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Bach-Flat Kähler Surfaces

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Abstract A Riemannian metric on a compact 4-manifold is said to be Bach-flat if it is a critical point for the L^2-norm of the Weyl curvature. When the Riemannian 4-manifold in question is a Kähler surface, we provide a rough classification of solutions, followed by detailed results regarding each case in the classification. The most mysterious case prominently involves 3-dimensional CR manifolds.

Keywords Riemannian 4-manifold · Bach tensor · Kähler metric · Weyl curvature · Einstein metric · Scalar curvature

1 Introduction

On a smooth connected compact 4-manifold M, the *Weyl functional*

$$\mathcal{W}(g) := \int_M \|W\|_g^2 \, d\mu_g, \tag{1}$$

quantifies the deviation of a Riemannian metric g from local conformal flatness. Here W denotes the *Weyl tensor* of g, which is the piece of the Riemann curvature of g complementary to the Ricci tensor, while the norm $\| \cdot \|_g$ and the volume form $d\mu_g$ in the integrand are those associated with the given metric g. The Weyl functional (1) is invariant not only under the action of the diffeomorphism group (via pull-backs),

In memoriam Gennadi Henkin.

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but also under the action of the smooth positive functions $f: M \to \mathbb{R}^+$ by conformal rescaling $g \rightsquigarrow fg$.

It is both natural and useful to study metrics that are critical points of the Weyl functional. The Euler–Lagrange equations for such a metric can be expressed [2,7] as $B = 0$, where the *Bach tensor* B is defined by

$$B_{ab} := \left(\nabla^c \nabla^d + \frac{1}{2} r^{cd}\right) W_{acbd},$$

so these critical metrics are said to be *Bach-flat*. For reasons reviewed in Sect. 2, every 4-dimensional conformally Einstein metric is Bach-flat, as is every anti-self-dual metric. Conversely, if the Bach-flat manifold (M^4, g) also happens to be *Kähler*, Derdziński [17, Prop. 4] discovered that the geometry of g must locally be of one of these two types near a generic point. Our purpose here is to sharpen this observation into a global classification of solutions. Our main result is the following:

Theorem A *Let (M^4, g, J) be a compact connected Bach-flat Kähler surface. Then g is either anti-self-dual or else is conformally Einstein on an open dense subset of M. Moreover, the geometric behavior of (M^4, g, J) fits into exactly one slot of the following classification scheme:*

I. *The scalar curvature satisfies $s > 0$ everywhere. In this case, there are just two possibilities:*
   (a) *(M, g, J) is Kähler–Einstein, with Einstein constant $\lambda > 0$; or else*
   (b) *$(M, s^{-2}g)$ is Einstein, with $\lambda > 0$, but has holonomy* $\mathbf{SO}(4)$.
II. *The scalar curvature satisfies $s \equiv 0$. There are again two possibilities:*
   (a) *(M, g, J) is Kähler–Einstein, with $\lambda = 0$; or else*
   (b) *(M, J) is a (possibly blown-up) ruled surface, and g is anti-self-dual, but M is not even homeomorphic to an Einstein manifold.*
III. *The scalar curvature satisfies $s < 0$ somewhere. Then there are again exactly two possibilities:*
   (a) *(M, g, J) is Kähler–Einstein, with $\lambda < 0$; or else*
   (b) *(M, J) is a (possibly blown-up) ruled surface, and s vanishes exactly along a smooth connected totally umbilic hypersurface $\mathcal{Z}^3 \subset M^4$. Moreover, $M - \mathcal{Z}$ has precisely two connected components, and on both of these $h := s^{-2}g$ is a complete Einstein metric with $\lambda < 0$.*

A great deal is already known about most cases in this classification:

- A complex surface admits [15,40,47] a metric of class I iff it has $c_1 > 0$. Moreover, this metric is always unique [3,34] up to complex automorphisms and homotheties.
  - The relevant metric is of type I(a) iff the complex automorphism group is reductive.
  - Up to isometry and rescaling, there are [34] exactly two solutions of type I(b).
- Metrics of class II are generally called scalar-flat Kähler metrics. These are exactly the Kähler metrics that are anti-self-dual, in the sense that the self-dual Weyl tensor W_+ vanishes identically.

  - The metrics of type II(a) are often called Calabi–Yau metrics. A Kähler-type complex surface admits such metrics [52] iff it has $c_1 = 0$ mod torsion. This happens exactly for the minimal complex surfaces of Kodaira dimension 0 for which b_1 is even. When a complex surface admits such a metric, there is then exactly one such metric in every Kähler class.
  - Any complex surface that admits a metric of type II(b) must be projective-algebraic and have Kodaira dimension $-\infty$; they all violate the Hitchin–Thorpe inequality, and most of them are non-minimal. Such solutions exist in great abundance [26,31,42]; in particular, any complex surface with b_1 even and Kodaira dimension $-\infty$ has blow-ups that admit metrics of this type.
- The two cases that make up class III are wildly different.
  - A complex surface admits a metric of type III(a) iff [1,51] it has $c_1 < 0$. This happens [4, Prop. VII.7.1] iff the complex surface is minimal, has Kodaira dimension 2, and contains no rational curve of self-intersection -2.
  - By contrast, any complex surface admitting a metric of type III(b) must have Kodaira dimension $-\infty$. Infinitely many solutions of this type are currently known [25,48]. However, the known solutions all display peculiar features that seem most likely to just be artifacts of the method of construction.

Our exposition begins, in Sect. 2, with a review of some key background facts that will help set the stage for our main results. We then develop the basic trichotomy of solutions in Sect. 3. The proof of Theorem A is then completed in Sect. 4 by carefully proving various auxiliary assertions regarding specific cases in our classification scheme. The article then concludes by proving some additional results about specific types of solutions, with a particular focus on interesting open problems.

2 The Weyl Functional

If (M, g) is any smooth compact oriented Riemannian 4-manifold, the Thom–Hirzebruch signature theorem $\mathbf{p_1} = 3\tau$ implies a Gauss–Bonnet-like integral formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|_g^2 - |W_+|_g^2 \right) d\mu_g \tag{2}$$

for the signature $\tau = b_+ - b_-$ of M. In this formula, $W_\pm = (W \pm \star W)/2$ denotes the self-dual (respectively, anti-self-dual) part of the Weyl curvature

$$W^{ab}{}_{cd} = \mathcal{R}^{ab}{}_{cd} - 2r^{[a}_{[c}\delta^{b]}_{d]} + \frac{s}{3}\delta^a_{[c}\delta^b_{d]}$$

of the given metric g, here expressed in terms of the Riemann curvature tensor $\mathcal{R}$, Ricci tensor r, and scalar curvature s. We remind the reader of the fundamental fact that $W^a{}_{bcd}$ is conformally invariant; this in particular explains the conformal invariance of the Weyl functional (1). But also notice that (2) allows one to re-express the

Weyl functional (1) as

$$\mathcal{W}(g) = -12\pi^2\tau(M) + 2\int_M |W_+|^2 \mathrm{d}\mu. \tag{3}$$

Now, for any smooth 1-parameter family of metrics

$$g_t := g + t\dot{g} + O(t^2)$$

the first variation of the Weyl functional is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g_t)\right|_{t=0} = -\int \dot{g}^{ab} B_{ab}\, \mathrm{d}\mu,$$

where [2,7] the *Bach tensor* B is given by

$$B_{ab} = \left(\nabla^c\nabla^d + \frac{1}{2}r^{cd}\right) W_{acbd} \tag{4}$$

Notice that the contracted Bianchi identity

$$\nabla^a W_{abcd} = \nabla_{[c}r_{d]b} + \tfrac{1}{6}g_{b[c}\nabla_{d]}s$$

implies that any Einstein metric satisfies the *Bach-flat* condition $B = 0$. However, since the conformal invariance of the Weyl functional also makes it clear that the Bach-flat condition is conformally invariant, it follows that any conformally Einstein 4-dimensional metric is automatically Bach-flat.

On the other hand, any oriented Riemannian 4-manifold satisfies the remarkable identity

$$\left(\nabla^a\nabla^b + \frac{1}{2}r^{ab}\right)(\star W)_{cabd} = 0,$$

which encodes the fact that the first variation of (2) is zero. This allows one to rewrite (4) as

$$B_{ab} = (2\nabla^c\nabla^d + r^{cd})(W_+)_{acbd}. \tag{5}$$

In particular, any metric with $W_+ \equiv 0$ is automatically Bach-flat. Indeed, Eq. (3) shows that such *anti-self-dual* metrics are actually minimizers of the Weyl functional, and so must satisfy the associated Euler–Lagrange equation $B = 0$.

The Bach tensor is automatically symmetric, trace-free, and divergence-free. This reflects the fact that $-B$ is the gradient of $\int |W|^2 \mathrm{d}\mu$, which is invariant under diffeomorphisms and rescalings. Since B must therefore be L^2-orthogonal to any tensor field of the form ug_{ab} or $\nabla_{(a}v_{b)}$, we have

$$B_{ab} = B_{ba}, \quad B_a{}^a = 0, \quad \nabla^a B_{ab} = 0 \tag{6}$$

for any 4-dimensional Riemannian metric.

We now narrow our discussion to the case of Kähler metrics. For any Kähler metric g on a complex surface (M, J), with the orientation induced by J, the self-dual Weyl tensor is given by

$$(W_+)_{ab}{}^{cd} = \frac{s}{12}\left[\omega_{ab}\omega^{cd} - \delta_a^{[c}\delta_b^{d]} + J_a{}^{[c}J_b{}^{d]}\right] \tag{7}$$

and so is completely determined by the scalar curvature and the Kähler form $\omega = g(J\cdot, \cdot)$. In particular, we therefore have

$$|W_+|^2 = \frac{s^2}{24}, \tag{8}$$

a key fact whose lack of conformal invariance ceases to seem paradoxical as soon as one recalls that the Kähler condition is not conformally invariant either. In conjunction, Eqs. (3) and (8) now tell us that any Bach-flat Kähler metric is a critical point of the Calabi functional

$$\mathcal{C}(\omega) = \int s^2 d\mu, \tag{9}$$

considered either as a functional on a fixed Kähler class $\Omega = [\omega]$ or on the entire space of Kähler metrics, with Ω allowed to vary. In particular, a conformally Einstein, Kähler metric g must be an extremal Kähler metric in the sense of Calabi [12]. One of several equivalent characterizations of an extremal metric is the requirement that $\xi := J\nabla s$ be a Killing field of g.

Plugging (7) into (5), we now obtain a concrete formula

$$B_{ab} = \frac{s}{6}\mathring{r}_{ab} + \frac{1}{4}J_a{}^c J_b{}^d \nabla_c\nabla_d s + \frac{1}{12}\nabla_a\nabla_b s + \frac{1}{12}g_{ab}\Delta s$$

for the Bach tensor of any Kähler metric, where $\mathring{r}$ denotes the trace-free part

$$\mathring{r}_{ab} = r_{ab} - \frac{s}{4}g_{ab}$$

of the Ricci curvature. Setting $J^*(B) = B(J\cdot, J\cdot)$, we next decompose the Bach tensor

$$B = B^{\boxplus} + B^{\boxminus}$$

into its J-invariant and J-anti-invariant parts, and observe that

$$\begin{aligned} B^{\boxplus} &:= \frac{1}{2}\left[B + J^*(B)\right] = \frac{1}{6}\left[s\mathring{r} + 2\,\mathrm{Hess}_0^{\boxplus}(s)\right] \\ B^{\boxminus} &:= \frac{1}{2}\left[B - J^*(B)\right] = \frac{1}{12}\left[\mathrm{Hess}(s) - J^*\,\mathrm{Hess}(s)\right]. \end{aligned} \tag{10}$$

Here Hess $= \nabla\nabla$ denotes the Hessian of a function, and $\text{Hess}_0^{\boxplus}$ is its trace-free, J-invariant part. Now notice that, since s is real-valued, $\text{Hess}(s) = J^* \,\text{Hess}(s)$ if and only if

$$\nabla_{\bar{\mu}}\nabla^{\nu}s = g^{\nu\bar{\lambda}}\nabla_{\bar{\mu}}\nabla_{\bar{\lambda}}s = 0,$$

and this is exactly Calabi's equation $\overline{\partial}\nabla^{1,0}s = 0$ for an extremal Kähler metric. Consequently, a Kähler metric g is extremal iff its Bach tensor B is J-invariant. When this happens, conditions (6) then tells us that $\psi = B(J\cdot, \cdot)$ is a harmonic anti-self-dual 2-form, and the symmetric tensors $g + tB$ are therefore J-compatible Kähler metrics for all small t. Since $\dot{g} = B$ for this variation of the metric, and since $-B$ is the gradient of $\mathcal{W}$, it therefore follows [15] that any critical point of the Calabi functional $\mathcal{C}$ on the space of all J-compatible Kähler metrics must actually be Bach-flat.

Revisiting (10) now reveals that a Kähler metric is Bach-flat iff it is extremal and satisfies

$$0 = s\mathring{r} + 2\,\text{Hess}_0(s). \tag{11}$$

However, the trace-free Ricci tensor $\mathring{r}$ always transforms [7] under conformal changes $g \rightsquigarrow u^2 g$ by

$$\mathring{r} \rightsquigarrow \hat{\mathring{r}} = \mathring{r} + 2u\,\text{Hess}_0(u^{-1}). \tag{12}$$

Thus, as was first pointed out by Derdziński [17], the peculiar conformal rescaling $h = s^{-2}g$ of a Bach-flat Kähler metric satisfies $\mathring{r} = 0$, and so is locally Einstein, on the (possibly empty) open set where $s \neq 0$. We will now begin to systematically explore the global ramifications of this observation.

3 The Basic Trichotomy

In this section, we will study the global behavior of the scalar curvature s on a compact Bach-flat Kähler surface. Our approach hinges on a general property of strictly extremal Kähler manifolds:

Lemma 1 *Let (M^4, g, J) be a compact connected extremal Kähler surface whose scalar curvature s is non-constant. Then $s : M \to \mathbb{R}$ is a generalized Morse function in the sense of Bott* [10]. *In other words, the locus where $\nabla s = 0$ is a disjoint union $\bigsqcup C_j \subset M$ of compact submanifolds, and the Hessian* $\text{Hess}(s) := \nabla\nabla s$ *is non-degenerate on the normal bundle $(TC_j)^{\perp}$ of each C_j. Moreover, each submanifold C_j is either a single point or a smooth compact connected complex curve.*

Proof Since s is non-constant, (M, g, J) is a *strictly* extremal Kähler manifold, and $\xi = J\nabla s$ is a *non-trivial* Killing field. The critical points of s are exactly the fixed points of the flow $\{\Phi_t : M \to M | t \in \mathbb{R}\}$ generated by ξ, and since the diffeomorphisms Φ_t are all isometries of (M, g), every connected component C_j of the fixed point set is [27] a totally geodesic submanifold.

At $p \in C_j$, let $v \in (T_pC_j)^{\perp}$ be a unit vector normal to C_j, and let $\gamma : \mathbb{R} \to M$ be the unit speed geodesic through $p = \gamma(0) \in C_j$ with initial tangent vector v.

Since $\xi = 0$ only at $\bigsqcup C_k$, we then have $\xi|_{\gamma(t)} \neq 0$ for all sufficiently small $t > 0$. However, since ξ is Killing, $\xi \circ \gamma$ is a Jacobi field along γ. Because $\xi|_{\gamma(0)} = 0$, we must therefore have $\nabla_{\gamma'(0)}\xi \neq 0$, as Jacobi's equation is a linear second-order ODE, and ξ would therefore vanish identically along γ if the initial value $(\xi|_p, (\nabla_v \xi)|_p)$ of this solution vanished. This shows that v cannot belong to the kernel of $v \mapsto (\text{Hess } s)(v, \cdot) = \omega(\cdot, \nabla_v \xi)$, and since v is an arbitrary unit normal vector, it follows that the restriction of the Hessian $\nabla\nabla s$ to the normal bundle $(TC_j)^\perp$ must be non-degenerate. This shows that $s : M \to \mathbb{R}$ is a Morse–Bott function, as claimed.

In particular, the tangent space TC_j of any component of the critical locus must exactly coincide with the kernel of the Hessian $\nabla\nabla s$ at any point. But since $\nabla^{1,0}s$ is a holomorphic vector field, this Hessian must be J-invariant. Hence $TC_j = \ker \text{Hess}(s)$ is also J-invariant, and $C_j \subset M$ is therefore a complex submanifold. Since s is non-constant by assumption, and since M is assumed to have complex dimension 2, the components C_j can only have complex dimension 0 or 1. Each component C_j of the critical locus is therefore either a single point or a totally geodesic compact complex curve. □

This immediately tells us something useful about the zero set

$$\mathcal{Z} := \{p \in M | s(p) = 0\}$$

of the scalar curvature.

Lemma 2 *Let (M, g, J) be a compact extremal Kähler manifold. If $s \not\equiv 0$, then the open subset $M - \mathcal{Z}$ is dense in M.*

Proof If s were a non-zero constant, $\mathcal{Z}$ would be empty, and there would be nothing to prove. We may thus assume from now on that s is non-constant. Lemma 1 then tells us that ∇s and $\nabla\nabla s$ can never vanish at the same point. In particular, if p is any point where $s(p) = 0$, Taylor's theorem with remainder allows us to construct a short embedded curve $\gamma : (-\varepsilon, \varepsilon) \to M$ through $p = \gamma(0)$ on which $s \circ \gamma$ vanishes only at the origin. Hence every point of $\mathcal{Z}$ belongs to the closure of $M - \mathcal{Z}$. This shows that $M - \mathcal{Z}$ is dense, as claimed. □

We now specialize to the case of *Bach-flat* Kähler surfaces. Since Eq. (10) tells us that these Kähler manifolds are in particular extremal, the above Lemmata therefore automatically apply. However, we have already observed that the equation $B = 0$ can be expressed as

$$0 = \mathring{r} + 2s^{-1} \text{ Hess}_0\, s$$

on the open set $M - \mathcal{Z}$ defined by $s \neq 0$, and, by Eq. (12), this is exactly equivalent to saying that the metric $h := s^{-2}g$ on $M - \mathcal{Z}$ satisfies $\mathring{r} = 0$. Since the doubly contracted Bianchi identity $2\nabla \cdot r = \nabla s$ implies that a 4-manifold with $\mathring{r} = 0$ must have locally constant scalar curvature, this means that the function κ defined by

$$\kappa = -6s\Delta s - 12|\nabla s|^2 + s^3 \tag{13}$$

is locally constant on $M - \mathcal{Z}$; indeed, on this open set

$$\kappa = s^3(6\Delta + s)s^{-1}$$

exactly represents the scalar curvature of the local Einstein metric $h = s^{-2}g$. On the other hand, since elliptic regularity implies [36] that any extremal Kähler metric is smooth with respect to the complex atlas, our definition (13) of κ certainly guarantees that it is a smooth function on all of M. These facts now allow us to deduce the following:

Lemma 3 *On any compact connected Bach-flat Kähler surface (M, g, J), the smooth function $\kappa : M \to \mathbb{R}$ defined by (13) is necessarily constant.*

Proof If $s \equiv 0$, Eq. (13) immediately tells us that $\kappa \equiv 0$, and we are done. We may therefore assume henceforth that $s \not\equiv 0$. Now notice that the smooth 1-form $d\kappa$ vanishes on the set $M - \mathcal{Z}$, since on this set κ is locally the scalar curvature of the Einstein metric $h = s^{-2}g$, and is therefore locally constant. But since $(M - \mathcal{Z}) \subset M$ is dense by Lemma 2, it therefore follows that $d\kappa \equiv 0$ by continuity. Integration on paths thus shows that κ is constant, as claimed. □

The sign of κ thus provides a basic trichotomy that will form the basis of our classification of these manifolds. However, the sign of κ also has a direct interpretation in terms of the behavior of the scalar curvature of the given Kähler metric:

Lemma 4 *On any compact connected Bach-flat Kähler surface (M, g, J), the minimum value $\min s$ of the scalar curvature of g has exactly the same sign (positive, negative, or zero) as the constant κ.*

Proof When the scalar curvature s is constant, (13) says $\kappa = s^3 = (\min s)^3$, so the claim obviously holds. Otherwise, s is non-constant, and Lemma 1 tells us that $\text{Hess}(s) \neq 0$ at any critical point of s. In particular, if $p \in M$ is a point where s achieves its minimum, $\Delta s := -\nabla^a \nabla_a s < 0$ at p, since we now know that $\text{Hess}\, s = \nabla\nabla s$ must be positive semi-definite and non-zero at a minimum. However, evaluation of Eq. (13) at p tells us that

$$\kappa = s(p)\left[s^2 - 6\Delta s\right](p),$$

since $|\nabla s|^2(p) = 0$. Since $[s^2 - 6\Delta s](p) > 0$, this shows that κ and $s(p) = \min s$ must have the same sign, and the result therefore follows. □

Our next result leads to a complete understanding of the $\kappa = 0$ case.

Proposition 1 *A compact connected Kähler surface (M, g, J) is Bach-flat and has $\kappa = 0$ if and only if its scalar curvature s vanishes identically.*

Proof If our Kähler surface (M, g, J) has $s \equiv 0$, it is anti-self-dual by (8), and therefore Bach-flat; the fact that such a manifold has $B = 0$ is also directly confirmed by (10). Inspection of Eq. (13) now reveals that it also has $\kappa = 0$.

Conversely, if (M, g, J) is a Bach-flat Kähler surface with $\kappa = 0$, then Lemma 4 tells us that $\min s = 0$. Thus, $\mathcal{Z} = s^{-1}(0)$ is non-empty, and every $p \in \mathcal{Z}$ is a minimum of s. We will now argue by contradiction, and assume that $s \not\equiv 0$. This implies that s is non-constant, so Lemma 1 now tells us that $\text{Hess}(s) := \nabla\nabla s$ must be non-zero at any point where $\nabla s = 0$. But since any point $p \in \mathcal{Z}$ is a minimum of s, this means that $\text{Hess}(s) \neq 0$ at every point of $\mathcal{Z}$. However, Eq. (11) tells us that the trace-free part $\text{Hess}_0(s)$ of the Hessian *does* vanish along the locus $\mathcal{Z}$ defined by $s = 0$. Thus, for every point $p \in \mathcal{Z}$, there is a constant $a = a(p) \neq 0$ such that

$$\nabla\nabla s = 2ag \tag{14}$$

at p. Since p is a minimum of s, we must moreover have $a > 0$. Hence every $p \in \mathcal{Z}$ is a non-degenerate local minimum of s, and it therefore follows that $\mathcal{Z}$ is discrete. Since M is compact, this then implies that $\mathcal{Z}$ is finite.

Now let $p \in \mathcal{Z}$ be any point where s vanishes, and let ϱ be the Riemannian distance from p in (M, g). Since (14) guarantees that $\nabla\xi = 2aJ$ at p, the isometry $\Phi_{\pi/2a}$, gotten by flowing along ξ for time $t = \pi/2a$, therefore fixes p, but reverses the direction of each geodesic through p. The Taylor expansion of s in geodesic normal coordinates x^j centered at p therefore contains only terms of even order, and (14) therefore tells us that

$$s = a\varrho^2 + O(\varrho^4).$$

On the other hand, we also have

$$g_{jk} = \delta_{jk} + O(\varrho^2)$$
$$g_{jk,\ell} = O(\varrho)$$

in geodesic normal coordinates. If we now pass to inverted coordinates $\tilde{x}^j = x^j/(a\varrho^2)$ and set $я := \sqrt{\sum_j (\tilde{x}^j)^2} = 1/(a\varrho)$, the metric $h = s^{-2}g$ thus satisfies

$$h_{jk} = \delta_{jk} + O(я^{-2})$$
$$h_{jk,\ell} = O(я^{-3})$$

so that $(M - \mathcal{Z}, h)$ is *asymptotically Euclidean*. However, since $\kappa = 0$, the Einstein metric h is actually Ricci-flat. This in particular implies [5] that each end of $(M - \mathcal{Z}, h)$ has mass zero. The positive mass theorem [43] therefore asserts that $(M - \mathcal{Z}, h)$ is isometric to Euclidean $\mathbb{R}^4$. In particular, g is conformally flat on $M - \mathcal{Z}$, and so has $W_+ \equiv 0$ on this open dense set. But by (8), this means that the Kähler metric g satisfies $s \equiv 0$ on $M - \mathcal{Z}$. But $M - \mathcal{Z}$ is by definition precisely the set where $s \neq 0$, so this is a contradiction! In other words, $M - \mathcal{Z}$ must actually be empty, and any compact Bach-flat Kähler surface with $\kappa = 0$ must therefore have $s \equiv 0$, as claimed. □

Remark The above proof can be recast in a way that avoids using the positive mass theorem. Indeed, the volume growth of an asymptotically Euclidean Ricci-flat manifold with one end must be exactly Euclidean in the large-radius limit, and would be even larger if there were several ends. The Bishop–Gromov inequality [8,23] thus forces the exponential map of any Ricci-flat asymptotically flat manifold to actually be an isometry. This shows that the only such manifold is Euclidean space.

The final contradiction could also have been rephrased so as to emphasize topology instead of geometry. For example, if $M - \mathcal{Z}$ were diffeomorphic to $\mathbb{R}^4$, it would only have one end, so $\mathcal{Z}$ would necessarily consist of a single point p, and M would have to be homeomorphic to $S^4 = \mathbb{R}^4 \cup \{p\}$. But this is absurd, because, for example, the Kähler class $[\omega]$ of (M, g, J) is a non-zero element of $H^2(M, \mathbb{R})$, while $H^2(S^4, \mathbb{R}) = 0$. Alternatively, one could obtain a contradiction at this same juncture by emphasizing that (M, J) is by hypothesis a complex surface, whereas S^4 does not [49] even admit an almost-complex structure.

It now remains for us to analyze the two cases $\kappa > 0$ and $\kappa < 0$. The first of these is simpler, and is quite thoroughly understood.

Proposition 2 *If the constant κ is positive, the scalar curvature s of the extremal Kähler manifold (M, g, J) is everywhere positive. Consequently, $\mathcal{Z}$ is empty, and (M, h) is a compact Einstein 4-manifold with positive Einstein constant.*

Proof If $\kappa > 0$, Lemma 4 tells us that $\min s > 0$, too. Thus $s > 0$ everywhere on M. Hence $\mathcal{Z} = s^{-1}(0) = \varnothing$, and $M - \mathcal{Z} = M$. Thus $h = s^{-2}g$ is a globally defined Einstein metric on M, with positive Einstein constant $\lambda = \kappa/4$. □

Previous results [34] therefore provide a complete classification of $\kappa > 0$ solutions. We will say more about this classification in Sect. 5 below.

By contrast, the $\kappa < 0$ case is distinctly more complicated:

Proposition 3 *If the constant κ is negative, and if (M, g, J) is not Kähler–Einstein, then $\mathcal{Z}$ is a smooth connected real hypersurface $\mathcal{Z}^3 \subset M^4$, and the complement $M - \mathcal{Z}$ of this hypersurface consists of exactly two connected components M_+ and M_-. The Einstein manifolds $(M_\pm, h := s^{-2}g)$ are both complete, and have negative Einstein constant. Moreover, these two manifolds are both Poincaré–Einstein, with the same conformal infinity $(\mathcal{Z}, [g|_{\mathcal{Z}}])$.*

Proof When $\kappa < 0$, Lemma 4 tells us that $\min s < 0$. If s is constant, the constant is thus negative, and s is nowhere zero. Equation (11) thus implies that the Kähler metric g is actually Einstein.

We may thus henceforth assume that s is non-constant. Lemma 1 therefore tells us that $s : M \to \mathbb{R}$ is a Morse–Bott function. In particular, Hess $s \neq 0$ at any critical point of s. If p is a point where s attains its minimum, we therefore have $\Delta s = -\nabla^a \nabla_a s < 0$ at p. However, since $\kappa < 0$, we also have $s(p) = \min s < 0$ by Lemma 4. Thus $s\Delta s > 0$ at p. On the other hand, since $\nabla s = 0$ at any minimum, (13) tells us that $6s\Delta s = s^3 - \kappa$ at p. Hence $\min(s^3 - \kappa) = [s^3 - \kappa](p) = [6s\Delta s](p) > 0$, and we therefore have $s^3 - \kappa > 0$ on all of M. Equation (13) now tells us that

$$s\Delta s + 2|\nabla s|^2 = \frac{s^3 - \kappa}{6} > 0$$

everywhere. In particular, we have $s\Delta s > 0$ at every critical point of s.

If q is now a point where s attains its maximum, our Morse–Bott argument predicts that $\Delta s = -\nabla^a \nabla_a s > 0$ at q, since Hess s is negative semi-definite and non-zero at q. Since we have shown that $s\Delta s > 0$ at every critical point, and so in particular at q, it therefore follows that $\max s = s(q) > 0$.

Since M is connected, and since s takes on both positive and negative values, the locus $\mathcal{Z}$ defined by $s = 0$ must therefore be non-empty. Moreover, since $s\Delta s > 0$ at every critical point of s, it follows that the locus $\mathcal{Z}$ defined by $s = 0$ cannot contain any critical points. In other words, 0 is a regular value of s, and $\mathcal{Z} = s^{-1}(0)$ is a therefore a smooth compact non-empty real hypersurface in M.

Now since $s : M \to \mathbb{R}$ is a Morse–Bott function, and since Hess s is J-invariant, any complex-codimension-one component C_j of the critical set must be a local maximum or a local minimum of s. The other critical points of s are isolated, and Hess s is non-degenerate at these critical points, with index 0, 2, or 4; those of index 0 are local minima, those of index 4 are local maxima, and those of index 2 are saddle points where the Hessian is of type $(++--)$. This dictates the manner in which the sub-level sets $M_t := s^{-1}((-\infty, t])$ can change as we increase t. Indeed, for regular values $t_1 < t_2$, the sub-level set M_{t_2} is obtained from the lower sub-level set M_{t_1} in a manner determined by the critical points with $t_1 < s < t_2$ and, up to homotopy, is gotten by adding

- a disjoint, unattached point for each isolated local minimum;
- a disjoint, unattached connected Riemann surface for each non-isolated local minimum;
- a 2-disk, attached along its S^1 boundary, for each saddle point;
- a 4-disk, attached along its S^3 boundary, for each isolated local maximum; and
- a 2-disk bundle over a connected Riemann surface, attached along its circle-bundle boundary, for each non-isolated local maximum.

Since these operations always entail adding a path-connected space along a path-connected boundary, different path components of M_{t_1} always survive as separate path components of M_{t_2}. It therefore follows that only one of the C_j can be a local minimum s, since two different local minima would necessarily end up in different connected components of the connected 4-manifold M. In particular, the set of all local minima of s is actually the set of all *global* minima, and must either be a connected compact complex curve or just a single point. Looking at the same picture upside down, we similarly see that the set of local maxima of s coincides with the set of global maxima, and must either be a single point or a connected compact complex curve.

The open set M_- defined by $s < 0$ is the interior of the compact manifold-with-boundary $M_0 := s^{-1}((-\infty, 0])$, and is therefore homotopy equivalent to it. On the other hand, since $M = M_t$ for any $t > \max s$, we see that M_0 and M have the same number of connected components. Hence M_- must be connected. However, since $-s : M \to \mathbb{R}$ is also a Morse–Bott function, the same reasoning also applies when we "turn the picture upside down." The set M_+ defined by $s > 0$ is therefore connected, too. Thus $M - \mathcal{Z} = M_- \sqcup M_+$ consists of exactly two connected components, as claimed.

Similar reasoning allows us to understand how ∂M_t changes as we vary t. In the region $\min s < s < \max s$, s is a Morse function in the standard sense, and only has critical points of index 2. If t_1 and t_2 are regular values of s with $\min s < t_1 < t_2 < \max s$, it therefore follows that ∂M_{t_2} is obtained from ∂M_{t_1} by performing surgeries in dimension $2 - 1 = 1$. In other words, every time one passes a critical point, one just modifies the 3-manifold ∂M_{t_1} by performing a *Dehn surgery*; that is, one just removes a solid torus $S^1 \times D^2$, and then glues it back in via a self-diffeomorphism of its $S^1 \times S^1$ boundary. Since M_t is connected when $t = \min s + \epsilon$ for sufficiently small $\epsilon > 0$, and because Dehn surgery on a connected 3-manifold always produces another connected 3-manifold, it follows that ∂M_t is connected for any regular value $t \in (\min s, \max s)$. In particular, since 0 is a regular value of $s : M \to \mathbb{R}$, it follows that $\mathcal{Z} = \partial M_0$ is connected, as claimed.

Since the metric $h = s^{-2} g$ on the interior M_- of the compact manifold-with-boundary M_0 is obtained by rescaling a smooth metric g by the inverse-square of a non-negative function $-s : M_0 \to \mathbb{R}$ which vanishes only at $\partial M_0 = \mathcal{Z}$ and has non-zero normal derivative along the boundary, the Einstein manifold (M_-, h) is conformally compact [20,29,37] and hence, in particular, is complete. By the same reasoning, (M_+, h) is also conformally compact, and therefore complete. Since, by construction, both of these manifolds have conformal infinity $(\mathcal{Z}, [g|_{\mathcal{Z}}])$ and Einstein constant $\kappa/4 < 0$, we have thus established all of our claims. □

Remark A key point in the above result is that $\max s > 0$ when $\kappa < 0$ and s is non-constant. This can also be proved in the following interesting way:

Since the flow of $\xi = J\nabla s$ preserves both g and the scalar curvature s of g, it therefore also preserves the conformally rescaled metric $h = s^{-2} g$ on the open set $M - \mathcal{Z}$ where h is defined. If we had $\max s < 0$ for a solution with s non-constant, $\mathcal{Z}$ would be empty, and (M, h) would be a compact Einstein manifold with Einstein constant $\kappa/4 < 0$ which supported a Killing field $\xi \not\equiv 0$. But any Killing field ξ satisfies the Bochner formula [9]

$$0 = \frac{1}{2}\Delta|\xi|^2 + |\nabla\xi|^2 - r(\xi, \xi) \qquad (15)$$

and since (M, h) would have negative Ricci curvature $r = \frac{\kappa}{4} h$, this immediately leads to a contradiction, as the right-hand side would be strictly positive at a maximum of $|\xi|^2$. Hence (M, g) must have $\max s \geq 0$. However, Eq. (13) tells us that $|\nabla s|^2 = -\kappa/12 > 0$ along the locus $\mathcal{Z}$ where $s = 0$. The maximum of s therefore cannot be achieved at a point where $s = 0$, and it thus follows that we must actually have $\max s > 0$, as claimed.

4 The Proof of Theorem A

The trichotomy laid out in the previous section proves most of Theorem A. It remains only to prove the auxiliary claims made regarding the second case of each of our three classes. In doing so, we will make repeated use of the following observation:

Lemma 5 *If the compact connected complex surface* (M^4, J) *admits a strictly extremal Kähler metric g, then* (M, J) *is ruled, and* $\tau(M) \leq 0$.

Proof By hypothesis, the scalar curvature s of g is non-constant, and $\xi = J\nabla s$ is a non-trivial Killing field. If $p \in M$ is a minimum of the scalar curvature s, it is fixed by the flow of ξ, and the action of ξ is therefore completely determined, via the exponential map of g, by the induced isometric action on T_pM generated by $\nabla\xi|_p$. However, the same observation allows us to identify the isotropy subgroup of p in the identity component $\mathrm{Iso}_0(M, g) \subset \mathrm{Aut}(M, J)$ of the isometry group with a subgroup of $\mathbf{U}(2)$. Since closure of the 1-parameter group of isometries generated by ξ is a closed connected Abelian subgroup of the isotropy group of p, and since $\mathbf{U}(2)$ has rank 2, this implies that either ξ is periodic, or that the closure of the group it generates is a 2-torus in $\mathrm{Iso}_0(M, g) \subset \mathrm{Aut}_0(M, J)$.

Let us first consider what happens if the isotropy group of p contains a 2-torus. Since the action of this torus on M is modeled, near p, on the action of $\mathbf{U}(1)\times\mathbf{U}(1) \subset \mathbf{U}(2)$ on $\mathbb{C}^2$, the generators give rise to two global holomorphic vector fields Ξ_1 and Ξ_2 which both vanish at p, but which are linearly independent at generic nearby points. Thus $\Theta = \Xi_1 \wedge \Xi_2$ is a holomorphic section of the anti-canonical bundle K^{-1} which vanishes at p, but which does not vanish identically. If ϕ is a holomorphic section of K^ℓ, $\ell > 0$, then the contraction $\langle\phi, \Theta^{\otimes\ell}\rangle$ is a global holomorphic function on M, and so must be constant. However, since $\langle\phi, \Theta^{\otimes\ell}\rangle$ certainly vanishes at p, this constant function consequently vanishes identically. Since $\Theta^{\otimes\ell}$ spans the fiber of $K^{-\ell}$ at a generic point, this shows that the holomorphic differential ϕ vanishes on an open set, and therefore vanishes identically. Thus, the plurigenera $p_\ell(M) = h^0(M, \mathcal{O}(K^\ell))$, $\ell > 0$, must all vanish, and the Kodaira dimension of (M, J) must be $-\infty$.

On the other hand, if $\xi = J\nabla s$ is instead periodic, then (M, J) contains a family of rational curves. Indeed, if ξ has period λ, then $2\nabla^{1,0}s = \nabla s - i\xi$ generates a holomorphic action of $\mathbb{C}/\langle i\lambda\rangle$, which we now identify with the punctured complex plane $\mathbb{C}^\times$ via $\zeta \mapsto \exp(2\pi\zeta/\lambda)$. We will next show that the generic orbit of the resulting $\mathbb{C}^\times$-action on (M, J) can be compactified by adding two fixed points, corresponding to 0 and ∞, to produce a rational curve in M. Indeed, we already saw in the proof of Proposition 3 that the only critical points of s that are not global maxima or minima are saddle points where Hess s is of type $(++--)$. Since only a 1-real-parameter family of trajectories of ∇s ascends to such a saddle point, the generic trajectory of ∇s misses every saddle point, and therefore flows from the minimum value to the maximum value of s. Now recall that $\min s$ is either attained at a unique point p_- or is attained along a single connected totally geodesic Riemann surface, which we will call Σ_-; similarly, $\max s$ is either attained at a unique point p_+ or is attained along a single connected totally geodesic Riemann surface Σ_+. The behavior of the flow lines of ∇s near $\Sigma_\pm$ is particularly simple; asymptotically, they simply approach $\Sigma_\pm$ orthogonally, at a unique point, since in exponential coordinates ξ just generates rotation in $\mathbb{C}^2 = \{(z, w)\}$ about the z-axis, and since $\nabla J = 0$, this implies that $\nabla^{1,0}s$ is therefore given in these coordinates by a constant times $w\frac{\partial}{\partial w} + O(|(z, w)|^3)$. On the other hand, near $p_\pm$, the periodicity of ξ analogously allows one to express $\nabla^{1,0}s$ as a constant times $kz\frac{\partial}{\partial z} + \ell w\frac{\partial}{\partial w} + O(|(z,w)|^3)$ for suitable non-zero integers k and ℓ. In either case, we not only see that a generic $\mathbb{C}^\times$-orbit may be completed as a holomorphic map $\mathbb{CP}_1 \to M$, but also that, as we vary the orbit by moving our initial data through a sufficiently small holomorphic disk D_ε transverse to a given generic $\mathbb{C}^\times$-orbit, the map $D_\varepsilon \times \mathbb{C}^\times \to M$ induced by the action extends continuously as a map $\Phi : D_\varepsilon \times \mathbb{CP}_1 \to M$, and a variant of the proof of the Riemann removable singularities theorem then shows that this continuous extension Φ is necessarily holomorphic. In particular, any $\phi \in \Gamma(M, \mathcal{O}(K^\ell))$ pulls back as a holomorphic section $\Phi^*\phi$ of $\mathcal{O}(K^\ell)$ on $D_\varepsilon \times \mathbb{CP}_1$. But the restriction of $K^\ell_{D_\varepsilon \times \mathbb{CP}_1}$ to any $\{\text{point}\} \times \mathbb{CP}_1$ has negative degree -2ℓ, so it follows that $\Phi^*\phi \equiv 0$. However, since Φ is a local biholomorphism on $D_\varepsilon \times \mathbb{C}^\times$, it follows that ϕ must itself vanish on an open set. Hence $\phi \equiv 0$ by uniqueness of analytic continuation. Thus, the plurigenera $p_\ell(M) = h^0(M, \mathcal{O}(K^\ell))$, $\ell > 0$, must all vanish, and we thus conclude that the Kodaira dimension of (M, J) must be $-\infty$, just as in the previous case.

Since (M, J) is of Kähler type, the Eniques–Kodaira classification [4,22] therefore implies that it is rational or ruled. However, since g is a strictly extremal Kähler metric on (M, J), the Futaki invariant of $(M, J, [\omega])$ must also be non-zero [13]. Since the Futaki invariant $\mathfrak{aut}(M) \to \mathbb{C}$ kills the derived Lie algebra $[\mathfrak{aut}(M), \mathfrak{aut}(M)]$, this means that $\mathrm{Aut}_0(M, J)$ cannot be semi-simple. Consequently, (M, J) cannot be $\mathbb{CP}_2$. Since (M, J) is rational or ruled, it is therefore either a geometrically ruled surface, or a blow-up of a geometrically ruled surface. In particular, $\tau(M) \leq 0$, as claimed. □

This now allows us to prove a useful fact regarding case I(b):

Proposition 4 *Let (M, g, J) be a compact connected Bach-flat Kähler surface with $\kappa > 0$. If g is not Kähler–Einstein, then the conformally related Einstein metric $h = s^{-2}g$ has holonomy* $\mathbf{SO}(4)$.

Proof By Proposition 2, (M^4, g, J) must be a strictly extremal Kähler surface, and Lemma 5 therefore tells us that (M, J) is a ruled surface with $\tau(M) \leq 0$. On the other hand, M also admits a globally defined Einstein metric $h = s^{-2}g$ with $\lambda > 0$, so Bochner's theorem [9] implies that $b_1(M) = 0$. Surface classification [4] therefore tells us that (M, J) is rational, and so in particular is simply connected.

Since M^4 is simply connected, the holonomy of any metric on M thus coincides with its restricted holonomy, and so is a compact connected Lie subgroup of $\mathbf{SO}(4)$. However, the action of $\mathbf{SO}(4)$ on $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ gives rise to an isomorphism $\mathbf{SO}(4)/\mathbb{Z}_2 = \mathbf{SO}(3) \times \mathbf{SO}(3)$, where the two copies of $\mathbf{SO}(3)$ act on Λ^+ and Λ^-, respectively, via the tautological 3-dimensional representation. Thus, if the holonomy group were smaller than $\mathbf{SO}(4)$, its image in at least one factor $\mathbf{SO}(3)$ would have to be contained in $\mathbf{SO}(2)$, and some non-zero self-dual or anti-self-dual 2-form would therefore be fixed by the holonomy group. Rescaling this 2-form α to have norm $\sqrt{2}$ with respect to h and then extending it as a parallel form to all of M, the almost-complex structure $\mathscr{J}$ defined by $\alpha = h(\mathscr{J}\cdot, \cdot)$ would then be integrable, and $(M, h, \mathscr{J})$ would then be a Kähler–Einstein manifold. More specifically, a parallel self-dual α would give rise to a $\mathscr{J}$ compatible with the same orientation as the original complex structure J, while a parallel anti-self-dual α would instead give rise to a $\mathscr{J}$ compatible with the opposite orientation.

Now the Bach-flat Kähler metric g is non-Einstein by assumption. Since $h = s^{-2}g$ is Einstein and is conformally related to g, it therefore follows that

$$\int_M s_g^2 \mathrm{d}\mu_g > \int_M s_h^2 \mathrm{d}\mu_h,$$

as can either be read off from the conformal invariance of

$$\int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2}\right) \mathrm{d}\mu = 8\pi^2\chi(M) - \int_M |W|^2 \mathrm{d}\mu$$

or deduced from the more general fact [39] that Einstein metrics are always Yamabe minimizers. Since the Kähler metric g satisfies (8), we thus have

$$\int_M |W_+|_h^2 \mathrm{d}\mu_h = \int_M |W_+|_g^2 \mathrm{d}\mu_g = \int_M \frac{s_g^2}{24} \mathrm{d}\mu_g > \int_M \frac{s_h^2}{24} \mathrm{d}\mu_h.$$

It therefore follows that h cannot satisfy (8), and so cannot be Kähler in a manner compatible with the given orientation of M. On the other hand, Eq. (2) tells us that

$$\int_M |W_-|_h^2 \mathrm{d}\mu_h = -12\pi^2\tau(M) + \int_M |W_+|_h^2 \mathrm{d}\mu_h$$

and since $\tau(M) \leq 0$, we therefore have

$$\int_M |W_-|_h^2 \mathrm{d}\mu_h \geq \int_M |W_+|_h^2 \mathrm{d}\mu_h > \int_M \frac{s_h^2}{24} \mathrm{d}\mu_h.$$

This shows that h cannot satisfy the reverse-oriented analog of (8), and hence cannot be Kähler with respect to a reverse-oriented complex structure, either. This proves that, when s_g is non-constant and positive, the holonomy group of (M, h) is exactly $\mathbf{SO}(4)$, as claimed. □

Concerning case II(b), we have the following:

Proposition 5 *If $\kappa = 0$ and (M, g, J) is not Kähler–Einstein, then (M, J) is a (possibly blown-up) ruled complex surface with $c_1^2 < 0$. Moreover, no 4-manifold homeomorphic to M ever admits an Einstein metric.*

Proof By Proposition 1, a Bach-flat Kähler surface (M, g, J) with $\kappa = 0$ must have $s \equiv 0$. Since one can decompose the Ricci form ρ of any Kähler surface as

$$\rho = \frac{s}{4}\omega + \mathring{\rho},$$

where $\mathring{\rho} \in \Lambda^- = \Lambda^{1,1}_{\mathbb{R}} \cap (\omega)^\perp$ is the primitive part of $\rho \in \Lambda^{1,1}$, we see that ρ is anti-self-dual whenever g is scalar-flat Kähler. It follows [30] that any scalar-flat Kähler surface satisfies

$$4\pi^2 c_1^2(M, J) = \int_M \mathring{\rho} \wedge \mathring{\rho} = -\int_M \mathring{\rho} \wedge \star\mathring{\rho} = -\int_M |\mathring{\rho}|^2 d\mu,$$

so that $c_1^2 \leq 0$, with equality if and only if g is Ricci-flat; cf. [28]. In particular, if (M, g, J) is not Kähler–Einstein, we have

$$(2\chi + 3\tau)(M) = c_1^2(M, J) < 0.$$

However, the Hitchin–Thorpe inequality [7,21,24,46] tells us that the homotopy invariant $(2\chi + 3\tau)(M)$ must be non-negative for any compact oriented 4-dimensional Einstein manifold. Thus, if $\kappa = 0$ and (M, g, J) is not Kähler–Einstein, M cannot even be homeomorphic to an Einstein manifold.

On the other hand, the plurigenera $p_\ell(M) = h^0(M, \mathcal{O}(K^\ell))$, $\ell > 0$, of a non-Ricci-flat scalar-flat Kähler surface must all vanish [30,50]. Indeed, if ϕ is a holomorphic section of K^ℓ, the Ricci form satisfies

$$\ell\rho = i\partial\bar{\partial} \log |\phi|^2 \tag{16}$$

away from the zero locus of ϕ. Taking the inner product of both sides with ω thus yields

$$0 = -\ell s = \Delta \log |\phi|^2$$

wherever $\phi \not\equiv 0$. Since $|\phi|^2$ must have a maximum on M, the strong maximum principle [19] therefore says that $\log |\phi|^2$ is constant away from the zero set of ϕ. If ϕ does not vanish identically, it therefore has constant non-zero norm, and Eq. (16) then says that (M, g) is Ricci-flat. If (M, g, J) is not Kähler–Einstein, its plurigenera must therefore all vanish, as claimed. The Kodaira dimension of (M, J) is therefore $-\infty$, and the Enriques–Kodaira classification [4] therefore tells us that (M, J) is rational or ruled. Moreover, since $c_1^2 < 0$, it certainly cannot be $\mathbb{CP}_2$. We thus conclude that (M, J) can actually be obtained from some ruled surface by blowing up ≥ 0 points. □

Putting these facts together, we now immediately have:

Proposition 6 *Suppose that (M, g, J) is a compact connected Bach-flat Kähler surface which is not Kähler–Einstein. Then (M, J) is ruled.*

Proof If g has $\mathring{r} \not\equiv 0$, Eq. (11) tells us that the Bach-flat Kähler manifold (M, g, J) has either s non-constant or $s \equiv 0$. If g has $s \equiv 0$, then Proposition 5 then tells us that (M, J) is ruled. Otherwise, g must be a strictly extremal Kähler metric, and Lemma 5 then again guarantees that (M, J) is ruled, as claimed. □

Finally, regarding solutions of type III(b), we have the following:

Proposition 7 *Let (M, g, J) be a compact connected Bach-flat Kähler surface with $\kappa < 0$ that is not Kähler–Einstein. Then the connected real hypersurface $\mathcal{Z} \subset M$ is totally umbilic with respect to g. Moreover, the Weyl curvature W of (M, g) vanishes identically along $\mathcal{Z}$.*

Proof While both of these features are general consequences [18,29] of the fact that $h = s^{-2}g$ is Poincaré–Einstein, we will give quick, self-contained proofs that supply further information in the present special context.

Let $\check{g} = g|_{\mathcal{Z}}$ denote the induced Riemannian metric (or *first fundamental form*) of our hypersurface, and let $\mathbb{I}$ denote the second fundamental form (or *shape tensor*) of $\mathcal{Z}$. To say that $\mathcal{Z}$ is totally umbilic just means that $\mathbb{I} = f\check{g}$ for some function $f : \mathcal{Z} \to \mathbb{R}$, which is then just the mean curvature of $\mathcal{Z}$. However, the second fundamental form can be expressed as $\mathbb{I} = (\nabla\nu)|_{\mathcal{Z}}$ in terms of any unit 1-form ν on M which is normal to $\mathcal{Z}$ along this hypersurface, and which is compatible with the given orientations of M and $\mathcal{Z}$. However, since $\mathcal{Z}$ is defined by $s = 0$, and since Eq. (13) tells us that $|\nabla s|^2 = -\frac{\kappa}{12}$ along $\mathcal{Z}$, it follows that $\nu = 2\sqrt{3/|\kappa|}\, ds$ is a valid choice for this unit co-normal field, provided we orient $\mathcal{Z}$ in the corresponding manner. Thus

$$\mathbb{I} = \left(2\sqrt{\frac{3}{|\kappa|}}\,\text{Hess } s\right)\Bigg|_{\mathcal{Z}}$$

gives us a convenient formula for the second fundamental form of $\mathcal{Z}$. However, Eq. (10) tells us that $\text{Hess}_0\, s = 0$ along the locus where $s = 0$, so $\text{Hess}\, s = -(\Delta s/4)g$ along $\mathcal{Z}$, and hence

$$\mathbb{I} = -\left(\frac{\sqrt{3}}{2\sqrt{|\kappa|}}\Delta s\right)\check{g}. \tag{17}$$

This shows that $\mathcal{Z}$ is totally umbilic, as claimed.

On the other hand, since the Kähler metric g has $s = 0$ along $\mathcal{Z}$, Eq. (8) tells us that $W_+ = 0$ there, too. Thus $W = W_-$ at $\mathcal{Z}$. However, if we let ν denote the unit 1-form normal to $\mathcal{Z}$, there is a natural bundle isomorphism

$$\begin{aligned} T^*\mathcal{Z} &\longrightarrow \Lambda^-|_{\mathcal{Z}} \\ \theta &\longmapsto (\nu \wedge \theta) - \star(\nu \wedge \theta). \end{aligned}$$

Applying the identity

$$W_-(\varphi - \star\varphi, \varphi - \star\varphi) - W_+(\varphi + \star\varphi, \varphi + \star\varphi) = -4\mathcal{R}(\varphi, \star\varphi),$$

to $\varphi = \nu \wedge \theta$, and remembering that $W_+ = 0$ along $\mathcal{Z}$, we therefore see that $W = W_-$ is completely characterized along $\mathcal{Z}$ by knowing $\mathcal{R}^1{}_{234} = W^1{}_{234}$ for those orthonormal co-frame $\{e^1, \ldots, e^4\}$ in which $e^1 = \nu$. But we can easily understand these components, using the fact that W is conformally invariant. Indeed, since a conformal change $g \rightsquigarrow u^2 g$ changes the second fundamental form by $\mathbb{I} \rightsquigarrow u\mathbb{I} + \langle du, \nu\rangle\check{g}$, a hypersurface is totally umbilic iff it can be made totally geodesic by a conformal change. But the unit normal 1-form ν is parallel along any totally geodesic hypersurface. In such a conformal gauge, we therefore have $\mathcal{R}^1{}_{234} = W^1{}_{234} = 0$. Conformal invariance thus allows us to deduce that W vanishes identically along $\mathcal{Z}$, as claimed. □

Theorem A is now an immediate consequence. Indeed, the basic trichotomy into solution classes I, II, and III just reflects the sign of κ, or equivalently, by Lemma 4, the sign of $\min s$. Propositions 1, 2, and 3 then explain the basic features of each class. If the solution is Kähler–Einstein, it is then of type I(a), II(a), or III(a), depending on the sign of κ. Otherwise, the underlying complex surface is ruled by Proposition 6, and the remaining claims made about solutions of type I(b), II(b), and III(b) are then proved by Propositions 4, 5, and 7, respectively.

5 Problems and Perspectives

As mentioned in Sect. 1, solutions of class I have been completely classified [34]. One key fact that enabled this classification was the observation [32] that in this case (M, J) has $c_1 > 0$, and hence is a Del Pezzo surface [16]. Indeed, since $s \neq 0$, we can represent $2\pi c_1$ by $\tilde{\rho} := \rho + 2i\partial\bar{\partial}\log|s|$, where ρ is the Ricci form of (M, g, J), and Eq. (11) then tells us that $\tilde{\rho} = q(J\cdot, \cdot)$, where the symmetric tensor field q is given by

$$q = \frac{2s + \kappa s^{-2}}{12} g + s^{-2}|\nabla s|^2 g^{\perp}. \tag{18}$$

Here, $g^{\perp}$ is defined to be zero on the span of ∇s and $J\nabla s$, but to coincide with g on the orthogonal complement of this subspace; while this of course means that $g^{\perp}$ is only defined away from the critical points of s, the tensor field $|\nabla s|^2 g^{\perp}$ has a unique smooth extension across the critical points, which is explicitly given by declaring it to be zero at this exceptional set. Since $\kappa > 0$ implies that $s > 0$ everywhere, it follows that $q > 0$ for a solution of type I, and that (M, J) is therefore a Del Pezzo surface when κ is positive. Conversely, one can show [15,40,47] that every Del Pezzo (M, J) admits a J-compatible class-I solution, and that this solution is moreover unique [3,34] up to complex automorphism and rescaling. In fact, the solution is Kähler–Einstein except in exactly two cases, namely the blow-up of $\mathbb{CP}_2$ at one or two distinct points. Thus, solutions of type I(a) and I(b) are distinguished by whether or not the Lie algebra of holomorphic vector fields on (M, J) is reductive.

By further elaboration on this idea, one is led to the following result:

Proposition 8 *Let g and $\tilde{g}$ be two J-compatible Bach-flat Kähler metrics on the same compact complex surface (M, J). If these two solutions have different types, according to the classification scheme of Theorem A, then one is of type* II(b)*, and the other is of type* III(b).

Proof Solutions of type II(a) and III(a) are distinguished from the others by the Kodaira dimension of (M, J). On the other hand, we have just seen that solution of class I exist precisely on complex surfaces with $c_1 > 0$, and solutions of type I(a) are then distinguished from those of type I(b) by whether the Lie algebra of holomorphic vector fields is reductive. It thus only remains to show that the existence of solution of class I precludes the existence of a solution of type II(b) or III(b). However, any extremal Kähler metric on a Del Pezzo surface has positive scalar curvature [35, Lemmata A.2 and B.2]. Consequently, the presence of a Bach-flat Kähler metric of class I automatically precludes the existence of a solution of any other class. □

This makes the following piece of speculation seem irresistible:

Conjecture 1 *On a fixed compact complex surface (M, J), any pair of J-compatible Bach-flat Kähler metrics are necessarily of the same type, in the sense of Theorem A.*

Although the currently known solutions of type III(b) will almost certainly turn out to be atypical in many respects, these known examples all live on geometrically ruled surfaces, which necessarily have $\tau = 0$. The following result thus explains why Conjecture 1 is supported by all known examples:

Proposition 9 *No compact complex surface (M, J) of signature $\tau = 0$ can admit a pair of J-compatible Bach-flat Kähler metrics that have different types, in the sense of Theorem A.*

Proof By Proposition 8, it suffices to show that there cannot simultaneously be a solution g of type II(b) and a solution $\tilde{g}$ of type III(b). However, any solution of class II is scalar-flat Kähler, and therefore has $W_+ = 0$ by Eq. (8), and such a solution is of type II(b) iff it is not Ricci-flat. If we now assume that $\tau(M) = 0$, Eq. (2) then tells us that $W_- = 0$, making (M, g) is conformally flat, as well as scalar-flat. The Weitzenböck formula for 2-forms thus simplifies to say that the Hodge Laplacian on 2-forms coincides with the Bochner Laplacian $\nabla^*\nabla$, and any harmonic 2-form must therefore be parallel. Since the assumption that $\tau = 0$ also forces $b_- = b_+ \neq 0$, our metric g is Kähler with respect to both orientations, and, since g is not flat, it follows [11,30] that (M, J) is therefore a geometrically ruled surface of the form $\Sigma \times_\gamma \mathbb{CP}_1$ for some representation $\gamma : \pi_1(\Sigma) \to \mathbf{PSU}(2)$. The given scalar-flat metric g is then a twisted product metric, obtained by equipping Σ and $\mathbb{CP}_1$ with metrics of constant (and opposite) Gauss curvature, and thus belongs to a 2-parameter family of constant-scalar-curvature metrics gotten by rescaling Σ and $\mathbb{CP}_1$ by arbitrary positive constants. Since $b_2(M) = 2$, this shows that the Futaki invariant is identically zero on an open set of the Kähler cone. However, the Futaki invariant is quite generally a real-analytic function of the Kähler class, as can be seen by locally sweeping out the Käher cone by real-analytic families of real-analytic Kähler metrics. Hence the Futaki invariant of (M, J) vanishes for every Kähler class, and it therefore follows [13] that any extremal Kähler metric on (M, J) must have constant scalar curvature. Consequently, (M, J) cannot admit a strictly extremal Kähler metric, and no J-compatible solution $\tilde{g}$ of type III(b) can therefore exist on (M, J). □

There is a more fundamental reason to hope that Conjecture 1 might be true. Recall that Bach-flat metrics are critical points of the Weyl functional, and that Bach-flat Kähler metrics are therefore, in particular, critical points of the Calabi energy (9) on the space of Kähler metrics compatible with a given complex structure. However, the known examples are always [44] actually *absolute minima* for the latter problem.

Conjecture 2 *Let g be a Bach-flat Kähler metric on a compact complex surface (M, J). Then g is an absolute minimizer of the Calabi energy $\mathcal{C}$ on the space of all J-compatible Kähler metrics on M.*

This conjecture is an easy exercise for solutions of type I(a), II(a), II(b), and II(a). It is moreover also true for solutions of type I(b), although the proof [34] is much more subtle in this case. By contrast, we do not currently know whether Conjecture 2 holds for *general* solutions of this type III(b), although it does in fact hold [44] for all *known* solutions. Notice that Conjecture 2 would certainly imply Conjecture 1, since any solution of type III(b) necessarily has $\mathcal{C} > 0$, whereas any solution of type II(b) obviously has $\mathcal{C} = 0$.

These remarks make it obvious that solutions of type III(b) represent the area where our understanding of the subject remains most deficient. Still, it is not hard to prove a bit more about them. For example:

Proposition 10 *Let (M, g, J) be a solution of type* III(b). *Then the complete Einstein Hermitan manifold (M_-, h, J) of Proposition 3 has numerically positive canonical line bundle K_{M_-}. In other words, every compact holomorphic curve $C \subset M_- \subset M$ satisfies $c_1 \cdot C < 0$.*

Proof The flow of $-\nabla s = J\xi$ for positive time is holomorphic, and preserves the region $M_- \subset M$ where $s < 0$; moreover, s is non-increasing under the flow. However, since the holomorphic vector field $\nabla s - i\xi$ has zeroes, its contraction with any holomorphic 1-form vanishes identically, and the induced action on the Albanese torus is therefore zero. By duality, the induced action on the Picard torus is also trivial, so the action sends any holomorphic curve to a curve to which it is linearly equivalent. In particular, any compact holomorphic curve $C \subset M_- \subset M$ gives rise to the same divisor line bundle $L \to M$ as any of its images under the downward flow of the gradient vector field of s. Taking the limit in $\mathbb{P}[\Gamma(M, \mathcal{O}(L))]$, we can thus represent the limit of the images of C under the downward flow by a (typically singular) curve in M_- which is sent to itself by the action of $\mathbb{C}^\times$. Such a curve is a sum, with non-negative integer coefficients, of the curve Σ_- at which the minimum of s is achieved (assuming the minimum does not occur at an isolated point) and of "vertical" rational curves arising from flow lines descending from saddle points of s in $M_- \subset M$. It therefore suffices to check that c_1 is negative on Σ_- and on any curve tangent to ∇s and $\xi = J\nabla s$. However, we can represent $2\pi c_1$ on M_- by $\tilde{\rho} := \rho + 2i\partial\bar{\partial}\log|s|$, where ρ is the Ricci form of (M, g, J), and the corresponding symmetric tensor field is once again given by (18). At critical points of s, or in directions tangent to the space of ∇s and ξ, this expression simplifies to just become $(2s + \kappa s^{-2})g/12$, which is negative-definite on the region M_- given by $s < 0$, since we also have $\kappa < 0$ for a solution of type III(b). This shows that the given curve C is homologous to a curve $\tilde{C}$ on which $c_1 \cdot \tilde{C} < 0$, and we therefore have $c_1 \cdot C < 0$, too, as claimed. □

On the other hand, this says nothing at all about (M_+, J), and this is definitely not a mere matter of accident. For example, when (M, J) is a Hirzebruch surface, M_- is a tubular neighborhood of a rational curve on which c_1 is negative, while M_+ is a tubular neighborhood of a curve on which c_1 is positive. Curiously enough, (M_-, h)

and (M_+, h) are in fact actually isometric[1] in these examples; however, this is not a paradox, because the relevant isometry is orientation-reversing, and so does not intertwine the given complex structures on $M_\pm$ in any direct manner.

We can also prove some things about the real hypersurface $\mathcal{Z} \subset M$:

Proposition 11 *Let (M, g, J) be a Bach-flat Kähler surface of type* III(b)*, and let $\mathcal{Z} \subset M$ be the smooth real hypersurface given by $s = 0$. Then the compact connected 3-manifold $\mathcal{Z}$ is Seifert-fibered. Moreover, the restriction of ξ to $\mathcal{Z}$ is a non-trivial Killing field of constant length with respect to the induced metric $\check{g} = g|_{\mathcal{Z}}$, and its orbits are therefore geodesics of $\check{g}$. The flow of ξ moreover preserves the CR structure induced on $\mathcal{Z}$ by (M, J), and, at any $p \in \mathcal{Z}$, the following are equivalent:*

- *the Levi form of the induced CR structure is non-degenerate;*
- *the Ricci curvature of $\check{g}$ is positive in the direction of ξ;*
- *the second fundamental form $\mathrm{I\!I}$ of $\mathcal{Z} \subset M$ is non-zero; and*
- *the extrinsic Laplacian $\Delta_g s$ of the scalar curvature is non-zero.*

Proof Equation (13) tells us that $|\xi|^2 = |\nabla s|^2 = -\kappa/12 > 0$ along $\mathcal{Z}$, so the restriction of the Killing field ξ to $\mathcal{Z}$ does indeed have constant, non-zero length. Since the closure of the group of isometries generated by ξ is a compact connected Abelian Lie group, and hence a torus, we can approximate ξ uniformly by non-zero periodic Killing fields, and the choice of such an approximation then endows $\mathcal{Z}$ with a circle action for which all isotropy groups are finite, thereby giving it a Seifert-fibered structure. Since ξ is a Killing field of constant length with respect to $\check{g}$, we also have

$$\xi^a \nabla_a \xi_b = -\xi^a \nabla_b \xi_a = -\frac{1}{2} \nabla_b |\xi|^2 = 0,$$

on $(\mathcal{Z}, \check{g})$, and the trajectories of ξ are therefore geodesic. Finally, since the flow of ξ on M preserves both J and s, the flow acts on the hypersurface $s = 0$ by CR automorphisms.

Since s is a non-degenerate defining function for $\mathcal{Z}$, the restriction of $i\partial\bar{\partial}s$ to the CR tangent space of $\mathcal{Z}$ exactly represents the Levi form. On the other hand, Eq. (11) tells us that $i\partial\bar{\partial}s$ is a multiple of the Kähler form ω along the locus $s = 0$, so we therefore conclude that the Levi form is non-degenerate exactly at these points of $\mathcal{Z}$ where $\Delta s \neq 0$. On the other hand, Eq. (17) tells us that the second fundamental form of $\mathcal{Z}$ is also non-zero exactly at points where $\Delta s \neq 0$. Finally, since the unit normal vector field is a constant multiple of $J\xi$, the restriction of $\mathrm{I\!I}(J\cdot, \cdot)$ to $\xi^\perp \subset T\mathcal{Z}$ is a non-zero constant times the intrinsic covariant derivative $\nabla\xi$, and since $\mathcal{Z}$ is umbilic, it therefore follows that $\mathrm{I\!I} \neq 0$ exactly when $|\nabla\xi|^2 \neq 0$; but the Bochner Weitzenböck formula (15) for a Killing field tells us that $|\nabla\xi|^2 \equiv r(\xi, \xi)$ on $(\mathcal{Z}, \check{g})$, so the positivity of the Ricci curvature of $\check{g}$ in the direction of ξ is also equivalent to all the other conditions under discussion. □

[1] These complete Einstein manifolds seem to have been first discovered by Bérard-Bergery [6], who made a systematic study of cohomogeneity-one Einstein metrics. They have subsequently been rediscovered several times by various groups of physicists [14,41].

In the known examples, $\mathcal{Z}$ is actually strictly pseudo-convex. Is this a general feature of all solutions, or is it a mere artifact, reflecting the fact that the known solutions have universal covers of cohomogeneity one?

As long as we only consider Bach-flat Kähler metrics that are compatible with some fixed complex structure J on M, Conjecture 1 claims that the solution type, as per Theorem A, should be completely determined by (M, J). While there is a preponderance of evidence in favor of such a conjecture, it is also important to notice that the type of the solution is certainly not just determined by the diffeotype of M alone. For example, while the smooth manifolds $S^2 \times S^2$ and $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ each support a unique complex structure with $c_1 > 0$, each also carries an infinite number of other complex structures realized by the various Hirzebruch surfaces $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(\ell)) \to \mathbb{CP}_1$, $\ell \geq 0$. Now, every Hirzebruch surface with $\ell > 2$ carries [25] a Bach-flat Kähler metric of type III(b), in contrast to the solutions of type I(b) that instead exist when $\ell = 0$ and 1. Similarly, the 4-manifolds arising as S^2-bundles over curves of genus ≥ 2 carry solutions of both of type II(b) and III(b), but this form of peaceful co-existence is once again only made possible by allowing the complex structure to vary.

Nonetheless, Theorem A does have consequences that do primarily reflect the differential topology of the underlying 4-manifold:

Proposition 12 *Let M be the underlying smooth 4-manifold of a non-minimal compact complex surface of Kodaira dimension ≥ 0. Then there is no complex structure J on M for which (M, J) admits a Bach-flat Kähler metric. Moreover, for complex surfaces of Kodaira dimension 1, the existence of Bach-flat Kähler metrics is similarly obstructed even when the surface is minimal.*

Proof On a compact complex surface (M, J) of Kodaira dimension ≥ 0, Theorem A tells us that any Bach-flat Kähler metric must be Kähler–Einstein, with Einstein constant $\lambda \leq 0$. This in particular either means that $c_1^{\mathbb{R}} = 0$ or $c_1 < 0$. Hence (M, g) must be minimal and have Kodaira dimension 0 or 2. However, for complex surfaces of Kähler type, Seiberg–Witten theory implies [33,38] that Kodaira dimension is a diffeomorphism invariant, and that non-minimality is moreover a diffeomorphism invariant whenever the Kodaira dimension is ≥ 0. Thus the operative obstruction really just reflects the differential topology of M, in a manner that is insensitive to the detailed complex geometry of the given J. □

Finally, it should perhaps also be emphasized that Proposition 12 certainly does not obstruct the existence of more general Bach-flat metrics. Indeed, a result of Taubes [45] implies that any complex surface has blow-ups that admit anti-self-dual metrics. Thus, there are certainly many non-minimal complex surfaces of each possible Kodaira dimension ≥ 0 that do indeed admit Bach-flat metrics; it is just that these metrics do not happen to be conformally Kähler!

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