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Hyperbolic Evolution Equations, Lorentzian Holonomy, and Riemannian Generalised Killing Spinors

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Abstract We prove that the Cauchy problem for parallel null vector fields on smooth Lorentzian manifolds is well-posed. The proof is based on the derivation and analysis of suitable hyperbolic evolution equations given in terms of the Ricci tensor and other geometric objects. Moreover, we classify Riemannian manifolds satisfying the constraint conditions for this Cauchy problem. It is then possible to characterise certain holonomy reductions of globally hyperbolic manifolds with parallel null vector in terms of flow equations for Riemannian special holonomy metrics. For exceptional holonomy groups these flow equations have been investigated in the literature before in other contexts. As an application, the results provide a classification of Riemannian manifolds admitting imaginary generalised Killing spinors. We will also give new local normal forms for Lorentzian metrics with parallel null spinor in any dimension.

Keywords Lorentzian geometry · Holonomy groups · Parallel null vector field · Cauchy problem · Killing spinors · Parallel spinors · Symmetric hyperbolic system

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1 Background and Main Results

Lorentzian manifolds with parallel null vector fields or parallel null spinor fields arise naturally in geometric as well as physical contexts. In general relativity they occur as wave-like solutions to the Einstein equations and in string theory they constitute supergravity backgrounds with a high degree of supersymmetry. In geometry they form a class of Lorentzian manifolds with *special holonomy*, i.e. whose holonomy group is reduced but the manifold is not locally a product. The holonomy algebras associated to Lorentzian manifolds with special holonomy were classified in [12,30] and *local metrics* realising the given holonomy algebras are constructed in [24]. More recently, the interplay between special Lorentzian holonomy, or more specifically, the existence of parallel null vector fields, and *global* geometric properties has become the focus of research [9,11,15,19,31,35]. In the present paper, we address the problem of constructing *globally hyperbolic* Lorentzian manifolds by solving a Cauchy problem that arises from the existence of a parallel null vector field. Another motivation arises from spin geometry: the existence of a parallel null spinor implies the existence of a parallel null vector field and it turns out that the associated Cauchy problem provides some interesting relations to Riemannian spin geometry. In fact, it naturally leads to a classification result for (complete) Riemannian manifolds admitting the so-called *imaginary generalised Killing spinors*.

As in a preceding paper [10], we have shown that the Cauchy problem for parallel null vector fields is well-posed for *real analytic data* and that it always has a globally hyperbolic solution. The proof rested on the derivation of suitable evolution equations which can be analysed using the Cauchy–Kowalevski Theorem. More precisely, let $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ be a Lorentzian manifold admitting a nontrivial vector field $V \in \mathfrak{X}(\overline{\mathcal{M}})$ which is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{\mathbf{g}}$ and satisfies $\overline{\mathbf{g}}(V, V) = 0$. Suppose further that $(\mathcal{M}, \mathbf{g})$ is a spacelike hypersurface of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ which embeds into $\overline{\mathcal{M}}$ with Weingarten tensor W. As seen in [10], requiring the vector field V to be parallel, imposes on $(\mathcal{M}, \mathbf{g})$ the constraint

$$\nabla U + u\mathrm{W} = 0, \tag{1.1}$$

where $U \in \mathfrak{X}(\mathcal{M})$ is the vector field given by the negative of the projection of V onto $T\mathcal{M}$ and $u = \sqrt{\mathbf{g}(U, U)}$. Then, using the Cauchy–Kowalevski Theorem, in [10] it was shown that *real analytic* initial data $(\mathcal{M}, \mathbf{g}, U, \mathrm{W})$ satisfying the constraint (1.1) can be extended to a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ with parallel null vector field V.

This result immediately suggests the question whether the Cauchy problem for a parallel null vector field is also well-posed for *smooth* data. For a parallel null spinor, this was verified in [33] using techniques surrounding the Cauchy problem for the vacuum Einstein equations. It turns out that these techniques can also be applied here—after overcoming some difficulties explained below—allowing us to prove our main theorem that the Cauchy problem for parallel null vector fields is also well-posed for *smooth data*:

Theorem 1 *Let* $(\mathcal{M}, \mathbf{g})$ *be a smooth Riemannian manifold admitting a nontrivial vector field* U *solving* (1.1) *for some symmetric endomorphism* W *on* $\mathcal{M}$. *Moreover,* *let* $\lambda \in C^\infty(\mathcal{M}, \mathbb{R}^*)$ *be a given function. Then there exists an open neighbourhood* $\overline{\mathcal{M}}$ *of* $\{0\} \times \mathcal{M}$ *in* $\mathbb{R} \times \mathcal{M}$ *and a Lorentzian metric of the form*

$$\overline{\mathbf{g}} = -\widetilde{\lambda}^2 dt^2 + \mathbf{g}_t,$$

where $\mathbf{g}_t$ *is a family of Riemannian metrics on* $\mathcal{M}$ *and* $\widetilde{\lambda}$ *is a positive function on* $\overline{\mathcal{M}}$ *with*

$$\mathbf{g}_0 = \mathbf{g}, \qquad \widetilde{\lambda}|_{\mathcal{M}} = \lambda,$$

such that U *extends to a parallel null vector field on* $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$*. Moreover,* $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ *can be chosen to be globally hyperbolic with spacelike Cauchy hypersurface* $\mathcal{M}$*, i.e.* $\mathcal{M}$ *is met by every inextendible timelike curve in* $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ *exactly once.*

The proof of Theorem 1 in Sect. 3 is based on the theory of quasilinear symmetric hyperbolic PDEs as known from general relativity. Let us point out, however, one *fundamental difference* to similar Cauchy problems in general relativity or in [33]:

Considering the Cauchy problem for the Einstein equations in general relativity, it follows by definition of the problem that evolution equations for the metric are given in terms of the Ricci (or Einstein) tensor. Similarly, there are integrability conditions for parallel null spinors on Lorentzian manifolds formulated in terms of the Ricci tensor (the Ricci tensor is nilpotent [5]) and they lead to obvious evolution equations for the metric in the Cauchy problem for parallel null spinor fields [33]. It is important for a smooth solution theory to have evolution equations in terms of the Ricci tensor in these cases because in Lorentzian signature, the resulting PDEs can be reformulated as hyperbolic systems, see [22] for instance. In contrast to that, the existence of a parallel null vector field on a Lorentzian manifold yields hardly any nontrivial information about the Ricci tensor. Thus, it is not obvious at all that the methods that work for the Cauchy problem for the Einstein equations or a parallel null spinor field also work for a parallel null vector field and that Theorem 1 can be proved by deriving an evolution equation for the metric $\overline{\mathbf{g}}$ in terms of the Ricci tensor. The key idea here is to simply introduce the Ricci tensor as new unknown $\mathrm{Z} = \mathrm{Ric}(\overline{\mathbf{g}})$, consider this as an evolution equation for the metric $\overline{\mathbf{g}}$, and then *close the system* by further differentiation that results in a first-order equation for Z. The resulting PDE turns out to be hyperbolic and is a key ingredient for the proof of Theorem 1. We believe that this approach can be used in other settings, for example, for the Einstein equations with complicated energy momentum tensor.

With the result of Theorem 1 at hand, it is natural to search for Riemannian manifolds solving the constraint equations (1.1), in particular for *geodesically complete* solutions. As a first step, we exhibit the local structure of solutions to (1.1). Using the flow of the vector field U in (1.1), which defines a *closed* one-form, it easily follows that the metric $\mathbf{g}$ can be brought into an adapted normal form:

Theorem 2 *Any solution* $(\mathcal{M}, \mathbf{g})$ *to* (1.1) *with nowhere vanishing* U *is locally isometric to*

$$\left(\mathcal{I} \times \mathcal{F}, \mathbf{g} = u^{-2} ds^2 + \mathbf{h}_s\right), \tag{1.2}$$

where $\mathcal{I} \subset \mathbb{R}$ is an interval, $\mathbf{h}_s$ is a family of Riemannian metrics on some manifold $\mathcal{F}$ parametrised by $s \in \mathcal{I}$, and $u^2 = \mathbf{g}(U, U)$. Under this isometry, $U = u^2 \partial_s$ and in the decomposition $T\mathcal{M} = \mathbb{R}\partial_s \oplus T\mathcal{F}$ and with $u_s = u(s, \cdot)$ we have

$$\mathrm{W} = -\frac{1}{u_s}\mathbf{g}(\nabla U, \cdot) = \begin{pmatrix} \partial_s(\frac{1}{u_s}) & grad^{\mathbf{h}_s}(\frac{1}{u_s}) \\ d(\frac{1}{u_s}) & -\frac{u_s}{2}\mathcal{L}_{\partial_s}\mathbf{h}_s \end{pmatrix}, \tag{1.3}$$

where $\mathcal{L}_{\partial_s}$ denotes the Lie derivative in s-direction. Moreover, if the vector field $\frac{1}{u^2}U$ is complete, then the universal cover of $\mathcal{M}$ is globally isometric to a manifold of the form (1.2) with $\mathcal{I} = \mathbb{R}$ and $\mathcal{F}$ simply connected.

Conversely, given $(\mathcal{M}, \mathbf{g})$ as in (1.2) with $\mathcal{I} = \mathbb{R}$ or $\mathcal{I} = S^1$ a circle, the vector field $U = u^2\partial_s$ solves (1.1) for W *as in (1.3). If in addition $\mathcal{F}$ is compact and u bounded, then $(\mathcal{M}, \mathbf{g})$ is complete.*

This theorem gives a (local) classification of Riemannian manifolds satisfying the constraint. It also gives a method to construct solutions to the constraint equation, in particular *complete solutions*: for compact $\mathcal{F}$, if $\mathcal{I} = S^1$, the Riemannian manifold $(\mathcal{M}, \mathbf{g})$ is complete by compactness, but in Sect. 6.2, we show that this also holds when $\mathcal{I} = \mathbb{R}$ and u is bounded.

To any manifold $(\mathcal{M}, \mathbf{g})$ as in (1.2), we can apply Theorem 1. Let us from now on assume that $\mathcal{M}$ is oriented. It follows that there is a naturally induced orientation on the manifold $\overline{\mathcal{M}}$ arising via Theorem 1. As there is also a parallel null vector on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, it follows that the holonomy group $\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, i.e. the group of parallel transports along closed loops, satisfies

$$\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset \mathbf{SO}(n) \ltimes \mathbb{R}^n \subset \mathbf{SO}(1, n+1),$$

where $\mathbf{SO}(n) \ltimes \mathbb{R}^n$ is the stabiliser in $\mathbf{SO}(1, n+1)$ of a null vector. In this case, the main ingredient of $\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ then is the *screen holonomy*

$$G := \mathrm{pr}_{\mathbf{SO}(n)}\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset \mathbf{SO}(n).$$

In [30], it was shown that (the connected component of) the screen holonomy G is always a *Riemannian holonomy group*, and hence a product of the groups on Berger's lists [13,14]. It is now natural to ask whether one can prescribe G by imposing additional conditions on the initial data, i.e. on the family of metrics $\mathbf{h}_s$ on $\mathcal{F} \subset \mathcal{M}$. We show that this is indeed the case when G arises as stabiliser of some tensor:

Theorem 3 *Let $(\mathcal{M}, \mathbf{g}, \mathrm{W}, U)$ be given as in (1.2) and (1.3) and let $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ be the Lorentzian manifold arising from this choice of initial data via Theorem 1 (for arbitrary choice for λ). Then $G = \mathrm{pr}_{\mathbf{SO}(n)}\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset \mathbf{SO}(n)$ lies in the stabiliser of some tensor in $T^{k,l}\mathbb{R}^n$ if and only if there is an s-dependent and $\nabla^{\mathbf{h}_s}$-parallel family of tensor fields η_s on $\mathcal{F}$, of the same type and subject to the flow equation*

$$\dot{\eta}_s = -\frac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \eta_s. \tag{1.4}$$

Here, the dot denotes the Lie derivative of a tensor with respect to ∂_s, e.g., $\dot{\eta}_s := \mathcal{L}_{\partial_s}\eta_s$, $\dot{\mathbf{h}}^\sharp\bullet$ denotes the natural action of the endomorphism $\dot{\mathbf{h}}^\sharp \in End(T\mathcal{F})$ on tensors in $T^{k,l}\mathcal{F}$, and $\sharp$ indicates the dualisation with respect to $\mathbf{h}_s$. Moreover:

(1) *There are proper subgroups H_1 and H_2 of* $\mathbf{SO}(n)$ *such that $G \subset H_1 \times H_2$ if and only if there is a local metric splitting*

$$(\mathcal{F}, \mathbf{h}_s) \cong (\mathcal{F}_1 \times \mathcal{F}_2, \mathbf{h}_s^1 + \mathbf{h}_s^2) \tag{1.5}$$

with $\mathrm{Hol}(\mathcal{F}_i, \mathbf{h}_s^i) \subset H_i$.

(2) *If G is contained in one of* $\mathbf{SU}(m)$, $\mathbf{Sp}(k)$, $\mathbf{G}_2$, $\mathbf{Spin}(7)$ *or trivial, this translates into the conditions for Riemannian special holonomy metrics from Table 1.*

In Table 1, we write $\mathbf{h}_s = \mathbf{h}_s(\phi_s)$ and $\mathbf{h}_s = \mathbf{h}_s(\psi_s)$ to indicate that for families of $\mathbf{G}_2$ and $\mathbf{Spin}(7)$ structures, the metric $\mathbf{h}_s$ is defined algebraically in terms of a distinguished stable 3-form ϕ_s or a generic 4-form ψ_s, respectively. The explicit formulas can be found for example in [16,26,27]. In particular, Theorems 1 and 3 provide a construction principle for Lorentzian manifolds with reduced screen holonomy. Obviously, warped products $(\mathcal{I} \times \mathcal{F}, \mathbf{g} = ds^2 + f(s)\mathbf{h}_0)$ with $(\mathcal{F}, \mathbf{h}_0)$ being a Ricci-flat special holonomy manifold, i.e. $\mathrm{Hol}(\mathcal{F}, h_0) \in \{\mathbf{SU}(m), \mathbf{Sp}(k), \mathbf{G}_2, \mathbf{Spin}(7)\}$ or trivial, are obvious examples for the construction in Theorem 3.

In the final part of this article, we turn to applications of these results and constructions. As a first application of Theorems 1 and 3, we address a classification problem in *Riemannian* spin geometry. In doing so, we have to change our point of view slightly: so far, the object of interest was the Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ constructed from initial data $(\mathcal{M}, \mathbf{g}, \mathrm{W})$ via Theorem 1 or 3. In Sect. 8, however, $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ is regarded as an auxiliary object and we show how the detailed study of $\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ from the previous statements can in turn be used to prove a partial classification of Riemannian

Table 1 Equivalent characterisation of special screen holonomy for $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ in terms of flow equations for tensors on $\mathcal{F}$

$\dim(\mathcal{F})$	Condition on $\mathcal{F}$	$\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset$
$2m$	$(\mathcal{F}, \omega_s, J_s, \mathbf{h}_s = \omega_s(J_s\cdot, \cdot))$ Ricci-flat Kaehler, $\dot{J}_s = -\frac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet J_s,\ \delta^{\mathbf{h}_s}(\dot{\mathbf{h}}_s) = 0$	$\mathbf{SU}(m) \ltimes \mathbb{R}^{2m}$
$4k$	$(\mathcal{F}, \omega_s^i, J_s^i, \mathbf{h}_s = \omega_s^i(J_s^i\cdot, \cdot))_{i=1,2,3}$ hyper-Kaehler, $\dot{J}_s^i = -\frac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet J_s^i$	$\mathbf{Sp}(k) \ltimes \mathbb{R}^{4k}$
7	$(\mathcal{F}, \phi_s \in \Omega^3(\mathcal{F})), \mathbf{h}_s = \mathbf{h}_s(\phi_s)$ $\mathbf{G}_2$ metrics, $\dot{\phi}_s = -\frac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \phi_s$	$\mathbf{G}_2 \ltimes \mathbb{R}^7$
8	$(\mathcal{F}, \psi_s \in \Omega^4(\mathcal{F}), \mathbf{h}_s = \mathbf{h}_s(\psi_s))$ $\mathbf{Spin}(7)$ metrics, $\dot{\psi}_s = -\frac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \psi_s$	$\mathbf{Spin}(7) \ltimes \mathbb{R}^8$
n	$\mathbf{h}_s$ flat	$\mathbb{R}^n$

manifolds $(\mathcal{M}, \mathbf{g})$ admitting *imaginary* W-*Killing spinors*. By definition, these are sections φ of the complex spinor bundle $\mathbb{S}^{\mathbf{g}} \to \mathcal{M}$ of $(\mathcal{M}, \mathbf{g})$ satisfying

$$\nabla^{\mathbb{S}^{\mathbf{g}}}_X \varphi = \frac{\mathrm{i}}{2} \, \mathrm{W}(X) \cdot \varphi, \tag{1.6}$$

for some $\mathbf{g}$-symmetric endomorphism W. Here, $\cdot$ denotes Clifford multiplication. Clearly, condition (1.6) arises as a generalisation of the equation for imaginary Killing spinors, for which $\mathrm{W} = \frac{\mathrm{i}}{2}$ Id, see [8]. Moreover, solutions to Eq. (1.6) are the counterpart to real generalised Killing spinors which have been in the focus of recent research, for example in [1,2]. Given a solution to (1.6), we denote by $U_\varphi \in \mathfrak{X}(\mathcal{M})$ the Dirac current of φ, given by

$$\mathbf{g}(U_\varphi, X) = -i \, (X \cdot \varphi, \varphi), \quad \text{for all } X \in T\mathcal{M}, \tag{1.7}$$

and assume that φ solves the algebraic constraint

$$U_\varphi \cdot \varphi = \mathrm{i} \, u_\varphi \, \varphi, \tag{1.8}$$

where $u_\varphi = \sqrt{\mathbf{g}(U_\varphi, U_\varphi)} = \|\varphi\|^2$. This constraint is known to hold for imaginary Killing spinors, i.e. $\mathrm{W} = \lambda$ Id, and it can also be motivated from the perspective of Lorentzian manifolds with parallel null spinors, cf. [10]. We obtain the following classification result which generalises results from [6,7], see also [8], where it is shown that in the complete case and for $\mathrm{W} = f$ Id, $(\mathcal{M}, \mathbf{g})$ is necessarily isometric to a warped product.

Theorem 4 *Let* $(\mathcal{M}, \mathbf{g})$ *be a Riemannian spin manifold admitting an imaginary* W-*Killing spinor* φ *satisfying the algebraic equation* (1.8)*. Then:*

(1) $(\mathcal{M}, \mathbf{g})$ *is locally isometric to*

$$(\mathcal{M}, \mathbf{g}) = \left(\mathcal{I} \times \mathcal{F}_1 \times \cdots \times \mathcal{F}_k, \mathbf{g} = \frac{1}{u^2} ds^2 + \mathbf{h}^1_s + \cdots + \mathbf{h}^k_s \right) \tag{1.9}$$

for Riemannian manifolds $(\mathcal{F}_i, \mathbf{h}^i_s)$ *of dimension* n_i, $u = ||\varphi||^2$, $\mathcal{I}$ *an interval, and under this isometry* W *is given by* (1.3)*. Moreover, for each* $i = 1, \ldots, k$, *each* h^i_s *is a family of special holonomy metrics to which exactly one of the cases of Table* 1 *applies.*

(2) *If* $(\mathcal{M}, \mathbf{g})$ *is simply connected and the vector field* $\frac{1}{u_\varphi^2} U_\varphi$ *is complete, the isometry* (1.9) *is global with* $\mathcal{I} = \mathbb{R}$.

(3) *Conversely, every Riemannian manifold* $(\mathcal{M}, \mathbf{g})$ *of the form* (1.9) *with* $\mathcal{I} \in \{S^1, \mathbb{R}\}$, *where* u *is any positive function and* $(\mathcal{F}_i, \mathbf{h}^i_s)$ *are families of special holonomy metrics subject to the flow equations in Table* 1 *is spin and admits an imaginary* W-*Killing spinor* φ *for* W *given by* (1.3) *with* $u = ||\varphi||^2$. φ *solves Eq.* (1.8).

As a second application, we give a local normal form for Lorentzian metrics admitting a parallel *null* spinor fields. To this end, one uses a relation between spinor fields and vector fields on Lorentzian manifolds provided by the *Dirac current*: for any spinor field ϕ on a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, its Dirac current V_ϕ is given by

$$\overline{\mathbf{g}}(X, V_\phi) = -\langle X \cdot \phi, \phi \rangle. \tag{1.10}$$

The zeroes of V_ϕ and ϕ coincide and if ϕ is parallel then so is V_ϕ. We say that φ is *null* if V_φ is a null vector. We show:

Theorem 5 *Let* $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ *be a Lorentzian manifold admitting a parallel null spinor field. Then* $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ *is locally isometric to*

$$(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \cong (\mathbb{R} \times \mathbb{R} \times \mathcal{F}_1 \times \cdots \times \mathcal{F}_m,\ 2dvdw + \mathbf{h}^1_w + \cdots + \mathbf{h}^m_w), \tag{1.11}$$

for some integer m*, manifolds* $\mathcal{F}_i$ *for* $i = 1, \ldots, m$*, where each* $\mathbf{h}^i_w$ *is a* w*-dependent family of Riemannian metrics on* $\mathcal{F}_i$ *to which exactly one of the cases in Table 1 applies. Conversely, every manifold as in* (1.11) *satisfying these conditions admits a parallel null spinor.*

Note that the normal forms in Theorem 5 need not be the most general ones. For example, in signature (10, 1), in [17] it is shown that a term $H_w dw^2$, where H is an arbitrary function not depending on v can be added to (1.11). However, the analysis of normal forms for metrics with parallel spinor in [17] rests on the known orbit structure of the action of $\mathbf{Spin}(1, n)$ in for small n, whereas Theorem 5 covers all dimensions.

This paper is organised as follows. In Sect. 2, we recall and collect basic formulas and invariants related to the geometry of spacelike hypersurfaces in Lorentzian manifolds. Together with the local existence and uniqueness theorem for solutions to quasilinear first-order symmetric hyperbolic systems they are the key ingredients for the *proof of Theorem* 1, which occupies a large part of the paper and consists of three main steps:

(1) In Sect. 3, we derive a first-order quasilinear symmetric hyperbolic system with solutions $(\overline{\mathbf{g}}, \alpha, Z)$, where $\alpha \in \Omega^*(\overline{\mathcal{M}})$ is a differential form, $\overline{\mathbf{g}}$ is a Lorentzian metric and Z a symmetric bilinear form on $\overline{\mathcal{M}}$.
(2) As a result, we will obtain a vector field V and a 1-form E on $\overline{\mathcal{M}}$ such that $\overline{\mathrm{Ric}} = Z - \mathrm{Sym}(\overline{\nabla} E)$, where $\overline{\mathrm{Ric}}$ is the Ricci tensor and $\overline{\nabla}$ the Levi-Civita connection of $\overline{\mathbf{g}}$. In Sect. 4, we will then derive a wave equation for E and $\overline{\nabla} V$ and determine suitable initial conditions for all data along $\mathcal{M}$. Using the constraint equations (1.1) this will imply that that E and $\overline{\nabla} V$ vanish on $\overline{\mathcal{M}}$ with the conclusion that V is parallel.
(3) Since the solutions obtained in Steps (1) and (2) are only local, in Sect. 5 we will show how to obtain a globally hyperbolic Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ from these local solutions.

Then in Sect. 6, we study Riemannian manifolds satisfying the constraint (1.1) and prove Theorem 2. The proof of Theorem 3 is given in Sect. 7 where we also study the relation to Lorentzian holonomy. Using this, the two applications in Theorems 4 and 5 are obtained in Sect. 8.

2 Preliminaries

Let $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ be a time-oriented Lorentzian manifold of dimension $(n+1)$ with global unit timelike vector field $T \in \mathfrak{X}(\overline{\mathcal{M}})$. Let us now additionally assume that $\mathcal{M} \subset \overline{\mathcal{M}}$ is a spacelike hypersurface with induced Riemannian metric $\mathbf{g}$ and that T restricts to the future-directed unit normal vector field along $\mathcal{M}$. We will use the following index conventions:

- Latin indices $i, j, k, \ldots$ run from 1 to n.
- Greek indices $\mu, \nu, \rho, \ldots$ run from 0 to n. We will use Greek indices whenever we restrict ourselves to local considerations on $\overline{\mathcal{M}}$, which is topologically an open neighbourhood of $\mathcal{M}$ in $\mathbb{R} \times \mathcal{M}$. In this situation, we may fix adapted coordinates $(x^0 = t, x^1, \ldots, x^n)$, where the t-coordinate refers to the $\mathbb{R}$-factor, the Greek indices $\mu, \nu, \ldots$ then refer to the coordinates $(x^0, \ldots, x^n)$ on $\overline{\mathcal{M}}$, and Latin indices $i, j, k, \ldots$ to the spatial coordinates $(x^1, \ldots, x^n)$ on $\mathcal{M}$. We may also use this index convention when fixing a local orthonormal frame $(T = s_0, s_1, \ldots, s_n)$ with $\overline{\mathbf{g}}(s_\mu, s_\nu) = \epsilon_\mu \delta_{\mu\nu}$ and $s_i \in T\mathcal{M}$. It will be clear from the context whether the indices refer to coordinates or an orthonormal frame.
- We will use indices $a, b, c, \ldots$ as *abstract indices*, i.e. only indicating the valence of a tensor. For example, a vector field B is denoted by B^a and a 1-form by B_a. We will however abuse this abstract index notation slightly when writing 0 for a contraction $B(T, \ldots)$ of a tensor B with the timelike unit vector field T,

  $$B_{0b\ldots} := T^a B_{ab\ldots},$$

  but also when using indices $i, j, k, \ldots = 1, \ldots, n$ for referring to directions in $T\mathcal{M}$.
- We raise and lower indices with respect to a metric. Sometimes, we also use the musical notation $\flat$ and $\sharp$ for dualising a tensor with a metric. It will be clear from the context with which metric we are working. Throughout the paper, we will also use Einstein summation convention, i.e. summing over the same upper and lower index.

By $\overline{\nabla}$ we denote the Levi-Civita connection of $\overline{\mathbf{g}}$. Moreover, $\delta = \delta^{\overline{\mathbf{g}}}$ denotes the divergence operator, i.e. given a $(p, 0)$ tensor field B on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, the divergence is the $(p-1, 0)$-tensor

$$\delta B = -\sum_{\mu=0}^{n} \epsilon_\mu \left(\overline{\nabla}_{s_\mu} B\right)(s_\mu, \ldots),$$

with an orthonormal basis s_μ, or with abstract index notation

$$(\delta B)_{b\ldots c} = -\nabla_a B^a{}_{b\ldots c}.$$

For a vector field V, we have $\mathrm{div}^{\overline{\mathbf{g}}} V = -\delta V^\flat$, or in indices $\mathrm{div}^{\overline{\mathbf{g}}}(V) = \nabla_a V^a$. For $X, Y \in T\mathcal{M}$ we denote by

$$\mathrm{W}(X, Y) := -\mathbf{g}(\overline{\nabla}_X T, Y)$$

the second fundamental form of $(\mathcal{M}, \mathbf{g}) \subset (\overline{\mathcal{M}}, \overline{\mathbf{g}})$, i.e. we have

$$\overline{\nabla}_X Y = \nabla_X Y - \mathrm{W}(X, Y)T, \tag{2.1}$$

where barred objects refer to data on $\overline{\mathcal{M}}$ and unbarred objects to data on $\mathcal{M}$. The dual of the second fundamental form is the (symmetric) *Weingarten operator*, also denoted by W, i.e. $\mathrm{W}(X, Y) = \mathbf{g}(\mathrm{W}(X), Y)$. It holds that $\mathrm{W} = -\overline{\nabla} T|_{T\mathcal{M}}$. The curvature tensors of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ and $(\mathcal{M}, \mathbf{g})$ are related via the Gauß, Codazzi and Mainardi equation. Here we need the following contracted version: let

$$\overline{G} = \overline{\mathrm{Ric}} - \frac{1}{2}\overline{\mathrm{scal}} \cdot \overline{\mathbf{g}}$$

denote the divergence-free Einstein tensor of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, where $\overline{\mathrm{Ric}}$ is the Ricci tensor and $\overline{\mathrm{scal}}$ the scalar curvature of $\overline{\mathbf{g}}$. Then we have on $\mathcal{M}$:

$$\begin{aligned} \overline{G}(T, T) &= \frac{1}{2}\left(\mathrm{scal}^{\mathbf{g}} - \mathrm{tr}_{\mathbf{g}}(\mathrm{W}^2) + (\mathrm{tr}_{\mathbf{g}}\mathrm{W})^2\right), \\ \overline{G}(T, X) &= (\delta^{\mathbf{g}}\mathrm{W})(X) + d(\mathrm{tr}_{\mathbf{g}}\mathrm{W})(X), \ \text{for all } X \in T\mathcal{M}. \end{aligned} \tag{2.2}$$

Now we specialise the discussion to the case that $\overline{\mathcal{M}}$ is topologically an open subset of $\mathbb{R} \times \mathcal{M}$ for some manifold $\mathcal{M}$. We assume that ∂_t is timelike everywhere on $\overline{\mathcal{M}}$ and set

$$T = \frac{1}{-\sqrt{\overline{\mathbf{g}}(\partial_t, \partial_t)}}\partial_t$$

and $\overline{\mathbf{g}}$ restricted to $T\mathcal{M}$ is then positive definite. Note that writing $T\mathcal{M}$ in this context refers more precisely to the pullback bundle $\pi^* T\mathcal{M} \to \overline{\mathcal{M}}$, where $\pi : \overline{\mathcal{M}} \to \mathcal{M} \cong \{0\} \times \mathcal{M}$ denotes the projection.

Next, suppose that $V \in \mathfrak{X}(\overline{\mathcal{M}})$ is a *null vector field* on $\overline{\mathcal{M}}$, i.e. a nonvanishing smooth vector field V such that $\overline{\mathbf{g}}(V, V) = 0$. We decompose V with respect to the splitting $T\overline{\mathcal{M}} = \mathbb{R}\partial_t \oplus T\mathcal{M}$, which need not be $\overline{\mathbf{g}}$-orthogonal, into

$$V = vT - U = v\,(T - N), \tag{2.3}$$

with $v \in C^\infty(\overline{\mathcal{M}})$ a smooth function, $U \in \Gamma(\pi^* T\mathcal{M})$ and $N = \frac{1}{v}U$. It is $\overline{\mathbf{g}}(N, N) = \frac{u^2}{v^2}$, where $u^2 = \overline{\mathbf{g}}(U, U)$. We also write U_t and N_t in order to emphasise the t-dependence. Note that $V \neq 0$ and $\overline{\mathbf{g}}(V, V) = 0$ requires that v, u and U do not vanish at any point. We emphasise that $\overline{\mathbf{g}}(T, U)$ is not necessarily zero on $\overline{\mathcal{M}}$. However, we have that $\overline{\mathbf{g}}(T, U)_{|\{0\}\times\mathcal{M}} = 0$ as T was assumed to be the unit normal vector field along $\mathcal{M}$. We identify $\mathcal{M}$ within $\overline{\mathcal{M}}$ as $\{0\} \times \mathcal{M}$. It follows that $v|_{\mathcal{M}} = u|_{\mathcal{M}}$, i.e. along $\mathcal{M}$, N is a unit vector field.

Finally, we assume that the null vector field V is *parallel*, i.e. that $\overline{\nabla} V \equiv 0$. As a consequence of (2.1), we obtain for every $X \in T\mathcal{M}$

$$0 = \mathrm{pr}_{T\mathcal{M}}(\overline{\nabla}_X V)|_{\mathcal{M}} = -v|_{\mathcal{M}} \mathrm{W}(X) - \nabla_X U = -u|_{\mathcal{M}} \mathrm{W}(X) - \nabla_X U, \quad (2.4)$$

which is precisely the constraint equation (1.1).

For the proof of Theorem 1, we will have to analyse various PDEs. As it turns out, they can all be locally reduced to a *first-order quasilinear symmetric hyperbolic system*. We collect some standard facts on that:

Consider an equation of the form

$$A^0(t, x, w)\partial_t w = A^i(t, x, w)\partial_i w + b(t, x, w), \quad (2.5)$$

for k real functions on $\mathbb{R} \times \mathbb{R}^n$ which are collected in a vector valued function $w(t, x) \in \mathbb{R}^k$. The solution will be defined on an appropriate subset of $\mathbb{R} \times \mathbb{R}^n$ and (t, x) denotes a point in $\mathbb{R} \times \mathbb{R}^n$. Equation (2.5) is called *quasilinear symmetric hyperbolic* if the matrices A^0 and A^i, which may depend on the point (t, x) as well as on the unknown w itself, are symmetric and A^0 is positive definite. For the given smooth initial data and smooth coefficients, there is a well-established local existence and uniqueness result for smooth solutions w to (2.5) which we shall use repeatedly. For details, we refer to [37, Sect. 16] or [23] and references therein.

3 Proof of Theorem 1: The Quasilinear Symmetric Hyperbolic System

As indicated in the introduction, for clarity the proof is subdivided into various steps: first, we find evolution equations whose solutions define the metric $\overline{\mathbf{g}}$ and the vector field V locally; then we show that the constructed vector field V is indeed parallel and patch the locally defined solutions together and discuss global properties. In this section, we deal with the evolution equations.

3.1 Finding the Evolution Equations

In order to get an idea of how to obtain the desired metric $\overline{\mathbf{g}}$ and the vector field V from the data $(\mathcal{M}, \mathbf{g}, U)$, suppose for a moment that we already have a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ such that

- $\overline{\mathcal{M}}$ is an open subset of $\mathcal{M} \cong \{0\} \times \mathcal{M}$ in $\mathbb{R} \times \mathcal{M}$,
- $\widetilde{\lambda}^2 := -\overline{\mathbf{g}}(\partial_t, \partial_t) > 0$, where $\partial_t = \partial_0$ refers to the vector field corresponding to the t-coordinate. This defines a timelike unit vector field $T = \widetilde{\lambda}^{-1}\partial_t$.
- There exists a parallel null vector field $V \in \mathfrak{X}(\overline{\mathcal{M}})$. This defines a spacelike vector field N by relation (2.3), i.e. by $V = v(T - N)$. Along $\mathcal{M}$, N is a unit vector field.

We derive some evolution equations as consequences:

Let $\alpha = V^\flat$ denote the $\overline{\mathbf{g}}$-dual 1-form to V and consider α as a section in the exterior algebra, i.e. $\alpha \in \Omega^*(\overline{\mathcal{M}})$. Since V is parallel, α is parallel and we have

$$(d + \delta^{\overline{\mathbf{g}}})\alpha = 0, \tag{3.1}$$

where $d + \delta^{\overline{\mathbf{g}}} : \Omega^*(\overline{\mathcal{M}}) \to \Omega^*(\overline{\mathcal{M}})$ is the de Rham operator. It is given by

$$d + \delta = c \circ \nabla,$$

where $c : T\overline{\mathcal{M}} \otimes \Lambda^*\overline{\mathcal{M}} \to \Lambda^*\overline{\mathcal{M}}$ denotes Clifford multiplication by forms, i.e.

$$c(X)\omega = X^\flat \wedge \omega - \iota_X\omega, \text{ for all } X \in T\overline{\mathcal{M}},$$

where $\iota_X\omega = \omega(X, \ldots)$ denotes the interior product. Symbolically, this can be written as

$$c(X) = (X^\flat \wedge) - \iota_X, \text{ for all } X \in T\overline{\mathcal{M}}.$$

The de Rham operator in (3.1) is of Dirac type, which suggests that it is hyperbolic. We will explicitly verify this later.

Next, as V is parallel, it annihilates the curvature tensor $\overline{\mathrm{R}}$ of $\overline{\nabla}$, i.e. $\overline{\mathrm{R}}(V, ., ., .) = 0$. In particular,

$$\overline{\mathrm{Ric}}(V, \cdot) = 0. \tag{3.2}$$

To evaluate this further, we denote by $\mathrm{pr}_{T\mathcal{M}}$ the standard and $\overline{\mathbf{g}}$-independent projection

$$\mathrm{pr}_{T\mathcal{M}} : T\overline{\mathcal{M}} = \mathbb{R}\partial_t \oplus T\mathcal{M} \to T\mathcal{M}$$

onto the second factor. Moreover, the metric $\overline{\mathbf{g}}$ defines a (nonorthogonal) splitting

$$T\overline{\mathcal{M}} = T\mathcal{M} \oplus \mathbb{R}V \tag{3.3}$$

of bundles over $\overline{\mathcal{M}}$. We introduce the $\overline{\mathbf{g}}$- and V-dependent projection

$$\mathrm{pr}^{\overline{\mathbf{g}},V}_{T\mathcal{M}} : T\overline{\mathcal{M}} \to T\mathcal{M}$$

onto the first factor in the splitting (3.3). That this projection is dependent on $\overline{\mathbf{g}}$ and V becomes evident when it is written as

$$\mathrm{pr}^{\overline{\mathbf{g}},V}_{T\mathcal{M}} = \mathrm{Id}_{T\overline{\mathcal{M}}} + \tfrac{1}{v}\left(\overline{\mathbf{g}}(T, .) - \overline{\mathbf{g}}(T, \mathrm{pr}_{T\mathcal{M}}(.))\right) V.$$

or, written in local coordinates $(x^0 = t, x^1, \ldots, x^n)$ with $(x^1, \ldots, x^n)$ coordinates on $\mathcal{M}$,

$$\left(\mathrm{pr}^{\overline{\mathbf{g}},V}_{T\mathcal{M}}\right)_\mu^{\ \nu} = \delta_\mu^{\ \nu} + (\widetilde{\lambda}v)^{-1}\left(\overline{\mathbf{g}}_{0\mu} - \overline{\mathbf{g}}_{0i}\delta^i_{\ \mu}\right) V^\nu.$$

Then Eq. (3.2) is equivalent to

$$\overline{\mathrm{Ric}} = \mathrm{Z} \circ \mathrm{pr}^{\overline{\mathbf{g}},V}_{T\mathcal{M}}, \tag{3.4}$$

where Z is a symmetric bilinear form on $T\mathcal{M}$, i.e. $Z \in \Gamma(\pi^*(T^*\mathcal{M} \otimes T^*\mathcal{M}))$, which is trivially extended to a symmetric bilinear form on $T\overline{\mathcal{M}} = T\mathcal{M} \oplus \mathbb{R}V$. Finally, a first-order equation for Z is then derived as follows. As every expression of the form $\overline{\mathrm{R}}(V, \cdot, \cdot, \cdot, \cdot)$ vanishes identically on $\overline{\mathcal{M}}$, it follows from the second Bianchi identity that $\overline{\nabla}_V \overline{\mathrm{Ric}} = 0$, which in particular implies that

$$\overline{\nabla}_V \mathrm{Z} = 0. \tag{3.5}$$

Seeking for a different formulation of this condition, we use the splitting (2.3) of V into T and N, both depending also on t, to see that (3.5) becomes

$$0 = (\overline{\nabla}_{\partial_t} \mathrm{Z}) - \widetilde{\lambda}\, \overline{\nabla}_N \mathrm{Z}. \tag{3.6}$$

Now, for brevity we rewrite this condition in local coordinates $(x^0 = t, x^1, \dots, x^n)$ with $(x^1, \dots, x^n)$ coordinates on $\mathcal{M}$. We obtain that Eq. (3.5) is equivalent to

$$\partial_t \mathrm{Z}_{kl} = \widetilde{\lambda} N^i \partial_i \mathrm{Z}_{kl} + 2\overline{\Gamma}^i_{0(k} \mathrm{Z}_{l)i} - 2\widetilde{\lambda} N^i \overline{\Gamma}^j_{i(k} \mathrm{Z}_{l)j}, \tag{3.7}$$

in which $\widetilde{\lambda} = \sqrt{-\overline{\mathbf{g}}(\partial_t, \partial_t)} > 0$, the unit vector field N depends on V via relation (2.3), and the round brackets denote the symmetrisation of indices.

The advantage of this formulation is that (3.7) is manifestly a $\overline{\mathbf{g}}$- and $\partial\overline{\mathbf{g}}$-dependent t-evolution equation for a t-dependent family of symmetric endomorphisms $\mathrm{Z}_t \in \Gamma(\mathcal{M}, T^*\mathcal{M} \otimes T^*\mathcal{M})$ on $\mathcal{M}$.

3.2 Hyperbolic Reduction

Let $(\mathcal{M}, \mathbf{g})$ be a Riemannian manifold and let (U, W) be a nontrivial solution to (1.1). The idea is to impose Eqs. (3.1), (3.4) and (3.7) locally as a coupled PDE system of first-order evolution equations for the unknowns $w = (\alpha, \overline{\mathbf{g}}, \partial\overline{\mathbf{g}}, \mathrm{Z})$ defined on a neighbourhood of $\mathcal{M}$ in $\mathbb{R} \times \mathcal{M}$ with initial data to be specified.

More precisely, we would like to rewrite (3.1), (3.4) and (3.7) locally as a *first*-order quasilinear symmetric hyperbolic PDE of the form (2.5). A well-studied technical problem is that $\overline{\mathrm{Ric}}$ is not hyperbolic when being considered as differential operator acting on the metric. There is a standard tool used for the Cauchy problem for the Einstein equations in general relativity how to overcome this, which is referred to as *hyperbolic reduction* and explained in detail in [34]. To this end, we bring into play a fixed background metric

$$h := -\lambda^2 dt^2 + \mathbf{g}. \tag{3.8}$$

on $\mathbb{R}\times\mathcal{M}$, where λ is the prescribed function from Theorem 1. Given local coordinates $(x^0,\dots,x^n)$, we denote by $\widetilde{\Gamma}^\mu_{\alpha\beta}$ the Christoffel symbols of h. For any metric $\overline{\mathbf{g}}$ on $\mathbb{R}\times\mathcal{M}$ with Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ we then introduce the difference tensor $A^\mu_{\alpha\beta}=\Gamma^\mu_{\alpha\beta}-\widetilde{\Gamma}^\mu_{\alpha\beta}$ and let

$$\begin{aligned} F_\nu &= g_{\mu\nu}g^{\alpha\beta}\widetilde{\Gamma}^\mu_{\alpha\beta},\\ E_\nu &= -g_{\mu\nu}g^{\alpha\beta}A^\mu_{\alpha\beta}. \end{aligned} \tag{3.9}$$

We denote by $\mathrm{Sym}(\overline{\nabla}E)[\overline{\mathbf{g}}]$ the symmetrisation of the $(2,0)$-tensor $\overline{\mathbf{g}}(\overline{\nabla}E,\cdot)$ for any given Lorentzian metric $\overline{\mathbf{g}}$, i.e. $\mathrm{Sym}(\overline{\nabla}E)(X,Y)=\frac{1}{2}\big((\overline{\nabla}_X E)(Y)+(\overline{\nabla}_Y E)(X)\big)$. Then the operator

$$\widehat{\mathrm{Ric}}[\overline{\mathbf{g}}] := \mathrm{Ric}[\overline{\mathbf{g}}]+\mathrm{Sym}(\overline{\nabla}E)[\overline{\mathbf{g}}]$$

is in coordinates given by

$$\widehat{\mathrm{Ric}}_{\mu\nu} = -\frac{1}{2}\overline{\mathbf{g}}^{\alpha\beta}\partial_\alpha\partial_\beta\overline{\mathbf{g}}_{\mu\nu}+\nabla_{(\mu}F_{\nu)}+\underbrace{\overline{\mathbf{g}}^{\alpha\beta}\overline{\mathbf{g}}^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu}+\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta}+\Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}]}_{=:H_{\mu\nu}[\overline{\mathbf{g}},\partial\overline{\mathbf{g}}]}, \tag{3.10}$$

where we use the standard notation for symmetrisation $\overline{\nabla}_{(\mu}F_{\nu)}=\frac{1}{2}\left(\overline{\nabla}_\mu F_\nu+\overline{\nabla}_\nu F_\mu\right)$. The crucial point is that second-order derivatives of $\overline{\mathbf{g}}$ appear only in the first term of $\widehat{\mathrm{Ric}}[\overline{\mathbf{g}}]$ (assured by addition of E, F depends only on $\overline{\mathbf{g}}$ and not on its derivatives). Hence, in the following, we will replace Eq. (3.4) by the equation

$$\overline{\mathrm{Ric}} = Z\circ \mathrm{pr}^{\overline{\mathbf{g}},V}_{T\mathcal{M}}-\mathrm{Sym}(\overline{\nabla}E), \tag{3.11}$$

where we abbreviate the Ricci tensor of $\overline{\mathbf{g}}$ as $\overline{\mathrm{Ric}}=\mathrm{Ric}[\overline{\mathbf{g}}]$. Of course, eventually we will construct a solution and then show that $E=0$.

3.3 Local Evolution Equations as First-Order Symmetric Hyperbolic System

After these preparations, we are now able to show:

Theorem 3.1 *Under the assumptions of Theorem 1 with given data $(\mathcal{M},\mathbf{g})$, λ and U, every point $p\in\mathcal{M}$ admits an open neighbourhood $\mathcal{V}_p$ in $\mathbb{R}\times\mathcal{M}$ on which the Eqs. (3.1), (3.7) and (3.11), considered as coupled PDE for the unknowns $(\overline{\mathbf{g}},\alpha,Z)$, are locally equivalent to a first-order quasilinear symmetric hyperbolic PDE of the form (2.5) provided that*

$$\overline{\mathbf{g}}|_{\mathcal{M}}=h \quad \textit{and} \quad \alpha|_{\mathcal{M}}=(h(\tfrac{u}{\lambda}\partial_t-U,\cdot))|_{\mathcal{M}},$$

where h is the background metric defined from λ and $\mathbf{g}$ in Eq. (3.8).

Proof Note first that for each choice of the unknowns $\alpha = (\alpha_0, \ldots, \alpha_n) \in \Omega^*(\overline{\mathcal{M}})$, where $\alpha_i \in \Omega^i(\overline{\mathcal{M}})$ and $\overline{\mathbf{g}}$ we can form vector fields $V \in \mathfrak{X}(\overline{\mathcal{M}})$, $U \in \mathfrak{X}(\overline{\mathcal{M}})$ algebraically by $V = V[\alpha, \overline{\mathbf{g}}] = \alpha_1^\sharp$ and $U = U[\alpha, \overline{\mathbf{g}}] = -\mathrm{pr}_{T\mathcal{M}} V$. Moreover, we set $u^2 = u^2[\alpha, \overline{\mathbf{g}}] = \overline{\mathbf{g}}(U, U)$. Now fix $p \in \mathcal{M}$ and choose $\mathcal{V}_p$ to be a coordinate neighbourhood of $p \in \mathbb{R} \times \mathcal{M}$ with coordinates $(x^0 = t, x^1, \ldots, x^n)$. We define the following open subset in the space of Lorentzian metrics on $\mathcal{V}_p$:

$$\mathcal{G}_p := \{\overline{\mathbf{g}} \mid \overline{\mathbf{g}}(\partial_0, \partial_0) < 0,\ dt(\mathrm{grad}^{\overline{\mathbf{g}}} t) < 0,\ \overline{\mathbf{g}}_{|T\mathcal{M} \otimes T\mathcal{M}} > 0\} \tag{3.12}$$

Note that $h \in \mathcal{G}_p$. Given any metric $\overline{\mathbf{g}} \in \mathcal{G}_p$, we fix a $\overline{\mathbf{g}}$-dependent pseudo-orthonormal basis $(s_0, \ldots, s_n)$ for $\overline{\mathbf{g}}$, i.e. $\overline{\mathbf{g}}(s_a, s_b) = \epsilon_a \delta_{ab}$, by applying the Gram–Schmidt procedure to $(\partial_t, \partial_1, \ldots, \partial_n)$. That is, $s_0 = \frac{1}{\sqrt{(-\overline{\mathbf{g}}(\partial_t, \partial_t))}} \partial_t = T$ and for $i > 0$

$$s_i = s_i[\overline{\mathbf{g}}] = \sum_{\mu=0}^{n} \zeta_i^\mu [\overline{\mathbf{g}}] \partial_\mu \tag{3.13}$$

on $\mathcal{V}_p$ for certain coefficients $\zeta_i^\mu[\mathbf{g}]$ which depend smoothly and only algebraically on $\overline{\mathbf{g}}$. Note that choosing $\overline{\mathbf{g}}$ from $\mathcal{G}_p$ ensures that the Gram–Schmidt algorithm is well defined for $(\partial_t, \partial_1, \ldots, \partial_n)$. By the special form of the fixed background metric $h = -\lambda^2 dt^2 + \mathbf{g}$ we have that $\zeta_{i>0}^0[h] = 0$. For any $\overline{\mathbf{g}} \in \mathcal{G}_p$ we then rewrite Eqs. (3.1), (3.7) and (3.11) on $\mathcal{V}_p$ as follows:

Local Reformulation of Equation (3.11)

In analogy to [22], for any Lorentzian metric $\overline{\mathbf{g}} \in \mathcal{G}_p$ and quantities $k_{\mu\nu}$ and $\overline{\mathbf{g}}_{\mu\nu,i}$ we consider the system

$$\begin{aligned}
\partial_t \overline{\mathbf{g}}_{\mu\nu} &= k_{\mu\nu}, && (3.14)\\
\overline{\mathbf{g}}^{ij} \partial_t \overline{\mathbf{g}}_{\mu\nu,i} &= \overline{\mathbf{g}}^{ij} \partial_i k_{\mu\nu}, && (3.15)\\
-\overline{\mathbf{g}}^{00} \partial_t k_{\mu\nu} &= 2\overline{\mathbf{g}}^{0j} \partial_j k_{\mu\nu} + \overline{\mathbf{g}}^{ij} \partial_j \overline{\mathbf{g}}_{\mu\nu,i} - 2H_{\mu\nu}[\overline{\mathbf{g}}, k] - 2\nabla_{(\mu} F_{\nu)}[\overline{\mathbf{g}}, k] \\
&\quad + 2\,(\mathrm{Z} \circ \mathrm{pr}^{\overline{\mathbf{g}}, V}_{T\mathcal{M}})_{\mu\nu}, && (3.16)
\end{aligned}$$

with initial conditions $\overline{\mathbf{g}}|_{\mathcal{M}} = h|_{\mathcal{M}}$ and

$$\overline{\mathbf{g}}_{\mu\nu,i}|_{t=0} = \partial_i \overline{\mathbf{g}}_{\mu\nu}|_{t=0} = \partial_i h_{\mu\nu}|_{(t)=0} \tag{3.17}$$

This system with the given initial condition is equivalent to Eq. (3.11).[1] Indeed, let a triple $(\overline{\mathbf{g}}_{\mu\nu}, k_{\mu\nu}, \overline{\mathbf{g}}_{\mu\nu,i})$ solve system (3.14)–(3.17). As $\overline{\mathbf{g}}^{ij}$ is invertible for $\overline{\mathbf{g}}$ sufficiently close to h, Eq. (3.15) is the same as $\partial_t \overline{\mathbf{g}}_{\mu\nu,i} = \partial_i k_{\mu\nu}$, and Eq. (3.14) then gives

$$\partial_t (\overline{\mathbf{g}}_{\mu\nu,i} - \partial_i \overline{\mathbf{g}}_{\mu\nu}) = 0.$$

[1] This has been shown in [22] for the vacuum Einstein equations $\overline{\mathrm{Ric}} = 0$ and remains valid in our setting, as here the Z-term in (3.11) enters only algebraically in the b_1-term.

Initial condition (3.17) ensures $\overline{\mathbf{g}}_{\mu\nu,i} - \partial_i \overline{\mathbf{g}}_{\mu\nu} = 0$ at $t = 0$ and thus everywhere. Then Eq. (3.16) is nothing but Eq. (3.11). Hence, for any fixed Z, the system (3.14)–(3.16) can be rewritten as

$$A_1^0(t, x, w^1)\partial_0 w^1 = A_1^i(t, x, w^1)\partial_i w^1 + b_1(t, x, w^1, \mathrm{Z}, \alpha), \tag{3.18}$$

where $w_1 = (\overline{\mathbf{g}}_{\mu\nu}, (\overline{\mathbf{g}}_{\mu\nu,i})_{i=1,\ldots,n}, k_{\mu\nu})_{\mu,\nu=0,\ldots,n}$. Moreover, the matrices A_1^0 and A_1^i are symmetric and $A_1^0(t, x, w_1)$ is positive definite for $\overline{\mathbf{g}} = h$, and hence in a neighbourhood of h. In fact, they can be written as

$$A_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g^{00} & 0 \\ 0 & 0 & -g_i{}^j \end{pmatrix}, \qquad A_1^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2g^{0i} & g^{ij} \\ 0 & g^{ij} & 0 \end{pmatrix}.$$

Local Reformulation of Equation (3.1)

Using the orthonormal basis s_μ, we can identify $\overline{\nabla}\alpha \in T^*M \otimes \Omega^*(M)$ with

$$-s_0 \otimes \overline{\nabla}_{s_0}\alpha + \sum_{k=1}^{n} s_k \otimes \overline{\nabla}_{s_k}\alpha \;\in\; TM \otimes \Omega^*(M).$$

With this identification, Eq. (3.1) writes as

$$0 = -c(s_0)\overline{\nabla}_{s_0}\alpha + \sum_{k=1}^{n} c(s_k)\overline{\nabla}_{s_k}\alpha.$$

Using the fundamental Clifford identity

$$c(X) \circ c(Y) + c(Y) \circ c(X) = -2\overline{\mathbf{g}}(X, Y) \cdot 1$$

for the $\overline{\mathbf{g}}$-dependent operator c, Eq. (3.1) for $\overline{\mathbf{g}} \in \mathcal{G}_p$ is equivalent to

$$\begin{aligned} \tfrac{1}{\lambda}\overline{\nabla}_{\partial_t}\alpha &= \sum_{k=1}^{n} c(s_0) \circ c(s_k)\overline{\nabla}_{s_k}\alpha \\ &= \sum_{k=1}^{n} \zeta_k^0[\overline{\mathbf{g}}]\, c(s_0) \circ c(s_k)\overline{\nabla}_{\partial_t}\alpha + \sum_{i,k=1}^{n} \zeta_k^i[\overline{\mathbf{g}}]\, c(s_0) \circ c(s_k)\overline{\nabla}_{\partial_i}\alpha, \end{aligned}$$

which can be re arranged to

$$\left(\tfrac{1}{\lambda} - \sum_{k=1}^{n} \zeta_k^0[\overline{\mathbf{g}}]\, c(s_0) \circ c(s_k)\right)\overline{\nabla}_{\partial_t}\alpha = \sum_{i,k=1}^{n} \zeta_k^i[\overline{\mathbf{g}}]\, c(s_0) \circ c(s_k)\,\overline{\nabla}_{\partial_i}\alpha. \tag{3.19}$$

By means of the fixed coordinates, we identify α with a smooth map $\alpha : \mathcal{V}_p \to \Lambda^*\mathbb{R}^{n+1} \cong \mathbb{R}^{2^{n+1}}$. In this identification, $\overline{\nabla}_{\partial_\mu} = \partial_\mu + \Gamma$, for an endomorphism Γ which depends on the Christoffel symbols of $\overline{\mathbf{g}}$. Then Eq. (3.19) becomes equivalent to a system

$$A_2^0(t, x, \overline{\mathbf{g}}, \alpha)\partial_t \alpha = \sum_{i=1}^n A_2^i(t, x, \overline{\mathbf{g}}, \alpha)\partial_i \alpha + b_2(t, x, \alpha, \overline{\mathbf{g}}, \partial\overline{\mathbf{g}}). \tag{3.20}$$

We claim that the matrices A_2^0 and A_2^i are symmetric. To see this, let $(e_0, \dots, e_n)$ denote the standard basis of $\mathbb{R}^{n+1}$, and consider the operator $c(e_\mu) = (e_\mu^\flat \wedge) - \iota_{e_\mu}$, where the dual is formed using the standard Minkowski inner product on $\mathbb{R}^{n+1}$. Now let σ^μ be the (algebraically) dual basis to e_μ, i.e. with $\sigma^\mu(e_\nu) = \delta^\mu_\nu$ and with $e_\mu^\flat = \epsilon_\mu \sigma^\mu$, where $\epsilon_0 = -1, \epsilon_{i>0} = 1$. Furthermore, let $\langle \cdot, \cdot \rangle$ be the standard *positive definite* inner product on $\Lambda^* \mathbb{R}^{n+1}$, i.e. with $\sigma^\mu, \sigma^\mu \wedge \sigma^\nu, \dots, \sigma^0 \wedge \dots \wedge \sigma^n$ as orthonormal basis. Then elementary linear algebra shows that

$$\left\langle c(e_\mu)\gamma, \delta \right\rangle = -\epsilon_\mu \left\langle \gamma, c(e_\mu)\delta \right\rangle, \text{ for all } \gamma, \delta \in \Lambda^*\mathbb{R}^{n+1}.$$

It follows from the Clifford identity for c that for $i > 0$

$$\langle (c(e_0) \circ c(e_i))\gamma, \delta \rangle = \langle \gamma, (c(e_0) \circ c(e_i))\delta \rangle,$$

which proves symmetry of the linear map $c(e_0) \circ c(e_\mu)$ and hence of the matrices A_2^μ. Moreover, for $\overline{\mathbf{g}} = h$, $A_2^0(t, x, h)$ reduces to a positive multiple of the identity matrix. Thus, A_2^0 is positive definite in a neighbourhood $\mathcal{V}_p$ of the initial data if these initial data are chosen as required in the theorem.

Local Reformulation of (3.7)

Locally, the t-dependent symmetric bilinear form Z on $T\mathcal{M}$ can be rewritten as $\mathrm{Z} = \mathrm{Z}_{kl} dx^k dx^l$ for t- and x dependent coefficients Z_{kl}. One verifies immediately that (3.7) is of the form

$$A_3^0(t, x)\partial_t(\mathrm{Z}_{kl})_{k,l>0} = \sum_{i=1}^n A_3^i(t, x, \overline{\mathbf{g}}, \alpha)\partial_i(\mathrm{Z}_{kl})_{k,l>0} + b_3(t, x, \mathrm{Z}, \overline{\mathbf{g}}, \partial\overline{\mathbf{g}}, \alpha), \tag{3.21}$$

where $A_3^0(t, x) = \mathrm{Id}$ is simply the identity matrix and $A_3^i(t, x, u)$ are multiples of the identity matrix.

Combining (3.18), (3.20) and (3.21) gives a coupled PDE of the form (2.5) with matrices A^0 and A^i being block diagonal with blocks A_1^0, A_2^0 and A_3^0, and blocks A_1^i, A_2^i and A_3^i, respectively. The unknowns are $w = (w_1, w_2, w_3)$, with $w_1 = (\overline{\mathbf{g}}_{\mu\nu}, (\overline{\mathbf{g}}_{\mu\nu,i}), k_{\mu\nu})$, $w_2 = \alpha$, $w_3 = \mathrm{Z}_{kl}$, and the inhomogeneity is $b = (b_1, b_2, b_3)$, which is defined in a neighbourhood of the initial data. Moreover, the previous discussion regarding the blocks of A^0 and the A^i's shows that A^0 and A^i are symmetric and A^0 is positive definite at least in a local neighbourhood of the initial data. □

3.4 Initial Data

Here we specify a *full set of initial data* for the first-order PDE for the quantities $(\overline{\mathbf{g}}_{\mu\nu}, \overline{\mathbf{g}}_{\mu\nu,i}, k_{ij}, \alpha, Z_{kl})$ on $\mathcal{V}_p$ derived in Theorem 3.1. Initial data for $\overline{\mathbf{g}}$ and α were already given in Theorem 3.1 and are needed to ensure that the PDE is indeed hyperbolic. Moreover, as seen in the proof of Theorem 3.1, to ensure that the system (3.14)-(3.16) is equivalent to (3.11), we were forced to set

$$\overline{\mathbf{g}}_{\mu\nu,i\,|t=0} = \partial_i \overline{\mathbf{g}}_{\mu\nu\,|t=0}$$

as initial condition $\overline{\mathbf{g}}_{\mu\nu,i}$.

Regarding $k_{\mu\nu}$, we observe that $\frac{1}{\lambda}\partial_t$ is the unit normal vector field with respect to h along $\mathcal{M}$ and set

$$k_{ij\,|t=0} = -2\lambda|_{\mathcal{M}} \mathrm{W}(\partial_i, \partial_j). \tag{3.22}$$

This is required, of course, by the fact that $(\mathcal{M}, \mathbf{g})$ should eventually embed into the solution $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ with Weingarten tensor being the given W. The initial data for k_{i0} and k_{00} are uniquely determined by the natural requirement

$$(E_\mu)_{|t=0}=0$$

for any solution $\overline{\mathbf{g}}$. It is by definition of E straightforward to compute, see [34], that this is the case if and only if

$$\begin{aligned} k_{00|t=0} &= -2\lambda|^2_{\mathcal{M}}\, F_{0|t=0} + 2\lambda|^3_{\mathcal{M}}\, \mathrm{tr}_{\mathbf{g}}\mathrm{W}, \\ k_{0i\,|t=0} &= \lambda|^2_{\mathcal{M}} \left[-F_i + \frac{1}{2}\mathbf{g}^{jk}(2\partial_j \mathbf{g}_{ki} - \partial_i \mathbf{g}_{jk}) + \partial_i(\log\lambda|_{\mathcal{M}}) \right]_{|t=0}. \end{aligned} \tag{3.23}$$

Note that it makes sense here to write $F_{|t=0}$, as by Eq. (3.9) the F-dependence on $\overline{\mathbf{g}}$ is only algebraic and $\overline{\mathbf{g}}_{|t=0}$ has already been specified. Moreover, for the background metric h as in (3.8) and initial conditions for $\overline{\mathbf{g}}$ as in Theorem 3.1, the initial conditions (3.23), simplify to

$$\begin{aligned} k_{00\,|t=0} &= -2\lambda|^2_{\mathcal{M}}\, \dot{\lambda}|_{\mathcal{M}} + 2\lambda|^3_{\mathcal{M}}\, \mathrm{tr}_{\mathbf{g}}\mathrm{W}, \\ k_{0i\,|t=0} &= 0. \end{aligned} \tag{3.24}$$

This makes evident that the initial conditions (3.23) are independent of the chosen coordinates.

Next, we give initial data for the symmetric bilinear form $\mathrm{Z}|_{\mathcal{M}}$ on $\mathcal{M}$. Their origin is not very transparent at this point, but we shall see in a later step of the proof that the following initial data for Z are demanded by requiring that $\overline{\nabla}V = 0$. We set

$$\begin{aligned} \mathrm{Z}|_{\mathcal{M}}(U, \cdot) &= u(d(\mathrm{tr}_{\mathbf{g}}\mathrm{W}) + \delta^{\mathbf{g}}\mathrm{W}), \\ \mathrm{Z}|_{\mathcal{M}}(X, Y) &= \mathrm{Ric}(X, Y) - \mathrm{R}(X, N, N, Y) - \mathrm{W}^2(X, Y) + \mathrm{W}(X, Y)\mathrm{tr}_{\mathbf{g}}\mathrm{W} \\ &\quad + \mathrm{W}(X, N)\mathrm{W}(Y, N) - \mathrm{W}(X, Y)\mathrm{W}(N, N), \end{aligned} \tag{3.25}$$

for all $X, Y \in U^{\perp}$ and where as usual $N = \frac{1}{u}U$.

3.5 Solving the Evolution Equation

Combining the choice of initial data with Theorem 3.1 we find, using the existence and uniqueness result for symmetric hyperbolic systems as discussed earlier, a neighbourhood $\mathcal{U}_p \subset \mathcal{V}_p$ of p in $\mathbb{R} \times \mathcal{M}$ such that the system (3.1), (3.7) and (3.11) has a unique smooth solution on $\mathcal{U}_p$ which coincides on $\mathcal{M} \cap \mathcal{U}_p$ with the initial data.

Given this solution $\big(\overline{\mathbf{g}}_{\mu\nu}, \ \overline{\mathbf{g}}_{\mu\nu,i}, \ k_{\mu\nu}, \ \alpha, \ \mathrm{Z}_{kl}\big)$, we define with the coordinates x^{μ} on $\mathcal{V}_p$ specified earlier the bilinear form $\overline{\mathbf{g}} = \overline{\mathbf{g}}^{\mathcal{U}_p} = \overline{\mathbf{g}}_{\mu\nu}dx^{\mu}dx^{\nu}$ on $\mathcal{U}_p$. Furthermore, after restricting $\mathcal{U}_p$ if necessary we may assume that $\overline{\mathbf{g}}$ is of Lorentzian signature on $\mathcal{U}_p$ and an element of $\mathcal{G}_p$ as this holds for the initial datum h. Moreover $\mathrm{Z} = \mathrm{Z}^{\mathcal{U}_p} = \mathrm{Z}_{kl}dx^k dx^l$ defines a symmetric bilinear form on $\mathcal{U}_p$ and the solution gives $\alpha = \alpha^{\mathcal{U}_p} \in \Omega^*(\mathcal{U}_p)$.

For reasons related to global hyperbolicity, which become clear in the last step of the proof, we restrict the solution domain $\mathcal{U}_p$ further as follows. Let

$$F^{\mathcal{U}_p} := \frac{1}{dt(\mathrm{grad}^{\overline{\mathbf{g}}}(t))}\mathrm{grad}^{\overline{\mathbf{g}}}(t) \in \mathfrak{X}(\mathcal{U}_p),$$

where t denotes the function $(t, x) \mapsto t$, and denote by $\phi^{\mathcal{U}_p}$ the flow of F. We restrict $\mathcal{U}_p$ to an open neighbourhood of p in $\mathbb{R} \times \mathcal{M}$, denoted with the same symbol, such that

$$\forall q \in \mathcal{U}_p : \exists \tau = \tau(q) \in \mathbb{R} : \ \phi^{\mathcal{U}_p}_{-\tau}(q) \in (\{0\} \times \mathcal{M}) \cap \mathcal{U}_p. \tag{3.26}$$

It is possible to restrict $\mathcal{U}_p$ further (denoted by the same symbol) such that the spacelike hypersurface $\mathcal{M}_p := \mathcal{M} \cap \mathcal{U}_p$ is a Cauchy hypersurface in $(\mathcal{U}_p, g)$, for details see [3, Chapter A.5]. By construction of the initial data (3.22) and as $k_{\mu\nu} = \partial_t \overline{\mathbf{g}}_{\mu\nu}$, $(\mathcal{M}_p, \mathbf{g})$ embeds into $\mathcal{U}_p$ with Weingarten tensor (the restriction of) W.

4 Proof of Theorem 1: The Wave Equation

In this section, we continue the proof of Theorem 1 by deriving a linear wave equation on E and ∇V, the solutions obtained in the previous section, as well as appropriate initial conditions that will ensure that $E = 0$ and $\nabla V = 0$.

4.1 Fundamental Properties of the Solution

Let $(\overline{\mathbf{g}}, \alpha, \mathrm{Z})$ denote the local solution to the system (3.1), (3.7) and (3.11). A priori, it is not clear that α is a 1-form and defines via $\overline{\mathbf{g}}$ a vector field. However, if we consider the Hodge-Laplacian $\Delta^{HL} = (d+\delta)^2$ on forms and decompose the solution α as $\alpha = \alpha_0 + \cdots + \alpha_{n+1} \in \Omega^0(\mathcal{U}_p) \oplus \cdots \oplus \Omega^{n+1}(\mathcal{U}_p)$, we get as a trivial consequence of $(d+\delta)\alpha = 0$ that

$$\Delta^{HL}\alpha_i = 0, \text{ for all } i = 0, \ldots, n+1. \tag{4.1}$$

Moreover, our choice of initial data and $(d+\delta)\alpha = 0$ guarantees that for $i \neq 1$

$$\begin{aligned} (\alpha_i)_{|\mathcal{M}_p} &= 0, \\ (\overline{\nabla}_{\partial_t}\alpha_i)_{|\mathcal{M}_p} &= 0, \end{aligned} \tag{4.2}$$

where $\mathcal{M}_p = \mathcal{U}_p \cap \mathcal{M}$ as before. By the main result of [3], the Cauchy problem for the normally hyperbolic operator Δ^{HL} is well-posed and as $\mathcal{M}_p \subset \mathcal{U}_p$ is a Cauchy hypersurface, we conclude that $\alpha_i = 0$ for all $i \neq 1$. Thus, the solution α is equivalently encoded in the vector field V such that

$$V^\flat = \alpha_1 = \alpha \in \mathfrak{X}(\mathcal{U}_p). \tag{4.3}$$

We decompose $V = v(T - N)$ as in the splitting (2.3) and may assume, after further restricting $\mathcal{U}_p$ if necessary, that the projections of V onto both summands of $T\overline{\mathcal{M}} = \mathbb{R}\partial_t \oplus \mathcal{M}$ are nontrivial as this holds for the initial data.

Next we extend the symmetric bilinear form $\mathrm{Z} \in \Gamma(\mathcal{U}_p, T^*\mathcal{M}_p \otimes T^*\mathcal{M}_p)$ uniquely to a section $\mathrm{Z} \in \Gamma(\mathcal{U}_p, T^*\mathcal{U}_p \otimes T^*\mathcal{U}_p)$, by demanding that V inserts trivially into Z. For this extended Z, the evolution equation (3.7) which was used to define Z then becomes equivalent to (3.5) as follows from combining (3.6) and (3.7). In summary, we have constructed $(\overline{\mathbf{g}}, V, \mathrm{Z})$ on $\mathcal{U}_p$ which satisfy the equations

$$\overline{\mathrm{Ric}} = Z - \mathrm{Sym}(\overline{\nabla}E), \tag{4.4}$$

$$(d + \delta^{\overline{\mathbf{g}}})V^\flat = 0, \tag{4.5}$$

$$(\overline{\nabla}_V Z)(A, B) = 0 \text{ for all } A, B \in T\mathcal{M}, , \tag{4.6}$$

$$\mathrm{Z}(V, \cdot) = 0. \tag{4.7}$$

On $\mathcal{U}_p$ we fix from now on a local $\overline{\mathbf{g}}$-pseudo-ONB $s = (s_0, \ldots, s_n)$ as constructed in (3.13). That means, $T = s_0 = \frac{1}{\sqrt{-\overline{\mathbf{g}}(\partial_t, \partial_t)}}\partial_t$ is a unit timelike vector field on $\mathcal{U}_p$ which restricts on $\mathcal{M}_p$ to $\frac{1}{\lambda|_{\mathcal{M}}}\partial_t$, the unit normal vector field to $\mathcal{M}_p$ with respect to $\mathbf{g}$. Moreover, as $h(\partial_t, X) = 0$ for $X \in T\mathcal{M}$, it follows that the $(s_1, \ldots, s_n)$ restricted to $\mathcal{M}$ are tangent to $\mathcal{M}$ and form a pointwise ONB for $(T\mathcal{M}_p, \mathbf{g})$.

In the subsequent calculations, we simplify and abbreviate our notation for some—otherwise very lengthy—formulas as follows: writing

$$A \equiv B \mod (\ldots),$$

where A, B are tensor fields of the same type over $\overline{\mathcal{M}}$ indicates that $A = B$ up to the addition of terms which are *linear* in the quantities specified in the bracket (or contractions of these quantities). The explicit formulas for these linear terms are straightforward to compute in each case but turn out to be irrelevant for our purposes. By $\overline{\nabla}$ we also denote the covariant derivative on tensor fields induced by the Levi-Civita connection of $\overline{\mathbf{g}}$. It follows from linearity and the product rule for $\overline{\nabla}$ that

$$A \equiv 0 \mod (C) \quad \text{implies} \quad \overline{\nabla} A = 0 \mod (C, \overline{\nabla} C).$$

As an example, Eqs. (4.6) and (4.7) imply that

$$\overline{\nabla}_V \mathrm{Z} \equiv 0 \mod (\overline{\nabla} V). \tag{4.8}$$

Indeed, the nonvanishing terms of $(\overline{\nabla}_V \mathrm{Z})$ are $(\overline{\nabla}_V \mathrm{Z})(T, X)$ for X a vector field on $\mathcal{U}_p$ which is tangent to $\mathcal{M}$ and $(\overline{\nabla}_V \mathrm{Z})(T, T)$. Both can be expressed in terms of $\overline{\nabla} V$ using $V = \frac{1}{u}(T - N)$ and Eqs. (4.6) and (4.7):

$$\begin{aligned}
(\overline{\nabla}_V \mathrm{Z})(T, X) &= (\overline{\nabla}_V \mathrm{Z})(N, X) - \tfrac{1}{u}\mathrm{Z}(\overline{\nabla}_V V, X) \;=\; -\tfrac{1}{u}\mathrm{Z}(\overline{\nabla}_V V, X),\\
(\overline{\nabla}_V \mathrm{Z})(T, T) &= (\overline{\nabla}_V \mathrm{Z})(N, N) - \tfrac{2}{u}\mathrm{Z}(\overline{\nabla}_V V, N) \;=\; -\tfrac{2}{u}\mathrm{Z}(\overline{\nabla}_V V, N).
\end{aligned}$$

4.2 PDEs for $\overline{\nabla} V$ and E

In the terminology of the previous subsection, we next show that the data $\overline{\nabla} V$ and E vanish on $\mathcal{U} = \mathcal{U}_p$ by showing that they solve a linear PDE for which uniqueness of solutions is guaranteed. All calculations and operators are with respect to the metric $\overline{\mathbf{g}} = \overline{\mathbf{g}}^{\mathcal{U}_p}$ on $\mathcal{U}_p$ as just specified.

We denote with $\Delta = \overline{\nabla}^2$ the Bochner Laplacian (or connection Laplacian) for $\overline{\mathbf{g}}$ acting on tensors, as $\Delta B_{b\ldots c} = \overline{\nabla}^a \overline{\nabla}_a B_{b\ldots c}$, in particular on 1-forms or vector fields. When acting on 1-forms, it is related to the Hodge Laplacian Δ^{HL} on 1-forms via the Weitzenböck formula

$$\Delta^{HL} = \Delta^{\overline{\nabla}} + \overline{\mathrm{Ric}}, \tag{4.9}$$

where depending on the situation we consider $\overline{\mathrm{Ric}}$ as $(2, 0)$ or $(1, 1)$ tensor.

Now we aim for a second-order equation for $\overline{\nabla} V$. For this we will prove a series of Lemmas. *The general assumption in these lemmas is that the system of equations* (4.4)-(4.7) *is satisfied*. For brevity in the proofs, we will now use indices $a, b, c, \ldots$ as *abstract indices*, i.e. only indicating the valence of a tensor. The Bochner Laplacian applied to a vector field X is denoted by $\Delta X^a = \overline{\nabla}^b \overline{\nabla}_b X^a$, where we use Einstein's summation convention. The identity (4.9), for example, reads as

$$\Delta^{HL} X_a = \overline{\nabla}^b \overline{\nabla}_b X_a + \overline{\mathrm{Ric}}_a{}^b X_b.$$

We will also use expressions such as $\overline{\nabla}E(V)$ or $\overline{\nabla\nabla}_V E$. These are meant to be as V inserted into the tensor $\overline{\nabla}E \in \otimes^2 T^*\mathcal{U}$ and into $\overline{\nabla}\,\overline{\nabla}E \in \otimes^3 T^*\mathcal{U}$ in the respective slot, e.g., $\overline{\nabla}E(V) \in \otimes T^*\mathcal{U}$ and $\overline{\nabla}\,\overline{\nabla}_V E \in \otimes^2 T^*\mathcal{U}$. Expressed with indices, this would be $V^a\overline{\nabla}_b E_a$ and $V^a\overline{\nabla}_c\overline{\nabla}_b E_a$, respectively. We also use Δ as acting on arbitrary tensors.

Lemma 4.1 *The tensor $\overline{\nabla}V \in T^*\mathcal{U} \otimes T\mathcal{U}$ satisfies*

$$\Delta\overline{\nabla}V \equiv 0 \mod (\overline{\nabla}V, (\overline{\nabla\nabla}E)(V), \overline{\nabla}\,\overline{\nabla}_V E, \overline{\nabla}_V\overline{\nabla}E). \tag{4.10}$$

Proof Using abstract index notation and successively interchanging covariant derivatives using the curvature tensor we obtain

$$\begin{aligned}
\Delta\overline{\nabla}_a V^b &= \overline{\nabla}^c\overline{\nabla}_a\overline{\nabla}_c V^b + V^d\overline{\nabla}^c\overline{\mathrm{R}}_{ca\ d}^{\ \ b} + \overline{\mathrm{R}}_{ca\ d}^{\ \ b}\overline{\nabla}^c V^d\\
&= \overline{\nabla}_a\Delta V^b - \overline{\mathrm{Ric}}_a^{\ c}\overline{\nabla}_c V^b + \overline{\mathrm{R}}^{d\ b}_{\ a\ c}\overline{\nabla}_d V^c + V^c\overline{\nabla}^d\overline{\mathrm{R}}_{da\ c}^{\ \ b} + \overline{\mathrm{R}}_{ca\ d}^{\ \ b}\overline{\nabla}^c V^d\\
&\equiv \overline{\nabla}_a\Delta V^b + V^c\overline{\nabla}^d\overline{\mathrm{R}}_{da\ c}^{\ \ b} \quad \mod \overline{\nabla}_c V^d.
\end{aligned}$$

To deal with the first remaining term we use Eqs. (4.4), (4.7) and (4.5) and its consequence $0 = \Delta^{HL}V^b = \Delta V^b + \overline{\mathrm{Ric}}^b_{\ c}V^c$:

$$\begin{aligned}
\overline{\nabla}_a\Delta V^b &= -\overline{\nabla}_a\mathrm{Ric}^b_{\ c}V^c\\
&= -\overline{\nabla}_a\mathrm{Z}^b_{\ c}V^c + \tfrac{1}{2}V^c(\overline{\nabla}_a\overline{\nabla}^b E_c + \overline{\nabla}_a\overline{\nabla}_c E^b)\\
&\equiv 0 \quad \mod (\overline{\nabla}_c V^d, V^e\overline{\nabla}_c\overline{\nabla}^d E_e, V^e\overline{\nabla}_c\overline{\nabla}_e E^d).
\end{aligned}$$

Finally, we use the symmetries of R_{abcd} to deal with the term $\overline{\nabla}^d\overline{\mathrm{R}}_{da\ c}^{\ \ b}V^c$:

$$\begin{aligned}
&V^c\overline{\nabla}^d\overline{\mathrm{R}}_{da\ c}^{\ \ b} = V^c\overline{\nabla}^d\overline{\mathrm{R}}^b_{\ cda}\\
&= -V^c\left(\overline{\nabla}^b\overline{\mathrm{R}}_{c\ da}^{\ d} + \overline{\nabla}_c\overline{\mathrm{R}}^{db}_{\ \ da}\right)\\
&= +V^c\left(\overline{\nabla}^b\overline{\mathrm{Ric}}_{ca} - \overline{\nabla}_c\overline{\mathrm{Ric}}_a^{\ b}\right)\\
&= V^c\left(\overline{\nabla}^b\mathrm{Z}_{ca} - \overline{\nabla}_c\mathrm{Z}_a^b\right) + \frac{1}{2}V^c\left(\overline{\nabla}^b\overline{\nabla}_c E_a + \overline{\nabla}^b\overline{\nabla}_a E_c - \overline{\nabla}_c\overline{\nabla}_a E^b - \overline{\nabla}_c\overline{\nabla}^b E_a\right)\\
&\equiv 0 \quad \mod (\overline{\nabla}_c V^d, V^e\overline{\nabla}_c\overline{\nabla}^d E_e, V^e\overline{\nabla}_c\overline{\nabla}_e E^d, V^e\overline{\nabla}_e\overline{\nabla}_c E^d),
\end{aligned}$$

because of Eqs. (4.4), (4.6), (4.7), and (4.8). This verifies the lemma. □

The idea is now to prolong Eq. (4.10), i.e. to derive linear equations for the E-dependent quantities in the brackets, which should all vanish, until we obtain a closed linear PDE system. We start with deriving a second-order equation for E. To this end, we introduce the "Einstein tensor" of Z, i.e.

$$L := \mathrm{Z} - \frac{1}{2}\mathrm{tr}_{\overline{\mathbf{g}}}(\mathrm{Z})\,\overline{\mathbf{g}}.$$

Let $\overline{G} = \overline{\mathrm{Ric}} - \frac{\overline{\mathrm{scal}}}{2}\overline{\mathbf{g}}$ denote the Einstein tensor of $(\mathcal{U}_p, \overline{\mathbf{g}})$. By Eq. (4.4) we get for $X, Y \in T\mathcal{U}_p$

$$\overline{G} = -\mathrm{Sym}(\overline{\nabla} E) + \mathrm{Z} + \tfrac{1}{2}(\mathrm{tr}_{\overline{\mathbf{g}}}(\overline{\nabla} E) - \mathrm{tr}_{\overline{\mathbf{g}}} Z) \cdot \overline{\mathbf{g}} \tag{4.11}$$

This equation implies

Lemma 4.2 *The* 1*-form* E *satisfies*

$$0 = \Delta E - \overline{\mathrm{Ric}}(E^{\sharp}, .) - 2\delta^{\overline{\mathbf{g}}} L. \tag{4.12}$$

Proof Taking the divergence $\delta^{\overline{\mathbf{g}}}$ on both sides of (4.11) yields

$$\begin{aligned} 0 &= \delta L_b - \tfrac{1}{2}\overline{\nabla}^a \overline{\nabla}_a E_b - \tfrac{1}{2}\overline{\nabla}^a \overline{\nabla}_b E_a + \tfrac{1}{2}\overline{\nabla}_b \overline{\nabla}^a E_a \\ &= \delta L_b - \tfrac{1}{2}\Delta E_b + \tfrac{1}{2}\, \overline{\mathrm{R}}_{b\ a}^{\ a\ c} E_c \\ &= \delta L_b - \tfrac{1}{2}\Delta E_b + \tfrac{1}{2}\, \overline{\mathrm{Ric}}_b^{\ c} E_c, \end{aligned}$$

which proves the statement. □

Next, we investigate the quantity $\overline{\nabla}_V E$ and prove

Lemma 4.3 *The* 1*-form* $\overline{\nabla}_V E$ *satisfies*

$$\Delta(\overline{\nabla}_V E) = \overline{\nabla}_V(\delta^{\overline{\mathbf{g}}} L) \mod (E, \overline{\nabla} E, \overline{\nabla} V). \tag{4.13}$$

Proof Again we commute covariant derivatives using abstract indices

$$\begin{aligned} \overline{\nabla}^c \overline{\nabla}_c (V^d \overline{\nabla}_d E^a) &\equiv V^d \overline{\nabla}^c \overline{\nabla}_c \overline{\nabla}_d E^a \mod (\overline{\nabla}_c V^d, \overline{\nabla}_c E^d) \\ &\equiv V^d \overline{\nabla}_d \Delta E^a \mod (\overline{\nabla}_c V^d, E^d, \overline{\nabla}_c E^d) \\ &\equiv V^d \overline{\nabla}_d (\delta L)^a \mod (\overline{\nabla}_c V^d, E^d, \overline{\nabla}_c E^d) \end{aligned}$$

by Eq. (4.12). □

Next, we find a second-order equation for $\overline{\mathbf{g}}(\overline{\nabla} E, V)$, i.e. for $V^b \overline{\nabla}_a E_b$.

Lemma 4.4 *The* 1*-form* $(\overline{\nabla} E)(V)$ *satisfies*

$$\Delta(\overline{\nabla} E(V)) = 0 \mod (\overline{\nabla} V, E, \overline{\nabla} E). \tag{4.14}$$

Proof Similarly as before, we have

$$\begin{aligned} \Delta(V^b \overline{\nabla}_a E_b) &\equiv V^b \overline{\nabla}^c \overline{\nabla}_c \overline{\nabla}_a E_b \mod (\overline{\nabla}_c V^d, \overline{\nabla}_c E^d) \\ &\equiv V^b \overline{\nabla}_a \Delta E_b \mod (\overline{\nabla}_c V^d, \overline{\nabla}_c E^d) \\ &\equiv V^b \overline{\nabla}_a (\delta L)_b \mod (\overline{\nabla}_c V^d, E^d, \overline{\nabla}_c E^d). \end{aligned}$$

Calculating $\mod (\overline{\nabla} V)$ we get

$$V^b (\delta L)_b = V^b \overline{\nabla}^c \mathrm{Z}_{cb} - \tfrac{1}{2} V^b \overline{\nabla}_b (\mathrm{tr}(\mathrm{Z})) \equiv 0 \mod (\overline{\nabla} V),$$

using the definition of L and $V^b Z_{ba} = 0$. □

Finally, we derive an equation for the 1-form $\delta^{\overline{\mathbf{g}}} L$.

Lemma 4.5 *The* 1*-form* $\overline{\nabla}_V \delta L$ *satisfies*

$$\overline{\nabla}_V(\delta L) \equiv 0 \mod (\overline{\nabla} V, \overline{\nabla}\,\overline{\nabla} V, \overline{\nabla} E). \tag{4.15}$$

Proof Using the definition of L and of $\delta^{\overline{\mathbf{g}}}$ we compute

$$\begin{aligned} V^b \overline{\nabla}_b(\delta L)_a &= V^b \left(\overline{\nabla}_b \overline{\nabla}^c Z_{ca} - \tfrac{1}{2} \overline{\nabla}_b \overline{\nabla}_a (Z_c{}^c) \right) \\ &\equiv V^b \overline{\nabla}_b \overline{\nabla}^c Z_{ca} \mod (\overline{\nabla} V) \\ &\equiv V^b \overline{\nabla}^c \overline{\nabla}_b Z_{ca} + V^b \overline{\mathrm{R}}_{b\ c}^{\ c\ d} Z_{da} + V^b \overline{\mathrm{R}}_{b\ a}^{\ c\ d} Z_{cd} \mod (\overline{\nabla} V) \\ &\equiv V^b \overline{\mathrm{Ric}}_b{}^d Z_{da} + V^b \overline{\mathrm{R}}_{b\ a}^{\ c\ d} Z_{cd} \mod (\overline{\nabla} V) \\ &\equiv V^b\, \overline{\mathrm{R}}_{a\ b}^{\ d\ c} Z_{cd} \mod (\overline{\nabla} V, \overline{\nabla} E), \end{aligned}$$

because of Eq. (4.4). The term $V^b\, \overline{\mathrm{R}}_{a\ b}^{\ d\ c}$, however, is a linear expression in the second and first covariant derivatives of V, and hence the claim follows. □

4.3 Reformulation of the PDEs in Terms of Differential Operators

Now we want to use the PDEs derived in the previous section as a "wave equation", i.e. in terms of a differential operator involving Δ. We introduce the following vector bundle over $\mathcal{U}_p$:

$$\mathcal{E} := (T^*\mathcal{M} \otimes T\mathcal{M}) \oplus T^*\mathcal{M} \oplus T^*\mathcal{M} \oplus T^*\mathcal{M}.$$

The vector bundle $\mathcal{E}$ carries a covariant derivative naturally induced by $\overline{\nabla}$ and denoted by the same symbol. Moreover, there is an operator Δ of Laplace type on $\mathcal{E}$ which is given by taking the Bochner-Laplacian Δ in each summand and letting this operator act diagonally on sections, i.e.

$$\Delta = \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix}.$$

Using the solutions of the previous section, we define the sections $\eta \in \Gamma(\mathcal{E}|_{\mathcal{U}})$ and $\xi \in \Gamma(T^*\mathcal{U})$ by

$$\eta := \left(\overline{\nabla} V, E, \overline{\nabla}_V E, (\overline{\nabla} E)(V) \right), \qquad \xi := \delta^{\overline{\mathbf{g}}} L. \tag{4.16}$$

Combining the equations in Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5 we obtain

Proposition 4.1 *The sections η and ξ as defined in (4.16) solve the coupled linear PDE*

$$\Delta\eta = F(\eta, \overline{\nabla}\eta, \xi), \tag{4.17}$$
$$\overline{\nabla}_V \xi = H(\eta, \overline{\nabla}\eta), \tag{4.18}$$

where F and H are certain sections of $\mathcal{E}|_{\mathcal{U}}$ and $T^*\mathcal{U}$ which depend linearly *on the indicated quantities.*

Now suppose that η and ξ are arbitrary sections of $\mathcal{E}|_{\mathcal{U}}$ and $T^*\mathcal{U}$ and interpret the left-hand side of (4.17) as a linear differential operator acting on these sections. Moreover, we trivialise the bundles $\mathcal{E}|_{\mathcal{U}}$ and $T^*\mathcal{U}$ with respect to the fixed coordinates $(x^0, \ldots, x^n)$ on $\mathcal{U}$ and view in terms of this identification $\eta \in C^\infty(\mathcal{U}, \mathbb{R}^N)$, where $N = n^2 + 3n$, and $\xi \in C^\infty(\mathcal{U}, \mathbb{R}^{n+1})$.

Proposition 4.2 *In the fixed local trivialisation, Eqs. (4.17) and (4.18) imply a linear symmetric hyperbolic first-order PDE*

$$A^0(t, x, \eta, \partial\eta, \xi)\partial_0 \begin{pmatrix} \eta \\ \partial\eta \\ \xi \end{pmatrix} = A^i(t, x, \eta, \partial\eta, \xi)\partial_i \begin{pmatrix} \eta \\ \partial\eta \\ \xi \end{pmatrix} + b(t, x, \eta, \partial\eta, \xi) \tag{4.19}$$

for η and ξ, i.e. b depends linearly on $(\eta, \partial\eta, \xi)$.

Proof The proof uses only that the linear second-order operator

$$P := \Delta - F(\cdot, \cdot, \xi) \tag{4.20}$$

acting on $\mathcal{E}|_{\mathcal{U}}$ is *normally hyperbolic* for each ξ. In general, given any tensor bundle $\mathcal{E} \to \mathcal{U}$ trivialised by the coordinates x^i, i.e. $\mathcal{E} \cong \mathcal{U}_p \times \mathbb{R}^N$, and any linear second-order differential operator $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, we say that P is normally hyperbolic if its principal symbol is given by the metric, i.e. in the local trivialisation

$$P = -\overline{\mathbf{g}}^{\mu\nu}(p)\frac{\partial^2}{\partial_\mu \partial_\nu} + M^\mu(p)\frac{\partial}{\partial x^\mu} + K(p)$$

for matrix-valued coefficients M_μ and K depending smoothly on p. Note that in our case the term F in (4.20) only affects the matrices M and K but not the symbol. If $\eta = (\eta_1, \ldots, \eta_N) \in C^\infty(\mathcal{U}_p, \mathbb{R}^N)$ is arbitrary, the equation $P\eta = 0$ can be rewritten as linear first-order equation by applying formally the same steps as before when (3.11) was rewritten as first-order equation: For $A = 0, \ldots, N$, we introduce the quantities $k_A := \partial_t \eta_A$ and $\eta_{A,i} := \partial_i \eta_A$. In terms of these quantities, $P\eta = 0$ implies that

$$\partial_t \eta_A = k_A, \tag{4.21}$$
$$\overline{\mathbf{g}}^{ij}\partial_t \eta_{A,i} = \overline{\mathbf{g}}^{ij}\partial_i k_A, \tag{4.22}$$

$$-\overline{\mathbf{g}}^{00}\partial_t k_A = 2\overline{\mathbf{g}}^{0j}\partial_j k_A + \overline{\mathbf{g}}^{ij}\partial_j \eta_{A,i} + H_0^{AB} k_B + \sum_{i=1}^{n} H_i^{AB}\eta_{B,i} + K^{AB}\eta_B, \quad (4.23)$$

holds.[2] Equations (4.21)–(4.23) applied to our operator (4.20) and sections ζ_i yield

$$A_1^0(t,x)\partial_0 \begin{pmatrix} \eta \\ \partial\eta \end{pmatrix} = A_1^i(t,x)\partial_i \begin{pmatrix} \eta \\ \partial\eta \end{pmatrix} + b_1(t,x,\eta,\partial\eta,\xi). \quad (4.24)$$

It is easy to read off an explicit form of the matrices A_1^μ and to see that they are symmetric and that A_1^0 is positive definite as $\overline{\mathbf{g}} \in \mathcal{G}_p$.

We turn to Eq. (4.18). We write $\xi = \xi^\mu \partial_\mu$ and $V = u_t(T - N_t) = u_t(\frac{1}{\sqrt{-\overline{\mathbf{g}}(\partial_t,\partial_t)}}\partial_t - N_t^i \partial_i)$. In terms of these quantities, Eq. (4.18) is equivalent to

$$\partial_t(\xi^\mu)_{\mu=0,\ldots,n} = \sqrt{-\overline{\mathbf{g}}(\partial_t,\partial_t)}N_t^i \partial_i(\xi^\mu)_{\mu=0,\ldots,n} + b_2(t,x,\eta,\partial\eta,\xi), \quad (4.25)$$

where b_2 depends linearly on $(\eta, \partial\eta)$ via H and linearly on ξ via contractions of ξ with Christoffel symbols for $\overline{\mathbf{g}}$ which results from writing $\nabla = \partial + \Gamma$. Combining (4.24) and (4.25) gives (4.19). □

4.4 Initial Data and the Vanishing of $\overline{\nabla}V$ and E

In this section, we will show that E and $\overline{\nabla}V$ vanish everywhere on $\mathcal{U}$ and that V is a null vector field. We will achieve this by showing that the data η, ξ and $\overline{\nabla}\eta$ as defined in (4.16), and containing the tensors $\overline{\nabla}V$ and E, vanish on $\mathcal{U}$. The data η, ξ and $\overline{\nabla}\eta$ were solutions of the linear system (4.17) and (4.18). Hence, using the uniqueness result for solutions to (4.19), it suffices to show that η, ξ and $\overline{\nabla}\eta$ vanish on $\mathcal{M}_p$ (which for simplicity we will denote by $\mathcal{M}$ in the following) in order to obtain that $\overline{\nabla}V$ and that $E = 0$. Moreover, we show that V is null on $\mathcal{M}$ which will imply, by V being parallel, that V is null everywhere.

Proposition 4.3 *The vector field V defined in Eq.* (4.3) *and the sections defined in Eq.* (4.16) *of Sect.* 4.3 *satisfy equations along the initial hypersurface $\mathcal{M}$,*

$$\overline{\mathbf{g}}(V,V)|_{\mathcal{M}} = 0, \qquad \eta|_{\mathcal{M}} = 0, \qquad \overline{\nabla}_T\eta|_{\mathcal{M}} = 0, \qquad \xi|_{\mathcal{M}} = 0. \quad (4.26)$$

In particular, $\overline{\nabla}V$ and E vanish on $\mathcal{M}$.

Proof In this proof, all equations are understood as being evaluated on $\mathcal{M}$, more precisely on $\mathcal{M}_p = \mathcal{M} \cap \mathcal{U}_p$ only, i.e. we do not always write the restriction $|_{\mathcal{M}}$ after

[2] This system is even equivalent to $P\eta = 0$. Indeed, let a triple $(\eta_A, k_A, \eta_{A,i})$ solve (4.21)–(4.23). As g^{ij} is invertible for g sufficiently close to h, (4.22) is the same as $\partial_t \eta_{A,i} = \partial_i k_A$, and (4.21) then gives $\partial_t(\eta_{A,i} - \partial_i\eta_A) = 0$. Appropriate choice of initial data ensures $\eta_{A,i} = \partial_i\eta_A$ at $t = 0$ and thus equality everywhere. Then (4.23) is nothing but $P\eta = 0$.

each expression here. Recall that η and ξ were defined as

$$\eta = \left(\overline{\nabla} V, E, \overline{\nabla}_V E, (\overline{\nabla} E)(V)\right), \qquad \xi = \delta^{\overline{\mathbf{g}}} L, \quad \text{with } L = \mathrm{Z} - \frac{1}{2}\mathrm{tr}_{\overline{\mathbf{g}}}(\mathrm{Z})\,\overline{\mathbf{g}}.$$

In the following, the order in which the initial conditions are verified turns out to be very important. First note that along $\mathcal{M}$ we have that $\overline{\mathbf{g}} = h$, where h is the background metric, and hence that $T\mathcal{M}$ is orthogonal to T. Moreover, we will not distinguish between E and $E^\sharp$. It follows from the identity (2.4) and initial data for V, i.e. from

$$V|_{\mathcal{M}} = uT - U, \tag{4.27}$$

that the imposed constraint equation (1.1) is equivalent to $\mathrm{pr}_{T\mathcal{M}}(\overline{\nabla}_X V)|_{\mathcal{M}} = 0$, i.e. to

$$\overline{\mathbf{g}}(\overline{\nabla}_X V, Y)|_{\mathcal{M}} = 0, \qquad \text{for } X, Y \in T\mathcal{M}.$$

Moreover, Eq. (4.27) implies that

$$\overline{\mathbf{g}}(V, V)|_{\mathcal{M}} = 0. \tag{4.28}$$

Differentiating this in the direction of $X \in T\mathcal{M}$ yields

$$0 = \overline{\mathbf{g}}(\overline{\nabla}_X V, V) = u\overline{\mathbf{g}}(\overline{\nabla}_X V, T),$$

from which follows that $\overline{\nabla}_X V = 0$ on $\mathcal{M}$ for $X \in T\mathcal{M}$. The evolution equation $(d + \delta)V^\flat = 0$ reduces then on $\mathcal{M}$ to

$$c(T) \circ \overline{\nabla}_T V^\flat = 0.$$

Multiplying this from the left with $c(T)$ yields that $\overline{\nabla}_T V = 0$ on $\mathcal{M}$ and hence that

$$\overline{\nabla} V|_{\mathcal{M}} = 0. \tag{4.29}$$

Moreover, the initial data for $\overline{\mathbf{g}}$ were chosen in Sect. 3.4 precisely in such a way that

$$E|_{\mathcal{M}} = 0. \tag{4.30}$$

This also implies that

$$\overline{\nabla}_X E|_{\mathcal{M}} = 0, \text{ for } X \in T\mathcal{M}. \tag{4.31}$$

Showing that the remaining quantities in η, $\overline{\nabla}\eta$ and ξ vanish along $\mathcal{M}$ is rather involved. Again for brevity, we will use abstract index notation with indices $a, b, c, \ldots$ ranking from 0 to n. We will however abuse this abstract index notation as indicated before, when writing a 0 for a contraction $B(T,\ldots)$ of a tensor B with the vector field T,

$$T^a B_{ab\ldots} = B_{0b\ldots},$$

but also when using indices $i, j, k, \ldots$ ranging from 1 to n and referring to directions in $T\mathcal{M}$. Since along $\mathcal{M}$ the vector field T is orthogonal to $T\mathcal{M}$ we have that $\overline{\mathbf{g}}_{0i} = h_{0i} = 0$, as well as $\overline{\mathbf{g}}_{00} = -1$ and $\overline{\mathbf{g}}_{ij} = \mathbf{g}_{ij}$.

We will start by showing that the initial data specified for Z imply that $\overline{\nabla}_T E$ vanishes on $\mathcal{M}$, i.e. that $\overline{\nabla}_0 E_a = 0$ along $\mathcal{M}$. Starting point is Eq. (4.11), which in indices reads as

$$\overline{G}_{ab} = -\overline{\nabla}_{(a}E_{b)} + Z_{ab} + \tfrac{1}{2}(\overline{\nabla}_c E^c - Z_c{}^c)\overline{\mathbf{g}}_{ab}. \tag{4.32}$$

Evaluation on $\mathcal{M}$ using the hypersurface formula (2.2),

$$\overline{G}_{0i} = \nabla_k \mathrm{W}^k_{\ i} + \overline{\nabla}_i \mathrm{W}_k{}^k,$$

implies that

$$\tfrac{1}{2}\overline{\nabla}_0 E_i - \tfrac{1}{2}\overline{\nabla}_i E_0 \;=\; -\nabla_k \mathrm{W}^k_{\ i} + \overline{\nabla}_i \mathrm{W}_k{}^k - Z_{0i} \;=\; -\nabla_k \mathrm{W}^k_{\ i} + \overline{\nabla}_i \mathrm{W}_k{}^k - \tfrac{1}{u} U^k \mathrm{Z}_{ki} \;=\; 0, \tag{4.33}$$

because of $0 = \mathrm{Z}(V,\cdot) = u\mathrm{Z}(T,X) - \mathrm{Z}(U,X)$ and the first initial condition in (3.25) for Z. But now E_a is zero along $\mathcal{M}$ and hence is $\overline{\nabla}_i E_0$, and so we obtain that

$$\nabla_0 E_i = 0. \tag{4.34}$$

Hence, it remains to prove that also $\overline{\nabla}_0 E_0 = 0$.

To this end, recall the hypersurface formula (2.2) contracted with T twice,

$$\overline{G}_{00} = \tfrac{1}{2}(\mathrm{scal}^{\mathbf{g}} - \mathrm{W}_{ij}\mathrm{W}^{ij} + (W_i{}^i)^2),$$

and the above formula (4.32) to obtain

$$\begin{aligned}\tfrac{1}{2}(\mathrm{scal}^{\mathbf{g}} - \mathrm{W}_{ij}\mathrm{W}^{ij} + (W_i{}^i)^2) &= -\overline{\nabla}_0 E_0 + \mathrm{Z}_{00} - \tfrac{1}{2}\overline{\nabla}_c E^c + \tfrac{1}{2}\mathrm{Z}_c{}^c\\ &= -\tfrac{1}{2}\overline{\nabla}_0 E_0 + \tfrac{1}{2}\mathrm{Z}_{00} - \tfrac{1}{2}\overline{\nabla}_k E^k + \tfrac{1}{2}\mathrm{Z}_k{}^k.\end{aligned}$$

Hence, using again $\mathrm{Z}(V,.) = 0$ and $\overline{\nabla}_i E_j = 0$ along $\mathcal{M}$, we get

$$\tfrac{1}{2}\overline{\nabla}_0 E_0 \;=\; \tfrac{1}{2}N^i N^j \mathrm{Z}_{ij} + \tfrac{1}{2}\mathrm{Z}_k{}^k - \tfrac{1}{2}(\mathrm{scal}^{\mathbf{g}} - \mathrm{W}_{ij}\mathrm{W}^{ij} + (W_i{}^i)^2). \tag{4.35}$$

The next lemma shows that this term vanishes:

Lemma 4.6 *On $\mathcal{M}$ satisfying the constraint* (1.1) *it holds that*

$$\mathrm{scal}^{\mathbf{g}} - \mathrm{W}_{ij}\mathrm{W}^{ij} + (W_k{}^k)^2 \;=\; N^i N^j \mathrm{Z}_{ij} + \mathrm{Z}_k{}^k. \tag{4.36}$$

Proof On $\mathcal{M}$ we have $N = \frac{1}{u}U$ with $u^2 = \overline{\mathbf{g}}(U, U)$. An easy consequence of this and of the constraint (1.1), i.e. of $\nabla_i U_j + u\mathrm{W}_{ij} = 0$, is the formula

$$\nabla_i u = -uN^k \mathrm{W}_{ik}, \tag{4.37}$$

and the resulting

$$\nabla_i N_j = N^k \mathrm{W}_{ki} N_j - \mathrm{W}_{ij}. \tag{4.38}$$

Now we use both the initial conditions (3.25) for Z to first determine

$$\mathrm{Z}_k^{\ k} = \mathrm{scal}^{\mathbf{g}} - \mathrm{W}_{ij}\mathrm{W}^{ij} + (\mathrm{W}_k^{\ k})^2 + N^i N^j \left(2\mathrm{W}_{ki}\mathrm{W}^k_{\ j} - 2\mathrm{Ric}_{ij} - 2\mathrm{W}_k^{\ k}\mathrm{W}_{ij} + \mathrm{Z}_{ij}\right),$$

and then compute, using the constraint (1.1) and Eqs. (4.37) and (4.38), that

$$\begin{aligned} N^i N^j \mathrm{Z}_{ij} &= N^i \nabla_i \mathrm{W}_j^{\ j} - N^i \nabla^j \mathrm{W}_{ij} + \\ &= \tfrac{1}{u^2} N^i \nabla_i u \nabla_j U^j - \tfrac{1}{u} N^i \nabla_i \nabla_j U^j - \tfrac{1}{u^2} N^i \nabla_j u \nabla_i U^j + \tfrac{1}{u} N^i \nabla_j \nabla_i U^j \\ &= N^i N^k \mathrm{W}_{ik} \mathrm{W}_j^{\ j} - N^k \mathrm{W}_{kj} \mathrm{W}_i^{\ j} N^i + N^i N^k \mathrm{R}_{jik}^{\ \ \ \ j} \\ &= -N^i N^j \left(\mathrm{W}_{kj}\mathrm{W}_j^{\ k} - \mathrm{W}_{ij}\mathrm{W}_k^{\ k} - \mathrm{Ric}_{ik}\right). \end{aligned}$$

Putting these two equations together gives the desired Eq. (4.36). □

Hence we have established that $\overline{\nabla}_0 E_0 = 0$. Combined with (4.34) this yields $\overline{\nabla}_0 E|_{\mathcal{M}} = 0$. It then follows automatically that $V^a \overline{\nabla}_b E_a = V^a \overline{\nabla}_a E_b = 0$ on $\mathcal{M}$. Altogether we have now that

$$\overline{\nabla}_a E|_{\mathcal{M}} = 0.$$

Then Eq. (4.4) yields as immediate consequence that

$$V^a \overline{\mathrm{Ric}}_{ab}|_{\mathcal{M}} = 0. \tag{4.39}$$

We turn to $\overline{\nabla}_0 \overline{\nabla}_i V^a$-terms for $X \in T\mathcal{M}$ and want to show that such expressions vanish on $\mathcal{M}$. As $\overline{\nabla} V = 0$ on $\mathcal{M}$ we have that

$$\overline{\nabla}_0 \overline{\nabla}_i V^b = \overline{\mathrm{R}}_{0ia}^{\ \ \ \ b} V^a \tag{4.40}$$

on $\mathcal{M}$, and we have to show that this expression vanishes. Note that as a further consequence of $\overline{\nabla}_a V = 0$ on $\mathcal{M}$ we have

$$\overline{\mathrm{R}}_{ijb}^{\ \ \ \ a} V^b = 0 \tag{4.41}$$

on $\mathcal{M}$. We will now prove that $V^a \overline{\mathrm{R}}_{0abc} = 0$ on $\mathcal{M}$ for all $b, c = 0, \ldots, n$:

First we note that

$$V^a\overline{\mathrm{R}}_{0aij} = -V^a\overline{\mathrm{R}}_{ija0} = 0,$$

because of (4.41). Next we use $V^a\overline{\mathrm{Ric}}_{ab} = 0$ to get

$$V^a\overline{\mathrm{R}}_{0ai}{}^{0} = -V^a\overline{\mathrm{Ric}}_{ab} + V^a\overline{\mathrm{R}}_{jai}{}^{j} = 0,$$

again because of (4.41). This implies $V^a\overline{\mathrm{R}}_{0abc} = 0$ on $\mathcal{M}$.

Hence, the vanishing of the term (4.40) is equivalent to

$$0 = V^a\overline{\mathrm{R}}_{ia0j} \tag{4.42}$$

for all $i, j = 1, \ldots, n$. In fact, because of $V = u(T - N)$, it suffices to prove (4.42) for $X^i, Y^j \in N^\perp \subset T\mathcal{M}$, i.e. with $X^i N_i = Y^i N_i = 0$. We use now indices $r, s, t = 1, \ldots, n-1$ for tensors from $N^\perp$. Using this convention, we rewrite

$$V^a\overline{\mathrm{R}}_{ra0s} = -uN^i\overline{\mathrm{R}}_{ri0s} + \overline{\mathrm{R}}_{r00s} = -uN^iN^j\overline{\mathrm{R}}_{rijs} - u\overline{\mathrm{Ric}}_{rs} + u\overline{\mathrm{R}}_{ri}{}^{i}{}_{s}.$$

Because of $\overline{\nabla}E = 0$ and the resulting $\mathrm{Z} = \overline{\mathrm{Ric}}$ on $\mathcal{M}$, this means that (4.42) is verified if and only if

$$\mathrm{Z}_{rs} = -N^iN^j\overline{\mathrm{R}}_{rijs} + \overline{\mathrm{R}}_{ri}{}^{i}{}_{s}. \tag{4.43}$$

Now we use the Gauß equation

$$\overline{\mathrm{R}}_{ijkl} = \mathrm{R}_{ijkl} - \mathrm{W}_{i[k}\mathrm{W}_{l]j}$$

in order to rewrite the curvature terms in (4.43) in terms of data on $\mathcal{M}$. The rewritten Eq. (4.43) is precisely the defining initial condition for Z_{rs}, i.e. Eq. (3.25). This proves (4.42) and thus $\overline{\nabla}_0\overline{\nabla}_i V^b = 0$. As $V^a\overline{\mathrm{Ric}}_{ab} = 0$ on $\mathcal{M}$, we have that

$$0 = \Delta^{HL}V^b = \Delta V^b = \overline{\nabla}_a\overline{\nabla}^a V^b = -\overline{\nabla}_0\overline{\nabla}_0 V^b = 0,$$

because of $\overline{\nabla}_i\overline{\nabla}_j V^a = 0$ on $\mathcal{M}$.

The last and most complicated part of the proof now consists of showing that $\delta^{\overline{\mathbf{g}}}L_a$ and $V^a\overline{\nabla}_0\overline{\nabla}_a E_b$ vanish on $\mathcal{M}$. As a starting point, we take Eq. (4.4),

$$\overline{\mathrm{Ric}}_{ab} = \mathrm{Z}_{ab} - \overline{\nabla}_{(a}E_{b)}.$$

Differentiating both sides covariantly in direction of V, using that $\overline{\nabla}_a V^a \mathrm{Z}_{bc} = 0$ along $\mathcal{M}$ due to (4.8), yields

$$V^a\overline{\nabla}_a\overline{\mathrm{Ric}}_{bc} = V^a\overline{\nabla}_a\overline{\nabla}_{(b}E_{c)} = V^a\overline{\mathrm{R}}_{a(b}{}^{d}{}_{c)}E_d + V_a\overline{\nabla}_{(b}\overline{\nabla}^a E_{c)}. \tag{4.44}$$

Setting $b = 0$ and $c = j$ this and $E_d|_{\mathcal{M}} = 0$ and $\overline{\nabla}_a E_b|_{\mathcal{M}} = 0$ gives

$$V^a \overline{\nabla}_a \overline{\mathrm{Ric}}_{0j} = \tfrac{1}{2} V^a \overline{\nabla}_0 \overline{\nabla}_a E_j = \tfrac{u}{2} \overline{\nabla}_0 \overline{\nabla}_0 E_j - N^i \overline{\nabla}_i \overline{\nabla}_0 E_j = \tfrac{u}{2} \overline{\nabla}_0 \overline{\nabla}_0 E_j. \quad (4.45)$$

We now show that $V^a \overline{\nabla}_a \overline{\mathrm{Ric}}_{0j}$ vanishes on $\mathcal{M}$:

Using the second Bianchi identity, we find

$$V^a \overline{\nabla}_a \overline{\mathrm{Ric}}_{0j} = V^a \overline{\nabla}_a \overline{\mathrm{R}}_{b0j}{}^{b} = -V^a \overline{\nabla}_a \overline{\mathrm{R}}_{ji0}{}^{i} = -V^a \overline{\nabla}_j \overline{\mathrm{R}}_{ia0}{}^{i} - V^a \overline{\nabla}_i \overline{\mathrm{R}}_{aj0}{}^{i} = 0, \quad (4.46)$$

because of Eq. (4.42) along $\mathcal{M}$ and differentiation is along $\mathcal{M}$. Thus, (4.45) gives

$$\overline{\nabla}_0 \overline{\nabla}_0 E_j = 0. \quad (4.47)$$

This, as well as $E_a = 0$ and $\overline{\nabla}_a E_b = 0$ on $\mathcal{M}$, imply that also

$$\Delta E_i = -\overline{\nabla}_a \overline{\nabla}^a E_i = 0$$

on $\mathcal{M}$. But then, Eq. (4.12) immediately yields

$$(\delta^{\overline{\mathbf{g}}} L)_i = 0$$

for all $j = 1, \ldots, n$. Moreover, Lemma 4.5 says that $V^a (\delta^{\overline{\mathbf{g}}} L)_a$ is a linear expression in $\overline{\nabla}_b V^d$ and $\overline{\nabla}_b \overline{\nabla}_c V^d$ terms, which vanish on $\mathcal{M}$. It follows that on $\mathcal{M}$

$$(\delta^{\overline{\mathbf{g}}} L)_0 = \frac{1}{u} V^a (\delta^{\overline{\mathbf{g}}} L)_a + N^i (\delta^{\overline{\mathbf{g}}} L)_i = 0,$$

and as a consequence,

$$\delta^{\overline{\mathbf{g}}} L = 0 \text{ on } \mathcal{M}.$$

Again using formula (4.12) shows now that $\overline{\nabla}_0 \overline{\nabla}_0 E_b = 0$ on $\mathcal{M}$. Inserting this into (4.46) shows that

$$V^a \overline{\nabla}_0 \overline{\nabla}_a E_b = 0.$$

Hence, all covariant derivatives $\overline{\nabla}_a \overline{\nabla}_b E_c$ vanish, proving Eq. (4.26) in the proposition. □

5 Proof of Theorem 1: Global Aspects

5.1 From Local to Global

We next globalise the local development of the initial data. So far we have constructed for every $p \in \mathcal{M}$ the data $(\overline{\mathbf{g}}^{\mathcal{U}_p}, V^{\mathcal{U}_p}, \mathrm{Z}^{\mathcal{U}_p})$ defined on some open $\mathcal{U}_p \subset \mathbb{R} \times \mathcal{M}$

sufficiently small. Let $p, q \in \mathcal{M}$ and assume that $\mathcal{U}_p \cap \mathcal{U}_q \neq \emptyset$. Choose coordinates $(x^0, \dots, x^n)$ and $(y^0, \dots, y^n)$ on $\mathcal{U}_p$ and $\mathcal{U}_q$, respectively, as before. On $\mathcal{U}_p \cap \mathcal{U}_q$, we consider the coordinates given by restriction of the x^i. Then, with respect to these coordinates, the data

$$w_p = \left(\overline{\mathbf{g}}^{\mathcal{U}_p}_{\mu\nu},\ V^{\mathcal{U}_p}_{\mu},\ Z^{\mathcal{U}_p}_{ij}\right), \quad w_q = \left(\overline{\mathbf{g}}^{\mathcal{U}_q}_{\mu\nu},\ V^{\mathcal{U}_q}_{\mu},\ Z^{\mathcal{U}_q}_{ij}\right),$$

solve by construction the system (4.4)–(4.6) formulated locally in the x-coordinates. This follows as these equations are manifestly coordinate invariant. Moreover, the initial data $(w_p)_{|\mathcal{M}_p} = (w_q)_{|\mathcal{M}_q}$ coincide since they arise as restrictions of globally defined data on $\mathcal{M}$. It then follows from the uniqueness result for solutions of symmetric quasilinear hyperbolic systems that

$$w_p = w_q \text{ in } \mathcal{U}_p \cap \mathcal{U}_q. \tag{5.1}$$

We now set

$$\overline{\mathcal{M}} := \cup_{p \in \mathcal{M}} \mathcal{U}_p \subset \mathbb{R} \times \mathcal{M}.$$

As each $\overline{\mathbf{g}}^{\mathcal{U}_p}$ lies in $\mathcal{G}_p$, the $\overline{\mathbf{g}}^{\mathcal{U}_p}$ defines a global Lorentzian metric on $\mathcal{M}$ on which ∂_t is a timelike vector field. We equip $\mathcal{M}$ with the time orientation induced by ∂_t. By the previous local constructions, $(\mathcal{M}, \mathbf{g})$ embeds into $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ with Weingarten tensor W. Moreover, by (5.1) the locally defined vector fields $V^{\mathcal{U}_p}$ give rise to a vector field $V \in \mathfrak{X}(\overline{\mathcal{M}})$ which is parallel and of length zero as this holds locally.

5.2 $\mathcal{M} \subset \overline{\mathcal{M}}$ is a Spacelike Cauchy Hypersurface

To this end, let $\gamma : I \to \overline{\mathcal{M}} = \cup_{p \in \mathcal{M}} \mathcal{U}_p$ be an inextendible timelike curve and let $t^* \in I$ be any fixed parameter. Let $p \in \mathcal{M}$ such that $\gamma(t^*) \in \mathcal{U}_p$. For such fixed p, we consider the restricted curve

$$\gamma_{|\gamma^{-1}(\mathcal{U}_p)} : \gamma^{-1}(\mathcal{U}_p) \to \mathcal{U}_p,$$

which is an inextendible timelike curve in the globally hyperbolic manifold $(\mathcal{U}_p, g^{\mathcal{U}_p})$. Thus, the spacelike Cauchy hypersurface $\mathcal{M}_p \subset \mathcal{M}$ in $\mathcal{U}_p$ is met by $\gamma_{|\gamma^{-1}(\mathcal{U}_p)}$. It remains to show that γ meets $\mathcal{M}$ at most once. With respect to the splitting $\mathbb{R} \times \mathcal{M}$ we decompose

$$\gamma = (\gamma_t, \gamma_{\mathcal{M}})$$

and compute

$$0 > \overline{\mathbf{g}}(\dot{\gamma}_t \partial_t, \dot{\gamma}_t \partial_t) + 2 \cdot \overline{\mathbf{g}}(\dot{\gamma}_t \partial_t, \dot{\gamma}_{\mathcal{M}}) + \overline{\mathbf{g}}(\dot{\gamma}_{\mathcal{M}}, \dot{\gamma}_{\mathcal{M}}).$$

Let us assume that there is $\tau \in I$ with $\dot{\gamma}_t(\tau) = 0$. Let $q := \gamma(\tau)$. Then $0 > \overline{\mathbf{g}}^{\mathcal{U}_q}(\dot{\gamma}_{\mathcal{M}}(\tau), \dot{\gamma}_{\mathcal{M}}(\tau))$. This, however, contradicts the condition $\overline{\mathbf{g}} \in \mathcal{G}_q$ imposed on $\mathcal{U}_q$ and $\overline{\mathbf{g}}^{\mathcal{U}_q}$ in the second step of the proof. Consequently, $\gamma_t : I \to \mathbb{R}$ is strictly monotone, and thus $\gamma_t = 0$ has at most one solution. In total, γ intersects $\mathcal{M}$ exactly once. It follows that $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ is globally hyperbolic with Cauchy hypersurface $\mathcal{M}$ and parallel null vector V.

5.3 The Metric is of the Form $-\widetilde{\lambda}^2 dt^2 + \mathbf{g}_t$

Here we will prove the last aspect of Theorem 1, namely that the metric $\overline{\mathbf{g}}$ obtain in this way is of the form

$$\overline{\mathbf{g}} = -\widetilde{\lambda}^2 dt^2 + \mathbf{g}_t.$$

Let t denote the function $\overline{\mathcal{M}} \ni (t, x) \mapsto t$. By construction, the vector field $\mathrm{grad}^{\overline{\mathbf{g}}}(t)$ is a global timelike vector field on $\overline{\mathcal{M}}$ and the leaves of the integrable distribution $(\mathrm{grad}^{\overline{\mathbf{g}}}(t))^{\perp}$ are the t-levels $\{t\} \times \mathcal{M} =: \mathcal{M}_t$. Let $F \in \mathfrak{X}(\overline{\mathcal{M}})$ denote the vector field that is proportional to $\mathrm{grad}^{\overline{\mathbf{g}}}(t)$ and such that $dt(F) \equiv 1$, i.e.

$$F = \frac{1}{dt(\mathrm{grad}^{\overline{\mathbf{g}}}(t))} \mathrm{grad}^{\overline{\mathbf{g}}}(t),$$

and denote by ϕ its flow. Note that ϕ sends level sets to level sets, i.e. $\phi_s(p) \in \mathcal{M}_{t(p)+s}$. Indeed, for each $p \in \overline{\mathcal{M}}$, the function $f(s) := t(\phi_s(p)) \in \mathbb{R}$ satisfies

$$f'(s) = dt|_{\phi_s(p)}(F) \equiv 1,$$

and hence $f(s) = t(\phi_s(p)) = s + t(p)$. We further define two open neighbourhoods $\overline{\mathcal{M}}^{1,2}$ of $\mathcal{M}$ in $\overline{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}$,

$$\begin{aligned} \overline{\mathcal{M}}^1 &:= \{(t, x) \in \overline{\mathcal{M}} \mid \phi_t(x) \text{ exists}\}, \\ \overline{\mathcal{M}}^2 &:= \{p \in \overline{\mathcal{M}} \mid \exists \tau \text{ such that } \phi_{-\tau}(p) \in \mathcal{M}_0\}, \end{aligned}$$

where we identify $x \in \mathcal{M}$ with $(0, x) \in \mathcal{M}_0$. Note that for each $p = (t, x) \in \overline{\mathcal{M}}^2$ the number $\tau = \tau(p)$ is uniquely determined. Namely if $\phi_{-\tau_1}(p), \phi_{-\tau_2}(p) \in \mathcal{M}_0$ it follows from applying the function t that

$$0 = t(\phi_{-\tau_1}(p)) - t(\phi_{-\tau_2}(p)) = t(p) - \tau_1 - (t(p) - \tau_2).$$

Moreover, as $\overline{\mathcal{M}} = \cup_{x \in \mathcal{M}} \mathcal{U}_x$ and each $\mathcal{U}_x$ satisfies by construction (3.26), it simply follows that $\overline{\mathcal{M}}^2 = \overline{\mathcal{M}}$. Then we have a well-defined diffeomorphism

$$\Psi : \overline{\mathcal{M}}^1 \ni (t, x) \mapsto \phi_t(x) \in \overline{\mathcal{M}},$$

with $\Psi((t,x)) \in \mathcal{M}_t$. Its inverse is given by

$$\Psi^{-1}(p) = \big(\tau(p), \phi_{-\tau}(p)\big).$$

It also satisfies

$$d\Psi_{(\widetilde{t},x)}(\partial_t) = F|_{\phi_{\widetilde{t}}(x)}.$$

Therefore, for the pulled back metric we have at each $(\widetilde{t},x) \in \overline{\mathcal{M}}^1$ and for every vector field X on $\overline{\mathcal{M}}^1$ with values in $T\mathcal{M}_\tau$ for some fixed τ

$$(\Psi^*\overline{\mathbf{g}})_{(\widetilde{t},x)}(\partial_t,\partial_t) = \overline{\mathbf{g}}_{\phi_{\widetilde{t}}(x)}(F,F) \;=\; \frac{1}{\overline{\mathbf{g}}(\mathrm{grad}^{\overline{\mathbf{g}}}t, \mathrm{grad}^{\overline{\mathbf{g}}}t)}, \tag{5.2}$$

$$(\Psi^*\overline{\mathbf{g}})_{(\widetilde{t},x)}(\partial_t,X) = \overline{\mathbf{g}}_{\phi_{\widetilde{t}}(x)}(F, d\psi_{(\widetilde{t},x)}(X)) \;=\; 0, \tag{5.3}$$

since $d\psi_{(\widetilde{t},x)}(X)$ is tangential to $\mathcal{M}_{\widetilde{t}+\tau}$ and F is a multiple of the gradient of t. Hence, ∂_t is orthogonal to $T\mathcal{M}$ with respect to $\Psi^*\overline{\mathbf{g}}$, showing that

$$\Psi^*\overline{\mathbf{g}} = -\widetilde{\lambda}^2 dt^2 + \mathbf{g}_t \tag{5.4}$$

for some t-dependent family of metrics on $\mathcal{M}$. As Ψ restricts to the identity on $\mathcal{M} \subset \overline{\mathcal{M}}^{1,2}$ it follows that $\mathbf{g}_0 = \mathbf{g}$. Moreover, by Eq. (5.2),

$$\widetilde{\lambda} = \frac{1}{\sqrt{\overline{\mathbf{g}}(\mathrm{grad}^{\overline{\mathbf{g}}}t, \mathrm{grad}^{\overline{\mathbf{g}}}t)}}.$$

On $\mathcal{M} = \mathcal{M}_0$ we have $\widetilde{\lambda}|_{\mathcal{M}} = \frac{1}{\sqrt{h(\mathrm{grad}^h t, \mathrm{grad}^h t)|_{\mathcal{M}}}} = \lambda|_{\mathcal{M}}$. In summary, passing from $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ to $(\overline{\mathcal{M}}^1, \psi^*\overline{\mathbf{g}})$ via ψ yields an open neighbourhood of $\mathcal{M}$ in $\mathbb{R} \times \mathcal{M}$ with parallel null vector field, metric of the form (5.4), and as ψ restricts to the identity on $\mathcal{M}$, we deduce that $\mathcal{M}$ is also a spacelike Cauchy hypersurface for $(\overline{\mathcal{M}}^1, \psi^*\overline{\mathbf{g}})$. This finishes the proof of Theorem 1. □

Remark 5.1 The proof of Theorem 1 shows that $\overline{\mathbf{g}}$ depends on the background metric h which was introduced in the proof in terms of the following PDE system: the contracted difference tensor E of the Levi-Civita connections of $\overline{\mathbf{g}}$ and h vanishes, i.e.

$$E(X) = -\mathrm{tr}_{\overline{\mathbf{g}}}\left(\overline{\mathbf{g}}(A(\cdot,\cdot), X)\right) = 0 \text{ for all } X \in TM, \tag{5.5}$$

where $A(Y,Z) := \overline{\nabla}_Y Z - \nabla^h_Y Z$ for $Y, Z \in TM$. Imposing this extra condition in Theorem 1 for the solution $\overline{\mathbf{g}}$ for a fixed background metric h determines $\overline{\mathbf{g}}$ uniquely for each choice of h.

6 Riemannian Manifolds Satisfying the Constraint

In this section, we study Riemannian manifolds $(\mathcal{M}, \mathbf{g})$ satisfying the constraint condition (1.1), which in fact means that there is a nonzero vector field U such that ∇U is a symmetric endomorphism of $(T\mathcal{M}, \mathbf{g})$.

6.1 The Local Structure and the Proof of Theorem 2

The condition (1.1) is equivalent to $\nabla U^\flat = \mathbf{g}(\nabla U, \cdot)$ being symmetric, which in turn is equivalent to $dU^\flat = 0$. Now we can argue analogously as in [31, Proposition 8]:

Locally near some fixed $x_0 \in \mathcal{M}$ we have that $U = \mathrm{grad}^{\mathbf{g}}(z)$ for some function z on $\mathcal{V} \subset \mathcal{M}$ with $z(x_0) = 0$. The leafs of the integrable distribution $U^\perp = \ker(dz)$ are given by the level sets

$$\mathcal{U}_c = \{x \in \mathcal{V} \mid z(x) = c\}.$$

Let $Z \in \Gamma(\mathcal{V})$ denote the vector field that is proportional to U and such that $\mathbf{g}(U, Z) = dz(Z) \equiv 1$, i.e.

$$Z = \frac{1}{dz(\mathrm{grad}^{\mathbf{g}} z)} \mathrm{grad}^{\mathbf{g}} z = \frac{1}{\mathbf{g}(U, U)} U,$$

and denote by ϕ its flow. Choose $\epsilon > 0$ and an open subset $\mathcal{W} \subset \mathcal{M}$ centered around x_0 such that ϕ is defined on $(-\epsilon, \epsilon) \times \mathcal{W}$. We now restrict the levels $\mathcal{U}_c$ to their intersections with $\mathcal{W}$, denoted with the same symbol. Since

$$\mathcal{L}_Z U^\flat = dU^\flat(Z, .) = 0,$$

the flow sends level sets to level sets, i.e. $\phi_s(x) \in \mathcal{U}_{z(x)+s}$. Indeed, for each $x \in \mathcal{U}_{z(x)}$, the function $f(s) := z(\phi_s(x)) \in \mathbb{R}$ satisfies

$$f'(s) = df_s(\partial_s) = dz|_{\phi_s(x)}(Z) \equiv 1,$$

and hence $f(s) = z(\phi_s(x)) = s + z(x)$. Then we have a diffeomorphism

$$\begin{aligned} \Psi : (-\epsilon, \epsilon) \times \mathcal{U}_t &\longrightarrow \{y \in \mathcal{W} \mid |z(y)| < \epsilon\} \subset \mathcal{W}, \\ (s, x) &\longmapsto \phi_s(x), \end{aligned}$$

with $\Psi(s, x) \in \mathcal{U}_s$. Its inverse is given by

$$\Psi^{-1}(x) = \big(z(x), \phi_{-z(x)}(x)\big) \in \mathcal{I} \times \mathcal{U}_0.$$

It also satisfies

$$d\Psi_{(s,x)}(\partial_s) = Z|_{\phi_s(x)}.$$

Therefore, for the pulled back metric we have

$$\Psi^*\mathbf{g}(\partial_s, \partial_s) = \mathbf{g}_{\phi_s(x)}(Z, Z) = \frac{1}{\mathbf{g}(\mathrm{grad}^{\mathbf{g}} z, \mathrm{grad}^{\mathbf{g}} z)} = \frac{1}{\mathbf{g}(U, U)},$$
$$\Psi^*\mathbf{g}(\partial_s, X) = \mathbf{g}_{\phi_s(x)}(Z, d\Psi_{(s,x)}(X)) = 0,$$

since $d\Psi_{(t,x)}(X)$ is tangential to a level set whenever X is, and Z is a multiple of the gradient of z. Finally, $\mathbf{h}_s$ is given by

$$\mathbf{h}_s(X, Y)|_x := \Psi^*\mathbf{g}(X, Y) = \mathbf{g}_{\Psi_s(x)}(d\phi_s|_x(X), d\phi_s|_x(Y)),$$

for $X, Y \in \mathcal{U}_0$. Hence, $\Psi^*\mathbf{g} = \mu^2 ds^2 + \mathbf{h}_s$ with

$$\mu = \frac{1}{\sqrt{\mathbf{g}(\mathrm{grad}^{\mathbf{g}} z, \mathrm{grad}^{\mathbf{g}} z)}} = \frac{1}{u}, \qquad \partial_s = \frac{1}{dz(\mathrm{grad}^{\mathbf{g}} z)} \mathrm{grad}^{\mathbf{g}} z = \frac{1}{u^2} U.$$

Setting $\mathcal{F} := \mathcal{U}_0 = z^{-1}(0)$, this gives the local form of the metric (1.2).

Moreover, if $Z = \frac{1}{u^2} U$ is complete, [31, Proposition 8] shows that the flow of the lift of Z to the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ defines a *global* diffeomorphism Ψ between $\widetilde{\mathcal{M}}$ and $\mathbb{R} \times \widetilde{\mathcal{F}}$, where $\widetilde{F}$ is the universal cover of a leaf $\mathcal{F}$ of $U^{\perp}$.

Finally, we compute for $(\mathcal{M}, \mathbf{g})$ as in formula (1.2) and $X, Y \in T\mathcal{F}$ the symmetric bilinear form $\mathrm{W} = -\frac{1}{u} \nabla U$ as follows:

$$\mathrm{W}(\partial_s, \partial_s) = \partial_s \left(\tfrac{1}{u}\right),$$
$$\mathrm{W}(\partial_s, X) = -u \cdot \mathbf{g}(\nabla_{\partial_s} \partial_s, X) = X \left(\tfrac{1}{u}\right),$$
$$\mathrm{W}(X, Y) = -u \cdot \mathbf{g}(\nabla_X \partial_s, Y) = -\tfrac{u}{2} \dot{\mathbf{h}}_s(X, Y).$$

Clearly, this is equivalent to Eq. (1.3). This finishes the proof of Theorem 2. □

6.2 Complete Riemannian Manifolds Satisfying the Constraint

In order to obtain complete Riemannian manifolds satisfying the constraint, we will use the following lemma, which is a weaker version of forthcoming results in [32], see also [20, Lemma 2].

Lemma 6.1 *Let $\mathcal{F}$ be a compact manifold with a s-dependent family of Riemannian metrics $\mathbf{h}_s$ and let u be a bounded, positive smooth function on $\mathcal{M} = \mathbb{R} \times \mathcal{F}$. Then the metric*

$$\mathbf{g} = \frac{1}{u^2} ds^2 + \mathbf{h}_s$$

on $\mathcal{M}$ is complete.

Proof According to the decomposition $\mathcal{M} = \mathbb{R} \times \mathcal{F}$ we can write every curve $\gamma : [a, b) \to \mathcal{M}$ as $\gamma(t) = (s(t), x(t))$ with $s : [a, b) \to \mathbb{R}$. Hence,

$$\mathbf{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = \left(\frac{\dot{s}(t)}{u(\gamma(t))}\right)^2 + \mathbf{h}_{s(t)}(\dot{x}(t), \dot{x}(t)).$$

For a curve with $\mathbf{g}(\dot{\gamma}, \dot{\gamma}) \equiv c \in \mathrm{R}$ constant, e.g. a geodesic, $\mathbf{h}_s$ being positive definite shows that $0 \leq \mathbf{h}_s(\dot{x}, \dot{x}) \leq c$ is bounded, and u bounded implies

$$(\dot{s})^2 = (u \circ \gamma)^2(c - \mathbf{h}_u(\dot{x}, \dot{x})) \leq c(u \circ \gamma)^2 \leq c \sup u,$$

showing that also $\dot{s} : [a, b) \to \mathbb{R}$ is bounded. Hence, if $b \in \mathbb{R}$, the function $s : [a, b) \to \mathbb{R}$ is bounded and its image lies in a compact set in $\mathbb{R}$. Hence $s(b) = \lim_{t \to b} s(t) \in \mathrm{R}$ is well defined.

Now assume that $(\mathcal{M}, \mathbf{g})$ is incomplete. Let $\gamma : [a, b) \to \mathcal{M}$ be a maximal geodesic with $b \in \mathbb{R}$. Then γ leaves every compact set in $\mathcal{M}$. Indeed, if $\gamma(t)$ remained in a compact set, then $\{\gamma(t_n)\}_{n \in \mathbb{N}}$ with $t_n \to b^-$ would have a convergent subsequence. However, $\{\gamma(t_n)\}$ is a Cauchy sequence for the metric $d_{\mathbf{g}}$ induced by the Riemannian metric $\mathbf{g}$. Hence $\{\gamma(t_n)\}$ converges, and thus γ could be extended beyond b. On the other hand, we have seen that the image of s lies in a compact set in $\mathbb{R}$. Hence, that γ leaves every compact set in $\mathcal{M} = \mathbb{R} \times \mathcal{F}$ is a contradiction to $\mathcal{F}$ compact. □

7 Special Lorentzian Holonomy and Families of Riemannian Metrics

Based on the classification of indecomposable holonomy groups of Lorentzian manifolds with parallel null vector field [12, 30], we will now show how we can use Theorem 1 to construct Lorentzian manifolds with prescribed holonomy from families of Riemannian metrics. Our aim in this section is to prove Theorem 3.

7.1 The Screen Bundle of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$

To every Lorentzian manifold with parallel null vector field, in particular to the data $(\overline{\mathcal{M}}, \overline{\mathbf{g}}, V)$ constructed via Theorem 1, we can associate the *screen bundle*

$$\mathcal{S} := V^{\perp}/V \to \overline{\mathcal{M}}$$

equipped with covariant derivative $\nabla^{\mathcal{S}}_X[Y] := \left[\overline{\nabla}_X Y\right]$. In contrast to the general case, however, our setting always yields a canonical realisation of $\mathcal{S}$ as a subbundle S of $T\overline{\mathcal{M}}$ by means of the natural vector bundle isomorphism

$$\begin{array}{ccc} T\overline{\mathcal{M}} \supset S := T^{\perp} \cap V^{\perp} \ni Y & \longmapsto & [Y] \in \mathcal{S}. \\ \searrow & & \swarrow \\ & \overline{\mathcal{M}} & \end{array}$$

This isomorphism pulls back ∇^S to the covariant derivative

$$\nabla^S := \mathrm{pr}_S \circ \overline{\nabla}_{|S},$$

which in turn is metric with respect to the positive definite screen metric $\overline{\mathbf{g}}^S := \overline{\mathbf{g}}_{S\times S}$. Having these identifications in mind, we also refer to S as the screen. The screen construction is a useful tool when analysing the holonomy of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, which by construction is contained in the stabiliser of a null vector, i.e. (up to conjugation), we have

$$\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset \mathbf{SO}(n) \ltimes \mathbb{R}^n \subset \mathbf{SO}(1, n+1).$$

(Note that the Lorentzian manifolds arising via Theorem 1 are time-orientable.) For any subgroup $\mathbf{G} \subset \mathbf{SO}(n) \ltimes \mathbb{R}^n$, let $\mathrm{pr}_{\mathbf{SO}(n)}\mathbf{G}$ denote its projection onto the $\mathbf{SO}(n)$-part. Then we have by construction

$$\mathrm{Hol}(S, \nabla^S) \cong \mathrm{pr}_{\mathbf{SO}(n)}\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}). \tag{7.1}$$

Recall that on $\overline{\mathcal{M}}$ the parallel null vector field V decomposes into $V = \overline{u}T - N$. We next list useful formulas for the screen covariant derivative ∇^S and the screen curvature R^S. By trivial extension, we will often view a section of $S \to \overline{\mathcal{M}}$ equivalently as element of $\mathfrak{X}(\overline{\mathcal{M}}) = \Gamma(T\overline{\mathcal{M}})$ which is everywhere orthogonal to T and V and denote it with the same symbol.

Lemma 7.1 *Let $\sigma \in \Gamma(S)$ and let $X, Y, Z \in \Gamma(T\overline{\mathcal{M}})$. The following hold:*

$$\begin{aligned}
\nabla^S_Y\sigma &= \overline{\nabla}_Y\sigma - \frac{1}{\overline{u}}\overline{\mathbf{g}}(\sigma, \overline{\nabla}_Y T)\, V,\\
\mathrm{R}^S(X, Y)\sigma &= \mathrm{pr}_S(\overline{\mathrm{R}}(X, Y)\sigma),\\
0 &= (\nabla^S_Z \mathrm{R}^S)(X, Y) + (\nabla^S_X \mathrm{R}^S)(Y, Z) + (\nabla^S_Y \mathrm{R}^S)(Z, X).
\end{aligned}$$

Proof These are straightforward calculations following directly from the various definitions, parallelity of V as well as the symmetries and second Bianchi identity for $\overline{\mathrm{R}}$. □

For the data $(\overline{\mathcal{M}}, \overline{\mathbf{g}}, V)$ constructed via Theorem 1, let

$$S^{r,s} = \otimes^{r,s} S := \underbrace{S^* \otimes \cdots \otimes S^*}_{r\times} \otimes \underbrace{S \otimes \cdots \otimes S}_{s\times} \to \overline{\mathcal{M}}$$

denote the (r, s)-screen tensor bundle with covariant derivative induced by ∇^S and denoted with the same symbol. R^S also denotes the curvature operator of $(T^{r,s}, \nabla^S)$.

Finally, we need to understand the pullback $S|_{\mathcal{M}} \to \mathcal{M}$ of the screen bundle $S \to \overline{\mathcal{M}}$ by means of the inclusion $\mathcal{M} = \{0\} \times \mathcal{M} \hookrightarrow \overline{\mathcal{M}}$. By construction, it follows that $S|_{\mathcal{M}} = U^\perp \to \mathcal{M}$, whence the restriction $\sigma|_{\mathcal{M}} \in \Gamma(S|_{\mathcal{M}})$ of any $\sigma \in \Gamma(S)$ to $\mathcal{M}$ can be regarded as vector field on $\mathcal{M}$ which is orthogonal to U. On the vector bundle

$U^\perp \to \mathcal{M}$ we have the connection that is induced by the Levi-Civita connection of $\mathbf{g}$, $\nabla^\perp = \mathrm{pr}_{U^\perp} \circ \nabla^{\mathbf{g}}$. Then

Lemma 7.2 *For each $X \in \mathfrak{X}(\mathcal{M})$ and $\sigma \in \Gamma(S)$, we have*

$$\left(\nabla^S_X \sigma\right)|_{\mathcal{M}} = \nabla^\perp_X \sigma|_{\mathcal{M}}. \tag{7.2}$$

Proof It follows from formula (2.1) that

$$\nabla^S_X \sigma|_{\mathcal{M}} = \mathrm{pr}_S(\overline{\nabla}_X \sigma|_{\mathcal{M}}) = \mathrm{pr}_S(\nabla_X \sigma|_{\mathcal{M}}) = \mathrm{pr}_{U^\perp}(\nabla_X \sigma|_{\mathcal{M}}).$$

□

Now we describe the parallelity of a section of $S^{r,s} \to \overline{\mathcal{M}}$ in terms of the corresponding section of the pulled back bundle $S|_{\mathcal{M}}$.

Proposition 7.1 *Let $(\mathcal{M}, \mathbf{g}, U)$ be a Riemannian manifold satisfying the constraint (1.1) and $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ the Lorentzian manifold arising via Theorem 1. Then the ∇^S-parallel sections of the bundle $S^{r,s} \to \overline{\mathcal{M}}$ are in one-to-one correspondence with $\nabla^\perp$-parallel sections ζ of the pulled back bundle $S^{r,s}|_{\mathcal{M}} \to \mathcal{M}$, i.e. with*

$$\nabla^\perp_X \zeta = 0, \quad \textit{for all } X \in T\mathcal{M}. \tag{7.3}$$

Proof By the previous lemma, it is clear that a parallel section of $T^{r,s}S \to \overline{\mathcal{M}}$ satisfies (7.3).

On the other hand, let us assume condition (7.3). We extend ζ to a section of $T^{r,s}S \to \overline{\mathcal{M}}$ by parallel transport in V-direction, i.e. such that $\nabla^S_V \zeta = 0$. It then suffices to show that $\nabla^S_X \zeta = 0$ for $X \in T^\perp$. To this end, we introduce the bundle

$$H := (T^\perp)^* \otimes T^{r,s}S \to \overline{\mathcal{M}}$$

as well as the section $A \in \Gamma(H)$, given by

$$A(X) := \nabla^S_X \zeta.$$

Clearly, there are naturally induced covariant derivatives on H. For $X \in T^\perp$ we compute, using the identities from Lemma 7.1 as well as $\overline{\nabla} V = 0$, that

$$\begin{aligned}
(\nabla^S_V A)(X) &= \nabla^S_V (A(X)) - A(\overline{\nabla}_V X) \\
&= \mathrm{R}^S(V, X)\zeta + \nabla^S_X \nabla^S_V \zeta + \nabla^S_{[V,X]}\zeta - \nabla^S_{\overline{\nabla}_V X}\zeta \\
&= \mathrm{R}^S(V, X)\zeta \\
&= 0.
\end{aligned}$$

Thus, $\overline{\nabla}^S_V A = 0$ which is a linear symmetric hyperbolic first-order PDE for A. As $A|_{\mathcal{M}} = 0$ by assumption, we conclude $A \equiv 0$. □

Now we specify Proposition 7.1 for the situation when $(\mathcal{M}, g)$ is globally given as in Theorem 2, i.e. when $\mathcal{M} = \mathcal{I} \times \mathcal{F}$, where $s \in \mathcal{I}$ with $\mathcal{I}$ and interval, $\mathbb{R}$ or the circle, $\mathcal{F}$ is a smooth manifold, and $\mathbf{g} = u^{-2}ds^2 + \mathbf{h}_s$ with a smooth nonvanishing function u on $\mathcal{M}$. Then, $U = u^2\partial_s$ and we can express a section Z of the bundle $U^\perp \to \mathcal{M}$ by an s-dependent family of sections Z_s of $T\mathcal{F} \to \mathcal{F}$. Differentiating such a section Z_s in direction $X \in T\mathcal{F}$, we get the identity

$$\nabla^\perp_X Z = \nabla^{\mathbf{h}_s}_X Z,$$

where $\nabla^{\mathbf{h}_s}$ is the Levi-Civita connection of the metric $\mathbf{h}_s$. Differentiating in s-direction, by the Koszul formula, we get for each $X \in T\mathcal{F}$ that

$$2\mathbf{g}(\nabla^\perp_{\partial_s} Z_s, X) = \partial_s(\mathbf{g}(Z_s, X)) + \mathbf{g}([\partial_s, Z_s], X) = (\mathcal{L}_{\partial_s}\mathbf{g})(Z_s, X) + 2\mathbf{g}([\partial_s, Z_s], X), \tag{7.4}$$

where $\mathcal{L}_{\partial_s}$ denotes the Lie derivative with respect to ∂_s and where we assume that $[\partial_s, X] = 0$. However, we have that $(\mathcal{L}_{\partial_s}\mathbf{g})(X, Y) = (\mathcal{L}_{\partial_s}\mathbf{h}_s)(X, Y)$, whenever X and Y are tangential to $\mathcal{F}$. Hence, when dualising Eq. (7.4) with the metric $\mathbf{h}_s$ we get that

$$\nabla^\perp_{\partial_s} Z_s = \tfrac{1}{2}(\mathcal{L}_{\partial_s}\mathbf{h}_s)^\sharp(Z_s) + [\partial_s, Z_s],$$

where $\sharp$ denotes the (s-dependent) dualisation with respect to $\mathbf{h}_s$. Introducing the notation $\dot{\mathbf{h}}_s$ for $\mathcal{L}_{\partial_s}\mathbf{h}_s$ we can write this concisely as

$$\nabla^\perp_{\partial_s} Z_s = [\partial_s, Z_s] + \tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp(Z_s).$$

Using this, for a family of 1-forms $\sigma_s \in \Gamma(T^*\mathcal{F})$ we get

$$(\nabla^\perp_{\partial_s}\sigma_s)(X) = \partial_s(\sigma_s(X)) - \sigma_s(\nabla^\perp_{\partial_s} X) = (\mathcal{L}_{\partial_s}\sigma_s)(X) - \tfrac{1}{2}\sigma_s\left((\dot{\mathbf{h}}_s)^\sharp(X)\right),$$

for all $X \in T\mathcal{F}$, i.e. that

$$\nabla^\perp_{\partial_s}\sigma_s = \dot{\sigma}_s + \tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \sigma_s,$$

where $\bullet$ denotes the natural action of endomorphisms on 1-forms and $\dot{\mathrm{s}}_s := \mathcal{L}_{\partial_s}\sigma_s$ is the Lie derivative.

This relation generalises to families of tensor fields σ_s of higher rank and we obtain:

Corollary 7.1 *Let $\mathcal{F}$ be a smooth manifold and $\mathbf{h}_s$ be a family of Riemannian metrics on $\mathcal{F}$, where $s \in \mathcal{I}$ with $\mathcal{I}$ being an interval, $\mathbb{R}$ or the circle, u a nonzero smooth function on $\mathcal{M} = \mathcal{I} \times \mathcal{F}$ and $\mathbf{g} = u^{-2}ds^2 + \mathbf{h}_s$ be the Riemannian manifold defined in (1.2). Moreover, let $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ the Lorentzian manifold arising from $(\mathcal{M}, \mathbf{g})$ via Theorem 1. Then there is a one-to-one correspondence between*

(1) *sections $\overline{\sigma}$ of the bundle $S^{k,l} \to \overline{\mathcal{M}}$ such that $\nabla^S_X\overline{\sigma} = 0$ for all $X \in T\overline{\mathcal{M}}$;*
(2) *sections σ of of the bundle $\otimes^{k,l}U^\perp \to \mathcal{M}$ such that $\nabla^\perp_Y\sigma = 0$ for all $Y \in T\mathcal{M}$;*

(3) *s-dependent families of sections* σ_s *of* $\otimes^{k,l} T\mathcal{F} \to \mathcal{F}$ *with*

$$\nabla^{\mathbf{h}_s}_Z \sigma_s = 0, \quad \text{for all } Z \in T\mathcal{F}, \tag{7.5}$$

$$\dot{\sigma}_s = -\tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \sigma_s, \tag{7.6}$$

where $\nabla^{\mathbf{h}_s}$ *is the Levi-Civita connection of the metric* $\mathbf{h}_s$*, the dot denotes the Lie derivative in s-direction,* $\sharp$ *the dualisation with respect to* $\mathbf{h}_s$*, and* $\bullet$ *is the natural action of an endomorphism field on* $\otimes^{k,l} T\mathcal{F}$*., i.e.*

$$(\dot{\mathbf{h}}_s^\sharp \bullet \sigma_s)(X_1, \dots, X_k) = \dot{\mathbf{h}}_s^\sharp(\sigma_s(X_1, \dots, X_k)) \\ -\sigma_s(\dot{\mathbf{h}}_s^\sharp(X_1), X_2, \dots, X_k) - \dots - \sigma_s(X_1, \dots, X_{k-1}, \dot{\mathbf{h}}_s^\sharp(X_k)),$$

for $X_1, \dots, X_k \in T\mathcal{F}$.

This corollary will now provide us with a proof of Theorem 3.

7.2 Lorentzian Special Holonomy and the Proof of Theorem 3

Here we use the result of Sect. 7.1 to obtain a proof of Theorem 3. In the setting of Theorem 3 start with data $(\mathcal{M}, \mathbf{g}, \mathrm{W}, U)$ satisfying the initial condition (1.1) and then first apply Theorem 2 to conclude that these data are given as

$$(\mathcal{M} = L \times \mathcal{F}, \mathbf{g} = \frac{1}{u^2} ds^2 + \mathbf{h}_s, U = u^2 \partial_s)$$

solving (1.1) for W as in (1.3). Thus the existence of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ with parallel null vector and initial data for $\mathbf{g}_t$ and $\dot{\mathbf{g}}_t$ as desired follows from Theorem 1. Next, it follows from Sect. 7.1, in particular from Proposition 7.1, that $\mathrm{pr}_{\mathbf{SO}(n)}\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) = \mathrm{Hol}(S, \nabla^S)$ fixes an element in $T^{k,l}\mathbb{R}^n$ if and only if there is $\sigma \in \Gamma(\mathcal{M}, T^{k,l}U^\perp)$ solving $\nabla^\perp \sigma = 0$. Using the explicit form of $(\mathcal{M}, \mathbf{g})$ and U from Theorem 2, σ can be equivalently viewed as s-dependent family of tensor fields $\sigma_s \in \Gamma(\mathcal{F}, T^{k,l}\mathcal{F})$. By Corollary 7.1, equation $\nabla^\perp \sigma = 0$ is then equivalent to Eqs. (7.5) and (7.6). This proves the first statement in Theorem 3 and it remains to verify the statements in Table 1. For this, we first consider the situation that the screen holonomy is in $\mathbf{U}(\frac{n}{2})$, i.e. that

$$\mathrm{Hol}(S, \nabla^S) = \mathrm{pr}_{\mathbf{SO}(n)}\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset \mathbf{U}(\tfrac{n}{2}).$$

By Eq. (7.5), this case requires $\mathrm{Hol}(\mathcal{F}, \mathbf{h}_s) \subset \mathbf{U}(\frac{n}{2})$. In other words, there are families of complex structures J_s, Kaehler forms ω_s on $\mathcal{F}$ which are parallel with respect to

$$\mathbf{h}_s = \omega_s(J_s \cdot, \cdot) \tag{7.7}$$

and satisfy the flow equations (1.4). Hence, $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{U}(\frac{n}{2})$ is equivalent to $\mathrm{Hol}(\mathcal{F}, \mathbf{h}_s) \subset \mathbf{U}(\frac{n}{2})$ and equations

$$\dot{J}_s + \tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet J_s = 0, \qquad \dot{\omega}_s + \tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \omega_s = 0 \tag{7.8}$$

for all s and where the dot again denotes the Lie derivative with respect to the parameter s.

Now we turn to those holonomy groups in Table 1 that are defined as the stabiliser of one or more tensors, i.e. to $\mathbf{Sp}(\frac{n}{4})$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$:

The case $n = 4k$ and constraints for $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{Sp}(k)$ is in complete analogy to the $\mathbf{U}(\frac{n}{2})$-case, characterised by families of hyper-Kähler metrics $\mathbf{h}_s$ on $\mathcal{F}$ with corresponding compatible parallel almost complex structures (J_s^1), $i = 1, 2, 3$, i.e. $J_s^1 J_s^2 = J_s^3$ and Kaehler forms ω_s^i such that $\mathbf{h}_s = \omega_s(J_s\cdot, \cdot)$ satisfying the corresponding flow equations (7.8).

For the case $n = 7$ and constraints for $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{G}_2$ recall that the exceptional group $\mathbf{G}_2 \subset \mathbf{SO}(7)$ can be realised as the stabiliser subgroup of a stable 3-form in $\mathbb{R}^7$, see for example [16,21,25,26] more details. Hence, by Corollary 7.1 the case $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{G}_2$ is characterised by a family of associated stable 3-forms $\phi_s \in \Omega^3(\mathcal{F})$ on $\mathcal{F}$ evolving according to Eq. (7.6) with associated family $\mathbf{h}_s$ of $\mathbf{G}_2$. This implies the corresponding entry in Table 1.

For the case $n = 8$ and constraints for $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{Spin}(7)$ recall the algebraic properties of the group $\mathbf{Spin}(7) \subset \mathbf{SO}(8)$ and its realisation in terms of the stabiliser of a generic 4-form, again see [16,26] for details but also [28]. The discussion then is completely analogous to the $\mathbf{G}_2$ case and the constraint equations are equivalent to the existence of a family of parallel $\mathbf{Spin}(7)$-structures ψ_s on $\mathcal{F}$ evolving under the flow equation (7.6).

Now we turn to the case that is n even and *the screen holonomy is special unitary*, i.e. $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{SU}(\frac{n}{2})$. This is the most difficult case because this reduction is not simply given as the stabiliser of a tensor, but rather by a trace condition in addition to the reduction to $\mathbf{U}(n)$.

The parallel almost complex structures J_s coming from the reduction $\mathrm{Hol}(S, \nabla^S) \subset \mathbf{SU}(\frac{n}{2})$ give a $\nabla^\perp$ parallel almost complex structure $J \in \Gamma(\mathcal{M}, \mathrm{End}(U^\perp))$. By Proposition 7.1, J gives via ∇^S-parallel translation a ∇^S-parallel almost complex structure J^S on the screen $S \to \overline{\mathcal{M}}$. From now on, we will work on the Lie algebra level. The holonomy algebra $\mathfrak{hol}(S, \nabla^S)$ is contained in $\mathfrak{su}(\frac{n}{2})$ if and only if $\mathfrak{hol}(S, \nabla^S)\mathfrak{u}(\frac{n}{2})$ and each of its elements A satisfies

$$\mathrm{tr}(J^S \circ A) = 0,$$

where we identify elements in the holonomy algebra with endomorphism of a fibre of the screen bundle S. Now we apply the Ambrose–Singer holonomy theorem to the holonomy algebra $\mathfrak{hol}(S, \nabla^S)$ at $p \in \mathcal{M}$, which states that

$$\mathfrak{hol}(S, \nabla^S) = \mathrm{span}\{(P_\gamma^S)^{-1} \circ \mathrm{R}^S(X, Y) \circ P_\gamma^S \mid \gamma : [0, 1] \to \overline{\mathcal{M}},\ \gamma(0) = p, X, Y \in T_{\gamma(1)}\overline{\mathcal{M}}\},$$

where R^S is the curvature of the screen bundle S and P_γ^S is the parallel transport along a curve γ. Since J^S is parallel, it commutes with all parallel transports P_γ^S and we obtain

$$\begin{aligned}\mathrm{tr}(J^S \circ (P_\gamma^S)^{-1} \circ \mathrm{R}^S(X,Y) \circ P_\gamma^S) &= \mathrm{tr}((P_\gamma^S)^{-1} \circ J^S \circ \mathrm{R}^S(X,Y) \circ P_\gamma^S) \\ &= \mathrm{tr}(J^S \circ \mathrm{R}^S(X,Y)).\end{aligned}$$

Using this, the Ambrose–Singer theorem and the fact that $\mathrm{R}^S(V,\cdot) = 0$ we obtain that $\mathfrak{hol}(S,\nabla^S) \subset \mathfrak{su}(\frac{n}{2})$ if and only if J^S additionally satisfies

$$\mathrm{tr}\left(J^S \circ \mathrm{R}^S(X,Y)\right) = 0, \quad \text{for all} \quad X,Y \in T^\perp \to \overline{\mathcal{M}}. \tag{7.9}$$

Let us now consider the left side of condition (7.9) as section C in the bundle $\Lambda^2 T^\perp \to \overline{\mathcal{M}}$, which in turn carries a covariant derivative induced by $\overline{\nabla}$. We have by parallelity of J^S and V and Lemma 7.1 that

$$\begin{aligned}(\overline{\nabla}_V C)(X,Y) &= \mathrm{tr}\left(J^S \circ (\nabla_V^S R^S)(X,Y)\right) \\ &= \mathrm{tr}\left(J^S \circ (\nabla_X^S R^S)(V,Y)\right) + \mathrm{tr}\left(J^S \circ (\nabla_Y^S R^S)(X,V)\right) \\ &= 0.\end{aligned}$$

Hence, $C \equiv 0$ if and only if

$$C|_{\mathcal{M}} = 0, \tag{7.10}$$

which in turn is evaluated by using the Gauß equation

$$\overline{\mathrm{R}}(X,Y,Z,L) = \mathrm{R}(X,Y,Z,L) - \mathrm{W}(X,Z)\mathrm{W}(Y,L) + \mathrm{W}(X,L)\mathrm{W}(Y,Z), \tag{7.11}$$

for all $X,Y,Z,L \in T\mathcal{M}$. Let s_i be a local orthonormal basis of $S|_{\mathcal{M}} = U^\perp \to \mathcal{M}$ and $X,Y \in T\mathcal{M}$. Then we have

$$\begin{aligned}&-\mathrm{tr}\left(J \circ \mathrm{R}^S(X,Y)\right)|_{\mathcal{M}} = \sum_i \mathbf{g}(\mathrm{R}^S(X,Y)s_i, J^S(s_i)) \\ &= \sum_i \overline{\mathrm{R}}(X,Y,s_i,J^S(s_i)) \\ &= \sum_i \mathrm{R}(X,Y,s_i,J(s_i)) - \mathrm{W}(X,a_i)\mathrm{W}(Y,J(s_i)) + \mathrm{W}(X,J(s_i))\mathrm{W}(Y,s_i) \\ &= -\mathrm{tr}\,(J \circ \mathrm{R}(X,Y)) - \mathrm{W}(Y,J(\mathrm{W}(X))) + \mathrm{W}(X,J(\mathrm{W}(Y))).\end{aligned}$$

Therefore, the additional condition on $(\mathcal{M}, \mathbf{g}, \mathrm{W}, U, J)$ ensuring special unitary screen holonomy is

$$\mathrm{tr}\,(J \circ \mathrm{R}(X, Y)) = -\mathrm{W}(Y, J(\mathrm{W}(X))) + \mathrm{W}(X, J(\mathrm{W}(Y))). \tag{7.12}$$

For Theorem 3, one has to evaluate Eq. (7.12) in terms of data on $(\mathcal{F}, \mathbf{h}_s)$ as in Theorem 2. Let $\mathrm{W}_s = \mathrm{W}_{|\mathcal{F}_s \times \mathcal{F}_s}$. Then one finds for the embedding $(\mathcal{F}, \mathbf{h}_s) \hookrightarrow (\mathcal{M}, \mathbf{g})$ with unit normal $u\,\partial_s$ along $\mathcal{F}$ that

$$\nabla_X Y = \nabla^{\mathbf{h}_s}_X Y + \mathrm{W}_s(X, Y) u \cdot \partial_s, \ \forall X, Y \in T\mathcal{F}.$$

That is W_s is actually the Weingarten tensor of this embedding. Thus, using a Riemannian version of the Gauß equation, the curvature R_s of $\mathbf{h}_s$ is for $X, Y, Z, L \in T\mathcal{F}$ related to that of $(\mathcal{M}, \mathbf{g})$ via

$$\mathrm{R}(X, Y, Z, L) = \mathrm{R}_s(X, Y, Z, L) + \mathrm{W}_s(X, Z)\mathrm{W}_s(Y, L) - \mathrm{W}_s(X, L)\mathrm{W}_s(Y, Z) \tag{7.13}$$

and the Codazzi equation

$$\mathrm{R}(X, u\partial_s, Y, Z) = \left(d^{\nabla^{\mathbf{h}_s}} \mathrm{W}_s\right)(Y, Z, X) := \left(\nabla^{\mathbf{h}_s}_Y \mathrm{W}_s\right)(Z, X) - \left(\nabla^{\mathbf{h}_s}_Z \mathrm{W}_s\right)(Y, X). \tag{7.14}$$

Inserting Eq. (7.13) into (7.12), we obtain for $X, Y \in T\mathcal{F}$ after a straightforward calculation

$$0 = \mathrm{tr}(J_s \circ \mathrm{R}_s(X, Y)) \overset{\nabla^{\mathbf{h}_s} J_s = 0}{=} -2\mathrm{Ric}_s(X, J_s(Y)).$$

On the other hand, we also need to evaluate

$$\mathrm{tr}\,(J \circ \mathrm{R}(X, \partial_s)) = -\mathrm{W}(\partial_s, J(\mathrm{W}(X))) + \mathrm{W}(X, J(\mathrm{W}(\partial_s))). \tag{7.15}$$

The right side of (7.15) is calculated using (1.3) and is equal to $-2\mathbf{g}(\mathrm{grad}^{\mathbf{g}}(\frac{1}{u}), J(\mathrm{W}(X)))$. For the left side, (7.12) and (7.14) yield with a straightforward computation

$$-(\delta^{\mathbf{h}_s}\dot{\mathbf{h}}_s)(J_s(X)) + -2\mathbf{g}\left(\mathrm{grad}^{\mathbf{g}}\left(\frac{1}{u}\right), J(\mathrm{W}(X))\right)$$

Thus, (7.15) is equivalent to

$$(\delta^{\mathbf{h}_s}\dot{\mathbf{h}}_s) = 0 \tag{7.16}$$

Hence, the constraints for special unitary screen holonomy are equivalent to the existence of Ricci-flat Kähler metrics $(J_s, \omega_s, \mathbf{h}_s = \omega_s(J_s\cdot, \cdot))$ on $\mathcal{F}$ satisfying the flow equation (7.6) and additionally solve (7.16).

Finally we consider to the case when *the screen holonomy splits or is trivial*: Suppose that there are proper subgroups H_1 and H_2 of $\mathbf{SO}(n)$ such that

$$\mathrm{Hol}(S, \nabla^S) \subset H_1 \times H_2 \subset \mathbf{SO}(n) \tag{7.17}$$

Equivalently, there is a nontrivial, decomposable and ∇^S-parallel form in the screen bundle. Thus, by Theorem 1.4 and the holonomy principle, (7.17) is equivalent to a local metric splitting

$$(\mathcal{F}, \mathbf{h}_s) \cong (\mathcal{F}_1 \times \mathcal{F}_2, \mathbf{h}_s^1 + \mathbf{h}_s^2) \tag{7.18}$$

with $\mathrm{Hol}(\mathcal{F}_i, \mathbf{h}_s^i) \subset H_i$ and additionally the volume forms $\mathrm{vol}^{\mathbf{h}_s^i}$ of the metrics $\mathbf{h}_s^i$, $i = 1, 2$ evolve according to

$$\mathcal{L}_{\partial s} \mathrm{vol}^{\mathbf{h}_s^i} = -\tfrac{1}{2} \dot{\mathbf{h}}_s^{i,\sharp} \bullet \mathrm{vol}^{\mathbf{h}_s^i}. \tag{7.19}$$

However, it is well known and straightforward to compute that (7.19) holds for any time-evolving metric with associated family of volume forms.

Finally, let us now consider the special case that the screen is flat, i.e. the standard representation of $\mathrm{Hol}(S, \nabla^S)$ decomposes into n trivial subrepresentations. It follows immediately from an iterated version of the statement in the case where the screen holonomy splits hat this is equivalent to $(\mathcal{F}, \mathbf{h}_s)$ being a family of flat metrics. This proves Theorem 3. □

7.3 One-Parameter Families of Special Riemannian Structures

Here, we reformulate the evolutions equations (7.6) in the case of Kähler and $\mathbf{G}_2$-structures further and formulate a question. We focus on one-parameter families of Kähler structures.

Lemma 7.3 *Let $(\mathcal{F}, h_s, J_s, \omega_s)$ be an s-dependent family of Riemannian Kähler structures on $\mathcal{F}$, i.e. with parallel complex structures J_s and $\omega_s = h_s(J_s\cdot, .)$ and set*

$$\Lambda^{1,1}(\mathcal{F}, J_s) := \{\phi \in \Lambda^2(\mathcal{F}) \mid \phi(J_s X, J_s Y) = \phi(X, Y)\}.$$

Then ω and J satisfy the flow equations

$$\dot{J}_s + \tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet J_s = 0, \qquad \dot{\omega}_s + \tfrac{1}{2}\dot{\mathbf{h}}_s^\sharp \bullet \omega_s = 0 \tag{7.20}$$

if and only if

$$\dot{\omega}_s \in \Lambda^{1,1}(\mathcal{F}, J_s). \tag{7.21}$$

Proof For brevity, we write drop the index s indicating the s-dependence and write a dot for the Lie derivative with respect to ∂_s, i.e. $\dot{\omega} = \mathcal{L}_{\partial_s}\omega$, etc.

First compute

$$(\dot{\mathbf{h}}^\sharp \bullet \omega)(X, Y) = -\omega(\dot{\mathbf{h}}^\sharp X, Y) - \omega(X, \dot{\mathbf{h}}^\sharp Y) = \dot{h}(X, JY) - \dot{h}(JX, Y), \qquad (7.22)$$

which implies that $(\dot{\mathbf{h}}^\sharp \bullet \omega) \in \Lambda^{1,1}(\mathcal{F}, J_s)$. This shows that Eq. (7.20) implies relation (7.21).

Secondly, Lie-differentiating and skew-symmetrising the relation $0 = \omega - \mathbf{h}(J., .)$ yields

$$\begin{aligned}\dot{\omega}(X, Y) &= \tfrac{1}{2}\left(\dot{\mathbf{h}}(JX, Y) - \dot{\mathbf{h}}(X, JY) + \mathbf{h}(\dot{J}X, Y) - \mathbf{h}(X, \dot{J}Y)\right)\\ &= -\tfrac{1}{2}(\dot{\mathbf{h}}^\sharp \bullet \omega) + \tfrac{1}{2}\left(\mathbf{h}(\dot{J}X, Y) - \mathbf{h}(X, \dot{J}Y)\right).\end{aligned}$$

by (7.22). We have seen that $(\dot{\mathbf{h}}^\sharp \bullet \omega) \in \Lambda^{1,1}(\mathcal{F}, J_s)$ and we claim that

$$\mathbf{h}(\dot{J}X, Y) - \mathbf{h}(X, \dot{J}Y) \in \Lambda^2_-(\mathcal{F}, J_s) := \{\phi \in \Lambda^2(\mathcal{F}) \mid \phi(J_s X, J_s Y) = -\phi(X, Y)\}, \qquad (7.23)$$

which shows that relation (7.21) implies Eq. (7.20). To prove claim (7.23), we differentiate $0 = \omega(X, Y) - \omega(JX, JY)$ as in [36, Lemma 4.3] and use $\mathbf{h}(X, Y) = -\omega(JX, Y) = \omega(X, JY)$ to obtain that

$$\begin{aligned}0 &= \dot{\omega}(X, Y) - \dot{\omega}(JX, JY) - \omega(\dot{J}X, JY) - \omega(JX, \dot{J}Y)\\ &= \dot{\omega}(X, Y) - \dot{\omega}(JX, JY) - \mathbf{h}(\dot{J}X, Y) + h(X, \dot{J}Y).\end{aligned}$$

This proves claim (7.23) and because of $\Lambda^2(\mathcal{F}) = \Lambda^{1,1}(\mathcal{F}, J_s) \oplus \Lambda^2_-(\mathcal{F}, J_s)$ establishes the desired equivalence. □

Note that not every family of Kähler structures $(\mathbf{h}_s, J_s)$ satisfies Eq. (7.20): for example, for the constant family of flat metrics $\mathbf{h} \equiv \mathbf{h}_s$ in even dimension the compatible complex structures are parametrised by the homogeneous space $\mathbf{GL}_n\mathbb{C}/\mathbf{U}(n)$. Taking a nonconstant curve of $\mathbf{h}$-parallel, i.e. constant, complex structures J_s gives a Kähler structure $(\mathbf{h}, J_s)$ with $\dot{\mathbf{h}} = 0$ but $\dot{J}_s \neq 0$, which contradicts (7.20).[3] Of course, a constant family of constant complex structures $J_s \equiv J$ always satisfies Eq. (7.20) for the flat metric $\mathbf{h}$. Clearly this suggests the following question: *given a family of Kähler metrics, is there a family of complex structures J_s, or of Kähler forms ω_s, such that condition* (7.21), *and hence flow equation* (7.20) *is satisfied?*

A difficulty when analysing Eq. (7.21) arises from the fact that for an s-dependent family of complex structures J_s, the algebraic splitting of the two forms into $\Lambda^{1,1}$ and Λ^2_- *depends on the parameter s.*

For $\mathbf{G}_2$-structures the situation is similar. Let ϕ_t be a family of $\mathbf{G}_2$-structures defining the family of Riemannian holonomy $\mathbf{G}_2$ metrics $\mathbf{h}_s$. Since the tangent space at a stable three-form ϕ splits under $\mathbf{G}_2$ into three irreducible components

$$\begin{aligned}\mathbb{R} \oplus Sym^2_0(\mathbb{R}^7) \oplus \mathbb{R}^7 &\simeq \Lambda^3\mathbb{R}^7\\ (r, S, X) &\mapsto r\phi + S^\sharp \bullet \phi + X \lrcorner (*\phi)\end{aligned} \qquad (7.24)$$

[3] We would like to thank Vincente Cortés for alerting us to this example.

it follows that

$$\dot{\phi} = S^{\sharp} \bullet \phi + X \lrcorner (*\phi),$$

for a family of symmetric bilinear forms, whereas the associated metric satisfies

$$\dot{\mathbf{h}} = 2S,$$

see [18,27,29]. Hence, similarly to the Kähler case, for the curve ϕ_t the equation that results from Corollary 7.1,

$$\dot{\phi} + \tfrac{1}{2}\dot{h}^{\sharp} \bullet \phi = 0 \tag{7.25}$$

is equivalent to the condition

$$\dot{\phi} \in \mathbb{R} \oplus Sym_0(\mathbb{R}^7),$$

i.e. that $\dot{\phi}$ has no $\mathbb{R}^7$-component in the decomposition. Again, it remains the question whether for a given family of parallel $\mathbf{G}_2$-structures $\mathbf{h}_s$ we can always find a corresponding family of stable 3-forms ϕ_t satisfying this condition. This suggest to formulate the following

Open Questions *Let $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ be a Lorentzian manifold obtained from a Riemannian manifold $(\mathcal{M}, \mathbf{g})$ satisfying the constraints via Theorem 1 and with screen holonomy $G = \mathrm{pr}_{\mathbf{SO}(n)}\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}})$.*

(1) *If $G \subset \mathbf{U}(\frac{n}{2})$, does there always exists a ∇^S-parallel complex structure J on $\mathcal{S}$ such that the associated family of $\mathbf{h}_s$-parallel and compatible complex structures J_s satisfies the flow equation $\dot{J}_s + \frac{1}{2}\dot{h}^{\sharp} \bullet J_s = 0$?*
(2) *If $G \subset \mathbf{G}_2$ does there always exists a ∇^S-parallel stable 3-form ϕ on $\mathcal{S}$ such that the associated family of $\mathbf{h}_s$-parallel stable 3-forms ϕ_s satisfies the flow equation $\dot{\phi}_s + \frac{1}{2}\dot{h}^{\sharp} \bullet \phi_s = 0$?*

Remark 7.1 Interestingly, the flow equation (7.25) for the $\mathbf{G}_2$-case appears in [29] in a completely unrelated context as $\mathbf{G}_2$-flow equation for not necessarily parallel 1-parameter families of $\mathbf{G}_2$-structures $\alpha_s \in \Omega^3(\mathcal{F})$ on $\mathcal{F}$. In fact, let $A_{ij} = A_{ij}(s)$ be any s-dependent family of symmetric (2, 0) tensors on $\mathcal{F}$ and consider the equation

$$\partial_s \alpha_{ijk} = A_i^l \alpha_{ljk} + A_i^l \alpha_{ilk} + A_k^l \alpha_{ijl} \tag{7.26}$$

for some given initial generic 3-form $\alpha_{s=0}$. As $\mathbf{G}_2 \subset \mathbf{SO}(7)$, every generic 3-form in dimension 7 yields a metrics $\mathbf{h}_s = \mathbf{h}_s(\alpha_s)$ in a natural way and [29] then shows the relation

$$\partial_s h_{ij} = 2A_{ij},$$

which provides the link to our situation. However, it remains unclear under which conditions a parallel $\mathbf{G}_2$-structure α_0 on $\mathcal{F}$ evolves under the flow equations (7.26) to a parallel family of $\mathbf{G}_2$-structures as required here. In general, we have that (see [29])

$$\partial_s(\nabla_l^s \alpha_{ijk}) = A_i^m(\nabla_l^s \alpha_{mjk}) + A_j^m(\nabla_l^s \alpha_{imk}) + A_k^m(\nabla_l^s \alpha_{ijm}) + (\nabla^s A)\text{-terms.}$$

To assure that a parallel $\mathbf{G}_2$-structure remains parallel under the flow, one would thus have to control the $\nabla^s A$-terms. This lies beyond the scope of this paper. The same discussion is possible on the level of $\mathbf{Spin}(7)$ structures and their flow equations which appeared in an unrelated context in [28].

8 Applications to Riemannian and Lorentzian Spinor Field Equations

Here, we will use the previous results in order to obtain the two classification statements from Theorems 4 and 5 in the introduction.

8.1 Generalised Imaginary Killing Spinors on Riemannian Manifolds and the Proof of Theorem 4

Let us first suppose that $(\mathcal{M}, \mathbf{g})$ admits an imaginary W-Killing spinor. Differentiating (1.8), it is easy to calculate that its Dirac current $U = U_\varphi$ defined by relation (1.7) satisfies Eq. (1.6). Thus, Theorem 1 applies and there is a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ in which $(\mathcal{M}, \mathbf{g})$ embeds with second fundamental form W and $u\partial_t - U$ extends to a parallel null vector field V on $\overline{\mathcal{M}}$. We extend the spinor φ to a spinor ϕ on $\overline{\mathcal{M}}$ by parallel translation in direction of V, i.e. with $\overline{\nabla}_V \phi = 0$. Setting

$$A(X) := \overline{\nabla}_X \phi$$

for $X \in \partial_t^\perp$, we find using parallelity of V as well as the relations between spinorial and Riemannian curvature (for details see [4]) that

$$(\overline{\nabla}_V A)(X) = \overline{\nabla}_V \overline{\nabla}_X \phi - \overline{\nabla}_{\overline{\nabla}_V X} \phi = \overline{\mathrm{R}}^{S^{\overline{\mathbf{g}}}}(V, X)\phi = \tfrac{1}{2}\overline{\mathrm{R}}(V, X) \cdot \phi = 0.$$

The well-known hypersurface formulas for the spinor covariant derivative [2] imply that the generalised Killing spinor equation for φ is equivalent to $A|_{\mathcal{M}} = 0$ and as A solves a linear first-order symmetric hyperbolic PDE we conclude that $A \equiv 0$, and hence $\overline{\nabla}\phi = 0$. In particular, $\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}}) \subset \mathbf{SO}(n) \ltimes \mathbb{R}^n$ fixes not only a parallel vector but also a nontrivial spinor. However, results in [9,30] show that this can only happen if the screen holonomy satisfies that

$$\mathrm{pr}_{\mathbf{SO}(n)}(\mathrm{Hol}(\overline{\mathcal{M}}, \overline{\mathbf{g}})) \subset H_1 \times \cdots \times H_k, \tag{8.1}$$

with H_i is equal to $\mathbf{SU}(m_i)$, $\mathbf{Sp}(k_i)$, $\mathbf{G}_2$, $\mathbf{Spin}(7)$, or trivial. By Theorem 2, $(\mathcal{M}, \mathbf{g})$ is locally of the form $(\mathbb{R} \times \mathcal{F}, \frac{1}{u}^2 ds^2 + \mathbf{h}_s)$. Condition (8.1) yields that locally $(\mathcal{F}_s, \mathbf{h}_s)$ splits into a metric product $(\mathcal{F}_s^1, h_s^1) \times \cdots \times (\mathcal{F}_s^k, h_s^k)$ with $\mathrm{Hol}(\mathcal{F}_s^k, h_s^k) \subset H_k$. Moreover, Theorem 3 applied to this situation yields the evolution equations for $\mathbf{h}_s^i$ as given in Theorem 4. This proves (1) in Theorem 4. The proof of the global version in (2) follows directly from the global statement in Theorem 2.

Conversely, assume that $(\mathcal{M}, \mathbf{g})$ is given as in the formulation of Theorem 4. As an immediate consequence of Theorem 3, $(\mathcal{M}, \mathbf{g})$ embeds into a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ with parallel null vector field V. Moreover, we have that condition (8.1) holds for the screen holonomy of $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ which follows from Theorem 3. However, in [9,30] it is shown that each such Lorentzian holonomy group fixes a spinor whose Dirac current as defined in relation (1.10) is up to constant the null vector stabilised by $\mathbf{SO}(n) \ltimes \mathbb{R}^n$. By the holonomy principle, there thus exists a parallel spinor ϕ with $V = V_\phi$. The well-known hypersurface formulas for the spinor covariant derivative in [2] imply that ϕ restricts to an imaginary W-Killing spinor $\varphi = \phi|_{\mathcal{M}} \in \Gamma(\mathbb{S}^{\mathbf{g}})$ on $\mathcal{M}$ with W being the Weingarten tensor of $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ as given in Sect. 2. It has been shown in [10] that

$$U = \mathrm{pr}_{T\mathcal{M}} V|_{\mathcal{M}} = \mathrm{pr}_{T\mathcal{M}} V_\phi|_{\mathcal{M}} = U_\varphi.$$

As V_ϕ is null, it follows that $V_\phi \cdot \phi = 0$, which evaluated on $\mathcal{M}$ gives precisely Eq. (1.8). This shows (3) in Theorem 4 and finishes the proof. □

8.2 Lorentzian Holonomy and the Proof of Theorem 5

For a manifold of the form (1.11) set $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$, $\mathbf{h}_w = \mathbf{h}_w^1 + \cdots + \mathbf{h}_w^m$. Now introduce new coordinates by setting $v = -t + s$ and $w = t + s$, i.e. the metric in (1.11) becomes

$$\overline{\mathbf{g}} = -dt^2 + ds^2 + \mathbf{h}_{t+s} =: -dt^2 + \mathbf{g}_t. \tag{8.2}$$

The metric (8.2) admits a parallel null spinor if and only if $(\mathcal{M} := \mathbb{R} \times \mathcal{F}, \mathbf{g}_0)$ admits an imaginary generalised W-Killing spinor additionally solving Eq. (1.8). Indeed, it is clear that a parallel spinor restricts to an imaginary W-Killing spinor on $(\mathcal{M}, \mathbf{g}_0)$. On the other hand, if a imaginary W-Killing spinor φ additionally solving Eq. (1.8) exists on $(\mathcal{M}, \mathbf{g}_0)$ we use that $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ admits a parallel null vector and exactly the same argument as in the proof of Proposition 7.1 or in Sect. 8.1 extend φ to a parallel null spinor for $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$. But with this equivalence, the statement follows immediately from the local classification result in Theorem 4. □

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