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Rigidity Phenomena in Manifolds with Boundary Under a Lower Weighted Ricci Curvature Bound

Yohei Sakurai[1]

Abstract We study Riemannian manifolds with boundary under a lower N-weighted Ricci curvature bound for N at most 1, and under a lower weighted mean curvature bound for the boundary. We examine rigidity phenomena in such manifolds with boundary. We conclude a volume growth rigidity theorem for the metric neighborhoods of the boundaries, and various splitting theorems. We also obtain rigidity theorems for the smallest Dirichlet eigenvalues for the weighted p-Laplacians.

Keywords Manifold with boundary · Weighted Ricci curvature · Weighted p-Laplacian

Mathematics Subject Classification Primary 53C20

1 Introduction

In this paper, we study Riemannian manifolds with boundary under a lower weighted Ricci curvature bound, and under a lower weighted mean curvature bound for the boundary. We develop the preceding studies of the author [33]. As explained below, we examine rigidity phenomena in such manifolds with boundary beyond the usual weighted setting.

For $n \geq 2$, let M be an n-dimensional Riemannian manifold with or without boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function. We

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denote by Ric_g the Ricci curvature defined by g, by ∇f the gradient of f, and by Hess f the Hessian of f. For $N \in (-\infty, \infty]$, the *N-weighted Ricci curvature* Ric_f^N is defined as

$$\mathrm{Ric}_f^N := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{\nabla f \otimes \nabla f}{N - n} \tag{1.1}$$

if $N \in (-\infty, \infty) \setminus \{n\}$; otherwise, if $N = \infty$, then $\mathrm{Ric}_f^N := \mathrm{Ric}_g + \mathrm{Hess}\, f$; if $N = n$, and if f is a constant function, then $\mathrm{Ric}_f^N := \mathrm{Ric}_g$; if $N = n$, and if f is not constant, then $\mathrm{Ric}_f^N := -\infty$ [2]. We notice that the parameter N has been usually chosen from $[n, \infty]$.

On manifolds without boundary under a lower N-weighted Ricci curvature bound, many results have been already known in the usual weighted case of $N \in [n, \infty]$ (see, e.g., [20–22,30,34,35,38]). Recently, in the complemental weighted case of $N \in (-\infty, n)$, several geometric properties have begun to be studied (see [16,17,23,25,26], [27,40]). Wylie [39] has obtained a splitting theorem of Cheeger–Gromoll type (cf. [8]) in the complementary weighted case of $N \in (-\infty, 1]$, and asked a question whether the splitting theorem can be extended to the remaining case of $N \in (1, n)$.

For manifolds with boundary under a lower N-weighted Ricci curvature bound, and under a lower weighted mean curvature bound for the boundary, the author [33] has studied rigidity phenomena in the usual weighted case of $N \in [n, \infty]$. In the present paper, we produce rigidity theorems in the complementary weighted case of $N \in (-\infty, 1]$. Our rigidity theorems in the case of $N \in (-\infty, 1]$ give natural extensions of the corresponding results in [33].

To prove our rigidity theorems, we develop comparison theorems. We prove Laplacian comparison theorems for the distance function from the boundary, and volume comparison theorems for metric neighborhoods of the boundary. The author [33] has shown such comparison theorems in the usual weighted case of $N \in [n, \infty]$. For manifolds with boundary of non-negative N-weighted Ricci curvature, and of non-negative weighted mean curvature for the boundary, Wylie [39] has shown a Laplacian comparison inequality for the distance function from a connected component of the boundary in the weighted case of $N \in (-\infty, 1]$. To conclude our comparison theorems, we need slightly more complicated calculations than that done by the author [33], and by Wylie [39]. Under an assumption concerning a subharmonicity of the distance function from the boundary, we derive our rigidity theorems from studies of the equality cases in our comparison theorems.

1.1 Setting

We summarize our setting as follows: For $n \geq 2$, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. The boundary ∂M is assumed to be smooth. We denote by d_M the Riemannian distance on M induced from the length structure determined by g. Let $f : M \to \mathbb{R}$ be a smooth function. For the Riemannian volume measure vol_g on M, let

$$m_f := e^{-f}\, \mathrm{vol}_g . \tag{1.2}$$

For $N \in (-\infty, \infty]$, we denote by Ric_f^N the N-weighted Ricci curvature (see (1.1)). We note that for $N_1, N_2 \in (-\infty, \infty] \setminus \{n\}$ with $N_1 \leq N_2$, if $N_1, N_2 \in (n, \infty]$ or $N_1, N_2 \in (-\infty, n)$, then $\mathrm{Ric}_f^{N_1} \leq \mathrm{Ric}_f^{N_2}$; if $N_1 \in (-\infty, n)$ and $N_2 \in (n, \infty]$, then $\mathrm{Ric}_f^{N_2} \leq \mathrm{Ric}_f^{N_1}$. We denote by $\mathrm{Ric}_{f,M}^N$ the infimum of Ric_f^N on the unit tangent bundle on the interior $\mathrm{Int}\, M$ of M. For $x \in \partial M$, we denote by u_x the unit inner normal vector on ∂M at x. Let H_x denote the mean curvature of ∂M at x defined as the trace of the shape operator of u_x. The f-*mean curvature* $H_{f,x}$ at x is defined by

$$H_{f,x} := H_x + g\left((\nabla f)_x, u_x\right). \tag{1.3}$$

We put $H_{f,\partial M} := \inf_{x \in \partial M} H_{f,x}$. Our main subject is a weighted Riemannian manifold (M, d_M, m_f) with boundary such that for $\kappa, \lambda \in \mathbb{R}$ and for $N \in (-\infty, 1]$ we have $\mathrm{Ric}_{f,M}^N \geq \kappa$ and $H_{f,\partial M} \geq \lambda$.

1.2 Volume Growth Rigidity

Let $\rho_{\partial M} : M \to \mathbb{R}$ be the distance function from ∂M defined as $\rho_{\partial M}(p) := d_M(p, \partial M)$. For $r \in (0, \infty)$, we put $B_r(\partial M) := \{\, p \in M \mid \rho_{\partial M}(p) \leq r \,\}$. For $x \in \partial M$, let $\gamma_x : [0, T) \to M$ be the geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) = u_x$. We define a function $\tau : \partial M \to \mathbb{R} \cup \{\infty\}$ by

$$\tau(x) := \sup\{t \in (0, \infty) \mid \rho_{\partial M}(\gamma_x(t)) = t\}. \tag{1.4}$$

We define a function $F_x : [0, \tau(x)] \setminus \{\infty\} \to (0, \infty)$ by

$$F_x(t) := e^{\frac{f(\gamma_x(t)) - f(x)}{n-1}}. \tag{1.5}$$

Notice that if f is constant, then F_x is equal to 1. For $\kappa, \lambda \in \mathbb{R}$, we say that κ and λ satisfy the *subharmonic condition* if

$$\inf_{x \in \partial M} \inf_{t \in (0, \tau(x))} \kappa \int_0^t F_x^2(s)\, ds \geq -\lambda.$$

We remark that if κ and λ satisfy the subharmonic condition, then subharmonicity of $\rho_{\partial M}$ is derived from $\mathrm{Ric}_{f,M}^N \geq \kappa$ and $H_{f,\partial M} \geq \lambda$ in the case of $N \in (-\infty, 1]$ (see Lemma 3.1). Note that if $\kappa, \lambda \in [0, \infty)$, then they satisfy the subharmonic condition. We denote by h the induced Riemannian metric on ∂M. For the Riemannian volume measure vol_h on ∂M induced from h, we put $m_{f,\partial M} := e^{-f|_{\partial M}} \mathrm{vol}_h$.

For an interval I, and for a Riemannian manifold M_0 with Riemannian metric g_0, let $\Phi : I \times M_0 \to \mathbb{R}$ be a positive smooth function. For each $x \in M_0$, let $\Phi_x : I \to \mathbb{R}$ be the function defined as $\Phi_x(t) := \Phi(t, x)$. We say that a Riemannian manifold $(I \times M_0, dt^2 + \Phi_x^2(t)\, g_0)$ is a *twisted product space*. When τ is infinity on ∂M, we define $[0, \infty) \times_F \partial M$ as the twisted product space $([0, \infty) \times \partial M, dt^2 + F_x^2(t)\, h)$.

For the metric neighborhoods of the boundaries, we prove an absolute volume comparison theorem of Heintze–Karcher type, and a relative volume comparison theorem

of Bishop–Gromov type (see Sects. 4.1 and 4.2). We obtain rigidity results concerning the equality cases in those comparison theorems (see Sect. 4.3).

We conclude the following volume growth rigidity theorem:

Theorem 1.1 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\operatorname{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. If we have*

$$\liminf_{r \to \infty} \frac{m_f(B_r(\partial M))}{r} \geq m_{f,\partial M}(\partial M), \tag{1.6}$$

then (M, d_M) is isometric to $([0, \infty) \times_F \partial M, d_{[0,\infty) \times_F \partial M})$. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, \infty)$; in particular, (M, d_M) is isometric to $([0, \infty) \times \partial M, d_{[0,\infty) \times \partial M})$.

When $\kappa = 0$ and $\lambda = 0$, Theorem 1.1 has been proved in the unweighted case in [32], and in the usual weighted case in [33].

Remark 1.1 We do not know whether Theorem 1.1 can be extended to the weighted case of $N \in (1, n)$.

Remark 1.2 Under the same setting as in Theorem 1.1, we always have the following inequality (see Lemma 4.1):

$$\limsup_{r \to \infty} \frac{m_f(B_r(\partial M))}{r} \leq m_{f,\partial M}(\partial M). \tag{1.7}$$

Theorem 1.1 is concerned with rigidity phenomena.

We have the following corollary of Theorem 1.1:

Corollary 1.2 *Under the same setting as in Theorem 1.1, if $N = 1$ and $\kappa = 0$, and if we have (1.6), then there exist a function $f_0 : [0, \infty) \to \mathbb{R}$ and a Riemannian metric h_0 on ∂M such that M is isometric to a warped product space $([0, \infty) \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0)$.*

1.3 Splitting Theorems

In our setting, we show Laplacian comparison theorems for $\rho_{\partial M}$, and study the equality cases (see Sect. 3).

By using a Laplacian comparison theorem for $\rho_{\partial M}$, and that for Busemann functions, we prove the following splitting theorem:

Theorem 1.3 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function such that $\sup f(M) < \infty$. For $N \in (-\infty, 1]$ we suppose $\operatorname{Ric}_{f,M}^N \geq 0$, and $H_{f,\partial M} \geq 0$. If for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$, then (M, d_M) is isometric to $([0, \infty) \times_F \partial M, d_{[0,\infty) \times_F \partial M})$. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, \infty)$; in particular, (M, d_M) is isometric to $([0, \infty) \times \partial M, d_{[0,\infty) \times \partial M})$.*

In the unweighted case, Kasue [13] has proved Theorem 1.3 under the compactness assumption for the boundary (see also the work of Croke and Kleiner [9]). Theorem 1.3 itself has been proved in the unweighted case in [32], and in the usual weighted case in [33].

Remark 1.3 We do not know whether Theorem 1.3 can be extended to the weighted case of $N \in (1, n)$.

As a corollary of Theorem 1.3, we see the following:

Corollary 1.4 *Under the same setting as in Theorem* 1.3, *if* $N = 1$, *and if for some* $x_0 \in \partial M$ *we have* $\tau(x_0) = \infty$, *then there exist a function* $f_0 : [0, \infty) \to \mathbb{R}$ *and a Riemannian metric* h_0 *on* ∂M *such that* M *is isometric to* $([0, \infty) \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0)$.

In Theorem 1.3, by applying the Wylie splitting theorem in [39] to the boundary, we obtain a multi-splitting theorem (see Sect. 5.3). We also generalize a splitting theorem studied in [13] (and [9,12]) for the case where boundaries are disconnected (see Sect. 5.4).

1.4 Eigenvalue Rigidity

For $p \in [1, \infty)$, the $(1, p)$-*Sobolev space* $W_0^{1,p}(M, m_f)$ *on* (M, m_f) *with compact support* is defined as the completion of the set of all smooth functions on M whose support is compact and contained in Int M with respect to the standard $(1, p)$-Sobolev norm. We denote by $\| \cdot \|$ the standard norm induced from g, and by div the divergence with respect to g. For $p \in [1, \infty)$, the (f, p)-*Laplacian* $\Delta_{f,p} \varphi$ for $\varphi \in W_0^{1,p}(M, m_f)$ is defined by

$$\Delta_{f,p} \varphi := -e^f \, \mathrm{div} \left(e^{-f} \|\nabla \varphi\|^{p-2} \nabla \varphi \right)$$

as a distribution on $W_0^{1,p}(M, m_f)$. A real number μ is said to be an (f, p)-*Dirichlet eigenvalue* for $\Delta_{f,p}$ on M if there exists $\varphi \in W_0^{1,p}(M, m_f) \setminus \{0\}$ such that $\Delta_{f,p}\varphi = \mu |\varphi|^{p-2} \varphi$ holds on Int M in a distribution sense on $W_0^{1,p}(M, m_f)$. For $p \in [1, \infty)$, the *Rayleigh quotient* $R_{f,p}(\varphi)$ for $\varphi \in W_0^{1,p}(M, m_f) \setminus \{0\}$ is defined as

$$R_{f,p}(\varphi) := \frac{\int_M \|\nabla \varphi\|^p \, d m_f}{\int_M |\varphi|^p \, d m_f}.$$

We put $\mu_{f,1,p}(M) := \inf_\varphi R_{f,p}(\varphi)$, where the infimum is taken over all non-zero functions in $W_0^{1,p}(M, m_f)$. The value $\mu_{f,1,2}(M)$ is equal to the infimum of the spectrum of $\Delta_{f,2}$ on (M, m_f). If M is compact, and if $p \in (1, \infty)$, then $\mu_{f,1,p}(M)$ is equal to the infimum of the set of all (f, p)-Dirichlet eigenvalues on M.

Let $p \in (1, \infty)$. For $D \in (0, \infty)$, let $\mu_{p,D}$ be the positive minimum real number μ such that there exists a function $\varphi : [0, D] \to \mathbb{R}$ satisfying

$$\left(|\varphi'(t)|^{p-2} \varphi'(t) \right)' + \mu \, |\varphi(t)|^{p-2} \varphi(t) = 0, \quad \varphi(0) = 0, \quad \varphi'(D) = 0. \tag{1.8}$$

In the case where $p = 2$, we see $\mu_{2,D} = \pi^2 (2D)^{-2}$.

For a positive number $D \in (0, \infty)$, and for a connected component ∂M_1 of ∂M, we denote by $[0, D] \times_F \partial M_1$ the twisted product space $([0, D] \times \partial M_1, dt^2 + F_x^2(t) h)$, where for every $x \in \partial M_1$ the function $F_x : [0, D] \to (0, \infty)$ is defined as (1.5). The *inscribed radius of M* is defined as

$$D(M, \partial M) := \sup_{p \in M} \rho_{\partial M}(p).$$

Suppose that M is compact. We say that the metric space (M, d_M) is an *F-model space* if M is isometric to either (1) for a connected component ∂M_1 of ∂M, the twisted product space $[0, 2D(M, \partial M)] \times_F \partial M_1$; or (2) for an involutive isometry σ of ∂M without fixed points, the quotient space $([0, 2D(M, \partial M)] \times_F \partial M)/G_\sigma$, where G_σ is the isometry group on $[0, 2D(M, \partial M)] \times_F \partial M$ of the identity and the involute isometry $\hat{\sigma}$ defined by $\hat{\sigma}(t, x) := (2D(M, \partial M) - t, \sigma(x))$. If (M, d_M) is an F-model space, and if for every $x \in \partial M$ the function F_x is equal to 1 on $[0, D(M, \partial M)]$, then we call the F-model space (M, d_M) an *equational model space*. The notion of the equational model spaces coincides with that of the $(0, 0)$-equational model spaces introduced in [33].

We prove the following rigidity theorem for $\mu_{f,1,p}$:

Theorem 1.5 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1, \infty)$, and let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. For $D \in (0, \infty)$ we assume $D(M, \partial M) \leq D$. Then*

$$\mu_{f,1,p}(M) \geq \mu_{p,D}. \tag{1.9}$$

If the equality in (1.9) holds, then $D(M, \partial M) = D$, and the metric space (M, d_M) is an F-model space. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, D]$; in particular, (M, d_M) is an equational model space.

In the unweighted case, Li and Yau [18] have obtained the estimate (1.9), and Kasue [14] has proved Theorem 1.5 when $p = 2$, $\kappa = 0$, and $\lambda = 0$. In [33], the author has proved Theorem 1.5 in the usual weighted case when $\kappa = 0$ and $\lambda = 0$.

Remark 1.4 We do not know whether Theorem 1.5 can be extended to the weighted case of $N \in (1, n)$.

Suppose that M is compact. We say that the metric space (M, d_M) is a *warped model space* if there exist a function $f_0 : [0, 2D(M, \partial M)] \to \mathbb{R}$ and a Riemannian metric h_0 on ∂M such that M is isometric to either (1) for a connected component ∂M_1 of ∂M, the warped product space $([0, 2D(M, \partial M)] \times \partial M_1, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0)$; or (2) for an involutive isometry σ of ∂M without fixed points, the quotient space $([0, 2D(M, \partial M)] \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0)/G_\sigma$, where G_σ is the isometry group on $([0, 2D(M, \partial M)] \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0)$ of the identity and the involute isometry $\hat{\sigma}$ defined as $\hat{\sigma}(t, x) := (2D(M, \partial M) - t, \sigma(x))$.

We obtain the following corollary of Theorem 1.5:

Corollary 1.6 *Under the same setting as in Theorem* 1.5*, if $N = 1$ and $\kappa = 0$, and if the equality in* (1.9) *holds, then the metric space (M, d_M) is a warped model space.*

1.5 Organization

In Sect. 2, we prepare some notations and recall the basic facts for Riemannian manifolds with boundary. In Sect. 3, we show Laplacian comparison results for $\rho_{\partial M}$. In Sect. 4, we show volume comparison results, and conclude Theorem 1.1 and Corollary 1.2. In Sect. 5, we prove Theorem 1.3 and Corollary 1.4, and discuss its variants. In Sect. 6, we prove Theorem 1.5 and Corollary 1.6. We also obtain an explicit lower bound for $\mu_{f,1,p}$ (see Sect. 6.3).

2 Preliminaries

We refer to [31] for the basics of Riemannian manifolds with boundary (cf. Sect. 2 in [32], and in [33]).

2.1 Riemannian Manifolds with Boundary

For $n \geq 2$, let M be an n-dimensional, connected Riemannian manifold with (smooth) boundary with Riemannian metric g. For a point $p \in \text{Int } M$, let $T_p M$ be the tangent space at p on M, and let $U_p M$ be the unit tangent sphere at p on M. We denote by $\|\cdot\|$ the standard norm induced from g. If $v_1, \ldots, v_k \in T_p M$ are linearly independent, then it holds that $\|v_1 \wedge \cdots \wedge v_k\| = \sqrt{\det(g(v_i, v_j))}$.

Let d_M be the Riemannian distance on M induced from the length structure determined by g. For an interval I, we say that a curve $\gamma : I \rightarrow M$ is a *normal minimal geodesic* if for all $s, t \in I$ we have $d_M(\gamma(s), \gamma(t)) = |s - t|$, and γ is a *normal geodesic* if for each $t \in I$ there exists an interval $J \subset I$ with $t \in J$ such that $\gamma|_J$ is a normal minimal geodesic. If M is complete with respect to d_M, then the Hopf–Rinow theorem for length spaces (see, e.g., Theorem 2.5.23 in [5]) tells us that the metric space (M, d_M) is a proper, geodesic space, namely all closed bounded subsets of M are compact, and for every pair of points in M there exists a normal minimal geodesic connecting them.

For $i = 1, 2$, let M_i be connected Riemannian manifolds with boundary with Riemannian metric g_i. For each i, the boundary ∂M_i carries the induced Riemannian metric h_i. We say that a homeomorphism $\Phi : M_1 \rightarrow M_2$ is a *Riemannian isometry with boundary* from M_1 to M_2 if Φ satisfies the following conditions:

(1) $\Phi|_{\text{Int } M_1} : \text{Int } M_1 \rightarrow \text{Int } M_2$ is smooth, and $(\Phi|_{\text{Int } M_1})^*(g_2) = g_1$;
(2) $\Phi|_{\partial M_1} : \partial M_1 \rightarrow \partial M_2$ is smooth, and $(\Phi|_{\partial M_1})^*(h_2) = h_1$.

If $\Phi : M_1 \rightarrow M_2$ is a Riemannian isometry with boundary, then the inverse Φ^{-1} is also a Riemannian isometry with boundary. Notice that there exists a Riemannian isometry with boundary from M_1 to M_2 if and only if the metric space (M_1, d_{M_1}) is isometric to (M_2, d_{M_2}) (see, e.g., Sect. 2 in [32]).

2.2 Jacobi Fields Orthogonal to the Boundary

Let M be a connected Riemannian manifold with boundary with Riemannian metric g. For a point $x \in \partial M$, and for the tangent space $T_x \partial M$ at x on ∂M, let $T_x^\perp \partial M$ be the orthogonal complement of $T_x \partial M$ in the tangent space at x on M. Take $u \in T_x^\perp \partial M$. For the second fundamental form S of ∂M, let $A_u : T_x \partial M \to T_x \partial M$ be the *shape operator* for u defined as

$$g(A_u v, w) := g(S(v, w), u).$$

We denote by u_x the unit inner normal vector at x. The *mean curvature* H_x at x is defined as $H_x := \text{trace } A_{u_x}$. We denote by $\gamma_x : [0, T) \to M$ the normal geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) = u_x$. We say that a Jacobi field Y along γ_x is a ∂M-*Jacobi field* if Y satisfies the following initial conditions:

$$Y(0) \in T_x \partial M, \quad Y'(0) + A_{u_x} Y(0) \in T_x^\perp \partial M.$$

We say that $\gamma_x(t_0)$ is a *conjugate point* of ∂M along γ_x if there exists a non-zero ∂M-Jacobi field Y along γ_x with $Y(t_0) = 0$. We denote by $\tau_1(x)$ the first conjugate value for ∂M along γ_x. It is well known that for all $x \in \partial M$ and $t > \tau_1(x)$, we have $t > \rho_{\partial M}(\gamma_x(t))$.

For the normal tangent bundle $T^\perp \partial M := \bigcup_{x \in \partial M} T_x^\perp \partial M$ of ∂M, let $0(T^\perp \partial M)$ be the zero-section $\bigcup_{x \in \partial M} \{ 0_x \in T_x^\perp \partial M \}$ of $T^\perp \partial M$. On an open neighborhood of $0(T^\perp \partial M)$ in $T^\perp \partial M$, the normal exponential map $\exp^\perp$ of ∂M is defined as $\exp^\perp(x, u) := \gamma_x(\|u\|)$ for $x \in \partial M$ and $u \in T_x^\perp \partial M$.

For $x \in \partial M$ and $t \in [0, \tau_1(x))$, we denote by $\theta(t, x)$ the absolute value of the Jacobian of $\exp^\perp$ at $(x, t u_x) \in T^\perp \partial M$. For each $x \in \partial M$, we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. For each i, let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x} e_{x,i}$. Note that for all $x \in \partial M$ and $t \in [0, \tau_1(x))$, we have $\theta(t, x) = \|Y_{x,1}(t) \wedge \cdots \wedge Y_{x,n-1}(t)\|$. This does not depend on the choice of the orthonormal bases.

2.3 Cut Locus for the Boundary

We recall the basic properties of the cut locus for the boundary. The basic properties seem to be well known. We refer to [32] for the proofs.

Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g. For $p \in M$, we call $x \in \partial M$ a *foot point* on ∂M of p if $d_M(p, x) = \rho_{\partial M}(p)$. Since (M, d_M) is proper, every point in M has at least one foot point on ∂M. For $p \in \text{Int } M$, let $x \in \partial M$ be a foot point on ∂M of p. Then there exists a unique normal minimal geodesic $\gamma : [0, l] \to M$ from x to p such that $\gamma = \gamma_x|_{[0,l]}$, where $l = \rho_{\partial M}(p)$. In particular, $\gamma'(0) = u_x$ and $\gamma|_{(0,l]}$ lies in $\text{Int } M$.

Let $\tau : \partial M \to \mathbb{R} \cup \{\infty\}$ be the function defined as (1.4). By the property of τ_1, for all $x \in \partial M$ we have $0 < \tau(x) \leq \tau_1(x)$. For the inscribed radius $D(M, \partial M)$ of

M, from the definition of τ, we have $D(M, \partial M) = \sup_{x \in \partial M} \tau(x)$. The function τ is continuous on ∂M.

The continuity of τ implies the following (see, e.g., Sect. 3 in [32]):

Lemma 2.1 *Suppose that ∂M is compact. Then $D(M, \partial M)$ is finite if and only if M is compact.*

We put

$$TD_{\partial M} := \bigcup_{x \in \partial M} \{t\, u_x \in T_x^{\perp} \partial M \mid t \in [0, \tau(x))\},$$

$$T\mathrm{Cut}\, \partial M := \bigcup_{x \in \partial M} \{\tau(x)\, u_x \in T_x^{\perp} \partial M \mid \tau(x) < \infty\},$$

and define $D_{\partial M} := \exp^{\perp}(TD_{\partial M})$ and $\mathrm{Cut}\, \partial M := \exp^{\perp}(T\mathrm{Cut}\, \partial M)$. We call $\mathrm{Cut}\, \partial M$ the *cut locus for the boundary* ∂M. From the continuity of τ, the set $\mathrm{Cut}\, \partial M$ is a null set of M. Furthermore, we have

$$\mathrm{Int}\, M = (D_{\partial M} \setminus \partial M) \sqcup \mathrm{Cut}\, \partial M, \quad M = D_{\partial M} \sqcup \mathrm{Cut}\, \partial M.$$

This implies that if $\mathrm{Cut}\, \partial M = \emptyset$, then ∂M is connected. The set $TD_{\partial M} \setminus 0(T^{\perp} \partial M)$ is a maximal domain in $T^{\perp} \partial M$ on which $\exp^{\perp}$ is regular and injective.

In [33], we have already known the following:

Lemma 2.2 *If there exists a connected component ∂M_0 of ∂M such that for all $x \in \partial M_0$ we have $\tau(x) = \infty$, then ∂M is connected and $\mathrm{Cut}\, \partial M = \emptyset$.*

The function $\rho_{\partial M}$ is smooth on $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$. For each $p \in \mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$, the gradient vector $\nabla \rho_{\partial M}(p)$ of $\rho_{\partial M}$ at p is given by $\nabla \rho_{\partial M}(p) = \gamma'(l)$, where $\gamma : [0, l] \to M$ is the normal minimal geodesic from the foot point on ∂M of p to p.

For $\Omega \subset M$, we denote by $\bar{\Omega}$ the closure of Ω in M, and by $\partial \Omega$ the boundary of Ω in M. For a domain Ω in M such that $\partial \Omega$ is a smooth hypersurface in M, we denote by $\mathrm{vol}_{\partial \Omega}$ the canonical Riemannian volume measure on $\partial \Omega$.

We have the following fact to avoid the cut locus for the boundary that has been stated in [33] (see Lemma 2.6 in [33]):

Lemma 2.3 *Let Ω be a domain in M such that $\partial \Omega$ is a smooth hypersurface in M. Then there exists a sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ of closed subsets of $\bar{\Omega}$ such that for every $k \in \mathbb{N}$, the set $\partial \Omega_k$ is a smooth hypersurface in M except for a null set in $(\partial \Omega, \mathrm{vol}_{\partial \Omega})$ satisfying the following properties:*

(1) *for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$;*
(2) *$\bar{\Omega} \setminus \mathrm{Cut}\, \partial M = \bigcup_{k \in \mathbb{N}} \Omega_k$;*
(3) *for every $k \in \mathbb{N}$, and for almost every point $p \in \partial \Omega_k \cap \partial \Omega$ in $(\partial \Omega, \mathrm{vol}_{\partial \Omega})$, there exists the unit outer normal vector for Ω_k at p that coincides with the unit outer normal vector on $\partial \Omega$ for Ω at p;*
(4) *for every $k \in \mathbb{N}$, on $\partial \Omega_k \setminus \partial \Omega$, there exists the unit outer normal vector field ν_k for Ω_k such that $g(\nu_k, \nabla \rho_{\partial M}) \geq 0$.*

Moreover, if $\bar{\Omega} = M$, then for every $k \in \mathbb{N}$, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfies $\partial\Omega_k \cap \partial M = \partial M$.

As noticed in [33], for the cut locus for a single point, we have known a similar fact to Lemma 2.3 (see, e.g., Theorem 4.1 in [7]). One can prove Lemma 2.3 by a similar method to the case of the cut locus for a single point.

2.4 Busemann Functions and Asymptotes

Let M be a connected complete Riemannian manifold with boundary. A normal geodesic $\gamma : [0, \infty) \to M$ is said to be a *ray* if for all $s, t \in [0, \infty)$ it holds that $d_M(\gamma(s), \gamma(t)) = |s - t|$. For a ray $\gamma : [0, \infty) \to M$, the *Busemann function* $b_\gamma : M \to \mathbb{R}$ *of γ* is defined as

$$b_\gamma(p) := \lim_{t \to \infty} (t - d_M(p, \gamma(t))).$$

Take a ray $\gamma : [0, \infty) \to M$ and a point $p \in \text{Int } M$, and choose a sequence $\{t_i\}$ with $t_i \to \infty$. For each i, we take a normal minimal geodesic $\gamma_i : [0, l_i] \to M$ from p to $\gamma(t_i)$. Since γ is a ray, it follows that $l_i \to \infty$. Take a sequence $\{T_j\}$ with $T_j \to \infty$. Using the fact that M is proper, we take a subsequence $\{\gamma_{1,i}\}$ of $\{\gamma_i\}$, and a normal minimal geodesic $\gamma_{p,1} : [0, T_1] \to M$ from p to $\gamma_{p,1}(T_1)$ such that $\gamma_{1,i}|_{[0,T_1]}$ uniformly converges to $\gamma_{p,1}$. In this manner, take a subsequence $\{\gamma_{2,i}\}$ of $\{\gamma_{1,i}\}$ and a normal minimal geodesic $\gamma_{p,2} : [0, T_2] \to M$ from p to $\gamma_{p,2}(T_2)$ such that $\gamma_{2,i}|_{[0,T_2]}$ uniformly converges to $\gamma_{p,2}$, where $\gamma_{p,2}|_{[0,T_1]} = \gamma_{p,1}$. By means of a diagonal argument, we obtain a subsequence $\{\gamma_k\}$ of $\{\gamma_i\}$ and a ray γ_p in M such that for every $t \in (0, \infty)$ we have $\gamma_k(t) \to \gamma_p(t)$ as $k \to \infty$. We call such a ray γ_p an *asymptote for γ from p*.

The following lemmas have been shown in [32].

Lemma 2.4 *Suppose that for some $x \in \partial M$ we have $\tau(x) = \infty$. Take $p \in \text{Int } M$. If $b_{\gamma_x}(p) = \rho_{\partial M}(p)$, then $p \notin \text{Cut } \partial M$. Moreover, for the unique foot point y on ∂M of p, we have $\tau(y) = \infty$.*

Lemma 2.5 *Suppose that for some $x \in \partial M$ we have $\tau(x) = \infty$. For $l \in (0, \infty)$, put $p := \gamma_x(l)$. Then there exists $\epsilon \in (0, \infty)$ such that for all $q \in B_\epsilon(p)$, all asymptotes for the ray γ_x from q lie in $\text{Int } M$.*

2.5 Weighted Riemannian Manifolds with Boundary

Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. For a smooth function φ on M, the *weighted Laplacian* $\Delta_f \varphi$ *for φ* is defined by

$$\Delta_f \varphi := \Delta\varphi + g(\nabla f, \nabla\varphi),$$

where $\Delta\varphi$ is the Laplacian for φ defined as the minus of the trace of its Hessian. Note that Δ_f coincides with the $(f, 2)$-Laplacian $\Delta_{f,2}$.

It seems that the following formula of Bochner type is well known (see [19], and Chap. 14 in [37]).

Proposition 2.6 [19] *For every smooth function φ on M, we have*

$$-\frac{1}{2} \Delta_f \|\nabla\varphi\|^2 = \mathrm{Ric}_f^{\infty}(\nabla\varphi) + \|\mathrm{Hess}\,\varphi\|^2 - g\left(\nabla\Delta_f\,\varphi, \nabla\varphi\right),$$

where $\|\mathrm{Hess}\,\varphi\|$ is the Hilbert–Schmidt norm of $\mathrm{Hess}\,\varphi$.

Notice that for every $x \in \partial M$, and for every $t \in (0, \tau(x))$, the value $\Delta\rho_{\partial M}(\gamma_x(t))$ is equal to the mean curvature $H_{x,t}$ of the t-level set of $\rho_{\partial M}$ at $\gamma_x(t)$ toward $\nabla\rho_{\partial M}$. In our weighted case, by the definition of the weighted Laplacian, we see the following:

Lemma 2.7 *Take $x \in \partial M$. Then for every $t \in (0, \tau(x))$, the value $\Delta_f\rho_{\partial M}(\gamma_x(t))$ is equal to the f-mean curvature $H_{f,x,t}$ of the t-level set of $\rho_{\partial M}$ at $\gamma_x(t)$ toward $\nabla\rho_{\partial M}$ defined as*

$$H_{f,x,t} := H_{x,t} + g(\nabla f, \nabla\rho_{\partial M})(\gamma_x(t)).$$

In particular, $\Delta_f\rho_{\partial M}(\gamma_x(t))$ tends to $H_{f,x}$ as $t \to 0$, where $H_{f,x}$ is the f-mean curvature of ∂M at x defined as (1.3).

For $x \in \partial M$ and $t \in [0, \tau(x))$, we put

$$\theta_f(t, x) := e^{-f(\gamma_x(t))} \theta(t, x), \tag{2.1}$$

where $\theta(t, x)$ is the absolute value of the Jacobian of the map $\exp^{\perp}$ at $(x, tu_x) \in T^{\perp}\partial M$. For all $x \in \partial M$ and $t \in (0, \tau(x))$, it holds that

$$\Delta_f\,\rho_{\partial M}(\gamma_x(t)) = -(\log\theta(t, x))' + f(\gamma_x(t))' = -\frac{\theta_f'(t, x)}{\theta_f(t, x)}. \tag{2.2}$$

Let $\bar{\theta}_f : [0, \infty) \times \partial M \to \mathbb{R}$ be a function defined by

$$\bar{\theta}_f(t, x) := \begin{cases} \theta_f(t, x) & \text{if } t < \tau(x), \\ 0 & \text{if } t \geq \tau(x). \end{cases} \tag{2.3}$$

The following has been shown in [33]:

Lemma 2.8 *If ∂M is compact, then for all $r \in (0, \infty)$*

$$m_f(B_r(\partial M)) = \int_{\partial M} \int_0^r \bar{\theta}_f(t, x)\, dt\, d\,\mathrm{vol}_h,$$

where m_f denotes the weighted measure on M defined as (1.2)*, and h denotes the induced Riemannian metric on ∂M.*

2.6 Twisted and Warped Product Spaces

In [39], for the proof of a splitting theorem of Cheeger–Gromoll type, Wylie has proved that a twisted product space over $\mathbb{R}$ becomes a warped product space under a non-negativity of the 1-weighted Ricci curvature (see Proposition 2.2 in [39]). The proof is based on a pointwise calculation, and the same holds true for a twisted product space over an arbitrary interval.

From the argument in the proof of Proposition 2.2 in [39], we can derive the following in our setting:

Proposition 2.9 [39] *Let M be a Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that there exist an interval I in the form of $[0, \infty)$ or $[0, D]$ for some positive number D, and a connected component ∂M_1 of ∂M such that M is isometric to a twisted product space $I \times_F \partial M_1$. If $\mathrm{Ric}^1_{f,M} \geq 0$, then there exist functions $f_0 : I \to \mathbb{R}$ and $f_1 : \partial M_1 \to \mathbb{R}$ such that for all $t \in I$ and $x \in \partial M_1$ we have $f(\gamma_x(t)) = f_0(t) + f_1(x)$; in particular, M is isometric to a warped product space $(I \times \partial M_1, dt^2 + e^{\frac{2f_0(t)}{n-1}} h_0)$, where for the induced metric h on ∂M_1 we put $h_0 := e^{2\frac{f_1 - (f|_{\partial M_1})}{n-1}} h$.*

Notice that Proposition 2.9 has been implicitly used in the proof of Theorem 5.1 in [39].

3 Laplacian Comparisons

In this section, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

3.1 Basic Comparisons

Recall that for $x \in \partial M$, the function $F_x : [0, \tau(x)] \setminus \{\infty\} \to (0, \infty)$ is defined as (1.5). For $\kappa, \lambda \in \mathbb{R}$, we define a function $F_{\kappa,\lambda,x} : [0, \tau(x)] \setminus \{\infty\} \to \mathbb{R}$ by

$$F_{\kappa,\lambda,x}(t) := \kappa \int_0^t F_x^2(s)\, ds + \lambda. \tag{3.1}$$

Note that if κ and λ satisfy the subharmonic condition, then for every $x \in \partial M$ the function $F_{\kappa,\lambda,x}$ is non-negative.

We have the following Laplacian comparison inequality for $\rho_{\partial M}$:

Lemma 3.1 *Take $x \in \partial M$. For $\kappa, \lambda \in \mathbb{R}$ and for $N \in (-\infty, 1]$ we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma_x'(t)) \geq \kappa$, and suppose $H_{f,x} \geq \lambda$. Then for all $t \in (0, \tau(x))$ we have*

$$\Delta_f\, \rho_{\partial M}(\gamma_x(t)) \geq F_x^{-2}(t)\, F_{\kappa,\lambda,x}(t). \tag{3.2}$$

In particular, if κ and λ satisfy the subharmonic condition, then for all $t \in (0, \tau(x))$ we have $\Delta_f \, \rho_{\partial M}(\gamma_x(t)) \geq 0$.

Proof The function $\rho_{\partial M} \circ \gamma_x$ is smooth on $(0, \tau(x))$. We put $h_{f,x} := (\Delta_f \, \rho_{\partial M}) \circ \gamma_x$. We first show that for all $s \in (0, \tau(x))$

$$\left(F_x^2(s) \, h_{f,x}(s) - \kappa \int_0^s F_x^2(u) \, du \right)' \geq 0. \tag{3.3}$$

Fix $s \in (0, \tau(x))$, and put $f_x := f \circ \gamma_x$. We apply Proposition 2.6 to the function $\rho_{\partial M}$. Since $\|\nabla \rho_{\partial M}\| = 1$ along $\gamma_x|_{(0,\tau(x))}$, it holds that

$$0 = \mathrm{Ric}_f^\infty(\gamma_x'(s)) + \|\mathrm{Hess}\, \rho_{\partial M}\|^2 (\gamma_x(s)) - g\left(\nabla \Delta_f \rho_{\partial M}, \nabla \rho_{\partial M}\right)(\gamma_x(s))$$
$$= \left(\mathrm{Ric}_f^N(\gamma_x'(s)) + \frac{f_x'(s)^2}{N-n} \right) + \|\mathrm{Hess}\, \rho_{\partial M}\|^2 (\gamma_x(s)) - h_{f,x}'(s).$$

From the assumption $\mathrm{Ric}_f^N(\gamma_x'(s)) \geq \kappa$, we deduce

$$0 \geq \kappa + \frac{f_x'(s)^2}{N-n} + \|\mathrm{Hess}\, \rho_{\partial M}\|^2 (\gamma_x(s)) - h_{f,x}'(s). \tag{3.4}$$

By the Cauchy–Schwarz inequality, we have

$$\|\mathrm{Hess}\, \rho_{\partial M}\|^2 (\gamma_x(s)) \geq \frac{(\Delta \rho_{\partial M}(\gamma_x(s)))^2}{n-1} = \frac{\left(h_{f,x}(s) - f_x'(s)\right)^2}{n-1}. \tag{3.5}$$

Combining (3.4) and (3.5), we see

$$0 \geq \kappa + \frac{f_x'(s)^2}{N-n} + \frac{\left(h_{f,x}(s) - f_x'(s)\right)^2}{n-1} - h_{f,x}'(s)$$
$$= \kappa + \frac{(1-N)\, f_x'(s)^2}{(n-1)(n-N)} + \frac{h_{f,x}(s)^2}{n-1} - \left(\frac{2h_{f,x}(s)\, f_x'(s)}{n-1} + h_{f,x}'(s) \right). \tag{3.6}$$

In the right-hand side of the Eq. (3.6), by $N \leq 1$, the second term is non-negative. The third one is non-negative. The last one satisfies

$$\frac{2h_{f,x}(s)\, f_x'(s)}{n-1} + h_{f,x}'(s) = e^{-\frac{2f_x(s)}{n-1}} \left(e^{\frac{2f_x(s)}{n-1}} \, h_{f,x}(s) \right)'$$
$$= F_x^{-2}(s) \left(F_x^2(s)\, h_{f,x}(s) \right)'.$$

Hence, we have $0 \geq \kappa - F_x^{-2}(s) \left(F_x^2(s)\, h_{f,x}(s) \right)'$. This implies (3.3).

We see that $F_x(s)$ tends to 1 as $s \to 0$. Furthermore, by Lemma 2.7, $h_{f,x}(s)$ tends to $H_{f,x}$ as $s \to 0$. It follows that

$$F_x^2(s)\, h_{f,x}(s) - \kappa \int_0^s F_x^2(u)\, du \to H_{f,x} \tag{3.7}$$

as $s \to 0$. By (3.3) and (3.7), for all $s, t \in (0, \tau(x))$ with $s \le t$

$$F_x^2(t)\, h_{f,x}(t) - \kappa \int_0^t F_x^2(u)\, du$$

$$\ge F_x^2(s)\, h_{f,x}(s) - \kappa \int_0^s F_x^2(u)\, du \ge H_{f,x} \ge \lambda. \tag{3.8}$$

Thus, we arrive at (3.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.1 Under the same setting as in Lemma 3.1, Wylie [39] has shown a Laplacian comparison inequality for the distance function from a connected component of the boundary that is similar to (3.2) when $\kappa = 0$ and $\lambda = 0$ (see the proof of Theorem 5.1 in [39]).

Remark 3.2 Assume that for some $t_0 \in (0, \tau(x))$ the equality in (3.2) holds. Then (3.8) implies that $F_x^2 h_{f,x} = F_{\kappa,\lambda,x}$ and $\left(F_x^2 h_{f,x}\right)' = \kappa F_x^2$ on $(0, t_0)$. Hence, for every $t \in (0, t_0)$, the equality in the Cauchy–Schwarz inequality in (3.5) holds; in particular, there exists a function φ on $\gamma_x((0, t_0))$ such that at each point on $\gamma_x((0, t_0))$ we have Hess $\rho_{\partial M} = \varphi\, g$ on the orthogonal complement of $\nabla \rho_{\partial M}$. Furthermore, for every $t \in (0, t_0)$, the second term and the third one in the right-hand side of (3.6) are equal to 0; in particular, $(1 - N)(f_x')^2 = 0$ on $(0, t_0)$.

From Lemma 3.1, we derive the following:

Lemma 3.2 *Take $x \in \partial M$. For $\kappa, \lambda \in \mathbb{R}$ and for $N \in (-\infty, 1]$ we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma_x'(t)) \ge \kappa$, and suppose $H_{f,x} \ge \lambda$. Then for all $s, t \in [0, \tau(x))$ with $s \le t$ we have*

$$\theta_f(t, x) \le e^{-\int_s^t F_x^{-2}(u) F_{\kappa,\lambda,x}(u)\, du}\, \theta_f(s, x),$$

where $\theta_f(t, x)$ is defined as (2.1). In particular, if κ and λ satisfy the subharmonic condition, then for all $s, t \in [0, \tau(x))$ with $s \le t$ we have $\theta_f(t, x) \le \theta_f(s, x)$.

Proof By (2.2) and Lemma 3.1, for all $t \in (0, \tau(x))$

$$\frac{d}{dt} \log \frac{e^{-\int_0^t F_x^{-2}(u) F_{\kappa,\lambda,x}(u)\, du}}{\theta_f(t, x)} = -F_x^{-2}(t) F_{\kappa,\lambda,x}(t) + \Delta_f\, \rho_{\partial M}(\gamma_x(t)) \ge 0.$$

It follows that for all $s, t \in [0, \tau(x))$ with $s \le t$

$$\frac{\theta_f(t, x)}{\theta_f(s, x)} \le \frac{e^{-\int_0^t F_x^{-2}(u) F_{\kappa,\lambda,x}(u)\, du}}{e^{-\int_0^s F_x^{-2}(u) F_{\kappa,\lambda,x}(u)\, du}}.$$

Therefore, we have the lemma. $\qquad\square$

3.2 Equality Cases

We recall the following radial curvature equation (see, e.g., Theorem 2 in [29]):

Lemma 3.3 *Let ρ be a smooth function defined on a domain in M such that $\|\nabla\rho\| = 1$. Let X be a parallel vector field along an integral curve of $\nabla\rho$ that is orthogonal to $\nabla\rho$. Then we have*

$$g(R(X, \nabla\rho)\nabla\rho, X) = g(\nabla_{\nabla\rho}A_{\nabla\rho}X, X) - g(A_{\nabla\rho}A_{\nabla\rho}X, X),$$

where R is the curvature tensor induced from g, and $A_{\nabla\rho}$ is the shape operator of the level set of ρ toward $\nabla\rho$. In particular, if there exists a function φ defined on the domain of the integral curve such that $A_{\nabla\rho}X = -\varphi\, X$, then we have

$$g(R(X, \nabla\rho)\nabla\rho, X) = -(\varphi' + \varphi^2)\|X\|^2.$$

For the equality case of Lemma 3.1, we have the following:

Lemma 3.4 *Take $x \in \partial M$. For $\kappa, \lambda \in \mathbb{R}$ and for $N \in (-\infty, 1]$ we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma_x'(t)) \geq \kappa$, and suppose $H_{f,x} \geq \lambda$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x}e_{x,i}$. Assume that for some $t_0 \in (0, \tau(x))$ the equality in (3.2) holds. Then $\kappa = 0$ and $\lambda = 0$, and for all i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, t_0]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_x$ is constant on $[0, t_0]$; in particular, $Y_{x,i} = E_{x,i}$.*

Proof By the equality assumption, there exists a function φ on the set $\gamma_x((0, t_0))$ such that at each point on $\gamma_x((0, t_0))$ we have $\mathrm{Hess}\,\rho_{\partial M} = \varphi\, g$ on the orthogonal complement of $\nabla\rho_{\partial M}$ (see Remark 3.2). Put $\varphi_x := \varphi \circ \gamma_x$. For each i, it holds that

$$g(A_{\nabla\rho_{\partial M}}E_{x,i}, E_{x,i}) = -\mathrm{Hess}\,\rho_{\partial M}(E_{x,i}, E_{x,i}) = -\varphi_x.$$

It follows that $A_{\nabla\rho_{\partial M}}E_{x,i} = -\varphi_x E_{x,i}$. From Lemma 3.3, we derive

$$R(E_{x,i}, \nabla\rho_{\partial M})\nabla\rho_{\partial M} = -(\varphi_x' + \varphi_x^2)E_{x,i}. \tag{3.9}$$

Put $f_x := f \circ \gamma_x$ and $h_{f,x} := (\Delta_f \rho_{\partial M}) \circ \gamma_x$. By the equality assumption, we have $F_x^2 h_{f,x} = F_{\kappa,\lambda,x}$ on $[0, t_0]$ (see Remark 3.2). Fix $t \in [0, t_0]$. Since $h_{f,x}(t) = F_x^{-2}(t) F_{\kappa,\lambda,x}(t)$, we have

$$\Delta\rho_{\partial M}(\gamma_x(t)) = -f_x'(t) + F_x^{-2}(t) F_{\kappa,\lambda,x}(t).$$

On the other hand, from Hess $\rho_{\partial M} = \varphi\, g$, we deduce $\Delta\rho_{\partial M}(\gamma_x(t)) = -(n-1)\varphi_x(t)$. Hence, $\varphi_x(t)$ is equal to

$$(n-1)^{-1}\left(f_x(t) - f(x) - \int_0^t F_x^{-2}(s)\, F_{\kappa,\lambda,x}(s)\, ds\right)'.$$

Now, define a function $\mathcal{F}_x : [0, t_0] \to (0, \infty)$ by

$$\mathcal{F}_x(t) := e^{\int_0^t \varphi_x(s)\, ds} = e^{-\frac{\int_0^t F_x^{-2}(s) F_{\kappa,\lambda,x}(s)\, ds}{n-1}} F_x(t).$$

Note that if $\kappa = 0$ and $\lambda = 0$, then $\mathcal{F}_x = F_x$. By (3.9), a vector field $\tilde{Y}_{x,i}$ along $\gamma_x|_{[0,t_0]}$ defined by $\tilde{Y}_{x,i} := \mathcal{F}_x E_{x,i}$ is a ∂M-Jacobi field along $\gamma_x|_{[0,t_0]}$ with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Therefore, $Y_{x,i}$ coincides with $\tilde{Y}_{x,i}$ on $[0, t_0]$.

From (3.9) and $Y_{x,i} = \mathcal{F}_x E_{x,i}$, we derive $R(E_{x,i}, \nabla\rho_{\partial M})\nabla\rho_{\partial M} = -\mathcal{F}''_x \mathcal{F}_x^{-1} E_{x,i}$. This implies that for each $t \in (0, t_0)$

$$\begin{aligned}
\mathrm{Ric}_g(\gamma'_x(t)) &= -(n-1)\mathcal{F}''_x(t)\,\mathcal{F}_x^{-1}(t) \\
&= -f''_x(t) - \frac{f'_x(t)^2}{n-1} + \kappa - \frac{F_x^{-4}(t)\, F_{\kappa,\lambda,x}^2(t)}{n-1};
\end{aligned}$$

in particular, $\mathrm{Ric}_f^1(\gamma'_x(t)) = \kappa - (n-1)^{-1} F_x^{-4}(t) F_{\kappa,\lambda,x}^2(t)$. By the monotonicity of Ric_f^N with respect to N, we see

$$\kappa \le \mathrm{Ric}_f^N(\gamma'_x(t)) \le \mathrm{Ric}_f^1(\gamma'_x(t)) = \kappa - \frac{F_x^{-4}(t)\, F_{\kappa,\lambda,x}^2(t)}{n-1},$$

and hence $F_{\kappa,\lambda,x}(t) = 0$. We obtain $\kappa = 0$ and $\lambda = 0$, and $Y_{x,i} = F_x E_{x,i}$.

Now, we have $(1 - N)(f'_x)^2 = 0$ on $[0, t_0]$ (see Remark 3.2). If N is smaller than 1, then $f'_x = 0$ on $[0, t_0]$; in particular, f_x is constant on $[0, t_0]$. This completes the proof. □

By Lemma 3.4, we have the following:

Lemma 3.5 *Take $x \in \partial M$. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma'_x(t)) \ge \kappa$, and suppose $H_{f,x} \ge \lambda$. We choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Assume that for some $t_0 \in (0, \tau(x))$ we have $\Delta_f \rho_{\partial M}(\gamma_x(t_0)) = 0$. Then $\kappa = 0$ and $\lambda = 0$, and for all i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, t_0]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_x$ is constant on $[0, t_0]$; in particular, $Y_{x,i} = E_{x,i}$.*

Proof The assumption $\Delta_f \rho_{\partial M}(\gamma_x(t_0)) = 0$ implies that the equality in (3.2) holds. Lemma 3.4 leads to the lemma. □

3.3 Distributions

From Lemma 3.1, we derive the following:

Lemma 3.6 *Take $x \in \partial M$. Let $p \in (1, \infty)$, and let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose that for all $t \in (0, \tau(x))$ we have $\mathrm{Ric}_f^N(\gamma_x'(t)) \geq \kappa$, and suppose $H_{f,x} \geq \lambda$. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a monotone increasing smooth function. Then for all $t \in (0, \tau(x))$*

$$\Delta_{f,p}(\varphi \circ \rho_{\partial M})(\gamma_x(t)) \geq - \left((\varphi')^{p-1} \right)' (t). \tag{3.10}$$

Proof For all $t \in (0, \tau(x))$ we see

$$\Delta_{f,p}(\varphi \circ \rho_{\partial M})(\gamma_x(t)) = - \left((\varphi')^{p-1} \right)' (t) + \Delta_{f,2}\, \rho_{\partial M}(\gamma_x(t))\, \varphi'(t)^{p-1}.$$

This together with Lemma 3.1 implies (3.10). □

Remark 3.3 The equality case in Lemma 3.6 corresponds to that in Lemma 3.1 (see Lemma 3.5).

From Lemma 3.6, we deduce the following:

Proposition 3.7 *Let $p \in (1, \infty)$, and let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. For a monotone increasing smooth function $\varphi : [0, \infty) \to \mathbb{R}$, we put $\Phi := \varphi \circ \rho_{\partial M}$. Then*

$$\Delta_{f,p}\, \Phi \geq - \left((\varphi')^{p-1} \right)' \circ \rho_{\partial M}$$

in a distribution sense on M. More precisely, for every non-negative smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in Int M, *we have*

$$\int_M \|\nabla \Phi\|^{p-2} g\, (\nabla \psi, \nabla \Phi)\, d m_f \geq \int_M \psi \left(- \left((\varphi')^{p-1} \right)' \circ \rho_{\partial M} \right) d m_f. \tag{3.11}$$

Proof By Lemma 2.3, there exists a sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ of closed subsets of M such that for every k, the set $\partial \Omega_k$ is a smooth hypersurface in M satisfying the following: (1) for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \mathrm{Cut}\, \partial M = \bigcup_k \Omega_k$; (3) $\partial \Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial \Omega_k \setminus \partial M$, there exists the unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla \rho_{\partial M}) \geq 0$. For the canonical Riemannian volume measure vol_k on $\partial \Omega_k \setminus \partial M$, put $m_{f,k} := e^{-f|_{\partial \Omega_k \setminus \partial M}}\, \mathrm{vol}_k$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in Int M. By the Green formula, and by $\partial \Omega_k \cap \partial M = \partial M$,

$$\int_{\Omega_k} \|\nabla \Phi\|^{p-2} g\, (\nabla \psi, \nabla \Phi)\, d m_f$$

$$= \int_{\Omega_k} \psi\, \Delta_{f,p} \Phi\, d m_f + \int_{\partial \Omega_k \setminus \partial M} \|\nabla \Phi\|^{p-2}\, \psi\, g\, (\nu_k, \nabla \Phi)\, d m_{f,k}.$$

From Lemma 3.6 and $g(v_k, \nabla \rho_{\partial M}) \geq 0$, we derive

$$\int_{\Omega_k} \|\nabla \Phi\|^{p-2} g\,(\nabla \psi, \nabla \Phi)\; d\,m_f \geq \int_{\Omega_k} \psi\,\left(-\left((\varphi')^{p-1}\right)' \circ \rho_{\partial M}\right) d\,m_f.$$

Letting $k \to \infty$, we have the proposition. $\square$

Remark 3.4 In Proposition 3.7, we assume that the equality in (3.11) holds. Then for every $x \in \partial M$, and for every $t \in (0, \tau(x))$, the equality in (3.10) also holds. The equality case in Proposition 3.7 corresponds to that in Lemma 3.6 (see Remark 3.3).

Remark 3.5 Perales [28] has shown a Laplacian comparison inequality for the distance function from the boundary in a barrier sense for manifolds with boundary of non-negative Ricci curvature. We can prove that the Laplacian comparison inequality for $\rho_{\partial M}$ in Lemma 3.1 globally holds on M in a barrier sense.

4 Volume Comparisons

Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

4.1 Absolute Volume Comparisons

We have the following absolute volume comparison inequality of Heintze–Karcher type (cf. [11]):

Lemma 4.1 *Suppose that ∂M is compact. For $\kappa, \lambda \in \mathbb{R}$, and for $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Then for all $r \in (0, \infty)$ we have*

$$m_f(B_r(\partial M)) \leq \int_{\partial M} \int_0^{\min\{r, \tau(x)\}} e^{-\int_0^t F_x^{-2}(u)\, F_{\kappa,\lambda,x}(u)\, du}\; dt\; dm_{f,\partial M},$$

where $F_{\kappa,\lambda,x}$ is the function defined as (3.1). In particular, if κ and λ satisfy the sub-harmonic condition, then for all $r \in (0, \infty)$ we have $m_f(B_r(\partial M)) \leq r\, m_{f,\partial M}(\partial M)$, and hence (1.7).

Proof Define a function $\tilde{\theta} : [0, \infty) \times \partial M \to \mathbb{R}$ by

$$\tilde{\theta}(t, x) := \begin{cases} e^{-\int_0^t F_x^{-2}(u)\, F_{\kappa,\lambda,x}(u)\, du} & \text{if } t < \tau(x), \\ 0 & \text{if } t \geq \tau(x). \end{cases}$$

By Lemma 3.2, for all $x \in \partial M$ and $t \in (0, \infty)$ we see $\bar{\theta}_f(t, x) \leq \tilde{\theta}(t, x)\, e^{-f(x)}$, where $\bar{\theta}_f$ is the function defined as (2.3). Integrate the both sides of the inequality over $(0, r)$ with respect to t, and then do that over ∂M with respect to x. From Lemma 2.8, we deduce

$$m_f(B_r(\partial M)) \leq \int_{\partial M} \int_0^r \tilde{\theta}(t, x)\, dt\, dm_{f,\partial M}.$$

This implies the lemma. □

Remark 4.1 Under a lower N-weighted Ricci curvature bound, Bayle [3] has stated an inequality of Heintze–Karcher type without proof in the case of $N \in [n, \infty)$. Morgan [24] has proved it in the case of $N = \infty$, and Milman [23] has done in the case of $N \in (-\infty, 1)$.

4.2 Relative Volume Comparisons

We have the following relative volume comparison theorem of Bishop–Gromov type:

Theorem 4.2 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Then for all $r, R \in (0, \infty)$ with $r \leq R$*

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{R}{r}. \tag{4.1}$$

Proof Lemma 3.2 implies that for all $s, t \in [0, \infty)$ with $s \leq t$ we have $\bar{\theta}_f(t, x) \leq \bar{\theta}_f(s, x)$, where $\bar{\theta}_f$ is the function defined as (2.3). By integrating the both sides over $(0, r)$ with respect to s, and then doing that over (r, R) with respect to t, we see

$$r \int_r^R \bar{\theta}_f(t, x)\, dt \leq (R - r) \int_0^r \bar{\theta}_f(s, x)\, ds.$$

From Lemma 2.8, we derive

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} = 1 + \frac{\int_{\partial M} \int_r^R \bar{\theta}_f(t, x)\, dt\, d\,\mathrm{vol}_h}{\int_{\partial M} \int_0^r \bar{\theta}_f(s, x)\, ds\, d\,\mathrm{vol}_h} \leq 1 + \frac{R - r}{r} = \frac{R}{r}.$$

This proves the theorem. □

When $\kappa = 0$ and $\lambda = 0$, Theorem 4.2 has been proved in the unweighted case in [32], and in the usual weighted case in [33].

Remark 4.2 In [32], the author has proved a measure contraction inequality around the boundary in the unweighted case. We can prove a similar measure contraction inequality in our setting. The measure contraction inequality enables us to give another proof of Theorem 4.2.

4.3 Volume Growth Rigidity

We show the following:

Lemma 4.3 *Suppose that ∂M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Assume that there exists $R \in (0, \infty)$ such that for every $r \in (0, R]$ the equality in (4.1) holds. Then $\tau \geq R$ on ∂M.*

Proof The proof will be done by contradiction. Suppose that there exists $x_0 \in \partial M$ such that $\tau(x_0) < R$. Put $t_0 := \tau(x_0)$. Take $\epsilon \in (0, \infty)$ with $t_0 + \epsilon < R$. By the continuity of τ, there exists a closed geodesic ball B in ∂M centered at x_0 such that τ is smaller than or equal to $t_0 + \epsilon$ on B. Using Lemma 3.2, we see

$$m_f(B_R(\partial M)) \leq R\, m_{f,\partial M}(\partial M \setminus B) + (t_0 + \epsilon)\, m_{f,\partial M}(B) < R\, m_{f,\partial M}(\partial M).$$

On the other hand, $m_f(B_R(\partial M))/m_{f,\partial M}(\partial M)$ is equal to R. This is a contradiction. $\square$

Suppose that ∂M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Assume that there exists $R \in (0, \infty)$ such that for every $r \in (0, R]$ the equality in (4.1) holds. Then for each $r \in (0, R)$ the level set $\rho_{\partial M}^{-1}(r)$ is an $(n-1)$-dimensional submanifold of M (see Lemma 4.3). In particular, $(B_r(\partial M), g)$ is an n-dimensional (not necessarily, connected) complete Riemannian manifold with boundary. We denote by $d_{B_r(\partial M)}$ and by $d_{[0,r]\times_F \partial M}$ the Riemannian distances on $(B_r(\partial M), g)$ and on $[0, r] \times_F \partial M$, respectively, where $[0, r] \times_F \partial M$ is the twisted product space $([0, r] \times \partial M, dt^2 + F_x^2(t)\, h)$.

Lemma 4.4 *Suppose that ∂M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Assume that there exists $R \in (0, \infty)$ such that for every $r \in (0, R]$ the equality in (4.1) holds. Then for each $r \in (0, R)$ the metric space $(B_r(\partial M), d_{B_r(\partial M)})$ is isometric to $([0, r] \times_F \partial M, d_{[0,r]\times_F \partial M})$. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, r]$; in particular, $(B_r(\partial M), d_{B_r(\partial M)})$ is isometric to $([0, r] \times \partial M, d_{[0,r]\times \partial M})$.*

Proof Since each connected component of ∂M one-to-one corresponds to the connected component of $B_r(\partial M)$, it suffices to consider the case where $B_r(\partial M)$ is connected. For each $x \in \partial M$ we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x} e_{x,i}$. Since the equality in (4.1) holds, for all $t \in [0, r]$ we see $\theta_f(t, x) = \theta_f(r, x)$. By (2.2), for all $t \in (0, r]$ we see $\Delta_f \rho_{\partial M}(\gamma_x(t)) = 0$. Lemma 3.5 implies that we have $\kappa = 0$ and $\lambda = 0$, and for all i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, r]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vectors field along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_x$ is constant on $[0, r]$. Define a map $\Phi : [0, r] \times \partial M \to B_r(\partial M)$ by $\Phi(t, x) := \gamma_x(t)$. For each $p \in (0, r) \times \partial M$ the differential map $D(\Phi|_{(0,r)\times\partial M})_p$ sends an orthonormal basis of $T_p([0, r] \times \partial M)$ to that of $T_{\Phi(p)} B_r(\partial M)$, and for each $x \in \{0, r\} \times \partial M$ the map $D(\Phi|_{\{0,r\}\times\partial M})_x$ sends an orthonormal basis of $T_x(\{0, r\} \times \partial M)$ to that of $T_{\Phi(x)} \partial(B_r(\partial M))$. Hence, Φ is a Riemannian isometry with boundary from $[0, r] \times_F \partial M$ to $B_r(\partial M)$. $\square$

Now, we are in a position to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 Suppose that ∂M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Furthermore, we assume (1.6). By Lemma 4.1 and Theorem 4.2, for every $R \in (0, \infty)$, and for every $r \in (0, R]$,

$$\frac{m_f(B_R(\partial M))}{R} = \frac{m_f(B_r(\partial M))}{r} = m_{f,\partial M}(\partial M);$$

in particular, the equality in (4.1) holds. From Lemma 4.3, we deduce $\tau = \infty$ on ∂M. We see Cut $\partial M = \emptyset$, and hence ∂M is connected. Take a sequence $\{r_i\}$ with $r_i \to \infty$. By Lemma 4.4, for every i there exists a Riemannian isometry $\Phi_i : [0, r_i] \times \partial M \to B_{r_i}(\partial M)$ with boundary from $[0, r_i] \times_F \partial M$ to $B_{r_i}(\partial M)$ defined by $\Phi_i(t, x) := \gamma_x(t)$. Moreover, if $N \in (-\infty, 1)$, then for each $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, r_i]$. Since Cut $\partial M = \emptyset$, we obtain a Riemannian isometry $\Phi : [0, \infty) \times \partial M \to M$ with boundary from $[0, \infty) \times_F \partial M$ to M defined by $\Phi(t, x) := \gamma_x(t)$ such that $\Phi|_{[0, r_i] \times \partial M} = \Phi_i$ for all i. Furthermore, if $N \in (-\infty, 1)$, then $f \circ \gamma_x$ is constant on $[0, \infty)$. Thus, we complete the proof of Theorem 1.1. $\qquad\square$

Proof of Corollary 1.2 Combining Theorem 1.1 and Proposition 2.9, we conclude Corollary 1.2. $\qquad\square$

5 Splitting Theorems

Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

5.1 Basic Splitting

Let $\varphi : M \to \mathbb{R}$ be a continuous function, and let U be a domain contained in Int M. For $p \in U$, and for a function ψ defined on an open neighborhood of p, we say that ψ is a *support function of φ at p* if we have $\psi(p) = \varphi(p)$ and $\psi \leq \varphi$. We say that φ is *f-subharmonic on U* if for every $p \in U$, and for every $\epsilon \in (0, \infty)$, there exists a smooth, support function $\psi_{p,\epsilon}$ of φ at p such that $\Delta_f \psi_{p,\epsilon}(p) \leq \epsilon$.

We recall the following maximal principle of Calabi type (see, e.g., [6], and Lemma 2.4 in [10]).

Lemma 5.1 *Let U be a domain contained in* Int M. *If a f-subharmonic function on U takes the maximal value at a point in U, then it must be constant on U.*

Wylie [39] has proved a subharmonicity of Busemann functions on manifolds without boundary (see Lemma 3.4 in [39]). In our case, under an assumption concerning asymptotes for a ray defined in Sect. 2.4, the subharmonicity holds in the following form:

Lemma 5.2 [39] *Assume* $\sup f(M) < \infty$. *For $N \in (-\infty, 1]$ we suppose* $\mathrm{Ric}^N_{f,M} \geq 0$. *Let $\gamma : [0, \infty) \to M$ be a ray that lies in* Int M, *and let U be a domain contained in*

Int M such that for each $p \in U$, there exists an asymptote for γ from p that lies in Int M. Then the Busemann function b_γ of γ is f-subharmonic on U.

Now, we prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3 Assume $\sup f(M) < \infty$. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq 0$, and $H_{f,\partial M} \geq 0$. Suppose that for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$.
For the connected component ∂M_0 of ∂M containing x_0, put

$$\Omega := \{y \in \partial M_0 \mid \tau(y) = \infty\}.$$

By the continuity of τ, the set Ω is a non-empty closed subset of ∂M_0.
We show the openness of Ω in ∂M_0. Fix $y_0 \in \Omega$. Take $l \in (0, \infty)$, and put $p_0 := \gamma_{y_0}(l)$. There exists an open neighborhood U of p_0 in Int M contained in $D_{\partial M}$. Taking U smaller, we may assume that for every $q \in U$ the unique foot point on ∂M of q belongs to ∂M_0. By Lemma 2.5, there exists $\epsilon \in (0, \infty)$ such that for all $q \in B_\epsilon(p_0)$, all asymptotes for γ_{y_0} from q lie in Int M. We may assume $U \subset B_\epsilon(p_0)$. By Lemma 5.2, $b_{\gamma_{y_0}}$ is f-subharmonic on U, and by Lemma 3.1, $\Delta_f \rho_{\partial M} \geq 0$ on U. Hence, $b_{\gamma_{y_0}} - \rho_{\partial M}$ is f-subharmonic on U. The function $b_{\gamma_{y_0}} - \rho_{\partial M}$ takes the maximal value 0 at p_0. Lemma 5.1 implies that $b_{\gamma_{y_0}} = \rho_{\partial M}$ on U. From Lemma 2.4, it follows that Ω is open in ∂M_0.
Since ∂M_0 is a connected component of ∂M, we have $\Omega = \partial M_0$. By Lemma 2.2, ∂M is connected and Cut $\partial M = \emptyset$. The equality in Lemma 3.1 holds on Int M. For each $x \in \partial M$ we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x} e_{x,i}$. By Lemma 3.4, for all i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, \infty)$, where $\{E_{x,i}\}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_x$ is constant on $[0, \infty)$. Define a map $\Phi : [0, \infty) \times \partial M \to M$ by $\Phi(t, x) := \gamma_x(t)$. For every $p \in (0, \infty) \times \partial M$ the differential map $D(\Phi|_{(0,\infty) \times \partial M})_p$ sends an orthonormal basis of $T_p((0, \infty) \times \partial M)$ to that of $T_{\Phi(p)} M$, and for every $x \in \{0\} \times \partial M$ the map $D(\Phi|_{\{0\} \times \partial M})_x$ sends an orthonormal basis of $T_x(\{0\} \times \partial M)$ to that of $T_{\Phi(x)} \partial M$. Then Φ is a Riemannian isometry with boundary from $[0, \infty) \times_F \partial M$ to M. This proves Theorem 1.3.   □

Proof of Corollary 1.4 From Theorem 1.3 and Proposition 2.9, we derive Corollary 1.4.   □

Lemma 2.1 and the continuity of τ imply that if M is non-compact and ∂M is compact, then for some $x \in \partial M$ we have $\tau(x) = \infty$. We have the following corollary of Theorem 1.3.

Corollary 5.3 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function such that $\sup f(M) < \infty$. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq 0$, and $H_{f,\partial M} \geq 0$. If M is non-compact and ∂M is compact, then (M, d_M) is isometric to $([0, \infty) \times_F \partial M, d_{[0,\infty) \times_F \partial M})$. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, \infty)$; in particular, (M, d_M) is isometric to $([0, \infty) \times \partial M, d_{[0,\infty) \times \partial M})$.*

5.2 Weighted Ricci Curvature on the Boundary

For $x \in \partial M$, we recall that u_x denotes the unit inner normal vector on ∂M at x.

The following seems to be well known, especially in a submanifold setting (see, e.g., Proposition 9.36 in [4], and Lemma 5.4 in [32]).

Lemma 5.4 *Take $x \in \partial M$, and a unit vector u in $T_x \partial M$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$ with $e_{x,1} = u$. Then we have*

$$\mathrm{Ric}_h(u) = \mathrm{Ric}_g(u) - K_g(u_x, u) + \mathrm{trace}\, A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{x,i})\|^2,$$

where h is the induced Riemannian metric on ∂M, and $K_g(u_x, u)$ is the sectional curvature at x in (M, g) determined by u_x and u.

By using Lemma 5.4, we have the following:

Lemma 5.5 *Take $x \in \partial M$, and a unit vector u in $T_x \partial M$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$ with $e_{x,1} = u$. Then for all $N \in (-\infty, \infty)$ we have*

$$\mathrm{Ric}_{f|\partial M}^{N-1}(u) = \mathrm{Ric}_f^N(u) + g((\nabla f)_x, u_x)\, g(S(u, u), u_x)$$

$$- K_g(u_x, u) + \mathrm{trace}\, A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{x,i})\|^2, \qquad (5.1)$$

where $K_g(u_x, u)$ is the sectional curvature at x in (M, g) determined by u_x and u.

Proof First, we assume $N \neq n$. We see

$$h((\nabla(f|_{\partial M}))_x, u) = g((\nabla f)_x, u),$$
$$\mathrm{Hess}(f|_{\partial M})(u, u) = \mathrm{Hess}\, f(u, u) + g((\nabla f)_x, u_x)\, g(S(u, u), u_x),$$

where h is the induced Riemannian metric on ∂M. Hence, we have

$$\mathrm{Ric}_{f|\partial M}^{N-1}(u) = \mathrm{Ric}_h(u) + \mathrm{Hess}(f|_{\partial M})(u, u) - \frac{h((\nabla(f|_{\partial M}))_x, u)^2}{(N-1) - (n-1)}$$

$$= \mathrm{Ric}_h(u) + \mathrm{Hess}\, f(u, u) + g((\nabla f)_x, u_x)\, g(S(u, u), u_x) - \frac{g((\nabla f)_x, u)^2}{N - n}.$$

From Lemma 5.4, we derive (5.1).

Next, we assume $N = n$. If f is constant, then $\mathrm{Ric}_{f|\partial M}^{N-1}(u) = \mathrm{Ric}_h(u)$ and $\mathrm{Ric}_f^N(u) = \mathrm{Ric}_g(u)$, and hence Lemma 5.4 implies (5.1). If f is not constant, then both the left-hand side of (5.1) and the right-hand side are equal to $-\infty$. Therefore, we complete the proof. □

From Lemma 5.5, we derive the following:

Lemma 5.6 *Take $x \in \partial M$, and a unit vector u in $T_x \partial M$. If (M, d_M) is isometric to $([0, \infty) \times_F \partial M, d_{[0,\infty) \times_F \partial M})$, then for all $N \in (-\infty, \infty)$*

$$\mathrm{Ric}_{f|\partial M}^{N-1}(u) = \mathrm{Ric}_f^N(u) + \frac{\mathrm{Hess}\, f(u_x, u_x)}{n-1}.$$

Proof Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$ with $e_{x,1} = u$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Since (M, d_M) is isometric to $([0, \infty) \times_F \partial M, d_{[0,\infty) \times_F \partial M})$, there exists a Riemannian isometry with boundary from M to $[0, \infty) \times_F \partial M$. In particular, for all i we see $Y_{x,i} = F_x E_{x,i}$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Hence, for all i

$$A_{u_x} e_{x,i} = -Y'_{x,i}(0) = -\frac{g((\nabla f)_x, u_x)}{n-1} e_{x,i}. \tag{5.2}$$

By (5.2), for all $i \neq 1$ we have $S(u, e_{x,i}) = 0_x$, and we have

$$S(u, u) = -\frac{g((\nabla f)_x, u_x)}{n-1} u_x, \quad \mathrm{trace}\, A_{S(u,u)} = \frac{g((\nabla f)_x, u_x)^2}{n-1}. \tag{5.3}$$

Furthermore, the sectional curvature $K_g(u_x, u)$ at x in (M, g) determined by u_x and u satisfies

$$K_g(u_x, u) = -g(Y''_{x,1}(0), u) = -\left(\frac{\mathrm{Hess}\, f(u_x, u_x)}{n-1} + \left(\frac{g((\nabla f)_x, u_x)}{n-1}\right)^2\right). \tag{5.4}$$

By Lemma 5.5, and by (5.3) and (5.4), we see

$$\begin{aligned}
\mathrm{Ric}_{f|\partial M}^{N-1}(u) &= \mathrm{Ric}_f^N(u) - \frac{g((\nabla f)_x, u_x)^2}{n-1} + \frac{\mathrm{Hess}\, f(u_x, u_x)}{n-1} \\
&\quad + \left(\frac{g((\nabla f)_x, u_x)}{n-1}\right)^2 + \frac{g((\nabla f)_x, u_x)^2}{n-1} - \left(\frac{g((\nabla f)_x, u_x)}{n-1}\right)^2 \\
&= \mathrm{Ric}_f^N(u) + \frac{\mathrm{Hess}\, f(u_x, u_x)}{n-1}.
\end{aligned}$$

This completes the proof.   □

5.3 Multi-splitting

Let M_0 be a connected complete Riemannian manifold (without boundary). A normal geodesic $\gamma : \mathbb{R} \to M_0$ is said to be a *line* if for all $s, t \in \mathbb{R}$ we have $d_{M_0}(\gamma(s), \gamma(t)) = |s - t|$.

Wylie [39] has proved the following splitting theorem of Cheeger–Gromoll type (see Theorem 1.2 and Corollary 1.3 in [39]):

Theorem 5.7 [39] *Let M_0 be a connected complete Riemannian manifold, and let $f : M_0 \to \mathbb{R}$ be a smooth function bounded from above. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f, M_0} \geq 0$. If M_0 contains a line, then there exists a connected complete Riemannian manifold N_0 such that M_0 is isometric to a warped product space over $\mathbb{R} \times N_0$. Moreover, if $N \in (-\infty, 1)$, then M_0 is isometric to the standard product $\mathbb{R} \times N_0$.*

Remark 5.1 For manifolds of non-negative N-weighted Ricci curvature, Lichnerowicz [19] has generalized the Cheeger–Gromoll splitting theorem in the case where $N = \infty$ and f is bounded. Fang, Li, and Zhang [10] have done in the case where $N \in [n, \infty)$, and in the case where $N = \infty$ and f is bounded above.

We obtain the following corollary of Theorem 1.3:

Corollary 5.8 *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function such that $\sup f(M) < \infty$. For $N \in (-\infty, 1)$ we suppose $\mathrm{Ric}^N_{f, M} \geq 0$, and $H_{f, \partial M} \geq 0$. Assume that for some $x_0 \in \partial M$ we have $\tau(x_0) = \infty$. Then there exist $k \in \{0, \dots, n-1\}$ and an $(n-1-k)$-dimensional, connected complete Riemannian manifold N_0 containing no line such that $(\partial M, d_{\partial M})$ is isometric to $(\mathbb{R}^k \times N_0, d_{\mathbb{R}^k \times N_0})$. In particular, (M, d_M) is isometric to $([0, \infty) \times \mathbb{R}^k \times N_0, d_{[0,\infty) \times \mathbb{R}^k \times N_0})$.*

Proof Due to Theorem 1.3, the metric space (M, d_M) is isometric to $([0, \infty) \times_F \partial M, d_{[0,\infty) \times_F \partial M})$. Moreover, for every $x \in \partial M$, the function $f \circ \gamma_x$ is constant on $[0, \infty)$; in particular, (M, d_M) is isometric to $([0, \infty) \times \partial M, d_{[0,\infty) \times \partial M})$. For every $x \in \partial M$, it holds that $\mathrm{Hess}\, f(u_x, u_x) = 0$. By Lemma 5.6, we have $\mathrm{Ric}^{N-1}_{f|_{\partial M}} = \mathrm{Ric}^N_f$ on the unit tangent bundle over ∂M. It follows that $\mathrm{Ric}^{N-1}_{f|_{\partial M}, \partial M} \geq 0$. Note that $N - 1$ is smaller than 1, and $\sup_{x \in \partial M} f(x)$ is finite. Applying Theorem 5.7 to ∂M inductively, we complete the proof. $\qquad \square$

5.4 Variants of the Splitting Theorem

We have already known several rigidity results studied in [13] (and [9, 12]) for manifolds with boundary whose boundaries are disconnected. In [33], the author has given generalizations of them in the usual weighted case (see Sect. 6.4 in [33]). We generalize one of them in our setting.

For $A_1, A_2 \subset M$, we put $d_M(A_1, A_2) := \inf_{p_1 \in A_1, p_2 \in A_2} d_M(p_1, p_2)$.

The following has been shown in [13] (see Lemma 1.6 in [13]):

Lemma 5.9 [13] *Suppose that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of ∂M. Assume that ∂M_1 is compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Then there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. Furthermore, for every $i = 1, 2$ there exists $x_i \in \partial M_i$ such that $d_M(x_1, x_2) = D$. The normal minimal geodesic $\gamma : [0, D] \to M$ from x_1 to x_2 is orthogonal to ∂M both at x_1 and at x_2, and the restriction $\gamma|_{(0,D)}$ lies in $\mathrm{Int}\, M$.*

We prove the following splitting theorem:

Theorem 5.10 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\ldots}$ denote the connected components of ∂M. Suppose that ∂M_1 is compact, and put $D := \inf_{i=2,3,\ldots} d_M(\partial M_1, \partial M_i)$. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Then (M, d_M) is isometric to $([0, D] \times_F \partial M_1, d_{[0,D] \times_F \partial M_1})$. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M_1$ the function $f \circ \gamma_x$ is constant on $[0, D]$; in particular, (M, d_M) is isometric to $([0, D] \times \partial M_1, d_{[0,D] \times \partial M_1})$.*

Proof By Lemma 5.9, there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. For each $i = 1, 2$, let $\rho_{\partial M_i} : M \to \mathbb{R}$ be the distance function from ∂M_i defined as $\rho_{\partial M_i}(p) := d_M(p, \partial M_i)$. Put

$$\Omega := \{p \in \mathrm{Int}\, M \mid \rho_{\partial M_1}(p) + \rho_{\partial M_2}(p) = D\}.$$

Lemma 5.9 implies that Ω is a non-empty closed subset of $\mathrm{Int}\, M$.

We show that Ω is open in $\mathrm{Int}\, M$. Take $p \in \Omega$. For each $i = 1, 2$, there exists a foot point $x_{p,i} \in \partial M_i$ on ∂M_i of p such that $d_M(p, x_{p,i}) = \rho_{\partial M_i}(p)$. From the triangle inequality, we derive $d_M(x_{p,1}, x_{p,2}) = D$. The normal minimal geodesic $\gamma : [0, D] \to M$ from $x_{p,1}$ to $x_{p,2}$ is orthogonal to ∂M at $x_{p,1}$ and at $x_{p,2}$. Furthermore, $\gamma|_{(0,D)}$ lies in $\mathrm{Int}\, M$ and passes through p. There exists an open neighborhood U of p such that U is contained in $\mathrm{Int}\, M$ and $\rho_{\partial M_i}$ is smooth on U. By using Lemma 3.2, we see $\Delta_f \rho_{\partial M_i} \geq 0$ on U; in particular, $-(\rho_{\partial M_1} + \rho_{\partial M_2})$ is f-subharmonic on U. Lemma 5.1 implies that Ω is open in $\mathrm{Int}\, M$.

Since $\mathrm{Int}\, M$ is connected, we have $\mathrm{Int}\, M = \Omega$. For each $x \in \partial M_1$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Using Lemma 3.5, for all i we see $Y_{x,i} = F_x E_{x,i}$ on $[0, D]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M_1$ the function $f \circ \gamma_x$ is constant on $[0, D]$. Define a map $\Phi : [0, D] \times \partial M_1 \to M$ by $\Phi(t, x) := \gamma_x(t)$. The map Φ is a Riemannian isometry with boundary from $[0, D] \times_F \partial M_1$ to M.   □

Remark 5.2 Wylie [39] has proved the same result as Theorem 5.10 when $\kappa = 0$ and $\lambda = 0$ (see Theorem 5.1 in [39]).

6 Eigenvalue Rigidity

Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function.

6.1 Lower Bounds

In [33], the author has shown the following Picone-type inequality proved by Allegretto and Huang [1] in the Euclidean case (see Theorem 1.1 in [1], and Lemma 7.1 in [33]).

Lemma 6.1 [33] *Let φ and ψ be functions on M that are smooth on a domain U in M, and satisfy $\varphi > 0$ and $\psi \geq 0$ on U. Then for all $p \in (1, \infty)$ we have the following inequality on U:*

$$\|\nabla \psi\|^p \geq \|\nabla \varphi\|^{p-2} g\left(\nabla\left(\psi^p\,\varphi^{1-p}\right), \nabla\varphi\right). \tag{6.1}$$

Moreover, if the equality in (6.1) holds on U, then for some constant $c \neq 0$ we have $\psi = c\,\varphi$ on U.

We now give a proof of the inequality (1.9) in Theorem 1.5.

Proposition 6.2 *Suppose that M is compact. Let $p \in (1, \infty)$. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. For $D \in (0, \infty)$ we assume $D(M, \partial M) \leq D$. Then we have (1.9).*

Proof Let $\varphi_{p,D} : [0, D] \to \mathbb{R}$ be a function satisfying (1.8) for $\mu = \mu_{p,D}$. We may assume $\varphi_{p,D}|_{(0,D]} > 0$. Then we see $\varphi'_{p,D}|_{[0,D)} > 0$. Put $\Phi := \varphi_{p,D} \circ \rho_{\partial M}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in Int M. By Lemma 6.1

$$\|\nabla \psi\|^p \geq \|\nabla \Phi\|^{p-2} g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right), \nabla\Phi\right) \tag{6.2}$$

on Int $M \setminus$ Cut ∂M. From (6.2) and Proposition 3.7, we derive

$$\int_M \|\nabla \psi\|^p \, dm_f \geq \int_M \|\nabla \Phi\|^{p-2} g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right), \nabla\Phi\right) dm_f$$
$$\geq \int_M \left(\psi^p\,\Phi^{1-p}\right)\left(-\left(\left(\varphi'_{p,D}\right)^{p-1}\right)' \circ \rho_{\partial M}\right) dm_f$$
$$= \mu_{p,D} \int_M \psi^p \, dm_f.$$

It follows that $R_{f,p}(\psi) \geq \mu_{p,D}$. Hence, we arrive at (1.9). $\qquad \square$

Remark 6.1 In Proposition 6.2, we assume that there exists a non-negative, non-zero smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in Int M such that $R_{f,p}(\psi) = \mu_{p,D}$. Then the equality in (6.2) holds on Int $M \setminus$ Cut ∂M, and hence for some constant $c \neq 0$ we have $\psi = c\,\Phi$ on M (see Lemma 6.1). The equality case in (3.11) also happens (see Remark 3.4).

6.2 Equality Cases

For a positive number $D \in (0, \infty)$, we put $S_D(\partial M) := B_D(\partial M) \setminus U_D(\partial M)$.

For the proof of Theorem 1.5, we show the following lemma concerning the F-model space introduced in Sect. 1.4:

Lemma 6.3 *Suppose that M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Assume that for some $D \in (0, \infty)$ we have $\mathrm{Cut}\,\partial M = S_D(\partial M)$. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x}e_{x,i}$. Assume further that for all i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, D]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Then (M, d_M) is an F-model space.*

Proof First, we assume that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\ldots}$ be the connected components of ∂M. Put $D_1 := \inf_{i=2,3,\ldots} d_M(\partial M_1, \partial M_i)$. By Theorem 5.10, there exists a connected component ∂M_1 such that (M, d_M) is isometric to $([0, D_1] \times_F \partial M_1, d_{[0,D_1]\times_F \partial M_1})$. From $\mathrm{Cut}\,\partial M = S_D(\partial M)$, it follows that $D(M, \partial M) = D$ and $D_1 = 2D$, and hence (M, d_M) is an F-model space.

Next, we assume that ∂M is connected. By $\mathrm{Cut}\,\partial M = S_D(\partial M)$, we have $D(M, \partial M) = D$. From the property of Jacobi fields, $S_D(\partial M)$ is a smooth hypersurface in M, and every point in $S_D(\partial M)$ has two distinct foot points on ∂M. For every $x \in \partial M$, the vector $\gamma'_x(D)$ is orthogonal to $S_D(\partial M)$, and hence the number of foot points on ∂M of $\gamma_x(D)$ is equal to two. Now, we define an involutive isometry $\sigma : \partial M \to \partial M$ without fixed points by $\sigma(x) := y$, where y is the foot point on ∂M of $\gamma_x(D)$ that is different from x. We also define a map $\Phi : [0, 2D] \times \partial M \to M$ as follows: If $t \in [0, D)$, then $\Phi(t, x) := \gamma_x(t)$; if $t \in (D, 2D]$, then $\Phi(t, x) := \gamma_{\sigma(x)}(2D - t)$. We see that Φ is surjective and continuous. For all $x \in \partial M$ and $t \in [0, 2D]$ we have $\Phi(t, x) = \Phi(2D - t, \sigma(x))$. Since for all $x \in \partial M$ and i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, D]$, the map $\Phi|_{[0,D)}$ gives an isometry between $(U_D(\partial M), g)$ and the twisted product space $[0, D) \times_F \partial M$. Therefore, M is isometric to the quotient space $([0, 2D] \times_F \partial M)/G_\sigma$, where G_σ is the isometry group on $[0, 2D] \times_F \partial M$ of the identity and the involute isometry $\hat{\sigma}$ defined by $\hat{\sigma}(t, x) := (2D - t, \sigma(x))$. This implies that (M, d_M) is an F-model space. We complete the proof. $\square$

Furthermore, we recall the following fact concerning eigenfunctions for the (f, p)-Laplacian (see, e.g., [33,36]):

Proposition 6.4 *Suppose that M is compact. Let $p \in (1, \infty)$. Then there exists a non-negative, non-zero function Ψ in $W_0^{1,p}(M, m_f)$ such that $R_{f,p}(\Psi) = \mu_{f,1,p}(M)$. Moreover, for some $\alpha \in (0, 1)$ the function Ψ is $C^{1,\alpha}$-Hölder continuous on M.*

Now, we prove Theorem 1.5 and Corollary 1.6.

Proof of Theorem 1.5 Suppose that M is compact. Let $p \in (1, \infty)$, and let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. For $D \in (0, \infty)$, we assume $D(M, \partial M) \leq D$. From Proposition 6.2, we derive (1.9).

We assume that the equality in (1.9) holds. Proposition 6.4 implies that there exists a non-negative, non-zero function Ψ in $W_0^{1,p}(M, m_f)$ such that $R_{f,p}(\Psi) = \mu_{p,D}$ and Ψ is $C^{1,\alpha}$-Hölder continuous on M. Let $\varphi_{p,D} : [0, D] \to \mathbb{R}$ be a function satisfying

(1.8) for $\mu = \mu_{p,D}$, and let $\varphi_{p,D}|_{(0,D]} > 0$. Putting $\Phi := \varphi_{p,D} \circ \rho_{\partial M}$, we see that Φ coincides with a constant multiplication of Ψ on M (see Remark 6.1); in particular, Φ is also $C^{1,\alpha}$-Hölder continuous on M.

For each $x \in \partial M$, we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. Let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Then for all i we have $Y_{x,i} = F_x E_{x,i}$ on $[0, \tau(x)]$, where $\{E_{x,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_x$ is constant on $[0, \tau(x)]$ (see Remark 6.1).

We show Cut $\partial M = S_D(\partial M)$. From $D(M, \partial M) \leq D$ we deduce $S_D(\partial M) \subset$ Cut ∂M. We prove the opposite. Take $p_0 \in$ Cut ∂M. By the property of Jacobi fields, $\rho_{\partial M}$ is not differentiable at p_0. From the regularity of Φ, it follows that $\varphi'_{p,D}(\rho_{\partial M}(p_0)) = 0$; in particular, $\rho_{\partial M}(p_0) = D$. Hence, Cut $\partial M = S_D(\partial M)$. This implies $D(M, \partial M) = D$. By Lemma 6.3, we complete the proof of Theorem 1.5. □

Proof of Corollary 1.6 By Theorem 1.5 and Proposition 2.9, we complete the proof of Corollary 1.6. □

By Theorem 1.5 and $\mu_{2,D} = \pi^2 (2D)^{-2}$, we have the following:

Corollary 6.5 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}_{f,M}^N \geq \kappa$, and suppose $H_{f,\partial M} \geq \lambda$. For $D \in (0, \infty)$ we assume $D(M, \partial M) \leq D$. Then*

$$\mu_{f,1,2}(M) \geq \frac{\pi^2}{4D^2}. \tag{6.3}$$

If the equality in (6.3) holds, then $D(M, \partial M) = D$, and the metric space (M, d_M) is an F-model space. Moreover, if $N \in (-\infty, 1)$, then for every $x \in \partial M$ the function $f \circ \gamma_x$ is constant on $[0, D]$; in particular, (M, d_M) is an equational model space.

6.3 Explicit Lower Bounds

Let Ω be a relatively compact domain in M such that $\partial \Omega$ is a smooth hypersurface in M satisfying $\partial \Omega \cap \partial M = \emptyset$. For the canonical Riemannian volume measure $\mathrm{vol}_{\partial \Omega}$ on $\partial \Omega$, let $m_{f,\partial \Omega} := e^{-f|\partial \Omega} \, \mathrm{vol}_{\partial \Omega}$. Put

$$\delta_1(\Omega) := \inf_{p \in \Omega} \rho_{\partial M}(p), \quad \delta_2(\Omega) := \sup_{p \in \Omega} \rho_{\partial M}(p). \tag{6.4}$$

In the usual weighted case, the author [33] has proved the following volume estimate (see Propositions 8.1 and 8.2 in [33]):

Proposition 6.6 [33] *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. For $N \in [n, \infty]$*

we suppose $\mathrm{Ric}^N_{f,M} \geq 0$, and $H_{f,\partial M} \geq 0$. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega \cap \partial M = \emptyset$. Then

$$m_f(\Omega) \leq m_{f,\partial\Omega}(\partial\Omega)\,(\delta_2(\Omega) - \delta_1(\Omega)),$$

where $\delta_1(\Omega)$ and $\delta_2(\Omega)$ are the values defined as (6.4).

Kasue [15] has obtained Proposition 6.6 in the unweighted case.

In our setting, we have the following volume estimate:

Proposition 6.7 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega \cap \partial M = \emptyset$. Then*

$$m_f(\Omega) \leq m_{f,\partial\Omega}(\partial\Omega)\,(\delta_2(\Omega) - \delta_1(\Omega)),$$

where $\delta_1(\Omega)$ and $\delta_2(\Omega)$ are the values defined as (6.4).

We can prove Proposition 6.7 only by replacing the role of the comparison result in the usual weighted case with that of Lemma 3.1 in the proof of Proposition 6.6. We omit the proof.

From Proposition 6.6, the author [33] has derived the following estimate in the usual weighted case (see Theorems 8.4 and 8.5 in [33]):

Theorem 6.8 [33] *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. For $N \in [n, \infty]$ we suppose $\mathrm{Ric}^N_{f,M} \geq 0$, and $H_{f,\partial M} \geq 0$. For $D \in (0, \infty)$ we assume $D(M, \partial M) \leq D$. Then we have $\mu_{f,1,p}(M) \geq (pD)^{-p}$.*

The author [32] has shown Theorem 6.8 in the unweighted case.

In our setting, we can prove the following result by using Proposition 6.7 instead of Proposition 6.6 in the proof of Theorem 6.8. The argument is in [33].

Theorem 6.9 *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$, and let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the subharmonic condition. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. For $D \in (0, \infty)$ we assume $D(M, \partial M) \leq D$. Then we have $\mu_{f,1,p}(M) \geq (pD)^{-p}$.*

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