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Regularity Scales and Convergence of the Calabi Flow

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Abstract We define regularity scales to study the behavior of the Calabi flow. Based on estimates of the regularity scales, we obtain convergence theorems of the Calabi flow on extremal Kähler surfaces, under the assumption of global existence of the Calabi flow solutions. Our results partially confirm Donaldson's conjectural picture for the Calabi flow in complex dimension 2. Similar results hold in high dimension with an extra assumption that the scalar curvature is uniformly bounded.

Keywords Calabi flow · Backward regularity improvement · Kähler geometry · Donaldson conjecture

Mathematics Subject Classification 53C44

In memory of Professor Weiyue Ding.

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1 Introduction

In the seminal work [5], E. Calabi studied the variational problem of the functional $\int_M (S - \underline{S})^2$, the Calabi energy, among Kähler metrics in a fixed cohomology class. The vanishing points of the Calabi energy are called the constant scalar curvature Kähler (cscK) metrics. The critical points of the Calabi energy are called the extremal Kähler (extK) metrics. To search such metrics, Calabi introduced a geometric flow, which is now well known as the Calabi flow. Actually, on a compact Kähler manifold (M^n, ω, J), the Calabi flow deforms the metric by

$$\frac{\partial}{\partial t} g_{i\bar{j}} = S_{,i\bar{j}}, \tag{1.1}$$

where g is the metric determined by $\omega(t)$ and J, and S is the scalar curvature of g. Note that in the class $[\omega]$, every metric form can be written as $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some smooth Kähler potential function φ. Therefore, on the Kähler potential level, the above equation reduces to

$$\frac{\partial}{\partial t}\varphi = S - \underline{S} = -g^{i\bar{j}} \left\{\log\det\left(g_{k\bar{l}} + \varphi_{k\bar{l}}\right)\right\}_{,i\bar{j}} - \underline{S}, \tag{1.2}$$

where $\underline{S}$ is the average of scalar curvature, which is a constant depending only on the class $[\omega]$. Note that equation (1.2) is a fourth-order fully non-linear PDE. This order incurs extreme technical difficulty. In spite of this difficulty, the short-time existence of equation (1.2) was proved by X.X. Chen and W.Y. He in [15]. Furthermore, they also proved the global existence of (1.2) under the assumption that the Ricci curvature is uniformly bounded.

About two decades after the birth of the Calabi flow, in [30], S.K. Donaldson (See [33] also) pointed out that the Calabi flow fits into a general frame of moment map picture. In fact, by fixing the underlying symplectic manifold (M, ω) and deforming the almost complex structures J along Hamiltonian vector fields, $C^\infty(M)$ has an infinitesimal action on the moduli space of almost complex structures. The function $S - \underline{S}$ can be regarded as the moment map of this action, where S is the Hermitian scalar curvature in general. Therefore, $\int_M (S - \underline{S})^2$ is the moment map square function, defined on the moduli space of almost complex structures. Then the downward gradient flow of the moment map norm square can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} J = -\frac{1}{2} J \circ \bar{\partial}_J X_S, \tag{1.3}$$

where X_S is the symplectic dual vector field of dS. When the flow path of (1.3) locates in the integrable almost complex structures, the Hermitian scalar coincides with the Riemannian scalar curvature. Therefore, the flow (1.3) is nothing but the classical Calabi flow (1.1) up to diffeomorphisms. Based on this moment map picture, Donaldson then described some conjectural behaviors of the Calabi flow.

Conjecture 1.1 (Donaldson [30]) *Suppose the Calabi flows have global existence. Then the asymptotic behavior of the Calabi flow starting from* (M, ω, J) *falls into one of the four possibilities.*

1. *The flow converges to a cscK metric on the same complex manifold* (M, J).
2. *The flow is asymptotic to a one-parameter family of extK metrics on the same complex manifold* (M, J)*, evolving by diffeomorphisms.*
3. *The manifold does not admit an extK metric but the transformed flow* J_t *on* $\mathcal{J}$ *converges to* J'*. Furthermore, one can construct a destabilizing test configuration of* (M, J) *such that* (M, J') *is the central fiber.*
4. *The transformed flow* J_t *on* $\mathcal{J}$ *does not converge in smooth topology and singularities develop. However, one can still make sufficient sense of the limit of* J_t *to extract a scheme from it, and this scheme can be fitted in as the central fiber of a destabilizing test configuration.*

Conjecture 1.1 has attracted a lot of attentions for the study of the Calabi flow. On the way to understand it, there are many important works. For example, Berman [3], He [36], and Streets [45] proved the convergence of the Calabi flow in various topologies, under different geometric conditions. Székelyhidi [48] constructed examples of global solutions of the Calabi flow which collapse at time infinity. A finite-dimensional approximation approach to study the Calabi flow was developed in [32] by Fine.

Note that the global existence of the Calabi flows is a fundamental assumption in Conjecture 1.1. On Riemann surfaces, the global existence and the convergence of the Calabi flow have been proved by Chrusical [26], Chen [12], and Struwe [47]. However, much less is known in high dimension. It was conjectured by Chen [13] that every Calabi flow has global existence. This conjecture sounds to be too optimistic at the beginning. However, there are positive evidences for it. In [44], J. Streets proved the global existence of the minimizing movement flow, which can be regarded as weak Calabi flow solutions. Therefore, the global existence of the Calabi flow can be proved if one can fully improve the regularity of the minimizing movement flow, although there exist terrific analytic difficulties to achieve this. In general, Chen's conjecture was only confirmed in particular cases. For example, if the underlying manifold is an Abelian surface and the initial metric is T-invariant, Huang and Feng proved the global existence in [37].

In short, the Calabi flow can be understood from two points of view: either as a flow of metric forms (Calabi's point of view) within a given cohomologous class on a fixed complex manifold, or as a flow of complex structures (Donaldson's point of view) on a fixed symplectic manifold. Let (M^n, ω, J) be a reference compact Kähler manifold, g be the reference metric determined by ω and J. If J is fixed, then the Calabi flow evolves in the space

$$\mathcal{H} \triangleq \left\{\omega_\varphi \,\middle|\, \varphi \in C^\infty(M), \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\right\}. \tag{1.4}$$

If ω is fixed, then the Calabi flow evolves in the space

$$\mathcal{J} \triangleq \left\{J' | J' \text{ is an integrable almost complex structure compatible with } \omega\right\}. \tag{1.5}$$

We equip both $\mathcal{H}$ and $\mathcal{J}$ with $C^{k,\frac{1}{2}}$ topology for some sufficiently large $k = k(n)$, with respect to the reference metric g. Each point of view of the Calabi flow has its own advantage. We shall take both points of view and may jump from one to the other without mentioning this explicitly.

Theorem 1.2 *Suppose (M^2, ω, J) is a compact extremal Kähler surface. Define*

$$\widetilde{\mathcal{LH}} \triangleq \left\{\omega_\varphi \in \mathcal{H} |\ \text{The Calabi flow initiating from } \omega_\varphi \text{ has global existence} \right\},$$
$$\mathcal{LH} \triangleq \text{ The path-connected component of } \widetilde{\mathcal{LH}} \text{ containing } \omega.$$

Then the modified Calabi flow (c.f. Definition 3.1) starting from any $\omega_\varphi \in \mathcal{LH}$ converges to $\varrho^*\omega$ for some $\varrho \in \mathrm{Aut}_0(M, J)$, in the smooth topology of Kähler potentials.

In the setup of Conjecture 1.1, we have $\mathcal{LH} = \widetilde{\mathcal{LH}} = \mathcal{H}$ automatically. Therefore, Theorem 1.2 confirms the first two possibilities of Conjecture 1.1 in complex dimension 2.

Theorem 1.3 *Suppose $\left\{(M^2, \omega, J_s), s \in D\right\}$ is a smooth family of compact Kähler surfaces parametrized by the disk $D = \{z | z \in \mathbb{C}, |z| < 2\}$, with the following conditions satisfied.*

- *There is a smooth family of diffeomorphisms $\{\psi_s : s \in D \backslash \{0\}\}$ such that*

$$[\psi_s^*\omega] = [\omega], \quad \psi_s^* J_s = J_1, \quad \psi_1 = Id.$$

- *$[\omega]$ is integral.*
- *(M^2, ω, J_0) is a cscK surface.*

Denote

$$\widetilde{\mathcal{LJ}} \triangleq \{J_s | s \in D,\ \text{the Calabi flow initiating from } (M, \omega, J_s) \text{ has global existence} \},$$
$$\mathcal{LJ} \triangleq \text{ The path-connected component of } \widetilde{\mathcal{LJ}} \text{ containing } J_0.$$

Then the Calabi flow starting from any $J_s \in \mathcal{LJ}$ converges to $\psi^*(J_0)$ in the smooth topology of sections of $TM \otimes T^*M$, where $\psi \in Symp(M, \omega)$ depends on J_s.

Theorem 1.3 partially confirms the third possibility of Conjecture 1.1 in complex dimension 2, in the case that the C^∞-closure of the $\mathcal{G}^{\mathbb{C}}$-leaf of J_1 contains a cscK complex structure, for a polarized Kähler surface. Note that by the integral condition of $[\omega]$ and reductivity of the automorphism groups of cscK complex manifolds, the construction of destabilizing test configurations follows from [29] directly.

Theorems 1.2 and 1.3 have high dimensional counterparts. However, in high dimension, due to the loss of scaling invariant property of the Calabi energy, we need some extra assumptions of scalar curvature to guarantee the convergence.

Theorem 1.4 *Suppose (M^n, ω, J) is a compact extremal Kähler manifold.*

For each big constant A, we set

$$\widetilde{\mathcal{LH}}_A \triangleq \left\{\omega_\varphi \in \mathcal{H} |\ \text{The Calabi flow initiating from } \omega_\varphi \text{ has global existence and } |S| \leq A \right\},$$

$$\mathcal{LH}_A \triangleq \text{The path-connected component of } \widetilde{\mathcal{LH}_A} \text{ containing } \omega,$$
$$\mathcal{LH}' \triangleq \bigcup_{A>0} \mathcal{LH}_A.$$

Then the modified Calabi flow starting from each $\omega_\varphi \in \mathcal{LH}'$ converges to $\varrho^*\omega$ for some $\varrho = \varrho(\varphi) \in \mathrm{Aut}_0(M, J)$, in the smooth topology of Kähler potentials.

Note that by Chen-He's stability theorem(c.f. [15]), the set $\mathcal{LH}_A$ is non-empty if A is large enough. Therefore, $\mathcal{LH}'$ is a non-empty subset of $\mathcal{LH}$. We have the relationships

$$\mathcal{LH}' \subset \mathcal{LH} \subset \mathcal{H}. \tag{1.6}$$

Therefore, in order to understand the global behavior of the Calabi flow, it is crucial to set up the equalities.

$$\mathcal{LH} = \mathcal{H}, \tag{1.7}$$
$$\mathcal{LH}' = \mathcal{LH}. \tag{1.8}$$

Equality (1.7) is nothing but the restatement of Chen's conjecture. Equality (1.8) is more or less a global scalar curvature bound estimate.

Theorem 1.5 *Suppose $\{(M^n, \omega, J_s), s \in D\}$ is a smooth family of compact Kähler manifolds parametrized by the disk $D = \{z | z \in \mathbb{C}, |z| < 2\}$, with the following conditions satisfied.*

- *There is a smooth family of diffeomorphisms $\{\psi_s : s \in D\backslash\{0\}\}$ such that*

$$[\psi_s^*\omega] = [\omega], \quad \psi_s^* J_s = J_1, \quad \psi_1 = Id.$$

- *$[\omega]$ is integral.*
- *(M^n, ω, J_0) is a cscK manifold.*

Denote

$$\widetilde{\mathcal{LJ}_A} \triangleq \{J_s | s \in D, \text{ the Calabi flow initiating from } J_s \text{has global existence and } |S| \leq A\},$$
$$\mathcal{LJ}_A \triangleq \text{The path-connected component of } \widetilde{\mathcal{LJ}_A} \text{ containing } J_0,$$
$$\mathcal{LJ}' \triangleq \bigcup_{A>0} \mathcal{LJ}_A.$$

Then the Calabi flow starting from any $J_s \in \mathcal{LJ}'$ converges to $\psi^*(J_0)$, in the smooth topology of sections of $TM \otimes T^*M$, where $\psi \in Symp(M, \omega)$ depends on J_s.

It is interesting to compare the Calabi flow and the Kähler Ricci flow on Fano manifolds at the current stage. For simplicity, we fix $[\omega] = 2\pi c_1(M, J)$. Modulo the pioneering work of H.D. Cao([9], global existence) and G. Perelman([43], scalar curvature bound), Theorems 1.4 and 1.5 basically says that the convergence of the Calabi flow can be as good as that for the Kähler Ricci flow on Fano manifolds, whenever some critical metrics are assumed to exist, in a broader sense. The Kähler Ricci flow version of Theorems 1.4 and 1.5 has been studied by Tian and Zhu in [51] and [52], based on Perelman's fundamental estimate. A more general approach was developed by Székelyhidi and Collins in [27]. Our proof of Theorems 1.4 and 1.5 uses a general continuity method, see for example, Tian-Zhu's work [52] in the setting of Kähler Ricci flow. However, the continuity method does not work without regularity improvement properties. Therefore, it becomes a key step to obtain such regularity improvement properties, which is one of our major contributions in this paper. We prove Theorems 2.22 and 2.23 for the Calabi flow as the regularity improvement properties.

If the flows develop singularity at time infinity, then the behavior of the Calabi flow and the Kähler Ricci flow seems much different. Based on the fundamental work of Perelman, we know collapsing does not happen along the Kähler Ricci flow. In [23] and [24], it was proved by Chen and the second author that the Kähler Ricci flow will converge to a Kähler Ricci soliton flow on a Q-Fano variety. A different approach was proposed in complex dimension 3 in [50], by Tian and Zhang. However, under the Calabi flow, Székelyhidi [48] has shown that collapsing may happen at time infinity, by constructing examples of global solutions of the Calabi flow on ruled surfaces. In this sense, the Calabi flow is much more complicated. Of course, this is not surprising since we do not specify the underlying Kähler class. A more fair comparison should be between the Calabi flow and the Kähler Ricci flow, in the same class $2\pi c_1(M, J)$, of a given Fano manifold. However, few is known about the Calabi flow in this respect, except the underlying manifold is a toric Fano surface (c.f. [17]).

Theorems 1.2 and 1.3 push the difficulty of the Calabi flow study on Kähler surfaces to the proof of global existence, i.e., Chen's conjecture. Theorems 1.4 and 1.5 indicate that the study of the Calabi flow with bounded scalar curvature is important. It is not clear whether the global existence always holds. If global existence fails, what will happen? In other words, what is the best condition for the global existence of the Calabi flow? Whether the scalar curvature bound is enough to guarantee the global existence? In order to answer these questions, we can borrow ideas from the study of the Ricci flow. In [42], N. Sesum showed that the Ricci flow exists as long as the Ricci curvature stays bounded. Same conclusion holds for the Calabi flow, due to the work [15] of X.X. Chen and W.Y. He. However, we can also translate Sesum's result into the Calabi flow along another route. Note that the Calabi flow satisfies equation (1.1). So the metrics evolve by $\nabla\bar{\nabla}S$, the complex Hessian of the scalar curvature. Correspondingly, under the Ricci flow, the metrics evolve by $-2R_{ij}$. Modulo constants, we can regard $\nabla\bar{\nabla}S$ as the counterpart of Ricci curvature in the Calabi flow. Consequently, one can expect that the Calabi flow has global existence whenever $|\nabla\bar{\nabla}S|$ is bounded. This is exactly the case. To state our results precisely, we introduce the notations

$$O_g(t) = \sup_M |S|_{g(t)}, \quad P_g(t) = \sup_M \left|\bar{\nabla}\nabla S\right|_{g(t)}, \quad Q_g(t) = \sup_M |Rm|_{g(t)}. \tag{1.9}$$

We shall omit g and t if they are clear in the context.

Theorem 1.6 *Suppose that* $\{(M^n, g(t)), -T \leq t < 0\}$ *is a Calabi flow solution on a compact Kähler manifold* M *and* $t = 0$ *is the singular time. Then we have*

$$\limsup_{t\to 0} P|t| \geq \delta_0, \tag{1.10}$$

where $\delta_0 = \delta_0(n)$*. Furthermore, for each* $\alpha \in (0, 1)$*, we have*

$$\limsup_{t\to 0} O^{\alpha} Q^{2-\alpha}|t| \geq C_0, \tag{1.11}$$

where $C_0 = C_0(n, \alpha)$*. In particular, if* $t = 0$ *is a singular time of type-I, then we have*

$$\limsup_{t\to 0} O^2|t| > 0. \tag{1.12}$$

Theorem 1.6 is nothing but the Calabi flow counterpart of the main theorems in [53] and [22]. The tools we used in the proof of Theorem 1.6 are motivated by the study of the analogue question of the Ricci flow by the second author in [22] and [53]. Actually, the methods in [53] and [22] were built in a quite general frame. It was expected to have its advantage in the study of the general geometric flows.

The paper is organized as follows. In Sect. 2, we develop two concepts—curvature scale and harmonic scale— to study geometric flows. Based on the analysis of these two scales under the Calabi flow, we show global backward regularity improvement estimates. In Sect. 3, we combine the regularity improvement estimates, the excellent behavior of the Calabi functional along the Calabi flow and the deformation techniques to prove Theorems 1.2–1.5. Moreover, we give some examples where Theorems 1.2–1.5 can be applied. In Sect. 4 we show Theorem 1.6 and in Sect. 5 we discuss some further research directions of Calabi flow.

2 Regularity Scales

2.1 Preliminaries

Let M^n be a compact Kähler manifold of complex dimension n and g a Kähler metric on M with the Kähler form ω. The Kähler class corresponding to ω is denoted by $\Omega = [\omega]$. The space of Kähler potentials is defined by

$$\mathcal{H}(M, \omega) = \left\{\varphi \in C^{\infty}(M) \;\middle|\; \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\right\}.$$

In [5] and [6], Calabi introduced the Calabi functional

$$Ca(\omega_\varphi) = \int_M \left(S(\omega_\varphi) - \underline{S}\right)^2 \omega_\varphi^n,$$

where $S(\omega_\varphi)$ denotes the scalar curvature of the metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\underline{S}$ is the average of the scalar curvature $S(\omega_\varphi)$. The gradient flow of the Calabi functional is called the Calabi flow, which can be written by a parabolic equation of Kähler potentials:

$$\frac{\partial \varphi}{\partial t} = S(\omega_\varphi) - \underline{S}.$$

The metrics evolve by the equation:

$$\frac{\partial}{\partial t} g_{i\bar{j}} = S_{,i\bar{j}}.$$

The evolution equation of curvature tensor can be written as follows(cf. [17]):

$$\frac{\partial}{\partial t}\mathrm{Rm} = -\nabla\bar{\nabla}\nabla\bar{\nabla}S = -\Delta^2\mathrm{Rm} + \nabla^2\mathrm{Rm} * \mathrm{Rm} + \nabla\mathrm{Rm} * \nabla\mathrm{Rm}, \tag{2.1}$$

where the operator $*$ denotes some contractions of tensors. Thus, we have the inequality

$$\frac{\partial}{\partial t}|\mathrm{Rm}| \leq \left|\nabla^4 S\right| + c(n)|\mathrm{Rm}|\left|\nabla^2 S\right|. \tag{2.2}$$

2.2 Estimates Based on Curvature Bound

The global high order regularity estimate of the Calabi flow was studied in [17], when Riemannian curvature and Sobolev constant are bounded uniformly. Taking advantage of the localization technique developed in [34] and [46], one can localize the estimate in [17].

Lemma 2.1 *Suppose* $\{(\Omega^n, g(t)), 0 \leq t \leq T\}$ *is a Calabi flow solution on an open Kähler manifold* Ω*, and* $\overline{B_{g(T)}(x,r)}$ *is a geodesic complete ball in* Ω*. Suppose*

$$C_S(B_{g(T)}(x,r), g(T)) \leq K_1, \tag{2.3}$$

$$\sup_{B_{g(T)}(x,r)\times[0,T]} \left\{|Rm| + \left|\frac{\partial}{\partial t}g\right| + \left|\nabla\frac{\partial}{\partial t}g\right|\right\} \leq K_2. \tag{2.4}$$

Then for every positive integer j*, there exists* $C = C(j, \frac{1}{r}, K_1, K_2)$ *such that*

$$\sup_{B_{g(T)}(x,0.5r)} |\nabla^j Rm|(\cdot, T) \leq C.$$

Proof This follows from the same argument as Theorem 4.4 of [46] and the Sobolev embedding theorem. □

Lemma 2.2 *Let* (M^n, g, J) *be a complete Kähler manifold with* $|\mathrm{Rm}|+|\nabla\mathrm{Rm}| \leq C_1$*. Then there exist positive constants* r_1, r_2 *depending only on* C_1, n *such that for each* $p \in M$ *there is a map* Φ *from the Euclidean ball* $\hat{B}(0, r_1)$ *in* $\mathbb{C}^n$ *to* M *satisfying the following properties.*

(1) Φ *is a local biholomorphic map from* $\hat{B}(0, r_1)$ *to its image.*
(2) $\Phi(0) = p$.
(3) $\Phi^*(g)(0) = g_E$, *where* g_E *is the standard metric on* $\mathbb{C}^n$.
(4) $r_2^{-1} g_E \leq \Phi^* g \leq r_2 g_E$ *in* $\hat{B}(0, r_1)$.

Proof This is only an application of Proposition 1.2 of Tian-Yau [49]. Similar application can be found in [10]. □

Theorem 2.3 (J.Streets [46]) *Suppose* $\{(M^n, g(t)), 0 \leq t \leq T\}$ *is a Calabi flow solution satisfying*

$$\sup_{M \times [0,T]} |\mathrm{Rm}| \leq K.$$

Then we have

$$\sup_{x \in M} \left|\nabla^l \mathrm{Rm}\right|(x, t) \leq C\left(K + \frac{1}{\sqrt{t}}\right)^{1+\frac{l}{2}}, \tag{2.5}$$

$$\sup_{x \in M} \left|\frac{\partial^l}{\partial t^l} \mathrm{Rm}\right|(x, t) \leq C\left(K + \frac{1}{\sqrt{t}}\right)^{1+2l}, \tag{2.6}$$

for every $t \in (0, T]$ *and positive integer* l. *Here* $C = C(l, n)$. *In particular, we have*

$$\sup_{x \in M} \left|\frac{\partial}{\partial t} |\mathrm{Rm}|\right|(x, T) \leq C\left(K^3 + \frac{1}{T^{\frac{3}{2}}}\right).$$

Proof By equation (2.1), we see that (2.6) follows from (2.5). We shall only prove (2.5).

We argue by contradiction. As in [46], we define function

$$f_l(x, t, g) = \sum_{j=1}^{l} \left|\nabla^j \mathrm{Rm}\right|_{g(t)}^{\frac{2}{2+j}}(x).$$

Suppose that (2.5) does not hold uniformly for some positive integer l. Then there exists a sequence of Calabi flow solutions $\{(M_i^n, g_i(t)), 0 \leq t \leq T\}$ satisfying the assumptions of the theorem and there are points $(x_i, t_i) \in M_i \times [0, T]$ such that

$$\lim_{i \to +\infty} \frac{f_l(x_i, t_i, g_i)}{K + t_i^{-\frac{1}{2}}} = \infty.$$

Suppose that the maximum of $\frac{f_l(x, t, g_i)}{K + t_i^{-\frac{1}{2}}}$ on $M \times (0, T]$ is achieved at (x_i, t_i). We can rescale the metrics by

$$\tilde{g}_i(x, t) \triangleq \lambda_i g\left(x, t_i + \lambda_i^{-2} t\right), \quad \lambda_i \triangleq f_l(x_i, t_i, g_i).$$

By construction, $t_i\lambda_i^2 \geq 1$ for i large and the flow $\tilde{g}_i(t)$ exists on the time period $[-1, 0]$. Moreover, it satisfies the following properties.

- $\lim_{i\to+\infty} \sup_{M_i\times[-1,0]} |\mathrm{Rm}|_{\tilde{g}_i} = 0$.
- $\sup_{M_i\times[-1,0]} f_l(\cdot, \cdot, \tilde{g}_i(t)) \leq 1$.
- $f_l(x_i, 0, \tilde{g}_i) = 1$.

By Lemma 2.2, we can construct local biholomorphic map Φ_i from a ball $\hat{B}(0, r) \subset \mathbb{C}^n$ to M_i, with respect to the metric $\tilde{g}_i(0)$ and base point x_i. Note that the radius r is independent of i. Let $\tilde{h}_i(t) = \Phi_i^*\tilde{g}_i(0)$. Then we obtain a sequence of Calabi flows $\{(\hat{B}(0, r), \tilde{h}_i(t)), -1 \leq t \leq 0\}$ satisfying (2.3) and (2.4), up to shifting of time. Furthermore, we have

$$\lim_{i\to\infty} |Rm|_{\tilde{h}_i(0)}(0) = 0, \quad \sum_{j=1}^{l} \left|\nabla^j \mathrm{Rm}\right|_{\tilde{h}_i(0)}^{\frac{2}{2+j}}(0) = 1. \tag{2.7}$$

By Lemma 2.1, we can take convergence in the smooth Cheeger–Gromov topology.

$$\left(\hat{B}(0, 0.5r), \tilde{h}_i(0)\right) \xrightarrow{Cheeger-Gromov-C^\infty} \left(\tilde{B}, \tilde{h}_\infty(0)\right).$$

On one hand, $\mathrm{Rm}_{\tilde{h}_\infty(0)} \equiv 0$ on $\tilde{B}$, which in turn implies that $\nabla^j Rm_{\tilde{h}_\infty(0)} \equiv 0$ on $\tilde{B}$ for each positive integer j. On the other hand, taking smooth limit of (2.7), we obtain

$$\sum_{j=1}^{l} \left|\nabla^j \mathrm{Rm}\right|_{\tilde{h}_\infty(0)}^{\frac{2}{2+j}}(0) = 1.$$

Contradiction. □

The proof of Theorem 2.3 follows the same line as that in [46] by J. Streets, we do not claim the originality of the result. We include the proof here for the convenience of the readers and to show the application of the local biholomorphic map Φ, which will be repeatedly used in the remainder part of this subsection. Actually, by delicately using interpolation inequalities, the constants in Theorem 2.3 can be made explicit.

Note that for a given Calabi flow, $S \equiv 0$ implies that all the high derivatives of S vanish. Therefore, for the Calabi flows with uniformly bounded Riemannian curvature and very small scalar curvature S, it is expected that the high derivatives of S are very small. In the remainder part of this subsection, we will justify this observation. Similar estimates for the Ricci flow were given by Theorem 3.2 of [53] and Lemma 2.1 of [22] by the parabolic Moser iteration. However, since the Calabi flow is a fourth-order parabolic equation, the parabolic Moser iteration in the case of the Ricci flow does not work any more. Here we use a different method to overcome this difficulty.

To estimate the higher derivatives of the curvature, we need the interpolation inequalities of Hamilton in [35]:

Lemma 2.4 ([35]) *For any tensor T and $1 \leq j \leq k-1$, we have*

$$\int_M |\nabla^j T|^{2k/j}\, \omega^n \leq C \cdot \max_M |T|^{\frac{2k}{j}-2} \cdot \int_M |\nabla^k T|^2\, \omega^n, \tag{2.8}$$

$$\int_M |\nabla^j T|^2\, \omega^n \leq C \cdot \left(\int_M |\nabla^k T|^2\, \omega^n\right)^{\frac{j}{k}} \cdot \left(\int_M |T|^2\, \omega^n\right)^{1-\frac{j}{k}}, \tag{2.9}$$

where $C = C(k, n)$ is a constant.

Combining Lemma 2.4 with Sobolev embedding theorem, we have the following result.

Lemma 2.5 *For any integer $i \geq 1$ and any Kähler metric ω, there exists a constant $C = C(C_S(\omega), i) > 0$ such that for any tensor T, we have*

$$\max_M \left|\nabla^i T\right|^2 \leq C \cdot \max_M\ |T|^{\frac{2n+1}{n+1}} \left(\int_M |T|^2\, \omega^n\right)^{\frac{1}{4(n+1)}} \cdot \left(\int_M \left|\nabla^{4(n+1)i} T\right|^2 \omega^n + \int_M \left|\nabla^{4(n+1)(i+1)} T\right|^2 \omega^n\right)^{\frac{1}{4(n+1)}}.$$

Proof Recall that the Sobolev embedding theorem implies that

$$\max_M |f| \leq C_S \left(\int_M (|f|^p + |\nabla f|^p)\, \omega^n\right)^{\frac{1}{p}}$$

for every smooth function f. Let $f = \left|\nabla^i T\right|^2$ and $p = 2(n+1)$ in the above inequality. Then we have

$$\max_M \left|\nabla^i T\right|^2 \leq C_S \left(\int_M \left(\left|\nabla^i T\right|^{4(n+1)} + |\nabla f|^{2(n+1)}\right) \omega^n\right)^{\frac{1}{2(n+1)}}.$$

The Kato's inequality implies that

$$|\nabla f| = 2\left|\nabla^i T\right| \cdot \left|\nabla \left|\nabla^i T\right|\right| \leq 2\left|\nabla^i T\right| \cdot \left|\nabla^{i+1} T\right| \leq \left|\nabla^i T\right|^2 + \left|\nabla^{i+1} T\right|^2.$$

Combining the above two inequalities, we obtain

$$\max_M \left|\nabla^i T\right|^2 \leq C \left(\int_M \left(\left|\nabla^i T\right|^{4(n+1)} + \left|\nabla^{i+1} T\right|^{4(n+1)}\right) \omega^n\right)^{\frac{1}{2(n+1)}}. \tag{2.10}$$

In inequality (2.8), let $k = 2(n+1)i$ and $j = i$, we have

$$\int_M \left|\nabla^i T\right|^{4(n+1)} \omega^n \leq C \cdot \max_M |T|^{4n+2} \cdot \int_M \left|\nabla^{2(n+1)i} T\right|^2 \omega^n.$$

If we let $k = 2(n+1)(i+1)$, $j = i+1$, then we have

$$\int_M \left|\nabla^{i+1}T\right|^{4(n+1)} \omega^n \le C \cdot \max_M |T|^{4n+2} \cdot \int_M \left|\nabla^{2(n+1)(i+1)}T\right|^2 \omega^n.$$

In inequality (2.9), let $k = 4(n+1)i$ and $j = 2(n+1)i$, then we have

$$\int_M \left|\nabla^{2(n+1)i}T\right|^2 \omega^n \le C \cdot \left(\int_M \left|\nabla^{4(n+1)i}T\right|^2\right)^{\frac{1}{2}} \cdot \left(\int_M |T|^2 \omega^n\right)^{\frac{1}{2}}.$$

Combining this with (2.10), we have

$$\begin{aligned}
&\max_M \left|\nabla^i T\right|^2 \\
&\le C \cdot \max_M |T|^{\frac{2n+1}{n+1}} \left(\int_M \left(\left|\nabla^{2(n+1)i}T\right|^2 + \left|\nabla^{2(n+1)(1+i)}T\right|^2\right) \omega^n\right)^{\frac{1}{2(n+1)}} \\
&\le C \cdot \max_M |T|^{\frac{2n+1}{n+1}} \left(\int_M |T|^2 \omega^n\right)^{\frac{1}{4(n+1)}} \left(\int_M \left|\nabla^{4(n+1)i}T\right|^2 \omega^n + \int_M \left|\nabla^{4(n+1)(i+1)}T\right|^2 \omega^n\right)^{\frac{1}{4(n+1)}}.
\end{aligned}$$

The lemma is proved. □

We next show some local estimates on the derivatives of the scalar curvature.

Lemma 2.6 *Fix any $\alpha \in (0,1)$ and $r > 0$. There exists an integer $N = N(\alpha) > 0$ such that if*

$$\sup_{B(p,2r)} \sum_{k=0}^{N} \left|\nabla^k \mathrm{Rm}\right| \le \Lambda_N \tag{2.11}$$

for some positive constant Λ_N, then we have the inequalities

$$\sup_{B(p,r)} \left|\nabla\bar{\nabla}S\right| \le C \sup_{B(p,2r)} |S|^\alpha, \tag{2.12}$$

$$\sup_{B(p,r)} \left|\nabla\bar{\nabla}\nabla\bar{\nabla}S\right| \le C \sup_{B(p,2r)} \left|\nabla\bar{\nabla}S\right|^\alpha, \tag{2.13}$$

for some constant $C = C\,(C_S(B(p,2r)), r, \Lambda_N, \alpha)$.

Proof Let $\eta \in C^\infty(\mathbb{R},\mathbb{R})$ be a cutoff function such that $\eta(s) = 1$ if $s \le 1$ and $\eta(s) = 0$ if $s \ge 2$. Moreover, we assume $\left|\eta'(s)\right| \le 2$. We define $\chi(x) = \eta\left(\frac{d_g(p,x)}{r}\right)$ for any $x \in M$, where $d_g(p,x)$ is the distance from p to x with respect to the metric g. Then $\chi = 1$ on $B(p,r)$ and $\chi = 0$ outside $B(p,2r)$. The derivatives of χ satisfy the inequalities

$$\left|\nabla_g^i \chi\right| \le C(\Lambda_i, r). \tag{2.14}$$

Using Lemma 2.5 for χS, we have the inequality

$$\sup_{B(p,r)} \left|\nabla^2 S\right|^2 \leq C(C_S) \cdot \sup_{B(p,2r)} |S|^{\frac{2n+1}{n+1}} \left(\int_{B(p,2r)} |S|^2 \omega^n\right)^{\frac{1}{4(n+1)}}$$
$$\cdot \left(\int_{B(p,2r)} \left|\nabla^{8(n+1)}(\chi S)\right|^2 \omega^n + \int_{B(p,2r)} \left|\nabla^{12(n+1)}(\chi S)\right|^2 \omega^n\right)^{\frac{1}{4(n+1)}}.$$

For any integer k we set

$$f(k) = \int_{B(p,2r)} \left|\nabla^k(\chi S)\right|^2 \omega^n.$$

Under the assumption (2.11), we have

$$\begin{aligned} f(k) &\leq \int_{B(p,2r)} |S| \left|\nabla^{2k}(\chi S)\right| \omega^n \\ &\leq \mathrm{Vol}(B(p,2r))^{\frac{1}{2}} \sup_{B(p,2r)} |S| \cdot f(2k)^{\frac{1}{2}} \\ &\leq \left(\mathrm{Vol}(B(p,2r))^{\frac{1}{2}} \sup_{B(p,2r)} |S|\right)^{2-2^{1-m}} f(2^m k)^{\frac{1}{2^m}} \\ &\leq C(\Lambda_{2^m k}, m, r)\left(\mathrm{Vol}(B(p,2r))^{\frac{1}{2}} \sup_{B(p,2r)} |S|\right)^{2-2^{1-m}}, \end{aligned}$$

where m is any positive integer. Here we used the fact that $\mathrm{Vol}(B(p,2r)$ is bounded from above by the volume comparison theorem. Combining the above inequalities, for any m we have the estimate

$$\sup_{B(p,r)} \left|\nabla^2 S\right| \leq C\left(C_S, r, m, \Lambda_{12\cdot 2^m(n+1)}\right) \sup_{B(p,2r)} |S|^{1-\frac{1}{(n+1)2^{m+2}}}.$$

Thus, (2.12) is proved. Applying Lemma 2.5 to $\nabla\bar{\nabla}S$ and using the same argument as above, we have the inequality (2.13). The lemma is proved. □

The following proposition is a weak version of the corresponding result for the Ricci flow in [22,53].

Proposition 2.7 *Fix $\alpha \in (0,1)$. If $\{(M, g(t)), -s \leq t \leq 0\}$ is a Calabi flow solution with $\sup_{[-s,0]} Q_g(t) \leq 1$, then there is a constant $C = C(s,\alpha) > 0$ such that*

$$\max_M \left|\nabla\bar{\nabla}S\right|(0) \leq C \max_M |S|^\alpha(0), \tag{2.15}$$
$$\max_M \left|\nabla\bar{\nabla}\nabla\bar{\nabla}S\right|(0) \leq C \max_M \left|\nabla\bar{\nabla}S\right|^\alpha(0). \tag{2.16}$$

Proof Since $\sup_{[-s,0]} Q_g(t) \le 1$, Theorem 2.3 applies. For any integer $k \ge 0$, there is a constant $C = C(k, s)$ such that

$$\sup_{M\times[-\frac{s}{2},0]} |\nabla^k \mathrm{Rm}| \le C(k, s). \tag{2.17}$$

By Lemma 2.2, for any $p \in M$ there is $r = r(n, s) > 0$ and a local biholomorphic map Φ from $\hat{B}(0, r) \subset \mathbb{C}^n$ to its image in $(M, g(0))$ and $\Phi(0) = p$. Define the pullback metrics $\hat{g}(t) = \Phi^* g(t)$ on $\hat{B}(0, r)$. Then the Kähler metric $\hat{g}(t)$ satisfies the equation of Calabi flow on $\hat{B}(0, r)$ and their injectivity radii on $\hat{B}(0, r)$ are bounded from below. Moreover, on $\hat{B}(0, r)$ the Sobolev constant and all the derivatives of the curvature tensor of $\hat{g}(t)$ for $t \in [-\frac{s}{2}, 0]$ are bounded by (2.17). By Lemma 2.6, there is a constant $C(s, \alpha) > 0$ such that

$$\sup_{\hat{B}(0,\frac{r}{2})} \left|\nabla\bar{\nabla} S\right|_{\hat{g}} (0) \le C \max_{\hat{B}(0,r)} |S|^{\alpha}_{\hat{g}}(0),$$

which implies that the metric $g(t)$ satisfies the inequality (2.15). Similarly, we can show the second inequality (2.16). The proposition is proved. □

2.3 From Metric Equivalence to Curvature Bound

If we regard curvature as the 4-th order derivative of Kähler potential function, then Theorem 2.3 can be roughly understood as from C^4-estimate to C^l-estimate, for each $l \ge 5$. In this subsection, we shall set up the estimate from C^2 to C^4 for the Calabi flow family.

Lemma 2.8 *For every $\delta > 0$, there exists a constant $\epsilon = \epsilon(n, \delta)$ with the following properties. If $\{(M^n, g(t)), t \in [-1, 0]\}$ and $\{(M^n, h(t)), t \in [-1, 0]\}$ are Calabi flow solutions such that*

- $Q_g(0) = 1$ *and* $Q_g(t) \le 2$ *for any* $t \in [-1, 0]$.
- $Q_h(t) \le 2$ *for any* $t \in [-1, 0]$.
- $e^{-\epsilon} g(0) \le h(0) \le e^{\epsilon} g(0)$.

Then we have

$$|\log Q_h(0)| < \delta.$$

Proof We follow the argument of Proposition 2.1 in [22]. Suppose not, there exist constants $\delta_0 > 0$, $\epsilon_i \to 0$ and two sequences of the Calabi flow solutions

$$\{(M_i, g_i(t)), t \in [-1, 0]\}, \quad \{(M_i, h_i(t)), t \in [-1, 0]\}$$

such that the following properties are satisfied.

(1) $Q_{g_i}(0) = 1$ and $Q_{g_i}(t) \le 2$ for any $t \in [-1, 0]$.

(2) $Q_{h_i}(t) \leq 2$ for any $t \in [-1, 0]$, and $|\log Q_{h_i}(0)| \geq \delta_0$.
(3) $e^{-\epsilon_i} g_i(0) \leq h_i(0) \leq e^{\epsilon_i} g_i(0)$.

By property (1) and (2), we claim that there is a point $z_i \in M_i$ such that

$$\max\{|\mathrm{Rm}|_{h_i(0)}(z_i), |\mathrm{Rm}|_{g_i(0)}(z_i)\} \geq e^{-2\delta_0}, \quad \left|\log \frac{|\mathrm{Rm}|_{h_i(0)}(z_i)}{|\mathrm{Rm}|_{g_i(0)}(z_i)}\right| \geq \frac{1}{2}\delta_0. \tag{2.18}$$

In fact, by property (2) we have two possibilities $Q_{h_i}(0) \geq e^{\delta_0}$ or $Q_{h_i}(0) \leq e^{-\delta_0}$. If $Q_{h_i}(0) \geq e^{\delta_0}$, then we assume that the point z_i achieves $Q_{h_i}(0)$:

$$|\mathrm{Rm}|_{h_i(0)}(z_i) = Q_{h_i}(0).$$

Then the first inequality of (2.18) obviously holds and the second also holds since

$$\log \frac{|\mathrm{Rm}|_{h_i(0)}(z_i)}{|\mathrm{Rm}|_{g_i(0)}(z_i)} \geq \log \frac{|\mathrm{Rm}|_{h_i(0)}(z_i)}{Q_{g_i}(0)} \geq \delta_0.$$

Now we consider the case when $Q_{h_i}(0) \leq e^{-\delta_0}$. We assume that the point z_i achieves $Q_{g_i}(0) = 1$:

$$|\mathrm{Rm}|_{g_i(0)}(z_i) = 1.$$

Then the first inequality of (2.18) obviously holds and the second also holds since

$$\log \frac{|\mathrm{Rm}|_{g_i(0)}(z_i)}{|\mathrm{Rm}|_{h_i(0)}(z_i)} \geq \log e^{\delta_0} = \delta_0.$$

Thus, (2.18) is proved.

By Theorem 2.3, we have the higher order curvature estimates for the metrics $g_i(0)$. By Lemma 2.2, there is $r > 0$ independent of i and a local biholomorphic map Φ_i from $\hat{B}(0, r) \subset \mathbb{C}^n$ to its image in M_i and $\Phi_i(0) = z_i$. Define the pullback metrics

$$\hat{g}_i(t) = \Phi_i^* g_i(t), \quad \hat{h}_i(t) = \Phi_i^* h_i(t). \tag{2.19}$$

Then the Kähler metrics $\hat{g}_i(t)$ and $\hat{h}_i(t)$ satisfy the equation of the Calabi flow and their injectivity radii at the point $0 \in \hat{B}(0, r)$ are bounded from below by a uniform positive constant which is independent of i. Moreover, all the derivatives of the curvature tensor of $\hat{g}_i(t)$ and $\hat{h}_i(t)$ are bounded by Theorem 2.3. Thus, we can take Cheeger–Gromov smooth convergence

$$\left(\hat{B}(0, r), 0, \hat{g}_i(t)\right) \xrightarrow{C.G.-C^\infty} \left(B', p', \hat{g}'\right), \quad \left(\hat{B}(0, r), 0, \hat{h}_i(t)\right) \xrightarrow{C.G.-C^\infty} \left(B'', p'', \hat{h}'\right). \tag{2.20}$$

Define the identity map F_i by

$$F_i : \left(\hat{B}(0,r), 0, \hat{g}_i(t)\right) \mapsto \left(\hat{B}(0,r), 0, \hat{h}_i(t)\right),$$
$$x \mapsto x.$$

By property (3), we see that

$$\left|\log \frac{d_{\hat{h}_i(0)}(x,y)}{d_{\hat{g}_i(0)}(F_i^{-1}(x), F_i^{-1}(y))}\right| = \left|\log \frac{d_{\hat{h}_i(0)}(x,y)}{d_{\hat{g}_i(0)}(x,y)}\right| \leq \epsilon_i \to 0, \quad \forall\, x, y \in \hat{B}(0,r).$$

Therefore, F_i converges to an isometry F_∞ :

$$F_\infty : \left(B', p', \hat{g}'\right) \to \left(B'', p'', \hat{h}'\right), \quad F_\infty(p') = p''.$$

By Calabi–Hartman's theorem in [8], the isometry F_∞ is smooth. Therefore, we have

$$|\mathrm{Rm}|_{\hat{g}'}(p') = |\mathrm{Rm}|_{\hat{h}'}(p''). \tag{2.21}$$

However, by the smooth convergence (2.20) and the inequalities (2.18) we obtain

$$\max\left\{|\mathrm{Rm}|_{\hat{g}'}(p'), |\mathrm{Rm}|_{\hat{h}'}(p'')\right\} \geq e^{-2\delta_0}, \quad \left|\log \frac{|\mathrm{Rm}|_{\hat{g}'}(p')}{|\mathrm{Rm}|_{\hat{h}'}(p'')}\right| \geq \frac{1}{2}\delta_0,$$

which contradicts (2.21). The lemma is proved. □

As a direct corollary, we have the next result.

Lemma 2.9 *There exists a constant $\epsilon_0 = \epsilon_0(n)$ with the following properties. Suppose $\{(M^n, g(t)), -1 \leq t \leq K, 0 \leq K\}$ is a Calabi flow solution such that*

$$Q_g(0) = 1, \quad Q_g(t) \leq 2, \quad \forall\, t \in [-1, 0],$$

and $T > 0$ is the first time such that $\left|\log Q_g(T)\right| = \log 2$, then we have

$$\int_0^T P_g(t)\, dt \geq \epsilon_0. \tag{2.22}$$

Proof Consider two solutions to the Calabi flow $\{(M, g(t)), -1 \leq t \leq 0\}$ and $\{(M, h(t)), -1 \leq t \leq 0\}$ where $h(x,t) = g(x, t+T)$. Note that $|\log Q_h(0)| = \log 2$. By Lemma 2.8 there exists a point $x \in M$ and a non-zero vector $V \in T_x M$ such that

$$\left|\log \frac{h(0)(V,V)}{g(0)(V,V)}\right| \geq \epsilon_0$$

for some $\epsilon_0 > 0$. Using the equation (2.2), we have

$$\epsilon_0 \le \left|\log \frac{g(T)(V,V)}{g(0)(V,V)}\right| \le \int_0^T P_g(t)\,dt.$$

Thus, (2.22) holds and the lemma is proved. □

2.4 Curvature Scale

Inspired by Theorem 2.3, we define a concept "curvature scale," to study improving regularity property of the Calabi flow.

Definition 2.10 Suppose $\{(M, g(t)), t \in I \subset \mathbb{R}\}$ is a Calabi flow solution. Define the curvature scale $F_g(t_0)$ of $t_0 \in I$ by

$$F_g(t_0) = \sup \left\{ s > 0 \,\middle|\, \sup_{M\times[t_0-s,t_0]} |\mathrm{Rm}|^2 \le s^{-1} \right\},$$

where we assume $\sup_{M\times[t_0-s,t_0]} |\mathrm{Rm}|^2 = \infty$ whenever $t_0 - s \notin I$. We denote the curvature scale of time t_0 by $F_g(t_0)$.

Suppose that $\{(M, \tilde{g}(t)), -2 \le t < T\}$ is a Calabi flow solution and $T > 0$ is the singular time. Since the curvature tensor will blowup at the singular time, we have $\lim_{t\to T} F_{\tilde{g}}(t) = 0$. Note that $F_{\tilde{g}}(t)$ is a continuous function, we can assume that $0 < F_{\tilde{g}}(t) < 1$ for any $t \in [0, T)$. Choose any $t_0 \in [0, T)$ and let A be a positive constant such that $A^2 = \frac{1}{F_{\tilde{g}}(t_0)} > 1$. Rescale the metric $\tilde{g}(t)$ by

$$g(x,t) = A\,\tilde{g}\left(x, \frac{t}{A^2} + t_0\right), \quad t \in \left[A^2(-2 - t_0), A^2(T - t_0)\right]. \tag{2.23}$$

Then we obtain a solution $g(t)$ with existence time period containing $[-2, K]$ for some positive K. Moreover, we have $F_g(0) = 1$. In the following, we will calculate the derivative of $F_g(t)$ with respect to t. Since $F_g(t)$ might not be differentiable, we will use the Dini derivative:

$$\frac{\mathrm{d}^-}{\mathrm{d}t} f(t) := \liminf_{\epsilon\to 0^+} \frac{f(t+\epsilon) - f(t)}{\epsilon}, \quad \frac{\mathrm{d}^+}{\mathrm{d}t} f(t) := \limsup_{\epsilon\to 0^+} \frac{f(t+\epsilon) - f(t)}{\epsilon}. \tag{2.24}$$

The following result is the key estimate on the curvature scale.

Lemma 2.11 *Suppose $\{(M, g(t)), -2 \le t \le K, 0 \le K\}$ is a Calabi flow solution, $F_g(0) = 1$. Then at time $t = 0$ we have*

$$\frac{\mathrm{d}^-}{\mathrm{d}t} F_g(t) \ge \min\left\{0, -2\frac{\mathrm{d}^+}{\mathrm{d}t} Q_g(t)\right\}. \tag{2.25}$$

Proof By the definition of curvature scale, there exists time $t \in [-1, 0]$ such that $Q_g(t) = 1$ and we denote by t_0 the maximal time $t \in [-1, 0]$ with this property. There are several cases to consider:

(1). Suppose $Q_g(0) < 1$ and $t_0 > -1$. Then there is a constant $\delta \in (0, t_0 + 1)$ such that $Q_g(t) < 1$ for all $t \in (0, \delta)$. Thus, for any $t \in [0, \delta)$ we have $F_g(t) = 1$. So we obtain the derivative of $F_g(t)$ at $t = 0$,

$$\frac{\mathrm{d}^-}{\mathrm{d}t} F_g(t) = 0. \tag{2.26}$$

(2). Suppose $Q_g(0) < 1$ and $t_0 = -1$. By the definition of t_0, there exists a small constant $\delta > 0$ such that for any $t \in (-1, \delta)$ we have $Q_g(t) < 1$. This implies that $F_g(t) > 1$ for $t \in (0, \delta)$ when δ small. In other words, we have

$$\left.\frac{\mathrm{d}^-}{\mathrm{d}t} F_g(t)\right|_{t=0} \geq 0. \tag{2.27}$$

(3). Suppose $Q_g(0) = 1$ and there is a small constant $\delta > 0$ and a sequence of times $t_i \in (0, \delta)$ with $t_i \to 0$ such that

$$Q_g(t_i) \geq 1, \quad Q_g(t_i)^2 F_g(t_i) = 1, \quad \forall\, i \geq 1. \tag{2.28}$$

Therefore, we have the equality

$$\begin{aligned}
\liminf_{i \to +\infty} \frac{F_g(t_i) - F_g(0)}{t_i} &= \liminf_{i \to +\infty} \frac{Q_g(0) + Q_g(t_i)}{Q_g(0)^2 Q_g(t_i)^2} \cdot \frac{Q_g(0) - Q_g(t_i)}{t_i} \\
&= -\frac{2}{Q_g(0)^3} \limsup_{i \to +\infty} \frac{Q_g(t_i) - Q_g(0)}{t_i} \\
&\geq -\frac{2}{Q_g(0)^3} \frac{d^+}{dt} Q_g\Big|_{t=0}.
\end{aligned}$$

(4). Suppose $Q_g(0) = 1$ and there is a small constant $\delta > 0$ and a sequence of times $t_i \in (0, \delta)$ with $t_i \to 0$ such that

$$Q_g(t_i) \geq 1, \quad Q_g(t_i)^2 F_g(t_i) < 1, \quad \forall\, i \geq 1. \tag{2.29}$$

Since $F_g(t)$ is a continuous function and $F_g(0) = 1$, we assume that

$$|F_g(t) - 1| < \epsilon_0, \quad \forall\, t \in (0, \delta) \tag{2.30}$$

for some small $\epsilon_0 > 0$. For each time t_i, (2.29) implies that

$$Q_g(t_i)^2 < \frac{1}{F_g(t_i)} = \sup_{[t_i - F_g(t_i), t_i]} Q_g(t)^2. \tag{2.31}$$

Since the constant δ is small and the inequality (2.30) holds, we have $t_i - F_g(t_i) < 0 < t_i$. Using the fact that $Q_g(0) = 1$ and (2.31) we have $\frac{1}{F_g(t_i)} \geq 1$ and so $F_g(t_i) \leq 1$.

If there exists time t_{i_0} such that $F_g(t_{i_0}) = 1$, then for all $t \in (0, t_{i_0})$ we have $F_g(t) = 1$. Thus, the inequality (2.25) holds. On the other hand, if for all integer $i \geq 1$ the inequality $F_g(t_i) < 1$ holds, then we can find a maximal time $t_i' \in (0, t_i)$ such that

$$Q_g(t_i') = \sup_{[t_i - F_g(t_i), t_i]} Q_g(t)^2. \tag{2.32}$$

Moreover, since $Q(t) \leq 1$ for $t \in [-1, 0]$ and $[t_i' - F(t_i), t_i'] \subset [-1, t_i]$, we have

$$Q_g(t_i') = \sup_{[t_i' - F_g(t_i), t_i']} Q_g(t)^2 = \frac{1}{F_g(t_i)},$$

where we used (2.31) and (2.32) in the last equality. Therefore, at time t_i' we have $F_g(t_i') = F_g(t_i)$ and the identity

$$Q_g(t_i')F_g(t_i') = Q_g(t_i')F_g(t_i) = 1.$$

Therefore, there exists a sequence of times $\{t_i'\}(t_i' \in (0, t_i))$ with $t_i' \to 0$ and satisfying the conditions (2.28) in item (3). Note that

$$\frac{F(t_i) - F(0)}{t_i} = \frac{F(t_i') - F(0)}{t_i'} \cdot \frac{t_i'}{t_i} \geq \frac{F(t_i') - F(0)}{t_i'},$$

where we used the fact that $F(t_i') < F(0) = 1$ and $t_i' < t_i$. Therefore, we have

$$\liminf_{i \to +\infty} \frac{F_g(t_i) - F_g(0)}{t_i} \geq \liminf_{i \to +\infty} \frac{F_g(t_i') - F_g(0)}{t_i'} \geq -\frac{2}{Q_g(0)^3} \frac{d^+}{dt} Q_g \bigg|_{t=0}.$$

(5). Suppose $Q_g(0) = 1$ and there is a small constant $\delta > 0$ and a sequence of times $t_i \in (0, \delta)$ with $t_i \to 0$ such that $Q_g(t_i) < 1$. Then by the definition of $F_g(t)$, we have $F_g(t_i) \geq 1$. Thus, we have the inequality

$$\liminf_{i \to +\infty} \frac{F_g(t_i) - F_g(0)}{t_i} \geq 0.$$

Combining the above cases, (2.25) holds and the theorem is proved.

Lemma 2.12 *Suppose* $\{(M, g(t)), -2 \leq t \leq K, 0 \leq K\}$ *is a Calabi flow solution with* $F_g(0) = 1$. *Then at time* $t = 0$ *we have*

$$\left| \frac{\mathrm{d}Q_g}{\mathrm{d}t} \right| \leq C \min\left\{1, P_g^\alpha, O_g^\alpha\right\},$$

where α *be any number in* $(0, 1)$ *and* $C = C(\alpha, n)$.

Proof Suppose that $F(0) = 1$. By the definition of curvature scale and Theorem 2.3, all higher order derivatives of the Riemannian curvature tensor are bounded for $t \in [-\frac{1}{2}, 0]$. By Proposition 2.7, for any $\alpha \in (0, 1)$ we have the estimates

$$\max_M |\nabla\bar{\nabla}\nabla\bar{\nabla}S| \leq C \max_M |\nabla\bar{\nabla}S|^{\alpha},$$
$$\max_M |\nabla\bar{\nabla}S| \leq C \max_M |S|^{\alpha}.$$

Therefore, by the inequality (2.2) of $|\mathrm{Rm}|$, adjusting α in different inequalities if necessary, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{Rm}|\right| \leq C \min\left\{1, \max_M |\nabla\bar{\nabla}S|^{\alpha}, \max_M |S|^{\alpha}\right\}.$$

The lemma is proved.

A direct corollary of the above results is

Lemma 2.13 *Let $\alpha \in (0, 1)$. Suppose $\{(M, \tilde{g}(t)), -2 \leq t < T, 0 \leq T\}$ is a Calabi flow solution. For any $t_0 \in [-1, T)$ with $F_{\tilde{g}}(t_0) \in (0, 1)$, we have the following inequality at time t_0*

$$\frac{\mathrm{d}^-}{\mathrm{d}t} F_{\tilde{g}} \geq -C \min\left\{1, P_{\tilde{g}}^{\alpha} Q_{\tilde{g}}^{2(1-\alpha)} F_{\tilde{g}}, O_{\tilde{g}}^{\alpha} Q_{\tilde{g}}^{2-\alpha} F_{\tilde{g}}\right\}, \tag{2.33}$$

where $C = C(\alpha, n) > 0$ is a constant.

Proof We rescale the metric $\tilde{g}(t)$ by (2.23) with $A^2 = \frac{1}{F_{\tilde{g}}(t_0)}$. Then we obtain a solution $g(s)$ to the Calabi flow with $F_g(0) = 1$. By Lemma 2.11 and Lemma 2.12, at $s = 0$ we have the following inequality for $g(s)$:

$$\frac{\mathrm{d}^-}{\mathrm{d}t} F_g \geq -C \min\left\{1, P_g^{\alpha} Q_g^{2(1-\alpha)} F_g, O_g^{\alpha} Q_g^{2-\alpha} F_g\right\}. \tag{2.34}$$

Clearly, (2.33) follows directly from (2.34) and rescaling, since both sides of (2.34) are scaling invariant.

Lemma 2.14 *Fix $\alpha \in (0, 1)$. Suppose $\{(M^n, g(t)), -1 \leq t \leq K, 0 \leq K\}$ is a Calabi flow solution such that*

(1) $Q(0) = 1$, *where* $Q = Q_g$;
(2) $Q(t) \leq 2$ *for all* $t \in [-1, 0]$;
(3) $|\log Q(t)| \leq \log 2$ *for every* $t \in [0, K]$ *and* $|\log Q(K)| = \log 2$.

Then there are positive constants $A = A(n, \alpha)$ and $\epsilon(n, \alpha)$ such that

$$\int_0^K O^{\alpha} Q^{2-\alpha} dt \geq \frac{\epsilon}{A}, \tag{2.35}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} Q(t)\right| \leq A O^{\alpha} Q^{3-\alpha}, \quad \forall\, t \in [0, K]. \tag{2.36}$$

Proof Under the assumptions, we have the inequalities for $t \in [0, K]$

$$P(t) \le CO^{\alpha}(t)Q^{2-\alpha}(t), \quad \max_M \left|\nabla^4 S\right| \le CO^{\alpha}(t)Q^{3-\alpha}(t).$$

which are the scaling invariant version of Proposition 2.7. Combining this with Lemma 2.9, we have the inequalities (2.35) and (2.36). The lemma is proved.

The next result shows that under the scalar curvature conditions, if the curvature at some time is large enough, then the curvature before that can be controlled.

Proposition 2.15 *If* $\{(M^n, g(t)), -1 \le t \le 0\}$ *is a Calabi flow solution with*

(1) *the scalar curvature* $|S(t)| \le 1$ *for every* $t \in [-1, 0]$*;*
(2) $Q(0)$ *is big enough, i.e.,*

$$Q(0) > \max\left\{2^{\frac{3-\alpha}{\alpha}} A^{\frac{1}{\alpha}}, 2^{\frac{2(1-\alpha)}{\alpha}} \left(\frac{A}{\epsilon}\right)^{\frac{1}{\alpha}}\right\},$$

where ϵ *and* A *are constants given by Lemma 2.14,* Q *is* Q_g*.*

Then we have the inequality

$$Q(t) < \frac{2}{\sqrt{Q(0)^{-2} + t}}, \quad \forall\, t \in \left[-Q(0)^{-2}, 0\right]. \tag{2.37}$$

Consequently, we have

$$F(0) \ge \frac{1}{5Q(0)^2}. \tag{2.38}$$

Proof Let $\frac{2}{\sqrt{Q(0)^{-2}+t}}$ be a barrier function. Clearly, this function bounds $Q(t)$ when $t = -Q(0)^{-2}$ and $t = 0$. Let I be the collection of $t \in [-Q(0)^{-2}, 0]$ such that

$$Q(t) \ge \frac{2}{\sqrt{Q(0)^{-2} + t}}.$$

We want to show that I is empty.

Suppose $I \ne \emptyset$. Clearly, I is bounded and closed. Let $\underline{t}$ be the infimum of I. Then we have $\underline{t} \in I$ and

$$Q(\underline{t}) = \frac{2}{\sqrt{Q(0)^{-2} + \underline{t}}}. \tag{2.39}$$

□

Claim 2.16 *For every $t \in I$, we have*

$$\sup_{s\in[t-Q(t)^{-2},t]} Q(s) < 2Q(t). \tag{2.40}$$

Proof First, we show that (2.40) holds at $\underline{t}$. In fact, by the definition of $\underline{t}$ for any $s \in [\underline{t} - Q(\underline{t})^{-2}, \underline{t}]$ we have

$$Q(s) < \frac{2}{\sqrt{Q(0)^{-2}+s}} \leq \frac{2}{\sqrt{Q(0)^{-2}+\underline{t}-Q(\underline{t})^{-2}}} = \frac{2}{\sqrt{3}}Q(\underline{t}),$$

where we used (2.39) in the last equality. Thus, (2.40) holds at $\underline{t}$.

We next define

$$t_0 \triangleq \sup\left\{t \,\middle|\, \text{Every } \ s \in I \cap [\underline{t}, t) \ \text{ satisfies (2.40)}\right\}.$$

Clearly, $t_0 \in I$. In order to prove the Claim, we have to show that $t_0 = \bar{t}$, which denotes the supreme of I. Actually, at time t_0, one of following cases must appear according to the definition of t_0.

Case 1. $\sup_{s\in[t_0-Q(t_0)^{-2},t_0]} Q(s) = 2Q(t_0)$.

Case 2. $Q(t_0) = \dfrac{2}{\sqrt{Q(0)^{-2}+t_0}}$.

We will show that both cases will never happen if $t_0 < \bar{t}$.

(1). If Case 1 happens, then for some $s_0 \in [t_0 - Q(t_0)^{-2}, t_0]$ we have

$$Q(s_0) = 2Q(t_0). \tag{2.41}$$

We now show that $s_0 \in I$. In fact, since $t_0 \in I$, we have

$$Q(s_0) = 2Q(t_0) \geq \frac{4}{\sqrt{Q(0)^{-2}+t_0}} \geq \frac{4}{\sqrt{Q(0)^{-2}+s_0+Q(t_0)^{-2}}}. \tag{2.42}$$

This implies that

$$Q(s_0) = 2Q(t_0) \geq \frac{2\sqrt{3}}{\sqrt{Q(0)^{-2}+s_0}} \geq \frac{2}{\sqrt{Q(0)^{-2}+s_0}}$$

Thus, we have $s_0 \in I$.

By the assumption of t_0, we have

$$\sup_{s\in[s_0-Q(s_0)^{-2},s_0]} Q(s) \leq 2Q(s_0).$$

Then Q drops from $Q(s_0) = 2Q(t_0)$ to $Q(t_0)$ in a time period $|t_0 - s_0|$. By Lemma 2.14 we have

$$\int_{s_0}^{t_0} O^\alpha Q^{2-\alpha} dt > \frac{\epsilon}{A}.$$

Note that $|t_0 - s_0| < Q(t_0)^{-2}$ and $O(t) \leq 1$, $Q \leq 2Q(t_0)$ for $t \in [s_0, t_0]$, we then obtain

$$2^{2-\alpha} Q(t_0)^{-\alpha} > \frac{\epsilon}{A}.$$

Combining this with the inequality $Q(t_0) \geq \frac{2}{\sqrt{Q(0)^{-2}+t_0}}$, we have

$$Q(0) < 2^{\frac{2(1-\alpha)}{\alpha}} \left(\frac{A}{\epsilon}\right)^{\frac{1}{\alpha}},$$

which contradicts the choice of $Q(0)$. Therefore, Case 1 cannot happen.

(2). If Case 2 happens, by the definition of t_0 we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Q \leq \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{\sqrt{Q(0)^{-2}+t}}\right)$$

at time t_0. Recall that $Q(t_0) = \frac{2}{\sqrt{Q(0)^{-2}+t_0}}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} Q \geq -AO^\alpha Q^{3-\alpha} \tag{2.43}$$

at time t_0. It follows that

$$-AO(t_0)^\alpha Q(t_0)^{3-\alpha} \leq -\frac{1}{\left(Q(0)^{-2}+t_0\right)^{\frac{3}{2}}},$$

which implies that

$$Q(0) \leq A^{\frac{1}{\alpha}} \cdot 2^{\frac{3-\alpha}{\alpha}}, \tag{2.44}$$

which contradicts the choice of $Q(0)$. Thus, Case 2 cannot happen and the Claim is proved. □

Since I is closed, we have the supreme $\bar{t} \in I$. It is clear that $\bar{t} < 0$ since

$$Q(0) < \frac{2}{\sqrt{Q(0)^{-2}+0}}.$$

Then at time $\bar{t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t) \le \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{\sqrt{Q(0)^{-2}+t}}\right).$$

Combining this with (2.43), we get the inequality (2.44) again. So this contradiction implies that our assumption $I \neq \emptyset$ is wrong. The Proposition is proved.

Using Proposition 2.15, we can estimate the curvature scale when the curvature at some time is not large.

Proposition 2.17 *Suppose* $\{(M^n, g(t)), -2 \le t \le 0\}$ *is a Calabi flow solution satisfying*

- *the scalar curvature* $|S|(t) \le 1$ *for all* $t \in [-2, 0]$.
- *the curvature tensor satisfies*

$$Q(0) \le \max\left\{2^{\frac{3-\alpha}{\alpha}}A^{\frac{1}{\alpha}}, 2^{\frac{2(1-\alpha)}{\alpha}}\left(\frac{A}{\epsilon}\right)^{\frac{1}{\alpha}}\right\}.$$

Then there is a constant $c_0 = c_0(n, \epsilon, A) > 0$ *such that*

$$F(0) \ge c_0(n, \epsilon, A). \tag{2.45}$$

Proof Let

$$L = 2\max\left\{2^{\frac{3-\alpha}{\alpha}}A^{\frac{1}{\alpha}}, 2^{\frac{2(1-\alpha)}{\alpha}}\left(\frac{A}{\epsilon}\right)^{\frac{1}{\alpha}}\right\} \tag{2.46}$$

and we define

$$s_1 = \begin{cases} \sup\{t \mid Q(t) = L, -1 \le t \le 0\}, & \text{if } \{t \mid Q(t) = L, -1 \le t \le 0\} \neq \emptyset, \\ -1, & \text{if } \{t \mid Q(t) = L, -1 \le t \le 0\} = \emptyset. \end{cases}$$

If $s_1 = -1$, then the theorem holds. Otherwise, by Proposition 2.15 the curvature satisfies

$$Q(t) \le \sqrt{5}L, \quad t \in \left[s_1 - \frac{1}{5L^2}, s_1\right]. \tag{2.47}$$

On the other hand, we have $Q(t) \le L$ for any $t \in [s_1, 0]$. Therefore, $Q(t) \le \sqrt{5}L$ holds for any $t \in \left[s_1 - \frac{1}{5L^2}, 0\right]$. Since the interval $\left[-\frac{1}{5L^2}, 0\right]$ is contained in $\left[s_1 - \frac{1}{5L^2}, 0\right]$, we get

$$Q(t) \le \sqrt{5}L, \quad t \in \left[-\frac{1}{5L^2}, 0\right].$$

The Proposition is proved. □

2.5 Harmonic Scale

In Propositions 2.15 and 2.17, we prove the "stability" of the curvature scale under the scalar bound condition. However, this condition is in general not available. We observe that Calabi energy is scaling invariant for complex dimension 2. For this particular dimension, scalar bound condition can more or less be replaced by Calabi energy small, whenever collapsing does not happen. For the purpose to rule out collapsing, we need a more delicate scale, which is the harmonic scale introduced in this subsection.

Definition 2.18 (cf. [1,2]) Let (M^n, g) be an n-dimensional Riemannian manifold. Given $p \in (n, \infty)$ and $Q > 1$, the $L^{1,p}$ harmonic radius $\mathrm{hr}(x, g)$ at the point $x \in M$ is the largest number r_0 such that on the geodesic ball $B = B_x(r_0)$ of radius r_0 in (M, g), there is a harmonic coordinate chart $U = \{u_i\}_{i=1}^n : B \to \mathbb{R}^n$, such that the metric tensor $g_{ij} = g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$ satisfies

$$Q^{-1}(\delta_{ij}) \le (g_{ij}) \le Q(\delta_{ij}), \quad r_0^{1-\frac{n}{p}} \|\partial g_{ij}\|_{L^p} \le Q - 1.$$

The harmonic radius $\mathrm{hr}_g(M)$ is defined by

$$\mathrm{hr}_g(M) = \inf_{x \in M} \mathrm{hr}(x, g).$$

For the harmonic radius, Anderson–Cheeger showed the following result.

Lemma 2.19 *(cf. [2]) Fix $Q > 1$. Let (M_i, g_i) be a sequence of Riemannian manifolds which converges strongly in $L^{1,p}$ topology to a limit $L^{1,p}$ Riemannian manifold (M, g). Then*

$$\mathrm{hr}_g(M) = \lim_{i \to \infty} \mathrm{hr}_{g_i}(M_i).$$

Moreover, for any $x_i \in M_i$ with $x_i \to x \in M$ we have

$$\mathrm{hr}(x, g) = \lim_{i \to \infty} \mathrm{hr}(x_i, g_i).$$

Next, we introduce the harmonic scale which will be used in the convergence of Calabi flow on Kähler surfaces.

Definition 2.20 Let (M, g) be a Riemannian manifold. The harmonic scale $H_g(M)$ is the supreme of r such that

$$\max_M |\mathrm{Rm}| < r^{-2}, \quad \mathrm{hr}(x, g) > r, \quad \forall\, x \in M.$$

In other words, the harmonic scale of (M, g) is defined by

$$H_g(M) = \min \left\{ \left(\sup_M |\mathrm{Rm}| \right)^{-\frac{1}{2}}, \quad \mathrm{hr}_g(M) \right\}.$$

Lemma 2.21 *There is a universal small constant* ϵ *with the following properties.*

Suppose $\{(M^2, g(t)), -K \le t \le 0\}$ *is a Calabi flow solution on a compact Kähler surface,* $K \ge 2$*. Then for every* $t \in [-1, 0]$*, we have*

$$H_{g(t)}(M) \ge \frac{1}{2}$$

whenever $H_{g(0)}(M) \ge 1$ *and* $Ca(-K) - Ca(0) < \epsilon$*.*

Proof We argue by contradiction. Suppose the statement were wrong. Then we can find a sequence of Calabi flows $\{(M_i, g_i(t)), -K_i \le t \le 0\}(K_i > 2)$ violating the statement with

$$Ca_{g_i}(-K_i) - Ca_{g_i}(0) < \epsilon_i, \quad \epsilon_i \to 0.$$

Let $\{(M, g(t)), -K \le t \le 0\}$ be one of such flows. We shall truncate a critical time interval from this flow. Check if there is a time such that $H(t) < \frac{1}{2}$ in $[-1, 0]$. If no, stop. Otherwise, choose the first time t such that $H(t) = \frac{1}{2}$ and denote it by t_1. Then check the interval $[t_1 - H^4(t_1), t_1]$ to see if there is a time such that $H(t) \le \frac{1}{2}H(t_1)$. If no such time exists, we stop. Otherwise, repeat the process. Note that

$$\begin{aligned} &H(t_k) = \frac{1}{2^k}, \\ &|t_{k+1} - t_k| \le \frac{1}{16^k}, \\ &|t_k| \le 1 + \frac{1}{16} + \frac{1}{16^2} + \cdots < \frac{16}{15}. \end{aligned}$$

This process happens in a compact smooth space time $M \times [-2, 0]$ with bounded geometry. In particular, the harmonic scale is bounded. After each step, the harmonic scale decreases one half. Therefore, it must stop in finite steps. Suppose it stops at $(k+1)$-step. Therefore, for some time $t_{k+1} \in [t_k - H^4(t_k), t_k]$, we have

$$\begin{aligned} &H(t_{k+1}) = \frac{1}{2}H(t_k), \\ &H(t) \ge \frac{1}{2}H(t_{k+1}), \quad \forall\, t \in \left[t_{k+1} - H^4(t_{k+1}), t_{k+1}\right]. \end{aligned}$$

Denote $r_k = H(t_k)$ and let $\tilde{g}(t) = r_k^{-2} g(t_k + r_k^4 t)$, $s = r_k^{-4}(t_{k+1} - t_k) \in [-1, 0)$. Then for the flow $\tilde{g}$, we have

$$\begin{aligned} &H(M, \tilde{g}(0)) = 1, \\ &H(M, \tilde{g}(s)) = \frac{1}{2}, \\ &H(M, \tilde{g}(t)) \ge \frac{1}{2}, \quad \forall\, t \in [s, 0], \end{aligned}$$

$$H(M, \tilde{g}(t)) \geq \frac{1}{32}, \quad \forall\, t \in \left[s - \frac{1}{16}, s\right],$$
$$Ca(M, \tilde{g}(-2)) - Ca(M, \tilde{g}(0)) < \epsilon.$$

Now for each flow g_i, we rearrange the base point and rescale the flow according to the above arrangement. Denote the new flows by $\{(M_i, \tilde{g}_i(t)), -1 \leq t \leq 0\}$. Then the above equations hold for each $\tilde{g}_i$ with some $s_i \in [-1, 0)$ and $\epsilon_i \to 0$. Let x_i be the point where $H(\tilde{g}_i, s_i)$ achieves value. In other words, we have

$$H(M, \tilde{g}_i(s_i)) = \frac{1}{2}, \tag{2.48}$$
$$hr(x_i, \tilde{g}_i(s_i)) = \frac{1}{2}, \quad \text{or} \quad |\mathrm{Rm}|_{\tilde{g}_i(s_i)}(x_i) = 4. \tag{2.49}$$

Let $\underline{s}$ be the limit of s_i. Then $\underline{s} \leq 0$. On $(\underline{s} - \frac{1}{16}, 0)$ we have uniform bound of H when time is uniformly bounded away from $\underline{s} - \frac{1}{16}$. Then we have bound of curvature, curvature higher derivatives, injectivity radius, etc. Therefore, we can take smooth convergence on time interval $(\underline{s} - \frac{1}{16}, 0)$,

$$\left\{(M, x_i, \tilde{g}_i(t)), \underline{s} - \frac{1}{16} < t \leq 0\right\} \xrightarrow{Cheeger-Gromov-C^{\infty}} \left\{\left(\underline{M}, \underline{x}, \underline{g}(t)\right), \underline{s} - \frac{1}{16} < t \leq 0\right\},$$

and the Calabi energy of the limit metric $\underline{g}(t)$ is static for all $t \in (\underline{s} - \frac{1}{16}, 0)$. Note that for each fixed compact set $\Omega \subset \underline{M}$, the integral of $|\nabla\nabla S|^2$ on $M \times [\underline{s} - \frac{1}{16}, 0]$ is dominated by

$$\lim_{i\to\infty} Ca(\tilde{g}_i(-2)) - Ca(\tilde{g}_i(0)) = 0.$$

Therefore, on the limit flow $\underline{g}(t)$, we have $|\nabla\nabla S| \equiv 0$. Every $\underline{g}(t)$ is an extK metric and $\underline{g}(t)$ evolves by the automorphisms group generated by ∇S. In particular, the intrinsic Riemannian geometry does not evolve along the flow. From $t = \underline{s}$ to $t = 0$, suppose the generated automorphism is ϱ, which is the identity map when $\underline{s} = 0$. Clearly, ϱ is the limit of diffeomorphisms ϱ_i, which is the integration of the real vector field ∇S_i from time $t = s_i$ to time $t = 0$.

Note that at time $\underline{s}$ and 0, we have a priori bound for all high curvature derivatives of curvature. By Lemma 2.19, the harmonic radius is continuous in the smooth convergence. Note that at time $t = 0$, harmonic scale is 1, which implies that

$$|\mathrm{Rm}|_{\tilde{g}_i(0)}(x) \leq 1, \quad \mathrm{hr}(x_i, \tilde{g}_i(0)) \geq 1.$$

Therefore, we have

$$\mathrm{hr}(\underline{x}, \underline{g}(0)) = \lim_{i\to\infty} \mathrm{hr}(x_i, \tilde{g}_i(0)) \geq 1,$$
$$\mathrm{hr}(\underline{x}, \underline{g}(\underline{s})) = \lim_{i\to\infty} \mathrm{hr}(x_i, \tilde{g}_i(s_i)).$$

Since $\underline{g}$ evolves by automorphisms, we have

$$\lim_{i\to\infty} \mathrm{hr}(x_i, \tilde{g}_i(s_i)) = hr(\underline{x}, \underline{g}(s)) = hr(\varrho(\underline{x}), \underline{g}(0)) = \lim_{i\to\infty} \mathrm{hr}(\varrho_i(x_i), \tilde{g}_i(0)) \geq 1.$$

On the other hand, it is clear that

$$\lim_{i\to\infty} |\mathrm{Rm}|_{\tilde{g}_i(s_i)}(x_i) = |\mathrm{Rm}|_{\underline{g}(\underline{s})}(\underline{x}) = |\mathrm{Rm}|_{\underline{g}(0)}(\varrho(\underline{x})) = \lim_{i\to\infty} |\mathrm{Rm}|_{\tilde{g}_i(0)}(\varrho_i(x_i)) \leq 1.$$

Therefore, for large i, we have

$$\mathrm{hr}(x_i, \tilde{g}_i(s_i)) > \frac{3}{4}, \quad \sup_{B_{\tilde{g}_i(s_i)}(x_i, \frac{1}{2})} |\mathrm{Rm}|_{\tilde{g}_i(s_i)} < \frac{4}{3},$$

which contradicts (2.49). The lemma is proved. □

2.6 Backward Regularity Improvement

We now can summarize the main results in Sect. 2 as the following backward regularity improvement theorems.

Theorem 2.22 *There is a $\delta = \delta(T_0, B, c_0)$ with the following properties.*

Suppose $T \geq T_0$ and $\{(M^2, \omega(t), J), 0 \leq t \leq T\}$ is a Calabi flow solution satisfying

$$\begin{aligned} &Ca(0) - Ca(T) < \epsilon, \\ &\sup_M |Rm|(\cdot, T) \leq B, \\ &inj(M, g(T)) \geq c_0, \end{aligned}$$

where ϵ is the universal small constant in Lemma 2.21. Then we have

$$\sup_{M\times[T-\delta,T]} |\nabla^l Rm| \leq C_l, \quad \forall\, l \in \mathbb{Z}^+ \cup \{0\}.$$

Proof Note that $H_{g(T)}(M)$ is uniformly bounded from below. Therefore, up to rescaling, we can apply Lemma 2.21 to obtain a δ such that $H_{g(t)}(M)$ is uniformly bounded from below whenever $t \in [T-2\delta, T]$. Then the statement follows from the application of Theorem 2.3.

Theorem 2.23 *There is a $\delta = \delta(n, T_0, A, B)$ with the following properties.*

Suppose $T \geq T_0$ and $\{(M^n, \omega(t), J), 0 \leq t \leq T\}$ is a Calabi flow solution satisfying

$$\sup_{M\times[0,T]} |S| < A,$$

$$\sup_M |Rm|(\cdot, T) \le B.$$

Then we have

$$\sup_{M\times[T-\delta,T]} |\nabla^l Rm| \le C_l, \quad \forall\, l \in \mathbb{Z}^+ \cup \{0\}.$$

The proof of Theorem 2.23 is nothing but a mild application of Propositions 2.15 and 2.17, with loss of accuracy. The reason for developing Proposition 2.15 and Proposition 2.17 with more precise statement is for the later use in Sect. 4, where we study the blowup rate of Riemannian curvature tensors. Note that Theorem 2.23 deals with collapsing case also. If we add a non-collapsing condition at time T, then the scalar curvature bound in Theorem 2.23 can be replaced by a uniform bound of $\|S\|_{L^p}$ for some $p > n$. This was pointed out by Donaldson [31].

Remark 2.24 All the quantities in Theorems 2.22 and 2.23 are geometric quantities, and consequently are invariant under the action of diffeomorphisms. Therefore, if we transform the Calabi flow by diffeomorphisms, then all the estimates in Theorems 2.22 and 2.23 still hold. In particular, they hold for the modified Calabi flow(c.f. Definition 3.1) and the complex structure Calabi flow(c.f. equation (1.3)).

3 Convergence of the Calabi Flow

3.1 Deformation of the Modified Calabi Flow Around extK Metrics

In this subsection, we fix the underlying complex manifold and evolve the Calabi flow in a fixed Kähler class.

Let ω be an extK metric, X to be the extremal vector field defined by the extremal Kähler metric ω, i.e.,

$$X = g^{i\bar{k}} \frac{\partial S}{\partial z^{\bar{k}}} \frac{\partial}{\partial z^i}. \tag{3.1}$$

It is well known that X is a holomorphic vector field, due to the work of Calabi [5]. Recall that for every holomorphic vector field V, the Futaki invariant is defined to be

$$\mathrm{Fut}(V, [\omega]) \triangleq \int_M V(f)\, \omega^n, \tag{3.2}$$

where f is the normalized scalar potential defined by

$$\Delta f = S - \underline{S}, \quad \int_M e^f \omega^n = \mathrm{vol}(M).$$

Note that (3.2) is well-defined since the right-hand side of (3.2) depends only on the Kähler class $[\omega]$. Let $V = X$, then we have

$$\mathrm{Fut}(X, [\omega]) = \int_M (S - \underline{S})^2 \omega^n = Ca(\omega).$$

According to [14], in the class $[\omega]$, the minimal value of the Calabi energy is achieved at ω. In other words, for every smooth metric ω_φ, we have

$$Ca(\omega_\varphi) \geq Ca(\omega),$$

with equality holds if and only if ω_φ is also an extK metric. Note that ω_φ is extremal if and only if $\varrho_0^* \omega_\varphi = \omega$ for some $\varrho_0 \in Aut_0(M, J)$, by the uniqueness theorem of extK metrics(c.f. [4,19,21]).

In the Kähler class $[\omega]$, consider a smooth family of Kähler metrics $g(t)$ satisfying

$$\frac{\partial}{\partial t} g_{i\bar{j}} = S_{,i\bar{j}} + L_{Re(X)} g_{i\bar{j}}, \tag{3.3}$$

where $Re(X)$ is the real part of the holomorphic vector field X defined in (3.1). The above flow was considered in [38] and Sect. 3.2 of [39].

Definition 3.1 Equation (3.3) is called the modified Calabi flow equation. Correspondingly, the functional $Ca(\omega_\varphi) - Ca(\omega)$ is called the modified Calabi energy.

The space $\mathcal{H}$ (c.f. equation (1.4)) has an infinitely dimensional Riemannian symmetric space structure, as described by Donaldson [28], Mabuchi [40], and Semmes [41]. Every two metrics $\omega_{\varphi_1}, \omega_{\varphi_2}$ can be connected by a weak $C^{1,1}$-geodesic, by the result of Chen [11]. Therefore, $\mathcal{H}$ has a metric induced from the geodesic distance d, which plays an important role in the study of the Calabi flow. For example, the Calabi flow decreases the geodesic distance in $\mathcal{H}$(c.f. [7]). Furthermore, by the invariance of geodesic distance up to automorphism action, the modified Calabi flow also decreases the geodesic distance. However, d is too weak for the purpose of improving regularity. Even if we know that $d(\omega_\varphi, \omega)$ is very small, we cannot obtain too much information of ω_φ. For the convenience of improving regularity, we introduce an auxiliary function $\hat{d}$ on $\mathcal{H}$.

Definition 3.2 For each ω_φ in the class $[\omega]$, we define

$$\hat{d}(\omega_\varphi) \triangleq \inf_{\varrho \in Aut_0(M,J)} \left\| \varrho^* \omega_\varphi - \omega \right\|_{C^{k,\frac{1}{2}}}.$$

Note that $\hat{d}$ is not really a distance function. The advantage of $\hat{d}$ is that if $\hat{d}$ is very small, then one can choose an automorphism ϱ such that $\varrho^* \omega_\varphi$ is around the extK metric ω, in the $C^{k,\frac{1}{2}}$-norm. Then regularity improvement of $\varrho^* \omega_\varphi$ becomes possible. Note that in the Kähler class $[\omega]$, a metric form ω_φ is extremal if and only if the modified Calabi energy vanishes, in light of the result of Chen in [14]. Then it follows from the uniqueness of extremal metrics that $\varrho_0^*\omega_\varphi = \omega$ for some $\varrho_0 \in Aut_0(M, J)$. By the definition of $\hat{d}$, we have

$$\hat{d}(\omega_\varphi) = \inf_{\varrho \in Aut_0(M,J)} \left\|\varrho^*\omega_\varphi - \omega\right\|_{C^{k,\frac{1}{2}}} \leq \left\|\varrho_0^*\omega_\varphi - \omega\right\|_{C^{k,\frac{1}{2}}} = 0.$$

In short, $Ca(\omega_\varphi) - Ca(\omega) = 0$ implies that $\hat{d}(\omega_\varphi) = 0$. The following lemma indicates that there is an almost version of this phenomenon.

Lemma 3.3 *For each $\epsilon > 0$, there is a $\delta = \delta(\omega, \epsilon)$ with the following property.*
If $\varphi \in \mathcal{H}$ satisfies

$$\left\|\omega_\varphi - \omega\right\|_{C^{k+1,\frac{1}{2}}} < 1, \quad Ca(\omega_\varphi) - Ca(\omega) < \delta,$$

then $\hat{d}(\omega_\varphi) < \epsilon$.

Proof For otherwise, there is an $\epsilon_0 > 0$ and a sequence of $\varphi_i \in \mathcal{H}$ satisfying

$$\left\|\omega_{\varphi_i} - \omega\right\|_{C^{k+1,\frac{1}{2}}} < 1, \quad Ca(\omega_{\varphi_i}) - Ca(\omega) < \delta_i \to 0, \tag{3.4}$$

$$\hat{d}(\omega_{\varphi_i}) \geq \epsilon_0. \tag{3.5}$$

Then we can assume that ω_{φ_i} converges to ω_{φ_∞} in the $C^{k,\frac{1}{2}}$-topology, which gives rise to $C^{k,\frac{1}{2}}$-metric with Calabi energy the same as $Ca(\omega)$. By the uniqueness theorem(c.f. [4,19,21]), we can find an automorphism $\varrho_\infty \in Aut_0(M, J)$ such that

$$\varrho_\infty^*\omega_{\varphi_\infty} = \omega.$$

Note that ϱ_∞ is automatically smooth. Then we have

$$\varrho_\infty^*\omega_{\varphi_i} \xrightarrow{C^{k,\frac{1}{2}}} \omega,$$

which contradicts (3.5). □

There is a modified Calabi flow version of the short-time existence theorem of Chen-He(c.f. [15]).

Lemma 3.4 *There is a $\xi_0 = \xi_0(\omega)$ with the following properties.*
Suppose $\left\|\omega_\varphi - \omega\right\|_{C^{k,\frac{1}{2}}} < \xi_0$. Then the modified Calabi flow starting from ω_φ exists on time interval $[0, 1]$ and we have

$$\sup_{\frac{1}{2} \leq t \leq 1} \left\|\omega_{\varphi(t)} - \omega\right\|_{C^{k+1,\frac{1}{2}}} < 1.$$

Now we fix ξ_0 in Lemma 3.4 and define $\delta_0 = \delta_0(\omega, \xi_0)$ as in Lemma 3.3.

Lemma 3.5 *Suppose $\eta_0 < \xi_0$ is small enough such that $Ca(\omega_{\omega_\varphi}) - Ca(\omega) < \delta_0$ for every $\varphi \in \mathcal{H}$ satisfying $\left\|\omega_\varphi - \omega\right\|_{C^{k,\frac{1}{2}}} < \eta_0$. Then the modified Calabi flow starting from ω_φ has global existence whenever $\left\|\omega_\varphi - \omega\right\|_{C^{k,\frac{1}{2}}} < \eta_0$.*

Proof Clearly, the flow exists on $[0, 1]$ and $\hat{d}(\omega_{\varphi(1)}) < \xi_0$. We continue to use induction to show that the flow exists on $[0, N]$ for each positive integer N and $\hat{d}(\omega_{\varphi(N)}) < \xi_0$.

Suppose the statement holds for N. Then at time N, we have $\hat{d}(\omega_{\varphi(N)}) < \xi_0$. Therefore, we can find an automorphism ϱ_N such that

$$\left\|\varrho_N^* \omega_{\varphi(N)} - \omega\right\|_{C^{k,\frac{1}{2}}} < \xi_0.$$

Then the modified Calabi flow starting from $\varrho_N^* \omega_{\varphi(N)}$ exists for another time period with length 1, in light of Lemma 3.4. Moreover, we have the $C^{k+1,\frac{1}{2}}$-bound of the metric at the end of this time period. Note that the modified Calabi energy is monotonically decreasing along the flow. Therefore, at the end of the time period 1, one can apply Lemma 3.3 to obtain that $\hat{d} < \xi_0$. Since the intrinsic geometry and $\hat{d}$ does not change under the automorphism action, it is clear that the modified Calabi flow starting from $\omega_{\varphi(N)}$ exists for another time length 1 with proper geometric bounds. Therefore, $\omega_{\varphi(t)}$ are well-defined smooth metrics for $t \in [N, N+1]$ and $\hat{d}(\omega_{\varphi(N+1)}) < \xi_0$. □

We observe that the automorphisms ϱ_i defined in the proof of Lemma 3.5 are uniformly bounded. Actually, note that the geodesic distance between two modified Calabi flows is non-increasing. By triangle inequality, we have

$$\begin{aligned} d(\omega, \omega_\varphi) &\geq d(\omega, \omega_{\varphi(i)}) \geq d(\omega, (\varrho_i^{-1})^* \omega) - d((\varrho_i^{-1})^* \omega, \omega_{\varphi(i)}) \\ &= d(\omega, (\varrho_i^{-1})^* \omega) - d(\omega, \varrho_i^* \omega_{\varphi(i)}). \end{aligned}$$

It follows that

$$d(\omega, (\varrho_i^{-1})^* \omega) \leq d(\omega, \omega_\varphi) + d(\omega, \varrho_i^* \omega_{\varphi(i)}) < C.$$

Therefore, ϱ_i must be uniformly bounded. Furthermore, since the modified Calabi energy always tends to zero as $t \to \infty$, by the results of Streets [45] and He [36], it is clear that

$$Ca(\omega_{\varphi(i)}) - Ca(\omega_{\varphi(i-1)}) \to 0.$$

Applying Lemma 3.3, we can choose $\xi_i \to 0$ such that

$$\left\|\varrho_i^* \omega_{\varphi(i)} - \omega\right\|_{C^{k,\frac{1}{2}}} < \xi_i \to 0, \quad \text{as } i \to \infty.$$

In particular, we have $d(\omega, \varrho_i^* \omega_{\varphi(i)}) \to 0$ as $i \to \infty$.

Theorem 3.6 *Suppose $\varphi \in \mathcal{H}$ satisfying $\left\|\omega_\varphi - \omega\right\|_{C^{k,\frac{1}{2}}} < \eta_0$. Then the modified Calabi flow starting from ω_φ converges to $\varrho^*\omega$ for some $\varrho \in$ $\mathrm{Aut}_0(M, J)$, in the smooth topology of Kähler potentials.*

Proof First, let us show the convergence in distance topology.

Let ϱ_j be the automorphisms defined in the proof of Lemma 3.5. From the above discussion, we see that ϱ_j are uniformly bounded, by taking subsequence if necessary, we can assume ϱ_j converges to ϱ_∞. Then

$$\lim_{j\to+\infty} d((\varrho_\infty^{-1})^*\omega, \omega_{\varphi(j)}) = \lim_{j\to+\infty} d((\varrho_j^{-1})^*\omega, \omega_{\varphi(j)}) = \lim_{j\to+\infty} d(\omega, (\varrho_j)^*\omega_{\varphi(j)}) = 0.$$

Note that $(\varrho_\infty^{-1})^*\omega$ is also an extremal Kähler metric and hence a fixed point of the modified Calabi flow. By the monotonicity of geodesic distance along the modified Calabi flow, we have

$$d((\varrho_\infty^{-1})^*\omega, \omega_{\varphi(t)}) \leq d((\varrho_\infty^{-1})^*\omega, \omega_{\varphi(j)})$$

whenever $t \geq j$. In the above inequality, let $t \to \infty$ and then let $j \to \infty$, we obtain

$$\lim_{t\to\infty} d((\varrho_\infty^{-1})^*\omega, \omega_{\varphi(t)}) = 0. \tag{3.6}$$

This means that $\omega_{\varphi(t)}$ converges to $(\varrho_\infty^{-1})^*\omega$ in the distance topology.

Second, we improve the convergence topology from distance topology in (3.6) to smooth topology.

Define $\varrho(t) = \varrho_j$ for each $t \in [j, j+1)$. In light of the proof of Lemma 3.5, both $\varrho(t)^*\omega_{\varphi(t)}$ and $\varrho(t)$ are uniformly bounded. It follows that $\omega_{\varphi(t)}$ are uniformly bounded in each C^l-topology. Let $t_i \to \infty$ be a time sequence such that

$$\omega_{\varphi(t_i)} \xrightarrow{C^\infty} \omega_{\varphi_\infty}, \quad \text{as} \quad i \to \infty.$$

In view of (3.6), we see that $d(\omega_{\varphi_\infty}, (\varrho_\infty^{-1})^*\omega) = 0$, which forces that $\omega_{\varphi_\infty} = (\varrho_\infty^{-1})^*\omega$. Since $\{t_i\}$ is arbitrary time sequence such that $\omega_{\varphi(t_i)}$ converges, the above discussion actually implies that

$$\omega_{\varphi(t)} \xrightarrow{C^\infty} (\varrho_\infty^{-1})^*\omega, \quad \text{as} \quad t \to \infty.$$

Let $\varrho = \varrho_\infty^{-1}$. Then the proof of the Theorem is complete. □

The convergence in Theorem 3.6 could be as precise as "exponential" if we have further conditions on ω or ω_φ.

Theorem 3.7 ([15,38]) *Same conditions as those in Theorem 3.6. Suppose one of the following conditions is satisfied.*

- *ω is a cscK metric.*
- *ω_φ is G-invariant, where G is a maximal compact subgroup of $Aut_0(M, J)$.*

Then the modified Calabi flow starting from ω_φ converges to $\varrho^*\omega$ exponentially fast in the smooth topology of Kähler potentials. In other words, for each positive integer l, there are constants C, θ depending on l such that

$$|\varphi(t) - \varphi_\infty|_{C^l(g)} \leq C_l \cdot e^{-\theta t}, \forall t \geq 0.$$

3.2 Convergence of Kähler Potentials

The key of this subsection is the following regularity improvement properties.

Proposition 3.8 *Suppose $\{(M^2, \omega_{\varphi(t)}, J), 0 \leq t < \infty\}$ is a modified Calabi flow solution satisfying*

$$\left\|\omega_{\varphi(t_0)} - \omega\right\|_{C^{k,\frac{1}{2}}} < \eta_0, \quad Ca(\omega_{\varphi(t_0-1)}) - Ca(\omega_{\varphi(t_0)}) < \epsilon \tag{3.7}$$

for some $t_0 \geq 1$, where η_0 is the constant in Theorem 3.6, ϵ is the constant in Lemma 2.21. Then for each positive integer $l \geq k$, there is a constant C_l depending on l and ω such that

$$\left\|\omega_{\varphi(t_0)} - \omega\right\|_{C^{l,\frac{1}{2}}} < C_l. \tag{3.8}$$

Proof The solution of the modified flow (3.3) and the Calabi flow differs only by an action of ϱ, which the automorphism generated by $Re(X)$ from time $t_0 - 1$ to t_0, where X is defined in (3.1). Since X is a fixed holomorphic vector field. It follows that $\|\varrho\|_{C^{l,\frac{1}{2}}}$ is uniformly bounded by A_l for each positive integer l. Therefore, in order to show (3.8) for modified Calabi flow, it suffices to prove it for unmodified Calabi flow.

Clearly, the Riemannian geometry of $\omega_{\varphi(t_0)}$ is uniformly bounded, by shrinking η_0 if necessary. We also have $Ca(\omega_{\varphi(t_0-1)}) - Ca(\omega_{\varphi(t_0)}) < \epsilon$. Therefore, Theorem 2.22 applies and we obtain uniform $|\nabla_\varphi^l Rm(\omega_\varphi)|_{g_\varphi}$ bound for each non-negative integer l. Due to the bound of curvature derivatives, one can obtain the metric equivalence for a fixed time period before t_0, say on $[t_0 - \frac{1}{4}, t_0]$. Then we see that $\omega_{\varphi(t_0-\frac{1}{4})}$ is uniformly equivalent to ω and has uniformly bounded Ricci curvature. This forces that $\omega_{\varphi(t_0-\frac{1}{4})}$ has uniform $C^{1,\frac{1}{2}}$-norm, due to Theorem 5.1 of Chen-He [15]. Consequently, (3.8) follows from the smoothing property of the Calabi flow(c.f. the proof of Theorem 3.3 of [15]). □

Proposition 3.9 *Suppose $\{(M^n, \omega_{\varphi(t)}, J), 0 \leq t < \infty\}$ is a modified Calabi flow solution satisfying*

$$\left\|\omega_{\varphi(t_0)} - \omega\right\|_{C^{k,\frac{1}{2}}} < \eta_0, \quad \sup_{M\times[t_0-1,t_0]} |S| < A, \tag{3.9}$$

for some $t_0 \geq 1$, where η_0 is the constant in Theorem 3.6. Then for each positive integer $l \geq k$, there is a constant C_l depending on l , A and ω such that

$$\left\|\omega_{\varphi(t_0)} - \omega\right\|_{C^{l,\frac{1}{2}}} < C_l.$$

Proposition 3.9 is the high dimension correspondence of Proposition 3.8. The proof of Proposition 3.9 is almost the same as that of Proposition 3.8, except that we use Theorem 2.23 to improve regularity, instead of Theorem 2.22.

Now we are ready to prove our main theorem for the extremal Kähler metrics.

Proof of Theorem 1.2 Pick $\omega_\varphi \in \mathcal{LH}$, we need to show that the modified Calabi flow starting from ω_φ converges to $\varrho^*(\omega)$ for some $\varrho \in Aut_0(M, J)$. For simplicity of notation, each flow mentioned in the remaining part of this proof is the modified Calabi flow. According to the definition of $\mathcal{LH}$, we can choose a path $\varphi_s,\ s \in [0, 1]$ connecting ω and ω_φ, i.e.,

$$\omega_{\varphi_0} = \omega, \quad \omega_{\varphi_1} = \omega_\varphi.$$

Let I be the collection of $s \in [0, 1]$ such that the flow starting from ω_{φ_s} converges. In order to show the convergence of the flow starting from ω_φ, it suffices to show the openness and closedness of I, since I obviously contains at least one element $s = 0$.

For each s, let $\varphi_s(t)$ be the the time-t-Kähler potential of the flow starting from ω_{φ_s}. Define function

$$\hat{d}(s, t) \triangleq \inf_{\varrho \in Aut_0(M,J)} \left\|\varrho^*\omega_{\varphi_s(t)} - \omega\right\|_{C^{k,\frac{1}{2}}}.$$

Note that bounded closed set in $Aut_0(M, J)$ is compact since $Aut_0(M, J)$ is a finite-dimensional Lie group. Therefore, we can always find a $\varrho_{s,t} \in Aut_0(M, J)$ such that

$$\hat{d}(s, t) = \left\|\varrho^*_{s,t}\omega_{\varphi_s(t)} - \omega\right\|_{C^{k,\frac{1}{2}}}.$$

Therefore, $\hat{d}$ is a well-defined function. It is also clear that $\hat{d}$ depends on s, t continuously. One can refer Lemma 3.13 and 3.14 for a similar, but more detailed discussion. By Theorem 3.6, if $\hat{d}(s_0, t_0) < \delta_0$ for some $s_0 \in [0, 1], t_0 \in [0, \infty)$, then the flow starting from $\omega_{\varphi_{s_0}(t_0)}$ converges. Consequently, the flow starting from $\omega_{\varphi_{s_0}(0)}$ converges. By continuous dependence of the metrics on the initial data, we see that I is an open set.

We continue to show that I is also closed. Without loss of generality, we can assume that $[0, \bar{s}) \subset I$ and it suffices to show that $\bar{s} \in I$.

For each $s \in [0, \bar{s})$, let T_s be the first time such that $\hat{d}(s, t) = 0.5\eta_0$. By the convergence assumption and the continuity of $\hat{d}$, each T_s is a bounded number.

Claim 3.10 *The bound of T_s is uniform, i.e.,* $\sup_{s\in[0,\bar{s})} T_s < \infty$.

If the Claim fails, we can find a sequence $s_i \to \bar{s}$ such that $T_{s_i} \to \infty$. For simplicity of notation, we denote T_{s_i} by T_i. Consider the flow starting from $\varphi_{\bar{s}}$. By the results of Streets(c.f. [36,45]), we see that

$$\lim_{t\to\infty} Ca(\omega_{\varphi_{\bar{s}}}(t)) = Ca(\omega).$$

Therefore, for each fixed ϵ, we can find L_ϵ such that $Ca(\omega_{\varphi_{\bar{s}}}(L_\epsilon)) < Ca(\omega) + \epsilon$. By continuity, we have

$$\lim_{i\to\infty} Ca(\omega_{\varphi_{s_i}(L_\epsilon)}) = Ca(\omega_{\varphi_{\bar{s}}}(L_\epsilon)) < Ca(\omega) + \epsilon. \tag{3.10}$$

Note that the Calabi energy is non-increasing along each flow. Since $T_i - 1 > L_\epsilon$ for large i, it follows from (3.10) that

$$\lim_{i\to\infty} Ca(\omega_{\varphi_{s_i}(T_i-1)}) \leq Ca(\omega) + \epsilon$$

for each fixed positive ϵ. Since the Calabi energy is always bounded from below by $Ca(\omega)$, it is clear that

$$\lim_{i\to\infty} \left| Ca(\omega_{\varphi_{s_i}(T_i-1)}) - Ca(\omega_{\varphi_{s_i}(T_i)}) \right| = \lim_{i\to\infty} \left(Ca(\omega_{\varphi_{s_i}(T_i-1)}) - Ca(\omega_{\varphi_{s_i}(T_i)}) \right) = 0. \tag{3.11}$$

Note that the harmonic scale of $\omega_{\varphi_{s_i}(T_i)}$ is uniformly bounded away from zero. By Theorem 2.22, we obtain all the curvature and curvature higher derivative bounds of $\omega_{\varphi_{s_i}(T_i)}$. Define

$$\varrho_i \triangleq \varrho_{s_i,T_i}, \quad \omega_{\hat{\varphi}_i} \triangleq \varrho_i^*(\omega_{\varphi_{s_i}(T_i)}).$$

Note that $\omega_{\hat{\varphi}_i}$ has uniform $C^{k,\frac{1}{2}}$-norm by the choice of T_i. Because of (3.11), we can apply Proposition 3.8 and see that $\omega_{\hat{\varphi}_i}$ has uniform $C^{l,\frac{1}{2}}$-norm for each integer $l \geq k$. Therefore, we can take the smooth convergence in fixed coordinate charts.

$$\hat{\varphi}_i \xrightarrow{C^\infty} \hat{\varphi}_\infty, \quad \omega_{\hat{\varphi}_i} \xrightarrow{C^\infty} \omega_{\hat{\varphi}_\infty}. \tag{3.12}$$

Since the Calabi energy converges in the above process, by (3.11) and the above inequality, we obtain that $\omega_{\hat{\varphi}_\infty}$ is an extK metric. By the uniqueness theorem of extK metrics, we obtain that there is a $\varrho_\infty \in Aut_0(M, J)$ such that

$$\omega_{\hat{\varphi}_\infty} = \varrho_\infty^* \omega.$$

Note that ϱ_∞ is automatically smooth. Now (3.12) can be rewritten as

$$\varrho_i^*(\omega_{\varphi_{s_i}(T_i)}) \xrightarrow{C^\infty} \varrho_\infty^*\omega, \quad \left(\varrho_i \circ \varrho_\infty^{-1}\right)^*(\omega_{\varphi_{s_i}(T_i)}) = \left(\varrho_\infty^{-1}\right)^* \circ \varrho_i^*(\omega_{\varphi_{s_i}(T_i)})$$
$$= \left(\varrho_\infty^{-1}\right)^* \omega_{\hat{\varphi}_i} \xrightarrow{C^\infty} \omega.$$

It follows from the definition of $\hat{d}$ that

$$\hat{d}(s_i, T_i) \le \left\| \left(\varrho_i \circ \varrho_\infty^{-1}\right)^*(\omega_{\varphi_{s_i}(T_i)}) - \omega \right\|_{C^{k,\frac{1}{2}}} \to 0,$$

which contradicts our choice of T_i, i.e., $\hat{d}(s_i, T_i) = 0.5\eta_0$. Therefore, the proof of Claim 3.10 is complete.

In light of Claim 3.10, we can choose a sequence of $s_i \to \bar{s}$ and $T_i = T_{s_i} \to \bar{T} < \infty$. By continuity of $\hat{d}$, we see that

$$\hat{d}(\bar{s}, \bar{T}) = 0.5\eta_0.$$

Consequently, Theorem 3.6 applies and the flow starting from $\omega_{\varphi_{\bar{s}}(\bar{T})}$ converges. Hence the flow starting from $\omega_{\varphi_{\bar{s}}(0)}$ converges and $\bar{s} \in I$. The proof of the Theorem is complete. □

Theorem 1.4 can be proved almost verbatim, except replacing Propsotion 3.8 by Proposition 3.9.

3.3 Convergence of Complex Structures

In this subsection, we regard the Calabi flow as the flow of the complex structures on a given symplectic manifold (M, ω). We also assume this symplectic manifold has a cscK complex structure J_0. Note that the uniqueness theorem of extK metrics in a given Kähler class of a fixed complex manifold plays an important role in the convergence of potential Calabi flow. Similar uniqueness theorem will play the same role in the convergence of complex structure Calabi flow. Actually, by the celebrated work of Chen-Sun(c.f. Theorem 1.3 of [20]), on each C^∞-closure of a $\mathcal{G}^{\mathbb{C}}$-leaf of a smooth structure J, there is at most one cscK complex structure(i.e., the metric determined by ω and J is cscK), if it exists.

It is important to note that the complex structure Calabi flow solution is invariant under Hamiltonian diffeomorphism. Suppose J_A and J_B are two isomorphic complex structures, i.e.,

$$\varphi^*\omega = \omega, \quad J_A = \varphi^* J_B$$

for some symplectic diffeomorphism φ. Let $J_A(t)$ be a Calabi flow solution starting from J_A, then $\varphi^* J_A(t)$ is a Calabi flow solution starting from J_B.

Let g_0 be the metric compatible with ω and J_0. Then it is clear that g_0 is cscK and therefore smooth metric. We can choose coordinate system of M such that g, J, ω are all smooth in each coordinate chart. We equip the tangent bundle and cotangent bundle and their tensor products with the natural metrics induced from g_0. Clearly, if a diffeomorphism φ preserves both ω and J_0, then it preserves g_0 and therefore locates in $\mathrm{ISO}(M, g_0)$, which is compact Lie group. Therefore, each φ is smooth and has a priori bound of $C^{l,\frac{1}{2}}$-norm for each positive integer l, whenever φ is regarded as a smooth section of the bundle $M \times M \to M$, equipped with natural metric induced from g_0. There is an almost version of this property. In other words, if φ preserves ω and the $C^{k,\frac{1}{2}}$-norm of $\varphi^* J$ is very close to J_0, then φ has a priori bound of $C^{k+1,\frac{1}{2}}$-norm. This is basically because of the improving regularity property of isometry(c.f. [8]).

Proposition 3.11 *Let* $\left\{U_x, \{x^i\}_{i=1}^m\right\}$ *and* $\left\{U_y, \{y^i\}_{i=1}^m\right\}$ *be two coordinates of an open Riemannian manifold* (V, ds^2)*. The Riemannian metric in these two coordinates can be written as*

$$ds^2 = g_{ij}(x)dx^i dx^j = \tilde{g}_{ij}(y)dy^i dy^j. \tag{3.13}$$

As subsets of $\mathbb{R}^m$*,* U_x*, and* U_y *satisfy*

$$\left\{x \,\middle|\, |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2} < \frac{1}{2}\right\} \subset U_x \subset \{x \,|\, |x| < 1\},$$
$$\left\{y \,\middle|\, |y| < \frac{1}{2}\right\} \subset U_y \subset \{y \,|\, |y| < 1\}.$$

In each coordinate, the metrics are uniform equivalent to Euclidean metrics.

$$\frac{1}{2}\delta_{ij} < g_{ij}(x) < 2\delta_{ij}, \quad \forall\, x \in U_x$$
$$\frac{1}{2}\delta_{ij} < \tilde{g}_{ij}(y) < 2\delta_{ij}, \quad \forall\, y \in U_y.$$

Suppose g_{ij} *and* $\tilde{g}_{ij}$ *are of class* $C^{k,\frac{1}{2}}$ *for some integer* $1 \le k \le \infty$*. Suppose the natural map* $x = f(y)$ *satisfies* $f(0) = 0$*. Then* f *is of class* $C^{k+1,\frac{1}{2}}$ *and*

$$\|f\|_{C^{k+1,\frac{1}{2}}(B_{\frac{1}{4}})} < C\left(m, k, \|g_{ij}\|_{C^{k,\frac{1}{2}}(U_x)}, \|\tilde{g}_{ij}\|_{C^{k,\frac{1}{2}}(U_y)}\right),$$

where $B_{\frac{1}{4}}$ *is the standard ball in* $\mathbb{R}^m$ *with radius* $\frac{1}{4}$ *and centered at* 0.

Proof Equation (3.13) can be rewritten as

$$\tilde{g}_{ij} = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}.$$

Denote Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ as the Christoffel symbol of ds^2 under the x and y coordinate, respectively. Then direct calculation implies that

$$\tilde{\Gamma}_{ij}^k = \frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\Gamma_{pq}^r\frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^u}\frac{\partial^2 x^u}{\partial y^i \partial y^j}.$$

In other words, the second derivatives of $x(y)$ can be expressed as

$$\frac{\partial^2 x^u}{\partial y^i \partial y^j} = \frac{\partial x^u}{\partial y^k}\tilde{\Gamma}^k_{ij} - \Gamma^u_{pq}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}. \tag{3.14}$$

Now suppose the metric $\tilde{g}_{ij}$ is $C^{k,\frac{1}{2}}$ for some integer $k \geq 1$. Moreover, let us assume

$$\|g_{ij}\|_{C^{k,\frac{1}{2}}(U_x)}, \quad \|\tilde{g}_{ij}\|_{C^{k,\frac{1}{2}}(U_y)} \leq A.$$

Therefore, $\tilde{\Gamma}^k_{ij}$ and Γ^k_{ij} are $C^{k-1,\frac{1}{2}}$ and has uniformly bounded $C^{k-1,\frac{1}{2}}$-norm in smaller balls. By bootstrapping argument, we obtain that

$$\|f\|_{C^{k+1,\frac{1}{2}}(B_{\frac{1}{4}})} \leq C(m,k,A).$$

□

Lemma 3.12 *Suppose* $\varphi \in Symp(M,\omega)$. *Then we have*

$$\|\varphi\|_{C^{k+1,\frac{1}{2}}} < C\left(n, k, \|\varphi_* J\|_{C^{k,\frac{1}{2}}}, \|J\|_{C^{k,\frac{1}{2}}}\right).$$

Proof Regard φ as an isometry from (M,ω,J) to $(M,\omega,\varphi_* J)$. Then the proof boils down to Proposition 3.11. Note that $\varphi_* J$ is the push forward of J, which is the same as $(\varphi^{-1})^* J$, and the default metric we take $C^{k,\frac{1}{2}}$-norm is the metric g_0. □

Fix J_0 as the cscK complex structure. For each complex structure J compatible with ω, we define

$$\tilde{d}(J) = \inf \left\|\varphi^* J - J_0\right\|_{C^{k,\frac{1}{2}}},$$

where infimum is taken among all symplectic diffeomorphisms with finite $C^{k,\frac{1}{2}}$-norm. Let φ_i be a minimizing sequence to approximate $\tilde{d}(J)$. By triangle inequality and Lemma 3.12, we see that φ_i has uniformly bounded $C^{k+1,\frac{1}{2}}$-norm. Therefore, by taking subsequence if necessary, we can assume φ_i converges, in the $C^{k+1,\frac{1}{3}}$-topology, to a limit symplectic diffeomorphism φ_∞. Although the convergence topology is weak, it follows from definition that φ_∞ has bounded $C^{k+1,\frac{1}{2}}$-norm. Therefore, we see that

$$\left\|\varphi^*_\infty J - J_0\right\|_{C^{k,\frac{1}{2}}} \leq \lim_{i\to\infty} \left\|\varphi^*_i J - J_0\right\|_{C^{k,\frac{1}{2}}} = \tilde{d}(J).$$

On the other hand, by definition, we have

$$\tilde{d}(J) \leq \left\|\varphi^*_\infty J - J_0\right\|_{C^{k,\frac{1}{2}}}.$$

Combining the above two inequalities, we obtain that $\tilde{d}(J)$ is achieved by $\varphi^*_\infty J$. We have proved the following Lemma.

Lemma 3.13 *For each smooth complex structure J compatible with ω, $\tilde{d}(J)$ is achieved by a diffeomorphism $\varphi \in C^{k+1,\frac{1}{2}}(M, M)$ and $\varphi^*\omega = \omega$.*

Lemma 3.14 *$\tilde{d}$ is a continuous function on the moduli space of complex structures, equipped with $C^{k,\frac{1}{2}}$-topology.*

Proof Fix J_A, let $J_B \to J_A$. Then we need to show that

$$\tilde{d}(J_A) = \lim_{J_B \to J_A} \tilde{d}(J_B).$$

On one hand, by Lemma 3.13, we can find $\varphi_A \in C^{k+1,\frac{1}{2}}(M, M)$ such that

$$\tilde{d}(J_A) = \left\| \varphi_A^* J_A - J_0 \right\|_{C^{k,\frac{1}{2}}}.$$

It follows that

$$\tilde{d}(J_B) = \inf_{\varphi \in Symp(M,\omega)} \left\| \varphi^* J_B - J_0 \right\|_{C^{k,\frac{1}{2}}} \leq \left\| \varphi_A^* J_B - J_0 \right\|_{C^{k,\frac{1}{2}}}.$$

Let $J_B \to J_A$ and take limit on both sides, we have

$$\limsup_{J_B \to J_A} \tilde{d}(J_B) \leq \limsup_{J_B \to J_A} \left\| \varphi_A^* J_B - J_0 \right\|_{C^{k,\frac{1}{2}}} = \left\| \varphi_A^* J_A - J_0 \right\|_{C^{k,\frac{1}{2}}} = \tilde{d}(J_A).$$

On the other hand, for each J_B, we have φ_B to achieve the $\tilde{d}$. Then we see that

$$\liminf_{J_B \to J_A} \tilde{d}(J_B) = \liminf_{J_B \to J_A} \left\| \varphi_B^* J_B - J_0 \right\|_{C^{k,\frac{1}{2}}} \geq \left\| \varphi_\infty^* J_A - J_0 \right\|_{C^{k,\frac{1}{2}}} \geq \tilde{d}(J_A),$$

where φ_∞ is a limit symplectic diffeomorphism of φ_B as $J_B \to J_A$. The existence and estimates of φ_∞ follow from Lemma 3.12.

Therefore, $\tilde{d}$ is continuous at J_A by combining the above two inequalities. Since J_A is chosen arbitrarily in the moduli space of complex structures, we finish the proof. □

Lemma 3.15 *Suppose $J_A = J_s(t)$ for some $s \in D$ and $t \geq 0$. There is a constant $\delta > 0$ such that if $\tilde{d}(J_A) < \delta$, then the Calabi flow starting from J_A has global existence and converges to $\psi^*(J_0)$ for some $\psi \in \mathrm{Symp}(M, \omega)$.*

Proof By Theorem 5.3 of [20], there is a small δ such that every Calabi flow starting from the δ-neighborhood of J_0, in $C^{k,\frac{1}{2}}$-topology, will converge to some cscK J'. Note that the Calabi flow solution always stays in the $\mathcal{G}^{\mathbb{C}}$-leaf of J_1, hence J' is in the C^∞-closure of J_1. Also, on the other hand, according to the conditions of J_s, we see that J_0 is also in the C^∞-closure of J_1. Therefore, one can apply the uniqueness degeneration theorem, Theorem 1.3 of [20] to obtain that (M, ω, J_0) and (M, ω, J') are isomorphic. Namely, $(\omega, J') = \eta^*(\omega, J_0)$ for some diffeomorphism η.

If $\tilde{d}(J_A) < \delta$, then we can find a diffeomorphism φ such that

$$\varphi^*\omega = \omega, \quad \left\|\varphi^* J_A - J_0\right\|_{C^{k,\frac{1}{2}}(M,g_0)} < \delta.$$

By previous argument, the Calabi flow initiated from $\varphi^* J_A$ will converge to $\eta^*(J_0)$, for some symplectic diffeomorphism η. Then the Calabi flow starting from J_A converges to $(\varphi^{-1})^*\eta^*(J_0)$. Let $\psi = \eta \circ \varphi^{-1}$, we then finish the proof. □

With these preparation, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let I to be the collection of $s \in [0, 1]$ such that the Calabi flow initiated from J_s converges to $\psi^*(J_0)$ for some symplectic diffeomorphism ψ. By abusing of notation, we denote

$$\tilde{d}(s, t) = \tilde{d}(J_s(t)).$$

In light of Lemma 3.15 and continuity of $\tilde{d}$, we see that I is open. In order to show $I = [0, 1]$, it suffices to show the following claim.

Claim 3.16 *Suppose* $[0, \bar{s}) \subset I$ *for some* $\bar{s} \in (0, 1]$*, then* $\bar{s} \in I$.

We argue by contradiction.

If the statement was wrong, then $\tilde{d}(\bar{s}, 1) \geq \delta$, due to Lemma 3.15. For each $s \in (0, \bar{s})$, we see $\tilde{d}(s, t)$ will converge to zero finally, while $\tilde{d}(s, 1) > 0.5\delta$ whenever s is very close to $\bar{s}$, say, for $s \in [\bar{s} - \xi, \bar{s})$. Therefore, for each s nearby $\bar{s}$, there is a T_s such that $\tilde{d}(s, T_s) = 0.5\delta$ for the first time. A priori, there are two possibilities for the behavior of T_s:

- There is a sequence of s_i such that $s_i \to \bar{s}$ and $T_{s_i} \to \infty$.
- $\sup_{s \in [\bar{s}-\xi, \bar{s})} T_s < A$ for some constant A.

We shall exclude the first possibility. The proof is parallel to that of Claim 3.10. Actually, if the first possibility appears, then we see that

$$\lim_{i \to \infty} Ca(J_{s_i}(T_{s_i})) = 0.$$

In light of the monotonicity of the Calabi energy along each flow, the continuous dependence of the Calabi energy on parameters s and t, and the fact that

$$\lim_{t \to \infty} Ca(J_{\bar{s}}(t)) = 0.$$

For simplicity of notations, we denote $J_i' = J_{s_i}(T_{s_i})$. Note that every J_i' can be pulled back to J_1' by some diffeomorphisms η_i, which may not preserve ω. In other words, we have

$$J_1' = \eta_i^* J_i', \quad \omega_i \triangleq \eta_i^* \omega, \tag{3.15}$$

where $\omega_i = \omega + \sqrt{-1}\partial\bar{\partial} f_i$ for some smooth functions f_i, with respect to the complex structure J_1'. Note also that (M, ω, J_i') has the same intrinsic geometry as (M, ω_i, J_1'), which has uniformly bounded Riemannian geometry and all high order curvature covariant derivatives, due to Theorem 2.22. Therefore, we can take Cheeger–Gromov convergence:

$$(M, \omega_i, J_1') \xrightarrow{Cheeger-Gromov-C^\infty} \left(M, \bar{\omega}, \bar{J}\right).$$

In other words, there exist smooth diffeomorphisms φ_i such that

$$\varphi_i^* \omega_i \xrightarrow{C^\infty} \bar{\omega}, \quad \varphi_i^* J_1' \xrightarrow{C^\infty} \bar{J}. \tag{3.16}$$

Note that everything converges smoothly, the limit Kähler manifold $(M, \bar{\omega}, \bar{J})$ has zero Calabi energy and consequently is cscK. Moreover, it is adjacent to $([\omega], J_1')$, in the sense of Chen-Sun(c.f. Definition 1.4 of [20]). Therefore, the unique degeneration theorem of Chen-Sun (Theorem 1.6 of [20]) applies, we see that there is a smooth diffeomorphism φ of M such that

$$\bar{\omega} = \varphi^* \omega, \quad \bar{J} = \varphi^* J_0. \tag{3.17}$$

Let $\psi_i = \varphi_i \circ \varphi^{-1}$, $\gamma_i = \eta_i \circ \psi_i$. Combining (3.15), (3.16) and (3.17), we obtain

$$\psi_i^* \omega_i \xrightarrow{C^\infty} \omega, \quad \psi_i^* J_1' \xrightarrow{C^\infty} J_0,$$
$$\gamma_i^* \omega \xrightarrow{C^\infty} \omega, \quad \gamma_i^* J_i' \xrightarrow{C^\infty} J_0.$$

Since $[\omega]$ is integral, we see $[\gamma_i^* \omega] = [\omega]$ for sufficiently large i. Composing with an extra convergent sequence of diffeomorphisms if necessary, we can assume that

$$\gamma_i^* \omega = \omega, \quad \gamma_i^* J_i' \xrightarrow{C^\infty} J_0.$$

Then we have $\tilde{d}(J_i') \to 0$ as $i \to \infty$, which contradicts our choice of J_i', namely, $\tilde{d}(J_i') = 0.5\delta$. This contradiction excludes the first possibility. Therefore, we have

$$\sup_{s \in [\bar{s}-\xi, \bar{s})} T_s < A$$

for some uniform constant A. Now we choose $s_i \in [0, \bar{s})$ such that $s_i \to \bar{s}$, we can assume $T_{s_i} \to T_{\bar{s}}$ for some finite $T_{\bar{s}}$ by the above estimate. In light of the continuous dependence of solutions to the initial data, we see that the complex structure $J_{\bar{s}}(T_{\bar{s}})$ is the limit of $J_{s_i}(T_{s_i})$. Hence

$$\tilde{d}(J_{\bar{s}}(T_{\bar{s}})) = \lim_{i \to \infty} \tilde{d}(J_{s_i}(T_{s_i})) = 0.5\delta < \delta.$$

Consequently, Lemma 3.15 can be applied for the Calabi flow started from $J_{\bar{s}}(T_{\bar{s}})$. Therefore, $\bar{s} \in I$ and we finish the proof of Claim 3.16. □

The proof of Theorem 1.5 is almost the same as Theorem 1.3. The only difference is that we use Theorem 2.23 to improve regularity, rather than Theorem 2.22.

3.4 Examples

In this subsection, we will give higher dimension examples with global existence. On such examples, our results and methods developed in previous sections can be applied.

Example 3.17 (cf. [17]) Let (M, J) be the blowup of $\mathbb{CP}^2$ at three generic points. Let the Kähler class of ω be

$$3[H] - \lambda([E_1] + [E_2] + [E_3]), \quad 0 < \lambda < \frac{3}{2}, \quad \lambda \in \mathbb{Q},$$

where H denotes the pullback of the hyperplane of $\mathbb{CP}^2$ in M and $E_i (1 \leq i \leq 3)$ denotes the exceptional divisors. Suppose ω is invariant under the toric action and the action of $\mathbb{Z}_3$ and satisfies

$$\int_M S^2 \, dV < 192\pi^2 + 32\pi^2 \frac{(3-\lambda)^2}{3-\lambda^2}. \tag{3.18}$$

Then the Calabi flow starting from ω exists for all time and converges to a cscK metric in the smooth topology of the Kähler potentials.

Example 3.18 (cf. [16]) Let (M, J) be a toric Fano surface. Let ω be a Kähler metric with positive extremal Hamiltonian potential. Suppose $[\omega]$ is rational and ω is invariant under the toric action and satisfies

$$\int_M S^2 \, dV < 32\pi^2 \left(c_1^2(M) + \frac{1}{3} \frac{(c_1(M) \cdot \Omega)^2}{\Omega^2} \right) + \frac{1}{3} \|\mathcal{F}\|^2,$$

where $\|\mathcal{F}\|^2$ is the norm of Calabi–Futaki invariant. Then the modified Calabi flow starting from ω exists for all time and converges to an extK metric in the smooth topology of Kähler potentials.

Actually, in [17] and [16], the global existence of the flows in Example 3.17 and 3.18 was already proved by energy method, based on the work of [18]. Furthermore, they showed the sequence convergence in the Cheeger–Gromov topology. The only new thing here is the improvement of the convergence topology. Let us sketch a proof of the statement of Example 3.18. By results of [16], we can take time sequence $t_i \to \infty$ such that

$$(M, \omega(t_i), J) \xrightarrow{Cheeger-Gromov-C^\infty} (M', \omega', J'),$$

where (M', ω', J') is an extK metric. In other words, there exist smooth diffeomorphisms ψ_i such that

$$(\psi_i^* \omega(t_i), \psi_i^* J) \xrightarrow{C^\infty} (\omega', J')$$

in fixed coordinates. The toric symmetry condition forces that J' is isomorphic to J, i.e., $J' = \psi^* J$ for some smooth diffeomorphism ψ. Let $\eta_i = \psi_i \circ \psi^{-1}$, we have

$$(\eta_i^* \omega(t_i), \eta_i^* J) \xrightarrow{C^\infty} ((\psi^{-1})^* \omega', J). \tag{3.19}$$

The rational condition of $[\omega]$ forces that $[\eta_i^* \omega(t_i)] = [(\psi^{-1})^* \omega']$. Therefore, there exists smooth diffeomorphisms $\rho_i \to Id$ such that

$$\rho_i^* \eta_i^* \omega(t_i) = (\psi^{-1})^* \omega'. \tag{3.20}$$

Then equation (3.19) becomes

$$\rho_i^* \eta_i^* J \xrightarrow{C^\infty} J. \tag{3.21}$$

We can write $\eta_i = \varrho_i \circ \sigma_i$ for $\varrho_i \in Aut(M, J)$ and $\sigma_i \to Id$. Then (3.19) can be rewritten as

$$(\sigma_i^* \varrho_i^* \omega(t_i), \sigma_i^* J) \xrightarrow{C^\infty} ((\psi^{-1})^* \omega', J),$$
$$(\varrho_i^* \omega(t_i), J) \xrightarrow{C^\infty} ((\psi^{-1})^* \omega', J).$$

Note that $Aut(M, J) = Aut_0(M, J)$ in our examples. Hence $[\varrho_i^* \omega(t_i)] = [(\psi^{-1})^* \omega'] = [\omega]$. For simplicity of notation, we denote $(\psi^{-1})^* \omega'$ by ω_{extK}. Then on the fixed complex manifold (M, J), within the Kähler class $[\omega]$, we have $\varrho_i^* \omega(t_i) \to \omega_{extK}$ in the smooth topology of Kähler potentials. For some large i, $\varrho_i^* \omega(t_i)$ locates in a tiny $C^{k, \frac{1}{2}}$-neighborhood of ω_{extK}. Then we can apply Theorems 3.6 and 3.7 to show the modified Calabi flow staring from $\varrho_i^* \omega(t_i)$ converges to an extK metric $\varrho^* \omega_{extK}$ exponentially fast, where $\varrho \in Aut_0(M, J)$.

4 Behavior of the Calabi Flow at Possible Finite Singularities

All our previous discussion in this paper is based on the global existence of the Calabi flow, and there do exist some non-trivial examples of global existent Calabi flow. However, it is not clear whether the global existence of the Calabi flow holds in general. Suppose the Calabi flow starting from ω_φ fails to have global existence. Then there must be a maximal existence time T. By the work of Chen-He [15], we see that Ricci curvature must blowup at time T. In this subsection, we will study the behavior of more geometric quantities at the first singular time T. For simplicity of notations, we often let the singular time T to be 0. Recall that P, Q, R are defined in equation (1.9).

Proposition 4.1 *Suppose* $\{(M^n, g(t)), -1 \le t \le K, 0 \le K\}$ *is a Calabi flow solution satisfying*

$$Q(0) = 1, \quad Q(t) \le 2, \quad \forall\, t \in [-1, 0].$$

Then we have

$$Q(K) < 2^{\frac{1}{\epsilon_0}\int_0^K P_g(t)\,dt + 1}, \tag{4.1}$$

where ϵ_0 *is the dimensional constant obtained in Lemma 2.9.*

Proof For any non-negative integer i, we define $s_i = \inf\left\{t \mid t \ge 0,\, Q(t) = 2^i\right\}$. Note that $s_0 = 0$. Thus, we have

$$s_i - \frac{1}{Q(s_i)^2} \ge -1, \qquad \sup_{\left[s_i - \frac{1}{Q(s_i)^2},\, s_{i+1}\right]} Q(t) = Q(s_{i+1}) = 2^{i+1}.$$

We rescale the metrics by

$$g_i(x, t) \triangleq Q(s_i) g\left(x, \frac{t}{Q(s_i)^2} + s_i\right).$$

Then the flow $\left\{(M, g_i(t)), -1 \le t \le Q(s_i)^2(K - s_i)\right\}$ is a Calabi flow solution satisfying

- $Q_{g_i}(0) = 1$,
- $Q_{g_i}(t) \le 1$ for all $t \in [-1, 0]$,
- $Q_{g_i}\left(Q(s_i)^2(s_{i+1} - s_i)\right) = 2$.

Thus, Lemma 2.9 applies and we have

$$\int_{s_i}^{s_{i+1}} P_g(t)\,dt = \int_0^{Q(s_i)^2(s_{i+1}-s_i)} P_{g_i}(t)\,dt \ge \epsilon_0.$$

Let N be the largest i such that $s_i \le K$. Then

$$N\epsilon_0 \le \int_0^{s_N} P_g(t)\,dt \le \int_0^K P_g(t)\,dt,$$

which implies that $N \le \frac{1}{\epsilon_0}\int_0^K P_g(t)\,dt$. It follows that for any $t \in [0, K]$,

$$Q(t) \le 2^{N+1} \le 2^{\frac{1}{\epsilon_0}\int_0^K P_g(t)\,dt + 1}.$$

Then (4.1) follows trivially and we finish the proof. □

Since the Riemannian curvature tensor blows up at the singular time along the Calabi flow, Proposition 4.1 directly implies the following result.

Corollary 4.2 *Suppose* $\{(M^n, g(t)), -1 \leq t < 0\}$ *is a Calabi flow solution. If* $t = 0$ *is the singular time, then we have*

$$\int_{-1}^{0} P_g(t)\, dt = \infty.$$

In particular, $P_g(t)$ *will blow up at the singular time* $t = 0$*:*

$$\lim_{t \to 0} P_g(t) = \infty.$$

Next, we would like to estimate $Q(t)$ near the singular time of the Calabi flow. Analogous results for Ricci flow are proved by the maximum principle (cf. for example, Lemma 8.7 of [25]). Here we show similar results for the Calabi flow using the higher order curvature estimates.

Lemma 4.3 *There exists a constant* $\delta_0 = \delta_0(n) > 0$ *with the following properties.*

Suppose $\{(M^n, g(t)), -1 \leq t \leq 0\}$ *is a Calabi flow solution and* $t = 0$ *is the singular time. Then*

$$\limsup_{t \to 0} Q(t)\sqrt{-t} \geq \delta_0. \tag{4.2}$$

Proof By Theorem 2.3, we can find a constant $\delta_1(n) > 1$ satisfying the following properties. If $\{(M, g(t)), -1 \leq t \leq -\frac{1}{2}\}$ is a Calabi flow solution with $|\mathrm{Rm}|(t) \leq 1$ for $t \in [-1, -\frac{1}{2}]$, then

$$\left| \frac{\partial}{\partial t} |\mathrm{Rm}| \right|_{t=-\frac{1}{2}} \leq \delta_1. \tag{4.3}$$

We claim that under the assumption of Lemma 4.3, we have

$$\sup_{[-1, -\frac{1}{2\delta_1}]} Q(t) \geq \frac{1}{2}. \tag{4.4}$$

Otherwise, we have $\sup_{[-1, -\frac{1}{2\delta_1}]} Q(t) < \frac{1}{2}$. We choose $t_0 \in [-1, 0)$ the first time such that $Q(t_0) = 1$. Clearly, $t_0 > -\frac{1}{2\delta_1}$. Since

$$Q(t_0) \leq Q\left(-\frac{1}{2\delta_1}\right) + \sup_{M \times [-\frac{1}{2\delta_1}, t_0]} \left| \frac{\partial}{\partial t} |\mathrm{Rm}| \right| \cdot \left(t_0 + \frac{1}{2\delta_1} \right),$$

we have

$$1 < \frac{1}{2} + \delta_1 \left(t_0 + \frac{1}{2\delta_1} \right) \le \frac{1}{2} + \delta_1 \cdot \frac{1}{2\delta_1} \le 1, \tag{4.5}$$

which is a contradiction. Here we used (4.3) in the first inequality of (4.5). The inequality (4.4) is proved.

Now we estimate $Q(s_0)$ for any $s_0 \in (-1, 0)$. Since $t = 0$ is the singular time, we can assume that $Q(t) \le Q(s_0)$ for all $t \in [-1, s_0]$. Rescale the metric by

$$\tilde{g}(x, t) = Q\, g\left(x, \frac{t}{Q^2} \right), \quad t \in [-Q^2, 0).$$

Choose Q such that $s_0 Q^2 = -\frac{1}{2\delta_1}$. If $|s_0| \le \frac{1}{2\delta_1}$, then we have $Q \ge 1$ and $\{(M, \tilde{g}(t)), -1 \le t < 0\}$ is a Calabi flow solution with the singular time $t = 0$. Thus, by (4.4) we have

$$Q_{\tilde{g}}\left(-\frac{1}{2\delta_1} \right) \ge \frac{1}{2},$$

which is equivalent to say

$$Q(s_0)\sqrt{-s_0} \ge \frac{1}{2\sqrt{2\delta_1}}, \quad \forall\, s_0 \in \left[-\frac{1}{2\delta_1}, 0 \right).$$

The lemma is proved. □

The next result gives an upper bound of Q near the singular time with the assumption on P.

Lemma 4.4 *Suppose that* $\{(M^n, g(t)), -1 \le t < 0\}$ *is a Calabi flow with singular time* $t = 0$. *If*

$$\limsup_{t \to 0} P(t)|t| = C < +\infty, \tag{4.6}$$

then we have

$$Q(t) = o\left(|t|^{-\lambda} \right) \tag{4.7}$$

for any constant $\lambda > \frac{C \log 2}{\epsilon_0}$. *Here* ϵ_0 *is the constant in Lemma 2.9.*

Proof Since $t = 0$ is the singular time, for any $\delta > 0$ we can choose $t_0 = t_0(g, \delta)$ such that the following properties hold:

$$P(t)|t| < C + \delta, \quad \forall\, t \in [t_0, 0), \tag{4.8}$$

$$Q(t) \le Q(t_0), \quad \forall\, t \in [-1, t_0], \tag{4.9}$$

$$Q^2(t_0)|1 + t_0| \ge 1. \tag{4.10}$$

Define $s_i = \inf \left\{t \mid t \geq t_0,\ \ Q(t) = 2^i Q(t_0)\right\}$ and we rescale the metrics by

$$h_i(x,t) = Q(s_i) g\left(x, \frac{t}{Q^2(s_i)} + s_i\right), \quad t \in \Big[-(1+s_i)Q^2(s_i),\, Q^2(s_i)|s_i|\Big).$$

Then by (4.9) and (4.10) the flow $\{(M, h_i(t)), -1 \leq t \leq 0\}$ satisfies

$$Q_{h_i}(0) = 1, \quad Q_{h_i}(t) \leq 1, \quad \forall\, t \in [-1, 0],$$

and $Q_{h_i}\Big(Q^2(s_i)|s_{i+1} - s_i|\Big) = 2$. By Lemma 2.9, we have

$$\int_{s_i}^{s_{i+1}} P\, dt = \int_0^{Q^2(s_i)|s_{i+1}-s_i|} P_{h_i}(t)\, dt \geq \epsilon_0. \tag{4.11}$$

On the other hand, by (4.8) we have

$$\int_{s_i}^{s_{i+1}} P\, dt \leq (C+\delta) \log \frac{|s_i|}{|s_{i+1}|}. \tag{4.12}$$

Combining the inequalities (4.11) with (4.12), we have

$$\frac{|s_{i+1}|}{|s_i|} \leq e^{-\frac{\epsilon_0}{C+\delta}}.$$

After iteration, we get the inequality

$$|s_i| \leq |s_0| e^{-\frac{\epsilon_0 i}{C+\delta}} = |t_0| e^{-\frac{\epsilon_0 i}{C+\delta}}. \tag{4.13}$$

Combining (4.13) with the definition of s_i gives the result

$$\lim_{i\to+\infty} Q(s_i)|s_i|^{\lambda} \leq \lim_{i\to+\infty} Q(t_0)|t_0|^{\lambda}\Big(2e^{-\frac{\lambda\epsilon_0}{C+\delta}}\Big)^i = 0,$$

where we choose λ such that $2e^{-\frac{\lambda\epsilon_0}{C+\delta}} < 1$. Thus, for any $t \in [s_i, s_{i+1}]$ we have

$$Q(t)|t|^{\lambda} \leq Q(s_{i+1})|s_i|^{\lambda} = 2Q(s_i)|s_i|^{\lambda} \to 0$$

as $i \to +\infty$. The lemma is proved. □

We now prove Theorem 1.6.

Proof of Theorem 1.6 After rescaling the flow, we can assume that $T \geq 1$. It is clear that (1.12) follows from the combination of inequality (1.11) and the definition of type-I singularity(c.f. [39]), i.e., $\limsup_{t\to 0} Q^2|t| < \infty$. Therefore, we only need to show (1.10) and (1.11), which will be dealt with separately.

1. Proof of inequality (1.10):

If $\limsup_{t\to 0} P|t| = C \geq 0$, by Lemma 4.3 and Lemma 4.4 we have

$$0 = \limsup_{t\to 0} Q(t)|t|^{\frac{(C+\delta)\log 2}{\epsilon_0}} \geq \delta_1 |t|^{-\frac{1}{2}+\frac{(C+\delta)\log 2}{\epsilon_0}},$$

where $\delta > 0$ is any small constant. It follows that

$$C \geq \frac{\epsilon_0}{2\log 2}.$$

Thus, (1.10) is proved.

2. Proof of inequality (1.11):

Since 0 is the singular time, we have $\lim_{t\to 0} F(t) = 0$. Thus, $F(t) \in (0, 1)$ when t is close to 0. It follows that

$$-F(t) = \lim_{s\to 0} F(s) - F(t) \geq -C|t|, \Rightarrow F(t) \leq C|t|, \tag{4.14}$$

where we used the inequality $\frac{d^-}{dt} F(t) \geq -C$ by Lemma 2.13. On the other hand, by Lemma 2.13 again, we have

$$\frac{d^-}{dt} F \geq -C\, O^\alpha Q^{2-\alpha} F.$$

Therefore, for any small $\delta > 0$ we have

$$\log F(-\delta) - \log F(-T) \geq -C \int_{-T}^{-\delta} O^\alpha Q^{2-\alpha}\, dt,$$

where $C = C(\alpha, n)$ is a constant. It follows that

$$\int_{-T}^{-\delta} O^\alpha Q^{2-\alpha}\, dt \geq \frac{1}{C} \log F(-T) - \frac{1}{C} \log F(-\delta) \geq \frac{1}{C} \log \frac{F(-T)}{C} - \frac{1}{C} \log \delta,$$

where we used (4.14). Suppose $\limsup_{t\to 0} O^\alpha Q^{2-\alpha}|t| < A$. Then we have

$$\frac{1}{C} \log \frac{F(-T)}{C} - \frac{1}{C} \log \delta \leq -A \log \delta + A \log T. \tag{4.15}$$

for every small δ. It forces that $A \geq \frac{1}{C}$. Replace A by $\limsup_{t\to 0} O^\alpha Q^{2-\alpha}|t| + \epsilon$ and let $\epsilon \to 0$. Then we have

$$\limsup_{t\to 0} O^\alpha Q^{2-\alpha}|t| \geq \frac{1}{C}.$$

Therefore, (1.11) is proved. □

5 Further Study

The methods and results in this paper can be generalized in the following ways.

1. The method we developed in this paper reduces the convergence of the flow to three important steps: uniqueness of critical metrics, regularity improvement, and good behavior of some functional along the flow. Theorems 1.3 and 1.5 can be proved for minimizing extK metrics in a general class, assuming a uniqueness theorem of minimizing extK metric in a fixed $\mathcal{G}^{\mathbb{C}}$-leaf's C^{∞}-closure, or a generalization of Theorem 1.3 of [20]. This will be discussed in a subsequent paper.

2. The fourth possibility of Donaldson's conjectural picture seems to be extremely difficult. By the example of G.Székelyhidi, the flow singularity at time infinity could be very complicated. However, if we assume the underlying Kähler class to be $c_1(M, J)$ which has definite sign or zero, then the limit should be a normal variety and could be used to construct a destabilizing test configuration.

3. Our deformation method could be applied to a more general situation. In Sect. 3, we deformed the complex structures and the metrics within a given Kähler class. Actually, even the underlying Kähler classes can be deformed. The general deformations will be discussed in a separate paper.

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