_\phi \circ [d, i_\phi].$$

Then similarly, one is able to prove the cases for $k \geq 3$ by induction.

Now we can prove (2.5). Actually, the Leibniz rule and the formulae (2.1) (2.4) tell us: for $k \geq 2$,

$$\left[d,\ i_\phi^k\right] = k i_\phi^{k-1} \circ [d,\ i_\phi] - \frac{k(k-1)}{2} i_\phi^{k-2} \circ i_{[\phi,\phi]},$$

which implies (2.5). □

From now on, one considers the smooth family

$$\pi : \mathcal{X} \to B$$

of n-dimensional compact complex manifolds over a small real domain with the central fiber

$$X_0 := \pi^{-1}(0)$$

and the general fibers denoted by

$$X_t := \pi^{-1}(t).$$

Assume that $k = 1$ for simplicity. We will use the standard notions in deformation theory as in the beginning of [37, Chapter 4]. Fix an open coordinate covering $\{\mathcal{U}_j\}$ of $\mathcal{X}$ so that

$$\begin{aligned} \mathcal{U}_j &:= \left\{(\zeta_j, t) = (\zeta_j^1, \ldots, \zeta_j^n, t) \mid |\zeta_j| < 1, |t| < \epsilon\right\}, \\ \pi(\zeta_j, t) &= t \end{aligned}$$

and

$$\zeta_j^\alpha = f_{jk}^\alpha(\zeta_k, t) \ on\ \mathcal{U}_j \cap \mathcal{U}_k,$$

where f_{jk} is holomorphic in ζ_k and smooth in t. By Ehresmann's theorem [18], $\mathcal{X}$ is diffeomorphic to $X \times B$, where X is the underlying differentiable manifold of X_0. Then

$$\mathcal{U}_j = U_j \times B,$$

where $U_j = \{\zeta_j \mid |\zeta_j| < 1\}$. Thus, we can consider X_t as a compact manifold obtained by glueing U_j with $t \in B$ by identifying $\zeta_k \in U_k$ with $\zeta_j = f_{jk}(\zeta_k, t) \in U_j$. We refer the readers to [27, §4.1.(b)] for more details on this description. If x is a point of the underlying differentiable manifold X of X_0 and $t \in \Delta_\epsilon$, we notice that

$$\zeta_j^\alpha = \zeta_j^\alpha(x, t)$$

is a differentiable function of (x, t). Use the holomorphic coordinates z of $X_0 = X$ as differentiable coordinates so that

$$\zeta_j^\alpha(x, t) = \zeta_j^\alpha(z, t),$$

where $\zeta_j^\alpha(z, t)$ is a differentiable function of (z, t). At $t = 0$, $\zeta_j^\alpha(z, t)$ is holomorphic in z and otherwise it is only differentiable.

Then a Beltrami differential $\varphi(t)$ can be calculated out explicitly on the above local coordinate charts. As we focus on one coordinate chart, the subscript is suppressed. From [37, Page 150],

$$\varphi(t) = \left(\frac{\partial}{\partial z}\right)^T \left(\frac{\partial \zeta}{\partial z}\right)^{-1} \overline{\partial}\zeta, \tag{2.6}$$

where $\frac{\partial}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \vdots \\ \frac{\partial}{\partial z^n} \end{pmatrix}$, $\overline{\partial}\zeta = \begin{pmatrix} \overline{\partial}\zeta^1 \\ \vdots \\ \overline{\partial}\zeta^n \end{pmatrix}$, $\frac{\partial \zeta}{\partial z}$ stands for the matrix $(\frac{\partial \zeta^\alpha}{\partial z^j})_{\substack{1 \le \alpha \le n \\ 1 \le j \le n}}$ and α, j are the row and column indices. Here $\left(\frac{\partial}{\partial z}\right)^T$ is the transpose of $\frac{\partial}{\partial z}$ and $\overline{\partial}$ denotes the Cauchy–Riemann operator with respect to the holomorphic structure on X_0.

Since $\varphi(t)$ is locally expressed as $\varphi_{\bar{j}}^i d\bar{z}^j \otimes \frac{\partial}{\partial z^i} \in A^{0,1}(T_{X_0}^{1,0})$, it can be considered as a matrix $(\varphi_{\bar{j}}^i)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$. By (2.6), this matrix can be explicitly written as

$$\varphi = (\varphi_{\bar{j}}^i)_{\substack{1 \le i \le n \\ 1 \le j \le n}} = \varphi(t)\left(\frac{\partial}{\partial \bar{z}^j}, dz^i\right) = \left(\left(\frac{\partial \zeta}{\partial z}\right)^{-1} \left(\frac{\partial \zeta}{\partial \bar{z}}\right)\right)_{\bar{j}}^{i}. \tag{2.7}$$

A fundamental fact is that the Beltrami differential $\varphi(t)$ defined as above satisfies the integrability:

$$\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]. \tag{2.8}$$

One needs the following crucial calculation:

Lemma 2.4

$$\begin{pmatrix} \frac{\partial z}{\partial \zeta} & \frac{\partial z}{\partial \bar{\zeta}} \\ \frac{\partial \bar{z}}{\partial \zeta} & \frac{\partial \bar{z}}{\partial \bar{\zeta}} \end{pmatrix} = \begin{pmatrix} (\mathbb{1} - \varphi\overline{\varphi})^{-1} \left(\frac{\partial \zeta}{\partial z}\right)^{-1} & -\varphi\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1} \overline{\left(\frac{\partial \zeta}{\partial z}\right)}^{-1} \\ -\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi} \left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \overline{(\mathbb{1} - \varphi\overline{\varphi})}^{-1} \overline{\left(\frac{\partial \zeta}{\partial z}\right)}^{-1} \end{pmatrix}.$$

Here $\varphi\overline{\varphi}$, $\overline{\varphi}\varphi$ stand for the two matrices $(\varphi^i_{\bar{k}}\overline{\varphi^k_{\bar{j}}})_{\substack{1\le i\le n\\ 1\le j\le n}}$, $(\overline{\varphi^i_{\bar{k}}}\varphi^k_{\bar{j}})_{\substack{1\le i\le n\\ 1\le j\le n}}$, respectively.

In many places, $\varphi\overline{\varphi}$ and $\overline{\varphi}\varphi$ can also be seen as $\varphi^i_{\bar{k}}\overline{\varphi^k_{\bar{j}}}dz^j \otimes \frac{\partial}{\partial z^i} \in A^{1,0}(T^{1,0}_{X_0})$ and $\overline{\varphi^i_{\bar{k}}}\varphi^k_{\bar{j}}d\bar{z}^j \otimes \frac{\partial}{\partial \bar{z}^i} \in A^{0,1}(T^{0,1}_{X_0})$. Actually, $\varphi\overline{\varphi} = \overline{\varphi}\lrcorner\varphi$, $\overline{\varphi}\varphi = \varphi\lrcorner\overline{\varphi}$, and $\mathbb{1}$ is the identity matrix.

Proof It is easy to see that $\begin{pmatrix} \frac{\partial z}{\partial \zeta} & \frac{\partial z}{\partial \bar{\zeta}} \\ \frac{\partial \bar{z}}{\partial \zeta} & \frac{\partial \bar{z}}{\partial \bar{\zeta}} \end{pmatrix}$ is the inverse matrix of $\begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \end{pmatrix}$. Then it follows,

$$\begin{pmatrix} \mathbb{1} & 0 \\ -\left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ 0 & \frac{\partial \bar{\zeta}}{\partial \bar{z}} - \left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1}\left(\frac{\partial \zeta}{\partial \bar{z}}\right) \end{pmatrix}. \tag{2.9}$$

Take the inverse matrices of both sides of (2.9), yielding

$$\begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ 0 & \frac{\partial \bar{\zeta}}{\partial \bar{z}} - \left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1}\left(\frac{\partial \zeta}{\partial \bar{z}}\right) \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ -\left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \mathbb{1} \end{pmatrix}. \tag{2.10}$$

From Linear Algebra, we have the basic equality below

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}, \tag{2.11}$$

where A, B are invertible matrices. Combine with (2.7) and (2.11) and go back to (2.10):

$$\begin{aligned} \begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ \frac{\partial \bar{\zeta}}{\partial z} & \frac{\partial \bar{\zeta}}{\partial \bar{z}} \end{pmatrix}^{-1} &= \begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ 0 & \left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)\left(\mathbb{1} - \left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)^{-1}\left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1}\left(\frac{\partial \zeta}{\partial \bar{z}}\right)\right) \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ -\left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}} \\ 0 & \left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)(\mathbb{1} - \overline{\varphi}\varphi) \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ -\left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{\partial \zeta}{\partial z}\right)^{-1} & -\varphi\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)^{-1} \\ 0 & (\mathbb{1} - \overline{\varphi}\varphi)^{-1}\left(\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\left(\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \zeta}{\partial z}\right)^{-1} & \mathbb{1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \left(\mathbb{1} + \varphi\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\right)\left(\frac{\partial\zeta}{\partial z}\right)^{-1} & -\varphi\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\left(\frac{\partial\bar{\zeta}}{\partial\bar{z}}\right)^{-1} \\ -\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\left(\frac{\partial\zeta}{\partial z}\right)^{-1} & (\mathbb{1} - \overline{\varphi}\varphi)^{-1}\left(\frac{\partial\bar{\zeta}}{\partial\bar{z}}\right)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbb{1} - \varphi\overline{\varphi})^{-1}\left(\frac{\partial\zeta}{\partial z}\right)^{-1} & -\varphi\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\left(\overline{\frac{\partial\zeta}{\partial z}}\right)^{-1} \\ -\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\left(\frac{\partial\zeta}{\partial z}\right)^{-1} & \left(\overline{\mathbb{1} - \varphi\overline{\varphi}}\right)^{-1}\left(\overline{\frac{\partial\zeta}{\partial z}}\right)^{-1} \end{pmatrix}.$$

□

We need a few more local formulae:

Lemma 2.5

$$\begin{cases} d\zeta^\alpha &= \frac{\partial\zeta^\alpha}{\partial z^i}\left(e^{i_\varphi}(dz^i)\right), \\ \frac{\partial}{\partial\zeta^\alpha} &= \left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\left(\frac{\partial\zeta}{\partial z}\right)^{-1}\right)_\alpha^j \frac{\partial}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\left(\frac{\partial\zeta}{\partial z}\right)^{-1}\right)_\alpha^{\bar{j}} \frac{\partial}{\partial\bar{z}^j}. \end{cases}$$

Proof For the first equality,

$$\begin{aligned} d\zeta^\alpha &= \frac{\partial\zeta^\alpha}{\partial z^i}dz^i + \frac{\partial\zeta^\alpha}{\partial\bar{z}^j}d\bar{z}^j \\ &= \frac{\partial\zeta^\alpha}{\partial z^i}\left(dz^i + \left(\left(\frac{\partial\zeta}{\partial z}\right)^{-1}\right)_\beta^i \frac{\partial\zeta^\beta}{\partial\bar{z}^j}d\bar{z}^j\right) \\ &= \frac{\partial\zeta^\alpha}{\partial z^i}\left(dz^i + \varphi_{\bar{j}}^i d\bar{z}^j\right) = \frac{\partial\zeta^\alpha}{\partial z^i}\left(e^{i_\varphi}(dz^i)\right). \end{aligned}$$

Then the second one follows from Lemma 2.4:

$$\begin{aligned} \frac{\partial}{\partial\zeta^\alpha} &= \frac{\partial z^i}{\partial\zeta^\alpha}\frac{\partial}{\partial z^i} + \frac{\partial\bar{z}^j}{\partial\zeta^\alpha}\frac{\partial}{\partial\bar{z}^j} \\ &= \left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\left(\frac{\partial\zeta}{\partial z}\right)^{-1}\right)_\alpha^j \frac{\partial}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\left(\frac{\partial\zeta}{\partial z}\right)^{-1}\right)_\alpha^{\bar{j}} \frac{\partial}{\partial\bar{z}^j}. \end{aligned}$$

□

Corollary 2.6

$$\frac{\partial\zeta^\alpha}{\partial z^i}\frac{\partial}{\partial\zeta^\alpha} = \left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\right)_i^j \frac{\partial}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\right)_i^{\bar{j}} \frac{\partial}{\partial\bar{z}^j}.$$

Proof It is a direct corollary of the second equality in Lemma 2.5. □

By the above preparation, we can reprove the following important proposition in deformation theory of complex structures, which can be dated back to [20] (see [39, Sect. 1] and also [37, pp. 151–152]).

Proposition 2.7 *The holomorphic structure on X_t is determined by $\varphi(t)$. More specifically, a differentiable function f defined on any open subset of X_0 is holomorphic with respect to the holomorphic structure of X_t if and only if*

$$\left(\overline{\partial} - \sum_i \varphi^i(t)\partial_i\right) f(z) = 0, \tag{2.12}$$

where $\varphi^i(t) = \sum_j \varphi(t)^i_{\bar{j}} d\bar{z}^j$, or equivalently,

$$\left(\overline{\partial} - \varphi(t) \lrcorner \partial\right) f(z) = 0.$$

Proof By use of Lemma 2.5 and Corollary 2.6, we get

$$\begin{aligned}
df &= \frac{\partial f}{\partial \zeta^\alpha} d\zeta^\alpha + \frac{\partial f}{\partial \bar{\zeta}^\beta} d\overline{\zeta}^\beta \\
&= \frac{\partial f}{\partial \zeta^\alpha} \frac{\partial \zeta^\alpha}{\partial z^i} \left(e^{i_\varphi}(dz^i)\right) + \frac{\partial f}{\partial \bar{\zeta}^\beta} \overline{\frac{\partial \zeta^\beta}{\partial z^i} \left(e^{i_\varphi}(dz^i)\right)} \\
&= \left(\left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\right)^j_i \frac{\partial f}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \overline{\varphi}\right)^{\bar{j}}_i \frac{\partial f}{\partial \bar{z}^j}\right) \left(e^{i_\varphi}(dz^i)\right) \\
&\quad + \left(\left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)^{\bar{j}}_{\bar{i}} \frac{\partial f}{\partial \bar{z}^j} - \left(\varphi\,(\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)^j_{\bar{i}} \frac{\partial f}{\partial z^j}\right) \left(\overline{e^{i_\varphi}(dz^i)}\right) \\
&= e^{i_\varphi}\left(\left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\right)^k_i \left(\frac{\partial f}{\partial z^k} - \overline{\varphi}^j_{\bar{k}} \frac{\partial f}{\partial \bar{z}^j}\right) dz^i\right) \\
&\quad + e^{i_{\overline{\varphi}}}\left(\left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)^{\bar{k}}_{\bar{i}} \left(\frac{\partial f}{\partial \bar{z}^k} - \varphi^j_{\bar{k}} \frac{\partial f}{\partial z^j}\right) d\bar{z}^i\right).
\end{aligned}$$

Now, let us calculate the second term in the bracket:

$$\begin{aligned}
&e^{i_{\overline{\varphi}}}\left(\left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)^{\bar{k}}_{\bar{i}} \left(\frac{\partial f}{\partial \bar{z}^k} - \varphi^j_{\bar{k}} \frac{\partial f}{\partial z^j}\right) d\bar{z}^i\right) \\
&\quad = e^{i_{\overline{\varphi}}}\Big((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \lrcorner \overline{\partial} f - (\mathbb{1} - \overline{\varphi}\varphi)^{-1} \lrcorner \varphi \lrcorner \partial f \Big) \\
&\quad = e^{i_{\overline{\varphi}}}\Big((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \lrcorner (\overline{\partial} - \varphi \lrcorner \partial) f \Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{\partial}_t f &= e^{i_{\overline{\varphi}}}\left(\left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)^{\bar{k}}_{\bar{i}} \left(\frac{\partial f}{\partial \bar{z}^k} - \varphi^j_{\bar{k}} \frac{\partial f}{\partial z^j}\right) d\bar{z}^i\right) \\
&= e^{i_{\overline{\varphi}}}\Big((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \lrcorner (\overline{\partial} - \varphi \lrcorner \partial) f \Big)
\end{aligned} \tag{2.13}$$

since df can be decomposed into $\partial_t f + \overline{\partial}_t f$ with respect to the holomorphic structure on X_t. Hence, the desired result follows from the invertibility of $e^{i_{\overline{\varphi}}}$ and $(\mathbb{1} - \overline{\varphi}\varphi)^{-1}$. □

See also another proof in [11, Proposition 3.1] and our proof gives an explicit expression of $\overline{\partial}_t$ on the differentiable functions as in (2.13). The formula used in the classical proof of Proposition 2.7 is

$$(\overline{\partial} - \varphi\lrcorner\partial)f = (\mathbb{1} - \overline{\varphi}\varphi)^{\bar{i}}_{\bar{j}}\, \overline{\partial_i \zeta^\alpha} d\bar{z}^j \frac{\partial f}{\partial \bar{\zeta}^\alpha},$$

which is just an equivalent version of (2.13)

$$(\overline{\partial} - \varphi\lrcorner\partial)f = (\mathbb{1} - \overline{\varphi}\varphi)\lrcorner e^{-i_{\overline{\varphi}}}(\overline{\partial}_t f)$$

by use of the first formula of Lemma 2.5.

By the Leibniz rule, one has

$$\frac{\partial z^k}{\partial \bar{\zeta}^\alpha} + \varphi^k_{\bar{i}} \frac{\partial \bar{z}^i}{\partial \bar{\zeta}^\alpha} = 0, \tag{2.14}$$

which is equivalent to the definition (2.7). In fact, if (2.7) is assumed, then the Leibniz rule yields that

$$\begin{aligned}
\frac{\partial z^k}{\partial \bar{\zeta}^\alpha} + \varphi^k_{\bar{i}} \frac{\partial \bar{z}^i}{\partial \bar{\zeta}^\alpha} &= \frac{\partial z^k}{\partial \bar{\zeta}^\alpha} + \left(\left(\frac{\partial \zeta}{\partial z}\right)^{-1}\right)^k_\beta \frac{\partial \zeta^\beta}{\partial \bar{z}^i} \frac{\partial \bar{z}^i}{\partial \bar{\zeta}^\alpha} \\
&= \frac{\partial z^k}{\partial \bar{\zeta}^\alpha} - \left(\left(\frac{\partial \zeta}{\partial z}\right)^{-1}\right)^k_\beta \frac{\partial \zeta^\beta}{\partial z^i} \frac{\partial z^i}{\partial \bar{\zeta}^\alpha} \\
&= 0;
\end{aligned}$$

while the converse is similar. Thus, when f satisfies (2.12), one has

$$\begin{aligned}
\frac{\partial f}{\partial \bar{\zeta}^\alpha} &= \frac{\partial f}{\partial z^k} \frac{\partial z^k}{\partial \bar{\zeta}^\alpha} + \frac{\partial f}{\partial \bar{z}^k} \frac{\partial \bar{z}^k}{\partial \bar{\zeta}^\alpha} \\
&= \frac{\partial f}{\partial z^k} \frac{\partial z^k}{\partial \bar{\zeta}^\alpha} + \frac{\partial f}{\partial z^i} \varphi^i_{\bar{k}} \frac{\partial \bar{z}^k}{\partial \bar{\zeta}^\alpha} \\
&= \frac{\partial f}{\partial z^i} \left(\frac{\partial z^i}{\partial \bar{\zeta}^\alpha} + \varphi^i_{\bar{k}} \frac{\partial \bar{z}^k}{\partial \bar{\zeta}^\alpha} \right) \\
&= 0.
\end{aligned} \tag{2.15}$$

Conversely, $\frac{\partial f}{\partial \bar{\zeta}^\alpha} = 0$ implies that f satisfies (2.12). Actually, we can substitute (2.14) into the first equality of (2.15) to get

$$\frac{\partial f}{\partial \bar{\zeta}^\alpha} = \frac{\partial \bar{z}^k}{\partial \bar{\zeta}^\alpha} \left(\frac{\partial f}{\partial \bar{z}^k} - \varphi^j_{\bar{k}} \frac{\partial f}{\partial z^j} \right).$$

By Lemma 2.4, one knows that $\frac{\partial \bar{z}^k}{\partial \bar{\zeta}^\alpha}$ is an invertible matrix as t is small. Hence, this is the third proof of Proposition 2.7, which is implicit in Newlander–Nirenberg's proof of their integrability theorem [39].

Let us recall the Newlander–Nirenberg integrability theorem. Let φ be a holomorphic tangent bundle-valued (0,1)-form defined on a domain U of $\mathbb{C}^n$ and $L_i = \overline{\partial}_i - \varphi_{\bar{i}}^{j} \partial_j$. Assume that $L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n$ are linearly independent, and that they satisfy the integrability condition (2.8). Then the system of partial differential equations

$$L_i f = 0, i = 1, \ldots, n, \tag{2.16}$$

has n linearly independent smooth solutions $f = \zeta^\alpha = \zeta^\alpha(z), \alpha = 1, \ldots, n$, in a small neighborhood of any point of U. Here the solutions $\zeta^1, \ldots, \zeta^n$ are said to be linearly independent if

$$\det \frac{\partial \left(\zeta^1, \ldots, \zeta^n, \overline{\zeta^1}, \ldots, \overline{\zeta^n}\right)}{\partial \left(z^1, \ldots, z^n, \overline{z^1}, \ldots, \overline{z^n}\right)} \neq 0,$$

which obviously implies

$$\det(\mathbb{1} - \overline{\varphi}\varphi) \left|\det \frac{\partial(\zeta^1, \ldots, \zeta^n)}{\partial(z^1, \ldots, z^n)}\right|^2 \neq 0$$

since the resolution of the system (2.16) of partial differential equations yields

$$\begin{pmatrix} \frac{\partial \zeta}{\partial z} & \frac{\partial \zeta}{\partial \bar{z}}\varphi \\ \overline{\left(\frac{\partial \zeta}{\partial z}\varphi\right)} & \frac{\partial \bar{\zeta}}{\partial \bar{z}} \end{pmatrix} \begin{pmatrix} \mathbb{1} & -\varphi \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \frac{\partial \zeta}{\partial z} & 0 \\ \frac{\partial \bar{\zeta}}{\partial z} & \frac{\partial \bar{\zeta}}{\partial \bar{z}}(\mathbb{1} - \overline{\varphi}\varphi) \end{pmatrix}.$$

This theorem, together with Proposition 2.7, is actually the starting point of Kodaira–Nirenberg–Spencer's existence theorem for deformations and a quite clear description can be found in [27, pp. 268–269]. We also find that the term $\mathbb{1} - \overline{\varphi}\varphi$ in Lemma 2.4 is natural.

Motivated by the new proof of Proposition 2.7, we introduce a map

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}} : A^{p,q}(X_0) \to A^{p,q}(X_t),$$

which plays an important role in this paper.

Definition 2.8 For $\sigma \in A^{p,q}(X_0)$, we define

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma) = \sigma_{i_1\ldots i_p \bar{j}_1\ldots\bar{j}_q}(z)\left(e^{i_{\varphi(t)}}\left(dz^{i_1} \wedge \cdots \wedge dz^{i_p}\right)\right) \wedge \left(e^{i_{\overline{\varphi(t)}}}\left(d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\right)\right),$$

where σ is locally written as

$$\sigma = \sigma_{i_1\ldots i_p \bar{j}_1\ldots\bar{j}_q}(z) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

and the operators $e^{i_{\varphi(t)}}$, $e^{i_{\overline{\varphi(t)}}}$ follow the convention:

$$e^{\spadesuit} = \sum_{k=0}^{\infty} \frac{1}{k!} \spadesuit^k, \tag{2.17}$$

where $\spadesuit^k$ denotes k-time action of the operator $\spadesuit$. Since the dimension of X is finite, the summation in the above formulation is always finite.

Then we have:

Lemma 2.9 *The extension map $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}} : A^{p,q}(X_0) \to A^{p,q}(X_t)$ is a linear isomorphism as t is arbitrarily small.*

Proof Notice that

$$\left(dz^1 + \varphi(t) \lrcorner dz^1, \ldots, dz^n + \varphi(t) \lrcorner dz^n\right) \quad \text{and} \quad \left(d\bar{z}^1 + \overline{\varphi(t)} \lrcorner d\bar{z}^1, \ldots, d\bar{z}^n + \overline{\varphi(t)} \lrcorner d\bar{z}^n\right)$$

are two local bases of $A^{1,0}(X_t)$ and $A^{0,1}(X_t)$, respectively, thanks to the first identity of Lemma 2.5 and the matrix $\left(\frac{\partial \zeta^\alpha}{\partial z^i}\right)$ therein is invertible as t is small. Then the map $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}$ is obviously well-defined since $\varphi(t)$ is a well-defined, global $(1, 0)$-vector valued $(0, 1)$-form on X_0 as on [37, pp. 150–151].

For the desired isomorphism, we define the inverse map

$$e^{-i_{\varphi(t)}|-i_{\overline{\varphi(t)}}} : A^{p,q}(X_t) \to A^{p,q}(X_0)$$

of $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}$ as

$$\begin{aligned} &e^{-i_{\varphi(t)}|-i_{\overline{\varphi(t)}}}(\eta) \\ &\quad = \eta_{i_1\ldots i_p \bar{j}_1\ldots\bar{j}_q}(\zeta)\Big(e^{-i_{\varphi(t)}}\left(\left(dz^{i_1} + \varphi(t)\lrcorner dz^{i_1}\right) \wedge \cdots \wedge \left(dz^{i_p} + \varphi(t)\lrcorner dz^{i_p}\right)\right) \\ &\quad \wedge e^{-i_{\overline{\varphi(t)}}}\left(\left(d\bar{z}^{j_1} + \overline{\varphi(t)}\lrcorner d\bar{z}^{j_1}\right) \wedge \cdots \wedge \left(d\bar{z}^{j_q} + \overline{\varphi(t)}\lrcorner d\bar{z}^{j_q}\right)\right)\Big), \end{aligned}$$

where $\eta \in A^{p,q}(X_t)$ is locally written as

$$\begin{aligned} \eta = \eta_{i_1\ldots i_p \bar{j}_1\ldots\bar{j}_q}(\zeta) &\left(dz^{i_1} + \varphi(t)\lrcorner dz^{i_1}\right) \wedge \cdots \wedge \left(dz^{i_p} + \varphi(t)\lrcorner dz^{i_p}\right) \\ \wedge &\left(d\bar{z}^{j_1} + \overline{\varphi(t)}\lrcorner d\bar{z}^{j_1}\right) \wedge \cdots \wedge \left(d\bar{z}^{j_q} + \overline{\varphi(t)}\lrcorner d\bar{z}^{j_q}\right), \end{aligned}$$

and the operators $e^{-i_{\varphi(t)}}$, $e^{-i_{\overline{\varphi(t)}}}$ also follow the convention (2.17). □

The dual version of the fact about the basis in the proof is used by Chan–Suen [11] to prove Proposition 2.7 and also by Huang in the second paragraph of [25, Sect. (1.2)]. Notice that the extension map $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}$ admits more complete deformation significance than $e^{i_{\varphi(t)}}$ which extends only the holomorphic part of a complex differential form.

Lemma 2.10 *The map* $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}: A^{p,q}(X_0) \to A^{p,q}(X_t)$ *is a real operator.*

Proof It suffices to prove, for any $\sigma \in A^{p,q}(X_0)$,

$$\overline{e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma)} = e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\overline{\sigma}).$$

In fact, let

$$\sigma = \sum_{|I|=p,|J|=q} \sigma_{I\bar{J}}(z)dz^I \wedge d\bar{z}^J$$

by multi-index notation and then

$$\begin{aligned}\overline{e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma)} &= \overline{\sigma_{I\bar{J}}(z)e^{i_{\varphi(t)}}(dz^I) \wedge e^{i_{\overline{\varphi(t)}}}(d\bar{z}^J)}\\ &= \overline{\sigma_{I\bar{J}}(z)}e^{i_{\overline{\varphi(t)}}}(d\bar{z}^I) \wedge e^{i_{\varphi(t)}}(dz^J)\\ &= \overline{\sigma_{I\bar{J}}(z)}(-1)^{|I|\cdot|J|}e^{i_{\varphi(t)}}(dz^J) \wedge e^{i_{\overline{\varphi(t)}}}(d\bar{z}^I)\\ &= e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(-1)^{|I|\cdot|J|}\overline{\sigma_{I\bar{J}}(z)}dz^J \wedge d\bar{z}^I\\ &= e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\overline{\sigma}).\end{aligned}$$

□

2.2 Obstruction Equation

This section is to obtain obstruction equation for $\overline{\partial}$-extension, i.e., obstruction equation for extending a $\overline{\partial}$-closed (p,q)-form on X_0 to the one on X_t.

Lemma 2.11

$$\begin{aligned}d\left(e^{i_\varphi}\lrcorner dz^i\right) &= \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\overline{\varphi}\right)_k^{\bar{l}} \frac{\partial\varphi_{\bar{l}}^i}{\partial z^j}\left(e^{i_\varphi}(dz^k)\right) \wedge \left(e^{i_\varphi}(dz^j)\right)\\ &\quad - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)_{\bar{k}}^{\bar{l}} \frac{\partial\varphi_{\bar{l}}^i}{\partial z^j}\left(\overline{e^{i_\varphi}(dz^k)}\right) \wedge \left(e^{i_\varphi}(dz^j)\right).\end{aligned}$$

Proof Here we use Proposition 2.3. By (2.2), one has

$$\begin{aligned}d\left(e^{i_\varphi}(dz^i)\right) &= (d \circ e^{i_\varphi} - e^{i_\varphi} \circ d)(dz^i)\\ &= e^{i_\varphi}(\partial \circ i_\varphi - i_\varphi \circ \partial)(dz^i)\\ &= \frac{\partial\varphi_{\bar{l}}^i}{\partial z^j}\left(e^{i_\varphi}(dz^j)\right) \wedge d\bar{z}^l.\end{aligned}$$

Moreover, we have

$$\begin{aligned} d\bar{z}^l &= \frac{\partial \bar{z}^l}{\partial \zeta^\alpha} d\zeta^\alpha + \frac{\partial \bar{z}^l}{\partial \bar{\zeta}^\beta} d\bar{\zeta}^\beta \\ &= \frac{\partial \bar{z}^l}{\partial \zeta^\alpha} \frac{\partial \zeta^\alpha}{\partial z^i} \left(e^{i_\varphi}(dz^i)\right) + \frac{\partial \bar{z}^l}{\partial \bar{\zeta}^\beta} \overline{\frac{\partial \zeta^\beta}{\partial z^i} \left(e^{i_\varphi}(dz^i)\right)} \\ &= -\left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\,\overline{\varphi}\right)_k^{\bar{l}} \left(e^{i_\varphi}(dz^k)\right) + \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)_{\bar{k}}^{\bar{l}} \left(\overline{e^{i_\varphi}(dz^k)}\right). \end{aligned} \tag{2.18}$$

□

For a general $\sigma \in A^{p,q}(X_0)$, Proposition 2.3 and the integrability condition (2.8) give

$$\begin{aligned} d(e^{i_\varphi | i_{\bar{\varphi}}}(\sigma)) &= d \circ e^{i_\varphi} \circ e^{-i_\varphi} \circ e^{i_\varphi | i_{\bar{\varphi}}}(\sigma) \\ &= e^{i_\varphi} \circ \left([\partial, i_\varphi] + \bar{\partial} + \partial\right) \circ e^{-i_\varphi} \circ e^{i_\varphi | i_{\bar{\varphi}}}(\sigma) \\ &= e^{i_\varphi | i_{\bar{\varphi}}} \circ \left(e^{-i_\varphi | -i_{\bar{\varphi}}} \circ e^{i_\varphi} \circ \left([\partial, i_\varphi] + \bar{\partial} + \partial\right) \circ e^{-i_\varphi} \circ e^{i_\varphi | i_{\bar{\varphi}}}(\sigma)\right). \end{aligned} \tag{2.19}$$

Here

$$e^{-i_{\varphi(t)} | -i_{\overline{\varphi(t)}}} : A^{p,q}(X_t) \to A^{p,q}(X_0)$$

is the inverse map of $e^{i_{\varphi(t)} | \iota_{\overline{\varphi(t)}}}$ as defined in the proof of Lemma 2.9. We introduce one more new notation $\exists$ to denote the *simultaneous contraction* on each component of a complex differential form as in [46, Sect. 2.1]. For example, $(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \exists \sigma$ means that the operator $(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})$ acts on σ simultaneously as

$$\begin{aligned} &(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \exists \left(f_{i_1 \ldots i_p \overline{j_1} \ldots \overline{j_q}} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\right) \\ &= \sigma_{i_1 \ldots i_p \overline{j_1} \ldots \overline{j_q}} (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_1} \wedge \cdots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_p} \\ &\quad \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_1} \wedge \cdots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_q}, \end{aligned} \tag{2.20}$$

if σ is locally expressed by:

$$\sigma = \sigma_{i_1 \ldots i_p \overline{j_1} \ldots \overline{j_q}} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

This new simultaneous contraction is well-defined since $\varphi(t)$ is a global $(1,0)$-vector valued $(0,1)$-form on X_0 (on [37, pp. 150–151]) as reasoned in the proof of Lemma 2.9. Using this notation, one can rewrite the extension map $e^{i_\varphi | i_{\bar{\varphi}}}$ in Definition 2.8:

$$e^{i_\varphi | i_{\bar{\varphi}}} = (\mathbb{1} + \varphi + \bar{\varphi}) \exists.$$

Then one has

Lemma 2.12 ([46, Lemmata 2.2+2.3]). *For any* $\sigma \in A^{p,q}(X_0)$,

$$e^{-i_\varphi} \circ e^{i_\varphi|i_{\bar\varphi}}(\sigma) = (\mathbb{1} - \bar\varphi\varphi + \bar\varphi)\lrcorner\sigma \tag{2.21}$$

and

$$e^{-i_\varphi|-i_{\bar\varphi}} \circ e^{i_\varphi}(\sigma) = \left((\mathbb{1} - \bar\varphi\varphi)^{-1} - (\mathbb{1} - \bar\varphi\varphi)^{-1}\bar\varphi\right)\lrcorner\sigma, \tag{2.22}$$

where $\left((\mathbb{1} - \bar\varphi\varphi)^{-1} - (\mathbb{1} - \bar\varphi\varphi)^{-1}\bar\varphi\right)$ *acts on* σ *just as (2.20).*

Proof Here we give a different proof from those in [46, Lemmata 2.2+2.3]. Locally set

$$\sigma = \sigma_{I_p\bar J_q} dz^{I_p} \wedge d\bar z^{J_q}$$

by multi-index notation. So

$$e^{i_\varphi|i_{\bar\varphi}}(\sigma) = \sigma_{I_p\bar J_q} e^{i_\varphi}(dz^{I_p}) \wedge e^{i_{\bar\varphi}}(d\bar z^{J_q})$$

and thus,

$$\begin{aligned} e^{-i_\varphi} \circ e^{i_\varphi|i_{\bar\varphi}}(\sigma) &= \sigma_{I_p\bar J_q} dz^{I_p} \wedge e^{-i_\varphi} \circ e^{i_{\bar\varphi}}(d\bar z^{J_q}) \\ &= \sigma_{I_p\bar J_q} dz^{I_p} \wedge (\mathbb{1} - \bar\varphi\varphi + \bar\varphi)\lrcorner(d\bar z^{J_q}). \end{aligned}$$

As for (2.22), (2.18) tells us that

$$\begin{aligned} e^{-i_\varphi|-i_{\bar\varphi}} \circ e^{i_\varphi}(\sigma) &= \sigma_{I_p\bar J_q} e^{-i_\varphi|-i_{\bar\varphi}}(e^{i_\varphi}(dz^{I_p}) \wedge d\bar z^{J_q}) \\ &= \sigma_{I_p\bar J_q} e^{-i_\varphi|-i_{\bar\varphi}}\left(e^{i_\varphi}(dz^{I_p}) \wedge e^{i_\varphi|i_{\bar\varphi}}\left((\mathbb{1} - \bar\varphi\varphi)^{-1} - (\mathbb{1} - \bar\varphi\varphi)^{-1}\bar\varphi\right)\lrcorner d\bar z^{J_q}\right) \\ &= \sigma_{I_p\bar J_q} dz^{I_p} \wedge \left((\mathbb{1} - \bar\varphi\varphi)^{-1} - (\mathbb{1} - \bar\varphi\varphi)^{-1}\bar\varphi\right)\lrcorner d\bar z^{J_q}. \end{aligned}$$

□

The following equivalence describes the $\bar\partial$-extension obstruction for (p, q)-forms of the smooth family.

Proposition 2.13 *For any* $\sigma \in A^{p,q}(X_0)$,

$$\bar\partial_t\left(e^{i_\varphi|i_{\bar\varphi}}(\sigma)\right) = 0$$

amounts to

$$([\partial, i_\varphi] + \bar\partial)(\mathbb{1} - \bar\varphi\varphi)\lrcorner\sigma = 0.$$

Proof Substituting (2.21) and (2.22) into (2.19), one has

$$
\begin{aligned}
&d(e^{i_\varphi|i_{\bar\varphi}}(\sigma))\\
&\quad= e^{i_\varphi|i_{\bar\varphi}}\left(\left((\mathbb{1}-\bar\varphi\varphi)^{-1}-(\mathbb{1}-\bar\varphi\varphi)^{-1}\bar\varphi\right)\lrcorner\left([\partial, i_\varphi]+\bar\partial+\partial\right)(\mathbb{1}-\bar\varphi\varphi+\bar\varphi)\lrcorner\sigma\right).
\end{aligned}
\tag{2.23}
$$

From (2.22), we know that

$$
e^{-i_\varphi|-i_{\bar\varphi}}\circ e^{i_\varphi} : A^{p,q}(X_0)\to \bigoplus_{i=0}^{\min\{q,n-p\}} A^{p+i,q-i}(X_0).
$$

Thus, by carefully comparing the form types in both sides of (2.23), we have

$$
\bar\partial_t(e^{i_\varphi|i_{\bar\varphi}}(\sigma)) = e^{i_\varphi|i_{\bar\varphi}}\left((\mathbb{1}-\bar\varphi\varphi)^{-1}\lrcorner([\partial, i_\varphi]+\bar\partial)(\mathbb{1}-\bar\varphi\varphi)\lrcorner\sigma\right),
$$

which implies the desired equivalence follows from the invertibility of the operators $e^{i_\varphi|i_{\bar\varphi}}$ and $(\mathbb{1}-\bar\varphi\varphi)^{-1}\lrcorner$. □

2.3 Kuranishi Family and Beltrami Differentials

By (the proof of) Kuranishi's completeness theorem [29], for any compact complex manifold X_0, there exists a complete holomorphic family $\varpi : \mathcal{K}\to T$ of complex manifolds at the reference point $0\in T$ in the sense that for any differentiable family $\pi : \mathcal{X}\to B$ with $\pi^{-1}(s_0)=\varpi^{-1}(0)=X_0$, there is a sufficiently small neighborhood $E\subseteq B$ of s_0, and smooth maps $\Phi : \mathcal{X}_E\to\mathcal{K}$, $\tau : E\to T$ with $\tau(s_0)=0$ such that the diagram commutes

$$
\begin{array}{ccc}
\mathcal{X}_E & \xrightarrow{\ \Phi\ } & \mathcal{K}\\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varpi}\\
(E, s_0) & \xrightarrow{\ \tau\ } & (T, 0),
\end{array}
$$

Φ maps $\pi^{-1}(s)$ biholomorphically onto $\varpi^{-1}(\tau(s))$ for each $s\in E$, and

$$
\Phi : \pi^{-1}(s_0)=X_0\to\varpi^{-1}(0)=X_0
$$

is the identity map. This family is called *Kuranishi family* and constructed as follows. Let $\{\eta_\nu\}_{\nu=1}^m$ be a basis for $\mathbb{H}^{0,1}(X_0, T^{1,0}_{X_0})$, where some suitable Hermitian metric is fixed on X_0 and $m\geq 1$; Otherwise the complex manifold X_0 would be *rigid*, i.e., for any differentiable family $\kappa : \mathcal{M}\to P$ with $s_0\in P$ and $\kappa^{-1}(s_0)=X_0$, there is a neighborhood $V\subseteq P$ of s_0 such that $\kappa : \kappa^{-1}(V)\to V$ is trivial. Then one can construct a holomorphic family

$$\varphi(t) = \sum_{|I|=1}^{\infty} \varphi_I t^I := \sum_{j=1}^{\infty} \varphi_j(t),\ I = (i_1, \ldots, i_m),\ t = (t_1, \ldots, t_m) \in \mathbb{C}^m,$$

for $|t| < \rho$ a small positive constant, of Beltrami differentials as follows:

$$\varphi_1(t) = \sum_{\nu=1}^{m} t_\nu \eta_\nu$$

and for $|I| \geq 2$,

$$\varphi_I = \frac{1}{2} \overline{\partial}^* \mathbb{G} \sum_{J+L=I} [\varphi_J, \varphi_L],$$

where $\mathbb{G}$ is the associated Green's operator. It is obvious that $\varphi(t)$ satisfies the equation

$$\varphi(t) = \varphi_1 + \frac{1}{2} \overline{\partial}^* \mathbb{G}[\varphi(t), \varphi(t)].$$

Let

$$T = \{t \mid \mathbb{H}[\varphi(t), \varphi(t)] = 0\},$$

where $\mathbb{H}$ is the associated harmonic projection. Thus, for each $t \in T$, $\varphi(t)$ satisfies

$$\bar{\partial} \varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)], \tag{2.24}$$

and determines a complex structure X_t on the underlying differentiable manifold of X_0. More importantly, $\varphi(t)$ represents the complete holomorphic family $\varpi : \mathcal{K} \to T$ of complex manifolds. Roughly speaking, Kuranishi family $\varpi : \mathcal{K} \to T$ contains all sufficiently small differentiable deformations of X_0. We call $\varphi(t)$ the *canonical family of Beltrami differentials* for this Kuranishi family.

By means of these, one can reduce our argument on the deformation invariance of Hodge numbers for a smooth family of complex manifolds to that of the Kuranishi family by shrinking E if necessary, that is, one considers the Kuranishi family with the canonical family of Beltrami differentials constructed as above. From now on, one uses $\varphi(t)$ and φ interchangeably to denote this holomorphic family of integrable Beltrami differentials, and assumes $m = 1$ for simplicity.

3 Deformation Invariance of Hodge Numbers and Its Applications

Throughout this section, one just considers the Kuranishi family $\pi : \mathcal{X} \to \Delta_\epsilon$ of n-dimensional complex manifolds over a small complex disk with the general fibers $X_t := \pi^{-1}(t)$ according to the reduction in Sect. 2.3 and fixes a Hermitian metric g on the central fiber X_0. As a direct application of the extension formulae developed in

Sect. 2, we obtain several deformation invariance theorems of Hodge numbers in this section.

3.1 Basic Philosophy, Main Results, and Examples

Now let us describe our basic philosophy to consider the deformation invariance of Hodge numbers briefly. The Kodaira–Spencer's upper semi-continuity theorem ([28, Theorem 4]) tells us that the function

$$t \longmapsto h^{p,q}_{\bar\partial_t}(X_t) := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial_t}(X_t)$$

is always upper semi-continuous for $t \in \Delta_\varepsilon$ and thus, to approach the deformation invariance of $h^{p,q}_{\bar\partial_t}(X_t)$, we only need to obtain the lower semi-continuity. Here our main strategy is a modified iteration procedure, originally from [34] and developed in [33,52,53,63], which is to look for an injective extension map from $H^{p,q}_{\bar\partial}(X_0)$ to $H^{p,q}_{\bar\partial_t}(X_t)$. More precisely, for a nice uniquely chosen representative σ_0 of the initial Dolbeault cohomology class in $H^{p,q}_{\bar\partial}(X_0)$, we try to construct a convergent power series

$$\sigma_t = \sigma_0 + \sum_{j+k=1}^{\infty} t^k \bar{t}^j \sigma_{k\bar{j}} \in A^{p,q}(X_0),$$

with σ_t varying smoothly on t such that for each small t:

(1) $e^{i_\varphi | i_{\overline{\varphi}}}(\sigma_t) \in A^{p,q}(X_t)$ is $\bar\partial_t$-closed with respect to the holomorphic structure on X_t;
(2) The extension map $H^{p,q}_{\bar\partial}(X_0) \to H^{p,q}_{\bar\partial_t}(X_t) : [\sigma_0]_{\bar\partial} \mapsto [e^{i_\varphi | i_{\overline{\varphi}}}(\sigma_t)]_{\bar\partial_t}$ is injective.

The key point is to solve the obstruction equation, induced by the canonical family $\varphi(t)$ of Beltrami differentials, for the $\bar\partial_t$-closedness in (1), and verification of the injectivity of the extension map in (2). Then we state the main theorem of this section, whose proof will be postponed to Sect. 3.2.

Theorem 3.1 *If the injectivity of the mappings $\iota^{p+1,q}_{BC,\partial}$, $\iota^{p,q+1}_{\bar\partial,A}$ on the central fiber X_0 and the deformation invariance of the $(p, q-1)$-Hodge number $h^{p,q-1}_{\bar\partial_t}(X_t)$ holds, then $h^{p,q}_{\bar\partial_t}(X_t)$ are deformation invariant.*

There are three conditions involved in the theorem above, namely the injectivity of the mappings $\iota^{p+1,q}_{BC,\partial}$, $\iota^{p,q+1}_{\bar\partial,A}$ and the deformation invariance of the $(p, q-1)$-Hodge number, to assure the deformation invariance of the one of (p, q)-type. Resorting to Hodge, Bott–Chern, and Aeppli numbers of manifolds in the Kuranishi family of the Iwasawa manifold (cf. [3, Appendix]), we find the following three examples that the deformation invariance of the (p, q)-Hodge number fails when one of the three conditions is not true, while the other two hold. It indicates that the three conditions above may not be omitted in order to state a theorem for the deformation invariance of all the (p, q)-Hodge numbers.

Let $\mathbb{I}_3$ be the Iwasawa manifold of complex dimension 3 with $\varphi^1, \varphi^2, \varphi^3$ denoted by the basis of the holomorphic one form $H^0(\mathbb{I}_3, \Omega^1)$ of $\mathbb{I}_3$, satisfying the relation

$$d\varphi^1 = 0,\ d\varphi^2 = 0,\ d\varphi^3 = -\varphi^1 \wedge \varphi^2.$$

And the convention $\varphi^{12\bar{1}\bar{3}} := \varphi^1 \wedge \varphi^2 \wedge \overline{\varphi}^1 \wedge \overline{\varphi}^3$ will be used for simplicity.

Example 3.2 (The case $(p, q) = (1, 0)$). The injectivity of $\iota^{1,1}_{\bar{\partial},A}$ holds on $\mathbb{I}_3$ with the deformation invariance of $h^{1,-1}_{\bar{\partial}_t}(X_t)$ trivially established but $\iota^{2,0}_{BC,\partial}$ is not injective. In this case, $h^{1,0}_{\bar{\partial}_t}(X_t)$ are deformation variant.

Proof It is revealed from [3, Appendix] that $h^{1,1}_{\bar{\partial}} = 6$, $h^{1,1}_A = 8$, and $h^{2,0}_{BC} = 3$, $h^{2,0}_{\bar{\partial}} = 2$. And thus $\iota^{2,0}_{BC,\partial}$ is not injective. It is easy to check that

$$H^{1,1}_{\bar{\partial}}(X) = \left\langle [\varphi^{1\bar{1}}]_{\bar{\partial}}, [\varphi^{1\bar{2}}]_{\bar{\partial}}, [\varphi^{2\bar{1}}]_{\bar{\partial}}, [\varphi^{2\bar{2}}]_{\bar{\partial}}, [\varphi^{3\bar{1}}]_{\bar{\partial}}, [\varphi^{3\bar{2}}]_{\bar{\partial}} \right\rangle,$$
$$H^{1,1}_A(X) = \left\langle [\varphi^{1\bar{1}}]_A, [\varphi^{1\bar{2}}]_A, [\varphi^{2\bar{1}}]_A, [\varphi^{2\bar{2}}]_A, [\varphi^{3\bar{1}}]_A, [\varphi^{3\bar{2}}]_A, [\varphi^{1\bar{3}}]_A, [\varphi^{2\bar{3}}]_A \right\rangle,$$

which implies the injectivity of $\iota^{1,1}_{\bar{\partial},A}$. The deformation variance of $h^{1,0}_{\bar{\partial}_t}(X_t)$ can be read from [3, Appendix]. □

Example 3.3 (The case $(p, q) = (2, 0)$). The injectivity of $\iota^{3,0}_{BC,\partial}$ holds on $\mathbb{I}_3$ with the deformation invariance of $h^{2,-1}_{\bar{\partial}_t}(X_t)$ trivially established but $\iota^{2,1}_{\bar{\partial},A}$ is not injective. In this case, $h^{2,0}_{\bar{\partial}_t}(X_t)$ are deformation variant.

Proof We know that $h^{3,0}_{BC} = 1$, $h^{3,0}_{\partial} = 1$, and $h^{2,1}_{\bar{\partial}} = 6$, $h^{2,1}_A = 6$ from [3, Appendix]. The bases of respective cohomology groups can be illustrated as follows:

$$H^{3,0}_{BC} = \left\langle [\varphi^{123}]_{BC} \right\rangle,\ H^{3,0}_{\partial} = \left\langle [\varphi^{123}]_{\partial} \right\rangle,$$
$$H^{2,1}_{\bar{\partial}} = \left\langle [\varphi^{12\bar{1}}]_{\bar{\partial}}, [\varphi^{12\bar{2}}]_{\bar{\partial}}, [\varphi^{13\bar{1}}]_{\bar{\partial}}, [\varphi^{13\bar{2}}]_{\bar{\partial}}, [\varphi^{23\bar{1}}]_{\bar{\partial}}, [\varphi^{23\bar{2}}]_{\bar{\partial}} \right\rangle,$$
$$H^{2,1}_A = \left\langle\ [\varphi^{13\bar{1}}]_A, [\varphi^{13\bar{2}}]_A, [\varphi^{23\bar{1}}]_A, [\varphi^{23\bar{2}}]_A, [\varphi^{13\bar{3}}]_A, [\varphi^{13\bar{2}}]_A \right\rangle,$$

which indicates the injectivity of $\iota^{3,0}_{BC,\partial}$ and non-injectivity of $\iota^{2,1}_{\bar{\partial},A}$. The deformation variance of $h^{2,0}_{\bar{\partial}_t}(X_t)$ can be also got from [3, Appendix]. □

Example 3.4 (The case $(p, q) = (2, 3)$). The mapping $\iota^{3,3}_{BC,\partial}$ is injective on $\mathbb{I}_3$ with the injectivity of $\iota^{2,4}_{\bar{\partial},A}$ trivially established but $h^{2,2}_{\bar{\partial}_t}(X_t)$ are deformation variant. In this case, $h^{2,3}_{\bar{\partial}_t}(X_t)$ are deformation variant.

Proof It is obvious that $\iota^{3,3}_{BC,\partial}$ is injective, since $h^{3,3}_{BC} = 1$, $h^{3,3}_{\partial} = 1$, and

$$H^{3,3}_{BC} = \left\langle [\varphi^{123\bar{1}\bar{2}\bar{3}}]_{BC} \right\rangle, H^{3,3}_{\partial} = \left\langle [\varphi^{123\bar{1}\bar{2}\bar{3}}]_{\partial} \right\rangle.$$

And [3, Appendix] conveys the fact of the deformation variance of $h^{2,2}_{\bar{\partial}_t}(X_t)$ and $h^{2,3}_{\bar{\partial}_t}(X_t)$. □

It is observed that the injectivity of $\iota^{p+1,q}_{BC,\partial}$ or $\iota^{p,q+1}_{\bar{\partial},A}$ is equivalent to a certain type of $\partial\bar{\partial}$-lemma, for which we introduce the following notations:

Notation 3.5 We say a compact complex manifold X satisfies $\mathbb{S}^{p,q}$ and $\mathbb{B}^{p,q}$, if for any $\bar{\partial}$-closed $\partial g \in A^{p,q}(X)$, the equation

$$\bar{\partial} x = \partial g \tag{3.1}$$

has a solution and a ∂-exact solution, respectively. Similarly, a compact complex manifold X is said to satisfy $\mathcal{S}^{p,q}$ and $\mathcal{B}^{p,q}$, if for any $\bar{\partial}$-closed $g \in A^{p-1,q}(X)$, the Eq. (3.1) has a solution and a ∂-exact solution, respectively.

The following implications clearly hold

$$\begin{array}{ccc} \mathbb{B}^{p,q} & \Rightarrow & \mathbb{S}^{p,q} \\ \Downarrow & & \Downarrow \\ \mathcal{B}^{p,q} & \Rightarrow & \mathcal{S}^{p,q}. \end{array}$$

And it is apparent that a compact complex manifold X, where the $\partial\bar{\partial}$-lemma holds, satisfies $\mathbb{B}^{p,q}$ for any (p, q). Here the $\partial\bar{\partial}$-*lemma* refers to: for every pure-type d-closed form on X, the properties of d-exactness, ∂-exactness, $\bar{\partial}$-exactness, and $\partial\bar{\partial}$-exactness are equivalent.

It is easy to check that the following equivalent statements:

$$\begin{aligned} \text{the injectivity of } \iota^{p,q}_{BC,\partial} \text{ holds on } X &\Leftrightarrow X \text{ satisfies } \mathbb{B}^{p,q}; \\ \text{the injectivity of } \iota^{p,q}_{\bar{\partial},A} \text{ holds on } X &\Leftrightarrow X \text{ satisfies } \mathbb{S}^{p,q}; \\ \text{the surjectivity of } \iota^{p-1,q}_{BC,\bar{\partial}} \text{ holds on } X &\Leftrightarrow X \text{ satisfies } \mathcal{B}^{p,q}. \end{aligned}$$

Details of the proofs of theorems in this section will frequently apply Notation 3.5 for the convenience of solving $\bar{\partial}$-equations.

The speciality of the types may lead to the weakening of the conditions in Theorem 3.1, such as $(p, 0)$ and $(0, q)$. Hence, another two theorems follow, whose proofs will be given in Sect. 3.3.

Theorem 3.6 *If the injectivity of the mappings* $\iota^{p+1,0}_{\bar{\partial},A}$ *and* $\iota^{p,1}_{\bar{\partial},A}$ *on* X_0 *holds, then* $h^{p,0}_{\bar{\partial}_t}(X_t)$ *are independent of* t.

Theorem 3.7 *If the surjectivity of the mapping $\iota^{0,q}_{BC,\bar\partial}$ on X_0 and the deformation invariance of $h^{0,q-1}_{\bar\partial_t}(X_t)$ holds, then $h^{0,q}_{\bar\partial_t}(X_t)$ are independent of t.*

Remark 3.8 In the case of $q=1$ of Theorem 3.7, the surjectivity of the mapping $\iota^{0,1}_{BC,\bar\partial}$ is equivalent to the ***sGG*** condition proposed by Popovici–Ugarte [41,45], from [45, Theorem 2.1 (iii)].

Hence, the **sGG** manifolds can be examples of Theorem 3.7, where the Frölicher spectral sequence does not necessarily degenerate at the E_1-level, by [45, Proposition 6.3]. Inspired by the deformation invariance of the $(0,1)$, $(0,2)$, and $(0,3)$-Hodge numbers of the Iwasawa manifold $\mathbb{I}_3$ shown in [3, Appendix], we prove

Corollary 3.9 *Let $X=\Gamma\backslash G$ be a complex parallelizable nilmanifold of complex dimension n, where G is a simply connected nilpotent Lie group and Γ is denoted by a discrete and co-compact subgroup of G. Then X is an **sGG** manifold. In addition, the $(0,q)$-Hodge numbers of X are deformation invariant for $1\le q\le n$.*

Proof It is well known from [50, Theorem 1] and [3, Theorem 3.8] that the isomorphisms

$$H^{p,q}_{BC}(X)\cong H^{p,q}_{BC}(\mathfrak{g},J),\ H^{p,q}_{\bar\partial}(X)\cong H^{p,q}_{\bar\partial}(\mathfrak{g},J),$$

hold on the complex parallelizable nilmanifold X, where $\mathfrak{g}$ is the corresponding Lie algebra of G and J denotes the complex parallelizable structure on $\mathfrak{g}$. Then from Theorem 3.7, the corollary amounts to the verification of the surjectivity of the mappings of $\iota^{0,q}_{BC,\bar\partial}$ on the level of the Lie algebra (g,J) for $1\le q\le n$, which is equivalent to that the conditions $\mathcal{B}^{1,q}$ hold on the Lie algebra (g,J) for $1\le q\le n$.

Since J is complex parallelizable, it yields that $d\mathfrak{g}^{*(1,0)}\subseteq\bigwedge^2\mathfrak{g}^{*(1,0)}$, which implies that $\partial\big(\bigwedge^q\mathfrak{g}^{*(0,1)}\big)=0$ for $1\le q\le n$, where $\mathfrak{g}^*_{\mathbb{C}}=\mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{C}=\mathfrak{g}^{*(1,0)}\oplus\mathfrak{g}^{*(0,1)}$ with respect to J. Therefore, the conditions $\mathcal{B}^{1,q}$ for $1\le q\le n$ are satisfied on the Lie algebra (g,J) and the corollary follows. □

Remark 3.10 The deformation invariance for the $(0,2)$-Hodge number of a complex parallelizable nilmanifold has been shown in [35, Corollary 4.3].

Since nilmanifolds with complex parallelizable structures and abelian complex structures are conjugate to some extent, it is tempting to consider the deformation invariance of the $(p,0)$-Hodge numbers of nilmanifolds with abelian complex structures for $1\le p\le n$ under the spirit of Corollary 3.9. The following example, inspired by Console–Fino–Poon [14, Sect. 6], is a holomorphic family of nilmanifolds of complex dimension 5, whose central fiber is endowed with an abelian complex structure. This family admits the deformation invariance of the $(p,0)$-Hodge numbers for $1\le p\le 5$, but not the $(1,1)$-Hodge number or $(1,1)$-Bott–Chern number, which shows the function of Theorem 3.6 possibly beyond Kodaira–Spencer's squeeze [28, Theorem 13] in this case.

Example 3.11 Let X_0 be the nilmanifold determined by a ten-dimensional 3-step nilpotent Lie algebra $\mathfrak{n}$ endowed with the complex structure $J_{s,t}$ for $s = 1, t = 0$, as in [14, Sect. 6]. The natural decompositions with respect to the complex structure $J_{1,0}$ yield

$$\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1};\ \mathfrak{n}^*_{\mathbb{C}} = \mathfrak{n}^* \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n}^{*(1,0)} \oplus \mathfrak{n}^{*(0,1)}.$$

By contrast with the basis $\omega^1, \ldots, \omega^5$ of $\mathfrak{n}^{*(1,0)}$ used in [14, Sect. 6], another basis $\tau^1, \ldots, \tau^5$ will be applied, with the transition formula given by

$$\tau^1 = \omega^1, \tau^2 = (1+i)\omega^2 - \omega^3, \tau^3 = -(1+i)\omega^2, \tau^4 = \omega^4, \tau^5 = \omega^5.$$

Hence, the structure equation with respect to $\{\tau^k\}_{k=1}^5$ follows

$$\begin{cases} d\tau^1 = d\tau^2 = d\tau^4 = 0, \\ d\tau^3 = -\big(\tau^1 \wedge \bar{\tau}^1 + (1+i)\tau^1 \wedge \bar{\tau}^4\big), \\ d\tau^5 = \frac{1}{2}\big(\tau^1 \wedge \bar{\tau}^3 + \tau^3 \wedge \bar{\tau}^1 - \tau^2 \wedge \bar{\tau}^2\big). \end{cases} \tag{3.2}$$

It is easy to see $d\bar{\tau}^5 = -d\tau^5$, which implies $\partial\bar{\tau}^5 = -\overline{\partial}\tau^5$. Denote the basis of $\mathfrak{n}^{1,0}$ dual to $\{\tau^k\}_{k=1}^5$ by $\theta_1, \ldots, \theta_5$. The equation $d\omega(\theta, \theta') = -\omega([\theta, \theta'])$ for $\omega \in \mathfrak{n}^*_{\mathbb{C}}$ and $\theta, \theta' \in \mathfrak{n}_{\mathbb{C}}$, establishes the equalities

$$[\bar{\theta}_1, \theta_4] = (1-i)\bar{\theta}_3,\ \ [\bar{\theta}_i, \theta_4] = 0 \ \text{ for } 2 \le i \le 5.$$

According to [14, Theorem 3.6], the linear operator $\overline{\partial}$ on $\mathfrak{n}^{1,0}$, defined in [14, Sect. 3.2] by

$$\overline{\partial} : \mathfrak{n}^{1,0} \to \mathfrak{n}^{*(0,1)} \otimes \mathfrak{n}^{1,0} :\ \overline{\partial}_{\bar{U}} V = [\bar{U}, V]^{1,0} \ \text{ for } U, V \in \mathfrak{n}^{1,0},$$

produces an isomorphism $H^1(X_0, T^{1,0}_{X_0}) \cong H^1_{\overline{\partial}}(\mathfrak{n}^{1,0})$. Therefore, from Kodaira–Spencer deformation theory, an analytic deformation X_t of X_0 can be constructed by use of the integrable Beltrami differential

$$\varphi(t) = t_1\bar{\tau}^5 \otimes \theta_4 + t_2\bar{\tau}^4 \otimes \theta_4$$

for t_1, t_2 small complex numbers and $t = (t_1, t_2)$, which satisfies $\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$ and the so-called Schouten–Nijenhuis bracket $[\cdot, \cdot]$ (cf. [14, Formula (4.1)]) works as

$$[\bar{\omega} \otimes V, \bar{\omega}' \otimes V'] = \bar{\omega}' \wedge i_{V'} d\bar{\omega} \otimes V + \bar{\omega} \wedge i_V d\bar{\omega}' \otimes V' \ \text{ for } \omega, \omega' \in \mathfrak{n}^{*(1,0)}, V, V' \in \mathfrak{n}^{1,0},$$

since $\overline{\partial}\theta_4 = 0$ and $i_{\theta_4} d\bar{\tau}^5 = i_{\theta_4} d\bar{\tau}^4 = 0$. Then the general fibers X_t are still nilmanifolds, determined by the Lie algebra $\mathfrak{n}$ and the decompositions

$$\mathfrak{n}^*_{\mathbb{C}} = \mathfrak{n}^* \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n}^{*(1,0)}_{\varphi(t)} \oplus \mathfrak{n}^{*(0,1)}_{\varphi(t)},$$

with the basis of $\mathfrak{n}^{*(1,0)}_{\varphi(t)}$ given by $\tau^k(t) = e^{i_{\varphi(t)}}(\tau^k) = (\mathbb{1} + \varphi(t))\lrcorner\tau^k$ for $1 \leq k \leq 5$. Hence, the structure equation of $\{\tau^k(t)\}_{k=1}^5$ amounts to

$$\begin{cases} d\tau^1(t) = d\tau^2(t) = 0, \\ d\tau^4(t) = -t_1 d\tau^5(t), \\ d\tau^3(t) = \frac{1+i}{1-|t_2|^2}\big(\bar{t}_2\tau^1(t) \wedge \tau^4(t) + \bar{t}_1\tau^1(t) \wedge \tau^5(t)\big) \\ \qquad\qquad -\tau^1(t) \wedge \bar{\tau}^1(t) - \frac{1+i}{1-|t_2|^2}\big(\tau^1(t) \wedge \bar{\tau}^4(t) + t_1\bar{t}_2\tau^1(t) \wedge \bar{\tau}^5(t)\big), \\ d\tau^5(t) = \frac{1}{2}\big(\tau^1(t) \wedge \bar{\tau}^3(t) + \tau^3(t) \wedge \bar{\tau}^1(t) - \tau^2(t) \wedge \bar{\tau}^2(t)\big). \end{cases} \tag{3.3}$$

The proof of Theorem 3.6, which is contained in Proposition 3.19, shows that the obstruction of the deformation invariance of the $(p, 0)$-Hodge numbers along the family determined by $\varphi(t)$ actually lies in the Eq. (3.13), where the differential forms involved are invariant ones in this case. For any $\bar{\partial}$-closed $\sigma_0 \in \bigwedge^p \mathfrak{n}^{*(1,0)}$, it is easy to check that

$$\sigma_t = \sigma_0 + t_1\tau^5 \wedge (\theta_4 \lrcorner \sigma_0)$$

solves the equation (3.13), due to the equalities $\partial\bar{\tau}^5 = -\bar{\partial}\tau^5$ and $d\tau^4 = 0$. However, based on the structure equations (3.2) and (3.3), it yields that

$$h^{1,1}_{\bar{\partial}}(X_0) = 14,\ h^{1,1}_{\bar{\partial}_t}(X_t) = 11 \quad \text{and} \quad h^{1,1}_{BC}(X_0) = 11,\ h^{1,1}_{BC}(X_t) = 9,$$

where $t_2 \neq 0$ and $t_1\bar{t}_2 - \bar{t}_1 \neq 0$.

3.2 Proofs of the Invariance of Hodge Numbers $h^{p,q}_{\bar{\partial}_t}(X_t)$

This subsection is to prove Theorem 3.1, which can be restated by the use of Notation 3.5: if the central fiber X_0 satisfies both $\mathbb{B}^{p+1,q}$ and $\mathbb{S}^{p,q+1}$ with the deformation invariance of $h^{p,q-1}_{\bar{\partial}_t}(X_t)$ established, then $h^{p,q}_{\bar{\partial}_t}(X_t)$ are independent of t.

The basic strategy is described at the beginning of Sect. 3.1 and obviously our task is divided into two steps (1) and (2), which are to be completed in Propositions 3.14 and 3.15, respectively.

To complete (1), we need a lemma due to [41, Theorem 4.1] or [46, Lemma 3.14] for the resolution of $\partial\bar{\partial}$-equations.

Lemma 3.12 *Let (X, ω) be a compact Hermitian complex manifold with any suitable pure-type complex differential forms x and y. Assume that the $\partial\bar{\partial}$-equation*

$$\partial\bar{\partial}x = y \tag{3.4}$$

admits a solution. Then an explicit solution of the $\partial\bar{\partial}$-equation (3.4) can be chosen as

$$(\partial\bar{\partial})^*\mathbb{G}_{BC}y,$$

which uniquely minimizes the L^2-norms of all the solutions with respect to ω.

Here $\mathbb{G}_{BC}$ is the associated Green's operator of the *first* 4*-th order Kodaira–Spencer operator* (also often called *Bott–Chern Laplacian*) given by

$$\square_{BC} = \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial.$$

We need one more lemma inspired by [43, Lemma 3.1].

Lemma 3.13 *Assume that a compact complex manifold X satisfies $\mathcal{B}^{p+1,q}$. Each Dolbeault class $[\sigma]_{\bar{\partial}}$ of the (p,q) type can be canonically represented by a uniquely chosen d-closed (p,q)-form γ_σ.*

Proof We first choose the unique harmonic representative of $[\sigma]_{\bar{\partial}}$, still denoted by σ. It is clear that the d-closed representative $\gamma_\sigma \in A^{p,q}(X)$ satisfies

$$\sigma + \bar{\partial}\beta_\sigma = \gamma_\sigma$$

for some $\beta_\sigma \in A^{p,q-1}(X)$. This is equivalent that some $\beta_\sigma \in A^{p,q-1}(X)$ solves the following equation

$$\partial\bar{\partial}\beta_\sigma = -\partial\sigma.$$

The existence of β_σ is assured by our assumption on X and uniqueness with L^2-norm minimum by Lemma 3.12, that is, one can choose β_σ as $-(\partial\bar{\partial})^*\mathbb{G}_{BC}\partial\sigma$. □

Proposition 3.14 *Assume that X_0 satisfies $\mathbb{B}^{p+1,q}$ and $\mathbb{S}^{p,q+1}$. Then for each Dolbeault class in $H^{p,q}_{\bar{\partial}}(X_0)$ with the unique canonical d-closed representative σ_0 given as Lemma 3.13, there exists a power series on X_0*

$$\sigma_t = \sigma_0 + \sum_{j+k=1}^{\infty} t^k \bar{t}^{\bar{j}} \sigma_{k\bar{j}} \in A^{p,q}(X_0),$$

such that σ_t varies smoothly on t and $e^{i_\varphi|i_{\bar{\varphi}}}(\sigma_t) \in A^{p,q}(X_t)$ is $\bar{\partial}_t$-closed with respect to the holomorphic structure on X_t.

Proof The construction of σ_t is presented at first. The canonical choice of the representative for the initial Dolbeault cohomology class is guaranteed by the assumption that X_0 satisfies $\mathbb{B}^{p+1,q}$, which implies that $\mathcal{B}^{p+1,q}$ holds, and Lemma 3.13. By Proposition 2.13, the desired $\bar{\partial}_t$-closedness is equivalent to the resolution of the equation

$$([\partial, i_\varphi] + \bar{\partial})(\mathbb{1} - \bar{\varphi}\varphi) \lrcorner \sigma_t = 0. \tag{3.5}$$

Set $\widetilde{\sigma}_t = (\mathbb{1} - \bar{\varphi}\varphi) \lrcorner \sigma_t$ and we just need to resolve the system of equations

$$\begin{cases} \partial\widetilde{\sigma}_t = 0, \\ \bar{\partial}\widetilde{\sigma}_t + \partial\big(\varphi(t) \lrcorner \widetilde{\sigma}_t\big) = 0. \end{cases} \tag{3.6}$$

An iteration method, developed in [33,34,46,47,52,53,63,64], will be applied to resolve this system. Let

$$\widetilde{\sigma}_t = \widetilde{\sigma}_0 + \sum_{j=1}^{\infty} \widetilde{\sigma}_j t^j$$

be a power series of (p, q)-forms on X_0. By substituting this power series into (3.6) and comparing the coefficients of t^k, we turn to resolving

$$\begin{cases} d\widetilde{\sigma}_0 = 0, \\ \bar{\partial}\widetilde{\sigma}_k = -\partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \widetilde{\sigma}_{k-i}\right), \qquad \text{for each } k \geq 1, \\ \partial\widetilde{\sigma}_k = 0, \qquad \text{for each } k \geq 1. \end{cases} \tag{3.7}$$

Notice that $\widetilde{\sigma}_0 = \sigma_0$ and thus $d\widetilde{\sigma}_0 = 0$ by the choice of the canonical d-closed representative for the initial Dolbeault class in $H^{p,q}_{\bar{\partial}}(X_0)$.

As for the second equation of (3.7), we may assume that $\widetilde{\sigma}_i$, satisfying $\partial\widetilde{\sigma}_i = 0$, has been resolved for $0 \leq i \leq k-1$, and then check

$$\bar{\partial}\partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \widetilde{\sigma}_{k-i}\right) = 0.$$

In fact, by the integrability (2.24) and the commutator formula (2.2), one has

$$\begin{aligned} &-\bar{\partial}\partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \widetilde{\sigma}_{k-i}\right) \\ &= \partial\left(\sum_{i=1}^{k} \bar{\partial}\varphi_i \lrcorner \widetilde{\sigma}_{k-i} + \sum_{i=1}^{k} \varphi_i \lrcorner \bar{\partial}\widetilde{\sigma}_{k-i}\right) \\ &= \partial\left(\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i-1}[\varphi_j, \varphi_{i-j}] \lrcorner \widetilde{\sigma}_{k-i} - \sum_{i=1}^{k} \varphi_i \lrcorner \partial\left(\sum_{j=1}^{k-i} \varphi_j \lrcorner \widetilde{\sigma}_{k-i-j}\right)\right) \\ &= \partial\left(\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\Big(-\partial\big(\varphi_{i-j} \lrcorner (\varphi_j \lrcorner \widetilde{\sigma}_{k-i})\big) - \varphi_{i-j} \lrcorner (\varphi_j \lrcorner \partial\widetilde{\sigma}_{k-i})\right. \\ &\quad \left. + \varphi_j \lrcorner \partial(\varphi_{i-j} \lrcorner \widetilde{\sigma}_{k-i}) + \varphi_{i-j} \lrcorner \partial(\varphi_j \lrcorner \widetilde{\sigma}_{k-i})\Big) - \sum_{i=1}^{k} \varphi_i \lrcorner \partial\left(\sum_{j=1}^{k-i} \varphi_j \lrcorner \widetilde{\sigma}_{k-i-j}\right)\right) \\ &= \partial\left(\sum_{1 \leq j < i \leq k} \varphi_j \lrcorner \partial(\varphi_{i-j} \lrcorner \widetilde{\sigma}_{k-i}) - \sum_{i=1}^{k}\sum_{j=1}^{k-i} \varphi_i \lrcorner \partial(\varphi_j \lrcorner \widetilde{\sigma}_{k-i-j})\right) \\ &= 0. \end{aligned} \tag{3.8}$$

Hence, one can obtain a canonical solution

$$\widetilde{\sigma_k^1} = -\overline{\partial}^* \mathbb{G}_{\overline{\partial}} \partial \left(\sum_{i=1}^{k} \varphi_i \lrcorner \widetilde{\sigma}_{k-i} \right)$$

by the assumption that X_0 satisfies $\mathbb{S}^{p,q+1}$ and the useful fact that $\overline{\partial}^* \mathbb{G}_{\overline{\partial}} y$ is the unique solution, minimizing the L^2-norms of all the solutions, of the equation

$$\overline{\partial} x = y$$

on a compact complex manifold if the equation admits one, where x, y are pure-type complex differential forms and the operator $\mathbb{G}_{\overline{\partial}}$ denotes the corresponding Green's operator of the $\overline{\partial}$-Laplacian $\square$.

To fulfill the third equation $\partial \widetilde{\sigma}_k = 0$, we try to find some $\widetilde{\sigma_k^2} \in A^{p,q-1}(X_0)$ such that

$$\partial \left(\widetilde{\sigma_k^1} + \overline{\partial} \widetilde{\sigma_k^2} \right) = 0. \tag{3.9}$$

Then the solution $\widetilde{\sigma}_k$ can be set as

$$\widetilde{\sigma}_k = \widetilde{\sigma_k^1} + \overline{\partial} \widetilde{\sigma_k^2},$$

which satisfies both the second and the third equation of (3.7). At this moment, the assumption $\mathbb{B}^{p+1,q}$ on X_0 and Lemma 3.13 will also provide us a solution of (3.9)

$$\widetilde{\sigma_k^2} = -(\partial \overline{\partial})^* \mathbb{G}_{BC} \partial \widetilde{\sigma_k^1},$$

which yields

$$\widetilde{\sigma}_k = -\overline{\partial}^* \mathbb{G}_{\overline{\partial}} \partial \left(\sum_{i=1}^{k} \varphi_i \lrcorner \widetilde{\sigma}_{k-i} \right) + \overline{\partial} (\partial \overline{\partial})^* \mathbb{G}_{BC} \partial \overline{\partial}^* \mathbb{G}_{\overline{\partial}} \partial \left(\sum_{i=1}^{k} \varphi_i \lrcorner \widetilde{\sigma}_{k-i} \right).$$

Finally we resort to the elliptic estimates for the regularity of $\widetilde{\sigma}_t$, which is quite analogous to that in [46, Theorems 2.12 and 3.11]. So we just sketch this argument, which is divided into two steps:

(i) $\| \sum_{j=1}^{\infty} \widetilde{\sigma}_j t^j \|_{k,\alpha} \ll A(t)$;
(ii) $\widetilde{\sigma}_t$ is a real analytic family of (p, q)-forms in t .

Here are explicit details for the first step (i). Consider an important power series in deformation theory of complex structures

$$A(t) = \frac{\beta}{16\gamma} \sum_{m=1}^{\infty} \frac{(\gamma t)^m}{m^2} := \sum_{m=1}^{\infty} A_m t^m, \tag{3.10}$$

where β, γ are positive constants to be determined. The power series (3.10) converges for $|t| < \frac{1}{\gamma}$ and has a nice property:

$$A^i(t) \ll \left(\frac{\beta}{\gamma}\right)^{i-1} A(t).$$

See [37, Lemma 3.6 and its Corollary in Chapter 2] for these basic facts. We use the following notation: For the series with real positive coefficients

$$a(t) = \sum_{m=1}^{\infty} a_m t^m, \quad b(t) = \sum_{m=1}^{\infty} b_m t^m,$$

say that $a(t)$ *dominates* $b(t)$, written as $b(t) \ll a(t)$, if $b_m \le a_m$. But for a power series of (bundle-valued) complex differential forms

$$\eta(t) = \sum_{m=0}^{\infty} \eta_m t^m,$$

the notation

$$\|\eta(t)\|_{k,\alpha} \ll A(t)$$

means

$$\|\eta_m\|_{k,\alpha} \le A_m$$

with the $C^{k,\alpha}$-norm $\|\cdot\|_{k,\alpha}$ as defined on [37, Page 159]. Recall that the canonical family of Beltrami differentials $\varphi(t)$ satisfies a nice convergence property:

$$\|\varphi(t)\|_{k,\alpha} \ll A(t)$$

as given in the proof of [37, Proposition 2.4 in Chapter 4]. We need three more a priori elliptic estimates as follows. For any complex differential form ϕ,

$$\begin{aligned} \|\bar{\partial}^* \phi\|_{k-1,\alpha} &\le C_1 \|\phi\|_{k,\alpha}, \\ \|\mathbb{G}_{\bar{\partial}} \phi\|_{k,\alpha} &\le C_{k,\alpha} \|\phi\|_{k-2,\alpha}, \end{aligned}$$

where $k > 1$, C_1 and $C_{k,\alpha}$ depend only on k and α, not on ϕ, as shown in [37, Proposition 2.3 in Chapter 4], and

$$\|\mathbb{G}_{BC} \phi\|_{k,\alpha} \le C_{k,\alpha} \|\phi\|_{k-4,\alpha},$$

where $k > 3$ and $C_{k,\alpha}$ depends on only on k and α, not on ϕ, as shown in [27, Appendix.Theorem 7.4] for example. Based on these, an inductive argument implies

$$\left\| \sum_{j=1}^{l} \widetilde{\sigma}_j t^j \right\|_{k,\alpha} \ll A(t)$$

for any large $l > 0$ and each $k > 3$. Then (i) follows.

We proceed to (ii) since there is possibly no uniform lower bound for the convergence radius obtained in the $C^{k,\alpha}$-norm as k converges to $+\infty$. Applying the $\overline{\partial}$-Laplacian $\square = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ to

$$\widetilde{\sigma}_t = -\overline{\partial}^*\mathbb{G}_{\overline{\partial}}\partial\,(\varphi\lrcorner\widetilde{\sigma}_t) + \overline{\partial}(\partial\overline{\partial})^*\mathbb{G}_{BC}\partial\overline{\partial}^*\mathbb{G}_{\overline{\partial}}\partial\,(\varphi\lrcorner\widetilde{\sigma}_t) + \sigma_0$$

and the proof of [27, Appendix.Theorem 2.3] or [46, Proposition 3.15], one proves the following result. For each $l = 1, 2, \ldots$, choose a smooth function $\eta^l(t)$ with values in $[0, 1]$:

$$\eta^l(t) \equiv \begin{cases} 1, & \text{for } |t| \le (\frac{1}{2} + \frac{1}{2^{l+1}})r, \\ 0, & \text{for } |t| \ge (\frac{1}{2} + \frac{1}{2^l})r, \end{cases}$$

where r is a positive constant to be determined. Inductively, for any $l = 1, 2, \ldots$, $\eta^{2l+1}\widetilde{\sigma}_t$ is $C^{k+l,\alpha}$, where r can be chosen independently of l. Since $\eta^{2l+1}(t)$ is identically equal to 1 on $|t| < \frac{r}{2}$ which is independent of l, $\widetilde{\sigma}_t$ is C^∞ on X_0 with $|t| < \frac{r}{2}$. Then $\widetilde{\sigma}_t$ can be considered as a real analytic family of (p, q)-forms in t and thus it is smooth on t. □

In the first version [47] of this paper, we resort to J. Wavrik's work [57, Sect. 3] for the above regularity.

To guarantee (2), it suffices to prove:

Proposition 3.15 *If the $\overline{\partial}$-extension of $H^{p,q}_{\overline{\partial}}(X_0)$ as in Proposition 3.14 holds for a complex manifold X_0, then the deformation invariance of $h^{p,q-1}_{\overline{\partial}_t}(X_t)$ assures that the extension map*

$$H^{p,q}_{\overline{\partial}}(X_0) \to H^{p,q}_{\overline{\partial}_t}(X_t) : [\sigma_0]_{\overline{\partial}} \mapsto [e^{i_\varphi|i_{\overline{\varphi}}}(\sigma_t)]_{\overline{\partial}_t}$$

is injective.

Proof Let us fix a family of smoothly varying Hermitian metrics $\{\omega_t\}_{t\in\Delta_\epsilon}$ for the infinitesimal deformation $\pi : \mathcal{X} \to \Delta_\epsilon$ of X_0. Thus, if the Hodge numbers $h^{p,q-1}_{\overline{\partial}_t}(X_t)$ are deformation invariant, the Green's operator $\mathbb{G}_t$, acting on the $A^{p,q-1}(X_t)$, depends differentiably with respect to t from [28, Theorem 7] by Kodaira and Spencer. Using this, one ensures that this extension map cannot send a non-zero class in $H^{p,q}_{\overline{\partial}}(X_0)$ to a zero class in $H^{p,q}_{\overline{\partial}_t}(X_t)$.

If we suppose that

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma_t) = \overline{\partial}_t\eta_t$$

for some $\eta_t \in A^{p,q-1}(X_t)$ when $t \in \Delta_\epsilon \setminus \{0\}$, the Hodge decomposition of $\overline{\partial}_t$ and the commutativity of $\mathbb{G}_t$ with $\overline{\partial}_t^*$ and $\overline{\partial}_t$ yield that

$$
\begin{aligned}
e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma_t) = \overline{\partial}_t \eta_t &= \overline{\partial}_t \big(\mathbb{H}_t(\eta_t) + \square_t \mathbb{G}_t \eta_t\big) \\
&= \overline{\partial}_t \big(\overline{\partial}_t^* \overline{\partial}_t \mathbb{G}_t \eta_t\big) \\
&= \overline{\partial}_t \mathbb{G}_t \big(\overline{\partial}_t^* \overline{\partial}_t \eta_t\big) \\
&= \overline{\partial}_t \mathbb{G}_t \big(\overline{\partial}_t^* e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma_t)\big),
\end{aligned}
$$

where $\mathbb{H}_t$ and $\square_t$ are the harmonic projectors and the Laplace operators with respect to (X_t, ω_t), respectively. Let t converge to 0 on both sides of the equality

$$
e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma_t) = \overline{\partial}_t \mathbb{G}_t \big(\overline{\partial}_t^* e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma_t)\big),
$$

which turns out that σ_0 is $\overline{\partial}$-exact on the central fiber X_0. Here we use that the Green's operator $\mathbb{G}_t$ depends differentiably with respect to t. □

Example 3.16 (The case $q = n$). The deformation invariance for $h^{p,n}_{\overline{\partial}_t}(X_t)$ can be obtained from the one for $h^{p,n-1}_{\overline{\partial}_t}(X_t)$.

Proof Actually, it is easy to see that $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma) \in A^{p,n}(X_t)$ for any $\sigma \in A^{p,n}(X_0)$. By the consideration of types, the equality

$$
\overline{\partial}_t(e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\sigma)) = 0 \tag{3.11}
$$

trivially holds, without the necessity of the choice of a canonical d-closed representative or solving the Eq. (3.11) as in Proposition 3.14. And thus, from Proposition 3.15, the extension map

$$
H^{p,n}_{\overline{\partial}}(X_0) \to H^{p,n}_{\overline{\partial}_t}(X_t) : [\sigma]_{\overline{\partial}} \mapsto [e^{i_\varphi|i_{\overline{\varphi}}}(\sigma)]_{\overline{\partial}}
$$

is injective. We can also revisit this example by [27, Formula (7.74)]

$$
h^{p,q}_{\overline{\partial}_t}(X_t) + \nu^q(t) + \nu^{q+1}(t) = h^{p,q}_{\overline{\partial}}(X),
$$

where $\nu^q(t)$ is the number of eigenvalues $\sigma^q_j(t)$ for the canonical base f^q_{tj} of eigenforms of the Laplacian $\square_t = \overline{\partial}_t\overline{\partial}_t^* + \overline{\partial}_t^*\overline{\partial}_t$ less than some fixed positive constant. Notice that $\nu^{n+1}(t) = 0$. For more details see [27, Sect. 7.2.(c)]. □

Proposition 3.15 and Example 3.16 are indeed inspired by Nakamura's work [38, Theorem 2], which asserts that all plurigenera are not necessarily invariant under infinitesimal deformations, particularly for the Hodge number $h^{n,0}_{\overline{\partial}}$ and thus $h^{0,n}_{\overline{\partial}}$, while the obstruction Eq. (3.11) for extending $\overline{\partial}_t$-closed $(0, n)$-forms is un-obstructed. This example actually tells us that deformation invariance of $h^{0,n}_{\overline{\partial}}$ relies on the one of $h^{0,n-1}_{\overline{\partial}}$.

Proposition 3.17 *If $h^{p,q+1}_{\overline{\partial}}(X_0) = 0$ and the deformation invariance of $h^{p,q-1}_{\overline{\partial}_t}(X_t)$ holds, then $h^{p,q}_{\overline{\partial}_t}(X_t)$ are deformation invariant.*

Proof With the notations in the proof of Proposition 3.14, we can resolve Eq. (3.5) directly, which is equivalent to the following equation:

$$\overline{\partial}\sigma_k = -\partial\left(\sum_{i=1}^{k}\varphi_i \lrcorner \sigma_{k-i}\right) + \sum_{i=1}^{k}\varphi_i \lrcorner \partial\sigma_{k-i} \quad \text{for each } k \geq 1, \tag{3.12}$$

by use of the assumption that $h_{\overline{\partial}}^{p,q+1}(X_0) = 0$. Also interestingly notice that we are not able to deal with this case by the system (3.7) of equations. Set

$$\tau_k = -\partial\left(\sum_{i=1}^{k}\varphi_i \lrcorner \sigma_{k-i}\right) + \sum_{i=1}^{k}\varphi_i \lrcorner \partial\sigma_{k-i},$$
$$\eta_k = -\partial\left(\sum_{i=1}^{k}\varphi_i \lrcorner \sigma_{k-i}\right).$$

When $k = 1$, we have

$$\begin{aligned}\overline{\partial}\tau_1 &= \overline{\partial}\big(-\partial(\varphi_1 \lrcorner \sigma_0) + \varphi_1 \lrcorner \partial\sigma_0\big)\\ &= \partial(\overline{\partial}\varphi_1 \lrcorner \sigma_0 + \varphi_1 \lrcorner \overline{\partial}\sigma_0) + \overline{\partial}\varphi_1 \lrcorner \partial\sigma_0 + \varphi_1 \lrcorner \overline{\partial}\partial\sigma_0\\ &= 0,\end{aligned}$$

since $\overline{\partial}\varphi_1 = 0$ and $\overline{\partial}\sigma_0 = 0$. The assumption $h_{\overline{\partial}}^{p,q+1}(X_0) = 0$ implies that the equation

$$\overline{\partial}\sigma_1 = \tau_1$$

has a solution σ_1.

Assume that the Eq. (3.12) is solved for all $k \leq l$. Based on the assumption $h_{\overline{\partial}}^{p,q+1}(X_0) = 0$, the equation

$$\overline{\partial}\sigma_{l+1} = \tau_{l+1}$$

will have a solution σ_{l+1}, after we verify

$$\overline{\partial}\tau_{l+1} = 0.$$

Hence, we check it as follows, by use of the calculation (3.8), which implies that

$$\begin{aligned}\overline{\partial}\eta_{l+1} &= \partial\left(-\frac{1}{2}\sum_{i=1}^{l+1}\sum_{j=1}^{i-1}\varphi_j \lrcorner(\varphi_{i-j} \lrcorner \partial\sigma_{l+1-i}) + \sum_{i=1}^{l+1}\sum_{j=1}^{l+1-i}\varphi_i \lrcorner(\varphi_j \lrcorner \partial\sigma_{l+1-i-j})\right)\\ &= \partial\left(\frac{1}{2}\sum_{i=1}^{l+1}\sum_{j=1}^{l+1-i}\varphi_i \lrcorner(\varphi_j \lrcorner \partial\sigma_{l+1-i-j})\right),\end{aligned}$$

in this case. Then it follows that

$$
\begin{aligned}
\bar{\partial}\tau_{l+1} &= \bar{\partial}\eta_{l+1} + \sum_{i=1}^{l+1} \bar{\partial}\varphi_i \lrcorner \partial\sigma_{l+1-i} - \sum_{i=1}^{l+1} \varphi_i \lrcorner \partial\bar{\partial}\sigma_{l+1-i} \\
&= \partial\left(\frac{1}{2}\sum_{i=1}^{l+1}\sum_{j=1}^{l+1-i} \varphi_i \lrcorner (\varphi_j \lrcorner \partial\sigma_{l+1-i-j})\right) + \sum_{i=1}^{l+1}\sum_{j=1}^{i-1} \frac{1}{2}[\varphi_j, \varphi_{i-j}] \lrcorner \partial\sigma_{l+1-i} \\
&\quad + \sum_{i=1}^{l+1} \varphi_i \lrcorner \partial \left(\partial\Big(\sum_{j=1}^{l+1-i} \varphi_j \lrcorner \sigma_{l+1-i-j}\Big) - \sum_{j=1}^{l+1-i} \varphi_j \lrcorner \partial\sigma_{l+1-i-j} \right) \\
&= \partial\left(\frac{1}{2}\sum_{i=1}^{l+1}\sum_{j=1}^{l+1-i} \varphi_i \lrcorner (\varphi_j \lrcorner \partial\sigma_{l+1-i-j})\right) \\
&\quad + \sum_{i=1}^{l+1}\sum_{j=1}^{i-1} \frac{1}{2}\Big(-\partial\Big(\varphi_j \lrcorner (\varphi_{i-j} \lrcorner \partial\sigma_{l+1-i})\Big) \\
&\quad + \varphi_j \lrcorner \partial\Big(\varphi_{i-j} \lrcorner \partial\sigma_{l+1-i}\Big) + \varphi_{i-j} \lrcorner \partial\Big(\varphi_j \lrcorner \partial\sigma_{l+1-i}\Big)\Big) \\
&\quad - \sum_{i=1}^{l+1}\sum_{j=1}^{l+1-i} \varphi_i \lrcorner \partial\Big(\varphi_j \lrcorner \partial\sigma_{l+1-i-j}\Big) \\
&= 0.
\end{aligned}
$$

Therefore, we can also resolve the Eq. (3.12) and extend $\bar{\partial}$-closed (p, q)-forms unobstructed under the assumption that $h^{p,q+1}_{\bar{\partial}}(X_0) = 0$. □

3.3 Proofs of the Invariance of Hodge Numbers $h^{p,0}(X_t)$, $h^{0,q}(X_t)$: special cases

This subsection is devoted to the deformation invariance of $(p, 0)$ and $(0, q)$-Hodge numbers as two special cases of Theorem 3.1.

Theorem 3.6 can be restated by use of Notation 3.5 as follows:

Theorem 3.18 *If the central fiber X_0 satisfies both $\mathbb{S}^{p+1,0}$ and $\mathbb{S}^{p,1}$, then $h^{p,0}_{\bar{\partial}_t}(X_t)$ are independent of t.*

According to the philosophy described in Sect. 3.1, Theorem 3.18 amounts to:

Proposition 3.19 *Assume that X_0 satisfies $\mathbb{S}^{p+1,0}$ and $\mathbb{S}^{p,1}$. Then for any holomorphic $(p, 0)$-form σ_0 on X_0, there exits a power series*

$$\sigma_t = \sigma_0 + \sum_{k=1}^{\infty} t^k \sigma_k \in A^{p,0}(X_0),$$

such that σ_t varies smoothly on t and $e^{i_{\varphi(t)}}(\sigma_t) \in A^{p,0}(X_t)$ is holomorphic with respect to the holomorphic structure on X_t.

Proof With the notations in the proof of Proposition 3.14, we just present the construction of σ_t since the regularization argument is quite similar. Obviously, under the assumption $\mathbb{S}^{p+1,0}$ on X_0, the holomorphic $(p, 0)$-form σ_0 is actually d-closed. By Proposition 2.13 and type-consideration, the desired holomorphicity is equivalent to the resolution of the equation

$$\left([\partial, i_\varphi] + \bar{\partial}\right)(\mathbb{1} - \bar{\varphi}\varphi) \lrcorner \sigma_t = ([\partial, i_\varphi] + \bar{\partial})\sigma_t = 0. \tag{3.13}$$

Let

$$\sigma_t = \sigma_0 + \sum_{j=1}^{\infty} \sigma_j t^j$$

be a power series of $(p, 0)$-forms on X_0.

We will also resolve (3.13) by an iteration method. It suffices to consider the system of equations

$$\begin{cases} \bar{\partial}\sigma_0 = 0, & \\ \bar{\partial}\sigma_k = -\partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \sigma_{k-i}\right), & \text{for each } k \geq 1, \\ \partial\sigma_k = 0, & \text{for each } k \geq 0, \end{cases} \tag{3.14}$$

after the comparison of the coefficients of t^k.

As for the second equation of (3.14), we may also assume that, for $i = 0, \ldots, k-1$, $\widetilde{\sigma_i}$ with $\partial\widetilde{\sigma_i} = 0$ has been resolved, and then check

$$\bar{\partial}\partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \sigma_{k-i}\right) = 0$$

as reasoned in (3.8). The assumption $\mathbb{S}^{p,1}$ enables us to obtain a canonical solution

$$\sigma_k = -\bar{\partial}^* \mathbb{G}_{\bar{\partial}} \partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \sigma_{k-i}\right).$$

Meanwhile, the third equation $\partial\sigma_k = 0$ holds, due to the assumption $\mathbb{S}^{p+1,0}$ and the equality

$$\bar{\partial}\partial\sigma_k = \partial\partial\left(\sum_{i=1}^{k} \varphi_i \lrcorner \sigma_{k-i}\right) = 0.$$

□

Corollary 3.20 (The case of $(p,q)=(1,0)$). *If the central fiber X_0 satisfies both $\mathbb{S}^{2,0}$ and $\mathcal{S}^{1,1}$, then $h^{1,0}_{\overline{\partial}_t}(X_t)$ are independent of t.*

Proof From Theorem 3.18, $h^{1,0}_{\overline{\partial}_t}(X_t)$ are independent of t when X_0 satisfies $\mathbb{S}^{2,0}$ and $\mathbb{S}^{1,1}$. The condition $\mathbb{S}^{1,1}$ can be replaced by a weaker one $\mathcal{S}^{1,1}$.

A close observation to (3.8) and the fact that σ_i are all of the special type $(1,0)$ show that

$$\begin{aligned}\overline{\partial}\left(\sum_{i=1}^{k}\varphi_i\lrcorner\sigma_{k-i}\right) &= \frac{1}{2}\sum_{i=1}^{l+1}\sum_{j=1}^{i-1}\Big(-\partial\big(\varphi_j\lrcorner(\varphi_{i-j}\lrcorner\sigma_{l+1-i})\big)-\varphi_j\lrcorner(\varphi_{i-j}\lrcorner\partial\sigma_{l+1-i})\\ &\qquad+\varphi_j\lrcorner\partial(\varphi_{i-j}\lrcorner\sigma_{l+1-i})+\varphi_{i-j}\lrcorner\partial(\varphi_j\lrcorner\sigma_{l+1-i})\Big)\\ &\qquad-\sum_{i=1}^{l+1}\varphi_i\lrcorner\partial\left(\sum_{j=1}^{l+1-i}\varphi_j\lrcorner\sigma_{l+1-i-j}\right)\\ &=\sum_{1\le j<i\le l+1}\varphi_j\lrcorner\partial(\varphi_{i-j}\lrcorner\sigma_{l+1-i})-\sum_{i=1}^{l+1}\sum_{j=1}^{l+1-i}\varphi_i\lrcorner\partial(\varphi_j\lrcorner\sigma_{l+1-i-j})\\ &=0\end{aligned}$$

for $k\ge 1$, by the induction method. Hence, it suffices to use the condition $\mathcal{S}^{1,1}$ to solve the second one of the system (3.14) of equations. □

Actually, by Example 3.16, we can get a more general result that the deformation invariance for $h^{p,0}$ of an n-dimensional compact complex manifold X can be obtained from the one for $h^{p,1}$.

Corollary 3.21 (The case $(p,q)=(n-1,0)$ or $(n,0)$). *For $p=n-1$ or n, the condition $\mathbb{S}^{p,1}$ on X_0 assures the deformation invariance of $h^{p,0}_{\overline{\partial}_t}(X_t)$.*

Proof Analogously to Kodaira [26, Theorem 1] or [38, Lemma 1.2] that any holomorphic $(n-1)$-form on an n-dimensional compact complex manifold is d-closed, one is able to prove that any d-closed ∂-exact $(n,0)$-form is zero. Hence, any compact complex manifold X_0 satisfies $\mathbb{S}^{n,0}$ and thus this corollary is proved by Theorem 3.18. □

One restates Theorem 3.7 by use of Notation 3.5:

Theorem 3.22 *If the central fiber X_0 satisfies $\mathcal{B}^{1,q}$ with the deformation invariance of $h^{0,q-1}_{\overline{\partial}_t}(X_t)$ established, then $h^{0,q}_{\overline{\partial}_t}(X_t)$ are independent of t.*

For Theorem 3.22, it suffices to prove:

Proposition 3.23 *Assume that X_0 satisfies $\mathcal{B}^{1,q}$. Then for each Dolbeault class in $H^{0,q}_{\overline{\partial}}(X_0)$ with the unique canonical d-closed representative σ_0 given as Lemma 3.13, there exists $\sigma_t\in A^{0,q}(X_0)$ varying smoothly on t and $e^{i_{\overline{\varphi}}}(\sigma_t)\in A^{0,q}(X_t)$ is $\overline{\partial}_t$-closed with respect to the holomorphic structure on X_t.*

Proof We just need to present the construction of σ_t. By Proposition 2.13 and type-consideration, the desired $\overline{\partial}_t$-closedness is equivalent to the resolution of the equation

$$([\partial, i_\varphi] + \bar{\partial})(\mathbb{1} - \bar{\varphi}\varphi)\lrcorner\sigma_t = \bar{\partial}((\mathbb{1} - \bar{\varphi}\varphi)\lrcorner\sigma_t) - \varphi\lrcorner\partial((\mathbb{1} - \bar{\varphi}\varphi)\lrcorner\sigma_t) = 0.$$

Therefore, it suffices to take $\sigma_t = (\mathbb{1} - \bar{\varphi}\varphi)^{-1}\lrcorner\sigma_0$. □

Corollary 3.24 *All the Hodge numbers on a compact complex surface X are deformation invariant.*

Proof From these standard results in [6, Sect. IV.2], the $\partial\overline{\partial}$-lemma holds on X for weight 2, and thus the Hodge numbers $h^{1,0}(X_t), h^{0,1}(X_t)$ of the small deformation of X is independent of t by Corollary 3.20 and Remark 3.8, respectively. The deformation invariance of the remaining Hodge numbers is obtained by Serre duality and the deformation invariance of the Euler–Poincaré characteristic (see, for example, [28, Theorem 14]). □

4 The Gauduchon Cone $\mathcal{G}_X$

In this section we will study the Gauduchon cone and its relation with the balanced one, to explore the deformation properties of an **sGG** manifold proposed by Popovici [41].

Let us first recall some notations. Aeppli cohomology groups $H^{p,q}_{\mathrm{A}}(X, \mathbb{C})$ and Bott–Chern cohomology groups $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})$ are defined on any compact complex manifold X, even on non-compact ones (cf. for instance, [3,41]). Accordingly, the *real Aeppli cohomology group* $H^{p,p}_{\mathrm{A}}(X, \mathbb{R})$ is defined by

$$H^{p,p}_{\mathrm{A}}(X, \mathbb{R}) := \frac{\left\{\partial\overline{\partial}\text{-closed smooth real } (p,\, p)\text{-forms}\right\}}{\left\{\partial\eta + \overline{\partial\eta} \;\middle|\; \eta \text{ is a smooth complex-valued } (p-1,\, p)\text{-forms}\right\}}.$$

And the *real Bott–Chern cohomology group* $H^{p,p}_{\mathrm{BC}}(X, \mathbb{R})$ is given by

$$H^{p,p}_{\mathrm{BC}}(X, \mathbb{R}) := \frac{\left\{d\text{-closed smooth real } (p,\, p)\text{-forms}\right\}}{\left\{\sqrt{-1}\partial\overline{\partial}\eta \;\middle|\; \eta \text{ is a smooth real } (p-1,\, p-1)\text{-forms}\right\}}.$$

Also, similar types of currents can represent Aeppli classes or Bott–Chern ones. By [48, Lemme 2.5] or [41, Theorem 2.1.(iii)], a canonical non-degenerate duality between $H^{n-p,n-p}_{\mathrm{A}}(X, \mathbb{C})$ and $H^{p,p}_{\mathrm{BC}}(X, \mathbb{C})$ is given by

$$\begin{array}{ccc} H^{n-p,n-p}_{\mathrm{A}}(X, \mathbb{C}) \times H^{p,p}_{\mathrm{BC}}(X, \mathbb{C}) & \longrightarrow & \mathbb{C} \\ \left([\Omega]_{\mathrm{A}}, [\omega]_{\mathrm{BC}}\right) & \longmapsto & \int_X \Omega \wedge \omega. \end{array}$$

The pairing $(\bullet, \bullet)$, restricted to real cohomology groups, also becomes the duality between the two corresponding groups.

The *Gauduchon cone* $\mathcal{G}_X$ is defined by

$$\mathcal{G}_X = \left\{ \left[\Omega\right]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \,\middle|\, \Omega \text{ is a } \partial\overline{\partial}\text{-closed positive } (n-1, n-1)\text{-form} \right\},$$

where $\omega = \Omega^{\frac{1}{n-1}}$ is called a *Gauduchon metric*. It is a known fact in linear algebra, by Michelsohn [36, the part after Lemma 4.8], that for every positive $(n-1, n-1)$-form Γ on X, there exists a unique positive $(1, 1)$-form γ such that $\gamma^{n-1} = \Gamma$. Thus, the symbol $\Omega^{\frac{1}{n-1}}$ makes sense. Gauduchon metric exists on any compact complex manifold; thanks to Gauduchon's work [23]. Hence, the Gauduchon cone $\mathcal{G}_X$ is never empty. Similarly, the *Kähler cone* $\mathcal{K}_X$ and the *balanced cone* $\mathcal{B}_X$ are defined as

$$\mathcal{K}_X = \left\{ \left[\omega\right]_{\mathrm{BC}} \in H_{\mathrm{BC}}^{1,1}(X, \mathbb{R}) \,\middle|\, \omega \text{ is a } d\text{-closed positive } (1, 1)\text{-form} \right\},$$
$$\mathcal{B}_X = \left\{ \left[\Omega\right]_{\mathrm{BC}} \in H_{\mathrm{BC}}^{n-1,n-1}(X, \mathbb{R}) \,\middle|\, \Omega \text{ is a } d\text{-closed positive } (n-1, n-1)\text{-form} \right\},$$

where $\Omega^{\frac{1}{n-1}}$ is called a *balanced metric*. And the three cones are open convex cones (cf. [41, Observation 5.2] for the Gauduchon cone).

The numerically effective (shortly *nef*) cone, can be defined as

$$\left\{ \left[\omega\right]_{\mathrm{BC}} \in H_{\mathrm{BC}}^{1,1}(X, \mathbb{R}) \middle| \forall \epsilon > 0, \exists \text{ a smooth real } (1, 1)\text{-form } \alpha_\epsilon \in \left[\omega\right]_{\mathrm{BC}}, \text{ such that } \alpha_\epsilon \geq -\epsilon \widetilde{\omega} \right\},$$

where $\widetilde{\omega}$ is a fixed Hermitian metric on the compact complex manifold X. And the nef cone is a closed convex cone by [15, Proposition 6.1]. When X is Kähler, the nef cone is the closure of the Kähler cone $\mathcal{K}_X$. Thus, we will use the symbol $\overline{\mathcal{K}}_X$ for the nef cone in any situation. Similar definitions adapt to $\overline{\mathcal{B}}_X$ and $\overline{\mathcal{G}}_X$, which are also closed convex cones. There are many studies, such as [9,15–17,22,41,45,58] on these cones and their relations.

Definition 4.1 Degenerate cones.

We say that the *Gauduchon cone* $\mathcal{G}_X$ *degenerates* when $\mathcal{G}_X = H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$, which comes from [41, Sect. 5]. Similarly, the *balanced cone* $\mathcal{B}_X$ *degenerates* if the equality $\mathcal{B}_X = H_{\mathrm{BC}}^{n-1,n-1}(X, \mathbb{R})$ holds.

4.1 The Kähler Case of $\mathcal{G}_X$

We will consider various cones on Kähler manifolds at first. Thus, let X be a compact Kähler manifold.

Lemma 4.2 *The Gauduchon cone $\mathcal{G}_X$ does not degenerate on the compact Kähler manifold X. Moreover, $\mathcal{G}_X$ lies in one open half semi-space determined by some linear subspace of codimension one in $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$.*

Proof X carries a Kähler metric ω_X. Then $\left[\omega_X\right]_{\mathrm{BC}}$ lives in the Kähler cone $\mathcal{K}_X$, which cannot be the zero class of $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$. This implies that

$$\dim_{\mathbb{R}} H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) = \dim_{\mathbb{R}} H_{\mathrm{BC}}^{1,1}(X, \mathbb{R}) \geq 1.$$

Thus, the Gauduchon cone $\mathcal{G}_X$ is a non-empty open cone in a vector space with the dimension at least one, which implies that $\mathcal{G}_X$ must contain a non-zero class.

Meanwhile, the Gauduchon $\mathcal{G}_X$ cannot degenerate. If $\mathcal{G}_X$ degenerates, i.e., $0 \in \mathcal{G}_X = H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$, X carries a Hermitian metric $\widetilde{\omega}$ such that $\widetilde{\omega}^{n-1}$ is the type of $\overline{\partial}\psi + \partial\overline{\psi}$, where ψ is a smooth $(n-1, n-2)$-form on X. It is easy to check that $\widetilde{\omega}^{n-1} \wedge \omega_X$ is d-exact but $\int_X \widetilde{\omega}^{n-1} \wedge \omega_X > 0$, where a contradiction emerges. As an easy consequence of this, the Gauduchon cone $\mathcal{G}_X$ cannot contain the origin of $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$.

It is easy to see that the Kähler class $\left[\omega_X\right]_{\mathrm{BC}}$ determines one open half semi-space $\mathbf{H}_{\omega_X}^{+}$ in $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$ given by

$$\mathbf{H}_{\omega_X}^{+} = \left\{ \left[\Omega\right]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \;\middle|\; \int_X \Omega \wedge \omega_X > 0 \right\},$$

which is clearly cut out by the linear subspace of codimension one

$$\mathbf{H}_{\omega_X} = \left\{ \left[\Omega\right]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \;\middle|\; \int_X \Omega \wedge \omega_X = 0 \right\}.$$

And the Gauduchon cone $\mathcal{G}_X$ obviously lies in $\mathbf{H}_{\omega_X}^{+}$. Hence the lemma is proved. □

Remark 4.3 It is well known that neither the Kähler cone $\mathcal{K}_X$ nor the balanced cone $\mathcal{B}_X$ degenerates on the Kähler manifold X .

It is known that the quotient topology of Bott–Chern groups induced by the Fréchet topology of smooth forms or the weak topology of currents is Hausdorff (cf. [15, the part before Definition 1.3]). And every Hausdorff finite-dimensional topological real vector space is isomorphic to $\mathbb{R}^n$ with the Euclidean topology. Then it is harmless to fix an inner product $\langle \bullet, \bullet \rangle$ on the real vector space $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$, which induces the given topology on $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$. The space $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$ can be viewed as the vector space of continuous linear functionals on $\left(H_{\mathrm{BC}}^{1,1}(X, \mathbb{R}), \langle \bullet, \bullet \rangle\right)$. By the finite-dimensional case of Riesz representation theorem, there is a canonical isomorphism from $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$ to $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$ with $\left[\Omega\right]_{\mathrm{A}}$ to $\left[\omega_\Omega\right]_{\mathrm{BC}}$. That is, for any $\left[\Omega\right]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$, there exists a unique $\left[\omega_\Omega\right]_{\mathrm{BC}} \in H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$, such that

$$\left(\left[\Omega\right]_{\mathrm{A}}, \left[\omega\right]_{\mathrm{BC}}\right) = \left\langle \left[\omega\right]_{\mathrm{BC}}, \left[\omega_\Omega\right]_{\mathrm{BC}} \right\rangle$$

for any $\left[\omega\right]_{\mathrm{BC}} \in H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$. Thus, this isomorphism enables us to define the dual inner product on $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$ by the equality

$$\left\langle \left[\Omega_1\right]_{\mathrm{A}}, \left[\Omega_2\right]_{\mathrm{A}} \right\rangle := \left\langle \left[\omega_{\Omega_1}\right]_{\mathrm{BC}}, \left[\omega_{\Omega_2}\right]_{\mathrm{BC}} \right\rangle.$$

Let $\left\{\left[\omega_i\right]_{\mathrm{BC}}\right\}_{i=1}^m$ be an orthonormal basis of $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$. Then, $\left\{\left[\Omega_{\omega_i}\right]_{\mathrm{A}}\right\}_{i=1}^m$, the inverse image of $\left\{\left[\omega_i\right]_{\mathrm{BC}}\right\}_{i=1}^m$ under the above canonical isomorphism, is also an orthonormal one of $H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R})$ under the dual metric. And $\left\{[\omega_i]_{\mathrm{BC}}, [\Omega_{\omega_i}]_{\mathrm{A}}\right\}_{i=1}^{m}$ become dual bases with respect to $(\bullet,\bullet)$.

Definition 4.4 The open circular cone $\mathcal{C}(v,\theta)$.

Let $(V_{\mathbb{R}}, \langle\bullet,\bullet\rangle)$ be a real vector space $V_{\mathbb{R}}$, which equips with an inner product $\langle\bullet,\bullet\rangle$. Denote the induced norm by $\|\bullet\|$. The *open circular cone* $\mathcal{C}(v,\theta)$ is determined by a non-zero vector v in $V_{\mathbb{R}}$ and an angle $\theta\in\left[0,\frac{\pi}{2}\right]$, given by

$$\mathcal{C}(v,\theta)=\left\{w\in V_{\mathbb{R}}\setminus 0 \;\middle|\; \frac{\langle w,v\rangle}{\|w\|\|v\|}>\cos\theta\right\}.$$

And 2θ is called the *cone angle*. It is clear that the cone $\mathcal{C}(v,\theta)$ does not change if v is replaced by any vector in $\mathbb{R}^{>0}v$.

As stated in the proof of Lemma 4.2, the Gauduchon cone $\mathcal{G}_X$ must contain a non-zero class. Let us fix a non-zero class $[\Omega_0]_{\mathrm{A}}\in\mathcal{G}_X$.

Proposition 4.5 *On a compact Kähler manifold X, there exists a small angle* $\tilde{\theta}\in\left(0,\frac{\pi}{2}\right)$ *such that*

$$\mathcal{C}\left([\Omega_0]_{\mathrm{A}},\tilde{\theta}\right)\subseteq\mathcal{G}_X\subseteq\mathcal{C}\left([\Omega_{\omega_X}]_{\mathrm{A}},\frac{\pi}{2}-\tilde{\theta}\right),$$

where the class $[\Omega_{\omega_X}]_{\mathrm{A}}$ *in* $H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R})$ *denotes the inverse image of the Kähler class* $[\omega_X]_{\mathrm{BC}}$ *under the canonical isomorphism discussed before Definition 4.4.*

Proof Since $[\Omega_0]_{\mathrm{A}}$ is a non-zero class of $\mathcal{G}_X$, there exists a neighborhood of $[\Omega_0]_{\mathrm{A}}$, belonging to $\mathcal{G}_X$, namely,

$$\left\{[\Omega]_{\mathrm{A}}\in H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R}) \;\middle|\; \left\|[\Omega]_{\mathrm{A}}-[\Omega_0]_{\mathrm{A}}\right\|<\epsilon\right\}\subseteq\mathcal{G}_X$$

for some $\epsilon>0$. Since $\mathcal{G}_X$ is an open convex cone, the inclusion follows

$$\mathcal{C}\left([\Omega_0]_{\mathrm{A}}, \arcsin\frac{\epsilon}{\left\|[\Omega_0]_{\mathrm{A}}\right\|}\right)\subseteq\mathcal{G}_X.$$

Similarly, there exists $\tilde{\epsilon}>0$, such that

$$\mathcal{C}\left([\omega_X]_{\mathrm{BC}}, \arcsin\frac{\tilde{\epsilon}}{\left\|[\omega_X]_{\mathrm{BC}}\right\|}\right)\subseteq\mathcal{K}_X.$$

It is easy to see that

$$\mathcal{G}_X\subseteq\bigcap_{[\omega]_{\mathrm{BC}}\in\mathcal{C}\left([\omega_X]_{\mathrm{BC}},\theta_0\right)}\mathbf{H}_{\omega}^{+},$$

where θ_0 can be chosen as $\arcsin \frac{\tilde{\epsilon}}{\left\| [\omega_X]_{\mathrm{BC}} \right\|}$. From the discussion before Definition 4.4, we know that

$$\bigcap_{[\omega]_{\mathrm{BC}} \in \mathcal{C}\left([\omega_X]_{\mathrm{BC}}, \theta_0\right)} \mathbf{H}_{\omega}^{+} = \mathcal{C}\left(\left[\Omega_{\omega_X}\right]_{\mathrm{A}}, \frac{\pi}{2} - \theta_0\right).$$

Let the angle $\tilde{\theta}$ be

$$\min\left(\arcsin \frac{\epsilon}{\left\| [\Omega_0]_{\mathrm{A}} \right\|},\ \arcsin \frac{\tilde{\epsilon}}{\left\| [\omega_X]_{\mathrm{BC}} \right\|}\right).$$

□

As in [41, Sect. 5], if the finite-dimensional vector space $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$ of a compact complex manifold X is endowed with the unique norm-induced topology, the *closure of the Gauduchon cone* in $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$ is defined by

$$\overline{\mathcal{G}}_X = \left\{\alpha \in H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \;\middle|\; \forall \epsilon > 0, \exists \text{ smooth } \Omega_\epsilon \in \alpha, \text{ such that } \Omega_\epsilon \geq -\epsilon\Omega\right\}, \tag{4.1}$$

where $\Omega > 0$ is a fixed smooth $(n-1, n-1)$-form on X with $\partial\overline{\partial}\Omega = 0$. This cone is convex and closed, which is shown in [15, Proposition 6.1.(i)].

Corollary 4.6 *The closure of the Gauduchon cone $\overline{\mathcal{G}}_X$ on the Kähler manifold X must lie in some closed circular cone with the cone angle smaller than π, for example, the closure of $\mathcal{C}\left(\left[\Omega_{\omega_X}\right]_{\mathrm{A}}, \frac{\pi}{2} - \tilde{\theta}\right)$.*

In a similar manner, we can also show that the Kähler cone $\mathcal{K}_X$ on a Kähler manifold X must lie in some open circular cone with the cone angle smaller than π in $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$.

The following definition is inspired by [41, Observation 5.7 and Question 5.9].

Definition 4.7 $\left(\mathcal{A}\right)^{\vee_o}$ and $\left(\mathcal{A}\right)^{\vee_c}$

Let $\mathcal{A}$ be a convex cone in a finite-dimensional vector space $W_{\mathbb{R}}$, whose dual vector space is denoted by $W_{\mathbb{R}}^{\vee}$.

(1) $\left(\mathcal{A}\right)^{\vee_o}$ denotes the set of linear functions in $W_{\mathbb{R}}^{\vee}$, evaluating positively on $\mathcal{A}$;
(2) $\left(\mathcal{A}\right)^{\vee_c}$ denotes the set of linear functionals in $W_{\mathbb{R}}^{\vee}$, evaluating non-negatively on $\mathcal{A}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two closed convex cones in the $W_{\mathbb{R}}$ and $W_{\mathbb{R}}^{\vee}$, respectively. We say that $\mathcal{P}$ and $\mathcal{Q}$ are *dual cones*, if $\mathcal{P} = \left(\mathcal{Q}\right)^{\vee_c}$ and $\mathcal{Q} = \left(\mathcal{P}\right)^{\vee_c}$.

The *pseudo-effective cone* $\mathcal{E}_X$, the set of classes in $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$ represented by d-closed positive $(1, 1)$-currents, is a closed convex cone when X is any compact complex manifold (cf. [15, Proposition 6.1]). The *big cone* $\mathcal{E}_X^{\circ}$, an open convex cone in $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$, is defined to be the interior of the pseudo-effective cone $\mathcal{E}_X$ when X is Kähler, in which classes are represented by Kähler $(1, 1)$-currents (cf. [17, Definition 1.6]).

Theorem 4.8 *For a compact Kähler manifold X,*

$$\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}} \subseteq \left(\mathcal{E}_X^{\circ}\right)^{\vee_o}$$

and thus $\mathcal{G}_X \subsetneq \left(\mathcal{E}_X^{\circ}\right)^{\vee_o}$.

Proof It is clear that each class in $\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}}$ evaluates non-negatively on the big cone $\mathcal{E}_X^{\circ}$. Suppose that some class $\left[\Omega\right]_{\mathrm{A}}$ in $\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}}$ does not evaluate positively on $\mathcal{E}_X^{\circ}$, i.e., there exists a class $\left[T(\Omega)\right]_{\mathrm{BC}} \in \mathcal{E}_X^{\circ}$, with $T(\Omega)$ a Kähler current, such that

$$\int_X \Omega \wedge T(\Omega) = 0.$$

Then note that the big cone $\mathcal{E}_X^{\circ}$ actually lies in the closed half semi-space $\mathbf{H}_{\Omega}^{+} \bigcup \mathbf{H}_{\Omega}$ of $H_{\mathrm{BC}}^{1,1}(X, \mathbb{R})$ with $\left[T(\Omega)\right]_{\mathrm{BC}}$ attached to the linear subspace $\mathbf{H}_{\Omega}$. But a small neighborhood of $\left[T(\Omega)\right]_{\mathrm{BC}}$ will run out of the closed half semi-space $\mathbf{H}_{\Omega}^{+} \bigcup \mathbf{H}_{\Omega}$ into the other open half $\mathbf{H}_{\Omega}^{-}$. Meanwhile, the neighborhood is still contained in $\mathcal{E}_X^{\circ}$, since the big cone $\mathcal{E}_X^{\circ}$ is an open convex cone. This contradiction tells us that each class in $\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}}$ evaluates positively on $\mathcal{E}_X^{\circ}$. Hence, we have

$$\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}} \subseteq \left(\mathcal{E}_X^{\circ}\right)^{\vee_o}.$$

It is clear that $\mathcal{G}_X \subseteq \left(\mathcal{E}_X^{\circ}\right)^{\vee_o}$. Now suppose that $\left(\mathcal{E}_X^{\circ}\right)^{\vee_o} = \mathcal{G}_X$. Then

$$\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}} \subseteq \left(\mathcal{E}_X^{\circ}\right)^{\vee_o} = \mathcal{G}_X$$

follows directly, which is equivalent to the equality

$$\overline{\mathcal{G}}_X = \mathcal{G}_X \bigcup \left[0\right]_{\mathrm{A}}.$$

Hence, the hyperplane $\mathbf{H}_{\omega_X}(1)$ in $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R})$, defined by

$$\mathbf{H}_{\omega_X}(1) = \left\{ \left[\Omega\right]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \,\middle|\, \int_X \Omega \wedge \omega_X = 1 \right\},$$

has the same intersection with $\mathcal{G}_X$ and $\overline{\mathcal{G}}_X$. This implies that the intersection $\mathcal{G}_X \bigcap \mathbf{H}_{\omega_X}(1)$ is both open and closed on the hyperplane $\mathbf{H}_{\omega_X}(1)$, which is clearly connected. Then, we get $\mathcal{G}_X \bigcap \mathbf{H}_{\omega_X}(1) = \mathbf{H}_{\omega_X}(1)$, which leads to the inclusion

$$\mathbf{H}_{\omega_X}(1) \subseteq \mathcal{G}_X.$$

Hence, the open half semi-space $\mathbf{H}_{\omega_X}^{+}$ is contained in the Gauduchon cone $\mathcal{G}_X$. However, from the proof of Proposition 4.5, we know that $\mathcal{G}_X$ actually lies in

$\mathcal{C}\Big(\big[\Omega_{\omega_X}\big]_{\mathrm{A}}, \frac{\pi}{2} - \tilde{\theta}\Big)$, which is strictly contained in $\mathbf{H}^{+}_{\omega_X}$. Here is a contradiction. So $\mathcal{G}_X \subsetneqq \big(\mathcal{E}^{\circ}_X\big)^{\vee_o}$. $\square$

Remark 4.9 It is shown that $\overline{\mathcal{G}}_X \setminus \big[0\big]_{\mathrm{A}} = \big(\mathcal{E}^{\circ}_X\big)^{\vee_o}$ in Remark 4.12.

4.2 The Relation Between Balanced Cone $\mathcal{B}_X$ and Gauduchon Cone $\mathcal{G}_X$

There exists a pair of diagrams $(\mathrm{D}, \overline{\mathrm{D}})$ on a compact Kähler manifold X as follows, which is inspired by Fu–Xiao's work [22]. The diagrams D reads

$$\begin{array}{ccc} \mathcal{B}_X & \xrightarrow{\ \mathbb{J}\ } & \mathcal{G}_X \\ {\scriptstyle \mathbb{I}}\nwarrow & & \nearrow{\scriptstyle \mathbb{K}} \\ & \mathcal{K}_X & \end{array} \qquad \begin{array}{ccc} [\omega^{n-1}]_{\mathrm{BC}} & \xrightarrow{\ \mathbb{J}\ } & \big[\omega^{n-1}\big]_{\mathrm{A}} \\ {\scriptstyle \mathbb{I}}\nwarrow & & \nearrow{\scriptstyle \mathbb{K}} \\ & [\omega]_{\mathrm{BC}} & \end{array},$$

and the diagram $\overline{\mathrm{D}}$ follows,

$$\begin{array}{ccc} \overline{\mathcal{B}}_X & \xrightarrow{\ \overline{\mathbb{J}}\ } & \overline{\mathcal{G}}_X \\ {\scriptstyle \overline{\mathbb{I}}}\nwarrow & & \nearrow{\scriptstyle \overline{\mathbb{K}}} \\ & \overline{\mathcal{K}}_X & \end{array} \qquad \begin{array}{ccc} [\omega^{n-1}]_{\mathrm{BC}} & \xrightarrow{\ \overline{\mathbb{J}}\ } & \big[\omega^{n-1}\big]_{\mathrm{A}} \\ {\scriptstyle \overline{\mathbb{I}}}\nwarrow & & \nearrow{\scriptstyle \overline{\mathbb{K}}} \\ & [\omega]_{\mathrm{BC}} & \end{array}.$$

The former consists of three mappings among Kähler cone $\mathcal{K}_X$, balanced cone $\mathcal{B}_X$, and Gauduchon cone $\mathcal{G}_X$. And the latter is actually the extension of the former to the closures of respective cones. It is easy to see that all the mappings are well-defined and both diagrams are commutative. The mappings $(\mathbb{I}, \overline{\mathbb{I}})$, $(\mathbb{J}, \overline{\mathbb{J}})$, and $(\mathbb{K}, \overline{\mathbb{K}})$ are the restrictions of three natural maps $\mathscr{I}$, $\mathscr{J}$, and $\mathscr{K}$, respectively, which are independent of the Kählerness of X. The three mappings are given as follows:

$$\begin{array}{rccl} \mathscr{I}: & H^{1,1}_{\mathrm{BC}}(X,\mathbb{R}) & \to & H^{n-1,n-1}_{\mathrm{BC}}(X,\mathbb{R}) \\ & \big[\omega\big]_{\mathrm{BC}} & \mapsto & \big[\omega^{n-1}\big]_{\mathrm{BC}}, \\ \mathscr{J}: & H^{n-1,n-1}_{\mathrm{BC}}(X,\mathbb{R}) & \to & H^{n-1,n-1}_{\mathrm{A}}(X,\mathbb{R}) \\ & \big[\Omega\big]_{\mathrm{BC}} & \mapsto & \big[\Omega\big]_{\mathrm{A}}, \\ \mathscr{K}: & H^{1,1}_{\mathrm{BC}}(X,\mathbb{R}) & \to & H^{n-1,n-1}_{\mathrm{A}}(X,\mathbb{R}) \\ & \big[\omega\big]_{\mathrm{BC}} & \mapsto & \big[\omega^{n-1}\big]_{\mathrm{A}}. \end{array}$$

Moreover, when X is a complex manifold satisfying $\partial\overline{\partial}$-lemma, the mapping $\mathscr{J}$ is an isomorphism and thus the mappings $(\mathbb{J}, \overline{\mathbb{J}})$ are injective.

By [22, Proposition 1.1 and Theorem 1.2], the mapping $\mathbb{I}$ is injective. Meanwhile, $\overline{\mathbb{I}}$, restricted to the intersection of the nef cone and the big cone $\overline{\mathcal{K}}_X \bigcap \mathcal{E}^{\circ}_X$, is also injective. This is true, even when X is in the *Fujiki class* $\mathcal{C}$ (i.e., the class of compact complex manifolds bimeromorphic Kähler manifolds), see [22, Corollary 2.7]. The existence of classes in $\overline{\mathbb{I}}(\partial\mathcal{K}_X)\bigcap\mathcal{B}_X$ implies that the mapping $\mathbb{I}$ is not surjective. In fact, the class $\left[\tilde{\omega}\right]_{\mathrm{BC}}\in\partial\mathcal{K}_X$, mapped into the balanced cone $\mathcal{B}_X$, necessarily lies in the big cone $\mathcal{E}_X^{\circ}$, by [22, Theorem 1.3]. Thus, the class $\overline{\mathbb{I}}(\left[\tilde{\omega}\right]_{\mathrm{BC}})$ in $\mathcal{B}_X$ cannot be mapped by a Kähler class, since $\overline{\mathbb{I}}$ is injective on the intersection cone $\overline{\mathcal{K}}_X\bigcap\mathcal{E}_X^{\circ}$. Besides, Theorem 1.3 in [22] gives a precise description of $\overline{\mathbb{I}}(\partial\mathcal{K}_{\mathrm{NS}})\bigcap\mathcal{B}_X$ when X is a projective Calabi–Yau manifold. The cone $\mathcal{K}_{\mathrm{NS}}$ denotes the intersection $\mathcal{K}_X\bigcap\mathrm{NS}_{\mathbb{R}}$, where $\mathrm{NS}_{\mathbb{R}}$ is the real Neron–Severi group of X.

Recall that [22, Lemma 3.3] states that a Bott–Chern class $\left[\Omega\right]_{\mathrm{BC}}\in H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R})$ on a compact complex manifold X, lives in the balanced cone $\mathcal{B}_X$ if and only if

$$\int_X \Omega\wedge T>0,$$

for every non-zero $\partial\overline{\partial}$-closed positive $(1,1)$-current T. Similarly, one has

Lemma 4.10 *Let X be a compact complex manifold and Ω a real $\partial\overline{\partial}$-closed $(n-1,n-1)$-form on X. Then the class $\left[\Omega\right]_{\mathrm{A}}$ lives in $\mathcal{G}_X$ if and only if*

$$\int_X \Omega\wedge T>0,$$

for every non-zero d-closed positive $(1,1)$-current T on X.

Proof We mainly follow the ideas of the proof of [22, Lemma 3.3]. The necessary part is quite obvious. As to the sufficient part, let $\mathfrak{D}'^{1,1}_{\mathbb{R}}$ be the set of real $(1,1)$-currents on X with the weak topology. Fix a Hermitian metric ω_X on X and apply the Hahn–Banach separation theorem, which originates from Sullivan's work [49]. See also in [22, Lemma 3.3] and [41, Proposition 5.4].

Set

$$\mathfrak{D}_1=\left\{T\in\mathfrak{D}'^{1,1}_{\mathbb{R}}\;\middle|\;\int_X\Omega\wedge T=0 \text{ and } dT=0\right\},$$
$$\mathfrak{D}_2=\left\{T\in\mathfrak{D}'^{1,1}_{\mathbb{R}}\;\middle|\;\int_X\omega_X^{n-1}\wedge T=1 \text{ and } T\geq 0\right\}.$$

It is easy to see that $\mathfrak{D}_1$ is a closed linear subspace of the locally convex space $\mathfrak{D}'^{1,1}_{\mathbb{R}}$, while $\mathfrak{D}_2$ is a compact convex one in $\mathfrak{D}'^{1,1}_{\mathbb{R}}$. Since a d-closed positive $(1,1)$-current T, satisfying $\int_X\Omega\wedge T=0$, has to be zero current from the assumption of the lemma, $\mathfrak{D}_1\bigcap\mathfrak{D}_2=\emptyset$ by $\int_X\omega_X^{n-1}\wedge T=1$. Then there exists a continuous linear functional on $\mathfrak{D}'^{1,1}_{\mathbb{R}}$, denoted by $\widetilde{\Omega}$, a real $(n-1,n-1)$-form, such that it vanishes on $\mathfrak{D}_1$, which contains all real $\partial\overline{\partial}$-exact $(1,1)$-currents, and evaluates positively on $\mathfrak{D}_2$. Hence, $\widetilde{\Omega}$ has to be a $\partial\overline{\partial}$-closed positive $(n-1,n-1)$-form.

The following mapping

$$\begin{aligned}\pi:\left\{T\in\mathfrak{D}'^{1,1}_{\mathbb{R}}\;\middle|\;dT=0\right\}&\to H_{\mathrm{BC}}^{1,1}(X,\mathbb{R})\\ T\qquad\qquad&\mapsto\quad\left[T\right]_{\mathrm{BC}}\end{aligned}$$

is a canonical projection. $\pi(\mathfrak{D}_1)$ is the null space determined by the linear functional $\big[\Omega\big]_{\mathrm{A}}$ on $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, namely,

$$\left\{\big[T\big]_{\mathrm{BC}} \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R}) \;\Big|\; \int_X \Omega \wedge T = 0\right\},$$

since the class $\big[\Omega\big]_{\mathrm{A}}$ belongs to $H^{n-1,n-1}_{\mathrm{A}}(X,\mathbb{R})$, which can be seen as the dual space of $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$. The linear functional $\big[\widetilde{\Omega}\big]_{\mathrm{A}}$ vanishes on the null space, which implies $\big[\widetilde{\Omega}\big]_{\mathrm{A}} = a\big[\Omega\big]_{\mathrm{A}}$ for some $a \in \mathbb{R}$.

If there exists no non-zero d-closed positive $(1,1)$-current on X, by [41, Proposition 5.4], the Gauduchon cone $\mathcal{G}_X$ will degenerate. Therefore, the class $\big[\Omega\big]_{\mathrm{A}}$ will surely lie in $\mathcal{G}_X$. Assume that there exists a non-zero d-closed positive $(1,1)$-current T. Clearly, $\int_X \widetilde{\Omega} \wedge T = a \int_X \Omega \wedge T$. Moreover, $\widetilde{\Omega}$ is positive on $\mathfrak{D}_2$, which implies $\int_X \widetilde{\Omega} \wedge T > 0$, and $\int_X \Omega \wedge T > 0$ by the assumption of the lemma. Thus $a > 0$. Therefore, $\big[\Omega\big]_{\mathrm{A}} = \frac{1}{a}\big[\widetilde{\Omega}\big]_{\mathrm{A}}$, with $\widetilde{\Omega}$ a positive form, lives in $\mathcal{G}_X$. □

The closure of the Gauduchon cone $\overline{\mathcal{G}}_X$ (cf. (4.1) and [41, the part before Proposition 5.8]) and the pseudo-effective cone $\mathcal{E}_X$ are closed convex cones when X is any compact complex manifold. By the use of Lemma 4.10, we can get the so-called *Lamari's duality*. See [30, Lemma 3.3] and [45, the remark before Theorem 1.8 and the proof of Theorem 5.9].

Proposition 4.11 *Let X be a compact complex manifold. Then $\overline{\mathcal{G}}_X$ and $\mathcal{E}_X$ are dual cones, i.e., $\big(\overline{\mathcal{G}}_X\big)^{\vee_c} = \mathcal{E}_X$ and $\big(\mathcal{E}_X\big)^{\vee_c} = \overline{\mathcal{G}}_X$.*

Proof It is clear that $\mathcal{E}_X \subseteq \big(\overline{\mathcal{G}}_X\big)^{\vee_c}$ and $\overline{\mathcal{G}}_X \subseteq \big(\mathcal{E}_X\big)^{\vee_c}$. Let $\big[\Omega\big]_{\mathrm{A}} \in \big(\mathcal{E}_X\big)^{\vee_c}$, where Ω is a real $\partial\overline{\partial}$-closed $(n-1,n-1)$-form. Fix one class $\big[\Omega_0\big]_{\mathrm{A}} \in \mathcal{G}_X$ with Ω_0 positive. Obviously, for any fixed $\epsilon > 0$, the integral

$$\int_X \big(\Omega + \epsilon\Omega_0\big) \wedge T = \int_X \Omega \wedge T + \epsilon \int_X \Omega_0 \wedge T > 0,$$

where T is a non-zero d-closed positive $(1,1)$-current. Hence, the class $\big[\Omega\big]_{\mathrm{A}} + \epsilon\big[\Omega_0\big]_{\mathrm{A}} \in \mathcal{G}_X$ by Lemma 4.10. Therefore, the class $\big[\Omega\big]_{\mathrm{A}} \in \overline{\mathcal{G}}_X$, which implies $\big(\mathcal{E}_X\big)^{\vee_c} = \overline{\mathcal{G}}_X$.

Now, let $\big[\omega\big]_{\mathrm{BC}} \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, which does not live in the pseudo-effective cone $\mathcal{E}_X$. The point $\big[\omega\big]_{\mathrm{BC}}$ and $\mathcal{E}_X$ are a compact convex subspace and a closed convex one, respectively, in the locally convex space $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$. From Hahn–Banach separation theorem, there exists a continuous linear functional, denoted by $\big[\widetilde{\Omega}\big]_{\mathrm{A}}$, a class in $H^{n-1,n-1}_{\mathrm{A}}(X,\mathbb{R})$, such that it evaluates non-negatively on $\mathcal{E}_X$ and takes a negative value on the point $\big[\omega\big]_{\mathrm{BC}}$. Thus, the class $\big[\widetilde{\Omega}\big]_{\mathrm{A}} \in \overline{\mathcal{G}}_X$, from the equality $\big(\mathcal{E}_X\big)^{\vee_c} = \overline{\mathcal{G}}_X$. And the inequality $\int_X \widetilde{\Omega} \wedge \omega < 0$ indicates the inclusion

$$H^{1,1}_{\mathrm{BC}}(X,\mathbb{R}) \setminus \mathcal{E}_X \subseteq H^{1,1}_{\mathrm{BC}}(X,\mathbb{R}) \setminus \big(\overline{\mathcal{G}}_X\big)^{\vee_c},$$

which implies that $\mathcal{E}_X = \left(\overline{\mathcal{G}}_X\right)^{\vee_c}$. □

Remark 4.12 Proposition 4.11 enhances the result in Theorem 4.8. In fact, any class in $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \setminus \overline{\mathcal{G}}_X$ must take a negative value on some class of $\mathcal{E}_X$, and evaluates negatively on some class in the interior $\mathcal{E}_X^{\circ}$ when X is Kähler. Thus, each class in $H_{\mathrm{A}}^{n-1,n-1}(X, \mathbb{R}) \setminus \overline{\mathcal{G}}_X$ does not live in $\left(\mathcal{E}_X^{\circ}\right)^{\vee_o}$. Therefore, $\overline{\mathcal{G}}_X \setminus \left[0\right]_{\mathrm{A}} = \left(\mathcal{E}_X^{\circ}\right)^{\vee_o}$.

Recall that a compact complex manifold is *balanced* if it admits a balanced metric and the *closure of its balanced cone* is defined similarly to the one of Gauduchon cone (4.1).

Proposition 4.13 *For a compact balanced manifold X, the convex cone $\mathcal{E}_{\partial\overline{\partial}} \subseteq H_{\mathrm{A}}^{1,1}(X, \mathbb{R})$, generated by Aeppli classes represented by $\partial\overline{\partial}$-closed positive $(1,1)$-currents, is closed. And when X also satisfies the $\partial\overline{\partial}$-lemma, the following three statements are equivalent:*

(1) *The mapping $\mathbb{J} : \mathcal{B}_X \to \mathcal{G}_X$ is bijective.*
(2) *The mapping $\overline{\mathbb{J}} : \overline{\mathcal{B}}_X \to \overline{\mathcal{G}}_X$ is bijective.*
(3) *The mapping $\mathfrak{j} : \mathcal{E}_X \to \mathcal{E}_{\partial\overline{\partial}}$ is bijective,*

where the mapping $\mathfrak{j}$ is the restriction of the natural isomorphism $\mathscr{L} : H_{\mathrm{BC}}^{1,1}(X, \mathbb{R}) \to H_{\mathrm{A}}^{1,1}(X, \mathbb{R})$, induced by the identity map, to the pseudo-effective cone $\mathcal{E}_X$.

Proof Fix a balanced metric ω_X on X. Let $\left\{\left[T_k\right]_{\mathrm{A}}\right\}_{k\in\mathbb{N}^+}$ be a sequence in the cone $\mathcal{E}_{\partial\overline{\partial}}$, where T_k are $\partial\overline{\partial}$-closed positive $(1,1)$-currents. And the sequence converges to an Aeppli class $\left[\alpha\right]_{\mathrm{A}}$ in $H_{\mathrm{A}}^{1,1}(X, \mathbb{R})$. It is clear that

$$\lim_{k\to+\infty} \int_X T_k \wedge \omega_X^{n-1} = \int_X \alpha \wedge \omega_X^{n-1}.$$

Thus, the sequence $\left\{T_k\right\}_{k\in\mathbb{N}^+}$ is bounded in mass, and therefore weakly compact. Denote the limit of a weakly convergent subsequence $\left\{T_{k_i}\right\}$ by T. It is easy to check that T is a $\partial\overline{\partial}$-closed positive $(1,1)$-current and $\left[T\right]_{\mathrm{A}} = \left[\alpha\right]_{\mathrm{A}}$. Hence, $\left[\alpha\right]_{\mathrm{A}} \in \mathcal{E}_{\partial\overline{\partial}}$, which implies that the convex cone $\mathcal{E}_{\partial\overline{\partial}}$ is closed.

It is obvious that the three mappings $\mathbb{J}$, $\overline{\mathbb{J}}$, and $\mathfrak{j}$ are injective, since $\mathscr{J}$ and $\mathscr{L}$ are isomorphisms as long as the complex manifold X satisfies the $\partial\overline{\partial}$-lemma.

(1) $\Rightarrow$ (2) : We need to show that the inverse $\mathscr{J}^{-1}$ of the mapping $\mathscr{J}$ maps the closure $\overline{\mathcal{G}}_X$ into the one $\overline{\mathcal{B}}_X$. To see this, let $\left[\Psi\right]_{\mathrm{A}} \in \overline{\mathcal{G}}_X$. Denote the inverse image $\mathscr{J}^{-1}(\left[\Psi\right]_{\mathrm{A}})$ of $\left[\Psi\right]_{\mathrm{A}}$ under the mapping $\mathscr{J}$ by $\left[\Omega\right]_{\mathrm{BC}}$. For any $\epsilon > 0$,

$$\mathscr{J}^{-1}(\left[\Psi\right]_{\mathrm{A}} + \epsilon\left[\omega_X^{n-1}\right]_{\mathrm{A}}) = \left[\Omega\right]_{\mathrm{BC}} + \epsilon\left[\omega_X^{n-1}\right]_{\mathrm{BC}} \in \mathcal{B}_X,$$

since $\mathbb{J}$ is bijective and thus $\mathscr{J}^{-1}(\mathcal{G}_X) \subseteq \mathcal{B}_X$. This implies that $\left[\Omega\right]_{\mathrm{BC}} \in \overline{\mathcal{B}}_X$. Then $\mathscr{J}^{-1}(\overline{\mathcal{G}}_X) \subseteq \overline{\mathcal{B}}_X$, namely, the mapping $\mathscr{J}^{-1} : \overline{\mathcal{G}}_X \to \overline{\mathcal{B}}_X$ is well-defined. Hence, $\mathscr{J}^{-1}$ is the inverse of the mapping $\overline{\mathbb{J}}$ and thus $\overline{\mathbb{J}}$ is bijective.

(2) $\Rightarrow$ (3) : $\overline{\mathcal{G}}_X$ and $\mathcal{E}_X$ are dual cones by Proposition 4.11. $\overline{\mathcal{B}}_X$ and $\mathcal{E}_{\partial\overline{\partial}}$ are also dual cones by [22, Lemma 3.3 and Remark 3.4]. Hence, the mapping $\mathbb{j}$ is bijective due to the bijectivity of $\overline{\mathbb{J}}$.

(3) $\Rightarrow$ (1) : It has to be shown that $\mathbb{J}$ is surjective. Let $\left[\Omega\right]_{\mathrm{BC}}$ be a class in $H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R})$, which is mapped into $\mathcal{G}_X$ by $\mathscr{J}$. Then there exists a $\partial\overline{\partial}$-closed positive $(n-1,n-1)$-form Ψ and an $(n-2,n-1)$-form Θ, such that

$$\Omega = \Psi + \partial\Theta + \overline{\partial\Theta}.$$

Let $\widetilde{T}$ be any fixed non-zero $\partial\overline{\partial}$-closed positive (1, 1)-current. From the bijectivity of $\mathbb{j}$, there exists a d-closed positive (1, 1)-current T and a (0, 1)-current S, such that

$$\widetilde{T} = T + \partial S + \overline{\partial S}.$$

The current T cannot be zero current. If not, $\widetilde{T} = \partial S + \overline{\partial S}$, which implies that the integral $\int_X \omega_X^{n-1} \wedge \widetilde{T}$ will be larger than 0 and also equal to 0. This is a contradiction. Hence,

$$\begin{aligned}\int_X \Omega \wedge \widetilde{T} &= \int_X \Omega \wedge (T + \partial S + \overline{\partial S}) = \int_X \Omega \wedge T = \int_X (\Psi + \partial\Theta + \overline{\partial\Theta}) \wedge T \\ &= \int_X \Psi \wedge T > 0.\end{aligned}$$

Therefore, the class $\left[\Omega\right]_{\mathrm{BC}}$ lies in the balanced cone $\mathcal{B}_X$ by [22, Lemma 3.3] and thus the mapping $\mathbb{J}$ is surjective. □

Definition 4.14 ([9, Definition 1.3.(ii)]). Movable cone $\mathcal{M}_X$

Define the *movable cone* $\mathcal{M}_X \subseteq H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R})$ to be the closure of the convex cone generated by classes of currents in the type

$$\mu_*(\widetilde{\omega}_1 \wedge \cdots \wedge \widetilde{\omega}_{n-1})$$

where $\mu : \widetilde{X} \to X$ is an arbitrary modification and $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$ for $1 \leq j \leq n-1$. Here, X is an n-dimensional compact Kähler manifold.

We restate a lemma hidden in [22, Appendix] and [56].

Lemma 4.15 *Let X be a compact Kähler manifold. There exist the following inclusions:*

$$\mathcal{E}_X \subseteq \mathscr{L}^{-1}(\mathcal{E}_{\partial\overline{\partial}}) \subseteq \left(\mathcal{M}_X\right)^{\vee_c},$$

where $\mathscr{L}^{-1}(\mathcal{E}_{\partial\overline{\partial}})$ denotes the inverse image of the cone $\mathcal{E}_{\partial\overline{\partial}}$ under the isomorphism $\mathscr{L}$. Note that $H_{\mathrm{BC}}^{1,1}(X,\mathbb{R})$ and $H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R})$ are dual vector spaces in the Kähler case.

Proof It is clear that the mapping $\mathscr{L}$ is an isomorphism from $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ to $H^{1,1}_{\mathrm{A}}(X,\mathbb{R})$ and $\mathbb{j}$ is injective in the Kähler case. Thus, $\mathcal{E}_X \subseteq \mathscr{L}^{-1}(\mathcal{E}_{\partial\bar{\partial}})$. Let $[\alpha]_{\mathrm{BC}}$ be a class in the cone $\mathscr{L}^{-1}(\mathcal{E}_{\partial\bar{\partial}})$ with α a smooth representative, which implies that $[\alpha]_{\mathrm{A}}$ contains a $\partial\bar{\partial}$-closed positive (1, 1)-current $\widetilde{T}$.

To see $\mathscr{L}^{-1}(\mathcal{E}_{\partial\bar{\partial}}) \subseteq (\mathcal{M}_X)^{\vee_c}$, we need to show that $\int_X \alpha\wedge\mu_*(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1}) \geq 0$ for arbitrary modification $\mu : \widetilde{X} \to X$ and Kähler forms $\widetilde{\omega}_j$ on $\widetilde{X}$. A result in [2] states that for arbitrary modification $\mu : \widetilde{X} \to X$ and any $\partial\bar{\partial}$-closed positive (1, 1)-current $\widetilde{T}$ on X, there exists a unique $\partial\bar{\partial}$-closed positive (1, 1)-current T' on $\widetilde{X}$ such that $\mu_* T' = \widetilde{T}$ and $T' \in \mu^*([\widetilde{T}]_{\mathrm{A}})$. Here, we choose $\tilde{T}$ to be the one in the Aeppli class $[\alpha]_{\mathrm{A}}$. Then, one has

$$\begin{aligned}\int_X \alpha\wedge\mu_*(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1}) &= \int_{\widetilde{X}} \mu^*\alpha\wedge\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1}\\ &= \int_{\widetilde{X}} T'\wedge\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1} \geq 0,\end{aligned}$$

where T' and $\mu^*\alpha$ belong to the same Aeppli class on $\widetilde{X}$. □

Corollary 4.16 ([44, Sect. 6]). *If Conjecture 1.10 is assumed to hold true, then for a complex manifold X in the Fujiki class $\mathcal{C}$,*

$$\mathscr{J}^{-1}(\mathcal{G}_X) = \mathcal{B}_X \tag{4.2}$$

and thus Conjecture 1.7 is true in this case.

Proof The argument is a bit different from that in [44, Sect. 6] (or [12, Sect. 2]) and we claim no originality here. That X is balanced is obviously a result of (4.2) since the Gauduchon cone of a compact complex manifold is never empty and $\mathscr{J}$ is an isomorphism from the $\partial\bar{\partial}$-lemma. Now let us prove (4.2) under the assumption of Conjecture 1.10. Without loss of generality, we can assume that X is Kähler and thus this equality is a direct corollary of Lemma 4.15 and Proposition 4.13. □

Boucksom–Demailly–Paun–Peternell have proved in [9, Theorem 10.12, Corollary 10.13] that Conjecture 1.10 is true, when X is a compact hyperkähler manifold or a compact Kähler manifold which is a limit by deformation of projective manifolds with Picard number $\rho = h^{1,1}$. It follows that $\mathbb{J}$ is bijective in these two cases. The qualitative part of Transcendental Morse Inequalities Conjecture for differences of two nef classes [9, Conjecture 10.1.(ii)] has been proved by Popovici [42] and Xiao [59]. And a partial answer to the quantitative part is given by [44], with the case of nef $T^{1,0}_X$ obtained in [60, Proposition 3.2].

The following theorem may provide some evidence for the assertion of Question 1.8 whether the mapping $\mathbb{J}$ is bijective from the balanced cone $\mathcal{B}_X$ to the Gauduchon cone $\mathcal{G}_X$ on the Kähler manifold X.

Let us recall several important results from [10,62] on solving complex Monge–Ampère equations on a compact Kähler manifold X.

Fix a Kähler metric ω_X, a nef and big class $[\alpha]_{\mathrm{BC}}$, and a volume form η on X. By Yau's celebrated results in [62], for $0 < t \le 1$, there exists a unique smooth function u_t, satisfying that $\sup_X u_t = 0$, such that $\alpha + t\omega + \sqrt{-1}\partial\overline{\partial}u_t$ is a Kähler metric and

$$\left(\alpha + t\omega_X + \sqrt{-1}\partial\overline{\partial}u_t\right)^n = c_t\eta,$$

where $c_t = \frac{\int_X(\alpha+t\omega_X)^n}{\int_X \eta}$. As in [10, Theorems B and C], when t is equal to 0, there exists a unique α-psh u, satisfying that $\sup_X u = 0$, such that

$$\left\langle(\alpha + \sqrt{-1}\partial\overline{\partial}u)^n\right\rangle = c\eta,$$

where $c = \frac{\int_X \alpha^n}{\int_X \eta}$ and the bracket $\langle\cdot\rangle$ denotes the non-pluripolar product of positive currents. Moreover, u has minimal singularities and is smooth on $\mathrm{Amp}(\alpha)$, which is a Zariski open set on X and only depends on the class $[\alpha]_{\mathrm{BC}}$.

These results above can be viewed in the following manner as stated in [22, the part after Lemma 2.3]. The family of solutions u_t is compact in $L^1(X)$-topology. Then there exists a sequence u_{t_k} such that

$$\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k} \to \alpha + \sqrt{-1}\partial\overline{\partial}u$$

in the sense of currents on X with $t_k \to 0$. Meanwhile, u_t is compact in $C^\infty_{loc}(\mathrm{Amp}(\alpha))$, which means uniform convergence on any compact subset of $\mathrm{Amp}(\alpha)$. Therefore, there exists a subsequence of u_{t_k}, still denoted by u_{t_k}, such that

$$\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k} \to \alpha + \sqrt{-1}\partial\overline{\partial}u$$

in the sense of $C^\infty_{loc}(\mathrm{Amp}(\alpha))$. Hence u is smooth on $\mathrm{Amp}(\alpha)$ and $\alpha + \sqrt{-1}\partial\overline{\partial}u$ is a Kähler metric on $\mathrm{Amp}(\alpha)$, since η is a volume form.

Theorem 4.17 *Let X be a compact Kähler manifold and $[\alpha]_{\mathrm{BC}}$ a nef class. Then $[\alpha^{n-1}]_{\mathrm{A}} \in \mathcal{G}_X$ implies that $[\alpha^{n-1}]_{\mathrm{BC}} \in \mathcal{B}_X$. Hence, $\overline{\mathbb{I}}(\overline{\mathcal{K}}_X) \bigcap \mathcal{B}_X$ and $\overline{\mathbb{K}}(\overline{\mathcal{K}}_X) \bigcap \mathcal{G}_X$ can be identified by the mapping $\mathbb{J}$.*

Proof Assume that $[\alpha^{n-1}]_{\mathrm{A}}$ belongs to $\mathcal{G}_X$, where $[\alpha]_{\mathrm{BC}}$ is a nef class. From Lemma 4.10, for any non-zero d-closed positive $(1,1)$-current T, the integral $\int_X \alpha^{n-1} \wedge T > 0$. Since the nef cone $\overline{\mathcal{K}}_X$ is contained in the pseudo-effective cone $\mathcal{E}_X$, the nef class $[\alpha]_{\mathrm{BC}}$ contains a d-closed positive $(1,1)$-current S, which cannot be the zero current. Otherwise, $[0]_{\mathrm{A}} \in \mathcal{G}_X$, which contradicts with Lemma 4.2. Then, the integral $\int_X \alpha^n = \int_X \alpha^{n-1} \wedge S > 0$, which implies that the class $[\alpha]_{\mathrm{BC}}$ is nef and big, by [17, Theorem 0.5].

Let Q be any fixed $\partial\overline{\partial}$-closed positive $(1,1)$-current on X. From the discussion before this theorem, it is clear that the sequence of positive measures

$$\left\{(\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k})^{n-1} \wedge Q\right\}_{k\in\mathbb{N}^+}$$

has bounded mass, for example,

$$\int_X (\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k})^{n-1} \wedge Q \leq \int_X (\alpha + \omega_X)^{n-1} \wedge Q.$$

Therefore, there exists a subsequence, still denoted by

$$\left\{(\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k})^{n-1} \wedge Q\right\}_{k\in\mathbb{N}^+},$$

weakly convergent to a positive measure on X, denoted by μ. It follows that

$$\int_X \mu = \int_X \alpha^{n-1} \wedge Q,$$

since the equalities hold

$$\begin{aligned}\int_X \mu &= \lim_{k\to+\infty}\int_X (\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k})^{n-1} \wedge Q = \lim_{k\to+\infty}\int_X (\alpha + t_k\omega_X)^{n-1} \wedge Q \\ &= \int_X \alpha^{n-1} \wedge Q.\end{aligned}$$

Note that

$$(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1} \wedge Q\Big|_{\mathrm{Amp}(\alpha)}$$

is a well-defined positive measure on $\mathrm{Amp}(\alpha)$, since $\alpha + \sqrt{-1}\partial\overline{\partial}u$ is a Kähler metric on $\mathrm{Amp}(\alpha)$. Moreover, μ is equal to

$$(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1} \wedge Q\Big|_{\mathrm{Amp}(\alpha)}$$

on $\mathrm{Amp}(\alpha)$. Actually, for any smooth function f with $\mathrm{Supp}(f) \subseteq \mathrm{Amp}(\alpha)$, one has

$$\begin{aligned}\int_{\mathrm{Amp}(\alpha)} f\mu &= \int_X f\mu \\ &= \lim_{k\to+\infty}\int_X f(\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k})^{n-1} \wedge Q \\ &= \int_X f(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1} \wedge Q \\ &= \int_{\mathrm{Amp}(\alpha)} f(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1} \wedge Q \\ &= \int_{\mathrm{Amp}(\alpha)} f\left((\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1} \wedge Q\Big|_{\mathrm{Amp}(\alpha)}\right),\end{aligned} \tag{4.3}$$

where the equality (4.3) results from that the sequence $f(\alpha + t_k\omega_X + \sqrt{-1}\partial\overline{\partial}u_{t_k})^{n-1}$ converges to $f(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1}$ in the sense of smooth $(n-1, n-1)$-forms on X due to the convergence result stated before this theorem, with all their supports always contained in $\mathrm{Amp}(\alpha)$.

It is obvious that the integral $\int_X \alpha^{n-1} \wedge Q \geq 0$ for $[\alpha]_{\mathrm{BC}}$ nef. Now suppose that $\int_X \alpha^{n-1} \wedge Q = 0$. Then we have $\int_X \mu = \int_X \alpha^{n-1} \wedge Q = 0$. And μ is equal to $(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1} \wedge Q\big|_{\mathrm{Amp}(\alpha)}$ on $\mathrm{Amp}(\alpha)$ with $(\alpha + \sqrt{-1}\partial\overline{\partial}u)^{n-1}$ a positive $(n-1, n-1)$-form on $\mathrm{Amp}(\alpha)$. Then $\mathrm{Supp}(Q) \subseteq X \setminus \mathrm{Amp}(\alpha)$, which is an analytic subvariety V on X with $\dim V \leq n-1$.

Denote the irreducible components with dimension $n-1$ of V by $\{V_i\}_{i=1}^m$. By [1, Theorem 1.5] and [22, Lemma 3.5], there exist constants $c_i \geq 0$ for $1 \leq i \leq m$ such that

$$Q - \sum_{i=1}^{m} c_i[V_i] = 0,$$

since V has no irreducible component of dimension larger than $n-1$. And we have $\int_X \alpha^{n-1} \wedge [V_i] > 0$, where $[V_i]$ are non-zero d-closed positive $(1,1)$-currents for $1 \leq i \leq m$. Then $\int_X \alpha^{n-1} \wedge Q = 0$ forces that the constants c_i are all equal to 0, namely Q a zero current. Hence, $[\alpha^{n-1}]_{\mathrm{BC}} \in \mathcal{B}_X$ from [22, Lemma 3.3].

It is clear that the restricted mapping $\mathbb{J}$, from $\overline{\mathbb{I}}(\overline{\mathcal{K}}_X) \bigcap \mathcal{B}_X$ to $\overline{\mathbb{K}}(\overline{\mathcal{K}}_X) \bigcap \mathcal{G}_X$, is injective. And the proof above shows that it is also surjective. Hence the restricted mapping $\mathbb{J}$ is bijective. □

We will describe the degeneration of balanced cones on compact complex manifolds, similar to the case of Gauduchon cones in [41, Proposition 5.4].

Lemma 4.18 *Let X be a compact complex manifold. Then the balanced cone $\mathcal{B}_X$ degenerates if and only if there exists no non-zero $\partial\overline{\partial}$-closed positive $(1,1)$-current T on X.*

Proof Assume that $\mathcal{B}_X = H_{\mathrm{BC}}^{n-1,n-1}(X, \mathbb{R})$. In particular, there exists a Hermitian metric $\widetilde{\omega}$ on X, such that $\widetilde{\omega}^{n-1}$ is $\partial\overline{\partial}$-exact. If T is a non-zero $\partial\overline{\partial}$-closed positive $(1,1)$-current on X, the integral $\int_X \widetilde{\omega}^{n-1} \wedge T$ has to be larger than 0 for the form $\widetilde{\omega}^{n-1}$ being positive and simultaneously equal to zero as $\widetilde{\omega}^{n-1}$ is $\partial\overline{\partial}$-exact. This contradiction leads to non-existence of such current T.

Conversely, assume that there exists no non-zero $\partial\overline{\partial}$-closed positive $(1,1)$-current T on X. Let $\mathfrak{D}'^{1,1}_{\mathbb{R}}$ be the set of real $(1,1)$-currents on X with the weak topology. Fix a Hermitian metric ω_X on X. Then apply the Hahn–Banach separation theorem.

Let us set

$$\mathfrak{D}_1 = \left\{ T \in \mathfrak{D}'^{1,1}_{\mathbb{R}} \;\middle|\; \partial\overline{\partial}T = 0 \right\},$$

$$\mathfrak{D}_2 = \left\{ T \in \mathfrak{D}'^{1,1}_{\mathbb{R}} \;\middle|\; \int_X \omega_X^{n-1} \wedge T = 1 \text{ and } T \geq 0 \right\}.$$

It is easy to see that $\mathfrak{D}_1$ is a closed linear subspace of the locally convex space $\mathfrak{D}'^{1,1}_{\mathbb{R}}$, while $\mathfrak{D}_2$ is a compact convex one in $\mathfrak{D}'^{1,1}_{\mathbb{R}}$. And $\mathfrak{D}_1 \bigcap \mathfrak{D}_2 = \emptyset$ from the assumption. Then there exists a continuous linear functional on $\mathfrak{D}'^{1,1}_{\mathbb{R}}$, denoted by Ω, a real $(n-1, n-1)$-form, such that it vanishes on $\mathfrak{D}_1$ and evaluates positively on $\mathfrak{D}_2$. Hence, Ω has to be a $\partial\bar{\partial}$-exact positive $(n-1, n-1)$-form. It follows that the class $[\Omega]_{BC}$ is the zero class in $H^{n-1,n-1}_{BC}(X, \mathbb{R})$, which also lives in the balanced cone $\mathcal{B}_X$, which implies that the balanced cone $\mathcal{B}_X$ degenerates. □

Remark 4.19 [41, Proposition 5.4] tells us the Gauduchon cone of a compact complex manifold X degenerates if and only if there exists no non-zero d-closed positive $(1,1)$-current on X, and, together with Proposition 4.18, implies that the Gauduchon cone of a compact balanced manifold will degenerate when its balanced cone does.

Question 4.20 Fu–Li–Yau [21] constructed a balanced threefold, which is a connected sum of k-copies of $S^3 \times S^3$ $(k \geq 2)$ and whose balanced cone degenerates (cf. [22]). Is it possible to find a balanced manifold such that its Gauduchon cone degenerates while its balanced cone does not ?

4.3 Deformation Results Related with $\mathcal{G}_X$

In this subsection, we will discuss several deformation results related with $\mathcal{G}_X$ in Theorems 4.22 and 4.23.

Firstly, let us review Demailly's regularization theorem [15], whose different versions have been used by various authors in the literature. Recall that a real $(1,1)$-current T is said to be *almost positive* if $T \geq \gamma$ for some real smooth $(1,1)$-form, and each d-closed almost positive $(1,1)$-current T on a compact complex manifold can be written as $\theta + \sqrt{-1}\partial\bar{\partial} f$, where θ is a d-closed smooth $(1,1)$-form with f almost plurisubharmonic (shortly almost psh) function (cf. [7, Sect. 2.1] and [17, Sect. 3]). We say that a d-closed almost positive $(1,1)$-current T has *analytic (or algebraic) singularities* along the analytic subvariety Y, if f does, i.e., f can be locally written as

$$\frac{c}{2}\log(|g_1|^2 + |g_2|^2 + \cdots + |g_N|^2) + h,$$

where $c > 0$ (or $c \in \mathbb{Q}^+$), $\{g_i\}_{i=1}^N$ are local generators of the ideal sheaf of Y and h is some smooth function. It is clear that T is smooth outside the singularity Y. Then the following formulation of *Regularization Theorem* will be applied:

Theorem 4.21 ([17, Theorem 3.2], [7, Theorem 2.4], [8, Theorem 2.1]). *Let $T = \theta + \sqrt{-1}\partial\bar{\partial} f$ be a d-closed almost positive $(1,1)$-current on a compact complex manifold X, satisfying that $T \geq \gamma$ for some real smooth $(1,1)$-form. Then there exists a sequence of functions f_k with analytic singularities Y_k converging to f, such that, if we set $T_k = \theta + \sqrt{-1}\partial\bar{\partial} f_k$, it follows that*

(1) *T_k weakly converges to T.*
(2) *$T_k \geq \gamma - \epsilon_k\omega$, where $\lim\limits_{k\to+\infty} \downarrow \epsilon_k = 0$ and ω is some fixed Hermitian metric.*

(3) *The Lelong numbers* $\nu(T_k, x)$ *increase to* $\nu(T, x)$ *uniformly with respect to* $x \in X$.
(4) *The analytic singularities increase with respect to k, i.e.,* $Y_k \subseteq Y_{k+1}$.

Denote the blow up of X along the singularity Y_k by $\mu_k : \tilde{X}_k \to X$, and we will see that $\mu_k^*(T_k)$ still acquires the analytic singularity $\mu_k^{-1}(Y_k)$, without irreducible components of complex codimensions at least 2, for each k. According to [8, Sect. 2.5], the Siu's decomposition [51] for $\mu_k^*(T_k)$ writes

$$\mu_k^*(T_k) = \tilde{R}_k + \sum_j \nu_{kj} \big[\tilde{Y}_{kj}\big], \tag{4.4}$$

where $\tilde{R}_k$ is a d-closed smooth (1, 1)-form, satisfying that $\tilde{R}_k \geq \mu_k^*(\gamma - \epsilon_k \omega)$, $\tilde{Y}_{kj}$ are irreducible components of complex codimension one of $\mu_k^{-1}(Y_k)$ for all j, and ν_{kj} are all positive numbers. It is obvious that the degree of μ_k is equal to 1 for each k. It follows that, after the push forward,

$$T_k = \mu_{k*}\big(\mu_k^*(T_k)\big) = \mu_{k*}(\tilde{R}_k) + \sum_j \nu_{kj} \big[Y_{kj}\big], \tag{4.5}$$

which is exactly the Siu's decomposition for T_k. Here, $\mu_{k*}(\tilde{R}_k)$ is a d-closed positive (1, 1)-current, which is smooth outside irreducible components of complex codimension at least 2 of Y_k and satisfies that $\mu_{k*}(\tilde{R}_k) \geq \gamma - \epsilon_k \omega$. The symbols Y_{kj} stand for the irreducible components of complex codimension one of Y_k, since the following equalities hold

$$\mu_{k*}\Big(\big[\tilde{Y}_{kj}\big]\Big) = \begin{cases} \big[\mu_k(\tilde{Y}_{kj})\big], & \text{when } \dim \mu_k(\tilde{Y}_{kj}) = n-1; \\ 0, & \text{when } \dim \mu_k(\tilde{Y}_{kj}) < n-1. \end{cases}$$

Meanwhile, Barlet's theory [5] of cycle spaces comes into play and let us follow the statements in Demailly–Paun's paper [17, Sect. 5]. Let $\pi : \mathcal{X} \to \Delta_\epsilon$ be a holomorphic family of Kähler fibers of complex dimension n. Then there is a canonical holomorphic projection

$$\pi_p : C^p(\mathcal{X}/\Delta_\epsilon) \to \Delta_\epsilon,$$

where $C^p(\mathcal{X}/\Delta_\epsilon)$ denotes the relative analytic cycle space of complex dimension p, i.e., all cycles contained in the fibers of the family $\pi : \mathcal{X} \to \Delta_\epsilon$. And it is known that the restriction of π_p to the connected components of $C^p(\mathcal{X}/\Delta_\epsilon)$ are proper maps by the Kähler property of the fibers. Also, there is a cohomology class map, commuting with the projection to Δ_ϵ, defined by

$$\begin{array}{rccc} \iota_p : C^p(\mathcal{X}/\Delta_\epsilon) & \to & \mathrm{R}^{2(n-p)}\pi_*\big(\mathbb{Z}_{\mathcal{X}}\big) \\ Z & \mapsto & \big[Z\big], \end{array}$$

which associates to every analytic cycle Z in X_t its cohomology class $\big[Z\big] \in H^{2(n-p)}(X_t, \mathbb{Z})$. Again by the Kählerness, the mapping ι_p is proper.

Denote the images in Δ_ϵ of those connected components of $C^p(\mathcal{X}/\Delta_\epsilon)$ which do not project onto Δ_ϵ under the mapping π_p by $\bigcup S_\nu$, namely a countable union of analytic subvarieties S_ν of Δ_ϵ, from the properness of the mapping π_p restricted to each component of $C^p(\mathcal{X}/\Delta_\epsilon)$ for $1 \leq p \leq n-1$ (cf. [17, proof of Theorem 0.8]). Clearly, each $S_\nu \subsetneq \Delta_\epsilon$. And thus, for $t \in \Delta_\epsilon \setminus \bigcup S_\nu$, every irreducible analytic subvariety of complex codimension $n-p$ in X_t can be extended into any other fiber in the family $\pi : \mathcal{X} \to \Delta_\epsilon$ with the invariance of its cohomology class.

Now, let us go back to the deformation of Gauduchon cone. An ***sGG** manifold* is a compact complex manifold, satisfying that each Gauduchon metric on it is strongly Gauduchon from the definition in [45, Lemma 1.2]. And the **sGG** property is open under small holomorphic deformations from [45, the remark after Theorem 1.5]. Thus, let us call the holomorphic family $\pi : \mathcal{X} \to \Delta_\epsilon$ with the central fiber X_0 being an **sGG** manifold an ***sGG** family*. Moreover, Popovici and Ugarte proved that the following inclusion holds

$$\mathcal{G}_{X_0} \subseteq \lim_{t\to 0} \mathcal{G}_{X_t}$$

when the family $\pi : \mathcal{X} \to \Delta_\epsilon$ is an **sGG** family in [45, Definition 5.6, Theorem 5.7]. The definition of $\lim_{t\to 0} \mathcal{G}_{X_t}$ is given by

$$\lim_{t\to 0} \mathcal{G}_{X_t} = \Big\{ \big[\Omega\big]_{\mathrm{A}} \in H_{\mathrm{A}}^{n-1,n-1}(X_0, \mathbb{R}) \,\Big|\, \mathrm{P}_t \circ \mathrm{Q}_0\Big(\big[\Omega\big]_{\mathrm{A}}\Big) \in \mathcal{G}_{X_t} \text{ for sufficiently small } t \Big\},$$

where the canonical mappings

$$\mathrm{P}_t : H_{\mathrm{DR}}^{2n-2}(X_t, \mathbb{R}) \to H_{\mathrm{A}}^{n-1,n-1}(X_t, \mathbb{R})$$

send the De Rham class $\big[\Theta\big]_{\mathrm{DR}}$ to the Aeppli class $\big[\Theta^{n-1,n-1}\big]_{\mathrm{A}}$, represented by the $(n-1, n-1)$-component of Θ on X_t, and the mapping

$$\mathrm{Q}_0 : H_{\mathrm{A}}^{n-1,n-1}(X_0, \mathbb{R}) \to H_{\mathrm{DR}}^{2n-2}(X_t, \mathbb{R}),$$

depends on a fixed Hermitian metric ω_0 on X_0 according to [45, Definition 5.3]. By [45, Proposition 5.1, Lemma 5.4], the canonical mappings P_t are surjective and the mapping Q_0 is injective, satisfying that

$$\mathrm{P}_0 \circ \mathrm{Q}_0 = \mathrm{id}_{H_{\mathrm{A}}^{n-1,n-1}(X,\mathbb{R})}.$$

The following theorem gives a bound from the other side.

Theorem 4.22 *Let $\pi : \mathcal{X} \to \Delta_\epsilon$ be a holomorphic family with a Kählerian central fiber. Then we have*

$$\lim_{t\to \tau} \mathcal{G}_{X_t} \subseteq \mathcal{N}_{X_\tau} \quad \textit{for each } \tau \in \Delta_\epsilon,$$

where $\mathcal{N}_{X_\tau}$ *is the convex cone generated by Aeppli classes of* $\partial_\tau\overline{\partial}_\tau$*-closed positive* $(n-1, n-1)$*-currents on* X_τ*. Moreover, the following inclusion holds,*

$$\lim_{t\to\tau} \mathcal{G}_{X_t} \subseteq \overline{\mathcal{G}}_{X_\tau} \quad \textit{for each} \;\; \tau \in \Delta_\epsilon \setminus \bigcup S_\nu,$$

where $\bigcup S_\nu$ *is explained above in this section.*

Proof It is clear that we can assume that each fiber of the family $\pi : \mathcal{X} \to \Delta_\epsilon$ is Kähler (apparently an **sGG** family) and $\{\omega_t\}_{t\in\Delta_\epsilon}$ is a family of Kähler metrics of the fibers, varying smoothly with respect to t, by use of the stability theorem of Kähler structures [28], after shrinking the disk Δ_ϵ.

For $\tau \in \Delta_\epsilon$, let $\left[\Omega\right]_{\mathrm{A}}$ be an element of $\lim_{t\to\tau} \mathcal{G}_{X_t}$, Ω its smooth representative, which indicates

$$\mathrm{P}_t \circ \mathrm{Q}_\tau\left(\left[\Omega\right]_{\mathrm{A}}\right) \in \mathcal{G}_{X_t} \quad \text{for} \;\; 0 < |t-\tau| < \delta_{[\Omega]_{\mathrm{A}}}$$

by definition. Set the positive representative of $\mathrm{P}_t \circ \mathrm{Q}_\tau(\left[\Omega\right]_{\mathrm{A}})$ as Ω_t. It is obvious that the following equality holds:

$$\lim_{t\to\tau} \int_{X_t} \Omega_t \wedge \omega_t = \int_{X_\tau} \Omega \wedge \omega_\tau,$$

since the integral just depends on the Aeppli class of Ω_t. This implies that

$$\{\Omega_t\}_{0<|t-\tau|<\delta_{[\Omega]_{\mathrm{A}}}}$$

have bounded mass, and thus the weak limit of a subsequence is a $\partial_\tau\overline{\partial}_\tau$-closed positive $(n-1, n-1)$-current, which lies in the Aeppli class $\left[\Omega\right]_{\mathrm{A}}$ on X_τ. Hence, this shows

$$\lim_{t\to\tau} \mathcal{G}_{X_t} \subseteq \mathcal{N}_{X_\tau}.$$

As to the second inclusion, let us fix $\tau \in \Delta_\epsilon \setminus \bigcup S_\nu$. Then the following integral should be considered

$$\int_{X_\tau} \Omega \wedge T,$$

where T is any fixed d-closed positive $(1, 1)$-current on X_τ. Apply Theorem 4.21 to T and we have a sequence of currents T_k with analytic singularities, denoted by Y_k, such that T_k always lies in the Bott–Chern class $\left[T\right]_{\mathrm{BC}}$ and $T_k \geq -\epsilon_k\omega_\tau$. From the very definition of $\bigcup S_\nu$, the singularity Y_k on X_τ, with possibly high codimensional irreducible components, can be extended into the other fibers of the family $\pi : \mathcal{X} \to \Delta_\epsilon$, for each k. The extension of Y_k is denoted by $\mathcal{Y}_k$, which is a relative analytic subvariety of the total space $\mathcal{X}$ of the family $\pi : \mathcal{X} \to \Delta_\epsilon$. Blow up $\mathcal{X}$ along $\mathcal{Y}_k$, and then we will obtain

$$\tilde{\mathcal{X}}_k \xrightarrow{\mu_k} \mathcal{X} \xrightarrow{\pi} \Delta_\epsilon.$$

The restriction of μ_k to the t-fiber is exactly the blow up $\mu_k(t) : \tilde{X}_k(t) \to X_t$ of X_t along $Y_k(t)$, with the exceptional divisor denoted by $\tilde{Y}_k(t)$, where $Y_k(t) = \mathcal{Y}_k \cap X_t$. Then we can apply Equalities (4.4) and (4.5) to T_k:

$$\begin{aligned}\int_{X_\tau} \Omega \wedge T &= \int_{X_\tau} \Omega \wedge T_k \\ &= \int_{X_\tau} \Omega \wedge \left(\mu_k(\tau)_*\big(\tilde{R}_k\big) + \sum_j \nu_{kj}\big[Y_{kj}\big]\right) \\ &= \int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \tilde{R}_k + \sum_j \nu_{kj} \int_{X_\tau} \Omega \wedge \big[Y_{kj}\big],\end{aligned} \tag{4.6}$$

where $\tilde{R}_k \geq -\epsilon_k \mu_k(\tau)^*\omega_\tau$, Y_{kj} are irreducible components of complex codimension one of Y_k and ν_{kj} are positive numbers for all j.

We claim the following two statements:

(1) $\int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \tilde{R}_k \geq -\epsilon_k \int_{X_\tau} \Omega \wedge \omega_\tau$;
(2) $\int_{X_\tau} \Omega \wedge \big[Y_{kj}\big] \geq 0.$

For the statement (1), we consider that

$$\begin{aligned}&\int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \tilde{R}_k \\ &= \int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \Big(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^*\omega_\tau\Big) \\ &\quad - 2\epsilon_k \int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \Big(\mu_k(\tau)^*\omega_\tau\Big) \\ &= \int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \Big(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^*\omega_\tau\Big) - 2\epsilon_k \int_{X_\tau} \Omega \wedge \omega_\tau.\end{aligned}$$

It should be noted that $\mu_k(\tau)^*\omega_\tau$ is a semi-positive $(1,1)$-form on $\tilde{X}_k(\tau)$ for each k. And thus, we can choose a sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}^+}$, converging to 0, such that $\mu_k(\tau)^*\omega_\tau - \lambda_k u_k$ is positive for each k, where u_k is some smooth form in the Bott–Chern cohomology class of $\big[\tilde{Y}_k(\tau)\big]$ (cf. [17, Lemma 3.5]). Hence, the integral above amounts to the following equalities:

$$\begin{aligned}&\int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \tilde{R}_k \\ &= \int_{\tilde{X}_k(\tau)} \Big(\mu_k(\tau)^*\Omega\Big) \wedge \Big(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^*\omega_\tau - \epsilon_k\lambda_k u_k\Big)\end{aligned}$$

$$+ \epsilon_k \lambda_k \int_{\tilde{X}_k(\tau)} \left(\mu_k(\tau)^* \Omega \right) \wedge u_k - 2\epsilon_k \int_{X_\tau} \Omega \wedge \omega_\tau$$
$$= \int_{\tilde{X}_k(\tau)} \left(\mu_k(\tau)^* \Omega \right) \wedge \left(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^* \omega_\tau - \epsilon_k \lambda_k u_k \right)$$
$$+ \epsilon_k \lambda_k \int_{\tilde{X}_k(\tau)} \left(\mu_k(\tau)^* \Omega \right) \wedge \left[\tilde{Y}_k(\tau) \right] - 2\epsilon_k \int_{X_\tau} \Omega \wedge \omega_\tau .$$

It is clear that

$$\left(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^* \omega_\tau - \epsilon_k \lambda_k u_k \right) = \left(\tilde{R}_k + \epsilon_k \mu_k(\tau)^* \omega_\tau \right) + \epsilon_k \left(\mu_k(\tau)^* \omega_\tau - \lambda_k u_k \right)$$

is a Kähler metric on $\tilde{X}_k(\tau)$ for each k. Then it follows that

$$\int_{\tilde{X}_k(\tau)} \left(\mu_k(\tau)^* \Omega \right) \wedge \left(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^* \omega_\tau - \epsilon_k \lambda_k u_k \right)$$
$$= \lim_{t \to \tau} \int_{\tilde{X}_k(t)} \left(\mu_k(t)^* \Omega_t \right) \wedge \tilde{\omega}_k(t) \geq 0,$$

where $\tilde{\omega}_k(t)$ is a family of Kähler metrics on $\tilde{X}_k(t)$, starting with

$$\left(\tilde{R}_k + 2\epsilon_k \mu_k(\tau)^* \omega_\tau - \epsilon_k \lambda_k u_k \right)$$

and varying smoothly with respect to t, from the stability theorem of Kähler structures [28]. Moreover, the integral $\int_{\tilde{X}_k(t)} \left(\mu_k(t)^* \Omega_t \right) \wedge \tilde{\omega}_k(t)$ only depends on the Aeppli class of $\mu_k(t)^* \Omega_t$ and $\left[\mu_k(t)^* \Omega_t \right]_{\mathrm{A}}$ converges to $\left[\mu_k(\tau)^* \Omega \right]_{\mathrm{A}}$ when $t \to \tau$. Similarly, we can get that

$$\epsilon_k \lambda_k \int_{\tilde{X}_k(\tau)} \left(\mu_k(\tau)^* \Omega \right) \wedge \left[\tilde{Y}_k(\tau) \right] = \epsilon_k \lambda_k \lim_{t \to \tau} \int_{\tilde{X}_k(t)} \left(\mu_k(\tau)^* \Omega_t \right) \wedge \left[\tilde{Y}_k(t) \right] \geq 0,$$

where $\tilde{Y}_k(t)$ is the extension of $\tilde{Y}_k(\tau)$ to the t-fiber $\tilde{X}_k(t)$ of the total space $\tilde{\mathcal{X}}_k$. Based on these two inequalities above, one has

$$\int_{\tilde{X}_k(\tau)} \left(\mu_k(\tau)^* \Omega \right) \wedge \tilde{R}_k \geq -\epsilon_k \int_{X_\tau} \Omega \wedge \omega_\tau .$$

Therefore, the statement (1) is proved.

For the statement (2), let us recall that every analytic irreducible subvariety of complex codimension $n - p$ in X_τ can be extended into any other fiber in the family $\pi : \mathcal{X} \to \Delta_\epsilon$ with the invariance of its cohomology class, from Barlet's theory of analytic cycle discussed above. Especially, the irreducible components Y_{kj} of complex codimension one of Y_k on X_τ can be extended to the ones $Y_{kj}(t)$ on the t-fiber X_t, which are contained in $Y_k(t)$. Then it is easy to see that

$$\int_{X_\tau} \Omega \wedge \left[Y_{kj}\right] = \lim_{t\to\tau} \int_{X_t} \Omega_t \wedge \left[Y_{kj}(t)\right] \geq 0.$$

The statement (2) is also proved.

Together with these two statements and (4.6), it is clear that

$$\int_{X_\tau} \Omega \wedge T \geq -\epsilon_k \int_{X_\tau} \Omega \wedge \omega_\tau,$$

for each k. Then it follows that

$$\int_{X_\tau} \Omega \wedge T \geq 0,$$

where T is any fixed d-closed positive (1, 1)-current on X_τ. Proposition 4.11 assures the inclusion: for $\tau \in \Delta_\epsilon \setminus \bigcup S_\nu$,

$$\lim_{t\to\tau} \mathcal{G}_{X_t} \subseteq \overline{\mathcal{G}}_{X_\tau}.$$

□

Theorem 4.23 *Let $\pi : \mathcal{X} \to \Delta_\epsilon$ be a holomorphic family with fibers all Kähler manifolds. For some $\tau \in \Delta_\epsilon$, the fiber X_τ admits the equality $\overline{\mathcal{K}}_{X_\tau} = \mathcal{E}_{X_\tau}$. Then the inclusion holds:*

$$\lim_{t\to\tau} \mathcal{G}_{X_t} \subseteq \overline{\mathcal{G}}_{X_\tau}.$$

In particular, the fiber X_τ with nef holomorphic tangent bundle $T^{1,0}_{X_\tau}$ satisfies the inclusion above.

Proof The condition $\overline{\mathcal{K}}_{X_\tau} = \mathcal{E}_{X_\tau}$ implies that, for any d-closed positive (1, 1)-current T and arbitrary $\delta > 0$, there exists a smooth (1, 1)-form α_δ, which lies in the Bott–Chern class $\left[T\right]_{\mathrm{BC}}$, such that

$$\alpha_\delta \geq -\delta\omega_\tau,$$

where ω_τ is the fixed Kähler metric of X_τ.

Fix an element $\left[\Omega\right]_{\mathrm{A}}$ of $\lim_{t\to\tau} \mathcal{G}_{X_t}$, which means that

$$\mathrm{P}_t \circ \mathrm{Q}_\tau\left(\left[\Omega\right]_{\mathrm{A}}\right) \in \mathcal{G}_{X_t} \quad \text{for } 0 < |t-\tau| < \delta_{[\Omega]_{\mathrm{A}}}.$$

Then for any d-closed positive (1, 1)-current T,

$$\begin{aligned}\int_{X_\tau} \Omega \wedge T &= \int_{X_\tau} \Omega \wedge \alpha_\delta \\ &= \int_{X_\tau} \Omega \wedge \left(\alpha_\delta + 2\delta\omega_\tau\right) - 2\delta \int_{X_\tau} \Omega \wedge \omega_\tau.\end{aligned}$$

It is clear that $\alpha_\tau + 2\tau\omega_\tau$ is a Kähler metric on X_τ, and thus, from the stability theorem of Kähler structures [28], there exists a family of Kähler metrics $\tilde{\alpha}_\delta(t)$ on X_t, starting with $\alpha_\tau + 2\delta\omega_\tau$ and varying smoothly with respect to t. It follows that

$$\int_{X_\tau} \Omega \wedge \left(\alpha_\delta + 2\delta\omega_\tau\right) = \lim_{t\to\tau} \int_{X_t} \Omega_t \wedge \tilde{\alpha}_\delta(t) \geq 0,$$

since the integral also depends on the Aeppli class of Ω_t, and Ω_t is the positive representative in $\mathrm{P}_t \circ \mathrm{Q}_\tau\left(\left[\Omega\right]_{\mathrm{A}}\right)$ for each $t \neq \tau$. As δ can be arbitrarily small, we have

$$\int_{X_\tau} \Omega \wedge T \geq 0,$$

which assures that $\left[\Omega\right]_{\mathrm{A}} \in \overline{\mathcal{G}}_{X_\tau}$ by Proposition 4.11. If a compact complex manifold has nef holomorphic tangent bundle, the nef cone and the pseudo-effective cone coincide by [15, Corollary 1.5]. Therefore, the proofs are completed. □

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