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Separation Theorems for Group Invariant Polynomials

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Abstract We study the existence of separation theorems by polynomials that are invariant under a group action. We show that if G is a finite subgroup of $GL(n, \mathbb{C})$, K is a set in $\mathbb{C}^n$ that is invariant under the action of G and z is a point in $\mathbb{C}^n \setminus K$ that can be separated from K by a polynomial Q, then z can be separated from K by a G-invariant polynomial P. Furthermore, if Q is homogeneous then P can be chosen to be homogeneous. As a particular case, if K is a symmetric polynomially convex compact set in $\mathbb{C}^n$ and $z \notin K$ then there exists a symmetric polynomial that separates z and K.

Keywords Group invariant polynomials · Separation theorem · Symmetric polynomials · Polynomial convexity

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1 Introduction

The complex Hahn–Banach Separation Theorem establishes that if $K \subset \mathbb{C}^n$ is a closed convex set and z is a point in $\mathbb{C}^n \setminus K$, then there exists a complex linear form f with $\sup_{w\in K}\{Re(f(w))\} < Re(f(z))$, see [3, Theorem 2.12] and [5, Theorem 3.4]. Thus, if we also assume that K is balanced we can replace the real part of f by the modulus of f in the inequality, obtaining

$$\sup_{w\in K} |f(w)| < |f(z)|. \tag{1.1}$$

If K is neither convex nor balanced, the above inequality does not hold in general. It seems natural to wonder if assuming a different geometric condition on the set K, it is possible to find a "good," perhaps non-linear, function f that still satisfies the inequality. For instance, if we assume that the set K is symmetric (i.e., if $w \in K$ then $-w \in K$) and $z \notin K$, is it possible to find a "good" function f satisfying equation (1.1) and such that $f(w) = f(-w)$ for all $w \in \mathbb{C}^n$?

In this work, we answer this question in the affirmative under the assumption that the set K is invariant under the action of a finite group of linear transformations. The function f will then turn out to be a polynomial.

Definition 1.1 We say that a polynomial P *separates a set* K *and a point* z if

$$\sup_{w\in K} |P(w)| < |P(z)|.$$

As usual, we denote by $\mathcal{P}(\mathbb{C}^n)$ the set of all complex-valued polynomials defined on $\mathbb{C}^n$ and by $GL(n,\mathbb{C})$ the general linear group of degree n, consisting of the set of $n \times n$ complex invertible matrices. We consider the natural group action of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$. For a subgroup $G \leq GL(n,\mathbb{C})$ and a set $K \subset \mathbb{C}^n$ we denote by

$$\langle G, K\rangle = \{gw : g \in G, w \in K\}$$

the action of the group G on the set K and we say that K is *invariant under the action of* G if $\langle G, K\rangle = K$. Given a finite subgroup $G \leq GL(n,\mathbb{C})$, and a polynomial $P \in \mathcal{P}(\mathbb{C}^n)$, we say that P is an *invariant polynomial* under the group G or G-invariant if $P(z) = P(gz)$ for all $g \in G$ and all $z \in \mathbb{C}^n$. For more details about the theory of invariant polynomials under the action of finite groups, we recommend [2, Chap. 7].

The main result of this paper will be presented in Sect. 2. We show that if K is a set in $\mathbb{C}^n$ which is invariant under the action of G and z is a point in $\mathbb{C}^n \setminus K$ that can be separated from K by a polynomial, then z can be separated from K by a G-invariant polynomial. Furthermore, if K and z can be separated by a homogeneous polynomial then K and z can be separated by a G-invariant homogenous polynomial. In Sect. 3, we study separation theorems for polynomially convex sets. Section 4 is devoted to some examples of sets K that are polynomially convex and invariant under the action of a group G and also to some specific examples of G-invariant polynomials that separate points of $\mathbb{C}^n \setminus K$ and K.

It is worth mentioning that Theorem 2.3 and Corollary 2.4 presented in Sect. 2 and the examples provided in Sect. 4 also hold in the real case with similar proofs.

2 *G*-Invariant Separating Polynomials and Finite Groups

First we show that given a finite subgroup G of $GL(n, \mathbb{C})$ we can find arbitrarily small convex balanced compact sets with non-empty interior that are invariant under the action of G. We denote by $B_{\ell_2^n}(R)$ the closed euclidean ball of center zero and radius $R > 0$ in $\mathbb{C}^n$. When $R = 1$ we write $B_{\ell_2^n}$ instead of $B_{\ell_2^n}(1)$.

Lemma 2.1 *Given a finite subgroup G of GL(n,* $\mathbb{C}$*) and a positive number R, there exists a convex balanced compact set* $K \subset \mathbb{C}^n$ *with non-empty interior that is invariant under the action of G and* $K \subset B_{\ell_2^n}(R)$.

Proof Consider $K' = \langle G, B_{\ell_2^n}\rangle$. Then, $\langle G, K'\rangle = K'$ and since G is finite, K' is compact and hence $K' \subset B_{\ell_2^n}(M)$ for some positive constant M. Then, the absolutely convex hull of K', $absconv(K')$, is convex, balanced, and satisfies that $absconv(K') \subset B_{\ell_2^n}(M)$. Now, for every $w \in absconv(K')$ there exists m vectors $w_1, \ldots, w_m \in K'$ and m complex numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ with $\sum_{j=1}^m |\lambda_j| \leq 1$ such that $w = \sum_{j=1}^m \lambda_j w_j$. Therefore, for every $g \in G$,

$$gw = g\left(\sum_{j=1}^m \lambda_j w_j\right) = \sum_{j=1}^m \lambda_j (g w_j) \in absconv(K').$$

Thus $absconv(K')$ is invariant under the action of G.

Notice that $absconv(K')$ contains $B_{\ell_2^n}$. Hence $absconv(K')$ has non-empty interior. Since K' is a compact set in a finite-dimensional space, $absconv(K')$ is compact. The set $K = \frac{R}{M} absconv(K')$ satisfies the desired conditions. □

The following is probably folkloric.

Lemma 2.2 *Given r complex numbers of modulus one,* $\{z_1 \ldots, z_r\}$*, there exists a strictly increasing sequence of natural numbers* $(m_k)_{k=1}^\infty$ *such that the sequence* $(z_j^{m_k})$ *converges to* 1 *for* $j = 1, \ldots, r$.

Proof Denote by S^r the cartesian product of r copies of the unit sphere $S = \{w \in \mathbb{C} : |w| = 1\}$. S^r is a compact subset of $\mathbb{C}^r$. Therefore given the sequence $\big((z_1^t, \ldots, z_r^t)\big)_{t=0}^\infty$ there exists a subsequence $\{(z_1^{t_k}, \ldots, z_r^{t_k})\}_{k=0}^\infty$ convergent to a point $(w_1, \ldots, w_r) \in S^r$. Thus, for every natural number n, there exists $k(n)$ such that $\|(z_1^{t_k}, \ldots, z_r^{t_k}) - (w_1, \ldots, w_r)\| < \frac{1}{2n}$ for every $k \geq k(n)$. Consider $k'(1) > k(1)$ and, by induction, choose $k'(n) > k(n)$ such that $m_n = t_{k'(n)} - t_{k(n)}$ satisfies $m_{n+1} > m_n$ for every n. Now, we have that

$$|z_j^{m_n} - 1| = |z_j^{t_{k'(n)}} - z_j^{t_{k(n)}}| \leq |z_j^{t_{k'(n)}} - w_j| + |w_j - z_j^{t_{k(n)}}| < \frac{1}{n},$$

for $j = 1, \ldots, r$ and every $n \in \mathbb{N}$. □

Now we present the main result of this note.

Theorem 2.3 *Let G be a finite subgroup of $GL(n, \mathbb{C})$ and K a set in $\mathbb{C}^n$ that is invariant under the action of G. If z is an element in $\mathbb{C}^n \setminus K$ that can be separated from K by a polynomial Q, then there exists a G-invariant polynomial P that separates z and K. Furthermore, if Q is homogeneous, then P can be chosen to be homogeneous.*

Proof Without loss of generality we can assume that

$$\sup_{w \in K} |Q(w)| \leq \delta < 1 \text{ and } |Q(z)| > 1$$

for some real number δ.

Let us consider the sequence of polynomials

$$P_m(w) = \sum_{g \in G} \big(Q(gw)\big)^m. \tag{2.1}$$

Then P_m is a G-invariant polynomial. Furthermore, if Q is a homogeneous polynomial, then P is also homogeneous. We will show that P_m separates z and K for some big non-negative entire number m.

On the one hand, since K is invariant under the action of G, for every w in K and every $g \in G$, we have that gw is in K. Therefore $|Q(gw)| \leq \delta < 1$ and $|\big(Q(gw)\big)^m| \leq \delta^m$ for every non-negative entire number m. Thus $\sup_{w \in K} |P_m(w)| \leq |G|\delta^m$ which converges to zero as m tends to infinity.

For each g in G, let θ_g be a real number such that $Q(gz) = |Q(gz)|e^{i\theta_g}$. Consider $0 < \epsilon < 1$. By Lemma 2.2, there exists an increasing sequence of natural numbers $\{m_k\}_{k=1}^{\infty}$ such that $|(e^{i\theta_g})^{m_k} - 1| < \epsilon$ for every g in the finite set G and every $k \in \mathbb{N}$. Thus, for every natural number k,

$$\begin{aligned}
|P_{m_k}(z)| &= \left| \sum_{g \in G} \big(Q(gz)\big)^{m_k} \right| = \left| \sum_{g \in G} |(Q(gz))|^{m_k} e^{i\theta_g m_k} \right| \\
&\geq \sum_{g \in G} \big|(Q(gz))\big|^{m_k} - \sum_{g \in G} \big|(Q(gz))\big|^{m_k} \big|e^{i\theta_g m_k} - 1\big| \\
&\geq (1 - \epsilon) \sum_{g \in G} \big|(Q(gz))\big|^{m_k} \\
&\geq (1 - \epsilon)|Q(z)|^{m_k}.
\end{aligned}$$

Since $|Q(z)| > 1$ and $\sup_{w \in K} |P_{m_k}(w)| < \big|G\big|\delta^{m_k}$, we have, for k big enough,

$$|P_{m_k}(z)| > \sup_{w \in K} |P_{m_k}(w)|.$$

□

Corollary 2.4 *Let G be a finite subgroup of $GL(n, \mathbb{C})$ and K a closed convex balanced subset of $\mathbb{C}^n$. If K is invariant under the action of G and z is an element in $\mathbb{C}^n \setminus K$, then there exists a homogeneous G-invariant polynomial P that separates z and K.*

Proof Since K is a closed convex set, there exists a complex linear form Q with

$$\sup_{w\in K} \{Re(Q(w))\} < Re(Q(z)). \tag{2.2}$$

Moreover, since K is balanced

$$\sup_{w\in K} |Q(w)| = \sup_{w\in K} \{Re(Q(w))\} < Re(Q(z)) \leq |Q(z)|.$$

By Theorem 2.3, we can find a homogeneous polynomial P_m invariant under the action of G that separates the point z and the set K. □

The condition of K being balanced in Corollary 2.4 cannot be removed in general. Indeed, if K is a convex compact set invariant under the action of G and z is a point not in K, as a consequence of Theorem 3.1 below, there exists a G-invariant polynomial that separates z and K. But, if K is not balanced, then there exists a point $z_0 \in K$ and a complex number λ of modulus less than or equal to one with $\lambda z_0 \notin K$. However, for any m-homogeneous polynomial $|P(\lambda z_0)| = |\lambda^m P(z_0)| \leq |P(z_0)| \leq \sup_{z\in K} |P(z)|$. Hence, any G-invariant separating polynomial of λz_0 and K cannot be homogeneous.

3 *G*-Invariant Separating Polynomials and Polynomially Convex Sets

The definition of separating polynomial is closely related to that of polynomially convex set. Recall that the *polynomially convex hull* of a compact subset K of $\mathbb{C}^n$ is the set

$$\widehat{K} = \left\{ z \in \mathbb{C}^n : |P(z)| \leq \sup_{w\in K} |P(w)| \text{ for every polynomial } P \right\}.$$

A compact set K of $\mathbb{C}^n$ is *polynomially convex* if $\hat{K} = K$. By definition we have that for a compact set $K \subset \mathbb{C}^n$, every point in $\mathbb{C}^n \setminus K$ can be separated from K by a polynomial if and only if the set K is polynomially convex. Some classical examples of polynomially convex sets are compact convex sets and the closure of polynomial polyhedrons. Here, for a set of polynomials $p_1, \ldots, p_m$ on $\mathbb{C}^n$ and a positive number r, the polynomial polyhedron defined by $p_1, \ldots, p_m$ of radius r is the set

$$K = \{(w_1, \ldots, w_n) \in \mathbb{C}^n : |w_j| < r, |p_j(w)| < r, 1 \leq j \leq n\}.$$

See [9] for a comprehensive reference about polynomial convexity.

Since we are working with polynomials of several complex variables, one can ask if it is possible to obtain a proof of Theorem 2.3 by using only techniques of complex analysis. The following result is a weaker version of Theorem 2.3 proved in this way.

Moreover, the proof presented here has some limitations. This proof only works for compact sets, and if we assume that the compact set K is convex and balanced, as in Corollary 2.4, we cannot ensure that the separating polynomials are homogeneous. Nevertheless, we feel that the argument is worth describing.

Theorem 3.1 *Let G be a finite subgroup of $GL(n, \mathbb{C})$. If K is a compact polynomially convex set that is invariant under the action of G and z is an element in $\mathbb{C}^n \setminus K$, then there exists a G-invariant polynomial P separating z and K.*

Proof Since G is a finite group, the orbit of the point z under the action of G, $Orb(z) = \{gz : g \in G\}$, has at most $|G|$ elements. In particular, $Orb(z)$ has zero length in $\mathbb{C}^n$ and $Orb(z) \cup K$ is compact. Then, by [9, Corollary 1.6.3], $Orb(z) \cup K$ is polynomially convex.

Since K is invariant under the action of the group G, the sets K and $Orb(z)$ are disjoint. Thus, as a consequence of the Oka–Weil theorem [9, Theorem 1.5.1], there exists a polynomial Q with $|Q(w) - 3/2| \leq \frac{1}{2}$ for all $w \in Orb(z)$ and $\sup_{w \in K} |Q(w)| \leq \delta < 1$. Since G is a subgroup of $GL(n, \mathbb{C})$, every element g of G is a linear transformation in $\mathbb{C}^n$. Therefore $Q \circ g$ is again a polynomial and since K is invariant under the action of G, $\sup_{w \in K} |(Q \circ g)(w)| < \delta$, for all $g \in G$. If we consider the polynomial

$$P(w) = \sum_{g \in G} (Q \circ g)(w)$$

we have that for every $w \in K$, $|P(w)| \leq \delta |G|$, and

$$|P(z)| \geq Re(P(z)) = Re \left(\sum_{g \in G} Q(gz) \right) \geq \sum_{g \in G} 1 = |G|.$$

Therefore P separates z and K. □

Remark 3.2 Let G be a finite subgroup of $GL(n, \mathbb{C})$ and K a compact set invariant under the action of G. The existence, for each $z \in \mathbb{C}^n \setminus K$, of a G-invariant polynomial that separates z and K is actually a characterization of K being polynomially convex. That is, $\widehat{K} = K$ if and only if every element of $\mathbb{C}^n \setminus K$ can be separated from K by a G-invariant polynomial. One implication is given by Theorem 3.1. It is easy to check that the converse implication also holds.

We finish this section by presenting a way to find polynomially convex sets that are invariant under the action of finite unitary reflection groups.

We recall that a *finite unitary reflection group* G is a finite subgroup of $GL(n, \mathbb{C})$ of unitary transformations that is generated by the reflections that it contains. By *reflection* we understand a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ that fixes pointwise only a hyperplane of dimension $n - 1$. The finite unitary groups were studied and classified by Shephard and Todd [8] and Flatto [4]. Even more, Hilbert proved that for any finite unitary reflection group G there exists a finite family of polynomials

$p_1, \ldots, p_r$ that are G-invariant and form a basis for the set of G-invariant polynomials. That is, given a G-invariant polynomial P, there exists a unique polynomial Q such that

$$P(w) = Q(p_1(w), \ldots, p_r(w)).$$

Chevalley showed that in fact for finite unitary reflection groups this basis can be chosen to be exactly n homogeneous G-invariant polynomials. See [2, §2.5 Theorem 4] and [1, Theorem (A)] for the details.

A classic example of finite unitary reflection group is the symmetric group of order n, consisting of the group of permutations of the set $\{1, \ldots, n\}$ and denoted by S_n. S_n can be naturally considered as a subgroup of $GL(n, \mathbb{C}^n)$, the action of a permutation $\sigma \in S_n$ on a point $(w_1, \ldots, w_n) \in \mathbb{C}^n$ being given by $\sigma(w_1, \ldots, w_n) = (w_{\sigma(1)}, \ldots, w_{\sigma(n)})$. For this group, a basis of S_n-invariant polynomials is given by the set of polynomials

$$p_m(w_1, \ldots, w_n) = \sum_{1 \le j_1 < \ldots < j_m \le n} w_{j_1} \cdots w_{j_m}, \quad j = 1, \ldots, n;$$

see [2, Theorem 3].

To prove the following proposition, we will use the definition of proper maps between open connected sets Ω and Ω' in $\mathbb{C}^n$. Recall that a map $F : \Omega \to \Omega'$ is said to be *proper* if for every compact set $C \subset \Omega'$, $F^{-1}(C)$ is compact in Ω; see, e.g., [6, Chap. 15].

Proposition 3.3 *Let G be a finite unitary reflection subgroup of $GL(n, \mathbb{C})$ with invariant polynomials $p_1, \ldots, p_n$. Consider the mapping F from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined by $F(w) = (p_1(w), \ldots, p_n(w))$. Then, for every polynomially convex compact set X in $\mathbb{C}^n$, the compact set $K = F^{-1}(X)$ is polynomially convex and invariant under the action of G.*

Proof Fix a polynomially convex compact set X in $\mathbb{C}^n$. The same proof used in [7, Proposition 2.1] can be used to show that the map $F : \mathbb{C}^n \mapsto \mathbb{C}^n$ is proper. Since X is polynomially convex, by [9, Theorem 1.6.24], K is a polynomially convex compact set. To finish we only need to show that $\langle G, K \rangle = K$. For this, fix $w \in K$. Then, for any $g \in G$ and any polynomial p_i, since p_i is G-invariant, we have that $p_i(gw) = p_i(w)$. Therefore, $gw \in F^{-1}(F(w)) \subseteq K$. □

4 Examples

In Theorem 2.3, we proved that if K is invariant under the action of a group G and if a point z can be separated from K by a polynomial Q, then there exists a G-invariant polynomial that separates z and K. Naturally the first idea to find this G-invariant polynomial is to consider the polynomial

$$P(w) = \sum_{g \in G} Q(gw).$$

Our first example shows that this approach does not work in general and that taking powers in equation (2.1) is necessary.

Example 1 Let us consider the set

$$K = \{(w, w) \in \mathbb{C}^2 : w \in \mathbb{C}\}.$$

Then K is invariant under the action of the group S_2. Furthermore, every point that does not belong to K can be separated from K by the polynomial $Q(w_1, w_2) = w_1 - w_2$. However the polynomial

$$P(w_1, w_2) = Q(w_1, w_2) + Q(w_2, w_1)$$

is identically zero. Hence P is S_2-invariant but does not separate any point in $\mathbb{C}^n \setminus K$ from K.

To continue, we present examples of sets that are invariant under the action of a group.

Example 2 Given a finite group $G \leq GL(n, \mathbb{C}^n)$, a set $\{p_1, \dots, p_r\}$ of G-invariant polynomials and r non-negative numbers $R_1, \dots, R_r$, the set

$$K = \cap_{j=1}^r p_j^{-1}(R_j \mathbb{D})$$

is invariant under the action of G. Here $R\mathbb{D}$ denotes the *closed* disk of center zero and radius $R > 0$ in $\mathbb{C}$. Furthermore, every point not in K can be separated from K by one of the G-invariant polynomials p_j, $j = 1, \dots, r$. Notice that K is not necessarily bounded.

For the rest of the examples that we consider, the sets K are polynomially convex.

Example 3 (*Finite unitary reflection groups*)

(a) For every $p \in [1, \infty)$, the set

$$B_{\ell_p^n} = \left\{ (w_1, \dots, w_n) \in \mathbb{C}^n : \|(w_1, \dots, w_n)\|_p = \left(\sum_{j=1}^n |w_i|^p \right)^{\frac{1}{p}} \leq 1 \right\}$$

is convex, hence polynomially convex and invariant under action of the symmetric group S_n.

(b) For every finite unitary reflection group G with basis of invariant polynomials $\{p_1, \dots, p_n\}$, the closure of the polynomial polyhedron defined by $p_1, \dots, p_n$ and radius r,

$$K = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |p_j(w)| \leq r, 1 \leq j \leq n\},$$

is polynomially convex and invariant under the action of G.

(c) Now we give an easy example of a set in $\mathbb{C}^2$ that is polynomially convex and balanced but not convex. As we mentioned before, the set of S_2-invariant polynomials is generated by the polynomials $p_1(w_1, w_2) = w_1 + w_2$ and $p_2(w_1, w_2) = w_1 w_2$. Then, for each $R > 0$, the set

$$K = (R\mathbb{D} \times \{0\}) \cup (\{0\} \times R\mathbb{D}),$$

is the inverse image of $X = R\mathbb{D} \times \{0\}$ by the proper mapping $F(w_1, w_2) = (w_1 + w_2, w_1 w_2)$. X is a convex set and hence polynomially convex in $\mathbb{C}^2$. Therefore, by Proposition 3.3, any point in $\mathbb{C}^2 \setminus K$ can be separated from K by a polynomial which is invariant under permutations.

Next, we provide an example where the group G is not a finite unitary reflection group.

Example 4 (*Symmetric sets*) If we consider any balanced, bounded, and convex set K in $\mathbb{C}^n$, then K is polynomially convex and invariant under the action of the group $G = \{Id, -Id\} \leq GL(n, \mathbb{C}^n)$, where Id denotes the identity permutation. Notice that since G only fixes the origin in $\mathbb{C}^n$, G is not a finite unitary reflection group for $n \geq 2$.

To continue, we present some specific examples of separating polynomials for common groups and polynomially convex sets in $\mathbb{C}^n$. The following example can be seen as a particular case of Proposition 3.3. However, here we present a direct proof of the result. The idea presented in this construction was the motivation for the proof of Theorem 2.3.

Example 5 (*The polydisk center zero and radius r*) Let K be the closed polydisk in $\mathbb{C}^n$ of radius $r > 0$ and center zero, $K = \{(w_1, \ldots, w_n) \in \mathbb{C}^n : |w_j| \leq r\}$, and G the group S_n. As we mentioned before, a polynomial P is invariant under the group S_n if for any permutation $\sigma \in S_n$ and any point $(w_1, \ldots, w_n)$,

$$P(w_1, \ldots, w_n) = P(w_{\sigma(1)}, \ldots, w_{\sigma(n)}).$$

For $j = 1, \ldots, n$ let us denote by Q_j the linear functional on $\mathbb{C}^n$ defined by $Q_j(w_1, \ldots, w_n) = \frac{w_j}{r}$. If $z = (z_1, \ldots, z_n)$ is a point in $\mathbb{C}^n$ not contained in K, then at least one of the coordinates z_{j_0} has modulus bigger than r. Then the linear functional Q_{j_0} satisfies that $|Q_{j_0}(z)| > 1 = \sup_{w \in K} |Q_{j_0}(w)|$.

Then, for every natural number m the polynomial

$$P_m(w) = \sum_{j=1}^{n} (Q_j(w))^m$$

is an m-homogeneous S_n-invariant polynomial. As a consequence of Lemma 2.2, and employing the same ideas used in the proof of Theorem 2.3, the lim sup of the sequence

$\{|P_m(z)|\}_{m=1}^{\infty}$ goes to infinity as m goes to infinity. Therefore there exists m_0 such that for for $m \geq m_0$ we have $|P_m(z)| > n$. However,

$$\sup_{w \in K} |P_m(w)| \leq n$$

for every natural number m. Hence for $m \geq m_0$ the m-homogeneous S_n-invariant polynomial P_m separates the point z and the set K.

In the following example, we explicitly give the separating polynomials for some of the sets presented in Example 3. It is worth mentioning that in the above example, all the separating polynomials depend on the function Q_1 and only the number m depends on the point z that we want to separate from K. In the following example, the separating polynomials cannot be obtained in this way and it is necessary to consider different linear forms Q and different numbers m, both of them depending on the point z that we want to separate from K.

Example 6 (*Finite unitary reflection groups*)

(a) If we consider $K = B_{\ell_2^n} \subset \mathbb{C}^n$ and the group S_n, then for every $(z_1, \ldots, z_n) \notin K$ we have that the linear form

$$Q(w_1, \ldots, w_n) = \frac{1}{\|z\|_2^2} \sum_{j=1}^{n} \overline{z_j} w_j$$

satisfies $Q(z) = 1 = \|Q\|$ and $\sup_{w \in K} |Q(w)| < 1$. Therefore a similar argument to the one given above shows that for a natural number m the m-homogeneous polynomial

$$P_m(w) = \sum_{g \in S_n} (Q(gw))^m$$

is S_n-invariant and separates the point z and the set K.

(b) Let us consider $G = S_3 \times S_2$. Then G is a finite unitary reflection group of order 12. Let us consider the set $K = B_{\ell_2^3} \times 2\mathbb{D}^2$, i.e.,

$$K := \{(w_1, \ldots, w_5) : |w_1|^2 + |w_2|^2 + |w_3|^2 \leq 1 \text{ and } |w_4|, |w_5| \leq 2\}.$$

Clearly $\langle G, K \rangle = K$ and K is polynomially convex. However, it is easy to see that K is not invariant under the group S_5.

Let z be a point not in K. Since z is not in K then $|z_1|^2 + |z_2|^2 + |z_3|^2 > 1$ or one of the coordinates z_4, z_5 has modulus bigger than two.

If $|z_1|^2 + |z_2|^2 + |z_3|^2 > 1$, then we consider the linear functional Q and the m-homogeneous polynomials P_m defined by

$$Q(w_1, \ldots, w_5) = \frac{1}{(|z_1|^2 + |z_2|^2 + |z_3|^2)^{\frac{1}{2}}} \sum_{j=1}^{3} \overline{z_j} w_j \quad \text{and}$$
$$P_m(w) = \sum_{g \in S_3} (Q(gw))^m.$$

If z_4 or z_5 has modulus bigger than two, we consider the m-homogeneous polynomials P_m defined by

$$P_m(w_1, \ldots, w_5) = \left(\frac{w_4}{2}\right)^m + \left(\frac{w_5}{2}\right)^m.$$

The same argument that was given in the above examples shows that for some natural number m the polynomial P_m separates the point z and the compact K.

Example 6(b) can be generalized in the following way. Let us consider the sets $K_1 \subset \mathbb{C}^{n_1}, \ldots, K_r \subset \mathbb{C}^{n_r}$ invariant under the action of the groups $G_1, \ldots, G_r$, respectively. Then, if $(z_1, \ldots, z_r)$ is a point not in $K_1 \times \cdots \times K_r$, for some $j \in \{1, \ldots, r\}$, $z_j \notin K_j$. If P_j is a polynomial in $\mathbb{C}^{n_j}$ that separates z_j from K_j, then, for some natural number m the polynomial

$$P(w) = \sum_{g \in G_j} (P_j(gw))^m$$

is a G_j-invariant polynomial that separates z_j and K_j.

By considering the natural embedding of the set K_j in $K_1 \times \cdots \times K_r$ and the natural embedding of the group G_j in the group $G_1 \times \cdots \times G_r$ we have that the polynomial associated to P is $G_1 \times \cdots \times G_r$-invariant and separates the point $(z_1, \ldots, z_r)$ and the set $K_1 \times \cdots \times K_r$.

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