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On Hyperbolic Metric and Invariant Beltrami Differentials for Rational Maps

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Abstract Given a rational map R, we consider the complement of the postcritical set S_R. In this paper we discuss the existence of invariant Beltrami differentials supported on an R invariant subset X of S_R. Under some geometrical restrictions on X, we show the absence of invariant Beltrami differentials with support intersecting X. In particular, we show that if X has finite hyperbolic area, then X cannot support invariant Beltrami differentials except in the case where R is a Lattès map.

Keywords Rational maps · hyperbolic metric · Teichmuller space

Mathematics Subject Classification 37F10 · 37F30 · 32A70

1 Introduction

This article is a complementary part to the work done in [3] with its own independent interest. We discuss geometric conditions under which there are no invariant Beltrami differentials supported on the dissipative set of a rational map R.

In this paper we will always assume that the Fatou set does not contain rotation domain cycles.

Now, let us introduce the geometric objects to be treated in this paper.

Denote by $J(R)$ the Julia set and by $P(R)$ the closure of the postcritical set of R. Consider the surface $S_R := \bar{\mathbb{C}} \backslash P(R)$, this surface is not always connected; however,

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on each connected component of S_R we fix a Poincaré hyperbolic metric and denote by λ the family of all these metrics.

Let $Q(S_R)$ be the subspace of $L_1(S_R)$ of holomorphic integrable functions on S_R.

A rational map R defines a complex pushforward map on $L_1(\mathbb{C})$, with respect to the Lebesgue measure m, which is a contracting endomorphism and is called the complex Ruelle–Perron–Frobenius, or the Ruelle operator for short. The Ruelle operator has the following formula:

$$R^*(\phi)(z) = \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{R'(y)^2} = \sum_i \phi(\zeta_i(z))(\zeta_i'(z))^2,$$

where ζ_i is a local complete system of branches of R^{-1}. The space $Q(S_R)$ is invariant under the action of the Ruelle operator. The Beltrami operator Bel : $L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C})$ given by

$$\text{Bel}(\mu) = \mu(R)\frac{\overline{R'}}{R'}$$

is dual to the Ruelle operator acting on $L_1(\mathbb{C})$.

The fixed point space $Fix(B)$ of the Beltrami operator is called the *space of invariant Beltrami differentials*. An element $\alpha \in L_\infty(\mathbb{C})$ is called non-trivial if and only if the functional given by

$$v_\alpha(\phi) = \int_{S_R} \phi(z)\alpha(z)|\text{d}z|^2$$

is non-zero on $Q(S_R)$. The norm of v_α in $Q^*(S_R)$, for a non-trivial element α, is called the *Teichmüller norm* of α and it is denoted by $\|\alpha\|_T$.

A non-trivial element α is called *extremal* if and only if $\|\alpha\|_\infty = \|\alpha\|_T$.

A sequence of unit vectors $\{\phi_i\}$ is called a *Hamilton–Krushkal* sequence for an extremal element α if and only if

$$\lim_{i\to\infty} |v_\alpha(\phi_i)| = \|\alpha\|_\infty.$$

A Hamilton–Krushkal sequence $\{\phi_i\}$ is called *degenerating* if it converges to 0 uniformly on compact sets.

Now we recall some basic concepts from ergodic theory which will be used throughout this text (see for example [6]). For $A, B \subset \bar{\mathbb{C}}$ measurable sets, the expressions $A = B$ and $A \cap B = \emptyset$ are understood up to sets of zero Lebesgue measure. A positive measure set $M \subset \bar{\mathbb{C}}$ is called *wandering* if the sets $\{R^{-k}(M)\}_k$ are pairwise disjoint Lebesgue measurable sets. Let $D(R)$ be the union of all wandering sets. The set $D(R)$ is called the *dissipative set* and the complement $C(R) = \mathbb{C} \backslash D(R)$ is the *conservative set*. Due to the classification of the components of the Fatou set, $C(R)$ intersects the Fatou set precisely at the union of rotation domain cycles. Hence, our assumption that there are no rotation domain cycles on the Fatou set implies that $C(R) \subset J(R)$.

Lemma 1 *We have the following facts:*

(1) *For every* $f \in L_1(\mathbb{C})$*, the sequence* $(R^*)^n(f)$ *converges to* 0 *almost everywhere on* $D(R)$.

(2) *If* $m(Y) < \infty$*, where* $Y = \bigcup_{n\geq 0} R^{-n}(W)$ *for a wandering set* W*, then for every* $f \in L_1(Y)$ *we have*

$$\lim_n \int_Y |(R^*)^n(f(z))||dz|^2 = 0.$$

(3) *If* $C(R) \neq \mathbb{C}$ *then for every* $\phi \in Q(S_R)$ *the sequence* $(R^*)^n(\phi)$ *converges pointwise to* 0 *on* S_R.

Proof Indeed, for the first part of the lemma it is sufficient to check that the series $\sum_{n=0}^{\infty}(R^*)^n(f)(z)$ converges absolutely almost everywhere on every wandering set W. In other words, it is enough to show that the function $\sum_{n=0}^{\infty}|(R^*)^n(f)|$ is integrable on W. But

$$\begin{aligned}
\int_W \sum_{n=0}^{\infty} |(R^*)^n(f)(z)||\mathrm{d}z|^2 &= \sum_{n=0}^{\infty}\int_W |(R^*)^n(f)(z)||\mathrm{d}z|^2 \\
&\leq \sum_{n=0}^{\infty}\int_{R^{-n}(W)} |f(z)||\mathrm{d}z|^2 \\
&\leq \int_{\bigcup_n R^{-n}(W)} |f(z)||\mathrm{d}z|^2 \\
&\leq \int_{\mathbb{C}} |f(z)||\mathrm{d}z|^2.
\end{aligned}$$

The assumption on the second part means that $m(R^{-n}(Y))$ converges to 0, then the inequality

$$\int_Y |(R^*)^n f(z)||\mathrm{d}z|^2 \leq \int_{R^{-n}(Y)} |f(z)||\mathrm{d}z|^2$$

finishes the proof.

For the last part, if a connected component $S \subset S_R$ intersects $D(R)$ on a set of positive Lebesgue measure, then for every $\phi \in Q(S_R)$, the sequence $(R^*)^n(\phi)$ converges to 0 pointwise on S. To check this, note that $(R^*)^n(\phi)$ is a sequence of holomorphic functions on S with uniformly bounded integrals

$$\int_S |(R^*)^n(\phi(z))||\mathrm{d}z|^2 \leq \|\phi\|_{L_1(S_R)}$$

hence by the mean value theorem $(R^*)^n(\phi)$ forms a normal family which converges to 0 on a set of positive measure.

Recall that $C(R) \neq \bar{\mathbb{C}}$ and $C(R) \subset J(R)$. If $J(R) \neq \bar{\mathbb{C}}$, then again by the classification of the components of the Fatou set, every component of S_R intersects the dissipative set. If $J(R) = \bar{\mathbb{C}}$, then $D(R)$ is a dense subset of $J(R)$ and hence $D(R)$ also intersects every component of S_R. □

Let $T : \mathcal{B} \to \mathcal{B}$ be a linear contraction of a Banach space $\mathcal{B}$. An element $b \in \mathcal{B}$ is called *mean ergodic* with respect to T if and only if the sequence of Cesàro averages with respect to T, given by $C_n(b) = \frac{1}{n}\sum_{i=0}^{n-1} T^i(b)$, forms a weakly precompact family. Indeed, if $y \in \mathcal{B}$ is a weak accumulation point of the sequence $C_n(b)$, then the $C_n(b)$ converges in norm to y and y is a fixed point for T (see Krengel [6]). If every element $b \in \mathcal{B}$ is mean ergodic with respect to T then the operator T is called *mean ergodic*.

By the Bers representation theorem, the space $Q^*(S_R)$ is linearly quasi-isometrically isomorphic to the *Bers* space $B(S_R)$ which is the space of holomorphic functions ϕ on S_R with the norm $\|\lambda^{-2}\phi\|_{L_\infty(S_R)}$.

In the case where S_R has finitely many components, a classical theorem states that $Q(S_R)$ is continuously included in $B(S_R)$ if and only if the infimum of the length of simple closed geodesics is bounded away from 0 (see for example [9] and references within). If S_R has infinitely many components, then one can still verify that if the infimum of the length of simple closed geodesics is bounded away from 0 then the inclusion $Q(S_R)$ in $B(S_R)$ is continuous. However, we do not know if there is an example of a rational map R such that S_R has infinitely many components.

2 Main Theorem

Let X be an R forward-invariant set of positive Lebesgue measure, then the set $W := \bigcup R^{-n}(X)$ is its saturation undertaking pre-images. In the following theorem we will only consider Cesàro averages with respect to the Ruelle operator R^* in $L_1(W)$.

Theorem 2 *Let $X \subset S_R$ be an R forward-invariant set of positive measure such that the restriction map $r(\phi) = \phi|_X$ from $Q(S_R)$ to $L_1(X)$ is weakly precompact. Then for every $\phi \in Q(S_R)$, the function $\phi|_W$ is mean ergodic with respect to R^* in $L_1(W)$.*

Proof If X is R forward-invariant then the Ruelle operator R^* defines an endomorphism of $L_1(X)$. Given $\phi \in Q(S_R)$, the family of Cesàro averages $C_n(\phi)$ restricted to X forms a weakly precompact subset of $L_1(X)$. We claim that $C_n(\phi)$ converges in norm on $L_1(X)$. Indeed, first we show that every weak accumulation point of $C_n(\phi)$ is a fixed point for the Ruelle operator. Let $f \in L_1(X)$ be the weak limit of $C_{n_i}(\phi)$, for some subsequence $\{n_i\}$, then $R^*(f)$ is the weak limit of $R^*(C_{n_i}(\phi))$. By the Fatou Lemma

$$\begin{aligned}\int_X |f - R^*(f)| &\le \liminf \int_X |C_{n_i}(\phi) - R^*(C_{n_i}(\phi))| \\ &\le \liminf \|C_{n_i}(\phi) - R^*(C_{n_i}(\phi))\|_{L_1(S_R)} \\ &\le \limsup \|C_{n_i}(I - R^*)(\phi)\|_{L_1(S_R)}.\end{aligned}$$

But

$$\|C_{n_i}(I - R^*)(\phi)\|_{L_1(S_R)} \le \frac{2}{n_i}\|\phi\|_{L_1(S_R)}.$$

Then f is a non-zero fixed point of the Ruelle operator. As in Lemma 11 of [8] we have that $|f|$ defines a finite absolutely continuous invariant measure, and even more, by Corollary 12 of [8] the set $\bigcup_n R^{-n}(\text{supp } f)$ supports an invariant Beltrami differential μ such that $\mu(z) = \frac{|f(z)|}{f(z)}$ almost everywhere on supp f. Since the support of f is a non-trivial subset of the conservative set of R, by the Poincaré recurrence theorem, almost every point of supp f is recurrent. By Lyubich's ergodicity theorem (see [7], Theorem 3.9 in [10]) and the fact that X does not intersect the postcritical set, we have supp $f = \mathbb{C}$ up to a set of measure 0. But f is non-zero, hence $\mu \not\equiv 0$. By Theorem 3.17 in [10], the map R is, so-called, a *flexible Lattès map*, in particular R is postcritically finite. In this situation, the space $Q(S_R)$ is finitely dimensional, the Ruelle operator R^* is a compact endomorphism of $Q(S_R)$, which implies that R^* is mean ergodic on $Q(S_R)$.

If R is not a flexible Lattès map then every weak limit of $C_n(\phi)$ is 0. Since the weak closure of convex bounded sets is equal to the closure in norm of convex bounded sets, we conclude our claim.

Now we promote the mean ergodicity of R^* on $L_1(X)$ to mean ergodicity on $L_1(W)$ as follows. Let $W_n = R^{-n}(X)$, one can inductively prove that $\phi|_{W_n}$ is mean ergodic with respect to R^* on $L_1(W_n)$. Indeed, let $\psi_n = \phi|_{W_n}$, since $R^* : L_1(W_n) \to L_1(W_{n-1}) \subset L_1(W_n)$ and $R^*(\psi_n) = R^*(\phi)|_{W_{n-1}}$, then by arguments above we are done.

Consider $\phi|_W - \phi|_{W_n}$, the L_1 norm of this difference converges to 0 in $L_1(W)$, since the Cesàro averages does not expand the L_1 norm we have

$$\|C_k(\phi|_W - \phi|_{W_n})\| \le \|\phi|_W - \phi|_{W_n}\|.$$

Hence $C_k(\phi|_W)$ converges to 0 and $\phi|_W$ is mean ergodic with respect to R^* on $L_1(W)$. □

Then we have the following immediate corollary.

Corollary 3 *The Ruelle operator R^* has a non-zero fixed point in $Q(S_R)$ if and only if R is a flexible Lattés map.*

Proof Recall that $C(R) \subset J(R)$. If $D(R)$ has positive Lebesgue measure, then by Lemma 1 for every $\phi \in Q(S_R)$ the Cesàro averages $C_n(\phi)$ converge to 0 pointwise on S_R. Hence, if $\phi \in Q(S_R)$ is a non-zero fixed point of R^* then $D(R)$ has zero Lebesgue measure. Thus, as in Theorem 2 by Corollary 12 in [8] there exist $\mu(z) = \frac{|\phi(z)|}{\phi(z)}$ invariant Beltrami differential and $supp(|\phi|) = \overline{C(R)} = J(R) = \bar{\mathbb{C}}$. Again by Theorem 3.17 in [10] the map R is a flexible Lattés map. □

Now we state our Main Theorem.

Theorem 4 *Let R be a rational map and let $X \subset S_R$ be an R forward-invariant set of positive Lebesgue measure. Assume that the restriction map $r(\phi) = \phi|_X$ from $Q(S_R)$ into $L_1(X)$ is weakly precompact. If μ is a non-trivial invariant Beltrami differential, then $m(supp\ \mu \cap X) > 0$ if and only if R is a flexible Lattès map.*

Proof Assume that R is a flexible Lattès map. Then R is ergodic on the Riemann sphere, and therefore, the support of every invariant Beltrami differential μ is the whole Riemann sphere. Hence, if X is a forward-invariant set of positive Lebesgue measure then $m(\text{supp }\mu \cap X) = m(X) > 0$.

Again let $W = \bigcup R^{-n}(X)$. Now let μ be a non-trivial invariant Beltrami differential supported on W. Then for every $\phi \in Q(S_R)$ we have

$$\int_{S_R} \mu(z)\phi(z)|\mathrm{d}z|^2 = \int_{S_R} \mu(z)C_k(\phi(z))|\mathrm{d}z|^2 = \int_W \mu(z)C_k(\phi(z))|\mathrm{d}z|^2.$$

By Theorem 2, if R is not Lattès, the right- hand side converges to 0 as k converges to ∞. Hence $\int \phi\mu = 0$ for every $\phi \in Q(S_R)$ and the functional $\phi \mapsto \int \phi\mu$ is 0 on $Q(S_R)$. This contradicts the assumption that μ is non-trivial. □

In the proofs of the previous theorems, the only ingredient was the precompactness of the Cesàro averages $C_n(\phi)$. Hence, it is enough to assume the weak precompactness only of Cesàro averages on elements of $Q(S_R)$. By results of the second author in [8], see also a related work in [3], it is enough to consider the Cesàro averages of rational functions in $Q(S_R)$ having poles only on the set of critical values.

3 Compactness

We will discuss the conditions under which the restriction map $\phi \mapsto \phi|_A$ is weakly precompact. Unfortunately, so far we have not found conditions where the restriction is weakly precompact but not compact. Let us start with the following definition.

Definition A rational map R satisfies the B-condition if and only if for every $\phi \in Q(S_R)$ we have

$$\|\lambda^{-2}(z)\phi(z)\|_{L_\infty(S_R)} \leq C\|\phi(z)\|_{L_1(S_R)},$$

where C is a constant independent of ϕ.

In other words, if R satisfies the B-condition, then $Q(S_R) \subset B(S_R)$ and the inclusion map $Q(S_R) \to B(S_R)$ is continuous. As it was noted in the introduction, this happens when the infimum of the length of simple closed geodesics on S_R is bounded away from 0.

Proposition 5 *If R satisfies the B-condition and $\text{Area}_\lambda(X) < \infty$ then the restriction map $r_X : \phi \mapsto \phi|_X$ from $Q(S_R)$ to $L_1(X)$ is a compact operator.*

Proof If R satisfies the B-condition then

$$\lambda^{-2}(z)|\phi(z)| \leq \sup_{z\in S_R} |\lambda^{-2}(z)\phi(z)| \leq C\|\phi\|_{L_1(S_R)},$$

hence $|\phi(z)| \leq C\|\phi\|_{L_1(S_R)}\lambda^2(z)$ for $z \in S_R$. If $\phi_i \in Q(S_R)$ is a sequence with $\|\phi_i\|_{L_1(S_R)} \leq 1$, then ϕ_i forms a normal family of holomorphic locally uniformly bounded functions on S_R. If ψ is a pointwise limit of a subsequence ϕ_{i_j}, then by Fatou's Lemma $\psi \in Q(S_R)$, and by the Lebesgue dominated convergence theorem, $r_X(\phi_{i_j})$ converges to $r_X(\psi)$ in norm in $L_1(X)$. Hence r_X is a compact operator. □

Combining Theorem 4 and Proposition 5 we have the following.

Corollary 6 *Assume that R satisfies the B-condition and X is a forward-invariant set of positive Lebesgue measure with* $Area_\lambda(X) < \infty$*. If* μ *is a non-trivial invariant Beltrami differential, then* $m(supp\ \mu \cap X) > 0$ *if and only if R is a flexible Lattès map.*

In general, the finiteness of the hyperbolic area of X does not imply the finiteness of hyperbolic area of W. Generically, it could be that the hyperbolic area of W is infinite regardless of the area of X.

On the other hand, by Corollary 6, if R satisfies the B-condition and the hyperbolic area $\text{Area}_\lambda(J(R))$ is bounded then R satisfies the Sullivan's conjecture, which states: *A rational map R admitting a non-zero invariant Beltrami differential supported on the Julia set is a flexible Lattès map.* However, in this situation, we believe that the following stronger statement holds true:

The $\text{Area}_\lambda(J(R)) < \infty$ if and only if either $m(J(R)) = 0$ or R is postcritically finite.

Let us discuss how the B-condition fits into the context of holomorphic dynamics.

First, due to McMullen (Theorem 8.4 in [11]) there exist a set of quadratic polynomials R such that $P(R) \cap J(R)$ is a Cantor set with bounded geometry, this is equivalent to the fact that the length of simple closed geodesics on S_R is bounded above and below. These polynomials are infinitely renormalizable maps with bounded combinatorics and definite moduli also known as polynomials with a priori bounds (see [11]). Even more, McMullen's rigidity result was based on the boundedness of the length of simple closed geodesics from above. Recently, Avila and Lyubich announced that among polynomials with a priori bounds there are polynomials with Julia set of positive Lebesgue measure (see [1]). Moreover, these examples form a set of positive Hausdorff dimension under a suitable parameterization.

Second, by a result due to Childers (see Theorem 1.2 in [4]) there are polynomials such that $J(R) \cap P(R)$ is a non-separating planar continuum; these examples include Cremer polynomials with only one recurrent critical point. In this case, the surface S_R is conformally equivalent to the punctured unit disk. Moreover, the well-known examples of Buff and Cheritat of polynomials with Julia set of positive Lebesgue measure are Cremer quadratic polynomials (see [2]).

By results of [9], the examples above satisfy the B-condition. However, if an infinite renormalizable quadratic polynomial satisfies the B-condition then not necessarily has a priori bounds.

In general, one of the obstacles for a planar Riemann surface to satisfy the B-condition is the existence of infinitely many cusps. As the arguments of Theorems 2 and 4 show, it is sufficient to have the B-condition on a small neighborhood of $J(R) \cap P(R)$. So, if R does not have parabolic periodic points, the behavior of the postcritical set on the Fatou set is irrelevant to study Beltrami differentials supported on the Julia set.

Now, let R be such that $S_R \cap J(R)$ contains infinitely many cusps of S_R, this is equivalent to say that there exists a critical point $c \in J(R)$ such that $R^n(c)$ is an infinite sequence of cusps of S_R. In this case, $R^n(c)$ is non-recurrent for every n and, using Shishikura and Lei generalization of Mañé's theorem [14], the series

$$\sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(c))}$$

is geometric, and in particular, is absolutely convergent with non-zero sum. Then by results of the second author, such a map R is not structurally stable (see [8]). In other words, this means that the dimension of the space of invariant Beltrami differentials on the Julia set is strictly smaller than the number of critical values on the Julia set. Therefore, possible candidates to satisfy the B-condition on the whole S_R are rational maps with finitely many postcritical points on the Fatou set.

Now we consider a more general condition when the restriction map r_X is compact and which reflects the geometry of the postcritical set.

On the product $S_R \times S_R \subset \mathbb{C}^2$ there exists a unique function $K(z, \zeta)$ which is characterized by the following conditions.

(1) $K(\zeta, z) = -\overline{K(z, \zeta)}$
(2) For every $\zeta_0 \in S_R$, the function $\phi_{\zeta_0}(z) = K(z, \zeta_0)$ belongs to the intersection $Q(S_R) \cap B(S_R)$.
(3) If z_0, ζ_0 belong to different components of S_R, then $K(z_0, \zeta_0) = 0$.
(4) The operator $P(f)(z) = \int \lambda^{-2}(\zeta)K(z, \zeta)f(\zeta)|d\zeta|^2$ from $L_1(S_R)$ to $L_1(S_R)$ is a continuous projection with $P(L_1(S_R)) = Q(S_R)$.

In fact, the function $K(z, \zeta)$ is defined on every planar hyperbolic Riemann surface S. For further details on this subject, see for example Chap. 3, §7 of the book of Kra [5].

Let us consider the function

$$\omega(\zeta, z) = \lambda^{-2}(\zeta)K(z, \zeta)$$

and

$$\alpha(z) = \sup_{\zeta \in S_R} |\omega(z, \zeta)|.$$

The following proposition is a consequence of the property (4) above.

Proposition 7 *If*

$$\int_X \alpha(z)|dz|^2 < \infty$$

then r_X is a compact operator.

Proof For every $\phi \in Q(S_R)$ and every $z \in S_R$ property (4) gives

$$\phi(z) = \int_{S_R} \lambda^{-2}(\zeta)K(z, \zeta)\phi(\zeta)|\mathrm{d}\zeta|^2$$

hence

$$|\phi(z)| \leq \sup_{\zeta \in S_R} |\lambda^{-2}(\zeta)K(z,\zeta)| \int_{S_R} |\phi(\zeta)||d\zeta|^2$$

thus

$$|\phi(z)| \leq \alpha(z)\|\phi\|_{L_1(S_R)}$$

for almost every $z \in S_R$.

When α is integrable on X, then again as in the proof of Proposition 5 and the Lebesgue dominated convergence theorem r_X is a compact operator. □

As a consequence we have:

Corollary 8 *If* $\int_{J(R)} \alpha(z)|dz|^2 < \infty$ *then* R *satisfies Sullivan's conjecture.*

Proof If an invariant Beltrami differential μ has support in $J(R)$ then μ is non-trivial (see Theorem 3 in [8]), now applying Theorem 4 and Proposition 7, we finish the proof. □

Let us note that Propositions 5 and 7 also imply that if X is a positive Lebesgue measure set, satisfying an integrability condition, then X cannot support extremal differentials with degenerating sequences.

Hence, Corollaries 6 and 8, in the case when X is a completely invariant, derive from results in [3].

Remark If S_R has finitely many components and R satisfies the B-condition then by classical results (see the comments before Proposition 1 in [12]), we have that $\alpha(z) \leq C\lambda^2(z)$ where C does not depend on z. In particular, if X has bounded hyperbolic area then $\alpha(z)$ is integrable on X. In this situation, Proposition 7 implies Proposition 5. As it is mentioned in [12], the conditions of Proposition 7 are strictly weaker than conditions of Proposition 5. Moreover, in general, the boundedness of the hyperbolic area is not a quasiconformal invariant (see Proposition 3 in [13]).

4 Quasi-Compactness

Let Y_n be an exhaustion of S_R by compact subsets such that $m(Y_{n+1}\backslash Y_n)$ converges to zero. Let P_n be the sequence of restrictions $P_n : L_1(S_R) \to L_1(S_R)$ given by $P_n(f) = \chi_n P(f)$ where χ_n is the characteristic function on Y_n. Immediately from the definition we have the following facts:

(1) For each n, the map P_n is a compact operator.
(2) The limit

$$\lim_{n\to\infty} \|P_n(f) - P(f)\|_{L_1(S_R)} \to 0$$

for all f on $L_1(S_R)$.

Definition A measurable set $X \subset S_R$ is called inefficient at infinity, or inefficient for short, if and only if there exists an exhaustion as above so that

$$\inf_n \|P_n - P\|_{L_1(X)} < 1.$$

In terms of operator theory, the operator $r_X \circ P$ is called a *quasi-compact* operator. This quasi-compactness, in general, does not imply that r_X is weakly compact.

We have the following theorem:

Theorem 9 *Let $\mu \neq 0$ be an invariant Beltrami differential such that supp μ is a finite measure subset of $J(R)$. If $X \subset S_R$ is an inefficient forward-invariant measurable set, then $m(supp\ \mu \cap X) = 0$.*

In fact, we can reformulate Theorem 9 as: the Ruelle operator $R^* : L_1(W) \to L_1(W)$ is mean ergodic, whenever $W = \bigcup R^{-n}(X)$ with a forward-invariant inefficient set X of positive Lebesgue measure. Compare with Theorems 2 and 4.

To prove the theorem we need the following general lemmas:

Lemma 10 *Let μ be an invariant Beltrami differential with supp $\mu \subset J(R) \cap S_R$ then μ is an extremal Beltrami differential.*

Proof First we show that μ admits an extremal representative. Indeed, if μ is not extremal then by the Banach extension theorem and Riesz representation theorem there exist another Beltrami differential α such that α is extremal, satisfies $\|\alpha\|_\infty = \|\mu\|_T < \|\mu\|_\infty$, and defines the same functional as μ in $Q(S_R)$. Let β be a $*$-weak limit of the Cesàro averages $C_n(\alpha) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha(R^i)\frac{\overline{(R^i)'}}{(R^i)'}$, then $\beta(R)\frac{\overline{R'}}{R'} = \beta$ almost everywhere and $\|\beta\|_\infty \leq \|\alpha\|_\infty$. Then we claim that $v_\beta = v_\mu$. Let $\{C_{n_i}(\alpha)\}$ be a sequence of averages $*$-weakly converging to β. For every $\gamma \in Q(S_R)$ we have

$$\int_{S_R} \gamma(z)\beta(z)|dz|^2 = \lim_i \int_{S_R} C_{n_i}(\alpha(z))\gamma(z)|dz|^2$$

by duality the previous limit is equal to

$$\lim_i \int_{S_R} \alpha(z)\frac{1}{n_i}\sum_{k=0}^{n_i-1} R^{*k}(\gamma(z))|dz|^2 = \lim_i \int_{S_R} \mu(z)\frac{1}{n_i}\sum_{k=0}^{n_i-1} R^{*k}(\gamma(z))|dz|^2$$

but μ is an invariant differential and again using duality the previous limit becomes

$$\lim_i \int_{S_R} \mu(z)\frac{1}{n_i}\sum_{k=0}^{n_i-1} R^{*k}(\gamma(z))|dz|^2 = \int_{S_R} \mu(z)\gamma(z)|dz|^2.$$

Since α is extremal we have $\|\beta\|_\infty = \|\alpha\|_\infty = \|\mu\|_T$. Thus β is the desired extremal invariant differential.

Now, let us consider $\beta = \beta_1 + \beta_2$, where β_1 and β_2 are the restrictions of β on the Fatou and Julia set, respectively, extended by 0 on its complements. Note that $\beta_2 \neq 0$, otherwise β is supported on the Fatou set and defines the same functional as μ which contradicts Theorem 3 in [8].

If $\beta_2 \neq \mu$ almost everywhere, then $v_{\beta_1} = v_{\mu-\beta_2}$ which again contradicts Theorem 3 in [8]. Then $\beta_2 = \mu$ and $v_{\beta_1} = 0$, hence μ is extremal. □

Lemma 11 *An inefficient set does not support extremal differentials with degenerating sequences.*

Proof We argue by contradiction. Let μ be an extremal differential supported on an inefficient set X. Suppose that $\{\phi_n\}$ is a degenerating sequence for μ.

By assumption, there exist n_0 such that

$$\sup_{f\in L_1(X),\|f\|=1} \int |P_{n_0}(f) - P(f)| = r < 1.$$

Since ϕ_n is degenerating and by the compactness of P_{n_0} we have that

$$\lim_{j\to\infty} \|P_{n_0}(\phi_j)\|_{L_1(S_R)} \to 0.$$

Hence

$$\begin{aligned}\|\mu\|_\infty &= \lim_j \left| \int_{\text{supp } \mu} \mu(z)\phi_j(z)|\mathrm{d}z|^2 \right| \\ &= \lim_j \left| \int_{\text{supp } \mu} \mu(z)(P_{n_0}(\phi_j(z)) - P(\phi_j(z)))|\mathrm{d}z|^2 \right| \\ &\leq \|\mu\|_\infty \sup_{f\in L_1(X),\|f\|=1} \int |P_{n_0}(f(z)) - P(f(z))||\mathrm{d}z|^2 = r\|\mu\|_\infty < \|\mu\|_\infty,\end{aligned}$$

which is a contradiction. □

As it is shown by the arguments of the previous lemma, it is sufficient to consider any measurable exhaustion where P_k are weakly compact operators. For instance, one can consider an exhaustion satisfying the conditions of Propositions 5 and 7. A simple corollary of the previous two lemmas is the following.

Corollary 12 *Let μ be an invariant Beltrami differential supported on $J(R)$, then the supp μ is inefficient if and only if R is a flexible Lattès map.*

Proof If R is a flexible Lattès map, then $Q(S_R)$ is finite dimensional then the operators P_n converge to P in norm. Hence, the infimum $\inf_n \|P_n - P\|_{L_1(\text{supp } \mu)} = 0$ and therefore supp μ is an inefficient set.

If supp μ is inefficient, then by Lemmas 10 and 11, μ is an extremal differential which does not accept degenerating sequences.

Let ϕ_j be a Hamilton–Krushkal sequence for μ and fix P_{n_0} and r as in the proof of Lemma 11. Choose $\epsilon = \frac{1-r}{2}$, then there exists an i_0 so that

$$(1-\epsilon)\|\mu\|_\infty \leq \left| \int_{S_R} \mu(z)\phi_{i_0}(z)|\mathrm{d}z|^2 \right|.$$

Let $\psi_k = C_{l_k}(\phi_{i_0})$ be a sequence of Cesàro averages which pointwise converge to a function ψ, then by the Fatou lemma $\psi \in Q(S_R)$. Let us show that $\psi \not\equiv 0$. Otherwise, ψ_k converges pointwise to 0 and $P_{n_0}(\psi_k)$ converges to 0 in norm in $L_1(\text{supp } \mu)$. By the invariance of μ we have

$$\begin{aligned}(1-\epsilon)\|\mu\|_\infty &\le \left|\int_{S_R} \mu(z)\phi_{i_0}(z)|\mathrm{d}z|^2\right| = \left|\int_{\text{supp } \mu} \mu(z)\psi_k(z)|\mathrm{d}z|^2\right| \\ &= \left|\int_{\text{supp } \mu} \mu(z)[P(\psi_k(z)) - P_{n_0}(\psi_k(z))]|\mathrm{d}z|^2\right. \\ &\quad \left. + \int_{\text{supp } \mu} \mu(z)P_{n_0}(\psi_k(z))|\mathrm{d}z|^2\right|\end{aligned}$$

then again, as in the Lemma 11, we get a contradiction for sufficiently large k. So, ψ is not identically 0. Since R^* is continuous with respect to the pointwise topology then $R^*(\psi) = \psi$. Now, by Corollary 3 we are done. □

In the situation of Corollary 12, we have that supp $\mu = \bar{\mathbb{C}}$ almost everywhere, and hence, every completely invariant set X of positive Lebesgue measure is a full measure subset of $\bar{\mathbb{C}}$.

If S_R is connected then Corollary 12 is a simple consequence of Lemma 11 and Teichmüller theory. In this case, μ is so-called a Teichmüller differential. Unfortunately, we did not find in the literature analogous statements for when S_R has infinitely many components.

Now we are ready to prove Theorem 9 using similar ideas as in the proof of Corollary 12.

Proof of Theorem 9 By contradiction, assume that $m(\text{supp } \mu \cap X) > 0$. Without loss of generality, we can assume that supp $\mu = \bigcup_{n\ge 0} R^{-n}(X)$. By Lemma 10, μ is an extremal differential. Let us show that μ does not accept degenerating sequences. If ϕ_i is a degenerating sequence for μ, then for every $\epsilon > 0$ there exists an i_0 such that

$$(1-\epsilon)\|\mu\|_\infty \le \left|\int_{\text{supp } \mu} \mu(z)\phi_{i_0}(z)|\mathrm{d}z|^2.\right|$$

By invariance of μ, the last expression is equal to

$$\lim_n \left|\int_{\text{supp } \mu} \mu(R^*)^n(\phi_{i_0}(z))|\mathrm{d}z|^2\right|.$$

But supp $\mu \backslash X = \bigcup_{n\ge 0} R^{-n}\left[R^{-1}(X)\backslash X\right]$. By Lemma 1,

$$\lim_n \int_{\text{supp } \mu} \mu(z)(R^*)^n(\phi_{i_0}(z))|\mathrm{d}z|^2 = \lim_n \int_X \mu(z)(R^*)^n(\phi_{i_0}(z))|\mathrm{d}z|^2.$$

If $m(\text{supp } \mu\backslash X) > 0$, then again by Lemma 1, $(R^*)^n(\phi_{i_0})$ converges pointwise to 0 on S_R. Since X is inefficient, there exist $\delta > 0$ and k_0 such that $\|P - P_{k_0}\|_{L_1(X)} \le 1-\delta$.

Since $(R^*)^n(\phi_{i_0})$ converges to 0 pointwise and P_{k_0} is compact then $P_{k_0}((R^*)^n(\phi_{i_0}))$ converges to 0 in the $L_1(X)$ norm. Therefore

$$\begin{aligned}(1-\epsilon)\|\mu\|_\infty &\leq \lim_n \left| \int_X \mu(z)(R^*)^n(\phi_{i_0})(z)|\mathrm{d}z|^2 \right| \\ &= \lim_n \left| \int_X \mu(z) \left[P((R^*)^n(\phi_{i_0}(z))) - P_{k_0}((R^*)^n(\phi_{i_0}(z))) \right] |\mathrm{d}z|^2 \right| \\ &\leq (1-\delta)\|\mu\|_\infty.\end{aligned}$$

Taking $\epsilon < \delta$ we get a contradiction.

If $m(\text{supp}\ \mu \backslash X) = 0$ then supp μ is inefficient and μ does not accept degenerating sequences by Lemma 11. Altogether μ does not accept degenerating sequences.

Now, as in the last part of the proof of Corollary 12, the map R is a flexible Lattès map and supp μ has full Lebesgue measure on $\mathbb{C}$, this is a contradiction to $m(\text{supp}\ \mu) < \infty$. □

The following proposition is an illustration of when the conditions of Theorem 9 are fulfilled.

Proposition 13 *If R is a rational map satisfying the B condition. If A is a measurable subset of S_R with*

$$\int_A \int_{S_R} |K(z,\zeta)||dz|^2|d\zeta|^2 < \infty$$

then for any exhaustion of S_R by measurable sets Y_n and operators P_n defined as above we have $\lim \|P_n - P\|_{L_1(A)} = 0$.

Proof Let Y_n be an exhaustion of measurable sets as above. Since $K(z,\zeta)$ is absolutely integrable on $A \times S_R$ then

$$|\chi_n K(z,\zeta)| \leq |K(z,\zeta)|$$

and $\chi_n K(z,\zeta) \to K(z,\zeta)$ pointwise on $A \times S_R$. By the Lebesgue dominated theorem

$$\inf_n \int_A \int_{S_R} |K(z,\zeta) - \chi_n K(z,\zeta)| = 0.$$

For all $\phi \in Q(S_R)$, we have

$$\begin{aligned}\|P_n(\phi) - P(\phi)\|_{L_1(A)} &\leq \int_A |P_n(\phi) - P(\phi)| \\ &\leq \int_A \int_{S_R} |\lambda^{-2}(\zeta)\phi(\zeta)(K(z,\zeta) - \chi_n K(z,\zeta))||\mathrm{d}\zeta|^2|\mathrm{d}z|^2 \\ &\leq \|\lambda^{-2}\phi\|_\infty \int_A \int_{S_R} |K(z,\zeta) - \chi_n K(z,\zeta)||\mathrm{d}\zeta|^2|\mathrm{d}z|^2\end{aligned}$$

then by the B-condition, the latter is bounded above by

$$C\|\phi\|_{L_1(S_R)}\int_A\int_{S_R}|K(z,\zeta)-\chi_n K(z,\zeta)||d\zeta|^2|dz|^2,$$

for some constant C which does not depend on ϕ.

Now let $f\in L_1(A)$, since P is a projection then $f=\phi+\omega$ where $\phi\in Q(S_R)$, $P(\omega)=P_n(\omega)=0$, and

$$\|\phi\|_{Q(S_R)}\leq\|P\|\|f\|_{L_1(A)}.$$

Hence $\lim\|P_n-P\|_{L_1(A)}=0$. □

To conclude, let us note that the arguments of the theorems in this paper work for entire and meromorphic functions in the class of Eremenko–Lyubich. This is the class of all entire or meromorphic functions with finitely many critical and singular values. It is not completely clear whether these arguments can be carried on entire or meromorphic functions whose asymptotic value set contains a compact set of positive Lebesgue measure.

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