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On Instability of the Nikodym Maximal Function Bounds over Riemannian Manifolds

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Abstract We show that, for odd d, the $L^{\frac{d+2}{2}}$ bounds of Sogge (J Am Math Soc 12:1–31, 1999) and Xi (Trans Am Math Soc 369:6351–6372, 2017) for the Nikodym maximal function over manifolds of constant sectional curvature are unstable with respect to metric perturbation, in the spirit of the work of Minicozzi and Sogge (Math Res Lett 4:221–237, 1997). A direct consequence is the instability of the bounds for the corresponding oscillatory integral operator. Furthermore, we extend our construction to show that the same phenomenon appears for any d-dimensional Riemannian manifold with a local totally geodesic submanifold of dimension $\lceil \frac{d+1}{2} \rceil$ if $d \geq 3$. In contrast, Sogge's $L^{\frac{7}{3}}$ bound for the Nikodym maximal function on 3-dimensional variably curved manifolds is stable with respect to metric perturbation.

Keywords Kakeya–Nikodym Problem · Nikodym maximal function · Oscillatory integral

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1 Introduction

A classical Nikodym set N in Euclidean space $\mathbb{R}^d$ is a measure one subset of the unit cube $[0, 1]^d$, which has the property that for each $x \in N$ there is a straight line γ_x such that $\gamma_x \cap N = \{x\}$. Because of these, the relative complement $\mathcal{N} = [0, 1]^d \setminus N$ must be a set of measure zero, which contains a line segment passing through each point of N.

It is implicit in the work of Córdoba that such a set $\mathcal{N}$ in $\mathbb{R}^2$ must have Hausdorff dimension 2. The Nikodym set conjecture asserts that all such set $\mathcal{N}$ in $\mathbb{R}^d$ should have Hausdorff dimension equal to d.

The so-called maximal Nikodym conjecture is actually a stronger conjecture that involves the following Nikodym maximal function:

$$f_\delta^{**}(x) = \sup \frac{1}{|T_x^\delta|} \int_{T_x^\delta} |f(y)| \mathrm{d}y,$$

where T_x^δ is a $1 \times \delta \times \cdots \times \delta$ tube with central axis γ_x passing through $x \in \mathbb{R}^d$. This maximal conjecture (formulated by Córdoba [3]) states for any $\epsilon > 0$

$$\| f_\delta^{**} \|_{L^d(\mathbb{R}^d)} \leq C_\epsilon \delta^{-\epsilon} \| f \|_{L^d(\mathbb{R}^d)}. \tag{1.1}$$

Interpolating with the trivial $L^1 \to L^\infty$ bound, we see that (1.1) is equivalent to

$$\| f_\delta^{**} \|_{L^q(\mathbb{R}^d)} \leq C_\epsilon \delta^{1-\frac{d}{p}-\epsilon} \| f \|_{L^p(\mathbb{R}^d)}, \tag{1.2}$$

where $1 \leq p \leq d$ and $q = (d-1)p'$.

Tao [11] showed that in $\mathbb{R}^d$ a bound like (1.2) is equivalent to the corresponding bound for the Kakeya maximal function f_δ^*, and thus the maximal Nikodym conjecture and the maximal Kakeya conjecture are equivalent in Euclidean space.

It is well known (see Lemma 2.15 in [1]) that for a given p (1.2) implies that the set $\mathcal{N}$ must have Hausdorff dimension at least p. For the case $d = p = 2$, (1.1) was proved by Córdoba [3]. However, it is still open for any $d \geq 3$. When $p = (d+1)/2$, $q = (d-1)p' = d+1$, (1.2) follows from Drury [4] in 1983. In 1991, Bourgain [1] improved this result for each $d \geq 3$ to some $p(d) \in ((d+1)/2, (d+2)/2)$ by the so-called bush argument, where Bourgain considered the "bush" where lots of tubes intersect at a given point. Four years later, Wolff [12] generalized Bourgain's bush argument to the hairbrush argument, by considering tubes with lots of "bushes" on them. Combining this hairbrush argument and the induction on scales, Wolff proved the following bound.

Theorem 1 (Wolff [12]) *The Nikodym maximal function satisfies*

$$\| f_\delta^{**} \|_{L^{\frac{(d-1)(d+2)}{d}}(\mathbb{R}^d)} \leq C_\epsilon \delta^{1-\frac{2d}{d+2}-\epsilon} \| f \|_{L^{\frac{d+2}{2}}(\mathbb{R}^d)}. \tag{1.3}$$

As mentioned before, (1.3) implies that the Hausdorff dimension of $\mathcal{N}$ is at least $(d+2)/2$. This is still the best result for the Nikodym set conjecture when $d = 3, 4$.

One can get better results for larger d or for the weaker Minkowski dimension; see, e.g., [2,5,6].

Even though the Kakeya set is not well defined for curved space, one can naturally extend the definition of the Nikodym set and the corresponding maximal function to manifolds.

Definition 1 For a Riemannian manifold (M, g), we call $\mathcal{N} \subset M$ a Nikodym-type set if there exists a set $\mathcal{N}^* \subset M$ with positive measure such that for each point $x \in \mathcal{N}^*$ there exists a geodesic γ_x passing through x with $|\gamma_x \cap \mathcal{N}| > 0$.

Definition 2 We define f_δ^{**} to be the Nikodym maximal function over a Riemannian manifold (M, g), such that

$$f_\delta^{**}(x) = \sup \frac{1}{|T_{\gamma_x}^\delta|} \int_{T_{\gamma_x}^\delta} |f(y)|‘\mathrm{d}y,$$

where $T_{\gamma_x}^\delta$ is the δ-neighborhood of a geodesic segment γ_x of length β that passes through x. Here β can be chosen to be any fixed number less than one half of the injectivity radius of (M, g).

In 1997, Minicozzi and Sogge [7] showed for a general manifold that Drury's result for $p = (d+1)/2$ still holds, but surprisingly counter examples were constructed to show that it is indeed sharp in odd dimensions.

Theorem 2 (Minicozzi and Sogge [7]) *Given (M^d, g) of dimension $d \geq 2$, then for f supported in a compact subset K of a coordinate patch and all $\epsilon > 0$*

$$\| f_\delta^{**} \|_{L^q(M^d, g)} \leq C_\epsilon \delta^{1-\frac{d}{p}-\epsilon} \| f \|_{L^p(M^d, g)},$$

where $p = \frac{d+1}{2}$ and $q = (d-1)p'$.

Furthermore, there exists an arbitrarily small perturbation of the Euclidean metric, $\tilde{g}$, such that over $(\mathbb{R}^d, \tilde{g})$ the above estimate breaks down for $(M^d, g) = (\mathbb{R}^d, \tilde{g})$ if $p > \lceil \frac{d+1}{2} \rceil$.[1]

In 1999, Sogge [8] managed to adapt Wolff's method for the generalized Nikodym maximal function to 3-dimensional manifolds with constant curvature. Combining a modified version of Wolff's multiplicity argument with an auxiliary maximal function, Sogge proved the following.

Theorem 3 (Sogge [8]) *Assume that (M^3, g) has constant sectional curvature. Then for f supported in a compact subset K of a coordinate patch and all $\epsilon > 0$*

$$\| f_\delta^{**} \|_{L^{\frac{10}{3}}(M^3, g)} \leq C_\epsilon \delta^{-\frac{1}{5}-\epsilon} \| f \|_{L^{\frac{5}{2}}(M^3, g)}. \tag{1.4}$$

[1] Here $\lceil \cdot \rceil$ is the usual ceiling function, i.e., $\lceil \frac{d+1}{2} \rceil$ denotes the smallest integer no less than $\frac{d+1}{2}$.

In his proof, Sogge was able to avoid the induction-on-scales argument, which is hard to perform in curved space. Xi managed to generalize Sogge's result to any dimension $d \geq 3$ [13]. Therefore, Wolff's bounds hold for all manifolds with constant curvature.

Theorem 4 (Xi [13]) *Assume that (M^d, g) has constant sectional curvature with $d > 3$. Then for f supported in a compact subset K of a coordinate patch and all $\epsilon > 0$*

$$\|f_\delta^{**}\|_{L^{\frac{(d-1)(d+2)}{d}}(M^d,g)} \leq C_\epsilon \delta^{1-\frac{2d}{d+2}-\epsilon} \|f\|_{L^{\frac{d+2}{2}}(M^d,g)}. \tag{1.5}$$

In this paper, we show that the bounds of Sogge [8] and Xi are also unstable with respect to metric perturbation.

Theorem 5 *Given (M^d, g) of dimension $d \geq 3$ with constant sectional curvature, for every $\varepsilon > 0$, there exists a metric g_ε, such that for any k, $\|g^{ij} - g_\varepsilon^{ij}\|_{C^k} \leq B_k \varepsilon$, for some positive constant B_k and over (M^d, g_ε), the estimate*

$$\|f_\delta^{**}\|_{L^q(M^d,g_\varepsilon)} \leq C_\epsilon \delta^{1-\frac{d}{p}-\epsilon} \|f\|_{L^p(M^d,g_\varepsilon)}$$

fails to hold if $p > \lceil \frac{d+1}{2} \rceil$ and $q = (d-1)p'$.

We prove this in the spirit of Minicozzi and Sogge [7], by constructing a metric perturbation so that the Nikodym-type set could be concentrated inside a submanifold of dimension $\lceil \frac{d+1}{2} \rceil$.

Indeed, we shall prove that this instability is generic, in the sense that for any Riemannian manifolds (M^d, g) with a local totally geodesic submanifold of dimension $\lceil \frac{d+1}{2} \rceil$, we can always perturb the metric locally, to make the trivial bound in Theorem 2 to be best possible.

Theorem 6 *Given (M^d, g) of dimension $d \geq 3$ such that M^d has a local totally geodesic submanifold of dimension $\lceil \frac{d+1}{2} \rceil$. Then for every $\varepsilon > 0$ there exists a metric g_ε, such that for any k, $\|g^{ij} - g_\varepsilon^{ij}\|_{C^k} \leq B_k \varepsilon$, for some positive constant B_k and over (M^d, g_ε), the estimate*

$$\|f_\delta^{**}\|_{L^q(M^d,g_\varepsilon)} \leq C_\epsilon \delta^{1-\frac{d}{p}-\epsilon} \|f\|_{L^p(M^d,g_\varepsilon)}$$

fails to hold if $p > \lceil \frac{d+1}{2} \rceil$ and $q = (d-1)p'$. Indeed, there exist Nikodym-type sets of dimension $\lceil \frac{d+1}{2} \rceil$ on (M^d, g_ε).

Even though the numerology here (and in the work of Sogge and Minicozzi) may seem a bit odd at first, it can be easily understood through a simple parameter counting. For a piece of totally geodesic submanifold $\mathcal{N}^n \subset M^d$ of dimension n to be a Nikodym-type set, there must be a collection of geodesic segments within $\mathcal{N}$ such that their extensions fill up a d-dimensional set $\mathcal{N}^*$. We know that the family of geodesics in $\mathcal{N}$ locally can be parametrized using $2n - 2$ parameters, together with the 1-parameter coming from each geodesic, and we must have $2n - 2 + 1 \geq d$, which clearly shows that $n = \lceil \frac{d+1}{2} \rceil$ is the smallest possible choice.

A direct consequence of the above results is that the corresponding oscillatory integral operator, which has the Riemannian distance function $d_g(x, y)$ as the phase function, cannot have preferable stable bound with respect to the metric perturbation. Also, in contrast, by a simple compactness argument, the $L^{\frac{7}{3}}$ bound of Sogge [8] is stable with respect to metric perturbation.

Our paper is organized as follows. In the next section, as a model case, we shall prove Theorem 6 for 3-dimensional manifolds with a 2-dimensional totally geodesic submanifold, since things are much simpler in this case, yet it still provides the essential insights into this problem. In Sect. 3, we shall prove Theorem 6 for $(2d+1)$-dimensional manifolds with the help of a simple ODE lemma. In Sect. 4, we finish the proof of Theorem 6 by pointing out how to easily generalize the proof to the even dimensional cases. Theorem 5 then follows as a corollary to Theorem 6. Finally, in the last section, we shall briefly discuss what Theorem 6 tells us about the related oscillatory integral operators.

2 Instability of Nikodym Bounds in Dimension 3

We work on a 3-dimensional Riemannian manifold (M^3, g) with a totally geodesic, two-dimensional submanifold N^2. In a local coordinate chart $(U_1, (x_1, x_2, x_3))$, without loss of generality, we can assume $N \cap U_1 = \{x_3 = 0\}$ and $\frac{\partial}{\partial x_3}$ is the unit normal vector of $N \cap U_1$. Further, we assume that in coordinates (x_1, x_2) on $N \cap U_1$ the cometric can be written as

$$ds^2 = dp_1^2 + \tilde{g}^{22}(x_1, x_2) dp_2^2 + dp_3^2, \quad \text{when } x_3 = 0,$$

where $\tilde{g}^{22}(x_1, x_2) = g^{22}(x_1, x_2, 0)$. This can be done, for example, by choosing the polar coordinates on N. Since $N \subset M$ is totally geodesic, the metric tensor must satisfy

$$\frac{\partial g^{ij}}{\partial x_3}\Big|_{x_3=0} = 0, \quad \text{for } 1 \le i, j \le 2.$$

Therefore, by taking the Taylor expansion of each $g^{ij}(x_1, x_2, x_3)$ at $x_3 = 0$,

$$ds^2 = dp_1^2 + \tilde{g}^{22}(x_1, x_2) dp_2^2 + dp_3^2 + x_3 \sum_{i=1}^{3} 2h^{i3} dp_i dp_3 + x_3^2 \sum_{1 \le i, j \le 2} 2 f^{ij} dp_i dp_j,$$

where h^{i3}, $1 \le i \le 3$, and f^{ij}, $1 \le i, j \le 3$, are certain smooth functions of variable $x = (x_1, x_2, x_3) \in U_1$ for $f^{ij} = f^{ji}$. The factor 2's in the last two terms are added only to simplify our calculations.

To prove Theorem 6 in this model case, we shall seek a small perturbation g_ε of the metric g such that in (M, g_ε) there exists a Nikodym-type set of dimension 2. For any small $\varepsilon > 0$, we set $\alpha = \alpha_\varepsilon = \varepsilon\rho$, where ρ is a fixed function in $C^\infty(\mathbb{R})$ such that

- $\rho(t) = 0$ for $t \leq 0$,
- $\rho(t) > 0$ for $t > 0$, and
- $|\rho(t)| < 1$ for any $t \in \mathbb{R}$.

Let $U \subset\subset U_1$ be a relative compact subset and let $\varphi(x) \in C_0^\infty(U_1)$ be a compactly supported bump function such that $\varphi|_U = 1$. Define

$$g_\varepsilon = g^{ij} dp_i dp_j + 2\varphi(x)\alpha_\varepsilon(x_1) dp_2 dp_3.$$

When ε is sufficiently small, g_ε is still positive definite and hence a Riemannian cometric on M. In the following lemma, we will investigate the geodesics in U with respect to g_ε. After a translation in x_1, we may assume $U = (-\delta_0, \delta_0)^3$.

Lemma 1 *For $x \in U$, let*

$$H(x, p) = \frac{1}{2}\left(p_1^2 + g^{22}(x_1, x_2)p_2^2 + p_3^2 + x_3 \sum_{i=1}^{3} 2h^{i3} p_i p_3 + x_3^2 \sum_{1 \leq i, j \leq 2} 2 f^{ij} p_i p_j\right) + \alpha(x_1) p_2 p_3$$

be the Hamiltonian associated to the cometric g_ε. Given $\theta \in (-1, 1)$, $a \in (-\delta_0, \delta_0)$, we denote the unique geodesic with initial position $x(0) = (0, a, 0)$ and initial momentum $p(0) = (\sqrt{1-\theta^2}, \theta, 0)$ as $x(a, \theta; s)$; then we have

$$x(a, 0; s) = (s, a, 0).$$

Furthermore, when $\theta = 0$ and $a = 0$, there exists $0 < s < \delta_0$, such that the Jacobian determinant of the map

$$(a, \theta, s) \to x(a, \theta, s)$$

is nonzero.

Proof To verify that the curves $x(a, 0; s) = (s, a, 0)$ are geodesics for our metric, we can look at the Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}s} = -\dfrac{\partial H}{\partial x}, \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\partial H}{\partial p}. \end{cases}$$

with initial data $x(0) = (0, a, 0)$, $p(0) = (1, 0, 0)$. This system generates the geodesic flow over the cotangent bundle; see, e.g., [9]. By a straightforward calculation, the Hamiltonian system becomes

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}s} = p_1 + x_3^2 \displaystyle\sum_{i=1}^{2} 2f^{i1} p_i + x_3 h^{13} p_3, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}s} = \tilde{g}^{22}(x_1, x_2) p_2 + x_3^2 \displaystyle\sum_{i=1}^{2} 2f^{i2} p_i + x_3 h^{23} p_3 + \alpha(x_1) p_3, \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}s} = p_3 + x_3 \displaystyle\sum_{i=1}^{3} h^{i3} p_i + x_3 h^{33} p_3 + \alpha(x_1) p_2, \\ \dfrac{\mathrm{d}p_1}{\mathrm{d}s} = -\dfrac{1}{2} \tilde{g}^{22}_{x_1}(x_1, x_2) p_2^2 - x_3^2 \displaystyle\sum_{1 \le i, j \le 2} f^{ij}_{x_1} p_i p_j - x_3 \sum_{i=1}^{3} h^{i3}_{x_1} p_i p_3 - \alpha'(x_1) p_2 p_3, \\ \dfrac{\mathrm{d}p_2}{\mathrm{d}s} = -\dfrac{1}{2} \tilde{g}^{22}_{x_2}(x_1, x_2) p_2^2 - x_3^2 \displaystyle\sum_{1 \le i, j \le 2} f^{ij}_{x_2} p_i p_j - x_3 \sum_{i=1}^{3} h^{i3}_{x_2} p_i p_3, \\ \dfrac{\mathrm{d}p_3}{\mathrm{d}s} = -x_3 \displaystyle\sum_{1 \le i, j \le 2} 2f^{ij} p_i p_j - x_3^2 \sum_{1 \le i, j \le 2} f^{ij}_{x_3} p_i p_j - \sum_{i=1}^{3} h^{i3} p_i p_3 - x_3 \sum_{i=1}^{3} h^{i3}_{x_3} p_i p_3. \end{cases} \tag{2.1}$$

It is then straightforward to check that

$$\begin{cases} x(a, 0; s) = (s, a, 0) \\ p(a, 0; s) = (1, 0, 0) \end{cases} \tag{2.2}$$

solve the system (2.1).

Now we verify the second part of Lemma 1. Note that

$$x(a, 0; s) = (x_1(a, 0; s), x_2(a, 0; s), x_3(a, 0; s)) = (s, a, 0).$$

Thus, when $\theta = 0$, the Jacobian is given by

$$|J| = \left| \frac{\partial x(a, \theta, s)}{\partial(\theta, s, a)} \right| = \left| \begin{bmatrix} \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial a} \\ \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial a} \\ \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial s} & \frac{\partial x_3}{\partial a} \end{bmatrix} \right| = \left| \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ \frac{\partial x_3}{\partial \theta} & 0 & 0 \end{bmatrix} \right| = \left| \frac{\partial x_3}{\partial \theta} \right|.$$

Now our goal is to find some $s > 0$ such that $\frac{\partial x_3}{\partial \theta} \neq 0$. By taking $\frac{\partial}{\partial \theta}$ on the third equation in (2.1) and restricting to $\theta = a = 0$, with the help of (2.2), we obtain

$$\frac{\partial}{\partial s} \frac{\partial x_3}{\partial \theta} = \frac{\partial p_3}{\partial \theta} + h^{13}(s, 0, 0) \frac{\partial x_3}{\partial \theta} + \alpha(s) \frac{\partial p_2}{\partial \theta}.$$

As $\frac{\partial p_2}{\partial \theta}$ and $\frac{\partial p_3}{\partial \theta}$ on the right-hand side above are unknown, we also take $\frac{\partial}{\partial \theta}$ on the last two equations in (2.1) and restrict to $\theta = a = 0$. Similarly, we obtain

$$\begin{cases} \frac{\partial}{\partial s}\frac{\partial p_2}{\partial \theta} = 0, \\ \frac{\partial}{\partial s}\frac{\partial p_3}{\partial \theta} = -2f^{11}(s,0,0)\frac{\partial x_3}{\partial \theta} - h^{13}(s,0,0)\frac{\partial p_3}{\partial \theta}. \end{cases}$$

The initial data $p_2(0) = \theta$ yield that $\frac{\partial p_2}{\partial \theta} = 1$ for any s, when $\theta = a = 0$. Hence, $\frac{\partial x_3}{\partial \theta}(0, 0; s)$ and $\frac{\partial p_3}{\partial \theta}(0, 0; s)$ satisfy the following ODE system:

$$\begin{cases} \frac{\partial}{\partial s}\frac{\partial x_3}{\partial \theta} = \frac{\partial p_3}{\partial \theta} + h^{13}(s,0,0)\frac{\partial x_3}{\partial \theta} + \alpha(s), \\ \frac{\partial}{\partial s}\frac{\partial p_3}{\partial \theta} = -2f^{11}(s,0,0)\frac{\partial x_3}{\partial \theta} - h^{13}(s,0,0)\frac{\partial p_3}{\partial \theta}. \end{cases} \tag{2.3}$$

Then we can argue that if $\frac{\partial x_3}{\partial \theta}$ is identically zero on a small interval $(0, \epsilon_0)$, then $\frac{\partial p_3}{\partial \theta}$ has to be identically zero by the second equation above and the fact that $\frac{\partial p_3}{\partial \theta}\big|_{s=0} = 0$. However, this leads to a contradiction since $\alpha(s) \neq 0$ for $s > 0$. □

Now we can use Lemma 1 to prove Theorem 6 for the 3-dimensional case. Notice that in our coordinate system g_ε agrees with g when $x_1 \leq 0$. Moreover, since $\alpha_\varepsilon(x_1) > 0$ when $x_1 > 0$, if we choose the point $(s, 0, 0)$ with $s > 0$ as in Lemma 1, then there is an open neighborhood $\mathcal{N}^*$ of the point $(s, 0, 0)$, such that if $x \in \mathcal{N}^*$ then there is a unique geodesic γ_x containing x and having the property that when $x_1 \leq 0$, γ_x is contained in the submanifold N. If we let

$$f^\delta(x) = \begin{cases} 1, & \text{if } x \in U,\ x_1 < 0, \text{ and } |x_3| < \delta, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

then it follows that for small fixed $x_1 > 0$, $(f^\delta)^{**}_\delta$ must be bounded from below by a positive constant on $\mathcal{N}^*$; therefore, for any $p,\ q \geq 1$

$$\|(f^\delta)^{**}_\delta\|_{L^q(\mathcal{N}^*)}/\|f^\delta\|_{L^p} \geq c_0\delta^{-1/p},$$

for some $c_0 > 0$ depending on $\mathcal{N}^*$. Since

$$3/p - 1 < 1/p \text{ when } p > 2,$$

we conclude that

$$\|f^{**}_\delta\|_{L^q(M^3, g_\varepsilon)} \leq C_\epsilon \delta^{1-\frac{3}{p}-\epsilon}\|f\|_{L^p(M^3, g_\varepsilon)}$$

cannot hold for $p > 2$.

3 Instability of Nikodym Bounds in Dimension $2d+1$

We work on a $2d+1$-dimensional Riemannian manifold (M^{2d+1}, g) with a totally geodesic, $d+1$-dimensional submanifold N^{d+1}. In a local coordinate chart $(U_1, (x_1, x_2, \cdots, x_{2d+1}))$, without loss of generality, we may assume $N \cap U_1 = \{x' = \vec{0}\}$ where $x' = (x_{d+2}, x_{d+3}, \cdots, x_{2d+1})$, and assume $\{\frac{\partial}{\partial x_{d+2}}, \frac{\partial}{\partial x_{d+3}}, \cdots, \frac{\partial}{\partial x_{2d+1}}\}$ form an orthonormal basis of the normal bundle of $N \cap U_1$. Further, we assume that $(x_1, x_2, \cdots, x_{d+1})$ give polar coordinate on $N \cap U_1$ around some point. Thus, when $x' = 0$, the cometric can be written as

$$\mathrm{d}s^2 = \mathrm{d}p_1^2 + \sum_{2\le i,j\le d+1} \tilde{g}^{ij}(x_1, \cdots, x_{d+1})\mathrm{d}p_i\mathrm{d}p_j + \sum_{d+2\le i\le 2d+1} \mathrm{d}p_i^2,$$

where $\tilde{g}^{ij}(x_1, \cdots, x_{d+1}) = g^{ij}(x_1, \cdots, x_{d+1}, 0, \cdots, 0)$. Indeed, one may use any coordinates that give the metric in the above form. Since $N \subset M$ is totally geodesic, the metric tensor must satisfy

$$\left.\frac{\partial g^{ij}}{\partial x_k}\right|_{x'=0} = 0, \quad \text{for } 1 \le i, j \le d+1,\ d+2 \le k \le 2d+1.$$

Therefore, by taking the Taylor expansion of each $g^{ij}(x)$ at $x' = 0$,

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}p_1^2 &+ \sum_{2\le i,j\le d+1} \tilde{g}^{ij}\mathrm{d}p_i\mathrm{d}p_j + \sum_{i=d+2}^{2d+1} \mathrm{d}p_i^2 \\ &+ \sum_{i=1}^{2d+1} \sum_{d+2\le k,l\le 2d+1} 2x_l h^{ikl}\mathrm{d}p_i\mathrm{d}p_k \\ &+ \sum_{1\le i,j\le d+1} \sum_{d+2\le k,l\le 2d+1} 2x_k x_l f^{ijkl}\mathrm{d}p_i\mathrm{d}p_j, \end{aligned} \tag{3.1}$$

where h^{ikl}, f^{ijkl} are certain smooth functions of variable $x \in U_1$ and $f^{ijkl} = f^{jikl} = f^{ijlk}$.

Our goal is to find a small perturbation g_ε of the metric g such that in (M, g_ε) there exists a Nikodym-type set of dimension $d+1$. For any small $\varepsilon > 0$, we set $\alpha = \alpha_\varepsilon = \varepsilon\rho$, where ρ is a fixed function in $C^\infty(\mathbb{R})$ such that

- $\rho(t) = 0$ for $t \le 0$,
- $\rho(t) > 0$ for $t > 0$, and
- $|\rho(t)| < 1$ for any $t \in \mathbb{R}$.

Let $U \subset\subset U_1$ be a relative compact subset and let $\varphi(x) \in C_0^\infty(U_1)$ be a compactly supported bump function such that $\varphi|_U = 1$. Define

$$g_\varepsilon = g^{ij}dp_i dp_j + 2\varphi(x)\alpha_\varepsilon(x_1)\sum_{i=2}^{d+1} p_i p_{d+i}. \tag{3.2}$$

When ε is sufficiently small, g_ε is still positive definite and hence a Riemannian cometric on M. In the following lemma, we will investigate the geodesics in U with respect to g_ε. After a translation in x_1, we may assume $U = (-\delta_0, \delta_0)^{2d+1}$ for some positive constant δ_0.

Lemma 2 *For $x \in U$, let*

$$
\begin{aligned}
H(x,p) = {} & \frac{1}{2}p_1^2 + \frac{1}{2}\sum_{2\le i,j\le d+1} \tilde{g}^{ij} p_i p_j + \frac{1}{2}\sum_{i=d+2}^{2d+1} p_i^2 \\
& + \sum_{i=1}^{2d+1} \sum_{d+2\le k,l\le 2d+1} x_l h^{ikl} p_i p_k + \sum_{1\le i,j\le d+1,\ d+2\le k,l\le 2d+1} x_k x_l f^{ijkl} p_i p_j \\
& + \alpha(x_1)\sum_{i=2}^{d+1} p_i p_{d+i}
\end{aligned}
$$

be the Hamiltonian associated to the cometric g_ε. Let $\vec{0}$ be the zero row vector in $\mathbb{R}^d$. Given $\theta = (\theta_2, \theta_3, \ldots, \theta_{d+1}) \in \mathbb{R}^d$ such that $|\theta|^2 = \sum_{i=2}^{d+1} \theta_i^2 < 1$ and $a = (a_2, a_3, \ldots, a_{d+1}) \in (-\delta_0, \delta_0)^d$, we denote the unique geodesic with initial position $x(0) = (0, a, \vec{0})$ and initial momentum $p(0) = (\sqrt{1-|\theta|^2}, \theta, \vec{0})$ as $x(a, \theta; s)$; then we have

$$x(a, \vec{0}; s) = (s, a, \vec{0}).$$

Furthermore, when $\theta = \vec{0}$ and $a = \vec{0}$, the absolute value of the Jacobian determinant of the map

$$(a, \theta, s) \to x(a, \theta, s)$$

is positive for s in some sufficiently small interval $(0, \epsilon_0)$.

Proof To verify that the curves $x(a, \vec{0}; s) = (s, a, \vec{0})$ are geodesics for our metric, we can look at the Hamiltonian system

$$
\begin{cases}
\dfrac{\mathrm{d}p}{\mathrm{d}s} = -\dfrac{\partial H}{\partial x}, \\
\dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\partial H}{\partial p}.
\end{cases}
$$

with initial data $x(0) = (0, \vec{a}, \vec{0})$, $p(0) = (1, \vec{0}, \vec{0})$. This system generates the geodesic flow over the cotangent bundle. In order to avoid tedium, we will adopt the Einstein summation convention. We assume that i, j, k, l, i', j', n are the indices within the following ranges: $1 \le i, j \le d+1$, $d+2 \le k, l \le 2d+1$, $2 \le i', j' \le d+1$, and $1 \le n \le 2d+1$. By a straightforward calculation, the Hamiltonian system becomes

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}s} = p_1 + x_l h^{1kl} p_k + x_k x_l f^{1jkl} p_j, & \\ \frac{\mathrm{d}x_m}{\mathrm{d}s} = \frac{1}{2}\tilde{g}^{mj'} p_{j'} + x_l h^{mkl} p_k + 2x_k x_l f^{mjkl} p_j + \alpha(x_1) p_{d+m}, & 2 \le m \le d+1, \\ \frac{\mathrm{d}x_m}{\mathrm{d}s} = p_m + x_l h^{mkl} p_k + x_l h^{nml} p_n + \alpha(x_1) p_{m-d}, & m \ge d+2, \\ \frac{\mathrm{d}p_1}{\mathrm{d}s} = -\frac{1}{2}\tilde{g}^{i'j'}_{x_1} p_{i'} p_{j'} - x_l h^{nkl}_{x_1} p_n p_k - x_k x_l f^{ijkl}_{x_1} p_i p_j - \alpha'(x_1) p_{i'} p_{d+i'}, & \\ \frac{\mathrm{d}p_m}{\mathrm{d}s} = -\frac{1}{2}\tilde{g}^{i'j'}_{x_m} p_{i'} p_{j'} - x_l h^{nkl}_{x_m} p_n p_k - x_k x_l f^{ijkl}_{x_m} p_i p_j, & 2 \le m \le d+1, \\ \frac{\mathrm{d}p_m}{\mathrm{d}s} = -h^{nkm} p_n p_k - x_l h^{nkl}_{x_m} p_n p_k - 2x_l f^{ijml} p_i p_j - x_k x_l f^{ijkl}_{x_m} p_i p_j, & m \ge d+2. \end{cases} \tag{3.3}$$

It is not hard to see that

$$\begin{cases} x(a, \vec{0}; s) = (s, a, \vec{0}) \\ p(a, \vec{0}; s) = (1, \vec{0}, \vec{0}) \end{cases} \tag{3.4}$$

solve the system (3.3).

Now we verify the second part of Lemma 2. Since $x(a, \vec{0}; s) = (s, a, \vec{0})$, when $\theta = \vec{0}$, the Jacobian is given by

$$|J| = \left| \begin{bmatrix} \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial a} \\ \vdots & \vdots & \vdots \\ \frac{\partial x_{2d+1}}{\partial \theta} & \frac{\partial x_{2d+1}}{\partial s} & \frac{\partial x_{2d+1}}{\partial a} \end{bmatrix} \right| = \left| \begin{bmatrix} * & 1 & \vec{0} \\ * & \vec{0}^\intercal & I_d \\ \left(\frac{\partial x_k}{\partial \theta_{j'}}\right)_{kj'} & \vec{0}^\intercal & \mathbf{0} \end{bmatrix} \right| = \left| \left(\frac{\partial x_k}{\partial \theta_{j'}}\right)_{kj'} \right|,$$

where $\mathbf{0}$ represents the zero $d \times d$ matrix. Now our goal is to calculate $\left(\frac{\partial x_k}{\partial \theta_{j'}}\right)_{kj'}$. By taking the gradient $\frac{\partial}{\partial \theta}$ of the equations with $\frac{dx_m}{ds}$ for $d+2 \le m \le 2d+1$ on the left-hand side in (3.3) and restricting to $\theta = a = 0$, with the help of (3.4), we obtain

$$\frac{\partial}{\partial s}\frac{\partial x_m}{\partial \theta} = \frac{\partial p_m}{\partial \theta} + \frac{\partial x_l}{\partial \theta} h^{1ml}(s, \vec{0}, \vec{0}) + \alpha(s)\frac{\partial p_{m-d}}{\partial \theta}, \quad \text{when } d+2 \le m \le 2d+1.$$

As the $\frac{\partial p_m}{\partial \theta}$ in the right-hand side above is unknown, we also take $\frac{\partial}{\partial \theta}$ on the last two lines of equations in (3.3) and restrict to $\theta = a = 0$. Similarly, we obtain

$$\begin{cases} \frac{\partial}{\partial s}\frac{\partial p_m}{\partial \theta} = 0, & \text{when } 2 \le m \le d+1, \\ \frac{\partial}{\partial s}\frac{\partial p_m}{\partial \theta} = -h^{1km}(s, \vec{0}, \vec{0})\frac{\partial p_k}{\partial \theta} - 2f^{11ml}(s, \vec{0}, \vec{0})\frac{\partial x_l}{\partial \theta}, & \text{when } d+2 \le m \le 2d+1. \end{cases}$$

For $1 \le i \le d$, let e_i be the unit vector in $\mathbb{R}^d$ whose i-th component is 1. The initial data $(p_2(0), \cdots, p_{d+1}(0)) = \theta$ yield that $\frac{\partial p_m}{\partial \theta} = e_{m-1}$ for $2 \le m \le d+1$ when $\theta = a = 0$. And thus $\frac{\partial x_m}{\partial \theta}(s, \vec{0}, \vec{0})$ and $\frac{\partial p_m}{\partial \theta}(s, \vec{0}, \vec{0})$ for $d+2 \le m \le 2d+1$ satisfy the following ODE system:

$$\begin{cases} \frac{\partial}{\partial s}\frac{\partial x_m}{\partial \theta} = \frac{\partial p_m}{\partial \theta} + h^{1ml}(s,\vec{0},\vec{0})\frac{\partial x_l}{\partial \theta} + \alpha(s)e_{m-d-1}, \\ \frac{\partial}{\partial s}\frac{\partial p_m}{\partial \theta} = -h^{1km}(s,\vec{0},\vec{0})\frac{\partial p_k}{\partial \theta} - 2f^{11ml}(s,\vec{0},\vec{0})\frac{\partial x_l}{\partial \theta}. \end{cases} \tag{3.5}$$

We shall then need the following ODE lemma.

Lemma 3 *For the ODE system*

$$\begin{cases} \dot{\xi}(s) = A(s)\xi(s) + \alpha(s)\begin{pmatrix} I_d \\ \mathbf{0} \end{pmatrix}, \\ \xi(0) = 0, \end{cases} \tag{3.6}$$

where

$$\xi(s) = \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix}$$

is a $2d \times d$ *matrix and* $A(s)$ *is a fixed* $2d \times 2d$ *matrix with smooth entries. The solution* $\xi(s)$ *then satisfies that*

$$(\det \xi_{11})(s) > \left(\frac{1}{2}\int_0^s \alpha(t)dt\right)^d$$

for s *in some sufficiently small interval* $(0, \epsilon_0)$.

Proof Let $Z(s)$ be the fundamental matrix for the homogeneous ODE system $\dot{\xi} = A\xi$. That is to say, $Z(s)$ is a $2d \times 2d$ invertible matrix that satisfies

$$\begin{cases} \dot{Z} = AZ, \\ Z(0) = I_{2d}. \end{cases} \tag{3.7}$$

Let $\eta(s)$ be a $2d \times d$ matrix-valued function such that $\xi = Z \cdot \eta$. Using this substitution, (3.6) simplifies to

$$\begin{cases} \dot{\eta}(s) = Z^{-1}(s)\alpha(s)\begin{pmatrix} I_d \\ \mathbf{0} \end{pmatrix}, \\ \eta(0) = 0. \end{cases}$$

We now integrate both sides of the equation and obtain the solution

$$\eta(s) = \int_0^s \alpha(t)Z^{-1}(t)\begin{pmatrix} I_d \\ \mathbf{0} \end{pmatrix} dt.$$

Let

$$\eta = \begin{pmatrix} \eta_{11} \\ \eta_{21} \end{pmatrix}, \quad \text{and } Z^{-1}(t) = \begin{pmatrix} w(t) & * \\ * & * \end{pmatrix},$$

where $w(t)$ is the up-left $d\times d$ matrix block. Thus $\eta_{11}(s)=\int_0^s\alpha(t)w(t)dt$. We denote $w(t)=(w_1(t),w_2(t),\cdots,w_d(t))$. By a straightforward calculation, we obtain

$$\begin{aligned}&\det\eta_{11}(s)\\&\quad=\int_0^s\cdots\int_0^s\alpha(t_1)\alpha(t_2)\cdots\alpha(t_d)\det\left(w_1(t_1),w_2(t_2),\cdots,w_d(t_d)\right)\mathrm{d}t_1\cdots\mathrm{d}t_d.\end{aligned}$$

As a fundamental matrix, Z satisfies that $Z(0)=I_{2d}$, whence $Z^{-1}(t)=I_{2d}+O(t)$ as $t\to0$. In particular, $w(t)=I_d+O(t)$. Recall that $\alpha(t)>0$ for any $t\in\mathbb{R}^+$. If we use α_{-1} to denote the anti-derivative of α, then clearly $\det\eta_{11}(s)>\left(\frac{1}{2}\alpha_{-1}(s)\right)^d>0$ for s in some sufficiently small interval $(0,\epsilon_0)$. As $\xi=Z(s)\cdot\eta=(I_{2d}+O(s))\cdot\eta$, we obtain $\xi_{11}=\eta_{11}+O(s)$. So the result follows by possibly choosing a smaller positive ϵ_0. □

Now we may apply the above lemma to finish the proof of Lemma 2. If we denote the $2d\times d$ matrix

$$\xi(s)=\begin{pmatrix}\xi_{11}\\\xi_{21}\end{pmatrix}=\begin{pmatrix}\left(\dfrac{\partial x_k}{\partial\theta_{j'}}\right)_{kj'}\\\left(\dfrac{\partial p_k}{\partial\theta_{j'}}\right)_{kj'}\end{pmatrix},$$

and the $2d\times 2d$ matrix

$$A(s)=\begin{pmatrix}\left(h^{1kl}\right)_{kl}&I_d\\-\left(2f^{11kl}\right)_{kl}&-\left(h^{1lk}\right)_{kl}\end{pmatrix},$$

then (3.6) is satisfied and we have $\det\left(\dfrac{\partial x_k}{\partial\theta_{j'}}\right)_{kj'}=\det(\xi_{11})>\left(\frac{1}{2}\int_0^s\alpha(t)dt\right)^d$, completing the proof of Lemma 2. □

Now we apply Lemma 2 to prove Theorem 6 for odd dimensions $2d+1$. Notice that in our coordinate system g_ε agrees with g when $x_1\le0$. Moreover, since $\alpha_\varepsilon(x_1)>0$ when $x_1>0$, if we fix a point $(s_0,\vec{0},\vec{0})$ with $s_0>0$, Lemma 2 implies that there is an open neighborhood $\mathcal{N}^*$ of the point $(s_0,\vec{0},\vec{0})$, such that if $x\in\mathcal{N}^*$, there is a unique geodesic γ_x containing x and having the property that when $x_1\le0$, γ_x is contained in the submanifold $\{x:x'=\vec{0}\}$. If we let

$$f^\delta(x)=\begin{cases}1,&x\in U,\ x_1<0,\ \text{and}|x'|=|(x_{d+2},\ldots,x_{2d+1})|<\delta,\\0,&\text{otherwise},\end{cases}\tag{3.8}$$

then it follows that for small fixed $x_1>0$, $(f^\delta)^{**}_\delta$ must be bounded from below by a positive constant on $\mathcal{N}^*$; therefore, for any $q,p\ge1$

$$\|(f^\delta)^{**}_\delta\|_{L^q(\mathcal{N}^*)}/\|f^\delta\|_{L^p}\ge c_0\delta^{-d/p},$$

for some $c_0 > 0$ depending on $\mathcal{N}^*$. Since

$$(2d+1)/p - 1 < d/p \text{ when } p > d+1,$$

we conclude that

$$\|f_\delta^{**}\|_{L^q(M^{2d+1},g_\varepsilon)} \leq C_\epsilon \delta^{1-\frac{2d+1}{p}-\epsilon} \|f\|_{L^p(M^{2d+1},g_\varepsilon)}$$

cannot hold for $p > d+1$.

4 Instability of Nikodym Bounds in Dimension 2*d*

In this section, we work on a $2d$-dimensional Riemannian manifold (M^{2d}, g) with a totally geodesic, $d+1$-dimensional submanifold N^{d+1}. As the construction is similar as in the odd dimensional case, we will only indicate the differences here. Using the same simplifications as before, like in (3.1), we can write the Riemannian cometric as

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}p_1^2 &+ \sum_{2\leq i,j\leq d+1} \tilde{g}^{ij}\mathrm{d}p_i\mathrm{d}p_j + \sum_{i=d+2}^{2d} \mathrm{d}p_i^2 \\ &+ \sum_{i=1}^{2d} \sum_{d+2\leq k,l\leq 2d} 2x_l h^{ikl}\mathrm{d}p_i\mathrm{d}p_k + \sum_{1\leq i,j\leq d+1} \sum_{d+2\leq k,l\leq 2d} 2x_k x_l f^{ijkl}\mathrm{d}p_i\mathrm{d}p_j, \end{aligned}$$

where h^{ikl}, f^{ijkl} are certain smooth functions of variable $x \in U_1$ and $f^{ijkl} = f^{jikl} = f^{ijlk}$. In this case, instead of (3.2), we take

$$g_\varepsilon = g^{ij}\mathrm{d}p_i\mathrm{d}p_j + 2\alpha_\varepsilon(x_1)\sum_{i=3}^{d+1}\mathrm{d}p_i\mathrm{d}p_{d-1+i}$$

to be the perturbed cometric. Therefore, by the proof of Lemma 2, given $\theta = (\theta_2, \theta_3, \ldots, \theta_d) \in \mathbb{R}^{d-1}$ with $|\theta|^2 = \sum_{i=2}^d \theta_i^2 < 1$ and $a = (a_2, a_3, \ldots, a_{d+1}) \in (-\delta_0, \delta_0)^d$, we denote the unique geodesic with initial position $x(0) = (0, a, \vec{0})$ and initial momentum $p(0) = (\sqrt{1-|\theta|^2}, \theta, \vec{0})$ as $x(a, \theta; s)$; then we have

$$x(a, \vec{0}; s) = (s, a, \vec{0}).$$

Furthermore, the absolute value of the Jacobian determinant of the map

$$(a, \theta, s) \to x(a, \theta, s)$$

is positive for $\theta = \vec{0}$, $a = \vec{0}$ and s in some sufficiently small interval $(0, \delta)$. Consequently, if we fix a point $(s_0, \vec{0}, \vec{0})$ with $s_0 > 0$, there is an open neighborhood $\mathcal{N}^*$ of the point $(s_0, \vec{0}, \vec{0})$, such that if $x \in \mathcal{N}^*$, there is a unique geodesic γ_x through x

and lying in the submanifold $N \cap U = \{x : x' = (x_{d+2}, x_{d+3}, \dots, x_{2d}) = \vec{0}\}$ when $x_1 \le 0$. As a result, if we put

$$f(x) = \begin{cases} 1, & \text{if } x \in U,\ x_1 < 0 \text{ and } |x'| = |(x_{d+2}, \dots, x_{2d})| < \delta, \\ 0, & \text{otherwise,} \end{cases}$$

it follows that for small fixed $x_1 > 0$, f_δ^{**} must be bounded from below by a positive constant on $\mathcal{N}^*$; therefore, for any $q, p \ge 1$

$$\|f_\delta^{**}\|_{L^q(\mathcal{N}^*)} / \|f\|_{L^p} \ge c_0 \delta^{-(d-1)/p},$$

for some $c_0 > 0$ depending on $\mathcal{N}^*$. Since

$$(2d)/p - 1 < (d-1)/p \text{ when } p > d+1,$$

we conclude that

$$\|f_\delta^{**}\|_{L^q(M^{2d}, g_\varepsilon)} \le C_\epsilon \delta^{1 - \frac{2d}{p} - \epsilon} \|f\|_{L^p(M^{2d}, g_\varepsilon)}$$

cannot hold for $p > d+1$.

5 Instability for Oscillatory Integral Bounds

Following the same strategy as in [10, p. 290], one may easily derive the following instability results for the related oscillatory integrals.

We consider the oscillatory integral operator

$$S_\lambda^g f(x) = \int_M e^{i\lambda d_g(x,y)} a(x, y) f(y)\, dy, \tag{5.1}$$

where $a(x, y) \in C_0^\infty$ vanishes near the diagonal. Then we have the following:

Corollary 1 *Given (M^d, g) of dimension $d \ge 3$ such that M^d has a local totally geodesic submanifold of dimension $\lceil \frac{d+1}{2} \rceil$, then for every $\varepsilon > 0$, such that for any k, $\|g^{ij} - g_\varepsilon^{ij}\|_{C^k} \le B_k \varepsilon$, for some positive constant B_k, and over (M^d, g_ε), the estimate*

$$\|S_\lambda^{g_\varepsilon} f\|_{L^q(M^d, g_\varepsilon)} \le C_\epsilon \lambda^{-\frac{d}{p} + \epsilon} \|f\|_{L^p(M^d, g_\varepsilon)}$$

fails to hold if

$$p > \begin{cases} \dfrac{2(3d+1)}{3(d-1)}, & \text{when } d \ge 3 \text{ is odd}, \\ \dfrac{2(3d+2)}{3d-2}, & \text{when } d \ge 4 \text{ is even}. \end{cases} \tag{5.2}$$

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