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Short-Time Existence of the Möbius-Invariant Willmore Flow

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Abstract In this article the author proves existence and uniqueness of a smooth short-time solution of the "Möbius-invariant Willmore flow" Eq. (9) starting in a C^∞-smooth immersion F_0 of a fixed smooth compact torus Σ into $\mathbb{R}^n$ without umbilic points. Hence, for some sufficiently small $T^* > 0$ there exists a unique smooth family $\{f_t\}$ of smooth immersions of the torus Σ into $\mathbb{R}^n$, with $f_0 = F_0$, which solve the evolution Eq. (9) for $t \in [0, T^*]$ and whose tracefree parts $A^0_{f_t}(x)$ of their second fundamental forms do not vanish in any $(x, t) \in \Sigma \times [0, T^*]$. The right-hand side of Eq. (9) has the specific property that any family $\{f_t\}$ of umbilic-free C^4-immersions $f_t : \Sigma \longrightarrow \mathbb{R}^n$ solves Eq. (9) if and only if its composition $\Phi(f_t)$ with any applicable Möbius-transformation Φ of $\mathbb{R}^n$ solves Eq. (9) as well.

Keywords Möbius-invariance · Willmore flow · Short-time existence · Parabolic Schauder theory

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1 Introduction and Main Result

The starting point of this article is the L^2-gradient-flow

$$\partial_t f_t = -\frac{1}{2}\big(\triangle^{\perp}_{f_t} H_{f_t} + Q(A^0_{f_t})(H_{f_t})\big) =: -\delta\mathcal{W}(f_t) \tag{1}$$

of the Willmore-functional

$$\mathcal{W}(f) := \frac{1}{4}\int_{\Sigma} \mid H_f \mid^2 \, d\mu_f,$$

defined on $W^{4,2}$-immersions $f : \Sigma \longrightarrow \mathbb{R}^n$ of a smooth, compact, orientable surface Σ without boundary into some $\mathbb{R}^n$. Here A_f denotes the second fundamental form of the immersion f, defined on pairs of tangent vector fields X, Y on Σ by

$$A_f(X, Y) := D_X(D_Y(f)) - P^{\mathrm{Tan}(f)}(D_X(D_Y(f))) \equiv (D_X(D_Y(f)))^{\perp_f}, \tag{2}$$

where $D_X(f) := Df(X)$ is the usual derivative of f in direction of X, $P^{\mathrm{Tan}(f)} : \mathbb{R}^n \to \mathrm{Tan}(f)$ denotes the bundle morphism which projects $\mathbb{R}^n$ orthogonally onto the tangent spaces $\mathrm{Tan}_{f(x)}(f(\Sigma))$ of the immersed surface $f(\Sigma)$ and where ${}^{\perp_f}$ abbreviates the bundle morphism $\mathrm{Id}_{\mathbb{R}^n} - P^{\mathrm{Tan}(f)}$. Furthermore, A^0_f denotes the tracefree part of A_f, i.e.

$$A^0_f(X, Y) := A_f(X, Y) - \frac{1}{2} g_f(X, Y)\, H_f,$$

and $H_f := \mathrm{Trace}(A_f) \equiv A_f(e_i, e_i)$ ("Einstein's summation convention") denotes the mean curvature of f, where $\{e_i\}$ is a local orthonormal frame of the tangent bundle of Σ. Finally $Q(A_f)$ operates on vector fields ϕ which are sections into the normal bundle of f, i.e. which are normal along f, by assigning $Q(A_f)(\phi) := A_f(e_i, e_j)\langle A_f(e_i, e_j), \phi\rangle$, which is again a section into the normal bundle of f, by definition of A_f. In fact, (1) is the L^2-gradient-flow of $\mathcal{W}$ since it is proved in [9] that for any vector field $\phi \in W^{4,2}(\Sigma, \mathbb{R}^n)$ which is normal along f, i.e. which satisfies $(\phi)^{\perp_f} = \phi$, there holds

$$\begin{aligned}\delta\mathcal{W}(f, \phi) &:= \frac{d}{ds}\mathcal{W}(f + s\phi)\mid_{s=0} = \frac{1}{2}\int_{\Sigma} \Big\langle \triangle^{\perp}_f H_f + Q(A^0_f)(H_f), \phi\Big\rangle d\mu_f \\ &= \int_{\Sigma} \langle \delta\mathcal{W}(f), \phi\rangle \, d\mu_f.\end{aligned} \tag{3}$$

Now, we consider a smooth Riemannian manifold M, endowed with a smooth metric g, and a two-dimensional smooth submanifold N of M, denote by ∇^g the unique Riemannian connection on M and by $A_g(X, Y) := (\nabla^g_X(Y))^{\perp_N}$ the second fundamental form of the pair $N \hookrightarrow M$, for any two tangent vector fields X, Y on N. We introduce a local smooth chart $\psi : \Omega \xrightarrow{\cong} N' \subset N$ of a coordinate neighbourhood N' of N, yielding partial derivatives ∂_1, ∂_2 on N', and denote $(A_g)_{ij} := A_g(\partial_i, \partial_j)$, $i, j = 1, 2$. One can easily verify the following transformation formulae

$$(A_g^0)_{ij} := A_g^0(\partial_i, \partial_j) = A_{\bar{g}}^0(\partial_i, \partial_j) =: (A_{\bar{g}}^0)_{ij} \tag{4}$$

and

$$\begin{aligned} \mid A_g^0 \mid^2 \sqrt{g} &:= g^{ik}\, g^{jl}\, g((A_g^0)_{ij}, (A_g^0)_{kl}) \sqrt{g} \\ &= \bar{g}^{ik}\, \bar{g}^{jl}\, \bar{g}((A_{\bar{g}}^0)_{ij}, (A_{\bar{g}}^0)_{kl}) \sqrt{\bar{g}} = \mid A_{\bar{g}}^0 \mid^2 \sqrt{\bar{g}} \end{aligned} \tag{5}$$

for the tracefree part A_g^0 of the second fundamental form A_g of $N \hookrightarrow M$ and for the product of their squared lengths $\mid A_g^0 \mid^2$ with $\sqrt{g} := \sqrt{\det(g(\partial_i, \partial_j))}$ subject to a conformal change of the metric, thus for a change of the Riemannian metric g to the Riemannian metric $\bar{g} := e^{2u}\, g$ (see [7], Chap. II, and [4]). By (2), (10)–(12) and these two formulae one can compute the following three results (see also Sect. 3, Proposition 13.6 and Lemma 13.7 in [12]):

Lemma 1 (1) *Let Σ be a smooth, compact, orientable surface without boundary. The Willmore-functional $\mathcal{W}$ is a "conformal invariant" on the set of C^4-immersions of Σ into $\mathbb{R}^n$. Precisely, this means that for any immersion $f \in C^4(\Sigma, \mathbb{R}^n)$ and for any conformal map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, for which $\Phi \circ f$ is well defined on Σ, there holds*

$$\mathcal{W}(\Phi(f)) = \mathcal{W}(f). \tag{6}$$

(2) *For scaling an immersion $f \in C^4(\Sigma, \mathbb{R}^n)$ by $\Phi(y) := \lambda\, y$, $\lambda \in \mathbb{R}_+$, there holds*

$$\delta\mathcal{W}(\Phi(f)) = \lambda^{-3} \delta\mathcal{W}(f)$$

on Σ, and for any immersion $f \in C^4(\Sigma, \mathbb{R}^n \setminus \{0\})$ there holds for the inversion $\Phi(y) := \frac{y}{|y|^2}$:

$$\delta\mathcal{W}(\Phi(f)) = \mid f \mid^8 \, D\Phi(f)(\delta\mathcal{W}(f))$$

on Σ, where $D\Phi(y) = \frac{1}{|y|^2}\big(\big(\delta_{ij} - 2\frac{y_i\, y_j}{|y|^2}\big)_{i,j=1,\dots,n}\big)$ and where $D\Phi(f(x))$ denotes the evaluation of this Jacobi-matrix in some image point $f(x)$ of the surface f, and $\delta\mathcal{W}(f) := \frac{1}{2}\big(\triangle_f^{\perp} H_f + Q(A_f^0)(H_f)\big)$ as in (1) or (3).

(3) *The differential operator $f \mapsto \mid A_f^0 \mid^{-4} \delta\mathcal{W}(f)$ (of fourth order) transforms "conformally invariantly" on umbilic-free immersions of any fixed compact torus into $\mathbb{R}^n$. Precisely this means: Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a umbilic-free C^4-immersion of a fixed compact torus into $\mathbb{R}^n$ and Φ be an arbitrary Möbius-transformation of $\mathbb{R}^n$ for which $\Phi \circ f$ is well defined on Σ. Then there holds the following transformation formula:*

$$\mid A_{\Phi(f)}^0 \mid^{-4} \delta\mathcal{W}(\Phi(f)) = D\Phi(f) \cdot \Big(\mid A_f^0 \mid^{-4} \delta\mathcal{W}(f) \Big). \tag{7}$$

Here in part (3), the surface Σ has to be confined to the class of smooth compact tori, since the assumption on f not to have any umbilic points on Σ forces the Euler-characteristic of Σ to vanish on account of the general Poincaré–Hopf Theorem for smooth sections with isolated zeroes into vector bundles over orientable, compact manifolds. Since for the differential operator ∂_t the chain rule applied to continuously differentiable families $\{f_t\}$ of C^4-immersions yields the same transformation formula as in (7), i.e. $\partial_t(\Phi(f_t)) = D\Phi(f_t) \cdot \partial_t(f_t)$, we achieve the following corollary of Lemma 1:

Corollary 1 (1) *If a family* $\{f_t\}$ *of* C^4*-immersions* $f_t : \Sigma \longrightarrow \mathbb{R}^n \setminus \{0\}$ *solves the Willmore flow equation* (1) *for* $t \in [0, T)$*, then its inversion* $\Phi(f_t) := \frac{f_t}{|f_t|^2}$ *solves the "inverse Willmore flow equation"*

$$\partial_t u_t = -\frac{1}{2} \mid u_t \mid^8 \left(\triangle^\perp_{u_t} H_{u_t} + Q(A^0_{u_t})(H_{u_t})\right) \equiv - \mid u_t \mid^8 \delta\mathcal{W}(u_t) \tag{8}$$

on $\Sigma \times [0, T)$.

(2) *Any family* $\{f_t\}$ *of* C^4*-immersions* $f_t : \Sigma \longrightarrow \mathbb{R}^n$ *without umbilic points, i.e. with* $\mid A^0_{f_t} \mid^2 > 0$ *on* Σ $\forall\, t \in [0, T)$*, solves the flow equation*

$$\begin{aligned} \partial_t f_t &= -\frac{1}{2} \mid A^0_{f_t} \mid^{-4} \left(\triangle^\perp_{f_t} H_{f_t} + Q(A^0_{f_t})(H_{f_t})\right) \\ &\equiv - \mid A^0_{f_t} \mid^{-4} \delta\mathcal{W}(f_t) \end{aligned} \tag{9}$$

if and only if its composition $\Phi(f_t)$ *with any applicable Möbius-transformation* Φ *of* $\mathbb{R}^n$ *solves the same flow equation and, thus, if and only if*

$$\partial_t(\Phi(f_t)) = - \mid A^0_{\Phi(f_t)} \mid^{-4} \delta\mathcal{W}(\Phi(f_t))$$

holds $\forall\, t \in [0, T)$ *and for every* $\Phi \in \text{Möb}(\mathbb{R}^n)$ *for which* $\Phi(f_t)$ *is well defined on* $\Sigma \times [0, T)$.

Part (2) of this corollary suggests to term the flow (9) "Möbius-invariant Willmore flow" (MIWF) and to find sufficient conditions on its initial immersion f_0 for its short-time existence and uniqueness, and also sufficient conditions for its global existence. So far, there has not been achieved any result at all about this flow. In [12] Mayer was able to obtain local bounds for the L^∞-norm of the second fundamental form A_{f_t} and its higher derivatives up to the existence time $T > 0$ of any maximal solution $\{f_t\}$ of the "inverse Willmore flow equation" (8) under appropriate smallness assumptions about the L^2-norms of A_{f_t} and its derivatives and about the L^∞-norm of f_t on small balls up to T and to use them in order to achieve a lower bound c for the maximal existence time T and an upper bound for the L^∞-norm of f_t up to the time $t = c$, which only depend on local L^2-bounds for A_{f_0} and its second derivatives and on the L^∞-norm of the initial surface f_0. The first achievement about the "MIWF" (9) consists of the proof of unique short-time existence for smooth initial data:

Theorem 1 (Main result) *Let Σ be a smooth compact torus. If $F_0 : \Sigma \longrightarrow \mathbb{R}^n$ is a C^∞-smooth immersion without umbilic points, thus with $| A^0_{F_0} |^2 > 0$ on Σ, then there exists some $T^* > 0$ such that the Möbius-invariant Willmore flow* (9) *possesses a unique solution $\{f_t\}$ on $\Sigma \times [0, T^*]$, depending smoothly on $t \in [0, T^*]$, which consists of C^∞-smooth, umbilic-free immersions $f_t : \Sigma \longrightarrow \mathbb{R}^n$, starting in F_0.*

We shall see below that the main tools of the existence proof are an adaption of the "DeTurck-Hamilton-Trick" (see also pp. 38–39 in [1] or the original source in [5] applied to Hamilton's investigation of the Ricci-flow in [8]) combined with parabolic Schauder a priori estimates for linear, uniformly parabolic operators L of fourth order "in diagonal form" with $C^{\alpha,\frac{\alpha}{4}}$-coefficients—which can be derived from Theorems 1,2 and 4,5 in [13]—the continuity method and the fact that the biharmonic heat operator

$$\partial_t + \triangle_G^2 : \{\{G_t\} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \mid G_0 = 0 \text{ on } \Sigma\}$$
$$\stackrel{\cong}{\longrightarrow} C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$$

yields an isomorphism, for any fixed smooth immersion G of the torus Σ into $\mathbb{R}^n$ and for any fixed $\alpha \in (0,1)$. For the definition of parabolic Hölder spaces, the reader is referred to pp. 44–45 in [6] and pp. 18–19 in [1]. In the final step, we shall adapt the "DeTurck-Hamilton-Trick" once again in order to prove *uniqueness* of the MIWF with given smooth start immersion. It should be pointed out that one could also use parabolic L^2- and Fredholm Theory, as developed by Mantegazza and Martinazzi in [11], instead of the author's "Schauder approach" in order to gain both the existence and uniqueness result of point (2) of Theorem 3 below. In a forthcoming article, the author plans to show a sufficient condition for global existence of the MIWF, to examine its singularities and to discuss some of its applications. In the appendix of this article the author explains, why the right-hand side of (9) is the "simplest" modification of the usual Willmore flow in order to achieve the desired "Möbius-invariance" of the resulting modified flow.

2 Preparations for the Proof of Theorem 1

For any fixed C^2-immersion $G : \Sigma \to \mathbb{R}^n$ and a smooth chart ψ of an arbitrary coordinate neighbourhood Σ' of a fixed smooth compact torus Σ, we will denote throughout this article the resulting partial derivatives on Σ' by ∂_i, $i = 1, 2$, the coefficients $g_{ij} := \langle \partial_i G, \partial_j G \rangle$ of the first fundamental form of G w.r.t. ψ and the associated Christoffel-symbols $(\Gamma_G)^m_{kl} := g^{mj} \langle \partial_{kl} G, \partial_j G \rangle$ of $(\Sigma', G^*(g_{\text{eu}}))$. Moreover, we define the first (covariant) derivatives by $\nabla_i^G(V) := \nabla_{\partial_i}^G(V) := \partial_i(V)$, $i = 1, 2$, and the second covariant derivatives by

$$\nabla^G_{kl}(V) \equiv \nabla^G_k \nabla^G_l(V) := \partial_{kl} V - (\Gamma_G)^m_{kl}\, \partial_m V \tag{10}$$

of any function $V \in C^2(\Sigma, \mathbb{R})$. Moreover, for any vector field $V \in C^2(\Sigma, \mathbb{R}^n)$ we define the projections of its first derivatives onto the normal bundle of the immersed torus $G(\Sigma)$ by

$$\nabla_i^{\perp_G}(V) \equiv (\nabla_i^G(V))^{\perp_G} := \nabla_i^G(V) - P^{\mathrm{Tan}(G)}(\nabla_i^G(V))$$

and the "normal second covariant derivatives" of V w.r.t. the immersion G by

$$\nabla_k^{\perp_G}\nabla_l^{\perp_G}(V) := \nabla_k^{\perp_G}(\nabla_l^{\perp_G}(V)) - (\Gamma_G)_{kl}^m \, \nabla_m^{\perp_G}(V).$$

Using these terms, we define the Beltrami–Laplacian w.r.t. G by $\triangle_G(V) := g^{kl}\nabla_{kl}^G(V)$, its projection $(\triangle_G V)^{\perp_G} := \left(g^{kl}\nabla_k^G\nabla_l^G(V)\right)^{\perp_G}$ onto the normal bundle of the surface $G(\Sigma)$ and the "normal Beltrami–Laplacian" by $\triangle_G^{\perp_G}(V) := g^{kl}\nabla_k^{\perp_G}\nabla_l^{\perp_G}(V)$. We shall note here, that Eqs. (2) and (10) together imply

$$(A_G)_{ij} = A_G(\partial_i, \partial_j) = \partial_{ij}G - (\Gamma_G)_{ij}^m \, \partial_m G = \nabla_i^G\nabla_j^G(G), \tag{11}$$

which shows that the second fundamental form A_G is a covariant tensor field of degree 2 and that there holds

$$H_G = g^{ij}\,(A_G)_{ij} = g^{ij}\,\nabla_i^G\nabla_j^G(G) = \triangle_G(G) \tag{12}$$

for the mean curvature of the immersion G.
The main problem about Eq. (9) is its non-parabolicity. We have

$$\triangle_f^{\perp} H_f + Q(A_f^0)(H_f) = (\triangle_f H_f)^{\perp_f} + 2\,Q(A_f)(H_f) - \frac{1}{2}\mid H_f\mid^2\; H_f$$

and by (12)

$$\begin{aligned}(\triangle_f H_f)^{\perp_f} &= g_f^{ij}\,g_f^{kl}\,\nabla_i^f\nabla_j^f\nabla_k^f\nabla_l^f(f)\\ &\quad - g_f^{ij}\,g_f^{kl}\,\langle\nabla_i^f\nabla_j^f\nabla_k^f\nabla_l^f(f), \partial_m f\rangle\, g_f^{mr}\,\partial_r(f)\end{aligned} \tag{13}$$

for any C^4-immersion $f : \Sigma \to \mathbb{R}^n$, which shows that the leading operator of the right-hand side of (9), i.e. of $\mid A_{f_t}^0\mid^{-4}\;\delta\mathcal{W}(f_t)$, is not uniformly elliptic (of fourth order), even if $\mid A_{f_t}^0\mid^2$ should stay positive on the torus Σ for all times $t \in [0, T]$. In order to overcome this unpleasant obstruction, we are going to adapt the "DeTurck-Hamilton-Trick" (see also pp. 38–39 in [1]), i.e. we fix some further C^4-immersion G of Σ into $\mathbb{R}^n$ and compute

$$\begin{aligned}\nabla_k^f\nabla_l^f(f) &= (\partial_{kl} f - (\Gamma_G)_{kl}^m\partial_m(f)) + C_{kl}^m(f, G)\,\partial_m(f)\\ &= \nabla_k^G\nabla_l^G(f) + C_{kl}^m(f, G)\,\partial_m(f)\end{aligned}$$

for $C_{kl}^m(f, G) := ((\Gamma_G)_{kl}^m - (\Gamma_f)_{kl}^m)$ on Σ'. It is important to note here that the difference $(\Gamma_G)_{kl}^m - (\Gamma_f)_{kl}^m$ is a tensor field of third degree and that therefore the difference

$$g_f^{ij}\,g_f^{kl}\left(\nabla_i^f\nabla_j^f\nabla_k^f\nabla_l^f(f) - \nabla_i^f\nabla_j^f(C_{kl}^m(f, G)\,\partial_m(f))\right) = g_f^{ij}\,g_f^{kl}\;\nabla_i^f\nabla_j^f\nabla_k^G\nabla_l^G(f)$$

is a "scalar", i.e. does not depend on the choice of the local chart ψ and thus yields a well-defined differential operator of fourth order on C^4-immersions $f : \Sigma \to \mathbb{R}^n$ again. Moreover, one can easily verify by (10) and the derivation formulae in (20) that $f \mapsto g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^G \nabla_l^G(f)$ is a non-linear operator of fourth order whose leading term is $g_f^{ij}\, g_f^{kl} \nabla_i^G \nabla_j^G \nabla_k^G \nabla_l^G(f)$, which is uniformly elliptic in the sense of (47) below, if $\| f - G \|_{C^1(\Sigma)}$ is sufficiently small. Neglecting derivatives of f_t of order < 4 we are thus led to firstly consider the evolution equation

$$\begin{aligned}\partial_t(f_t) &= -\frac{1}{2} \mid A^0_{f_t} \mid^{-4} \Big(2\,\delta\mathcal{W}(f_t) + g^{ij}_{f_t} g^{kl}_{f_t} g^{mr}_{f_t} \langle \nabla_i^{f_t} \nabla_j^{f_t} \nabla_k^{f_t} \nabla_l^{f_t}(f_t), \partial_m f_t\rangle\, \partial_r f_t \\ &\quad - g^{ij}_{f_t} g^{kl}_{f_t}\, \nabla_i^{f_t} \nabla_j^{f_t}\, ((\Gamma_{F_0})^m_{kl} - (\Gamma_{f_t})^m_{kl})\, \partial_m(f_t)\Big) \\ &=: \mathcal{D}_{F_0}(f_t),\end{aligned} \tag{14}$$

for any C^∞-smooth start immersion $F_0 : \Sigma \to \mathbb{R}^n$ satisfying $\mid A^0_{F_0} \mid > 0$ on Σ. The right-hand side $\mathcal{D}_{F_0}(f_t)$ of (14) can be expressed by

$$\begin{aligned}\mathcal{D}_{F_0}(f_t)(x) = -\frac{1}{2} \mid A^0_{f_t} \mid^{-4}\; g^{ij}_{f_t}\, g^{kl}_{f_t}\, \nabla_i^{f_t} \nabla_j^{f_t} \nabla_k^{F_0} \nabla_l^{F_0}(f_t)(x) \\ + B(x, D_x f_t, D_x^2 f_t, D_x^3 f_t),\end{aligned} \tag{15}$$

for $(x, t) \in \Sigma' \times [0, T]$, where the symbols $D_x f_t,\, D_x^2 f_t,\, D_x^3 f_t$ abbreviate the matrix-valued functions $(\partial_1 f_t, \partial_2 f_t)$, $(\nabla^{F_0}_{ij} f_t)_{i,j\in\{1,2\}}$ and $(\nabla^{F_0}_{ijk} f_t)_{i,j,k\in\{1,2\}}$ and where the n components of the lower order term $B(\,\cdot\,, D_x f_t, D_x^2 f_t, D_x^3 f_t)$ are "scalars", i.e. functions on $\Sigma' \times [0, T]$ that do not depend on the choice of the chart ψ of Σ'. Similarly as in the proof of Theorem 2 one can infer from this fact that there has to exist some well-defined function $B : \Sigma \times \mathbb{R}^{2n} \times \mathbb{R}^{4n} \times \mathbb{R}^{8n} \to \mathbb{R}^n$ whose n components are rational functions in their $14n$ real variables, such that (15) holds "globally" for any pair $(x, t) \in \Sigma \times [0, T]$. Hence, we arrive at the quasi-linear initial value problem

$$\begin{aligned}\partial_t(f_t) &= -\frac{1}{2} \mid A^0_{f_t} \mid^{-4}\; g^{ij}_{f_t}\, g^{kl}_{f_t}\, \nabla_i^{f_t} \nabla_j^{f_t} \nabla_k^{F_0} \nabla_l^{F_0}(f_t) + B(\,\cdot\,, D_x f_t, D_x^2 f_t, D_x^3 f_t), \\ f_0 &= F_0 \quad \text{on } \Sigma,\end{aligned} \tag{16}$$

of fourth order for C^4-immersions $f_t : \Sigma \to \mathbb{R}^n$, which is uniformly parabolic as long as $\| f_t - F_0 \|_{C^4(\Sigma)}$ remains sufficiently small, i.e. smaller than some $\delta > 0$ for any $t \in [0, T]$, since this guarantees that $\mid A^0_{f_t} \mid^2 > 0$ on Σ for $t \in [0, T]$, because of the assumption on F_0 to satisfy $\mid A^0_{F_0} \mid^2 > 0$ on the compact manifold Σ. This motivates to look for a short-time solution of (16) either within the open, convex subset

$$\begin{aligned}X_{F_0,\beta,\delta,T} := \Big\{ &\{f_t\} \in C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n) \mid \| f_t - F_0 \|_{C^4(\Sigma)} < \delta \text{ for } t \in [0, T], \\ &f_0 = F_0 \text{ on } \Sigma \Big\}\end{aligned}$$

of the affine closed subspace

$$Y_{\beta,T,F_0} := \{\{f_t\} \in C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \mid f_0 = F_0 \quad \text{on} \quad \Sigma\}$$

of the Banach space $C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ or within the closed convex subset

$$\tilde{X}_{F_0,\beta,\delta,T} := \Big\{\{f_t\} \in C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \mid \| f_t - F_0 \|_{C^4(\Sigma)} \leq \delta \text{ for } t \in [0,T], \\ f_0 = F_0 \quad \text{on} \quad \Sigma\Big\}$$

of $C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$, for some sufficiently small $\delta > 0$ and some arbitrary $\beta \in (0,1)$.

Now, since the operator $\mathcal{D}_{F_0}$ does not act linearly on the family $\{f_t\}$, one cannot immediately apply the standard theory for linear parabolic systems—as presented in [10], Chapters 4 and 5, or in Chap. 3 of [1] for linear parabolic systems of second resp. fourth order—in order to achieve existence of some solution of (16) on some sufficiently short time interval $[0, T^*]$. A first strategy to overcome this problem might consist in proving that for any element $\{G_t\} \in \tilde{X}_{F_0,\beta,\delta,T}$ the unique solution $\{f_t\}$ of the semi-linear equation

$$\partial_t(f_t) = -\frac{1}{2} \mid A^0_{G_t} \mid^{-4} \; g^{ij}_{G_t}\, g^{kl}_{G_t}\, \nabla^{G_t}_i \nabla^{G_t}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t) + B(x, D_x G_t, D^2_x G_t, D^3_x G_t) \tag{17}$$

with $f_0 = F_0$ on Σ is again contained in $\tilde{X}_{F_0,\beta,\delta,T}$ if $T > 0$ is chosen sufficiently small, which would define the map $\Psi : \tilde{X}_{F_0,\beta,\delta,T} \to \tilde{X}_{F_0,\beta,\delta,T}$, mapping $\{G_t\}$ onto the solution $\{f_t\}$, and then proving continuity of Ψ and that the closure of $\Psi(\tilde{X}_{F_0,\beta,\delta,T})$ is a compact subset of $C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ in order to derive the existence of some fixed point of Ψ, which would in fact be a solution of problem (16) on $\Sigma \times [0,T]$. But unfortunately, it is impossible to guarantee—by the use of the a priori estimates in Corollary 3 or in Proposition 3 below—the existence of some $T > 0$ such that the solution $\{f_t\}$ of equation (17) for every (!) given $\{G_t\} \in \tilde{X}_{F_0,\beta,\delta,T}$ again satisfies $\| f_t - F_0 \|_{C^4(\Sigma)} \leq \delta$ for all $t \in [0,T]$, implying $\{f_t\} \in \tilde{X}_{F_0,\beta,\delta,T}$ again. Hence, the construction of the map Ψ breaks down from the very beginning.

A more refined strategy is to consider the non-linear map $\phi : X_{F_0,\beta,\delta,T} \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$, defined by

$$\begin{aligned} \phi(f_t) &:= \partial_t(f_t) - \mathcal{D}_{F_0}(f_t) \\ &\equiv \partial_t(f_t) + \frac{1}{2} \mid A^0_{f_t} \mid^{-4} \; g^{ij}_{f_t} g^{kl}_{f_t}\, \nabla^{f_t}_i \nabla^{f_t}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t) \\ &\quad - B(x, D_x f_t, D^2_x f_t, D^3_x f_t), \end{aligned} \tag{18}$$

for some sufficiently small $\delta > 0$ and some arbitrarily fixed $T > 0$ and $\beta \in (0,1)$, and to prove the following two theorems, Theorems 2 and 3:

Theorem 2 *Let $F_0 : \Sigma \longrightarrow \mathbb{R}^n$ be a C^∞-smooth immersion of a smooth compact torus into $\mathbb{R}^n$ without umbilic points, thus with $\mid A^0_{F_0} \mid^2 > 0$ on Σ, and let $\beta \in (0, 1)$ and $T > 0$ be arbitrarily fixed. There is a sufficiently small $\delta > 0$ such that the following three statements hold:*

(1) *The map $\phi : X_{F_0,\beta,\delta,T} \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$, defined in (18), is of class C^1 on the open subset $X_{F_0,\beta,\delta,T} \subset Y_{\beta,T,F_0}$.*

(2) *In any fixed element $\{f_t\} \in X_{F_0,\beta,\delta,T}$ the Fréchet-derivative of ϕ is a linear, uniformly parabolic operator of order 4 whose leading operator of fourth order acts on each component of $f = \{f_t\}$ separately:*

$$D\phi(f).\eta = \partial_t(\eta) + \frac{1}{2} \mid A^0_{f_t} \mid^{-4} \; g^{ij}_{f_t}\, g^{kl}_{f_t}\, \nabla^{F_0}_{ijkl}(\eta)$$
$$+B^{ijk}_3 \cdot \nabla^{F_0}_{ijk}(\eta) + B^{ij}_2 \cdot \nabla^{F_0}_{ij}(\eta) + B^i_1 \cdot \nabla^{F_0}_i(\eta), \qquad (19)$$

on $\Sigma \times [0, T]$, for any element $\eta = \{\eta_t\}$ of the tangent space

$$T_f X_{F_0,\beta,\delta,T} = Y_{\beta,T,0} := \{\{\eta_t\} \in C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n) \mid \eta_0 = 0 \text{ on } \Sigma\},$$

where B^{ijk}_3, B^{ij}_2, B^i_1 are the coefficients of $\mathrm{Mat}_{n,n}(\mathbb{R})$-valued, contravariant tensor fields of degrees 3, 2 and 1, which depend on x, $D_x f_t$, $D^2_x f_t$, $D^3_x f_t$ and $D^4_x f_t$ and are of class $C^{\beta,\frac{\beta}{4}}$ on $\Sigma \times [0, T]$.

(3) *The Fréchet-derivative of ϕ yields an isomorphism*

$$D\phi(f) : Y_{\beta,T,0} \xrightarrow{\cong} C^{\beta,\frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$$

in any fixed $f \in X_{F_0,\beta,\delta,T}$.

Proof (1) We fix an arbitrary smooth chart $\psi : \Omega \xrightarrow{\cong} \Sigma'$ of an arbitrary coordinate neighbourhood Σ' of Σ, which yields partial derivatives ∂_m, $m = 1, 2$, on Σ'. For any C^2-immersion $G : \Sigma \to \mathbb{R}^n$ the choice of ψ yields the coefficients $g_{ij} := \langle \partial_i G, \partial_j G\rangle$ of the first fundamental form of G w.r.t. ψ and the associated Christoffel-symbols $(\Gamma_G)^m_{kl} := g^{mj} \langle \partial_{kl} G, \partial_j G\rangle$ of $(\Sigma', G^*(g_{\mathrm{eu}}))$. On account of (10) and by the general derivation formulae

$$\nabla^G_i(\omega_k) = \partial_i(\omega_k) - (\Gamma_G)^m_{ik}\omega_m$$
$$\nabla^G_i(\lambda_{jk}) = \partial_i(\lambda_{jk}) - (\Gamma_G)^m_{ij}\, \lambda_{mk} - (\Gamma_G)^m_{ik}\, \lambda_{jm}$$
$$\nabla^G_i(\zeta_{jkl}) = \partial_i(\zeta_{jkl}) - (\Gamma_G)^m_{ij}\, \zeta_{mkl} - (\Gamma_G)^m_{ik}\, \zeta_{jml} - (\Gamma_G)^m_{il}\, \zeta_{jkm} \qquad (20)$$

on Σ' for the coefficients ω_k, λ_{jk} and ζ_{jkl} (w.r.t. to the chart ψ) of covariant C^1-tensor fields ω, λ and ζ of degrees 1, 2 and 3, one can verify that for any C^4-immersion $f : \Sigma \to \mathbb{R}^n$, for any fixed C^∞-immersion $G : \Sigma \to \mathbb{R}^n$ and for fixed $i, j, k, l \in \{1, 2\}$ there is a unique rational function $P^G_{(ijkl)} \in C^\infty(\Sigma')[v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}, y_1, \ldots, y_{8n}]$ (with n components) in $2n + 4n +$

$8n = 14n$ real variables, whose coefficients are smooth functions which are rational functions of the partial derivatives $\partial_i(G^1), \dots, \partial_i(G^n), \partial_{ij}(G^1), \dots, \partial_{ij}(G^n), \partial_{ijk}(G^1), \dots, \partial_{ijk}(G^n)$ of the components of G up to third order, such that

$$\nabla_i^f \nabla_j^f \nabla_k^G \nabla_l^G(f) = \nabla_i^G \nabla_j^G \nabla_k^G \nabla_l^G(f) \\ +P^G_{(ijkl)}(\nabla_1^G(f^1), \dots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \dots, \nabla_{22}^G(f^n), \nabla_{111}^G(f^1), \dots, \nabla_{222}^G(f^n)) \tag{21}$$

holds on Σ'. We note here, that the terms
$P^G_{(ijkl)}(\nabla_1^G(f^1), \dots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \dots, \nabla_{22}^G(f^n), \nabla_{111}^G(f^1), \dots, \nabla_{222}^G(f^n))$
must be the coefficients of a covariant tensor field of fourth degree on Σ'. Moreover, since there holds

$$| A_f^0 |^2 = g_f^{ik}\, g_f^{jl}\, \langle (A_f^0)_{ij}, (A_f^0)_{kl} \rangle$$

and $(A_f)_{ij} = A_f(\partial_i, \partial_j) = \partial_{ij} f - (\Gamma_f)_{ij}^m\, \partial_m f$, one can verify only by (10) that there is a unique rational function $Q^G \in C^\infty(\Sigma')[v_1, \dots, v_{2n}, w_1, \dots, w_{4n}]$ in $6n$ real variables whose coefficients are smooth functions which are rational functions of the partial derivatives $\partial_i(G^1), \dots, \partial_i(G^n), \partial_{ij}(G^1), \dots, \partial_{ij}(G^n)$ of the components of G up to second order, such that

$$| A_f^0 |^4 = Q^G(\nabla_1^G(f^1), \nabla_1^G(f^2), \dots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \nabla_{11}^G(f^2), \dots, \nabla_{22}^G(f^n)) \tag{22}$$

holds on Σ', which is a "scalar", i.e which does not depend on the choice of the chart ψ of Σ'. Combining (21) and (22) and recalling that F_0 does not have any umbilic points on Σ, we obtain the existence of a unique rational function $R^{F_0} \in C^\infty(\Sigma')[v_1, \dots, v_{2n}, w_1, \dots, w_{4n}, y_1, \dots, y_{8n}]$ in $2n+4n+8n = 14n$ real variables, whose n components are rational functions in $14n$ real variables whose coefficients are rational functions of the partial derivatives of the components of F_0 up to third order, such that

$$\frac{1}{2} | A_{f_t}^0 |^{-4}\; g_{f_t}^{ij}\, g_{f_t}^{kl}\, \nabla_i^{f_t} \nabla_j^{f_t} \nabla_k^{F_0} \nabla_l^{F_0}(f_t) \\ = \frac{1}{2} | A_{f_t}^0 |^{-4}\; g_{f_t}^{ij}\, g_{f_t}^{kl}\, \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(f_t) \\ +R^{F_0}(\nabla_1^{F_0}(f_t^1), \dots, \nabla_2^{F_0}(f_t^n), \nabla_{11}^{F_0}(f_t^1), \dots, \nabla_{22}^{F_0}(f_t^n), \nabla_{111}^{F_0}(f_t^1), \\ \dots, \nabla_{222}^{F_0}(f_t^n)) \tag{23}$$

holds on $\Sigma' \times [0, T]$ for any family of immersions $\{f_t\} \in X_{F_0,\beta,\delta,T}$, if $\delta > 0$ is chosen sufficiently small. We should note here that the n components of $R^{F_0}(\nabla_1^{F_0}(f_t^1), \dots, \nabla_2^{F_0}(f_t^n), \nabla_{11}^{F_0}(f_t^1), \dots, \nabla_{22}^{F_0}(f_t^n), \nabla_{111}^{F_0}(f_t^1), \dots, \nabla_{222}^{F_0}(f_t^n))$ are "scalars" as well, i.e. do not depend on the choice of the chart ψ of Σ'. On account of the definition in (18) of the map ϕ we infer from (23) its representation

$$\begin{aligned}\phi(f_t)(x) &= \partial_t f_t(x) + \frac{1}{2} \mid A^0_{f_t}(x) \mid^{-4} \; g^{ij}_{f_t} \, g^{kl}_{f_t} \, \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t)(x) \\ &\quad + R^{F_0}(\nabla^{F_0}_1(f^1_t), \ldots, \nabla^{F_0}_2(f^n_t), \ldots, \nabla^{F_0}_{111}(f^1_t), \nabla^{F_0}_{111}(f^2_t), \ldots, \\ &\quad \nabla^{F_0}_{222}(f^n_t))(x) - B(x, D_x f_t(x), D^2_x f_t(x), D^3_x f_t(x)) \\ &= \partial_t f_t(x) + \frac{1}{2} \mid A^0_{f_t}(x) \mid^{-4} \; g^{ij}_{f_t} \, g^{kl}_{f_t} \, \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t)(x) \\ &\quad + \mathcal{F}(x, D_x f_t(x), D^2_x f_t(x), D^3_x f_t(x)) \end{aligned} \tag{24}$$

for any family of immersions $\{f_t\} \in X_{F_0,\beta,\delta,T}$ and for $(x, t) \in \Sigma' \times [0, T]$, if $\delta > 0$ is chosen sufficiently small. Here and in the sequel, the symbols $D_x f, D^2_x f, D^3_x f, \ldots$ abbreviate the matrix-valued functions $(\partial_1 f, \partial_2 f)$, $(\nabla^{F_0}_{ij} f)_{i,j\in\{1,2\}}$, $(\nabla^{F_0}_{ijk} f)_{i,j,k\in\{1,2\}}$, ... On account of the arbitrariness of the choice of the coordinate neighbourhood Σ' and its chart ψ and by the compactness of Σ, equation (24) gives rise to a unique and well-defined function $\mathcal{F} : \Sigma \times \mathbb{R}^{2n} \times \mathbb{R}^{4n} \times \mathbb{R}^{8n} \to \mathbb{R}^n$ whose n components are rational functions in their $14n$ real variables, such that (24) holds on $\Sigma \times [0, T]$, if $\delta > 0$ is chosen sufficiently small. Now, since F_0 is umbilic-free on Σ, i.e. since there holds $\mid A_{F_0} \mid^2 \geq c_0 > 0$ on Σ, and as Σ is compact, we know that the modulus of each denominator which appears in the fractions of $\mathcal{F}(x, D_x f_t(x), D^2_x f_t(x), D^3_x f_t(x))$ is bounded from below by some positive constant on $\Sigma \times [0, T]$, if $\{f_t\} \in X_{F_0,\beta,\delta,T}$ and $\delta > 0$ sufficiently small. Therefore, the rational function $(x, h) \mapsto \mathcal{F}(x, h)$ is of class C^∞ in an open neighbourhood $\mathcal{O}$ of the graph $\{(x, D_x F_0(x), \ldots, D^3_x F_0(x)) \mid x \in \Sigma\}$ in $\Sigma \times \mathbb{R}^M$, where we set $M := 14n$. For any $\delta > 0$ we define the open neighbourhood $\mathcal{V}_\delta$ of the function $(x, t) \mapsto (D_x F_0(x), \ldots, D^3_x F_0(x))$ in $C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M)$ by

$$h \in \mathcal{V}_\delta \Longleftrightarrow \| (D_x F_0, \ldots, D^3_x F_0) - h \|_{L^\infty(\Sigma\times[0,T])} < 2\delta.$$

[See pp. 44–45 in [6] for the definition of "$C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M)$".] Now, we choose $\delta > 0$ that small such that there holds $(x, h_t(x)) \in \mathcal{O}$ for any pair $(x, t) \in \Sigma \times [0, T]$ if $\{h_t\} \in \mathcal{V}_\delta$ and obtain that the partial derivatives of $\mathcal{F}$ w.r.t. the components of $h \in \mathbb{R}^M$ are of class C^∞ about every point $(x, h_t(x))$ $(\in \mathcal{O})$ if $\{h_t\}$ is contained in $\mathcal{V}_\delta$. Hence, fixing such a small $\delta > 0$, some $\{h_t\} \in \mathcal{V}_\delta$ and some $\eta = \{\eta_t\} \in C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M)$ with sufficiently small norm $0 < \| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^M)} < \tilde{\epsilon}$ we can derive from the classical mean value theorem:

$$\begin{aligned} &\mid D_h\mathcal{F}(x, h_t(x)) - D_h\mathcal{F}(x, h_t(x) + \eta_t(x)) \mid \\ &\quad \leq \mathrm{Const}(F_0, \delta, \tilde{\epsilon}) \mid \eta_t(x) \mid \leq \mathrm{Const}(F_0, \delta, \tilde{\epsilon}) \; \| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^M)} \end{aligned}$$

for any $t \in [0, T]$ and any $x \in \Sigma$, and thus (abbreviating $h := \{h_t\}$ and $\eta := \{\eta_t\}$)

$$\begin{aligned} &\| D_h\mathcal{F}(\,\cdot\,, h) - D_h\mathcal{F}(\,\cdot\,, h + \eta) \|_{L^\infty(\Sigma\times[0,T])} \\ &\quad \leq \mathrm{Const}(F_0, \delta, \tilde{\epsilon}) \; \| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^M)} . \end{aligned} \tag{25}$$

Moreover, as in the proof of the compactness of the embedding $C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])\hookrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])$, for $0<\beta<\alpha<1$, we can estimate for any fixed $x\in\Sigma$, any $\epsilon>0$ and any fixed $\alpha\in(\beta,1)$:

$$\begin{aligned}&\text{höl}_{\frac{\beta}{4}}\big(D_h\mathcal{F}(x,h_{(\,\cdot\,)}(x))-D_h\mathcal{F}(x,h_{(\,\cdot\,)}(x)+\eta_{(\,\cdot\,)}(x))\big)\\ &\le\epsilon^{\frac{\alpha-\beta}{4}}(\text{höl}_{\frac{\alpha}{4}}D_h\mathcal{F}(x,h_{(\,\cdot\,)}(x))+\text{höl}_{\frac{\alpha}{4}}D_h\mathcal{F}(x,h_{(\,\cdot\,)}(x)+\eta_{(\,\cdot\,)}(x)))\\ &\quad+2\,\epsilon^{-\frac{\beta}{4}}\ \|\ D_h\mathcal{F}(\,\cdot\,,h)-D_h\mathcal{F}(\,\cdot\,,h+\eta)\ \|_{L^\infty(\Sigma\times[0,T])}\end{aligned}\tag{26}$$

on $[0,T]$, and similarly for any fixed $t\in[0,T]$:

$$\begin{aligned}&\text{höl}_\beta\big(D_h\mathcal{F}(\,\cdot\,,h_t(\,\cdot\,))-D_h\mathcal{F}(\,\cdot\,,h_t(\,\cdot\,)+\eta_t(\,\cdot\,))\big)\\ &\quad\le\epsilon^{\alpha-\beta}(\text{höl}_\alpha D_h\mathcal{F}(\,\cdot\,,h_t(\,\cdot\,))+\text{höl}_\alpha D_h\mathcal{F}(\,\cdot\,,h_t(\,\cdot\,)+\eta_t(\,\cdot\,)))\\ &\quad+2\,\epsilon^{-\beta}\ \|\ D_h\mathcal{F}(\,\cdot\,,h)-D_h\mathcal{F}(\,\cdot\,,h+\eta)\ \|_{L^\infty(\Sigma\times[0,T])}\end{aligned}\tag{27}$$

on Σ. Combining (25), (26) and (27) we obtain again on account of the mean value theorem:

$$\begin{aligned}&\|\ D_h\mathcal{F}(\,\cdot\,,h)-D_h\mathcal{F}(\,\cdot\,,h+\eta)\ \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])}\\ &\quad=\|\ D_h\mathcal{F}(\,\cdot\,,h)-D_h\mathcal{F}(\,\cdot\,,h+\eta)\ \|_{L^\infty(\Sigma\times[0,T])}\\ &\qquad+\sup_{x\in\Sigma}\text{höl}_{\frac{\beta}{4}}(D_h\mathcal{F}(x,h_{(\,\cdot\,)}(x))-D_h\mathcal{F}(x,h_{(\,\cdot\,)}(x)+\eta_{(\,\cdot\,)}(x)))\\ &\qquad+\sup_{t\in[0,T]}\text{höl}_\beta(D_h\mathcal{F}(\,\cdot\,,h_t(\,\cdot\,))-D_h\mathcal{F}(\,\cdot\,,h_t(\,\cdot\,)+\eta_t(\,\cdot\,)))\\ &\quad\le\text{Const}(F_0,\delta,\tilde{\epsilon})\ \|\ \eta\ \|_{C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T])}\\ &\qquad+\epsilon^{\frac{\alpha-\beta}{4}}\ \text{Const}(F_0,\delta,\tilde{\epsilon})\Big(\ \|\ h\ \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])}+\|\ \eta\ \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])}\Big)\\ &\qquad+2\,\epsilon^{-\frac{\beta}{4}}\ \text{Const}(F_0,\delta,\tilde{\epsilon})\ \|\ \eta\ \|_{C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T])}\\ &\qquad+\epsilon^{\alpha-\beta}\ \text{Const}(F_0,\delta,\tilde{\epsilon})\Big(\epsilon^{1-\alpha}+\|\ h\ \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])}+\|\ \eta\ \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])}\Big)\\ &\qquad+2\,\epsilon^{-\beta}\ \text{Const}(F_0,\delta,\tilde{\epsilon})\ \|\ \eta\ \|_{C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T])}\\ &\quad\le\text{Const}(F_0,\delta,\tilde{\epsilon},h)\Big((1+\epsilon^{-\frac{\beta}{4}}+\epsilon^{-\beta})\ \|\ \eta\ \|_{C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T])}+\epsilon^{\frac{\alpha-\beta}{4}}\\ &\qquad+\epsilon^{\alpha-\beta}+\epsilon^{1-\beta}\Big),\end{aligned}\tag{28}$$

for any $\epsilon>0$ and any fixed $\alpha\in(\beta,1)$. Now, choosing exactly $\epsilon=\|\ \eta\ \|_{C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^M)}\in(0,\tilde{\epsilon})$ we arrive at the estimate:

$$\|\ D_h\mathcal{F}(\cdot,h)-D_h\mathcal{F}(\,\cdot\,,h+\eta)\ \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])}$$

$$\leq \text{Const}(F_0, \delta, \tilde{\epsilon}, h) \Bigg(\| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T])} + \| \eta \|^{1-\frac{\beta}{4}}_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T])}$$
$$+ \| \eta \|^{1-\beta}_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T])} + \| \eta \|^{\frac{\alpha-\beta}{4}}_{C^{1+\beta, \frac{\beta}{4}}(\Sigma\times[0,T])} + \| \eta \|^{\alpha-\beta}_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T])} \Bigg) \tag{29}$$

for any fixed $\alpha \in (\beta, 1)$. Now, denoting by

$$\mathcal{F}_\sharp : \mathcal{V}_\delta \subset C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M) \longrightarrow C^{\beta, \frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$$

the non-linear operator which maps $h = \{h_t\} \in \mathcal{V}_\delta$ to the function $\mathcal{F}(\,\cdot\,, h)$, we obtain by the classical mean value theorem for any $\eta = \{\eta_t\} \in C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M)$ with sufficiently small norm $0 < \| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^M)} < \tilde{\epsilon}$:

$$\begin{aligned}
&\mathcal{F}_\sharp(h + \eta) - \mathcal{F}_\sharp(h) - D_h\mathcal{F}(\,\cdot\,, h) \cdot \eta \\
&\quad = \mathcal{F}(\,\cdot\,, h + \eta) - \mathcal{F}(\,\cdot\,, h) - D_h\mathcal{F}(\,\cdot\,, h) \cdot \eta \\
&\quad = \int_0^1 D_h\mathcal{F}(\,\cdot\,, h + s\eta) \cdot \eta \, ds - D_h\mathcal{F}(\,\cdot\,, h) \cdot \eta \\
&\quad = \int_0^1 D_h\mathcal{F}(\,\cdot\,, h + s\eta) - D_h\mathcal{F}(\,\cdot\,, h) \, ds \cdot \eta
\end{aligned}$$

on $\Sigma \times [0, T]$, and therefore together with (29):

$$\begin{aligned}
&\| \mathcal{F}_\sharp(h + \eta) - \mathcal{F}_\sharp(h) - D_h\mathcal{F}(\,\cdot\,, h) \cdot \eta \|_{C^{\beta, \frac{\beta}{4}}(\Sigma\times[0,T])} \\
&\quad \leq M \int_0^1 \| D_h\mathcal{F}(\,\cdot\,, h+s\,\eta) - D_h\mathcal{F}(\,\cdot\,, h) \|_{C^{\beta, \frac{\beta}{4}}(\Sigma\times[0,T])} \, ds \; \| \eta \|_{C^{\beta, \frac{\beta}{4}}(\Sigma\times[0,T])} \\
&\quad \leq \text{Const}(F_0, \delta, \tilde{\epsilon}, h) \big(\| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}} + \| \eta \|^{1-\frac{\beta}{4}}_{C^{1+\beta, \frac{1+\beta}{4}}} + \| \eta \|^{1-\beta}_{C^{1+\beta, \frac{1+\beta}{4}}} \\
&\qquad + \| \eta \|^{\frac{\alpha-\beta}{4}}_{C^{1+\beta, \frac{1+\beta}{4}}} + \| \eta \|^{\alpha-\beta}_{C^{1+\beta, \frac{1+\beta}{4}}} \big) \cdot \| \eta \|_{C^{1+\beta, \frac{1+\beta}{4}}} .
\end{aligned}$$

This proves that the operator $\mathcal{F}_\sharp$ is Fréchet-differentiable in any fixed $h \in \mathcal{V}_\delta$ with Fréchet-derivative $D\mathcal{F}_\sharp(h) : C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M) \longrightarrow C^{\beta, \frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$ given by

$$D\mathcal{F}_\sharp(h).\eta = D_h\mathcal{F}(\,\cdot\,, h) \cdot \eta. \tag{30}$$

Moreover, (29) shows that the Fréchet-derivative $D\mathcal{F}_\sharp$ is continuous from $(\mathcal{V}_\delta, \| \cdot \|_{C^{1+\beta, \frac{1+\beta}{4}}(\Sigma\times[0,T])})$ into the Banach space of linear operators mapping $C^{1+\beta, \frac{1+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^M)$ to $C^{\beta, \frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$. Thus $\mathcal{F}_\sharp$ is a C^1-map from $(\mathcal{V}_\delta, \| \cdot$

$\|_{C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T])})$ to $C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T],\mathbb{R}^n)$. Denoting by $\mathcal{L}$ the linear mapping $f_t \mapsto (D_x f_t, D_x^2 f_t, D_x^3 f_t)$ we see that

$$\| (D_x F_0, \dots, D_x^3 F_0) - \mathcal{L}(f_t) \|_{L^\infty(\Sigma\times[0,T])} \leq \sup_{t\in[0,T]} \| F_0 - f_t \|_{C^3(\Sigma)} \leq \delta < 2\delta$$

for any family of surfaces $\{f_t\} \in X_{F_0,\beta,\delta,T}$, which shows that $\mathcal{L}$ maps $X_{F_0,\beta,\delta,T}$ into the neighbourhood $\mathcal{V}_\delta$.
Moreover, since F_0 is umbilic-free on Σ, i.e. since there holds $| A_{F_0} |^2 \geq c_0 > 0$ on Σ, and as $x \mapsto | A_f^0(x) |^2$ is a "scalar" on Σ for any C^2-immersion $f : \Sigma \to \mathbb{R}^n$, Eq. (22) yields a well-defined function $\mathcal{A} : \Sigma \times \mathbb{R}^{2n} \times \mathbb{R}^{4n} \to \mathbb{R}$, which is rational in its $6n$ real variables, such that there holds

$$\frac{1}{2} | A_f^0 |^{-4} = \mathcal{A}(\,\cdot\,, D_x f, D_x^2 f) \quad \text{on } \Sigma \tag{31}$$

for any C^2-immersion f with sufficiently small C^2-distance $\| f - F_0 \|_{C^2(\Sigma)}$. Again on account of $| A_{F_0} |^2 \geq c_0 > 0$ on Σ the rational function $(x,h) \mapsto \mathcal{A}(x,h)$ is of class C^∞ in an open neighbourhood $\mathcal{O}$ of the graph $\{(x, D_x F_0(x), D_x^2 F_0(x)) \mid x \in \Sigma\}$ in $\Sigma \times \mathbb{R}^M$, where we set now $M := 6n$. For any $\delta > 0$ we define the open neighbourhood $\tilde{\mathcal{V}}_\delta$ of the function $(x,t) \mapsto (D_x F_0(x), D_x^2 F_0(x))$ in $C^{2+\beta,\frac{2+\beta}{4}}(\Sigma \times [0,T],\mathbb{R}^M)$ by

$$h \in \tilde{\mathcal{V}}_\delta \Longleftrightarrow \| (D_x F_0, D_x^2 F_0) - h \|_{L^\infty(\Sigma\times[0,T])} < 2\delta.$$

Now, we choose $\delta > 0$ that small such that there holds $(x, h_t(x)) \in \mathcal{O}$ for any pair $(x,t) \in \Sigma \times [0,T]$ if $\{h_t\} \in \tilde{\mathcal{V}}_\delta$ and obtain that the partial derivatives of $\mathcal{A}$ w.r.t. the components of $h \in \mathbb{R}^M$ are of class C^∞ about every point $(x, h_t(x))$ $(\in \mathcal{O})$ if $\{h_t\}$ is contained in $\tilde{\mathcal{V}}_\delta$. Exactly as above we introduce the map $\mathcal{A}_\sharp : \tilde{\mathcal{V}}_\delta \subset C^{2+\beta,\frac{2+\beta}{4}}(\Sigma \times [0,T],\mathbb{R}^M) \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T],\mathbb{R})$ which maps $h = \{h_t\} \in \tilde{\mathcal{V}}_\delta$ to the function $\mathcal{A}(\,\cdot\,, h)$ and show by the above reasoning that the map $\mathcal{A}_\sharp$ is Fréchet-differentiable in any fixed $h \in \tilde{\mathcal{V}}_\delta$ with Fréchet-derivative $D\mathcal{A}_\sharp(h) : C^{2+\beta,\frac{2+\beta}{4}}(\Sigma \times [0,T],\mathbb{R}^M) \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T],\mathbb{R})$ given by

$$D\mathcal{A}_\sharp(h).\eta = D_h\mathcal{A}(\,\cdot\,, h) \cdot \eta \tag{32}$$

and that $D\mathcal{A}_\sharp$ is continuous from $(\tilde{\mathcal{V}}_\delta, \|\cdot\|_{C^{2+\beta,\frac{2+\beta}{4}}(\Sigma\times[0,T])})$ into the Banach space of linear operators mapping $C^{2+\beta,\frac{2+\beta}{4}}(\Sigma \times [0,T],\mathbb{R}^M)$ to $C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T],\mathbb{R})$, and thus that $\mathcal{A}_\sharp$ is a C^1-map from $(\tilde{\mathcal{V}}_\delta, \|\cdot\|_{C^{2+\beta,\frac{2+\beta}{4}}(\Sigma\times[0,T])})$ to $C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T],\mathbb{R})$.
We will need below that the linear map $\tilde{\mathcal{L}} : f_t \mapsto (D_x f_t, D_x^2 f_t)$ maps $X_{F_0,\beta,\delta,T}$ into the neighbourhood $\tilde{\mathcal{V}}_\delta$ on account of

$$\| (D_x F_0, D_x^2 F_0) - \tilde{\mathcal{L}}(f_t) \|_{L^\infty(\Sigma\times[0,T])} \leq \sup_{t\in[0,T]} \| F_0 - f_t \|_{C^2(\Sigma)} \leq \delta < 2\delta$$

for any family of surfaces $\{f_t\} \in X_{F_0,\beta,\delta,T}$. Finally, as the function

$$x \mapsto g_f^{ij}\, g_f^{kl}\, \nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}(f)(x)$$

is a scalar for any C^4-immersion $f : \Sigma \to \mathbb{R}^n$, there is a unique, well-defined function $\mathcal{Q} : \mathbb{R}^{2n} \times \mathbb{R}^{16n} \to \mathbb{R}^n$ whose n components are rational functions of its first $2n$ variables $(h^1, \dots, h^{2n}) =: \bar{h}$ and linear functions of its last $16n$ variables $(h^{2n+1}, \dots, h^{18n}) =: h^*$ such that there holds

$$g_f^{ij}\, g_f^{kl}\, \nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}(f) = \mathcal{Q}(D_x f, D_x^4 f) \quad \text{ on } \Sigma \tag{33}$$

for any C^4-immersion f. Firstly we infer from the "quasi-linearity" of $\mathcal{Q}$ that its derivative in some fixed point $h = (\bar{h}, h^*) \in \mathbb{R}^{2n} \times \mathbb{R}^{16n}$ in direction of an arbitrary $\eta = (\bar{\eta}, \eta^*) \in \mathbb{R}^{2n} \times \mathbb{R}^{16n}$ has the form

$$D_h \mathcal{Q}(h) \cdot \eta = D_{\bar{h}} \mathcal{Q}(\bar{h}, h^*) \cdot \bar{\eta} + \mathcal{Q}(\bar{h}, \eta^*). \tag{34}$$

As $D_{\bar{h}}\mathcal{Q}$ is linear in its last $16n$ variables as well, we note that there holds:

$$\begin{aligned} &D_{\bar{h}} \mathcal{Q}(\bar{h} + s\bar{\eta}, h^* + s\eta^*) - D_{\bar{h}} \mathcal{Q}(\bar{h}, h^*) \\ &\quad = D_{\bar{h}} \mathcal{Q}(\bar{h} + s\bar{\eta}, h^*) - D_{\bar{h}} \mathcal{Q}(\bar{h}, h^*) s\, D_{\bar{h}} \mathcal{Q}(\bar{h} + s\bar{\eta}, \eta^*), \end{aligned} \tag{35}$$

for any $s \in [0, 1]$. Now, we consider the product $X := C^{3+\beta, \frac{3+\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^{2n}) \times C^{\beta, \frac{\beta}{4}}(\Sigma \times [0, T], \mathbb{R}^{16n})$, which becomes a Banach space when equipped with the norm

$$\| h \|_X := \| \bar{h} \|_{C^{3+\beta, \frac{3+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^{2n})} + \| h^* \|_{C^{\beta, \frac{\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^{16n})},$$

for $h = (\bar{h}, h^*) \in X$. Moreover, we define for small $\delta > 0$ the open neighbourhood $\hat{\mathcal{V}}_\delta$ of the function $(x, t) \mapsto (D_x F_0(x), D_x^4 F_0(x))$ in X by

$$h = (\bar{h}, h^*) \in \hat{\mathcal{V}}_\delta \Longleftrightarrow \| D_x F_0 - \bar{h} \|_{L^\infty(\Sigma\times[0,T])} + \| D_x^4 F_0 - h^* \|_{L^\infty(\Sigma\times[0,T])} < 2\delta.$$

We firstly estimate for any fixed $h = (\bar{h}, h^*) \in \hat{\mathcal{V}}_\delta$ and any $\eta \in X$ with sufficiently small norm $0 < \| \eta \|_X < \tilde{\epsilon}$ by the mean value theorem and the "quasi-linearity" of $D_{\bar{h}}\mathcal{Q}$:

$$\begin{aligned} &\| D_{\bar{h}} \mathcal{Q}(\bar{h} + \bar{\eta}, h^*) - D_{\bar{h}} \mathcal{Q}(\bar{h}, h^*) \|_{L^\infty(\Sigma\times[0,T])} \\ &\quad \le \mathrm{Const}(F_0, \delta, \tilde{\epsilon}) \| \bar{\eta} \|_{L^\infty(\Sigma\times[0,T])} \| h^* \|_{L^\infty(\Sigma\times[0,T])} . \end{aligned}$$

Similarly as in (28) we can combine this estimate again with the mean value theorem in order to estimate:

$$\begin{aligned} &\| D_{\bar{h}} \mathcal{Q}(\bar{h} + \bar{\eta}, h^*) - D_{\bar{h}} \mathcal{Q}(\bar{h}, h^*) \|_{C^{\beta, \frac{\beta}{4}}(\Sigma\times[0,T])} \\ &= \| D_{\bar{h}} \mathcal{Q}(\bar{h} + \bar{\eta}, h^*) - D_{\bar{h}} \mathcal{Q}(\bar{h}, h^*) \|_{L^\infty(\Sigma\times[0,T])} \end{aligned}$$

$$
\begin{aligned}
&+\sup_{x\in\Sigma} \text{höl}_{\frac{\beta}{4}}(D_{\bar h}\mathcal{Q}(\bar h(x)+\bar\eta(x),h^*(x))-D_{\bar h}\mathcal{Q}(\bar h(x),h^*(x)))\\
&+\sup_{t\in[0,T]} \text{höl}_{\beta}(D_{\bar h}\mathcal{Q}(\bar h_t+\bar\eta_t,h^*_t)-D_{\bar h}\mathcal{Q}(\bar h_t,h^*_t))\\
\leq{}& \text{Const}(F_0,\delta,\tilde\epsilon)\ \| h^* \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])} \| \bar\eta \|_{L^\infty(\Sigma\times[0,T])}\\
&+\epsilon^{\frac{\alpha-\beta}{4}}\,\text{Const}(F_0,\delta,\tilde\epsilon)\Big(\| \bar h \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])} + \| \bar\eta \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])}\Big) \| h^* \|_{C^{\beta,\frac{\beta}{4}}}\\
&+2\,\epsilon^{-\frac{\beta}{4}}\,\text{Const}(F_0,\delta,\tilde\epsilon)\ \| h^* \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])} \| \bar\eta \|_{L^\infty(\Sigma\times[0,T])}\\
&+\epsilon^{\alpha-\beta}\,\text{Const}(F_0,\delta,\tilde\epsilon)(\| \bar h \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])} + \| \bar\eta \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])})\ \| h^* \|_{C^{\beta,\frac{\beta}{4}}}\\
&+2\,\epsilon^{-\beta}\,\text{Const}(F_0,\delta,\tilde\epsilon)\ \| h^* \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])} \| \bar\eta \|_{L^\infty(\Sigma\times[0,T])}\\
\leq{}& \text{Const}(F_0,\delta,\tilde\epsilon,h)\ \| h^* \|_{C^{\beta,\frac{\beta}{4}}} \Big((1+\epsilon^{-\frac{\beta}{4}}+\epsilon^{-\beta})\ \| \bar\eta \|_{L^\infty} + \epsilon^{\frac{\alpha-\beta}{4}} + \epsilon^{\alpha-\beta}\Big),
\end{aligned}
$$

for any $\epsilon > 0$ and any fixed $\alpha \in (\beta, 1)$. Now setting $\epsilon :=\| \eta \|_X > 0$ this estimate implies in particular that

$$
\begin{aligned}
&\| D_{\bar h}\mathcal{Q}(\bar h+\bar\eta,h^*)-D_{\bar h}\mathcal{Q}(\bar h,h^*) \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])}\\
&\quad\leq \text{Const}(F_0,\delta,\tilde\epsilon,h)\ \| h^* \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])}\\
&\Big(\| \eta \|_X + \| \eta \|_X^{1-\frac{\beta}{4}} + \| \eta \|_X^{1-\beta} + \| \eta \|_X^{\frac{\alpha-\beta}{4}} + \| \eta \|_X^{\alpha-\beta} \Big).
\end{aligned}
\tag{36}
$$

Exactly the same reasoning, now using the "quasi-linearity" of $\mathcal{Q}$ itself, yields the estimates

$$
\begin{aligned}
&\| \mathcal{Q}(\bar h+\bar\eta,\eta^*)-\mathcal{Q}(\bar h,\eta^*) \|_{L^\infty(\Sigma\times[0,T])}\\
&\quad\leq \text{Const}(F_0,\delta,\tilde\epsilon)\ \| \bar\eta \|_{L^\infty(\Sigma\times[0,T])} \| \eta^* \|_{L^\infty(\Sigma\times[0,T])}
\end{aligned}
$$

and

$$
\begin{aligned}
&\| \mathcal{Q}(\bar h+\bar\eta,\eta^*)-\mathcal{Q}(\bar h,\eta^*) \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])}\\
&\leq \text{Const}(F_0,\delta,\tilde\epsilon,h)\Big(\| \eta \|_X + \| \eta \|_X^{1-\frac{\beta}{4}} + \| \eta \|_X^{1-\beta} + \| \eta \|_X^{\frac{\alpha-\beta}{4}} + \| \eta \|_X^{\alpha-\beta} \Big)\\
&\| \eta^* \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])}.
\end{aligned}
\tag{37}
$$

As we also have

$$
\| D_{\bar h}\mathcal{Q}(\bar h+\bar\eta,\eta^*) \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])} \leq \text{Const}(F_0,\delta,\tilde\epsilon,h)\ \| \eta \|_X \tag{38}
$$

on account of the linearity of $D_{\bar h}\mathcal{Q}$ in its last $16n$ variables, we can finally combine (34)–(38) with the mean value theorem in order to conclude for the operator $\mathcal{Q}_\sharp$: $\hat{\mathcal{V}}_\delta \subset X \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^n)$ which maps $h=\{h_t\}\in\hat{\mathcal{V}}_\delta$ to the function $\mathcal{Q}(h)$:

$$
\begin{aligned}
&\| \mathcal{Q}_\sharp(h+\eta) - \mathcal{Q}_\sharp(h) - D_h \mathcal{Q}(h)\cdot\eta \|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])} \\
&\quad = \| \int_0^1 D_{\bar h}\mathcal{Q}(\bar h + s\,\bar\eta, h^*) \qquad - D_{\bar h}\mathcal{Q}(\bar h, h^*)\, ds\cdot\bar\eta \\
&\qquad + \int_0^1 s\, D_{\bar h}\mathcal{Q}(\bar h + s\,\bar\eta, \eta^*)\, ds\cdot\bar\eta \\
&\qquad + \int_0^1 \mathcal{Q}(\bar h + s\bar\eta, \eta^*) - \mathcal{Q}(\bar h, \eta^*)\, ds \,\|_{C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])} \\
&\quad \le \mathrm{Const}(F_0,\delta,\tilde\epsilon,h)\left(\|\eta\|_X + \|\eta\|_X^{1-\frac{\beta}{4}} + \|\eta\|_X^{1-\beta} + \|\eta\|_X^{\frac{\alpha-\beta}{4}} + \|\eta\|_X^{\alpha-\beta}\right) \\
&\quad \|\eta\|_X,
\end{aligned}
$$

for any fixed $h = (\bar h, h^*) \in \hat{\mathcal{V}}_\delta$. This shows that the operator $\mathcal{Q}_\sharp$ is Fréchet-differentiable in any fixed $h = (\bar h, h^*) \in \hat{\mathcal{V}}_\delta$ with Fréchet-derivative $D\mathcal{Q}_\sharp(h) : X \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^n)$ in direction of an arbitrary $\eta = (\bar\eta, \eta^*) \in X$ given by

$$
D\mathcal{Q}_\sharp(h).\eta = D_h\mathcal{Q}(h)\cdot\eta = D_{\bar h}\mathcal{Q}(\bar h, h^*)\cdot\bar\eta + \mathcal{Q}(\bar h, \eta^*). \tag{39}
$$

Furthermore (35)–(39) show that $D\mathcal{Q}_\sharp(h)$ is continuous in $h \in \hat{\mathcal{V}}_\delta$ w.r.t. $\|\cdot\|_X$. We finally let $\hat{\mathcal{L}}$ denote the linear map $f_t \mapsto (D_x f_t, D_x^4 f_t)$, which maps $X_{F_0,\beta,\delta,T}$ into the neighbourhood $\hat{\mathcal{V}}_\delta$ on account of

$$
\begin{aligned}
&\| D_x F_0 - \overline{\hat{\mathcal{L}}(f_t)} \|_{L^\infty(\Sigma\times[0,T])} + \| D_x^4 F_0 - \hat{\mathcal{L}}(f_t)^* \|_{L^\infty(\Sigma\times[0,T])} \\
&\le \sup_{t\in[0,T]} \| F_0 - f_t \|_{C^4(\Sigma)} \le \delta < 2\delta
\end{aligned}
$$

for any family of immersions $\{f_t\} \in X_{F_0,\beta,\delta,T}$. As (24) means by (31) and (33) that

$$
\phi = \partial_t + \mathcal{A}_\sharp\circ\tilde{\mathcal{L}}\cdot\mathcal{Q}_\sharp\circ\hat{\mathcal{L}} + \mathcal{F}_\sharp\circ\mathcal{L}, \tag{40}
$$

we can finally infer from the chain- and product rule for C^1-maps (on open subsets of Banach spaces) that ϕ is a C^1-map from $X_{F_0,\beta,\delta,T}$ to $C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^n)$, if $\delta > 0$ is chosen sufficiently small, as asserted in part (1) of the theorem.
(2) From (30)–(33) and (39)–(40) we infer again by the chain- and product rule for C^1-maps that the Fréchet-derivative of ϕ in any fixed $\{f_t\} \in X_{F_0,\beta,\delta,T}$ is given by

$$
\begin{aligned}
D\phi(f_t).\eta &= \partial_t\eta_t + D\mathcal{A}_\sharp(\tilde{\mathcal{L}}(f_t)).\tilde{\mathcal{L}}(\eta_t)\cdot\mathcal{Q}_\sharp(\hat{\mathcal{L}}(f_t)) \\
&\quad + \mathcal{A}_\sharp(\tilde{\mathcal{L}}(f_t))\big(D_{\bar h}\mathcal{Q}(\hat{\mathcal{L}}(f_t))\cdot\overline{\hat{\mathcal{L}}(\eta_t)} \\
&\quad + \mathcal{Q}(\overline{\hat{\mathcal{L}}(f_t)}, (\hat{\mathcal{L}}(\eta_t))^*)\big) + D\mathcal{F}_\sharp(\mathcal{L}(f_t)).\mathcal{L}(\eta_t) \\
&= \partial_t\eta_t + \frac{1}{2}\,|A^0_{f_t}|^{-4}\,\big(D_{\bar h}\mathcal{Q}(D_x f_t, D_x^4 f_t)\cdot D_x(\eta_t) \\
&\quad + g^{ij}_{f_t} g^{kl}_{f_t} \nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}(\eta_t)\big)
\end{aligned}
$$

$$+D_h\mathcal{A}(\cdot, D_x f_t, D_x^2 f_t)\cdot\left(D_x\eta_t, D_x^2\eta_t\right) g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}(f_t)$$
$$+D_h\mathcal{F}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t)\cdot\left(D_x\eta_t, D_x^2\eta_t, D_x^3\eta_t\right) \qquad (41)$$

on $\Sigma\times[0,T]$, for $\eta=\{\eta_t\}\in T_f X_{F_0,\beta,\delta,T}=Y_{\beta,T,0}$. First of all, this shows that the Fréchet-derivative $D\phi(f_t)$ of ϕ in any fixed $\{f_t\}\in X_{F_0,\beta,\delta,T}$ is a linear, uniformly parabolic differential operator of fourth order on $Y_{\beta,T,0}$ in "diagonal form", i.e. whose leading operator $\frac{1}{2}\mid A_{f_t}^0\mid^{-4}\ g_{f_t}^{ij} g_{f_t}^{kl}\nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}$ is uniformly elliptic in the sense of (47) for some sufficiently large ellipticity constant $\Lambda\geq 1$, depending on Σ, F_0 and on δ, and which acts on each component of $\{\eta_t\}\in Y_{\beta,T,0}$ separately, if $\delta>0$ is chosen sufficiently small. Furthermore, the coefficients of the rational function $\mathcal{F}$ are C^∞-smooth functions on Σ which only depend on the derivatives of F_0 up to 3rd order. As we also know that the modulus of each denominator which appears in the fractions of $D_h\mathcal{F}(x, D_x f_t(x), D_x^2 f_t(x), D_x^3 f_t(x))$ is bounded from below by some positive constant on $\Sigma\times[0,T]$, if $\{f_t\}\in X_{F_0,\beta,\delta,T}$ and $\delta>0$ sufficiently small, we can conclude by the mean value theorem that the composition $D_h\mathcal{F}(\cdot, D_x f, D_x^2 f, D_x^3 f)$ has to be of regularity class $C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^{14n})$ if $\{f_t\}\in X_{F_0,\beta,\delta,T}$ and $\delta>0$ sufficiently small; see again pp. 44–45 in [6] for the definition of "$C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R}^M)$". By the same reasoning, an appropriate choice of $\delta>0$ guarantees that the coefficients $\frac{1}{2}\mid A_{f_t}^0\mid^{-4}\ g_{f_t}^{ij} g_{f_t}^{kl}$ of the leading term of $D\phi(f_t)$ are of class $C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T],\mathbb{R})$, for any family $\{f_t\}\in X_{F_0,\beta,\delta,T}$. We can finally rewrite formula (41) and obtain that indeed the Fréchet-derivative $D\phi(f_t)$ in any fixed $\{f_t\}\in X_{F_0,\beta,\delta,T}$ is a linear, uniformly parabolic differential operator on $Y_{\beta,T,0}$ of the form

$$\begin{aligned}D\phi(f_t).\eta=\partial_t\eta_t+\frac{1}{2}\mid A_{f_t}^0\mid^{-4}\ g_{f_t}^{ij} g_{f_t}^{kl}\nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}(\eta_t)\\ +B_3^{ijk}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t)\cdot\nabla_{ijk}^{F_0}(\eta_t)\\ +B_2^{ij}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)\cdot\nabla_{ij}^{F_0}(\eta_t)\\ +B_1^{i}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)\cdot\nabla_{i}^{F_0}(\eta_t)\end{aligned}$$

on $\Sigma\times[0,T]$, if $\delta>0$ is chosen sufficiently small, whose leading operator $\frac{1}{2}\mid A_{f_t}^0\mid^{-4}$ $g_{f_t}^{ij} g_{f_t}^{kl}\nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}$ is uniformly elliptic in the sense of (47) and acts on each component of any $\{\eta_t\}\in Y_{\beta,T,0}$ separately, and where $B_3^{ijk}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t),\ldots$, $B_1^i(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ are coefficients of $\mathrm{Mat}_{n,n}(\mathbb{R})$-valued, contravariant tensor fields of degrees 3, 2 and 1, which are of regularity class $C^{\beta,\frac{\beta}{4}}$ on $\Sigma\times[0,T]$. This proves the assertions of part (2) of the theorem.

(3) Since for any fixed $\{f_t\}\in X_{F_0,\beta,\delta,T}$ the leading coefficient tensor $\frac{1}{2}\mid A_{f_t}^0\mid^{-4}$ $g_{f_t}^{ij} g_{f_t}^{kl}$ of $D\phi(f_t)$ is of class $C^{1+\beta,\frac{1+\beta}{4}}(\Sigma\times[0,T])$ and consequently of class $C^{\beta,\frac{\beta}{4}}(\Sigma\times[0,T])$, is uniformly elliptic (in the sense of (47)) on $\Sigma\times[0,T]$ and acts on each component of any $\{\eta_t\}\in Y_{\beta,T,0}$ separately, the result of part (2) shows

in particular that $D\phi(f_t) : Y_{\beta,T,0} \longrightarrow C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ meets all requirements of Proposition 1 resp. of Corollary 3 on $\Sigma \times [0,T]$, for some appropriate constant $\Lambda \geq 1$, depending on Σ, F_0 and δ, and with $\alpha = \beta$ and $G := F_0$. Hence, by Corollary 3 $D\phi(f_t)$ yields an isomorphism between $Y_{\beta,T,0}$ and $C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ in any fixed $\{f_t\} \in X_{F_0,\beta,\delta,T}$. □

Theorem 3 *Let $F_0 : \Sigma \longrightarrow \mathbb{R}^n$ be a C^∞-smooth immersion of some smooth compact torus into $\mathbb{R}^n$ without umbilic points, thus with $| A^0_{F_0} |^2 > 0$ on Σ, and let $\beta \in (0,1)$ be arbitrarily fixed and $\delta > 0$ be fixed as small as required in Theorem 2.*

(1) *There are sufficiently small $T > 0$ and $T^* \in (0,T)$ and a function $\chi_{T^*} \in C^{\gamma,\frac{\gamma}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$, for any $\gamma \in (0,\beta)$, satisfying $\chi_{T^*} \equiv 0$ on $\Sigma \times [0,T^*]$, such that χ_{T^*} is contained in the image of $\phi : X_{F_0,\gamma,\delta,T} \to C^{\gamma,\frac{\gamma}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$.*
(2) *For this $T^* > 0$ the initial value problem (16) has a unique and C^∞-smooth solution $\{f_t^*\} \in X_{F_0,\beta,\delta,T^*}$ on $\Sigma \times [0,T^*]$.*
(3) *This short-time solution $\{f_t^*\}$ of (16) solves the initial value problem*

$$\partial_t^{\perp_{f_t}}(f_t) = - | A^0_{f_t} |^{-4} \, \delta\mathcal{W}(f_t), \qquad f_0 = F_0 \tag{42}$$

on $\Sigma \times [0,T^*]$, where $\partial_t^{\perp_{f_t}}(f_t)(x)$ denotes the projection of the vector $\partial_t(f_t)(x)$ onto the normal space of the surface f_t at the point $f_t(x)$, for any $x \in \Sigma$.

Proof (1) Since

$$\begin{aligned} &\partial_t + \triangle^2_{F_0} : \{\{G_t\} \in C^{4+\beta,1+\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n) | G_0 = F_0 \text{ on } \Sigma\} \\ &\quad \stackrel{\cong}{\longrightarrow} C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \end{aligned} \tag{43}$$

yields an isomorphism by Corollary 3, there is exactly one family $\{h_t\} \in X_{F_0,\beta,\delta,T}$, for $T > 0$ sufficiently small, which solves the initial value problem

$$\partial_t h_t + \triangle^2_{F_0}(h_t) = \triangle^2_{F_0}(F_0) + \mathcal{D}_{F_0}(F_0), \qquad h_0 = F_0 \text{ on } \Sigma,$$

on $\Sigma \times [0,T]$, where we recall that $\mathcal{D}_{F_0} = \partial_t - \phi$ by definition of ϕ. We use this unique solution $\{h_t\}$ in order to define the function $\chi \in C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ by $\chi(t) := \phi(h_t)$ and see that

$$\begin{aligned} \chi(0) = \phi(h_0) &= \partial_t h_0 - \mathcal{D}_{F_0}(h_0) = \partial_t h_0 - \mathcal{D}_{F_0}(F_0) \\ &= \triangle^2_{F_0}(F_0) - \triangle^2_{F_0}(h_0) = 0 \qquad \text{on } \Sigma. \end{aligned}$$

Now, we introduce the functions χ_ρ on $\Sigma \times [0,T]$ by

$$\chi_\rho := \begin{cases} 0 & : \ t \in [0,\rho] \\ \chi(t-\rho) & : \ t \in [\rho, T] \end{cases}$$

for any $\rho \in [0, T)$. On account of $\chi(0) = 0$ on Σ we obtain immediately that $\chi_\rho \in C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ with $\| \chi_\rho \|_{C^{\beta,\frac{\beta}{4}}} \leq \| \chi \|_{C^{\beta,\frac{\beta}{4}}}$ for every $\rho \in [0, T)$. Since we also have

$$\chi_\rho(t)(x) \longrightarrow \chi(t)(x) \qquad \text{in each fixed } (x,t) \in \Sigma \times [0,T]$$

as $\rho \to 0$, we conclude from the compactness of the embedding

$$C^{\beta,\frac{\beta}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \hookrightarrow C^{\gamma,\frac{\gamma}{4}}(\Sigma \times [0,T], \mathbb{R}^n),$$

for any $\gamma \in (0, \beta)$, that

$$\chi_\rho \longrightarrow \chi \qquad \text{in } C^{\gamma,\frac{\gamma}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \tag{44}$$

for any $\gamma \in (0, \beta)$. Now, since $X_{F_0,\beta,\delta,T} \subset X_{F_0,\gamma,\delta,T}$, for any $\gamma < \beta$, Theorem 2 implies here for $f_t := h_t$ and with β replaced by γ that the Fréchet-derivative $D\phi(h)$ of ϕ in h yields an isomorphism

$$D\phi(h) : Y_{\gamma,T,0} = T_h X_{F_0,\gamma,\delta,T} \xrightarrow{\cong} C^{\gamma,\frac{\gamma}{4}}(\Sigma \times [0,T], \mathbb{R}^n),$$

for any $\gamma \in (0, \beta)$. Hence, as ϕ is a C^1-map on the open subset $X_{F_0,\gamma,\delta,T}$ of Y_{γ,T,F_0}, the "Inverse Function Theorem" for C^1-maps between Banach spaces implies that there are sufficiently small numbers $\tilde{\epsilon} > 0$ and $\tilde{\delta} > 0$ such that the open ball $B^{\gamma}_{\tilde{\delta}}(\chi)$ of radius $\tilde{\delta}$ about χ in $C^{\gamma,\frac{\gamma}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ is contained in the image $\phi(B^{4+\gamma}_{\tilde{\epsilon}}(h))$ of the open ball $B^{4+\gamma}_{\tilde{\epsilon}}(h) \subset X_{F_0,\gamma,\delta,T}$ of radius $\tilde{\epsilon}$ about h w.r.t. ϕ. Hence, by (44) there have to exist some $T^* \in (0, T)$ and some family $f^* = \{f_t^*\} \in X_{F_0,\gamma,\delta,T}$ such that $\phi(f^*) = \chi_{T^*}$, just as asserted in claim (1) of the theorem.

(2) Since there holds $\chi_{T^*} = 0$ on $\Sigma \times [0, T^*]$, the proved assertion of part (1) shows by definition of ϕ and by its reformulation in (24):

$$\begin{aligned} \partial_t(f_t^*)(x) &= -\frac{1}{2} \mid A^0_{f_t^*} \mid^{-4} \; g^{ij}_{f_t^*} g^{kl}_{f_t^*} \nabla_i^{f_t^*} \nabla_j^{f_t^*} \nabla_k^{F_0} \nabla_l^{F_0}(f_t^*)(x) \\ &\quad + B(x, D_x f_t^*(x), D_x^2 f_t^*(x), D_x^3 f_t^*(x)) \\ &= -\frac{1}{2} \mid A^0_{f_t^*} \mid^{-4} \; g^{ij}_{f_t^*} \, g^{kl}_{f_t^*} \, \nabla^{F_0}_{ijkl}(f_t^*)(x) \\ &\quad - \mathcal{F}(x, D_x f_t^*(x), D_x^2 f_t^*(x), D_x^3 f_t^*(x)) \end{aligned} \tag{45}$$

on $\Sigma \times [0, T^*]$, i.e. that $\{f_t^*\}$ is a solution of problem (16) on $\Sigma \times [0, T^*]$, where the composition $\mathcal{F}(\,\cdot\,, D_x f^*, D_x^2 f^*, D_x^3 f^*)$ is (at least) of class $C^{1+\gamma,\frac{1+\gamma}{4}}$ on $\Sigma \times [0, T^*]$. Furthermore, the linear operator

$$\begin{aligned} L := \partial_t + \frac{1}{2} \mid A^0_{f_t^*} \mid^{-4} \; g^{ij}_{f_t^*} \, g^{kl}_{f_t^*} \nabla^{F_0}_{ijkl} &: C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n) \\ &\to C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n) \end{aligned}$$

is uniformly parabolic on $\Sigma \times [0, T^*]$ on account of $\{f_t^*\} \in X_{F_0,\gamma,\delta,T}$ and is of diagonal form, and thus meets all conditions of Proposition 1 resp. of Corollary 3 for $T := T^* > 0$, for some appropriate constant $\Lambda \geq 1$ depending on Σ, F_0 and δ, and for any $\alpha \in (0,1)$. Hence, by Corollary 3 L yields an isomorphism between $Y_{\alpha,T^*,0}$ and $C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$ and therefore also an isomorphism

$$L : \{\{G_t\} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n) \mid G_0 = F_0 \text{ on } \Sigma\} \xrightarrow{\cong} C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$$

for any $\alpha \in (0,1)$. Since $\{f_t^*\}$ solves the reformulation

$$L(f^*)(x,t) = -\mathcal{F}(x, D_x f_t^*(x), D_x^2 f_t^*(x), D_x^3 f_t^*(x)) \quad \text{on } \Sigma \times [0,T^*] \quad \text{with } f_0^* = F_0 \text{ on } \Sigma, \tag{46}$$

of problem (16) and since the right-hand side of this equation is especially of class $C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$, for any $\alpha \in (0,1)$, this proves that $\{f_t^*\} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$, for any $\alpha \in (0,1)$, and thus that $\{f_t^*\}$ is indeed a $C^{4+\beta,1+\frac{\beta}{4}}$-solution of equation (16) on $\Sigma \times [0,T^*]$, and therefore in particular $\{f_t^*\} \in X_{F_0,\beta,\delta,T^*}$. In order to prove C^∞-regularity of the short-time solution $\{f_t^*\}$ of problem (16) on $\Sigma \times [0,T^*]$ we again consider the above linear, parabolic operator L. As we know now that $\{f_t^*\}$ is of regularity class $C^{4+\beta,1+\frac{\beta}{4}}$, we see as above and as in part (3) of the proof of Theorem 2 that L satisfies all requirements of the Regularity Theorem 3 for $k=1$, $\alpha = \beta$ and $T := T^* > 0$. Moreover, $\{f_t^*\}$ solves the reformulation (46) of equation (16) on $\Sigma \times [0,T^*]$ whose right-hand side is of class $C^{1+\beta,\frac{1+\beta}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$. Hence, we may apply the Regularity Theorem 3 and obtain that $\{f_t^*\}$ is of class $C^{5+\beta,\frac{5+\beta}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$. Thus, L, the right-hand side of equation (46) and the initial surface F_0 satisfy all requirements of Proposition 3 for $k=2$, $\alpha = \beta$ and $T := T^* > 0$. We can therefore repeat the above argument and obtain by induction that $\{f_t^*\}$ is of class $C^{4+k+\beta,1+\frac{k+\beta}{4}}(\Sigma \times [0,T^*], \mathbb{R}^n)$ for any $k \in \mathbb{N}_0$, i.e. that $\{f_t^*\} \in C^\infty(\Sigma \times [0,T^*], \mathbb{R}^n)$, just as asserted in part (2) of the theorem. Finally, having proved that any solution of equation (16) is smooth and since its right-hand side is quasi-linear, uniformly elliptic, also in the sense of the article [11], and in diagonal form, one can use the argument in the proof of Theorem 1.1 in [11], pp. 865–868, in order to prove also uniqueness of the solution $\{f_t^*\}$ of problem (16) within X_{F_0,β,δ,T^*}.
(3) Since the difference of the right-hand sides of the equations (42) and (14) resp. (16) is a section into the tangent bundle of their solutions $\{f_t\}$ in each time t, and since $\delta\mathcal{W}(f_t) = \frac{1}{2}\big(\triangle^\perp H_{f_t} + Q(A^0_{f_t})(H_{f_t})\big)$ is a section into the normal bundle of the surface f_t, the short-time solution $\{f_t^*\}$ of problem (16) has to be a smooth short-time solution of problem (42) on $\Sigma \times [0,T^*]$ as well. $\square$

For the proof of part (3) of Theorem 2, we need the following Schauder a priori estimates for uniformly parabolic operators of fourth order with $C^{\alpha,\frac{\alpha}{4}}$-coefficients and with uniformly elliptic leading operator in diagonal form, which can be derived from Theorems 1, 2 in [13] together with the compactness of the embedding

$$C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times I, \mathbb{R}^n) \hookrightarrow C^{4,1}(\Sigma \times I, \mathbb{R}^n)$$

and Ehrling's Lemma (see also Proposition 3.22 in [1]):

Proposition 1 *Let Σ be a smooth, compact, orientable surface without boundary, $G : \Sigma \to \mathbb{R}^n$ a C^∞-smooth immersion of Σ into $\mathbb{R}^n$, $\alpha \in (0, 1)$ arbitrarily fixed, $I = [a, b]$ a closed interval of length $T > 0$, and let $\psi : \Omega \xrightarrow{\cong} \Sigma'$ be a smooth chart of an arbitrary coordinate neighbourhood Σ' of Σ, yielding partial derivatives ∂_m, $m = 1, 2$, and the induced metric $g := G^*(g_{\text{eu}})$ with its coefficients $g_{ij} := \langle \partial_i G, \partial_j G\rangle$ on Σ'. Moreover, let*

$$L : C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times I, \mathbb{R}^n) \longrightarrow C^{\alpha,\frac{\alpha}{4}}(\Sigma \times I, \mathbb{R}^n)$$

be a linear differential operator of order 4 whose leading operator of fourth order acts "diagonally", i.e. acts on each component of f separately:

$$\begin{aligned} L(f)(x,t) := \partial_t(f)(x,t) \\ &+\big(A_4^{ijkl}(x,t)\, \nabla^G_{ijkl} + A_3^{ijk}\, \nabla^G_{ijk} + A_2^{ij}\, \nabla^G_{ij} \\ &+A_1^i\, \nabla^G_i + A_0(x,t)\big)(f)(x,t), \end{aligned}$$

in every pair $(x,t) \in \Sigma' \times I$, and L is to meet the following "principal requirements":

(1) *A_4^{ijkl}, A_3^{ijk}, A_2^{ij}, A_1^i, A_0 are the coefficients of contravariant tensor fields A_4, A_3, A_2, A_1, A_0 on $\Sigma \times I$ of degrees $4, 3, \dots, 0$ and of regularity class $C^{\alpha,\frac{\alpha}{4}}(\Sigma \times I)$. Moreover, the tensor A_4 has to be the square $E \otimes E$ of a contravariant real-valued symmetric tensor field E of order 2 and of regularity class $C^{\alpha,\frac{\alpha}{4}}(\Sigma \times I)$, i.e. $A_4^{ijkl}(x,t) = E^{ij}(x,t)\, E^{kl}(x,t)$ with $E^{ij}(x,t) = E^{ji}(x,t) \in \mathbb{R}$, and we assume that $A_3^{ijk}(x,t)$, $A_2^{ij}(x,t)$, $A_1^i(x,t)$, $A_0(x,t) \in \text{Mat}_{n,n}(\mathbb{R})$, in every pair $(x,t) \in \Sigma' \times I$ and for all indices $i, j, k, l \in \{1, 2\}$.*

(2) *The tensor field E is required to be uniformly elliptic on $\Sigma \times I$, i.e. there has to be a number $\Lambda \geq 1$ such that there holds*

$$E^{ij}(x,t)\, \xi_i \xi_j \geq \Lambda^{-1/2}\, g^{ij}(x)\, \xi_i \xi_j \tag{47}$$

for any vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and for any pair $(x,t) \in \Sigma' \times I$.

(3) *There holds $\| A_r \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times I)} \leq \Lambda$, for $r = 0, 1, \dots, 4$.*

Then, there exists some constant $C = C(\Sigma, G, T, \alpha, n, \Lambda) > 0$ such that

$$\| \eta \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times I)} \leq C \left(\| L(\eta) \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times I)} + \| \eta \|_{L^2(\Sigma\times I)} \right)$$

holds true for any family of surfaces $\{\eta_t\} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times I, \mathbb{R}^n)$ with $\eta_a = 0$ on Σ.

Using these a priori estimates we are able to prove the injectivity of the restriction of any such operator L to $Y_{\alpha,T,0}$, which still serves as a preparation for the proof of part (3) of Theorem 2 (see also p. 861 in [11] for a similar reasoning):

Proposition 2 *Let* $L : C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T],\mathbb{R}^n) \longrightarrow C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T],\mathbb{R}^n)$ *be a uniformly parabolic differential operator of order* 4*, which satisfies all requirements of Proposition 1 on* $I = [0,T]$*. Then its restriction to* $Y_{\alpha,T,0}$ *is injective.*

Proof Firstly, we show the subsequent improvement (50) of the a priori estimates of Proposition 1 for any $\eta \in Y_{\alpha,T,0}$ which satisfies $L(\eta) = 0$. To this end, we fix an arbitrary smooth chart $\psi : \Omega \xrightarrow{\cong} \Sigma'$ of a normal coordinate neighbourhood Σ' of Σ, some C^∞-smooth immersion $G : \Sigma \to \mathbb{R}^n$ of Σ into $\mathbb{R}^n$, and some $\eta \in Y_{\alpha,T,0}$ with $L(\eta) = 0$. We extend η to $\Sigma \times [-T,T]$ by $\bar{\eta}(x,t) \equiv 0$ on $\Sigma \times [-T,0]$, and extend the coefficient tensor fields A_4, A_3, A_2, A_1, A_0 of L to $\Sigma \times [-T,T]$ by setting

$$\bar{A}_r(x,-t) := A_r(x,t), \quad \text{for } r = 0,\dots,4$$

in every $x \in \Sigma$ and for any $t \in [0,T]$. Obviously, the extended differential operator $\bar{L}$ satisfies all requirements of Proposition 1 on $\Sigma \times [-T,T]$, i.e. for $a := -T$ and $b := T$. Now, since $\eta_0 \equiv 0$ on Σ and $\eta \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T],\mathbb{R}^n)$, we can firstly conclude that $\bar{\eta} \in C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [-T,T],\mathbb{R}^n)$ and that

$$\partial_{ijkl}\bar{\eta}_0 = 0, \quad \partial_{ijk}\bar{\eta}_0 = 0, \quad \partial_{ij}\bar{\eta}_0 = 0, \quad \partial_i\bar{\eta}_0 = 0 \qquad \text{on} \quad \Sigma'. \tag{48}$$

As we also know that $\partial_{ijkl}\eta \in C^{\alpha,\frac{\alpha}{4}}(\Sigma' \times [0,T],\mathbb{R}^n)$, $\partial_{ijk}\eta \in C^{1+\alpha,\frac{1+\alpha}{4}}(\Sigma' \times [0,T],\mathbb{R}^n)$, ..., $\partial_i\eta \in C^{3+\alpha,\frac{3+\alpha}{4}}(\Sigma' \times [0,T],\mathbb{R}^n)$ on account of the definition of parabolic Hölder spaces in [6], we see that

$$\partial_{ijkl}\bar{\eta} \in C^{\alpha,\frac{\alpha}{4}}(\Sigma' \times [-T,T],\mathbb{R}^n), \partial_{ijk}\bar{\eta} \in C^{1+\alpha,\frac{1+\alpha}{4}}(\Sigma' \times [-T,T],\mathbb{R}^n),$$
$$\dots, \partial_i\bar{\eta} \in C^{3+\alpha,\frac{3+\alpha}{4}}(\Sigma' \times [-T,T],\mathbb{R}^n). \tag{49}$$

Moreover, using that $L(\eta) = 0$ on $\Sigma \times [0,T]$ together with $\eta_0 \equiv 0$ on Σ and (48) we can compute by the mean value theorem that

$$\begin{aligned} \frac{d}{dt}\bar{\eta}_{(0+)}(x) = \frac{d}{dt}\eta_{(0+)}(x) &= \lim_{h \searrow 0} \frac{\eta_h(x)}{h} \\ &= -\big(A_4^{ijkl}(x,0)\,\nabla^G_{ijkl} + A_3^{ijk}(x,0)\,\nabla^G_{ijk} + A_2^{ij}(x,0)\,\nabla^G_{ij} \\ &\quad + A_1^i(x,0)\,\nabla^G_i + A_0(x,0)\big)(\eta_0)(x) = 0 \end{aligned}$$

for every $x \in \Sigma'$. As we trivially have $\frac{d}{dt}\bar{\eta}_{(0-)} = 0$ on Σ by definition of $\bar{\eta}$, we can infer the existence of $\partial_t\bar{\eta}_0$ with the value $\partial_t\bar{\eta}_0 = 0$ on Σ'. Together with (48) this proves $\bar{L}(\bar{\eta})(\,\cdot\,,0) = 0$ on Σ' and thus $\bar{L}(\bar{\eta}) = 0$ on $\Sigma' \times [-T,T]$, i.e.

$$\partial_t\bar{\eta} = -\big(\bar{A}_4^{ijkl}\,\nabla^G_{ijkl} + \bar{A}_3^{ijk}\,\nabla^G_{ijk} + \bar{A}_2^{ij}\,\nabla^G_{ij} + \bar{A}_1^i\,\nabla^G_i + \bar{A}_0\big)(\bar{\eta}) \;\text{ on }\; \Sigma' \times [-T,T].$$

Combining this with (49) and with the $C^{\alpha,\frac{\alpha}{4}}$-regularity of the coefficient tensor fields of $\bar{L}$ on $\Sigma \times [-T,T]$ and covering Σ by finitely many normal coordinate neighbourhoods, we infer that also $\partial_t \bar{\eta} \in C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [-T,T],\mathbb{R}^n)$ and therefore $\bar{\eta} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [-T,T],\mathbb{R}^n)$, together with

$$\| \bar{\eta} \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[-\epsilon,t])} = \| \eta \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[0,t])}$$

for any pair $\epsilon, t \in [0,T]$. Noting that $\bar{\eta}_{-T+t} = 0$ on Σ, for any fixed $t \in [0,T]$, we can apply Proposition 1 to $\bar{L}$ and $\bar{\eta}$ on the interval $I = [a,b] := [-T+t,t]$ of length T and obtain together with $\bar{L}(\bar{\eta}) = 0$ on $\Sigma \times [-T,T]$:

$$\begin{aligned} \| \eta \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[0,t])} &= \| \bar{\eta} \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[-T+t,t])} \\ \leq C(\Sigma,G,T,\alpha,n,\Lambda) \| \bar{\eta} \|_{L^2(\Sigma\times[-T+t,t])} &= C(\Sigma,G,T,\alpha,n,\Lambda) \| \eta \|_{L^2(\Sigma\times[0,t])}. \end{aligned} \tag{50}$$

Now, integrating the equation $\frac{d}{ds}(\langle \eta_s, \eta_s \rangle) = 2\langle \partial_s \eta_s, \eta_s \rangle$ over $\Sigma \times [0,t]$, for some fixed $t \in (0,T]$, and using $\eta_0 = 0$ on Σ and estimate (50) we obtain

$$\begin{aligned} \frac{1}{2} \int_\Sigma \langle \eta_t, \eta_t \rangle \, d\mu_G &= \int_0^t \int_\Sigma \langle \partial_s \eta_s, \eta_s \rangle \, d\mu_G \, ds \\ &\leq \| \eta \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[0,t])} \| \eta \|_{L^1(\Sigma\times[0,t])} \\ &\leq C(\Sigma,G,T,\alpha,n,\Lambda) \ \| \eta \|_{L^2(\Sigma\times[0,t])} \| \eta \|_{L^1(\Sigma\times[0,t])} \\ &\leq \sqrt{t\,\mu_G(\Sigma)}\, C(\Sigma,G,T,\alpha,n,\Lambda) \ \| \eta \|^2_{L^2(\Sigma\times[0,t])}. \end{aligned}$$

Hence, the continuous function $t \mapsto z(t) := \int_\Sigma \langle \eta_t, \eta_t \rangle \, d\mu_G$ satisfies the inequalities $0 \leq z(t) \leq 2\sqrt{t\,\mu_G(\Sigma)}\, C(\Sigma,G,T,\alpha,n,\Lambda) \int_0^t z(s)\,ds$, for any $t \in [0,T]$, and therefore Gronwall's Lemma finally yields $z \equiv 0$ on $[0,T]$, which proves the claimed injectivity of $L\,|_{Y_{\alpha,T,0}}$. □

A combination of this result with the a priori estimates of Proposition 1 and with the compactness of the embeddings

$$C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T],\mathbb{R}^n) \hookrightarrow C^{4,1}(\Sigma \times [0,T],\mathbb{R}^n)$$

and

$$C^{\alpha,\frac{\alpha}{4}}(\Gamma(\Sigma \times [0,T], \mathrm{Sym}^2((T\Sigma)^*))) \hookrightarrow C^0(\Gamma(\Sigma \times [0,T], \mathrm{Sym}^2((T\Sigma)^*)))$$

yields the following corollary by a standard contradiction-argument:

Corollary 2 *Let $L : C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T],\mathbb{R}^n) \longrightarrow C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T],\mathbb{R}^n)$ be a uniformly parabolic differential operator of order 4 satisfying all requirements of Proposition 1 on $I = [0,T]$, for some $T > 0$ and some $\Lambda \geq 1$. Then there exists some constant $C = C(\Sigma,G,T,\alpha,n,\Lambda) > 0$ such that*

$$\| \eta \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[0,T])} \leq C \ \| L(\eta) \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])}$$

holds true for any family of surfaces $\eta \in Y_{\alpha,T,0}$. □

Corollary 3 *Any uniformly parabolic differential operator*

$$L : C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \longrightarrow C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$$

of order 4 *which satisfies all conditions of Proposition 1 on* $I = [0,T]$ *yields an isomorphism between the Banach subspace* $Y_{\alpha,T,0}$ *of* $C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$ *and* $C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$.

Proof As is well known (see Proposition 3.23 in [1]), for any fixed smooth immersion G of Σ into $\mathbb{R}^n$, its associated biharmonic heat operator

$$\partial_t + \Delta_G^2 : Y_{\alpha,T,0} \xrightarrow{\cong} C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n) \tag{51}$$

is bijective and continuous and therefore an isomorphism. Moreover, this operator certainly satisfies requirements (2) and (3) of Proposition 1 for some constant $\Lambda_1 \geq 1$. If L satisfies requirements (2) and (3) of Proposition 1 with the constant $\Lambda_2 \geq 1$, then each convex combination

$$L_s := s\,L + (1-s)\,(\partial_t + \Delta_G^2) : C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$$
$$\longrightarrow C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n),$$

for $s \in [0,1]$, satisfies conditions (2) and (3) of Proposition 1 with the constant $\Lambda := \max\{\Lambda_1, \Lambda_2\}$. Hence, by Corollary 2 there exists some constant $C = C(\Sigma, G, T, \alpha, n, \Lambda) > 0$ such that the estimate

$$\| \eta \|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma\times[0,T])} \leq C \; \| L_s(\eta) \|_{C^{\alpha,\frac{\alpha}{4}}(\Sigma\times[0,T])} \tag{52}$$

holds true for any family of surfaces $\eta \in Y_{\alpha,T,0}$ and uniformly for each $s \in [0,1]$. Since each operator L_s maps $Y_{\alpha,T,0}$ continuously into $C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0,T], \mathbb{R}^n)$, the continuity method finally proves the claim of the corollary on account of (51) and (52). □

For the proof of part (2) of Theorem 3, we invoked the following classical "Schauder regularity theorem" for solutions of uniformly parabolic systems with leading term in diagonal form, which can be proved by induction, Proposition 1, the "method of difference quotients" and a precise use of the definition of parabolic Hölder spaces in [1], pp. 18–19:

Proposition 3 (Schauder Regularity Theorem) *Let* Σ *be a smooth, compact, orientable surface without boundary,* $G : \Sigma \to \mathbb{R}^n$ *a* C^∞*-smooth immersion of* Σ *into* $\mathbb{R}^n$*, and let* $T > 0$*,* $\alpha \in (0,1)$ *and* $k \in \mathbb{N}_0$ *be fixed arbitrarily, and* $\psi : \Omega \xrightarrow{\cong} \Sigma'$ *a smooth chart of an arbitrary coordinate neighbourhood* Σ' *of* Σ*, yielding partial* *derivatives* ∂_m, $m = 1, 2$, *and the induced metric* $g := G^*(g_{\text{eu}})$ *with its coefficients* $g_{ij} := \langle \partial_i G, \partial_j G \rangle$ *on* Σ'. *Moreover, let*

$$L : C^{4+\alpha, 1+\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbb{R}^n) \longrightarrow C^{\alpha, \frac{\alpha}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$$

be a linear differential operator of order 4 *whose leading operator of fourth order acts on each component of* f *separately:*

$$\begin{aligned} L(f)(x,t) := {} & \partial_t(f)(x,t) \\ & + \big(A_4^{ijkl}(x,t)\,\nabla^G_{ijkl} + A_3^{ijk}\,\nabla^G_{ijk} + A_2^{ij}\,\nabla^G_{ij} \\ & + A_1^i\,\nabla^G_i + A_0(x,t)\big)(f)(x,t), \end{aligned}$$

in every pair $(x, t) \in \Sigma' \times [0, T]$, *and L is to meet the following "principal requirements":*

(1) A_4^{ijkl}, A_3^{ijk}, A_2^{ij}, A_1^i, A_0 *are the coefficients of contravariant tensor fields* A_4, A_3, A_2, A_1, A_0 *on* $\Sigma \times [0, T]$ *of degrees* $4, 3, \ldots, 0$ *and of regularity class* $C^{k+\alpha, \frac{k+\alpha}{4}}(\Sigma \times [0, T])$. *Moreover, the tensor* A_4 *has to be the square* $E \otimes E$ *of a contravariant real-valued symmetric tensor field* E *of order* 2 *and of regularity class* $C^{k+\alpha, \frac{k+\alpha}{4}}(\Sigma \times [0, T])$, *i.e.* $A_4^{ijkl}(x, t) = E^{ij}(x, t)\, E^{kl}(x, t)$ *with* $E^{ij}(x, t) = E^{ji}(x, t) \in \mathbb{R}$, *and we assume that* $A_3^{ijk}(x, t)$, $A_2^{ij}(x, t)$, $A_1^i(x, t)$, $A_0(x, t) \in \mathrm{Mat}_{n,n}(\mathbb{R})$, *in every pair* $(x, t) \in \Sigma' \times [0, T]$ *and for all indices* $i, j, k, l \in \{1, 2\}$.

(2) *The tensor field* E *is required to be uniformly elliptic on* $\Sigma \times [0, T]$, *i.e. there has to be a number* $\Lambda \geq 1$ *such that there holds*

$$E^{ij}(x,t)\,\xi_i\xi_j \geq \Lambda^{-1/2}\, g^{ij}(x)\,\xi_i\xi_j$$

for any vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ *and for any pair* $(x, t) \in \Sigma' \times [0, T]$.

(3) *There holds* $\| A_r \|_{C^{k+\alpha, \frac{k+\alpha}{4}}(\Sigma \times [0,T])} \leq \Lambda$, *for* $r = 0, 1, \ldots, 4$.

If moreover $F_0 : \Sigma \to \mathbb{R}^n$ *is an immersion of class* $C^{4+k+\alpha}$ *and* $R \in C^{k+\alpha, \frac{k+\alpha}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$ *and if* $\eta \in C^{4+\alpha, 1+\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$ *is a solution of the initial value problem*

$$L(u) = R \ \text{ on } \ \Sigma \times [0, T], \qquad u_0 = F_0 \ \text{ on } \ \Sigma,$$

then there holds $\eta \in C^{4+k+\alpha, 1+\frac{k+\alpha}{4}}(\Sigma \times [0, T], \mathbb{R}^n)$, *and there exists some constant* $C = C(\Sigma, G, T, \alpha, k, n, \Lambda) > 0$ *such that*

$$\begin{aligned} & \| \eta \|_{C^{4+k+\alpha, 1+\frac{k+\alpha}{4}}(\Sigma \times [0,T])} \\ & \leq C \Big(\| R \|_{C^{k+\alpha, \frac{k+\alpha}{4}}} + \| \eta \|_{L^\infty(\Sigma \times [0,T])} + \| F_0 \|_{C^{4+k+\alpha}(\Sigma)} \Big) \end{aligned}$$

holds true.

3 Proof of Theorem 1

We now combine the "DeTurck-Hamilton-Trick" (see also pp. 38–39 in [1]) with the existence result of Theorem 3: By Theorem 3 there is some $T^* > 0$ and a family of surfaces $\{f_t^*\} \in X_{F_0,\beta,\delta,T^*} \cap C^\infty(\Sigma \times [0, T^*], \mathbb{R}^n)$ which solves the initial value problem (42) on $\Sigma \times [0, T^*]$. Moreover, we infer from $\| F_0 - f_t^* \|_{C^4(\Sigma)} < \delta$ for $t \in [0, T^*]$, on account of $\{f_t^*\} \in X_{F_0,\beta,\delta,T^*}$, that f_t^* is a (umbilic-free) smooth immersion from Σ to $\mathbb{R}^n$, just as F_0 is, if δ is sufficiently small. Therefore, in every pair $(x, t) \in \Sigma \times [0, T^*]$ there exists a unique tangent vector $\xi_t(x) \in \mathrm{Tan}_x \Sigma$ that satisfies

$$D_x f_t^*(x).(\xi_t(x)) = -P^{\mathrm{Tan}(f_t^*)}(\partial_t f_t^*(x)), \tag{53}$$

where $P^{\mathrm{Tan}(f_t^*)} : \mathbb{R}^n \to \mathrm{Tan}(f_t^*)$ denotes the bundle morphism which projects $\mathbb{R}^n$ orthogonally onto the tangent spaces $\mathrm{Tan}_{f_t^*(x)}(f_t^*(\Sigma))$ of the immersed surface $f_t^*(\Sigma)$, in each time $t \in [0, T^*]$. From the C^∞-smoothness of the family $\{f_t^*\}$, we infer the C^∞-smoothness of the vector field ξ on $\Sigma \times [0, T^*]$. Therefore, by classical existence and regularity theory of ordinary differential equations on smooth closed manifolds we know that ξ generates a C^∞-smooth flow $\Psi : \Sigma \times [0, T^*] \longrightarrow \Sigma$ of smooth automorphisms Ψ_t of Σ with $\Psi(\,\cdot\,, 0) = \mathrm{id}_\Sigma$, by setting $\Psi(x, t) := y_x(t)$, where $y_x(\,\cdot\,)$ is the unique, maximal solution of the initial value problem

$$y'(t) = \xi_t(y(t)), \qquad y(0) = x \tag{54}$$

for $t \in [0, T^*]$ and for any fixed $x \in \Sigma$. Finally we recall that the invariance of the Willmore-functional $\mathcal{W}$ w.r.t. diffeomorphic reparametrizations implies that

$$\delta\mathcal{W}(f_t^* \circ \Psi_t) = \delta\mathcal{W}(f_t^*) \circ \Psi_t,$$

for each $t \in [0, T^*]$, and that the second fundamental forms $A_{f_t^*}$ and their traceless parts $A^0_{f_t^*} = A_{f_t^*} - \frac{1}{2}\, g_{f_t^*}\, H_{f_t^*}$ are covariant tensor fields of degree 2, which implies

$$| A^0_{f_t^* \circ \Psi_t} |^2 = | A^0_{f_t^*} |^2 \circ \Psi_t$$

for each $t \in [0, T^*]$ in particular. Using the chain rule we can combine this with (56), (53) and (42) in order to compute for the composition $f_t := f_t^* \circ \Psi(\,\cdot\,, t)$:

$$\begin{aligned}
\partial_t f_t &= (\partial_t f_t^*) \circ \Psi_t + (D_x f_t^* \circ \Psi_t).(\partial_t \Psi_t) = (\partial_t f_t^*) \circ \Psi_t + D_x f_t^*(\xi_t) \circ \Psi_t \\
&= (\partial_t f_t^*) \circ \Psi_t - P^{\mathrm{Tan}(f_t^*)}(\partial_t f_t^*) \circ \Psi_t = \partial_t^\perp (f_t^*) \circ \Psi_t \\
&= -\big(| A^0_{f_t^*} |^{-4}\ \delta\mathcal{W}(f_t^*) \big) \circ \Psi_t = - | A^0_{f_t^* \circ \Psi_t} |^{-4}\ \delta\mathcal{W}(f_t^* \circ \Psi_t) \\
&= - | A^0_{f_t} |^{-4}\ \delta\mathcal{W}(f_t)
\end{aligned}$$

on $\Sigma \times [0, T^*]$ with $f_0 = F_0$ on Σ, which means that $\{f_t\}$ is indeed a smooth (umbilic-free) solution of the Möbius-invariant Willmore flow (9) on $\Sigma \times [0, T^*]$, starting in F_0. It remains to show uniqueness of this short-time solution. To this end, let $\{f_t\}$ be some arbitrary smooth family of umbilic-free C^∞-immersions which solve the Möbius-invariant Willmore flow (9) on $\Sigma \times [0, T^*]$ with initial condition $f_0 = F_0$ on Σ. The $\mathbb{R}^n$-valued function

$$X(x,t) := -\frac{1}{2} \mid A^0_{f_t}(x) \mid^{-4} \Big(g^{ij}_{f_t} g^{kl}_{f_t} g^{mr}_{f_t} \big\langle \nabla^{f_t}_i \nabla^{f_t}_j \nabla^{f_t}_k \nabla^{f_t}_l (f_t)(x), \partial_m f_t(x) \big\rangle \partial_r f_t(x) \\ - g^{ij}_{f_t} g^{kl}_{f_t} \nabla^{f_t}_i \nabla^{f_t}_j \left((\Gamma_{F_0})^m_{kl}(x) - (\Gamma_{f_t})^m_{kl}(x) \right) \partial_m (f_t)(x) \Big)$$

is a "scalar" on Σ and therefore a well-defined and smooth section of the tangent bundle $\mathrm{Tan}(f_t)$, for every $t \in [0, T^*]$. Hence, since the derivative $D_x f_t(x)$ is an isomorphism of the tangent space $\mathrm{Tan}_x(\Sigma)$ onto $\mathrm{Tan}_{f_t(x)}(f_t(\Sigma))$, in every fixed $x \in \Sigma$ and every $t \in [0, T^*]$, there has to exist a unique tangent vector $\bar{\xi}_t(x) \in \mathrm{Tan}_x(\Sigma)$ which satisfies

$$D_x f_t(x).(\bar{\xi}_t(x)) = X(x,t) \tag{55}$$

in every pair $(x,t) \in \Sigma \times [0, T^*]$. Since X and f are smooth, $\bar{\xi}$ is a smooth section of $\mathrm{Tan}(\Sigma)$. Therefore, as above, by the classical theory of ordinary differential equations we can construct a C^∞-smooth family $\Psi : \Sigma \times [0, T^*] \longrightarrow \Sigma$ of smooth automorphisms Ψ_t of Σ with $\Psi(\,\cdot\,, 0) = \mathrm{id}_\Sigma$ by setting $\Psi(x,t) := y_x(t)$, where $y_x(\,\cdot\,)$ is the unique, maximal solution of the initial value problem

$$y'(t) = \bar{\xi}_t(y(t)), \qquad y(0) = x \tag{56}$$

for $t \in [0, T^*]$ and for any fixed $x \in \Sigma$. Now, similarly to the above argument, we can use the definitions of $f_t, \Psi_t, X(\,\cdot\,, t)$ and $\mathcal{D}_{F_0}$ in order to compute for the composition $\tilde{f}_t := f_t \circ \Psi_t$ on account of the relation (55) between $\bar{\xi}$ and X:

$$\begin{aligned} \partial_t \tilde{f}_t &= (\partial_t f_t) \circ \Psi_t + (D_x f_t \circ \Psi_t).(\partial_t \Psi_t) = (\partial_t f_t) \circ \Psi_t + D_x f_t(\bar{\xi}_t) \circ \Psi_t \\ &= -\big(\mid A^0_{f_t} \mid^{-4} \, \delta \mathcal{W}(f_t) \big) \circ \Psi_t + X(\Psi_t, t) = \mathcal{D}_{F_0}(f_t) \circ \Psi_t \\ &= \mathcal{D}_{F_0}(f_t \circ \Psi_t) = \mathcal{D}_{F_0}(\tilde{f}_t) \end{aligned}$$

on $\Sigma \times [0, T^*]$ with $\tilde{f}_0 = F_0$ on Σ. Hence, on account of Theorem 3 the reparametrizations $\{\tilde{f}_t\}$ have to be the *unique* smooth solution to the DeTurck-IW-flow (16) on $\Sigma \times [0, T^*]$, starting in F_0.
Now, suppose there existed two smooth families $\{f^1_t\}$, $\{f^2_t\}$ of umbilic-free C^∞-immersions which solve the Möbius-invariant Willmore flow (9) on $\Sigma \times [0, T^*]$ with initial condition $f^1_0 = F_0 = f^2_0$ on Σ. As above, we can construct two smooth families $\Psi^1, \Psi^2 : \Sigma \times [0, T^*] \longrightarrow \Sigma$ of smooth automorphisms of Σ with $\Psi^i(\,\cdot\,, 0) = \mathrm{id}_\Sigma$, $i = 1, 2$, and such that $\{\tilde{f}^i_t\} := \{f^i_t \circ \Psi^i_t\}$ are both the *unique* smooth solution to the DeTurck-IW-flow (16), starting in F_0, which implies $f^1 \circ \Psi^1 = f^2 \circ \Psi^2$ on $\Sigma \times [0, T^*]$, i.e.

$$f^1_t = f^2_t \circ \Phi_t \qquad \text{on } \Sigma \times [0, T^*] \tag{57}$$

for $\Phi_t := \Psi^2_t \circ (\Psi^1_t)^{-1}$. Now we can use the chain rule again in order to compute by means of the definitions of f^1 and f^2, and by (9) and (57):

$$\begin{aligned} - \mid A^0_{f_t^1} \mid^{-4} \delta\mathcal{W}(f_t^1) = \partial_t f_t^1 &= \partial_t (f_t^2 \circ \Phi_t) \\ &= \partial_t (f_t^2) \circ \Phi_t + (D_x f_t^2 \circ \Phi_t).(\partial_t \Phi_t) \\ &= -\big(\mid A^0_{f_t^2} \mid^{-4} \delta\mathcal{W}(f_t^2)\big) \circ \Phi_t + (D_x f_t^2 \circ \Phi_t).(\partial_t \Phi_t) \\ &= - \mid A^0_{f_t^1} \mid^{-4} \delta\mathcal{W}(f_t^1) + (D_x f_t^2 \circ \Phi_t).(\partial_t \Phi_t) \end{aligned}$$

on $\Sigma \times [0, T^*]$, which implies $(D_x f_t^2 \circ \Phi_t).(\partial_t \Phi_t) \equiv 0$ and thus $\partial_t \Phi_t \equiv 0$ on $\Sigma \times [0, T^*]$, as $D_x f_t^2(x)$ is injective in every $x \in \Sigma$. As we know that $\Phi_0 = \mathrm{id}_\Sigma$, this shows that $\Phi_t \equiv \mathrm{id}_\Sigma$ for every $t \in [0, T^*]$ and thus in fact $f^1 = f^2$ on $\Sigma \times [0, T^*]$ by (57), which proves the entire assertion of Theorem 1. □

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Appendix

The aim of this section is to show that the right-hand side of (9) is the only modification of the usual Willmore gradient $\delta\mathcal{W}(f)$ by means of some scalar factor which only depends on f, Df and $D^2 f$, in order to obtain a Möbius-invariant flow. Let Σ be a surface without boundary and let $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^n)$ denote the open subset of $C^\infty(\Sigma, \mathbb{R}^n)$ which consists of all umbilic-free C^∞-immersions of Σ into $\mathbb{R}^n$. Firstly, we state

Proposition 4 *Let* $B : \mathbb{R}^n \times \mathbb{R}^{2n} \times \mathbb{R}^{4n} \to \mathbb{R}_+$ *denote some arbitrary real-analytic, positive function. The flow*

$$\partial_t f_t = -B(f_t, Df_t, D^2 f_t) \left(\triangle^\perp_{f_t} H_{f_t} + Q(A^0_{f_t})(H_{f_t}) \right) \quad \text{on} \quad \Sigma \times [0, T)$$

meets the property of Möbius-invariance—in the sense of Part 2 of Corollary 1—if and only if the composition $\varphi(f) := B(f, Df, D^2 f)$ *satisfies the following structure conditions:*

(1)
$$\begin{aligned} \varphi(\lambda f) &= \lambda^4 \varphi(f) \\ \varphi(M(f)) &= \varphi(f) \end{aligned} \tag{58}$$

$\forall \lambda > 0$ *and for all rigid motions* $M(y) = O(y) + c$, $O \in O(n)$, $c \in \mathbb{R}^n$.

(2) *There has to hold*

$$\varphi(I(f)) = |f|^{-8} \varphi(f) \qquad \text{on } \Sigma$$

for every immersion $f \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^n \setminus \{0\})$, *where* $I(y) := \frac{y}{|y|^2}$ *denotes inversion at the unit sphere* $\mathbb{S}^{n-1}$.

(3) *The map* $f \mapsto \varphi(f) \left(\triangle^\perp_f H_f + Q(A^0_f)(H_f) \right)$ *has to be a differential operator on* $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^n)$.

This proposition follows immediately from the statements of Lemma 1. We note that the function B does not depend on its 1st variable $y \in \mathbb{R}^n$ due to property (58) of φ. We now restrict our attention to the special case $n = 3$, because in this case we can use the precise knowledge of local, conformally invariant operators on $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3)$ due to Cairns et al. [2,3].

Theorem 4 *Let $\varphi : \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3) \to C^\infty(\Sigma, \mathbb{R}_+)$ be a real-analytic map of the form $\varphi(f) = B(Df, D^2 f)$ which satisfies the structure conditions (1)–(3) of Proposition 4. Then there holds*

$$\varphi(f) = c\,|A^0_f|^{-4}$$

for some $c > 0$ and for any $f \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3)$.

Proof By (4) and (5) one can easily compute that the map $f \mapsto |A^0_f|^{-4}$ satisfies the requirements (1)–(3) of Proposition 4 on $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3)$ and is "local of second order", which means that its value in any point $p \in \Sigma$ only depends on the first and second partial derivatives of f. Now, let $\varphi : \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3) \to C^\infty(\Sigma, \mathbb{R}_+)$ be an arbitrary map which is local of second order, i.e. of the form $\varphi(f) = B(Df, D^2 f)$, and which satisfies the conditions (1)–(3) of Proposition 4. We consider the quotient $Q(f) := \frac{\varphi(f)}{|A^0_f|^{-4}}$ and see that $f \mapsto Q(f)$ is a map from $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3)$ to $C^\infty(\Sigma, \mathbb{R}_+)$ which is again local of second order and satisfies $Q(\Phi(f)) = Q(f)$ for any Möbius-transformation Φ of $\mathbb{R}^3$ which is applicable to f. This means that Q is a local, conformally invariant operator from $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3)$ to $C^\infty(\Sigma, \mathbb{R}_+)$ of second order. Now, by Theorem 5.6 in [3] any non-constant, local and conformally invariant operator from $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbb{R}^3)$ to $C^\infty(\Sigma, \mathbb{R}_+)$ has to be at least of third order. Hence, we conclude $Q(f) \equiv \text{const.} > 0$, i.e. $\varphi(f) = c\,|A^0_f|^{-4}$, for some $c > 0$. □

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