J Geom Anal (2017) 27:1951–1994
DOI 10.1007/s12220-016-9747-2

On the Geometry of Metric Measure Spaces with Variable Curvature Bounds

Christian Ketterer[1]

Received: 14 September 2015 / Published online: 26 September 2016

Abstract Motivated by a classical comparison result of J. C. F. Sturm, we introduce a curvature-dimension condition $CD(k, N)$ for general metric measure spaces, variable lower curvature bound k and upper dimension bound $N \geq 1$. In the case of non-zero constant lower curvature, our approach coincides with the celebrated condition that was proposed by Sturm (Acta Math 196(1):133–177, 2006). We prove several geometric properties as sharp Bishop–Gromov volume growth comparison or a sharp generalized Bonnet–Myers theorem (Schneider's Theorem). In addition, the curvature-dimension condition is stable with respect to measured Gromov–Hausdorff convergence, and it is stable with respect to tensorization of finitely many metric measure spaces provided a non-branching condition is assumed. We also briefly describe possible extensions for variable dimension bounds.

Keywords Optimal transport · Curvature-dimension · Variable curvature · Generalized Myer theorem

Mathematics Subject Classification 51F02 · 53A02

1 Introduction

Metric measure spaces with generalized lower Ricci curvature bounds have become objects of interest in various fields of mathematics. Since Lott, Sturm, and Villani introduced the so-called curvature-dimension condition $CD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ via displacement convexity of the Shanon and Reny entropy on the

✉ Christian Ketterer
christian.ketterer@math.uni-freiburg.de

[1] Albert-Ludwigs-Universität Freiburg, Freiburg, Germany

L^2-Wasserstein space [24,33,34], a rather complete picture of the geometric and analytic properties of these spaces has been developed (e.g. [1,2,14,16,20–22,30]). Their approach is based on and inspired by recent fundamental breakthroughs in the theory of optimal transport (e.g. [7,9,25,27]).

However, the condition of lower-bounded Ricci curvature is also very restrictive. Neither non-compact smooth Riemannian manifolds do admit global lower curvature bounds in general, nor does Hamilton's Ricci flow in general. Moreover, one cannot exceed the information that is encoded by the constant curvature bound. Therefore, the regime of results is limited. However, in the context of smooth Riemannian manifolds, variable lower Ricci curvature plays an important role. One can deduce refined statements for the geometry of the space, e.g. [5,18,28,29,31,36]. Therefore, it seems natural to ask for an extension of the theory of Lott, Sturm, and Villani. For dimension-independent situations, a definition is proposed by Sturm in [35]. However, to deduce refined geometric statements, one also must bring a dimension bound into play.

In this article, we will focus on the finite dimensional case and introduce a curvature-dimension condition $CD(k,N)$ for metric measure spaces $(X, \mathrm{d}_X, \mathrm{m}_X)$, where the lower curvature bound $k : X \to \mathbb{R}$ is a lower semi-continuous function and $N = const \geq 1$ (variable dimension bounds will be discussed in Sect. 9). Before we describe our approach, let us recall that Lott, Sturm, and Villani define the curvature-dimension condition $CD(0,N)$ of an arbitrary metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ via displacement convexity for the N-Rény entropy functional

$$S_N(\varrho\, \mathrm{m}_X) = -\int_X \varrho^{1-\frac{1}{N}} d\, \mathrm{m}_X\, .$$

(The definitions in [24] and in [34] slightly differ.) In [34], Sturm gave a definition of $CD(K,N)$ for general $K \in \mathbb{R}$ via the so-called *distorted displacement convexity* (see also [37]). This approach involves the concept of modified volume distortion coefficients $\tau_{k,N}^{(t)}(\theta)$ that do not come from a linear ODE but are motivated by the geometry of Riemannian manifolds. They capture the geometric fact that Ricci curvature of a tangent vector v is the mean value of sectional curvatures of planes intersecting in v. Roughly speaking, non-zero curvature only happens perpendicular to v. Our idea is to introduce generalized volume distortion coefficients as follows. We define

$$\tau_{k_\gamma,N}^{(t)}(\theta) = t^{\frac{1}{N}} \left[\sigma_{k_\gamma,N-1}^{(t)}(\theta)\right]^{\frac{N-1}{N}}$$

where $k_\gamma(t\theta) = k \circ \gamma(t)$, $\gamma : [0,1] \to X$ is a constant-speed geodesic and $\sigma_{k_\gamma,N}^{(t)}(\theta)$ is the solution of

$$u''(t) + \tfrac{k(\gamma(t))}{N}\theta^2 u = 0 \tag{1}$$

with $u(0) = 0$ and $u(1) = 1$ where $\theta = |\dot\gamma|$. We remark, that in the case of constant curvature $k = K$ this yields $\sigma_{K,N}^{(t)}(|\dot\gamma|) = \sin_{K/N}(t|\dot\gamma|)/\sin_{K/N}(|\dot\gamma|)$ that is precisely the definition of Sturm in [34].

A key property of the distortion coefficients is their monotonicity w.r.t. k which is a consequence of a classical comparison result of J. C. F. Sturm for one-dimensional Sturm–Liouville-type operators.

Theorem 1.1 (Sturm's comparison theorem) *Let $k, k' : [a, b] \to \mathbb{R}$ be continuous function such that $k' \geq k$ on $[a, b]$ and $\mathfrak{s}_{k'} > 0$ on $(a, b]$. Then, $\mathfrak{s}_k \geq \mathfrak{s}_{k'}$ on $[a, b]$.*

$\mathfrak{s}_k$ is a solution of (1) with $\theta k/N = k$ and $\gamma(t) = t$, an initial condition $u(0) = 0$ and $u'(0) = 1$. The theorem is well known in the context of Riemannian manifolds and smooth Jacobi field calculus. Its geometric counterpart is the celebrated Rauch comparison theorem.

In particular, from generalized distortion coefficients, we also obtain a new characterization of the differential inequality $u'' \leq -ku$ (see Proposition 3.8) that appears naturally in connection with lower curvature bounds on smooth Riemannian manifolds.

Then our curvature-dimension condition takes the following form. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space as in Definition 2.1 and assume for simplicity that for m_X^2-a.e. pair (x, y) there exists a unique geodesic. Then $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the condition $CD(k, N)$ for $N \geq 1$ and a lower semi-continuous function $k : X \to \mathbb{R}$, if for any pair of absolutely continuous probability measures μ_0 and μ_1 on X with bounded support, there exists a dynamic optimal coupling $\Pi \in \mathcal{P}(\mathcal{G}(X))$ such that $(e_t)_\star \Pi = \mu_t$ is an L^2-Wasserstein geodesic in $\mathcal{P}^2(\mathrm{m}_X)$ and

$$\varrho_t(\gamma_t)^{-\frac{1}{N}} \geq \tau^{(1-t)}_{k^-_\gamma, N'}(|\dot{\gamma}|)\varrho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{k^+_\gamma, N}(|\dot{\gamma}|)\varrho_1(\gamma_0)^{-\frac{1}{N}}.$$

for all $t \in [0, 1]$ and Π-a.e. geodesic γ. Here $k^+_\gamma = k_\gamma$ and $k^-_\gamma = k_{\gamma^-}$ where γ^- is the time reverse reparametrization of γ. ϱ_t is the density of the push-forward of Π under the map $\gamma \mapsto \gamma_t$. If we replace $\tau_{k,N}$ by $\sigma_{k/N}$, we say X satisfies the reduced curvature-dimension condition $CD^*(k, N)$. Let us emphasize that we do not assume any non-branching assumption for the metric measure space in general, and we also do not assume a quadratic Cheeger energy as in [1] or an a priori lower curvature bound as in [35].

This is the first part of two articles where we investigate the geometric and and analytic consequences of our curvature-dimension condition. The main results in this article are

- The condition $CD(k, N)$ for $N \in [1, \infty)$ implies $CD(k, \infty)$ in the sense of [35] (Proposition 4.11).
- For Riemannian manifolds, the curvature-dimension condition $CD(k, N)$ is equivalent to a lower bound k for the Ricci tensor and an upper bound N for the dimension (Theorem 4.12).
- A generalized Brunn–Minkowski theorem and a generalized Bishop–Gromov comparison theorem hold (Theorems 5.1, 5.3, 5.9). The latter result in particular yields a local volume doubling property and finite Hausdorff dimension for the support of m_X.
- A generalized Bonnet–Myers theorem (Theorem 5.10). This is a non-smooth version of a result by Schneider [31] (see also [3,15]). It states that if the curvature

does not decrease too quickly for large distances from a point in the support of the measure, then the support is compact with explicit bound for the diameter. There are also similar statements in the context of smooth Finsler manifolds and for the Bakry–Emery–Ricci tensor in a smooth context [4,38].
- The curvature-dimension condition is stable with respect to measured Gromov convergence (Theorem 6.6). In particular, it implies that any family of compact Riemannian manifolds with uniform upper bound for the dimension, uniform upper bound for the diameter, and Ricci curvature uniformly bounded from below admit a converging subsequence such that the liminf function of the variable lower Ricci curvature bounds (that is the function that assigns to each point the smallest eigenvalue of the Ricci tensor) is a lower Ricci curvature bound for the limit space (Corollary 6.8).
- The curvature-dimension condition is stable under tensorization of finitely many metric measure spaces provided a non-branching assumption is satisfied (Theorem 7.4).
- The reduced curvature-dimension condition admits a globalization property (Theorem 8.3).

In the forthcoming addendum to this article [19], we also investigate variants of the condition $CD(k, N)$. Namely, following [14,26], we introduce an entropic curvature-dimension condition and a measure contraction property as well as an $EVI_{k,N}$-condition for gradient flows on metric spaces where k is a lower semi-continuous function. We will investigate their relation to each other and also to the reduced curvature-dimension condition presented in this paper. Given stronger regularity assumptions, we establish various equivalences and consequences.

In addition, considering the recent approach of Cavalletti and Mondino in [11] to prove isoperimetric inequalities and various other functional inequalities in the context of non-branching CD-spaces with constant curvature bound, our approach seems very well adapted for transforming their ideas to a non-constant curvature setting.

In the second section of this paper, we will present necessary preliminaries of optimal transport, Wasserstein calculus and geometry of metric spaces. In Sect. 3, we will introduce generalized distortion coefficients and we will present a new characterization of ku-convexity of a function u. In Sect. 4 we give the definition of $CD(k, N)$ in the general context of metric measure spaces, and in particular we will prove that is consistent with Sturm's definition in [35]. The topic of Sect. 5 will be geometric consequences of the curvature-dimension condition. In Sects. 6, 7, and 8, we will prove the stability property, the tensorization property under a branching assumption, and the globalization property of the reduced curvature-dimension condition, respectively.

In Sect. 9, we briefly discuss extensions of our approach that also capture a variable dimension bound. This is not obvious since the condition $CD(k, N)$ is defined via N-Reny entropy functionals where $N > 0$ has to be a constant parameter.

2 Preliminaries

Definition 2.1 *(Metric measure space)* Let (X, d_X) be a complete and separable metric space, and let m_X be a locally finite Borel measure on (X, d_X). That is, for all $x \in X$

there exists $r > 0$ such that $\mathrm{m}_X(B_r(x)) \in (0, \infty)$. Let $\mathcal{O}_X$ and $\mathcal{B}_X$ be the topology of open sets and the family of Borel sets, respectively. A triple $(X, \mathrm{d}_X, \mathrm{m}_X)$ will be called *metric measure space*. We assume that $\mathrm{m}_X(X) \neq 0$. If $\mathrm{m}_X(X) = 1$ we say $(X, \mathrm{d}_X, \mathrm{m}_X)$ is normalized.

(X, d_X) is called *length space* if $\mathrm{d}_X(x, y) = \inf \mathrm{L}(\gamma)$ for all $x, y \in X$, where the infimum runs over all rectifiable curves γ in X connecting x and y. (X, d_X) is called *geodesic space* if every two points $x, y \in X$ are connected by a curve γ such that $\mathrm{d}_X(x, y) = \mathrm{L}(\gamma)$. Distance minimizing curves of constant speed are called *geodesics*. A length space, which is complete and locally compact, is a geodesic space and proper [6, Theorem 2.5.23]. Rectifiable curves always admit a reparametrization proportional to arc length, and therefore become Lipschitz curves. In general, we assume that a geodesic $\gamma : [0, 1] \to X$ is parametrized proportional to its length, and the set of all such geodesics $\gamma : [0, 1] \to X$ is denoted with $\mathcal{G}(X)$. The set of all Lipschitz curves $\gamma : [0, 1] \to X$ parametrized proportional to arc-length is denoted with $\mathcal{LC}(X)$. (X, d_X) is called *non-branching* if for every quadruple (z, x_0, x_1, x_2) of points in X for which z is a midpoint of x_0 and x_1 as well as of x_0 and x_2, it follows that $x_1 = x_2$.

$\mathcal{P}(X)$ denotes the space of probability measures on $(X, \mathcal{B}_X)$, and $\mathcal{P}_2(X, \mathrm{d}_X) =: \mathcal{P}_2(X)$ denotes the L^2-*Wasserstein space* of probability measures μ on $(X, \mathcal{B}_X)$ with finite second moments, which means that $\int_X \mathrm{d}_X^2(x_0, x)d\mu(x) < \infty$ for some (hence all) $x_0 \in X$. The L^2-*Wasserstein distance* $\mathrm{d}_W(\mu_0, \mu_1)$ between two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is defined as

$$\mathrm{d}_W(\mu_0, \mu_1) := \mathrm{d}_{X,W}(\mu_0, \mu_1) := \sqrt{\inf_\pi \int_{X \times X} \mathrm{d}_X^2(x, y)\, d\pi(x, y)}. \tag{2}$$

Here the infimum ranges over all *couplings* of μ_0 and μ_1, i.e. over all probability measures on $X \times X$ with marginals μ_0 and μ_1. $(\mathcal{P}_2(X), \mathrm{d}_W)$ is a complete separable metric space. The subspace of m_X-absolutely continuous measures is denoted by $\mathcal{P}_2(X, \mathrm{m}_X) =: \mathcal{P}_2(\mathrm{m}_X)$. A minimizer of (2) always exists and is called *optimal coupling* between μ_0 and μ_1.

A probability measure Π on $\mathcal{G}(X)$ is called *dynamic optimal transference plan* if and only if the probability measure $(e_0, e_1)_*\Pi$ on $X \times X$ is an optimal coupling of the probability measures $(e_0)_*\Pi$ and $(e_1)_*\Pi$ on X. Here and in the sequel $e_t : \Gamma(X) \to X$ for $t \in [0, 1]$ denotes the evaluation map $\gamma \mapsto \gamma_t$. An absolutely continuous curve μ_t in $\mathcal{P}_2(X, \mathrm{m}_X)$ is a geodesic if and only if there is a dynamic optimal transference plan Π such that $(e_t)_*\Pi = \mu_t$. We write $\mathrm{DyCpl}(\mu_0, \mu_1)$ for the set of dynamic optimal transference plans between μ_0 and μ_1.

Let us recall the notion of *Markov kernel*. Let (Y, d_Y) be a separable and complete metric space. A Markov kernel is a map $Q : Y \times \mathcal{B}_Y \to [0, 1]$ with the following properties. $Q(y, \cdot)$ is a probability measure for each $y \in Y$. The function $Q(\cdot, A)$ is measurable for each $A \in \mathcal{B}_X$.

Lemma 2.2 *For each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there exists a dynamic optimal coupling Π such that*

$$\mathrm{d}_W(\mu_0, \mu_1)^2 = \int \mathrm{d}_X(\gamma(0), \gamma(1)) d\Pi(\gamma).$$

and there exist Markov kernels Π_{x_0,x_1}, Π_{x_0} *and* Π_{x_1} *such that*

$$d\Pi(\gamma) = d\Pi_{x_0,x_1}(\gamma)d\pi(x_0,x_1) = d\Pi_{x_0}(\gamma)d\mu_0(x_0) = d\Pi_{x_1}(\gamma)d\mu_1(x_1)$$

where $(e_0, e_1)_\star\Pi =: \pi$.

Proof For the existence of a dynamic optimal coupling, see [37]. The existence of the corresponding Markov kernels comes from the existence of regular conditional probability measures. □

3 *ku*-Convexity

Let $k : [a, b] \to \mathbb{R}$ be a continuous function. We study solutions of

$$v'' + kv = 0. \tag{3}$$

The generalized sin-functions $\mathfrak{s}_k : [a, b] \to \mathbb{R}$ is the unique solution of (3) such that $\mathfrak{s}_k(a) = 0$ and $\mathfrak{s}_k'(a) = 1$. The generalized cos-function is $\mathfrak{c}_k = \mathfrak{s}_k'$. Solutions of (3) depend continuously on the coefficient k. More precisely, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|k - k'|_\infty < \delta$ implies $|\,\mathfrak{s}_k - \mathfrak{s}_{k'}\,|_\infty < \epsilon$ where $k, k' : [a, b] \to \mathbb{R}$ are continuous. If $\gamma(t) = (1 - t)a + tb$ and $v : [a, b] \to \mathbb{R}$ is any solution of (3), then $v \circ \gamma = u : [0, 1] \to \mathbb{R}$ solves

$$u'' + k \circ \gamma|\dot{\gamma}|^2 u = 0. \tag{4}$$

In particular, $\mathfrak{s}_k(\gamma_t)$ solves (4) with $\mathfrak{s}_k(\gamma_0) = 0$ and $\frac{d}{dt}|_{t=0}\,\mathfrak{s}_k(\gamma_t) = |\dot{\gamma}(0)| = b - a$.
The next theorems are well-known.

Theorem 3.1 (J. C. F. Sturm's comparison theorem) *Let* $k, k' : [a, b] \to \mathbb{R}$ *be continuous function such that* $k' \geq k$ *on* $[a, b]$ *and* $\mathfrak{s}_{k'} > 0$ *on* $(a, b]$. *Then* $\mathfrak{s}_k \geq \mathfrak{s}_{k'}$.

Theorem 3.2 (Sturm–Picone oscillation theorem) *Let* $k, k' : [a, b] \to \mathbb{R}$ *be continuous such that* $k' \geq k$ *on* $[a, b]$. *Let* u *and* v *be solutions of* (3) *with respect to* k *and* k' *respectively. If* $u(a) = u(b) = 0$ *and* $u > 0$ *on* (a, b), *then either* $u = \lambda v$ *for some* $\lambda > 0$ *or there exists* $x_1 \in (a, b]$ *such that* $v(x_1) = 0$.

Definition 3.3 (generalized distortion coefficients) Consider $k : [0, L] \to \mathbb{R}$ that is continuous and $\theta \in (0, L]$. Then

$$\sigma_k^{(t)}(\theta) = \begin{cases} \frac{\mathfrak{s}_k(t\theta)}{\mathfrak{s}_k(\theta)} & \text{if } \mathfrak{s}_k\,|_{(0,\theta]} > c > 0, \\ \infty & \text{otherwise }. \end{cases}$$

We also define $\pi_k = \sup\{t \in [0, L] : \mathfrak{s}_k(s) > 0 \text{ for all } s \leq t\}$. If $\sigma_k^{(t)}(\theta) < \infty$, $t \mapsto \sigma_k^{(t)}(\theta)$ is a solution of

$$u''(t) + k(t\theta)\theta^2 u(t) = 0 \tag{5}$$

satisfying $u(0) = 0$ and $u(1) = 1$.

Proposition 3.4 *$\sigma_k^{(t)}(\theta)$ is non-decreasing with respect to $k : [0, \theta] \to \mathbb{R}$. More precisely $k(x) \geq k'(x)\ \forall x \in [0, \theta]$ implies $\sigma_k^{(t)}(\theta) \geq \sigma_{k'}^{(t)}(\theta)\ \forall t \in [0, 1]$.*

Proof Consider $\sigma_k^{(t)}(\theta)$ and $\sigma_{k'}^{(t)}(\theta)$ for k and k' such that $k(t) \geq k'(t)$ for all $t \in [0, 1]$. By Sturm–Picone oscillation theorem, $\sigma_k^{(t)}(\theta) = \infty$ implies $\sigma_{k'}^{(t)}(\theta) = \infty$. Hence, we only need to check the case when $\sigma_k^{(t)}(\theta) < \infty$ and $\sigma_{k'}^{(t)}(\theta) < \infty$.

We use the idea of the proof of Theorem 14.28 in [37]. We know that $\sigma_k^{(0)}(\theta) = \sigma_{k'}^{(0)}(\theta) = 0$ and $\sigma_k^{(1)}(\theta) = \sigma_{k'}^{(1)}(\theta) = 1$. Consider $\sigma_{k'}^{(t)}(\theta)/\sigma_k^{(t)}(\theta) =: h(t)$ for $t \in (0, 1]$. We know that $h(1) = 1$ and L'Hospital's rule yields

$$\lim_{t \downarrow 0} h(t) = \frac{\mathfrak{s}_k(\theta)}{\mathfrak{s}_{k'}(\theta)} \lim_{t \downarrow 0} \frac{\mathfrak{c}_{k'}(t\theta)}{\mathfrak{c}_k(t\theta)} = \frac{\mathfrak{s}_k(\theta)}{\mathfrak{s}_{k'}(\theta)} \leq 1.$$

Hence, it is sufficient to check that $h(t)$ has no local maximum in $(0, 1)$. For this reason, first we assume that $k > k'$. Set $\sigma_{k'}^{(t)}(\theta) = f$ and $\sigma_k^{(t)}(\theta) = g$. Assume there is a maximum in $t_0 \in (0, 1)$. Hence, $(f/g)'(t_0) = 0$ and $(f/g)''(t_0) \leq 0$. We compute the second derivative of f/g.

$$\begin{aligned}\left(\frac{f}{g}\right)'' &= \frac{f''g^3 - g''fg^2}{g^4} + \frac{2gg'fg' - 2g'f'g^2}{g^4} \\ &= -k'\theta^2 \frac{f}{g} + k\theta^2 \frac{f}{g} - \frac{2gg'}{g^2} \frac{f'g - fg'}{g^2}\end{aligned}$$

and therefore

$$\left(\frac{f}{g}\right)''(t_0) = (k'(t_0\,\theta) - k(t_0\theta))\theta^2 \frac{f(t_0)}{g(t_0)} > 0.$$

The case where $k \geq k'$ follows from that if we replace k by $k + \epsilon$. Then $\sigma_{k+\epsilon}^{(t)}(\theta)$ converges uniformly to $\sigma_k^{(t)}(\theta)$ if $\epsilon \to 0$. □

Proposition 3.5 *For $\theta \in (0, L]$ and $t \in (0, 1)$, the map $k \in (C([0, L]), |\cdot|_\infty) \mapsto \sigma_k^{(t)}(\theta) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is continuous where $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is equipped with the usual topology.*

Proof If all the distortion coefficients are finite, this follows from the stability of (3) under uniform changes of k. We only have to check the following. If $k_n \to k$ with respect to $|\cdot|_\infty$, and if $\sigma_k^{(t)}(\theta) = \infty$, then $\sigma_{k_n}^{(t)}(\theta) \uparrow \infty$. If $\sigma_k^{(t)}(\theta) = \infty$, then there exists $r \leq \theta$ such that $\mathfrak{s}_k(r) = 0$. If $r < \theta$, then by the stability property, $\mathfrak{s}_{k_n}(r_n) = 0$ for some $r_n < \theta$ and $n \in \mathbb{N}$ sufficiently large. Hence, $\sigma_{k_n}^{(t)}(\theta) = \infty$ for n sufficiently large. Otherwise, $r = \theta$ and $\mathfrak{s}_k > 0$ on $(0, \theta)$. Again by stability, it follows that $\mathfrak{s}_{k_n}(\theta) \to 0$ and $\mathfrak{s}_{k_n} \to \mathfrak{s}_k$ w.r.t. $|\cdot|_\infty$ if $n \to \infty$. Therefore, for any compact $J \subset (0, 1)$, there exists n_0 such that for each $n \geq n_0$, we have $\mathfrak{s}_{k_n}(\cdot\theta)|_J > c > 0$ for some $c > 0$. Hence, $\sigma_{k_n}^{(t)}(\theta) \uparrow \infty$ for each $t \in (0, 1)$. □

Lemma 3.6 *Let $a, b \in \mathbb{R}_{\geq 0}$ and $k : [0, \theta] \to \mathbb{R}$ as before. If $\sigma_k^{(t)}(\theta) < \infty$, then*

$$v(t) = \sigma_{k^-}^{(1-t)}(\theta)a + \sigma_{k^+}^{(t)}(\theta)b \tag{6}$$

solves (5) in the distributional sense satisfying $u(0) = a$ and $u(1) = b$.

Remark 3.7 Given k as above we set $k^- = k \circ \phi$ where $\phi(t) = b + a - t$. We also write $k =: k^+$. $\sigma_k^{(t)}(\theta) < \infty$ if and only if $\sigma_{k^-}^{(t)}(\theta) < \infty$. This follows from Sturm's oscillation theorem.

To see this, we assume $\sigma_k^{(t)}(\theta) = \infty$ and $\sigma_{k^-}^{(t)}(\theta)$ is finite. Then $\mathfrak{s}_k$ has a zero in $[0, \theta]$ and $\mathfrak{s}_{k^-}$ has no zero in $[0, \theta]$. However, $\mathfrak{s}_k(t)$ and $\mathfrak{s}_k(\theta - t)$ are solutions of $u'' + ku = 0$, and therefore Sturm's oscillation theorem yields a contradiction.

Proof We have

$$v''(t) = -k^-((1-t)\theta)\theta^2\sigma_{k^-}^{(1-t)}(\theta)a - k(t\theta)\theta^2\sigma_{k^+}^{(t)}(\theta)b$$

and

$$k^-((1-t)\theta) = k^+ \circ \phi((1-t)\theta) = k^+(\theta - (1-t)\theta)) = k^+(t\theta).$$

Hence (6) solves (5) in the classical sense with the right boundary condition. □

Proposition 3.8 *Let $k : [a, b] \to \mathbb{R}$ be continuous and $u : [a, b] \to \mathbb{R}_{\geq 0}$ be an upper semi-continuous. Then the following 4 statements are equivalent:*

(i) *$u'' + ku \leq 0$ in the distributional sense, that is*

$$\int_a^b \varphi''(t)u(t)dt \leq -\int_a^b \varphi(t)k(t)u(t)dt \tag{7}$$

for any $\varphi \in C_0^\infty((a, b))$ with $\varphi \geq 0$.

(ii) *It holds*

$$u(\gamma(t)) \geq (1-t)u(\gamma(0)) + tu(\gamma(1) + \int_0^1 g(t, s)k(\gamma(s))\theta^2 u(\gamma(s))ds \tag{8}$$

for any constant-speed geodesic $\gamma : [0, 1] \to [a, b]$ where $\theta = |\dot{\gamma}| = L(\gamma)$ with $g(s, t)$ being the Green function of $[0, 1]$.

(iii) *There is a constant $0 < L \leq b - a$ such that*

$$u(\gamma(t)) \geq \sigma_{k_\gamma^-}^{(1-t)}(\theta)u(\gamma(0)) + \sigma_{k_\gamma^+}^{(t)}(\theta)u(\gamma(1)) \tag{9}$$

for any constant-speed geodesic $\gamma : [0, 1] \to [a, b]$ with $\theta = |\dot{\gamma}| = L(\gamma) \leq L$. We set $k_\gamma = k \circ \bar{\gamma} : [0, \theta] \to \mathbb{R}$. $\bar{\gamma} : [0, \theta] \to [a, b]$ denotes the unit-speed reparametrization of γ. We use the convention $\infty \cdot 0 = 0$.

(iv) *The statement in (iii) holds for any geodesic* $\gamma : [0,1] \to [a,b]$.

Proof 1. First, we prove that (iii) implies (i). Since u is upper semi-continuous, it is bounded from above. Hence, $\sigma_{k_\gamma}^{(t)}(\theta) = \infty$ implies $u \circ \gamma(1) = 0$ for any geodesic γ. Therefore, one can find $L > L' > 0$ such that that $\mathfrak{s}_{k_\gamma} > 0$ on $(0, \theta]$ for any constant-speed geodesic $\gamma : [0,1] \to [a,b]$ with $\theta = |\dot\gamma| \le L'$. Otherwise $u = const = 0$. $\mathfrak{s}_{k_\gamma} > 0$ implies $\sigma_{k_\gamma}^{(t)}(\theta') < \infty$ for any $\theta' \in (0, \theta]$. □

Claim For k and t, fixed $f : h \mapsto \sigma_k^{(t)}(h)$ is twice differentiable at $h = 0$, and we have

$$h \in [0, L] \mapsto \sigma_k^{(t)}(h) = t\left[1 + \frac{1}{6}(1-t^2)k(0)h^2\right] + o(h^2)_k^t. \tag{10}$$

□

Proof of the claim We can compute the first and second derivative of f at 0 explicitly by application of L'Hospital's rule. Then we apply the Taylor expansion formula and the claim follows.

If $\overline{k} \ge k \ge \underline{k}$, then

$$\begin{aligned} o(h^2)_k^t &= \sigma_k^{(t)}(h) - \frac{1}{3}t(1-t^2)k(0)h^2 - t \\ &\le \sigma_{\overline{k}}^{(t)}(h) - t\left[\frac{1}{3}(1-t^2)\underline{k}h^2 + 1\right] = t\frac{1}{3}(1-t^2)(\overline{k} - \underline{k})h^2 + o(h^2)_{\overline{k}}^t \end{aligned}$$

and similarly,

$$o(h^2)_k^t \ge t\frac{1}{3}(1-t^2)(\underline{k} - \overline{k})h^2 + o(h^2)_{\underline{k}}^t.$$

Since k is uniformly continuous on $[a,b]$, we can choose $\overline{h} > 0$ and $(r_i)_{i=1,\dots,N}$ such that

$$\max k|_{[r_i - h, r_i + h]} - \min k|_{[r_i - h, r_i + h]} < \epsilon$$

for each $i = 1, \dots N$ and each $h \in [0, \overline{h}]$.

Upper semi-continuity of u together with the condition (9) yields continuity of u on $[a,b]$. We consider $s \in [a,b]$, $h > 0$ and a geodesic $\gamma : [0,1] \to [a,b]$ such that $\gamma_0 = s - h$, $\gamma_1 = s + h$ and $\gamma_{1/2} = s$ and $s \pm h \in [r_i - \underline{k}, r_i + \overline{k}]$ for some $i = 1, \dots N$. Then, from (10) and (9), it follows that

$$\frac{2u(s)-u(s-h)-u(s+h)}{h^2} \geq \underbrace{\frac{k(s-h)u(s-h)+k(s+h)u(s+h)}{2}}_{\to k(s)u(s)} - \epsilon$$
$$+ \underbrace{\frac{\min_{i=1,\dots,N} o(h^2)^t_{\min k|_{[r_i-h,r_i+h]}}}{h^2}}_{\to 0}.$$

Multiplication with $\phi \in C_0^\infty((a,b))$ such that $\phi \geq 0$, integration with respect to s, a change of variables and taking the limit $h \to 0$ yields

$$\int u(s)\phi''(s)ds \leq -\int k(s)u(s)\phi(s)ds + \epsilon \int \phi(s)ds.$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we obtain the result.

2. We prove the equivalence between (i) and (ii). We assume (i) holds. Consider $v(t) = \int_0^1 g(t,s)k(\gamma(s))\theta^2 u(\gamma(s))ds$. Then v solves

$$v''(t) = -k(\gamma(s))\theta^2 u(\gamma(s))$$

in distributional sense by definition of the Green function. Hence, $u \circ \gamma - v$ has non-positive derivative in the distributional sense, and it follows that $u \circ \gamma - v$ is concave (see Theorem 1.29 in [32]). This implies (ii). The backwards direction is straightforward and works like in the previous step.

3. We prove that (i) implies (iv). The implication (iv) $\Rightarrow$ (iii) is obvious. First, we assume that $u \in C([a,b]) \cap C^2((a,b))$. We consider the case when $\mathfrak{s}_{k_\gamma} > 0$ for any constant-speed geodesic $\gamma : [0,1] \to (a,b)$. The right-hand-side of (9) is denoted by $v(t)$ where $t \in [0,1]$. It is positive for any t and solves $v'' + k_\gamma \circ \gamma\ \theta^2 v = 0$ with boundary condition $v(0) = u(\gamma(0))$ and $v(1) = u(\gamma(1))$. Hence, it suffices to check that $\frac{u\circ\gamma}{v}$ has no local minimum in $(0,1)$. Otherwise, there is $\tau \in (0,1)$ such that $(\frac{u\circ\gamma}{v})'(\tau) = 0$ and $(\frac{u\circ\gamma}{v})''(\tau) \geq 0$. We can deduce a contradiction exactly like in the proof of Proposition 3.4. □

Next, we consider when there is a a constant-speed geodesic $\gamma : [0,1] \to (a,b)$ such that $\mathfrak{s}_{k_\gamma}(t_0) = 0$ for some $t_0 \in (0,\theta]$. Again we adapt parts of the proof of Theorem 14.28 in [37]. We show that $u = 0$. Let $v(t) = \mathfrak{s}_{k_\gamma}(\gamma(t))$ and $w(t) = u \circ \gamma(t)$. v satisfies $v'' + k_\gamma \circ \gamma\theta^2 v = 0$ and w satisfies $w'' + k_\gamma \circ \gamma\theta^2 \leq 0$. Consider $\frac{w}{v} =: h$. Then

$$\begin{aligned}(h'v^2)' = h''v^2 + 2vv'h' &= \left(\frac{w'v - v'w}{v^2}\right)' v^2 + 2vv'h' \\ &= \frac{w''v - v''w - v'w' + 2(v')^2uw}{v^2} + \frac{2v'w'v^2 - 2(v')^2vw}{v^2} \\ &\leq -k\theta^2\frac{w}{v} + k\theta^2\frac{w}{v} = 0\end{aligned}$$

Hence, $h'v^2$ is non-increasing. Suppose there is $\tau \in [0, 1]$ such that $h'(\tau) > 0$ then we also have that $h'v^2(\tau) > 0$ and $h'v^2 \geq C > 0$ on $[\tau, 1]$. for some constant $C > 0$. Hence $h' \geq C\frac{1}{v^2}$. $v = \mathfrak{s}_{k_\gamma} \circ \gamma$ is in $C^2([0, 1])$. Especially, it follows that $v(\delta) = \delta + o(\delta^2)$. Thus, $h'(h) \geq C\frac{1}{\delta^2}$. It follows

$$\int_\delta^\epsilon h'(\tau)d\tau = h(\epsilon) - h(\delta) \geq C\int_\delta^\epsilon \frac{1}{\tau^2}d\tau \to \infty \quad \text{if } \delta \to 0.$$

Hence $h(\delta) \to -\infty$ if $\delta \to 0$ which contradicts $h \geq 0$. On the other hand, if there is $\tau \in [0, 1]$ such that $h'(\tau) < 0$, the same argument yields $h(\epsilon) \to -\infty$ if $\delta \to 0$. It follows that $h' = 0$ and $w(t) = c \cdot \mathfrak{s}_{k_\gamma}(\gamma(t))$. Especially u is differentiable at $\gamma(1) \in (a, b)$ with $u|_{(\gamma(0),\gamma(1))} > 0$, $u(\gamma(1)) = 0$ and $u'(\gamma(1)) \neq 0$ if $u \neq 0$ since $u'(\gamma(1)) = 0$ would contradict the uniqueness of the solution of (3). However, $u(\gamma(1)) = 0$ and $u'(\gamma(1)) \neq 0$ yields $u(x) < 0$ for $x \geq \gamma(1)$ which is not possible. Hence, $u = 0$ and (9) holds.

Now, let u be just upper semi-continuous. The equivalence between (i) and (ii) yields that u is continuous. Consider $\phi \in C_0^\infty((0, 1))$ with $\int_0^1 \phi(t)dt = 1$ and $\phi_\epsilon(t) = \frac{1}{\epsilon}\phi(\frac{t}{\epsilon})$. $\phi_\epsilon \in C_0^\infty((0, \epsilon))$. We set

$$\tilde{u}(s) = u \star \phi_\epsilon(s) = \int_{-\epsilon}^0 \phi_\epsilon(-r)u(s-r)dr = \int_a^b \phi_\epsilon(t-s)u(t)dr$$

for $s \in [a, c]$ with $c < b$ such that $c + \epsilon \geq b$ and $\epsilon > 0$ sufficiently small. k is uniformly continuous on $[a, b]$. Hence, for $\delta > 0$, we can find $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$ we have $k(s - r) \leq k(s) + \delta$. Then

$$\begin{aligned}\tilde{u}''(s) = u \star \phi''_\epsilon(s) &= \int_a^b (\phi_\epsilon(t-s))''u(t)dt = \int_a^b \phi''_\epsilon(t-s)u(t)dt \\ &\leq -\int_a^b \phi_\epsilon(-r)k(s-r)u(r-s)dr \leq -(k(s)+\delta)\tilde{u}(s).\end{aligned}$$

Since $\tilde{u} \in C^2((a, c)) \cap C^0([a, c])$, the previous conclusion holds for $\tilde{u}$ and $\tilde{k} = k + \delta$. Now, since u is continuous, $\tilde{u} \to u$ with respect to uniform convergence on $[a, c]$. And since solutions of (3) change uniformly continuous if the coefficient k changes uniformly continuous on $[a, c]$, we obtain that $\mathfrak{s}_{\tilde{k}_\gamma} \to \mathfrak{s}_{k_\gamma}$ where γ is a geodesic in (c, b). Hence, in the case that where $\mathfrak{s}_{k_\gamma} > 0$ for any constant-speed geodesic $\gamma : [0, 1] \to (a, b)$, we obtain that $\mathfrak{s}_{\tilde{k}_\gamma} > 0$ for any constant-speed geodesic γ in (a, c) by Sturm's comparison theorem. It follows that (3) holds for $\tilde{u} : [a, c] \to [0, \infty)$ and by uniform convergence, it also holds for $u|_{[a,c]}$ if $\epsilon \to 0$. Then, it holds for u since c can be chosen arbitrarily close to b.

Finally, consider the case when there is a geodesic γ in (a, b) such that $\mathfrak{s}_{k_\gamma}(\gamma(1)) = 0$. Then we can choose c sufficiently close to b and $\epsilon > 0$ sufficiently small such that there is a geodesic $\tilde{\gamma}$ in (a, c) with $\mathfrak{s}_{\tilde{k}_\gamma}(\gamma(1)) = 0$. By the previous steps, it follows that $\tilde{u} = \phi_\epsilon \star u = 0$ that implies $u = 0$.

3.1 ku-Concavity in Metric Spaces

We consider a metric space (X, d_X) and a lower semi-continuous function $k : X \to \mathbb{R} \cup \{\infty\} = \bar{\mathbb{R}}$. $\bar{\mathbb{R}}$ is equipped with the usual topology. We define continuous functions $k_n : X \to \mathbb{R}$ that are bounded from above in the following way:

$$k_n(x) = \inf_{y \in X} \{\min(k(y) + n\, \mathrm{d}_X(x, y), n)\} \leq k(x).$$

We retain this notation for the rest of the article. k_n is monotone non-decreasing and converges pointwise to k as $n \to \infty$. For each k_n and for each Lipschitz curve $\gamma \in \mathcal{LC}(X)$, we can consider $\mathfrak{s}_{k_{n,\gamma}}$ where $k_{n,\gamma} = k_n \circ \bar{\gamma}$ and $\bar{\gamma} : [0, \mathrm{L}(\gamma)] \to X$ is the 1-speed reparametrization of γ. If $\mathfrak{s}_{k_{n,\gamma}} > 0$ for all n, the generalized sin-function $\mathfrak{s}_{k_{n,\gamma}} \geq 0$ is monotone non-increasing with respect to n. Hence, the limit exists pointwise in $[0, \mathrm{L}(\gamma)]$. It is again denoted with $\mathfrak{s}_{k_\gamma}$. $\mathfrak{s}_{k_\gamma}$ is upper semi-continuous and if k is continuous, $\mathfrak{s}_{k_\gamma}$ coincides with the previous definition. This follows since $k_{n,\gamma}$ converges uniformly to k_γ by Dini's theorem. Therefore, the stability of solutions of (3) under uniform changes of the coefficient k_γ implies that $\mathfrak{s}_{k_{n,\gamma}}$ converges uniformly to the solution of (3) with coefficient k_γ. We can see that $\mathfrak{s}_{k_\gamma} \geq \mathfrak{s}_{k'_\gamma}$ if $k, k' : X \to \mathbb{R}$ are lower semi-continuous and $k' \geq k$. In particular, we can consider $X = [a, b] \subset \mathbb{R}$.

Definition 3.9 Let $k : X \to \bar{\mathbb{R}}$ be lower semi-continuous and let $\gamma : [0, 1] \to X$ be in $\mathcal{LC}(X)$ with $|\dot{\gamma}| = \theta$. Consider the sequence k_n defined as above. Then $\sigma^{(t)}_{k_{n,\gamma}}(\theta)$ is monotone non-decreasing in $\mathbb{R} \cup \{\infty\}$. We define the *distortion coefficient with respect to* $k : X \to \mathbb{R}$ *along* γ as

$$\sigma^{(t)}_{k_\gamma}(\theta) := \lim_{n \to \infty} \sigma^{(t)}_{k_{n,\gamma}}(\theta) \in \mathbb{R} \cup \{\infty\} \quad \text{for } t \in [0, 1].$$

If k is continuous, the definition is consistent with the previous one. That is $\sigma^{(t)}_{k_\gamma}(\theta)$ equals $\sigma^{(t)}_{k \circ \bar{\gamma}}(\theta)$ as in Definition 3.3.

Lemma 3.10 *Let* $k : X \to \bar{\mathbb{R}}$ *be lower semi-continuous, and let* $\gamma \in \mathcal{LC}(X)$ *with* $|\dot{\gamma}| = \theta$. *If* $\sigma^{(t_0)}_{k_\gamma}(\theta) = \infty$ *for some* $t_0 \in (0, 1)$ *then* $\sigma^{(t)}_{k_\gamma}(\theta) = \infty$ *for any* $t \in (0, 1)$.

In particular, either one has $\sigma^{(t)}_{k_\gamma}(\theta) < \infty$ *for any* $t \in (0, 1)$ *and*

$$\sigma^{(t)}_{k_\gamma}(\theta) = \mathfrak{s}_{k_\gamma}(t\theta)/\mathfrak{s}_{k_\gamma}(\theta) \quad \textit{where} \;\; \mathfrak{s}_{k_\gamma}(\theta) \neq 0,$$

or $\sigma^{(\cdot)}_{k_\gamma}(\theta) \equiv \infty$.

Proof For the proof, we write $k_{n,\gamma} = k_n$ and $k_\gamma = k$. Assume $\lim_{n\to\infty} \sigma^{(t)}_{k_n}(\theta) < \infty$. We must have that $\mathfrak{s}_{k_n}(t_0\theta)/\mathfrak{s}_{k_n}(\theta) \to \infty$. Since $k_n \uparrow$, let $\underline{k} = const \leq k_n$ for all n. Hence, $\mathfrak{s}_{k_n}(t\theta)$ satisfies $u'' + \underline{k}u \leq 0$ for every n. By proposition 3.8, we have for every $s \in [0, 1]$ that

$$\mathfrak{s}_{k_n}(st_0\theta) \geq \sigma^{(s)}_{\underline{k}}(t_0\theta)\, \mathfrak{s}_{k_n}(t_0\theta) + \sigma^{(1-s)}_{\underline{k}}(t_0\theta)\, \mathfrak{s}_{k_n}(0) = \sigma^{(s)}_{\underline{k}}(t_0\theta)\, \mathfrak{s}_{k_n}(t_0\theta)$$

and

$$\mathfrak{s}_{k_n}(((1-s)t_0+s)\theta) \geq \sigma_{\underline{k}}^{(1-s)}(t_0\theta)\,\mathfrak{s}_{k_n}(t_0\theta) + \sigma_{\underline{k}}^{(s)}(t_0\theta)\,\mathfrak{s}_{k_n}(\theta) \geq \sigma_{\underline{k}}^{(1-s)}(t_0\theta)\,\mathfrak{s}_{k_n}(t_0\theta)$$

Hence, if we pick $t \in (0, 1)$, we can write $t = st_0$ or $t = (1-s)t_0 + s$. If $t = st_0$, it follows:

$$\mathfrak{s}_{k_n}(t\theta)/\,\mathfrak{s}_{k_n}(\theta) \geq \sigma_{\underline{k}}^{(s)}(t_0\theta)\,\underbrace{\mathfrak{s}_{k_n}(t_0\theta)/\,\mathfrak{s}_{k_n}(\theta)}_{\to\infty}.$$

and similarly, for $t = (1-s)t_0+s$. Thus, $\sigma_{k_n}^{(t)}(\theta) \to \infty$ for each $t \in (0, 1)$ if $n \to \infty$. □

Corollary 3.11 *Let $k : X \to \bar{\mathbb{R}}$ be lower semi-continuous, γ is a geodesic in X. Then $k \mapsto \sigma_{k_\gamma}^{(t)}(\theta)$ is monotone non-decreasing in the sense of Proposition 3.4.*

Proof If $k' \geq k$, let k'_n and k_n be the corresponding approximations. It is clear from the definition that $k'_{n,\gamma} \geq k_{n,\gamma}$. Hence, $\sigma_{k'_{n,\gamma}}^{(t)}(\theta) \geq \sigma_{k_{n,\gamma}}^{(t)}(\theta)$. Taking the limit $n \to \infty$ yields the result.

Remark 3.12 If $\gamma \in \mathcal{LC}(X)$, we define $\gamma^-(t) = \gamma(1-t)$, and we set

$$\sigma_{k_\gamma^-}^{(t)}(\theta) = \sigma_{k_{\gamma^-}}^{(t)}(\theta).$$

Therefore, one can see again that $\sigma_k^{(t)}(\theta) = \infty$ if and only if $\sigma_{k^-}^{(t)}(\theta) = \infty$.

Corollary 3.13 *Let $k : X \to \bar{\mathbb{R}}$ be lower semi-continuous, and let $u : X \to \mathbb{R}_{\geq 0}$ be upper semi-continuous. Then the following statements are equivalent:*

(i) *$(u\circ\bar{\gamma})'' + k_\gamma u\circ\bar{\gamma} \leq 0$ in the distributional sense for any constant-speed geodesic $\gamma : [0, 1] \to X$.*
(ii) *There is a constant $0 < L \leq b - a$ such that*

$$u(\gamma(t)) \geq \sigma_{k_\gamma^-}^{(1-t)}(\theta)u(\gamma(0)) + \sigma_{k_\gamma^+}^{(t)}(\theta)u(\gamma(1))$$

for any constant-speed geodesic $\gamma : [0, 1] \to X$ with $\theta = |\dot{\gamma}| = L(\gamma) \leq L$.
(iii) *The statement in (ii) holds for any geodesic $\gamma : [0, 1] \to X$.*

Proof If k is continuous, the result follows from Proposition 3.8. If k is lower semi-continuous, we consider k_n for $n \in \mathbb{N}$ as before. We set $u_\gamma(t) = u \circ \gamma(t)$.

(ii) ⇒ (i): Since $k_n \uparrow k$, we have $\sigma_{k_{n,\gamma}}^{(t)}(\theta) \uparrow \sigma_{k_\gamma}^{(t)}(\theta)$ for $t \in (0, 1)$. Then we can apply part **1.** of the proof of Proposition 3.8 to obtain (7) for u with k replaced by k_n. That is

$$-\int \phi''(t)u_\gamma(t)dt \geq \int \phi(t)k_{n,\gamma}(t)u_\gamma(t)dt$$
$$= \int \underbrace{[\phi(t)k_{n,\gamma}(t)u_\gamma(t)]_+}_{\nearrow} dt - \int \underbrace{[\phi(t)k_{n,\gamma}(t)u_\gamma(t)]_-}_{\leq C<\infty} dt.$$

for any $\phi \in C_c^\infty((0,|\dot{\gamma}|))$ where the left-hand side and C are independent of n. Recall that u_γ is non-negative and upper semi-continuous. Hence, by the monotone and dominated convergence theorem, the right-hand side converges to the integral of $\phi k_\gamma u_\gamma$.

(i) $\Rightarrow$ (iii): We can apply part **3.** from the proof of Proposition 3.8, and obtain (9) with k replaced by k_n. By the definition of distortion coefficients for general k, the result follows. □

Lemma 3.14 *Consider* $\lambda \in [0,1]$, $\theta > 0$, *a curve* $\gamma \in \mathcal{LC}(X)$ *with* $L(\gamma) = \theta$ *and* $k, k' : X \to \bar{\mathbb{R}}$ *lower semi-continuous. Then*

$$\sigma_{k_\gamma}^{(t)}(\theta)^{1-\lambda} \cdot \sigma_{k'_\gamma}^{(t)}(\theta)^\lambda \geq \sigma_{(1-\lambda)k_\gamma+\lambda k'_\gamma}^{(t)}(\theta).$$

Especially, $k \mapsto \log \sigma_{k_\gamma}$ *is convex.*

Proof For the proof, we write $k_{n,\gamma} = k_n$ and $k_\gamma = k$. Assume $\sigma_k^{(t)}(\theta) < \infty$ and $\sigma_{k'}^{(t)}(\theta) < \infty$ for each $t \in (0,1)$, since otherwise there is nothing to prove. We assume first that k and k' are continuous. $l: t \mapsto \log\left[\sigma_k^{(t)}(\theta)^{1-\lambda} \cdot \sigma_{k'}^{(t)}(\theta)^\lambda\right]$ solves

$$l'' \leq -(1-\lambda)k - \lambda k' - (l')^2.$$

Hence $\sigma_k^{(t)}(\theta)^{1-\lambda} \cdot \sigma_{k'}^{(t)}(\theta)^\lambda$ solves $v'' + \big((1-\lambda)k + \lambda k'\big)v \leq 0$ with boundary condition $v(0) = 0$ and $v(1) = 1$. The result follows by corollary 3.13.

If k and k' are lower semi-continuous, we consider again their approximations by k_n and k'_n. We easily obtain that

$$\sigma_k^{(t)}(\theta)^{1-\lambda} \cdot \sigma_{k'}^{(t)}(\theta)^\lambda \geq \sigma_{k_n}^{(t)}(\theta)^{1-\lambda} \cdot \sigma_{k'_n}^{(t)}(\theta)^\lambda \geq \sigma_{(1-\lambda)k_n+\lambda k'_n}^{(t)}(\theta).$$

We show that $\sigma_{(1-\lambda)k_n+\lambda k'_n}^{(t)}(\theta) \to \sigma_{(1-\lambda)k+\lambda k'}^{(t)}(\theta)$. One can check that $(1-\lambda)k_n + \lambda k'_n \leq ((1-\lambda)k + \lambda k')_n$. On the other hand, by continuity of the approximating sequence for all $n \in \mathbb{R}$ and for all $x \in [0,\theta]$, there exists $m_x \geq 2^n$ and $\delta_x > 0$ such that $(1-\lambda)k_{\bar{m}} + \lambda k'_{\bar{m}} \geq ((1-\lambda)k + \lambda k')_n$ on $B_{\delta_x}(x)$ for all $\bar{m} \geq m_x$. Hence, by compactness of $[0,\theta]$, we can choose $x_1, \ldots, x_n$ such that $[0,\theta] \subset \bigcup_{i=1,\ldots,n} B_{\delta_{x_i}}(x_i)$. Then $(1-\lambda)k_{m_n} + \lambda k'_{m_n} \geq ((1-\lambda)k + \lambda k')_n$ for $m_n := \max_i m_{x_i}$. Hence,

$$\underbrace{\sigma_{((1-\lambda)k+\lambda k')_{m_n}}^{(t)}(\theta)}_{\to \sigma_{(1-\lambda)k+\lambda k'}^{(t)}(\theta)} \leq \sigma_{(1-\lambda)k_{m_n}+\lambda k'_{m_n}}^{(t)}(\theta) \leq \underbrace{\sigma_{((1-\lambda)k+\lambda k')_n}^{(t)}(\theta)}_{\to \sigma_{(1-\lambda)k+\lambda k'}^{(t)}(\theta)}.$$

Hence, $\sigma_{(1-\lambda)k_n+\lambda k'_n}^{(t)}(\theta) \to \sigma_{(1-\lambda)k+\lambda k'}^{(t)}(\theta)$. □

Proposition 3.15 *Let $k : X \to \bar{\mathbb{R}}$ be continuous (lower semi-continuous). Let $t \in (0,1)$. Then the map*

$$\gamma \in (\mathcal{LC}(X), \mathrm{d}_\infty) \mapsto \sigma^{(t)}_{k_\gamma^{+/-}}(|\dot\gamma|) \in \mathbb{R} \cup \{\infty\}$$

is continuous (lower semi-continuous).

Proof If k is continuous, the result follows from Proposition 3.5. For k lower semi-continuous, we consider its continuous approximation k_n. Then, by definition for any Lipschitz curve $\gamma \in \mathcal{LC}(X)$,

$$\sigma^{(t)}_{k_{n,\gamma}^{+/-}}(|\dot\gamma|) \uparrow \sigma^{(t)}_{k_\gamma^{+/-}}(|\dot\gamma|).$$

In particular, $\gamma \mapsto \sigma^{(t)}_{k_\gamma^{+/-}}(|\dot\gamma|)$ is lower semi-continuous □

Definition 3.16 Consider a metric space (Y, d_Y) and a lower semi-continuous function $k : Y \to \bar{\mathbb{R}}$. We say an upper semi-continuous function $u : Y \to [0,\infty)$ is *ku-convex* if $u < \infty$ and for all geodesics $\gamma : [0,1] \to Y$

$$u(\gamma(t)) \geq \sigma^{(1-t)}_{k_\gamma^-}(\mathrm{L}(\gamma))u(\gamma(0)) + \sigma^{(t)}_{k_\gamma^+}(\mathrm{L}(\gamma))u(\gamma(1)) \tag{11}$$

where $k_\gamma = k \circ \bar\gamma : [0, \mathrm{L}(\gamma)] \to Y$ and $\bar\gamma$ is the unit-speed reparametrization of γ.

We say u is weakly ku-convex if $u < \infty$ and for all $x, y \in Y$ there exists a geodesic $\gamma : [0,1] \to Y$ between x and y such that (11) holds.

We say a function $f : Y \to \mathbb{R} \cup \{\pm\infty\}$ is (weakly) (k, N)-convex if $e^{-\frac{f}{N}} = u$ is (weakly) $\frac{k}{N}u$-concave. We use the convention $e^{\infty} = \infty$, $e^{-\infty} = 0$.

4 Curvature-Dimension Condition

Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space. Given a number $N \in \mathbb{R}$ with $N \geq 1$, we define the *N-Rényi entropy functional*

$$S_N(\,\cdot\,|\,\mathrm{m}_X) : \mathcal{P}_2(X) \to \mathbb{R}$$

with respect to m_X by

$$\nu = \varrho\,\mathrm{m}_X + \nu^s \mapsto S_N(\nu) := S_N(\nu|\,\mathrm{m}_X) := -\int_X \varrho^{-\frac{1}{N}}\,d\nu$$

where $\varrho\,\mathrm{m}_X + \nu^s$ is the Lebesgue decomposition of ν. S_N is lower semi-continuous for $N > 1$. If m_X is a finite measure for each $\nu \in \mathcal{P}_2(X)$, we have

$$\mathrm{Ent}(\nu|\,\mathrm{m}_X) = \lim_{N\to\infty} N(1 + S_N(\nu)),$$

where Ent is the *Boltzmann–Shanon entropy functional*, and this case Ent is lower semi-continuous.

We consider $k = k/N$ where $k : X \to \mathbb{R} \cup \{\infty\} =: \bar{\mathbb{R}}$ is a lower semi-continuous function and locally bounded from below, and we set $\sigma^{(t)}_{k_\gamma/N}(\theta) = \sigma^{(t)}_{\theta^2 k_\gamma(\cdot\theta)/N}(1) = \sigma^{(t)}_{k_\gamma,N}(\theta)$ where $\gamma \in \mathcal{LC}(X)$ and $\theta = |\dot\gamma|$.

Definition 4.1 Let $(X, \mathrm{d}_X, \mathrm{m}_X)$, k and γ as before. We define *generalized distortion coefficients with respect to k and N along* γ as

$$\tau^{(t)}_{k_\gamma,N}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } k > 0 \text{ and } N = 1 \\ t^{\frac{1}{N}} \left[\sigma^{(t)}_{k_\gamma,N-1}(\theta)\right]^{\frac{N-1}{N}} & \text{otherwise.} \end{cases}$$

We use the conventions $r \cdot \infty = \infty$ for $r > 0$, $0 \cdot \infty = 0$ and $(\infty)^\alpha = \infty$ for $\alpha > 0$. If $k > 0$, we have $\tau^{(t)}_{k_\gamma,1}(\theta) < \infty$ if and only if $\theta = 0$, and $\tau^{(t)}_{k_\gamma,1}(\theta) = t$ if $k \leq 0$.

Corollary 4.2 *For* $k, k' : [0, 1] \to \bar{\mathbb{R}}$, $N, N' > 0$, $t \in [0, 1]$, *and* $\theta > 0$,

$$\sigma^{(t)}_{k,N}(\theta)^N \sigma^{(t)}_{k',N'}(\theta)^{N'} \geq \sigma^{(t)}_{k+k',N+N'}(\theta)^{N+N'}$$

and, if $N \geq 1$,

$$\tau^{(t)}_{k,N}(\theta)^N \sigma^{(t)}_{k',N'}(\theta)^{N'} \geq \tau^{(t)}_{k+k',N+N'}(\theta)^{N+N'},$$

and in particular

$$\tau^{(t)}_{k,N}(\theta)^N \tau^{(t)}_{k',N'}(\theta)^{N'} \geq \tau^{(t)}_{k+k',N+N'}(\theta)^{N+N'}.$$

Proof The result follows directly from Lemma 3.14. □

Remark 4.3 For the rest of the article, we always assume that $(X, \mathrm{d}_X, \mathrm{m}_X)$ is a metric measure space and $k : X \to \bar{\mathbb{R}}$ is lower semi-continuous and locally bounded from below. In this case, we say that k is an *admissible function*. It follows from Proposition 3.15 that if k is continuous (lower semi-continuous), the map

$$\gamma \in \mathcal{G}(X) \mapsto \tau^{(t)}_{k^{+/-}_\gamma,N}(|\dot\gamma|) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is continuous (lower semi-continuous) for $t \in [0, 1]$. In particular, it is measurable and we can integrate it with respect to probability measures on $\mathcal{G}(X)$.

Definition 4.4 Consider an admissible function k, and $N \in [1, \infty)$. A metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the *curvature-dimension condition* $CD(k, N)$, if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$ with bounded support there exists a geodesic $(\nu_t)_{t\in[0,1]} \subset \mathcal{P}_2(X, \mathrm{m}_X)$ and a dynamic optimal coupling $\Pi \in \mathcal{P}(X)$ such that $(e_t)_\star \Pi = \nu_t$ and

$$S_{N'}(\nu_t) \leq -\int \left[\tau^{(1-t)}_{k^-_\gamma,N'}(|\dot\gamma|)\varrho_0\left(e_0(\gamma)\right)^{-\frac{1}{N'}} + \tau^{(t)}_{k^+_\gamma,N'}(|\dot\gamma|)\varrho_1\left(e_1(\gamma)\right)^{-\frac{1}{N'}}\right] d\Pi(\gamma) \tag{12}$$

for all $t \in [0,1]$ and all $N' \geq N$. $k_\gamma = k \circ \bar{\gamma}$ where $\gamma : [0,1] \to X$ is a geodesic and $\bar{\gamma}$ its 1-speed reparametrization.

Remark 4.5 The right-hand side in (12) is also denoted with $T^{(t)}_{k,N'}(\Pi|\,\mathrm{m}_X)$. If Π is the optimal dynamic coupling from the previous definition, let $\Pi'(x_0,x_1)(d\gamma) =: \Pi'_{x_0,x_1}(d\gamma)$ be its disintegration with respect to $(e_0,e_1)_\star\Pi = \pi$. One can reformulate (12) in the following way

$$S_{N'}(\nu_t) \leq -\int \left[\mathcal{T}^{(1-t)}_{k^-,N'}(\Pi'_{x_0,x_1})\varrho_0\,(x_0)^{-\frac{1}{N'}} + \mathcal{T}^{(t)}_{k^+,N'}(\Pi'_{x_0,x_1})\varrho_1\,(x_1)^{-\frac{1}{N'}}\right] d\pi(x_0,x_1) \tag{13}$$

where $\mathcal{T}^{(1-t)}_{k^-,N'}(\Pi') = \int \tau^{(1-t)}_{k^-_\gamma,N'}(|\dot{\gamma}|)d\Pi'(d\gamma)$.

Conversely, if there is a kernel $\Pi'_{x_0,x_1}(d\gamma)$ such that for μ_0 and μ_1 there exists a geodesic μ_t and an optimal coupling π with (13), then X satisfies $CD(k,N)$.

Remark 4.6 In the case where k is constant, the previous definition is equivalent to a variant of Sturm's curvature-dimension condition in [34] that is mostly used by other authors (for instance, in [30]). On the right-hand side, Sturm requires integration with respect to an optimal coupling π between ν_0 and ν_1 that is not necessarily related to μ_t. However, most authors assume π is induced by a dynamic coupling that also induces μ_t. In any case, our definition yields Sturm's definition for constant lower curvature bounds. On the other hand, if we consider a space that satisfies this variant of the curvature-dimension condition for constant lower curvature bound, it is exactly the condition that we propose.

Definition 4.7 Two metric measure spaces $(X,\mathrm{d}_X,\mathrm{m}_X)$ and $(X',\mathrm{d}_{X'},\mathrm{m}_{X'})$ are called *isomorphic* if there exists an isometry $\psi : \operatorname{supp}\mathrm{m}_X \to \operatorname{supp}\mathrm{m}_{X'}$ such that

$$\psi_\star\,\mathrm{m}_X = \mathrm{m}_{X'}\,.$$

Remark 4.8 As Sturm states in [34], one might not require that the geodesic ν_t is the projection of Π w.r.t. e_t. Consequently, $(e_t)_\star\Pi$ might not be necessarily m_X-absolutely continuous, or supported in $\operatorname{supp}\mathrm{m}_X$. In this case, we say $(X,\mathrm{d}_X,\mathrm{m}_X)$ satisfies a *modified curvature-dimension condition*. However, if $(e_t)_\star\Pi$ is not supported in $\operatorname{supp}\mathrm{m}_X$, the condition would not be a property of the isomorphism class of the metric measure space. In this case, the right notion of isomorphism would be to say $\psi : (X,\mathrm{d}_X) \to (X',\mathrm{d}_{X'})$ is an isometry and $\psi_\star\,\mathrm{m}_X = \mathrm{m}_{X'}$. Then, a modified curvature-dimension condition is stable w.r.t. ψ what is apparent from the proof of (i) in the next Proposition.

Proposition 4.9 *Let $(X,\mathrm{d}_X,\mathrm{m}_X)$ be a metric measure space which satisfies the condition $CD(k,N)$ for a continuous function $k : X \to \mathbb{R}$ and $N \geq 1$.*

(i) *If there is a strong isomorphism $\psi : (X,\mathrm{d}_X,\mathrm{m}_X) \to (X',\mathrm{d}_{X'},\mathrm{m}_{X'})$ onto a metric measure space $(X',\mathrm{d}_{X'},\mathrm{m}_{X'})$ then $(X',\mathrm{d}_{X'},\mathrm{m}_{X'})$ satisfies the condition $CD(\psi^\star k,N)$ with $\psi^\star k = k \circ \psi$.*

(ii) *For* $\alpha, \beta > 0$ *the rescaled metric measure space* $(X', \alpha\, \mathrm{d}_{X'}, \beta\, \mathrm{m}_{X'})$ *satisfies* $CD(\alpha^{-2}k, N)$.
(iii) *For each geodesically convex subset* $U \subset X$*, the metric measure space* $(U, \mathrm{d}_X|_{U\times U}, \mathrm{m}_X|_U)$ *satisfies* $CD(k|_U, N)$.

Proof (i) First, we observe that $\psi^\star k$ is still lower semi-continuous and locally bounded from below. ψ induces an isometry from $\mathcal{P}_2(X, \mathrm{m}_X)$ to $\mathcal{P}_2(X', \mathrm{m}_{X'})$, and the image of a geodesic in X is a geodesic in X'. Moreover,

$$\int_X \varrho_t^{-\frac{1}{N}+1} d\,\mathrm{m}_X = \int_{X'} (\varrho_t \circ \psi)^{-\frac{1}{N}+1} d\,\mathrm{m}_{X'}\,.$$

$\psi_\star \Pi$ is an optimal dynamic plan, if Π is so. Then the result follows.

(ii), (iii) The results follow easily. One can easily adapt the proofs of similar statements in [34].

Definition 4.10 [35] Let k be admissible. We say a metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ with $\mathrm{m}_X(X) = 1$ satisfies the condition $CD(k, \infty)$ if for $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$ there exists a W_2-geodesic $\mu_t \in \mathcal{P}^2(X, \mathrm{m}_X)$ and an optimal dynamic plan $\Pi \in \mathcal{P}(\mathcal{G}(X)$ such that $(e_t)_\star \Pi = \mu_t$ and

$$\mathrm{Ent}(\mu_t) \leq (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1) - \int_0^1 \int g(s,t) k(\gamma(s)) |\dot\gamma(s)|^2 d\Pi(\gamma) ds \tag{14}$$

for all $t \in [0, 1]$. $g(s,t) = \min\{s(1-t), t(1-s)\}$ is the *Green function* of $[0, 1]$.

Note that $\mathrm{m}_X(X) = 1$ guarantees that $\mathrm{Ent} : \mathcal{P}^2(X) \to \mathbb{R} \cup \{\infty\}$ is a well-defined, lower semi-continuous function. Otherwise, an exponential growth condition for m_X has to be assumed (for instance, see [17]).

Proposition 4.11 *Let* $(X, \mathrm{d}_X, \mathrm{m}_X)$ *be a metric measure space which satisfies the condition* $CD(k, N)$ *for a continuous function* $k : X \to \mathbb{R}$ *and* $N \geq 1$.

(i) *If* $k' : X \to \mathbb{R}$ *is a continuous function such that* $k' \leq k$*, and if* $N' \geq N$*, then* $(X, \mathrm{d}_X, \mathrm{m}_X)$ *also satisfies the condition* $CD(k', N')$.
(ii) *If* $(X, \mathrm{d}_X, \mathrm{m}_X)$ *has finite mass then it satisfies the condition* $CD(k, \infty)$ *in the sense of Sturm.*

Proof (i) is an immediate consequence of the monotonicity of $\sigma_k^{(t)}(\theta)$ w.r.t. k.

For (ii), it suffices to consider $\nu_0, \nu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$ with $\mathrm{Ent}(\nu_0|m_X) < \infty$ and $\mathrm{Ent}(\nu_1|m_X) < \infty$. In any other case, the right-hand side in (14) is ∞. By assumption, $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD(k, N)$. Hence, there exists a dynamic optimal transference plan Γ between ν_0 and ν_1 such that (12) is satisfied for $\forall N' \geq N$.

The assumption $\mathrm{m}_X(X) < \infty$ implies that $\mathrm{Ent}((e_t)_*\Gamma|\,\mathrm{m}_X) = \lim_{N'\to\infty}(1 + S_{N'}((e_t)_*\Gamma|\,\mathrm{m}_X)$ for $t \in [0, 1]$. It follows

$$
\begin{aligned}
&N'(1+S_{N'}((e_t)_*\Gamma|\,\mathrm{m}_X))\\
&\quad\le -N'\int\Big[-(1-t)+\tau^{(1-t)}_{k^-_\gamma,N'}(|\dot\gamma|)\varrho_0(e_0(\gamma))^{-\frac{1}{N'}}-t+\tau^{(t)}_{k^+_\gamma,N'}(|\dot\gamma|)\varrho_1(e_1(\gamma))^{-\frac{1}{N'}}\Big]\Gamma(\gamma)\\
&\quad\le (1-t)N'(1+S_{N'}((e_0)_*\Gamma|\,\mathrm{m}_X))+tN'(1+S_{N'}((e_1)_*\Gamma|\,\mathrm{m}_X))\\
&\qquad -N'\int\Big[\Big[(1-t)+\sigma^{(1-t)}_{k^-_\gamma,N'}(|\dot\gamma|)\Big]\varrho_0(e_0(\gamma))^{\frac{-1}{N'}}+\Big[t+\sigma^{(t)}_{k^+_\gamma,N'}(|\dot\gamma|)\Big]\varrho_1(e_1(\gamma))^{\frac{-1}{N'}}\Big]\Gamma(\gamma)\\
&\quad\le (1-t)N'(1+S_{N'}((e_0)_*\Gamma|\,\mathrm{m}_X))+tN'(1+S_{N'}((e_1)_*\Gamma|\,\mathrm{m}_X))\\
&\qquad -\int\underbrace{N'\Big[(1-\sigma^{(1-t)}_{k^-_\gamma,N'}(|\dot\gamma|)-\sigma^{(t)}_{k^+_\gamma,N'}(|\dot\gamma|)\Big]}_{=w(t)}\Gamma(\gamma)
\end{aligned}
$$

w solves $w''=-k_\gamma|\dot\gamma|^2(\sigma^{(1-t)}_{k^-_\gamma,N}(|\dot\gamma|)+\sigma^{(t)}_{k^+_\gamma,N}(|\dot\gamma|))$ with $w(0)=w(1)=0$. Hence

$$
w=\int_0^1\Big[g(s,t)k_\gamma|\dot\gamma|^2(\sigma^{(1-s)}_{k^-_\gamma,N}(|\dot\gamma|)+\sigma^{(s)}_{k^+_\gamma,N}(|\dot\gamma|))\Big]ds.
$$

Since $\sigma^{(1-t)}_{k^-_\gamma,N}(|\dot\gamma|)+\sigma^{(t)}_{k^+_\gamma,N}(|\dot\gamma|)\to 1$ if $N'\to\infty$ uniformly in $\gamma\in\mathcal{G}(X)$ for fixed t, it follows

$$
\begin{aligned}
&N'(1+S_{N'}((e_t)_*\Gamma|\,\mathrm{m}_X))\\
&\le (1-t)N'(1+S_{N'}((e_0)_*\Gamma|\,\mathrm{m}_X))+tN'(1+S_{N'}((e_1)_*\Gamma|\,\mathrm{m}_X))\\
&\qquad -\int\int_0^1\Big[g(s,t)k_\gamma|\dot\gamma|^2(\sigma^{(1-t)}_{k^-_\gamma,N}(|\dot\gamma|)+\sigma^{(t)}_{k^+_\gamma,N}(|\dot\gamma|))\Big]ds\Gamma(\gamma)\\
&\qquad \Big[\to -\int\int_0^1 g(s,t)k_\gamma|\dot\gamma|^2ds\Gamma(\gamma)\text{ if }N'\to\infty\Big]
\end{aligned}
$$

and this implies the result. □

Theorem 4.12 *Let $(M,g_M,V d\,\mathrm{vol}_M)$ be a weighted Riemannian manifold for a smooth function $V:M\to(0,\infty)$. Let $k:M\to\mathbb{R}$ be a lower semi-continuous function and $N\ge 1$.*

The metric measure space $(M,\mathrm{d}_M,V d\,\mathrm{vol}_M)$ satisfies the curvature-dimension condition $CD(k,N)$ if and only if $(M,g_M,V d\,\mathrm{vol}_M))$ has N-Ricci curvature bounded from below by k.

Remark 4.13 For each real number $N>n$, the N-Ricci tensor is defined as

$$
\mathrm{ric}^{N,V}(v)=\mathrm{ric}(v)-(N-n)\frac{\nabla^2 V^{\frac{1}{N-n}}(v)}{V^{\frac{1}{N-n}}(p)}
$$

where $v\in TM_p$. For $N=n$, we define

$$
\mathrm{ric}^{N,V}(v):=\begin{cases}\mathrm{ric}(v)+\nabla^2\log V(v) & \nabla\log V(v)=0\\ -\infty & \text{else.}\end{cases}
$$

For $1 \leq N < n$, we define $\mathrm{ric}^{N,\Psi}(v) := -\infty$ for all $v \neq 0$ and 0 otherwise.

Example 4.14 Let $\overline{(\alpha, \beta)} = I \subset \mathbb{R}$ be some interval where $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$. Let $k : I \to \mathbb{R}$ be a lower semi-continuous function and let $u : I \to [0, \infty)$ be a non-negative solution of $u'' + \frac{k}{N-1}u = 0$ for $N > 1$. Then, the metric measure space $(I, |\cdot|_2, u^{N-1}d\mathcal{L}^1)$ satisfies the curvature-dimension $CD(k, N)$.

Proof "$\Leftarrow$": Pick a point $p \in M$ and $\epsilon > 0$ such that $k|_{B_\epsilon(p)} \geq k_\epsilon$. There exists geodesically convex ball $B_\delta(p)$ for $0 < \delta < \epsilon$ around p. Hence,

$$(B_\delta(p), d_M|_{B_\delta(p)}, V d\,\mathrm{vol}_M|_{B_\delta})$$

satisfies the condition $CD(k_\epsilon, N)$. It follows that the N-Ricci tensor is bounded from below by k_ϵ (for instance see [34]). If ϵ goes to 0, we see that $k_\epsilon \to k(p)$ and the result follows.

"$\Rightarrow$": The proof goes exactly as the proof of the corresponding result in [24,34] or [9]. □

5 Geometric Consequences

Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be metric measure space. All the results of this section stay true if we replace the curvature-dimension condition by the modified curvature-dimension condition of Remark 4.8.

Theorem 5.1 (Brunn–Minkowski inequality) *Assume that the metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD(k, N)$ for k admissible and $N \geq 1$. Let $A_0, A_1 \subset X$ be bounded Borel sets such that $\mathrm{m}_X(A_0)\,\mathrm{m}_X(A_1) > 0$. We set $\mathcal{G}(A_0, A_1) = \{\gamma \in \mathcal{G}(X) : \gamma(i) \in A_i, i = 0, 1\}$. Then*

$$\mathrm{m}_X(A_t)^{\frac{1}{N}} \geq \inf_{\gamma \in \mathcal{G}(A_0, A_1)} \tau^{(1-t)}_{k^-_\gamma, N}(|\dot\gamma|)\,\mathrm{m}_X(A_0)^{\frac{1}{N}} + \inf_{\gamma \in \mathcal{G}(A_0, A_1)} \tau^{(t)}_{k^+_\gamma, N}(|\dot\gamma|)\,\mathrm{m}_X(A_1)^{\frac{1}{N}}. \tag{15}$$

where $\inf_{\gamma \in \mathcal{G}(A_0, A_1)} \tau^{(1-t/t)}_{k^{-/+}_\gamma, N}(|\dot\gamma|) \geq 0$.

Proof First, assume $\mathrm{m}(A_0), \mathrm{m}(A_1) < \infty$ and set $\mu_i = \mathrm{m}(A_i)^{-1}\,\mathrm{m}|_{A_i}$ for $i = 0, 1$. The curvature-dimension yields

$$\int_{A_t} \varrho_t^{\frac{1}{N'}} d\mu_t \geq \int \tau^{(1-t)}_{k^-_\gamma, N}(|\dot\gamma|)\,\mathrm{m}_X(A_0)^{1/N} + \tau^{(t)}_{k^+_\gamma, N}(|\dot\gamma|)\,\mathrm{m}_X(A_1)^{1/N}$$

where $(\mu_t = \varrho_t d\,\mathrm{m}_X)_t$ denotes the absolutely continuous geodesic that connects μ_0 and μ_1, and Π is an optimal dynamic plan. By Jensen's inequality, the left-hand side of the previous inequality is smaller than $\mathrm{m}_X(A_t)^{\frac{1}{N'}}$. The general case follows by approximation of A_i by sets of finite measure. □

Definition 5.2 *(Minkowski content)* Consider $x_0 \in \operatorname{supp} \mathrm{m}_X$ and $B_r(x_0) \subset X$. Set $v(r) = \mathrm{m}_X(\bar{B}_r(x_0))$. The Minkowski content of $\partial B_r(x_0)$ (the r-sphere around x_0) is defined as

$$s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} \mathrm{m}_X(\bar{B}_{r+\delta}(x_0) \backslash B_r(x_0)).$$

Theorem 5.3 *Assume $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD(k, N)$ for an admissible function k and $N \in [1, \infty)$. Then, (X, d_X) is a proper metric space, bounded sets have finite measure and satisfy a doubling property, and either m_X is supported by one point or all points and all sphere have mass* 0.

In particular, if $N > 1$ then for each $x_0 \in \operatorname{supp} \mathrm{m}_X$, for all $0 < r < R$ and $\underline{k} \in \mathbb{R}$ such that $k|_{B_R(x_0)} \geq \underline{k}$ and $R \leq \pi\sqrt{(N-1)/\underline{k} \vee 0}$, we have

$$\frac{s(r)}{s(R)} \geq \frac{\sin^{N-1}_{\underline{k}/(N-1)} r}{\sin^{N-1}_{\underline{k}/(N-1)} R} \quad \& \quad \frac{v(r)}{v(R)} \geq \frac{\int_0^r \sin^{N-1}_{\underline{k}/(N-1)} t dt}{\int_0^R \sin^{N-1}_{\underline{k}/(N-1)} t dt}. \tag{16}$$

If $N = 1$ and $k \leq 0$, then

$$\frac{s(r)}{s(R)} \geq 1, \quad \frac{v(r)}{v(R)} \geq \frac{r}{R}.$$

Proof 1. Let us fix a point $x_0 \in X$ such that $\mathrm{m}_X(\{x_0\}) = 0$, and let $R > 0$ be sufficiently small such that $k|_{B_{2R}(x_0)} \geq \underline{k}$ for some $\underline{k} \in \mathbb{R}$. Let $r \in (0, R)$ and put $t = r/R$. We choose $\epsilon > 0$ and $\delta > 0$ and define $A_0 = B_\epsilon(x_0)$ and $A_1 = \bar{B}_{R+\delta R}(x_0) \backslash B_R(x_0)$. By triangle inequality, one easily verifies that

$$A_t \subset \bar{B}_{r+\delta r+\epsilon r/R}(x_0) \backslash B_{r-\epsilon r/R}(x_0) \subset B_{2R}(x_0).$$

Hence, if we consider measures $\mu_i = \mathrm{m}_X(A_i)^{-1} \mathrm{m}_X|_{A_i}$ for $i = 0, 1$, the curvature-dimension condition, $\mathrm{m}_X(\{x_0\}) = \emptyset$, local finiteness of the reference measure, and the monotonicity of the distortion coefficients imply that

$$\mathrm{m}_X(\bar{B}_{(1+\delta)r}(x_0) \backslash B_r(x_0))^{1/N} \geq \tau^{(r/R)}_{\underline{k},N}((1 \pm \delta)R)\, \mathrm{m}_X(\bar{B}_{(1+\delta)R(x_0)} \backslash B_R(x_0))^{1/N}.$$

Since m_X is locally finite, we can assume that the right-hand side is finite.

2. Now, we can follow precisely the proof of Theorem 2.3 in [34] to obtain that $\mathrm{m}_X(\partial B_r(x_0)) = 0$ for $r \in (0, R)$, $\mathrm{m}_X(\{x\}) = 0$ for $x \in B_R(x_0) \backslash \{x_0\}$ and (16) for $r \in (0, R)$ and $R > 0$ as chosen like in the first step. If $\mathrm{m}_X(\{x_0\}) \neq 0$, we can choose a point x close to x_0 such that $\mathrm{m}_X(\{x\}) = 0$ and $B_R(x) \subset B_{2R}(x_0)$. This is implied by the local finiteness of m_X and the existence of ϵ-geodesics. If there is no such point x then necessarily $\operatorname{supp} \mathrm{m}_X = \{x_0\}$. We repeat the previous steps for x instead of x_0 and obtain that $\mathrm{m}_X(\{x_0\}) = 0$ unless $\operatorname{supp} \mathrm{m}_X = \{x_0\}$.

3. Hence, for any $x_0 \in X$, there is $R > 0$ (sufficiently small) such that d_X and m_X restricted to $\bar{B}_R(x_0)$ satisfy a doubling property provided the radius of balls is

sufficiently small, and therefore $\bar{B}_r(x_0)$ is compact for $r \in (0, R)$. In particular, X is locally compact. Then, since (X, d_X) is also a complete length space, the generalized Hopf–Rinow theorem (for instance, see Theorem 2.5.28 in [6]) implies (X, d_X) is a proper metric space. Therefore, any closed ball $\bar{B}_R(x_0)$ is compact, and we can repeat the previous step for any $0 < r < R$. In particular, it follows that (16) holds, and any bounded set has finite measure. □

Corollary 5.4 (Doubling) *For each metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfying the condition $CD(k, N)$ for an admissible k and $N \geq 1$, the doubling property holds on each bounded set $X' \subset \operatorname{supp} \mathrm{m}_X$, and in the case of $k \geq 0$, the doubling constant is $\leq 2^N$, and otherwise it can be estimated in terms of $\underline{k}$, N and L as follows*

$$C \leq 2^N \, \mathfrak{c}_{\underline{k}/(N-1)}^{N-1} \, L$$

where L is the diameter of the bounded set X', and $\underline{k} = \min_{X'} k$.

Proof The result follows from the previous theorem (see also [34]).

Corollary 5.5 (Hausdorff dimension) *For each metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfying a curvature-dimension condition $CD(k, N)$ for some admissible k and $N \geq 1$, the Hausdorff dimension of* $\operatorname{supp} \mathrm{m}_X$ *is $\leq N$.*

Definition 5.6 Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be any metric measure space, let $N \geq 1$ and let $k : X \to \mathbb{R}$ be admissible. We define the *effective diameter* of $(X, \mathrm{d}_X, \mathrm{m}_X)$ with respect to k and N as

$$\pi_{k/(N-1)} = \sup \left\{ \mathrm{d}_W(\mu_0, \mu_1) : \exists \Pi \in \mathrm{DyCpl}^{abs}(\mu_0, \mu_1) \;\text{ with }\; \mathcal{T}^{(t)}_{k^{+/-}, N}(\Pi) < \infty \right\}.$$

where $\Pi \in \mathrm{DyCpl}^{abs}(\mu_0, \mu_1)$ if $\Pi \in \mathrm{DyCpl}(\mu_0, \mu_1)$ with $(e_t)_\star \Pi$ m_X-absolutely continuous. By definition, we have $\pi_{k/(N-1)} \leq \operatorname{diam} \operatorname{supp} \mathrm{m}_X$.

Proposition 5.7 *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfy $CD(k, N)$ for an admissible function k and $N \geq 1$. Then $\pi_{k/(N-1)} = \operatorname{diam} \operatorname{supp} \mathrm{m}_X$.*

Proof Assume $\pi_{k/(N-1)} < \operatorname{diam} \operatorname{supp} \mathrm{m}_X$. Then, there are points $x, y \in \operatorname{supp} \mathrm{m}_X$ such that $\mathrm{d}_X(x, y) > c + \pi_{k/(N-1)}$ for some $c > 0$. Therefore, we can consider ϵ-balls $B_\epsilon(x) = A_0$ and $B_\epsilon(y) = A_1$ such that

$$\mathrm{d}_X(A_0, A_1) := \inf_{x_0 \in A_0, x_1 \in A_1} \mathrm{d}_X(x_0, x_1) > \pi_{k/(N-1)}.$$

If we define $\mu_{0/1} = \mathrm{m}_X(A_{0/1})^{-1} \mathrm{m}_X|_{A_{0/1}}$, we see that $\mathrm{d}_W(\mu_0, \mu_1) > \pi_{k/(N-1)}$. Hence, for each dynamical optimal plan $\Pi \in \mathrm{DyCpl}^{abs}(\mu_0, \mu_1)$

$$\infty \leq \int \tau^{(1-t)}_{k^-, N}(|\dot\gamma|) d\Pi(\gamma) \, \mathrm{m}_X(A_0)^{\frac{1}{N}} + \int \tau^{(t)}_{k^+, N}(|\dot\gamma|) d\Pi(\gamma) \, \mathrm{m}_X(A_1)^{\frac{1}{N}}.$$

However, by the curvature-dimension condition, the right-hand side is smaller than

$$-S_N(\mu_t|\,\mathrm{m}_X) \le \mathrm{m}_X(A_t)^{\frac{1}{N}} \le \mathrm{m}_X(B_R(o))^{\frac{1}{N}}$$

for some $o \in X$ and $R > 0$ sufficiently large such that $A_t \subset B_R(o)$. A_t is the set of all t-midpoints between A_0 and A_1. However, the Bishop–Gromov comparison tells us that balls have always finite measure. This results in a contradiction. □

Definition 5.8 Fix a point $x \in X$. Since $\partial B_r(x)$ is compact, we can consider $\min_{\partial B_r(x)} k = k_x(r)$ for $r < R_x$ where $R_x = \sup\{r > 0 : \partial B_r(x) \ne \emptyset\}$. Let $\underline{k}_x$ be the lower semi-continuous envelope of k_x. It is clear that $\underline{k}_x \le k$ and $\underline{k}_x$ induces a lower semi-continuous function on X - also denoted by $\underline{k}_x$-via

$$y \mapsto \underline{k}_x(y) := \underline{k}_x(\mathrm{d}_X(x, y)).$$

Theorem 5.9 *Let X be a metric measure space satisfying $CD(k, N)$. If $N > 1$ then for each $x_0 \in X$, for all $0 < r < R$ such that $R \le \pi_{\underline{k}_x/(N-1)}$, we have*

$$\frac{s(r)}{s(R)} \ge \frac{\sin^{N-1}_{\underline{k}_x/(N-1)} r}{\sin^{N-1}_{\underline{k}_x/(N-1)} R} \quad \& \quad \frac{v(r)}{v(R)} \ge \frac{\int_0^r \sin^{N-1}_{\underline{k}_x/(N-1)} t dt}{\int_0^R \sin^{N-1}_{\underline{k}_x/(N-1)} t dt}. \tag{17}$$

Proof First

$$\inf_{\mathrm{d}_X(x,z)} \left\{\underline{k}_x(\mathrm{d}_X(x, z)) + n|\,\mathrm{d}_X(x, z) - \mathrm{d}_X(x, y)|\right\} = \underline{k}_{x,n}(\mathrm{d}_X(x, y))$$

and since $\underline{k}_{x,n}(r) \uparrow \underline{k}_x(r)$ we have $\underline{k}_{x,n}(\mathrm{d}_X(x, y)) =: \underline{k}'_{x,n}(y) \uparrow \underline{k}_x(y)$. By monotonicity with respect to the curvature function, X satisfies $CD(\underline{k}'_{x,n}, N)$. Hence, if we consider $0 < r < R < R_x$, and A_i with μ_i for $i = 0, 1$ as in Theorem 5.3 (replace x_0 by x), then we obtain

$$\begin{aligned} &\mathrm{m}_X(\bar{B}_{(1+\delta+\epsilon)r}(x)\backslash B_{(1-\epsilon)r}(x))^{\frac{1}{N}} \\ &\quad\ge \int \tau^{(r/R)}_{\underline{k}_{x,n,\gamma},N}(|\dot\gamma|) d\Pi_{n,\epsilon,\delta}(\gamma)\, \mathrm{m}_X(\bar{B}_{(1+\delta)R}(x)\backslash B_R(x))^{\frac{1}{N}} \\ &\qquad + \int \tau^{(1-r/R)}_{\underline{k}_{x,n,\gamma},N}(|\dot\gamma|) d\Pi_{n,\epsilon,\delta}(\gamma)\, \mathrm{m}_X(\bar{B}_\epsilon(x))^{\frac{1}{N}} \end{aligned}$$

where $\Pi_{n,\epsilon,\delta}$ is an optimal dynamic plan between μ_0 and μ_1. Since the left-hand side is finite, the right-hand side is uniformly bounded, and the distortion coefficients are finite almost everywhere. If $\epsilon \to 0$, compactness of closed balls implies that we can find a subsequence of $\Pi_{n,\epsilon,\delta}$ that converges to $\Pi_{n,\delta}$ for $n \to \infty$ and with $(e_0)_*\Pi_{n,\delta} = \delta_x$. The previous inequality becomes

$$\mathrm{m}_X(\bar{B}_{(1+\delta)r}(x)\backslash B_r(x)) \ge \left(\int \tau^{(r/R)}_{\underline{k}_{x,n,\gamma},N}(|\dot\gamma|) d\Pi_{n,\epsilon,\delta}(\gamma)\right)^N \mathrm{m}_X(\bar{B}_{(1+\delta)R}(x)\backslash B_R(x))$$

We remark that $\gamma \mapsto \tau^{(r/R)}_{\underline{k}_{x,n,\gamma},N}(|\dot\gamma|)$ is bounded and continuous for geodesics γ in a sufficiently large ball. Similarly, if δ goes to 0, we can take another sub-sequence of $\Pi_{n,\delta}$ that converges to Π_n. If we divide both sides by δr and take $\delta \to 0$, the previous inequality becomes

$$s_x(r) \ge \left(\int \sigma^{(r/R)}_{\underline{k}_{x,n,\gamma}/(N-1)}(|\dot\gamma|)d\Pi_n(\gamma)\right)^N s_x(R).$$

$(e_0)_*\Pi_n = \delta_x$ and $(e_1)_*\Pi_n$ is a probability measure with $(e_1)_*\Pi_n(\partial B_R(x)) = 1$. Hence Π_n is supported on geodesics with $\gamma(0) = x$ and $|\dot\gamma| = R$, and by definition of $\underline{k}'_{x,n}$ we have that $\underline{k}'_{x,n} \circ \bar\gamma = \underline{k}'_{x,n}(\cdot R)$ is independent of γ. Therefore

$$\frac{s_x(r)}{s_x(R)} \ge \sigma^{(r/R)}_{\underline{k}_{x,n}/(N-1)}(R)^{N-1}.$$

Now, take $n \to \infty$. Since $\underline{k}_{x,n} \uparrow \underline{k}_x$, one can check as in Lemma 3.14 that - after choosing another subsequence - $\mathfrak{s}_{\underline{k}_{x,n}} \downarrow \mathfrak{s}_{\underline{k}_x}$. This is the first claim. The second one follows as in Theorem 5.3. □

Theorem 5.10 *Let* $(X, \mathrm{d}_X, \mathrm{m}_X)$ *be a metric measure space satisfying* $CD(k, N)$ *for* $N > 1$. *Assume there is point* $p \in \operatorname{supp} \mathrm{m}_X$, $\alpha > 0$ *and* $R > 0$ *such that*

$$k(x) \ge \left(\tfrac{1}{4}(N-1) + \alpha^2\right) \mathrm{d}_X(p, x)^{-2} \quad \textit{for all } x \in \operatorname{supp} \mathrm{m}_X \textit{ with } \mathrm{d}_X(p, x) > R.$$

Then $\operatorname{diam} \operatorname{supp} \mathrm{m}_X \le Re^{\frac{\pi}{\alpha}}$, *and* $\operatorname{supp} \mathrm{m}_X$ *is compact.*

Proof Assume the contrary. Then we can find a point $q \in X$ such that $\mathrm{d}_X(p, q) > (R+\delta)e^{\frac{\pi}{\alpha}}$ for some $0 < \delta < R$. We choose $\epsilon > 0$ such that $2\epsilon(2 - e^{-\frac{\pi}{\alpha}}) < \delta$ and

$$\min_{x \in B_\epsilon(p), y \in B_\epsilon(q)} \mathrm{d}_X(x, y) =: \mathrm{d}_X(\bar B_\epsilon(q), \bar B_\epsilon(p)) > (R+\delta)e^{\frac{\pi}{\alpha}}.$$

We set $\bar B_\epsilon(q) =: A_0$ and $\bar B_\epsilon(p) =: A_1$ and define probability measures

$$\mu_i = \mathrm{m}_X(A_i)^{-1} \mu_X|_{A_i}$$

where $i = 0, 1$. Let $q' \in \bar B_\epsilon(q)$ and $p' \in \bar B_\epsilon(p)$. We consider a geodesic $\gamma : [0, 1] \to X$ between q' and p' and estimate the curvature along γ as follows. Let $\bar\gamma$ be the unit speed reparametrization of γ. For $0 < t < [\mathrm{d}_X(q', p') + 2\epsilon](1 - e^{-\frac{\pi}{\alpha}})$ we have

$$\begin{aligned}
\mathrm{d}_X(p, \bar\gamma(t)) &\ge \mathrm{d}_X(p', \gamma(t)) - \mathrm{d}_X(p, p') \ge [\mathrm{d}_X(p', q') - t] - \epsilon \\
&> \mathrm{d}_X(q', p')e^{-\frac{\pi}{\alpha}} - 2\epsilon(1 - e^{-\frac{\pi}{\alpha}}) - \epsilon \\
&\ge (R+\delta) - 2\epsilon(2 - e^{-\frac{\pi}{\alpha}}) > R
\end{aligned}$$

Therefore

$$k(\bar{\gamma}(t)) \geq \frac{c}{\mathrm{d}_X(p,\bar{\gamma}(t))^2} \geq \frac{c}{(\mathrm{d}_X(p,p') + \mathrm{d}_X(p',\bar{\gamma}(t))^2}$$
$$\geq \left(\alpha^2 + \frac{1}{4}\right)(N-1)\frac{1}{(\epsilon + \mathrm{d}_X(q',p') - t)^2} =: k(t)(N-1)$$

We obtain a lower estimate for the modified distortion coefficient along γ. The generalized sin-function $\mathfrak{s}_{k\circ\bar{\gamma}/(N-1)}$ is bounded from below by $\mathfrak{s}_k$ which is given explicitly by

$$\mathfrak{s}_k(t) = C\sqrt{\epsilon + \mathrm{d}_X(p',q') - t}\,\sin\left[\alpha\log\left(\tfrac{\epsilon+\mathrm{d}_X(q',p')-t}{(\epsilon+\mathrm{d}_X(q',p'))e^{-\pi/\alpha}}\right)\right].$$

where C is a normalization constant. We see that the second zero of $\mathfrak{s}_k$ appears at

$$(\epsilon + \mathrm{d}_X(q',p'))(1-e^{-\frac{\pi}{\alpha}}) < \mathrm{d}_X(q',p') - R + \epsilon(1-e^{-\frac{\pi}{\alpha}}) < \mathrm{d}_X(q',p').$$

Therefore, the second zero of $\mathfrak{s}_{k\circ\bar{\gamma}}$ appears strictly before $t = \mathrm{d}_X(q,p)$. Consequently

$$\sigma^{(t)}_{k\circ\gamma,N-1}(\theta) \geq \sigma^{(t)}_k(\theta) = \infty.$$

We conclude that

$$\mathrm{m}_X(A_t)^{\frac{1}{N}} \geq \int \tau^{(1-t)}_{k^-_\gamma,N'}(|\dot{\gamma}|)d\Pi(\gamma)\,\mathrm{m}_X(A_0)^{\frac{1}{N'}} + \int \tau^{(t)}_{k^+_\gamma,N'}(|\dot{\gamma}|)d\Pi(\gamma)\,\mathrm{m}_X(A_1)^{\frac{1}{N'}}$$
$$= \infty.$$

A_t is again the set of all t-midpoints between A_0 and A_1, and Π is an optimal dynamic transference for μ_0 and μ_1. As in the previous Proposition, this yields a contradiction.

Example 5.11 The previous theorem is sharp in the sense that one cannot improve the result by replacing the lower bound $\frac{1}{4}(N-1)$ for c by a smaller lower bound. For instance, consider

$$([0,\infty), |\cdot|_2, \left(\sqrt{r}\right)^{N-1} dr).$$

Using Theorem 4.12 and Proposition 7.3 one can check that it satisfies the curvature-dimension $CD(k,N)$ for

$$k(r) = \frac{1}{4}(N-1)r^{-2}$$

k satisfies the assumption of the theorem for $c = \frac{1}{4}(N-1)$ and any $p \in [0,\infty)$ since $k(r) \sim \frac{1}{4}(N-1)|r-p|_2^{-2}$ for $r > 0$ sufficiently large but one cannot find a point $p \in [0,\infty)$, $c > \frac{1}{4}(N-1)$ and $R > 0$ such that $k(r)r^2 \geq c$ for $r > 0$ with $|r-p|_2 \geq R$. A Riemannian manifold of geometric dimension N satisfying this property can be constructed via warped products.

6 Stability

6.1 Measured Gromov–Hausdorff Convergence

A rectifiable curve $\gamma : [0, 1] \to X$ with constant-speed parametrization is called ϵ-geodesic if $L(\gamma) - \epsilon \le d_X(\gamma(0), \gamma(1))$. The family of all ϵ-geodesics is denoted with $\mathcal{G}^\epsilon(X)$, and it is equipped with the topology that comes from $d_\infty(\gamma, \tilde{\gamma}) = \sup_t d_X(\gamma(t), \tilde{\gamma}(t))$. Measurability is understood in the sense of this topology. Obviously, we have $\mathcal{G}^\epsilon(X) \subseteq \mathcal{G}^\eta(X)$ if $\epsilon \le \eta$ and $\mathcal{G}^0(X) = \mathcal{G}(X)$. If X is compact, then $\mathcal{G}^\epsilon(X)$ is compact with respect to d_∞. Since $L(\gamma) \le \epsilon + \mathrm{diam}_{X_i}$ for every $\gamma \in \mathcal{G}^\epsilon(X)$, this follows from Theorem 2.5.14 in [6].

A probability measure Π on $\mathcal{G}^\epsilon(X)$ is called *dynamic transference plan* between $(e_0)_\star\Pi$ and $(e_1)_\star\Pi$. If $k : X \to \mathbb{R} \cup \{\infty\}$ is an admissible function, we can consider k_γ for $\gamma \in \mathcal{G}^\epsilon(X)$ and the corresponding generalized sin-function and the modified distortion coefficient. One can check that $\gamma \mapsto \tau^{(t)}_{k_\gamma,N}(|\dot{\gamma}|)$ is a measurable function on $\mathcal{G}^\epsilon(X)$. The evaluation map $e_t : \gamma \mapsto \gamma(t)$ is continuous and hence measurable.

Definition 6.1 A sequence $(X_i, d_{X_i})_{i\in\mathbb{N}}$ of compact metric spaces converges in Gromov–Hausdorff sense to a compact metric space (X, d_X) if there is a compact metric space (Z, d_Z) and isometric embeddings $\iota_i : X_i \to Z$, $\iota : X \to Z$ such that $d_{Z,H}(\iota_i(X_i), \iota(X)) \to 0$ where $d_{Z,H}$ is the Haudorff distance w.r.t. to d_Z.

A sequence of compact metric measure spaces (X_i, d_{X_i}, m_{X_i}) converges in measured Gromov–Hausdorff sense to a compact metric measure space (X, d_X, m_X) if there exists a compact metric space (Z, d_Z) and isometric embeddings ι_i, ι as before such that the corresponding metric spaces converge in Gromov–Hausdorff sense and $(\iota_i)_\star m_{X_i} \to (\iota)_\star m_X$ with respect to weak convergence in Z.

A sequence of isomorphism classes $[(X_i, d_{X_i}, m_{X_i})]$ of metric measures spaces converges in measured Gromov sense to an isomorphism class of a normalized metric measure space (X, d_X, m_X) if there exists a metric space (Z, d_Z) and isometric embeddings $\iota_i : X_i \to Z$ and $\iota : X \to Z$ such that $(\iota_i)_\star m_{X_i} \to (\iota)_\star m_X$ with respect to weak convergence of finite measures in Z.

Remark 6.2 If a sequence (X_i, d_{X_i}, m_{X_i}) of compact metric measure spaces converges in measured Gromov–Hausdorff sense to (X, d_X, m_X) then this yields measured Gromov convergence of the corresponding isomorphism classes [33, Lemma 3.18], [17, Theorem 3.30]. The converse is not true in general. However, if the family $(X_i, d_{X_i}, m_{X_i})_{i\in\mathbb{N}}$ satisfies an uniform doubling property and the corresponding isomorphism classes converge in measured Gromov sense to $[(X, d_X, m_X)]$, then by Gromov's compactness theorem, one can extract a subsequence that converges in measured Gromov–Hausdorff sense to a limit space $(X', d_{X'}, m_{X'})$, and consequently $[(X, d_X, m_X)] = [(X', d_{X'}, m_{X'})]$. Therefore, if $\mathrm{supp}\, m_X = X$, then (X_i, d_{X_i}, m_{X_i}) converges in measured Gromov–Hausdorff sense to (X, d_X, m_X) [33, Lemma 3.18], [17, Theorem 3.33].

Lemma 6.3 *Let $k : X \to \mathbb{R} \cup \{\infty\}$ be admissible and $N > 1$. For dynamic couplings $(\Pi_n)_{n\in\mathbb{N}}$ supported on $\mathcal{G}^\eta(X)$ for some $\eta > 0$ with the same marginals $\mu_0, \mu_1 \in$*

$\mathcal{P}(X, \mathrm{m}_X)$ *which converge to a dynamic coupling* Π_∞*, it follows*

$$\limsup_{n\to\infty} T^{(t)}_{k,N}(\Pi_n|\,\mathrm{m}_X) \le T^{(t)}_{k,N}(\Pi_\infty|\,\mathrm{m}_X).$$

Proof First, we assume that k is continuous. We will show that

$$\liminf_{n\to\infty}\int \tau^{(t)}_{k^+_\gamma,N}(|\dot\gamma|)\varrho_0(\gamma_0)^{-\frac{1}{N}}\Pi_n(d\gamma) \ge \int \tau^{(t)}_{k^+_\gamma,N}(|\dot\gamma|)\varrho_0(\gamma_0)^{-\frac{1}{N}}\Pi_\infty(d\gamma).$$

Let $\Pi_{n,x_0}(d\gamma)$ be a disintegration of Π_n with respect to μ_0 for $n \in \mathbb{N}\cup\{\infty\}$, and let $C \in (0,\infty)$. We put

$$v^C_{0,n}(x_0) := \int \left[\tau^{(t)}_{k^+_\gamma,N}(|\dot\gamma|) \wedge C\right]\Pi_{n,x_0}(d\gamma).$$

where $n \in \mathbb{N}\cup\{\infty\}$. Since $C_b(X)$ is dense in $L^1(\mathrm{m}_X)$, and since $v^C_{0,n}$ is bounded by definition, for each $\epsilon > 0$, there is $\psi \in C_b(X)$ such that

$$\int v^C_{0,n}|\varrho_0^{-\frac{1}{N}} \wedge C - \psi|d\mu_0 < \epsilon \quad \text{for all} \quad n \in \mathbb{N}\cup\{\infty\} \tag{18}$$

if $C < \infty$. Weak convergence of $\Pi_n \to \Pi_\infty$ on $\mathcal{G}^\eta(X)$ implies that one can find n_ϵ such that for each $n \ge n_\epsilon$, one has

$$\int v^C_{0,\infty}\psi d\mu_0 \le \int v^C_{0,n}\psi d\mu_0 + \epsilon \tag{19}$$

Putting together (18) and (19) one gets

$$\int v^C_{0,\infty}[\varrho_0^{-\frac{1}{N}} \wedge C]d\mu_0 \le \int v^C_{0,n}[\varrho_0^{-\frac{1}{N}} \wedge C]d\mu_0 + 3\epsilon \le \int v^\infty_{0,n}\varrho_0^{-\frac{1}{N}}d\mu_0 + 3\epsilon.$$

It follows that for each $C > 0$,

$$\int v^C_{0,\infty}\varrho_0^{-\frac{1}{N}} \wedge C d\mu_0 \le \liminf_{n\to\infty}\int v^\infty_{0,n}\varrho_0^{-\frac{1}{N}}d\mu_0. \tag{20}$$

Finally, let $C \to \infty$

$$\int \tau^{(t)}_{k^+_\gamma,N}(|\dot\gamma|)\varrho_0(\gamma_0)^{-\frac{1}{N}}\Pi(d\gamma) \le \liminf_{n\to\infty}\int v^\infty_{0,n}\varrho_0^{-\frac{1}{N}}d\mu_0.$$

The same statement holds with ϱ_0 replaced by ϱ_1 and $\tau^{(t)}_{k^+_\gamma,N}$ replaced by $\tau^{(1-t)}_{k^-_\gamma,N}$.

Now, let k be lower semi-continuous, and let k_i be a sequence of continuous functions that converge pointwise monotone from below to k. By monotonicity of the distortion coefficients, we observe that

$$\tau^{(t)}_{k^+_{i,\gamma},N}(|\dot\gamma|) \uparrow \tau^{(t)}_{k^+_\gamma,N}(|\dot\gamma|)$$

for any $\gamma \in \mathcal{G}^\epsilon$. Therefore, $v^C_{0,\infty,i} \uparrow v^C_{0,\infty}$ and $v^\infty_{0,n,i} \uparrow v^\infty_{0,n}$ if $i \to \infty$. In particular, by the monotone convergence theorem for $\epsilon > 0$, we can choose $i_\epsilon \in \mathbb{N}$ such that

$$\int \left[v^C_{0,\infty} - v^C_{0,\infty,i} \right] \varrho_0^{-\frac{1}{N}} \wedge C d\mu_0 < \epsilon \quad \text{for } i \geq i_\epsilon.$$

Hence, together with (20) it follows that

$$\int v^C_{0,\infty} \varrho_0^{-\frac{1}{N}} \wedge C d\mu_0 - \epsilon \leq \liminf_{n\to\infty} \int v^\infty_{0,n,i} \varrho_0^{-\frac{1}{N}} d\mu_0 \leq \liminf_{n\to\infty} \int v^\infty_{0,n} \varrho_0^{-\frac{1}{N}} d\mu_0$$

and finally we let $C \to \infty$ and $\epsilon \to 0$, and the result follows as before. □

Proposition 6.4 *Let (X_i, d_{X_i}) be a sequence of compact length spaces converging in Gromov–Hausdorff sense to a length space (X, d_X). Then for all $\epsilon > 0$, there exists $i_\epsilon \in \mathbb{N}$ with the following property. If $i \geq i_\epsilon$, and $\gamma : [0, 1] \to X_i$ is a geodesic, then there exists a geodesic $\gamma' : [0, 1] \to X$ such that $\mathrm{d}_{Z,\infty}(\gamma', \gamma) < \epsilon$. Moreover, we can choose $\Phi_i : \gamma \mapsto \gamma'$ to be a measurable map from $\mathcal{G}(Y)$ to $\mathcal{G}(X)$.*

Proof We can use exactly the same argument as in the proof of a similar statement in the appendix of [23]. □

Definition 6.5 Let (X_i, d_{X_i}) be a sequence of compact metric spaces that converge to a compact metric space (X, d_X) in Gromov–Hausdorff sense. Let Z be a compact metric space where Gromov–Hausdorff convergence is realized. Let $k_i, k : X_i, X \to \mathbb{R}$ be admissible functions. We say $\liminf_{i\to\infty} k_i \geq k$ if for each $\eta > 0$ there exists $i_\eta \in \mathbb{N}$ and $\delta > 0$ such that $k_i(x) \geq k(y) - \eta$ if $i \geq i_\eta$, $x \in X_i$, $y \in X$ and $\mathrm{d}_Z(x, y) < \delta$. The definition does not depend on the choice of Z.

Theorem 6.6 *Let $(X_i, \mathrm{d}_{X_i}, \mathrm{m}_{X_i})_{i\in\mathbb{N}}$ be compact metric measure spaces satisfying $CD(k_i, N_i)$ respectively for admissible functions k_i and $N_i \in [1, \infty)$. Assume $(X_i, \mathrm{d}_{X_i}, \mathrm{m}_{X_i})$ converges in measured Gromov–Hausdorff sense to a compact metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$. Let $k : X \to \bar{\mathbb{R}}$ be an admissible function and $N \in [1, \infty)$ such that*

$$\liminf_{i\to\infty} k_i \geq k \;\;\&\;\; \limsup_{i\to\infty} N_i \leq N \;\;\&\;\; \mathrm{diam}_{X_i} \leq L.$$

Then $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD(k, N)$.

Remark 6.7 The previous stability theorem uses the notion of measured Gromov–Hausdorff convergence though it is not a property of isomorphism classes of metric measure spaces. The reason for that is that we will apply Proposition 6.4 that guarantees convergence of sequences of geodesics in (X_i, d_{X_i}) where the limit might not be in the support of m_X. If we assume that m_X has full support, then a suitable reformulation of the theorem holds for measured Gromov convergence as well by Remark 6.2. We want to emphasize that Gromov's precompactness theorem—that is a major source of geometric applications in finite dimensional context—yields measured Gromov–Hausdorff convergence in the first place anyway. See also the discussion at the beginning of Sect. 4.2 in [17].

Proof of Theorem 1. Without loss of generality, we assume that m_{X_i} has full support. Gromov–Hausdorff convergence yields that (X, d_X) is a geodesic space, and $\mathrm{diam}_X \le L$. The curvature-dimension condition yields a uniform doubling property for $\mathrm{supp}\,\mathrm{m}_{X_i}$. Therefore, by compactness of X_i, the measure m_{X_i} is finite. Moreover, $\mathrm{supp}\,\mathrm{m}_X$ satisfies a doubling property as well, and therefore m_X is finite as well. Hence, without loss of generality we normalize any reference measure. By Remark 6.2, measured Gromov–Hausdorff convergence yields measured Gromov convergence of the corresponding isomorphism classes. Let us fix a metric space (Z, d_Z) and isometric embeddings $\iota_i : X_i \to Z$ and $\iota : X \to Z$ where the measured Gromov and the measured Gromov–Hausdorff convergence are realized according to Definition 6.1. First, assume that k is continuous.

2. m_{X_i} converges weakly to m_X in Z as probability measures. Hence, let q_i be optimal couplings between m_{X_i} and m_X such that $\int \mathrm{d}_Z^2\, dq_i = \mathrm{d}_{Z,W}(\mathrm{m}_{X_i}, \mathrm{m}_X)^2 =: d_i^2 \to 0$. Therefore, we can choose i_ϵ such that for $i \ge i_\epsilon$ we have that $\mathrm{d}_i \le \epsilon$. Following [33], for fixed $i \in \mathbb{N}$, one can define a map $Q_i : \mathcal{P}_2(\mathrm{m}_X) \to \mathcal{P}_2(\mathrm{m}_{X_i})$ with

$$S_N(Q_i(\mu)|\,\mathrm{m}_{X_i}) \le S_N(\mu|\,\mathrm{m}_X) \;\;\&\;\; \mathrm{d}_{Z,W}^2(\mu, Q_i(\mu)) < \delta_i \tag{21}$$

where $\mathrm{d}_{Z,W}$ denotes the Wassertstein distance in (Z, d_Z) and $\delta_i \to 0$ for $i \to \infty$. Q_i is constructed by disintegration q_i with respect to m_{X_i}. More precisely, for $\mu = \varrho\,\mathrm{m}_X$, we set $\mu_i = Q_i(\mu) = \varrho_i d\,\mathrm{m}_{X_i}$ where

$$\varrho_i(y) = \int_X \varrho(x) Q_i(y, dx)$$

and Q_i is a disintegration of q_i w.r.t. m_X. Similarly, we define $Q^i : \mathcal{P}_2(\mathrm{m}_{X_i}) \to \mathcal{P}_2(\mathrm{m}_X)$ by $\mu^i = Q^i(\mu) = \varrho^i d\,\mathrm{m}_{X_i}$ where

$$\varrho^i(x) = \int_X \varrho(y) Q^i(x, dy)$$

and Q^i is a disintegration of q_i w.r.t. m_{X_i}. Again we have

$$S_N(Q^i(\mu)|\,\mathrm{m}_X) \le S_N(\mu|\,\mathrm{m}_{X_i}) \;\;\&\;\; \mathrm{d}_{Z,W}^2(\mu, Q^i(\mu)) < \delta^i \tag{22}$$

with $\delta^i \to 0$ if $i \to \infty$. Hence, we can choose i_ϵ such that $\delta^i, \delta_i \le \epsilon^2$ for $i \ge i_\epsilon$.

3. Pick measures $\mu_0 = \varrho_0\,\mathrm{m}_X$ and $\mu_1 = \varrho_1\,\mathrm{m}_X$ in $\mathcal{P}_2(X, \mathrm{m}_X)$ with bounded densities, and define $\mu_{j,i} = Q_i(\mu_j)$ for $j = 0, 1$. Due to the curvature-dimension condition for X_i, there exists a geodesic $(\mu_{t,i})_{t\in[0,1]}$ between $\mu_{0,i}$ and $\mu_{1,i}$ and a dynamic optimal plan Π_i such that $(e_t)_\star \Pi_i = \mu_{t,i}$ and

$$S_{N'}(\mu_{t,i}|\,\mathrm{m}_{X_i}) \le T^{(t)}_{k_i,N'}(\Pi_i|\,\mathrm{m}_{X_i}).$$

By (22), we know that Q^i decreases $S_{N'}$. Note that $Q^i(\mu_{t,i}) = \hat{\mu}_{t,i}$ satisfies

$$|\,\mathrm{d}_{X,W}(\mu_0, \hat{\mu}_{t,i}) - t\,\mathrm{d}_{X,W}(\mu_0, \mu_1)| \le 2\epsilon.$$

By compactness, $\hat{\mu}_{t,i}$ converges weakly in X for each rational $t \in [0, 1]$ for $i \to \infty$ after extracting a subsequence, and by Lipschitz continuity, the limit extends to a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}^2(X)$. For instance, see (vi) in the proof of Theorem 3.1 in [34]. For simplicity, we write N for N'.

4. In this step, we will generalize the map $\Phi_i : \mathcal{G}(X_i) \to \mathcal{G}(X)$. Increase i_ϵ such that $\mathrm{d}_{Z,\infty}(\gamma, \Phi_i(\gamma)) \leq \epsilon$ for $i \geq i_\epsilon$ and each geodesic $\gamma \in \mathcal{G}(X)$. Since X is a compact geodesic space, one can choose a measurable map $\Psi : X^2 \to \mathcal{G}(X)$ such that $\Psi(x, y)$ is a geodesic between x and y. For instance, this follows from a measurable selection theorem. Pick a geodesic $\gamma \in X_i$ and consider the geodesic $\Phi_i(\gamma)$ in X. Consider the map $\Xi_i : \mathcal{G}(X_i) \times X^2 \to \mathcal{G}^L(X)$ that is defined as follows

$$(\gamma, x_0, x_1) \mapsto \Psi(x_0, \Phi_i(\gamma)(0)) * \Phi(\gamma) * \Psi(\Phi_i(\gamma)(1), x_1) \in \mathcal{G}^L(X).$$

Here, the operator $*$ denotes the cancatenation of curves with constant-speed reparametrization on [0, 1]. It is clear from the construction that Ξ_i maps to $\mathcal{G}^L(X)$ and Ξ_i is measurable. We also set $\Xi_{i,\gamma}(\cdot) := \Xi_i(\gamma, \cdot)$ with $\Xi_{i,\gamma} : X^2 \to \mathcal{G}^L(X)$. Then we define $\mathcal{Q}(\gamma, d\sigma) = [(\Xi_{i,\gamma})_\star P_\gamma](d\sigma)$ where

$$P_\gamma(dx_0, dx_1) := \frac{\varrho_j(x_0)}{\varrho_{j,i}(\gamma(0))} Q_i(\gamma(0), dx_0) \otimes \frac{\varrho_j(x_1)}{\varrho_{j,i}(\gamma(1))} Q_i(\gamma(1), dx_1).$$

$\mathcal{Q} : \mathcal{G}(X) \times \mathcal{P}(\mathcal{G}^L(X)) \to \mathbb{R}$ is a Markov kernel. We define a dynamic plan $\hat{\Pi}_i$ via

$$\int_{\mathcal{G}(X_i)} \mathcal{Q}(\gamma, d\sigma)\Pi_i(d\gamma) = \hat{\Pi}_i(d\sigma) \in \mathcal{P}(\mathcal{G}^L(X)).$$

Set $(e_0, e_1)_\star \hat{\Pi}_i = \hat{\pi}_i$. If $f : X^2 \to \mathbb{R}$ is continuous and bounded, then we compute

$$\begin{aligned}
\int_{X^2} f(x_0, x_1)\hat{\pi}_i(dx_0, dx_1) &= \int_{\mathcal{G}(X_i)} \int_{\mathcal{G}^L(X)} f(e_0(\sigma), e_1(\sigma))\mathcal{Q}(\gamma, d\sigma)\Pi_i(d\gamma) \\
&= \int_{\mathcal{G}(X_i)} \int_{X^2} f(x_0, x_1) \frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(\gamma_0)\varrho_{1,i}(\gamma_1)} Q_i(\gamma_0, dx_0) Q_i(\gamma_1, dx_1)\Pi_i(d\gamma) \\
&= \int_{X_i^2} \int_{X^2} f(x_0, x_1) \frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(y_0)\varrho_{1,i}(y_1)} Q_i(y_0, dx_0) Q_i(y_1, dx_1)\pi_i(dy_0, dy_1) \\
&= \int_{X_i^2} \int_{X^2} f(x_0, x_1) \frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{1,i}(y_1)} Q_i(y_1, dx_1)\pi_i(y_0, dy_1) Q^i(x_0, dy_0)\, \mathrm{m}_X(dx_0)
\end{aligned} \tag{23}$$

Here $\pi_i(y_0, dy_1)$ is a disintegration of π_i w.r.t. $\mu_{0,i}$. The last equality follows from

$$\begin{aligned}
Q_i(y_0, dx_0)\pi_i(dy_0, dy_1) &= Q_i(y_0, dx_0)\pi_i(y_0, dy_1)\rho_{0,i}(y_0)\, \mathrm{m}_{X_i}(dy_0) \\
&= \rho_{0,i}(y_0)\pi_i(y_0, dy_1) Q_i(y_0, dx_0)\, \mathrm{m}_{X_i}(dy_0) \\
&= \rho_{0,i}(y_0)\pi_i(y_0, dy_1) Q^i(x_0, dy_0)\, \mathrm{m}_X(dx_0).
\end{aligned} \tag{24}$$

Since (23) holds for any f, we obtain an explicit formula for $\hat{\pi}_i$. If one chooses $f(x_0, x_1) = f_0(x_0)$ or $f(x_0, x_1) = f_1(x_1)$, one can see that that the first and the final marginal of $\hat{\Pi}_i$ are μ_0 and μ_1 respectively. Let $\hat{\Pi}_{i,x_0,x_1}(d\sigma)$ be a disintegration of $\hat{\Pi}_i$ with respect to $\hat{\pi}_i$.

Let $C > 0$ be a constant. For $\gamma \in \mathcal{G}(X_i)$ and for $\sigma \in \mathcal{G}^L(X)$, we define

$$\tau^{(1-t)/(t)}_{k^{-/+}_{i,\gamma},N}(|\dot{\gamma}|) =: b^{-/+}(\gamma) \in [0,\infty] \quad \text{and} \quad \tau^{(1-t)/(t)}_{k^{-/+}_{\sigma}-\eta,N}(|\dot{\sigma}|) \wedge C =: a^{-/+}(\sigma).$$

$\sigma \in \mathcal{G}^L \mapsto a^{-/+}(\sigma)$ is continuous function with respect to d_∞. The dependence of $a^{-/+}$ and $b^{-/+}$ on k, η, N and C is suppressed in our notation but we wil also write $a_k^{-/+}$ if necessary.

5. We consider $(e_0, e_1)_\star \Pi_i = \pi_i$, and $(e_0, e_1) : \Gamma_i \to \operatorname{supp} \pi_i \subset X \times X$. Let $\Pi_{i,y_0,y_1}(d\gamma)$ be the disintegration of Π_i with respect to π_i, and let $\pi_{j,i}(y', dy)$ be a disintegration of π_i with respect to $\mu_{j,i}$ for $j = 0, 1$. We put

$$v_0(y_0) := \int_{X_i} \int_{\mathcal{G}(X_i^2)} \tau^{(1-t)}_{k^-_{i,\gamma},N'}(|\dot{\gamma}|) \Pi_{i,y_0,y_1}(d\gamma) \pi_{0,i}(y_0, dy_1) = \int \tau^{(1-t)}_{k^-_{i,\gamma},N'}(|\dot{\gamma}|) \Pi_i(d\gamma)$$

and similarly, we define $v_1(y_1)$ replacing $\tau^{(1-t)}_{k^-_{i,\gamma},N'}(|\dot{\gamma}|)$ by $\tau^{(t)}_{k^+_{i,\gamma},N'}(|\dot{\gamma}|)$, and $\pi_{0,j}$ by $\pi_{1,j}$. We compute

$$\begin{aligned}
T^{(t)}_{k,N'}(\Pi_i | \mathrm{m}_{X_i}) &= \sum_{j=0,1} \int_{X_i} \left[\int_X \varrho_j(x_j) Q_i(y_j, dx_j) \right]^{1-\frac{1}{N}} v_j(y_j) \, \mathrm{m}_{X_i}(dy_j) \\
&\geq \sum_{j=0,1} \int_{X_i} \int_X \varrho_j(x_j)^{1-\frac{1}{N}} Q_i(y_j, dx_j) v_j(y_j) \, \mathrm{m}_{X_i}(dy_j) \\
&= \sum_{j=0,1} \int_{X_i} \int_X \varrho_j(x_j)^{-\frac{1}{N}} \frac{\varrho_j(x_j)}{\varrho_{j,i}(y_j)} Q_i(y_j, dx_j) v_j(y_j) \mu_i(dy_j) \\
&=: \sum_{i=0,1} (\dagger)_j
\end{aligned}$$

One has the following identity:

$$\begin{aligned}
(\dagger)_0 &= \int_{X_i} \int_X \varrho_0(x_0)^{-\frac{1}{N}} \frac{\varrho_j(x_0)}{\varrho_{0,i}(y_0)} Q_i'(y_0, dx_0) v_0(y_0) \mu_i(dy_0) \\
&= \int_{X_i^2} \int \left[\int_{X^2} \varrho_0(x_0)^{-\frac{1}{N}} \frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(y_0)\varrho_{1,i}(y_1)} Q_i(y_1, dx_1) Q_i(y_0, dx_0) b^-(\gamma) \right] \Pi_i(d\gamma) \\
&= \int_{X_i^2} \int \left[\int_{X^2} \varrho_0(x_0)^{-\frac{1}{N}} \frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(\gamma_0)\varrho_{1,i}(\gamma_1)} Q_i(\gamma_1, dx_1) Q_i(\gamma_0, dx_0) b^-(\gamma) \right] \Pi_i(d\gamma) \\
&= \int_{X_i^2} \int \int_{X^2} \varrho_0(x_0)^{-\frac{1}{N}} P_\gamma(d(x_0, x_1)) b^-(\gamma) \Pi_{i,y_0,y_1}(d\gamma) \pi_i(d(y_0, y_1)) = (\#)_0
\end{aligned}$$

In the third equality, we used that $(e_0, e_1)(\gamma) = (y_0, y_1)$ is constant on the support of $\Pi_{i,y_0,y_1}(d\gamma)$.

$$
\begin{aligned}
(\#)_0 &= \int_{X_i} \int \int_{X^2} \varrho_0(x_0)^{-\frac{1}{N}} \left[a^- \left((\Xi_{i,\gamma}(x_0, x_1)\right) + \left(b^-(\gamma) - a^-(\Xi_{i,\gamma}(x_0, x_1))\right)\right] \\
&\quad \times P_\gamma(d(x_0, x_1))\Pi_{i,y_0,y_1}(d\gamma)\pi_i(d(y_0, y_1)) \\
&= \Bigg\{ \int \int_{X^2} \varrho_0(e_0(\Xi_{i,\gamma}(x_0, x_1))^{-\frac{1}{N}} a^-(\Xi_{i,\gamma}(x_0, x_1))P_\gamma(d(x_0, x_1))\Pi_i(d\gamma) \\
&= (**)_0 \\
&\quad \Bigg\{ + \int_{X_i^2} \int \int_{X^2} \varrho_0(x_0)^{-\frac{1}{N}} \left(b^-(\gamma) - a^-(\Xi_{i,\gamma}(x_0, x_1))\right) P_\gamma(d(x_0, x_1))\Pi_i(d\gamma) \\
&= (*)_0
\end{aligned}
$$

and similarly for $(\dagger)_1$.

6. Consider the sequence of optimal couplings $(q_i)_{i\in\mathbb{N}}$ between m_X and m_{X_i}. We fix $\lambda > 0$. By Markov's inequality,

$$
q_i(\mathrm{d}_Z(x, y) > \lambda) \leq \frac{1}{\lambda^2} \int \mathrm{d}_Z^2(x, y) dq_i = \frac{d_i^2}{\lambda^2} \leq \frac{\epsilon^2}{\lambda^2} \quad \text{for } i \geq i_\epsilon.
$$

Hence, if we define $\mathcal{X}^\lambda = \{(x, y) \in X \times X_i : \mathrm{d}_Z(x, y) \leq \lambda\}$, we have $q_i((\mathcal{X}^\lambda)^c) \to 0$ for $i \to \infty$. Recall (24), consider $(*)_0$, and rewrite it as

$$
\begin{aligned}
&\int_{X_i^2} \int \int_{X^2} \varrho_0(x_0)^{-\frac{1}{N}} \left(b^-(\gamma) - a^-(\Xi_{i,\gamma}(x_0, x_1))\right) P_\gamma(d(x_0, x_1))\Pi_i(d\gamma) \\
&\quad = \int_{X\times X_i} \int_{X\times X_i} \int \left(b^-(\gamma) - a^-(\Xi_{i,\gamma}(x_0, x_1))\right) \Pi_{i,y_0,y_1}(d\gamma) \\
&\quad \times \frac{\varrho_0(x_0)^{1-\frac{1}{N}} \varrho_1(x_1)}{\varrho_{0,i}(y_0)\varrho_{1,i}(y_1)} Q_i(y_1, dx_1)Q_i(y_0, dx_0)\pi_i(dy_0, dy_1) \\
&\quad = \int_{X\times X_i} \int_{X\times X_i} \int \left(b^-(\gamma) - a^-(\Xi_{i,\gamma}(x_0, x_1))\right) \Pi_{i,y_0,y_1}(d\gamma) \\
&\quad \times \frac{\varrho_0(x_0)^{1-\frac{1}{N}} \varrho_1(x_1)}{\varrho_{1,i}(y_1)} Q_i(y_1, dx_1)\pi_i(y_0, dy_1)q_i(dx_0, dy_0) \\
&\quad = \left[\int_{\mathcal{X}^\lambda} \int_{\mathcal{X}^\lambda} \int \ldots\right] =: (II) \; + \; \left[\int_{\mathcal{X}^{\lambda,c}} \int_{\mathcal{X}^{\lambda,c}} \int \ldots\right] =: (I)
\end{aligned}
$$

First, a^- and ϱ_0 are bounded independent of i or λ, and b^- is non-negative. Hence, there is $M > 0$ such that

$$
\begin{aligned}
(I) &\geq -\int_{\mathcal{X}^{\lambda,c}}\int_{\mathcal{X}^{\lambda,c}}\int M\frac{\varrho_1(x_1)}{\varrho_{1,i}(y_1)}\Pi_{i,y_0,y_1}(d\gamma)Q_i(y_1,dx_1)\pi_i(y_0,dy_1)q_i(dx_0,dy_0)\\
&= -\int_{\mathcal{X}^{\lambda,c}}\int_{\mathcal{X}^{\lambda,c}} M\frac{\varrho_1(x_1)}{\varrho_{1,i}(y_1)}Q_i(y_1,dx_1)\pi_i(y_0,dy_1)q_i(dx_0,dy_0)\\
&\geq -\int_{\mathcal{X}^{\lambda,c}}\int_{X^i} M\pi_i(y_0,dy_1)q_i(dx_0,dy_0)\\
&= -M\int_{\mathcal{X}^{\lambda,c}} q_i(dx_0,dy_0) \geq -\frac{M}{\lambda}\,\mathrm{d}_{Z,W}(\mathrm{m}_{X_i},\mathrm{m}_X)^2 \geq -M\frac{\epsilon^2}{\lambda^2} \quad \text{for } i \geq i_\epsilon.
\end{aligned}
\tag{25}
$$

Second, we wan to estimate (II). Observe that $\mathrm{d}_{Z,\infty}(\gamma,\Xi_{i,\gamma}(x_0,x_1)) \leq 2\lambda + 2\epsilon$ and $\Xi_{i,\gamma}(x_0,x_1) \in \mathcal{G}^{2\lambda+2\epsilon}(X)$ provided we have $(x_0,\gamma_0),(x_1,\gamma_1) \in \mathcal{X}^\lambda$ and $i \geq i_\epsilon$. Hence, we fix $\eta > 0$ and since $\liminf k_i \geq k$, we choose $\lambda > 0$, $\epsilon > 0$ sufficiently small and $i_\eta \geq i_\epsilon$ sufficiently large such that $k_{i,\gamma} \geq k_\sigma - \eta$ whenever $\mathrm{d}_Z(\gamma,\sigma) \leq 2\lambda + 2\epsilon$ for every $\gamma \in X_i$ and every $\sigma \in X$ and for $i \geq i_\eta$. Hence, by monotonicity of distortion coefficients, we have $\tau^{(t)}_{k_\gamma,N}(|\dot\gamma|) \geq \tau^{(t)}_{k_\sigma-\eta,N}(|\dot\sigma|)$, and therefore $b^-(\gamma) - a^-(\Xi_{i,\gamma}(x_0,x_1)) \geq 0$ provided $(x_0,\gamma_0),(x_1,\gamma_1) \in \mathcal{X}^\lambda$. Hence, $(II) \geq 0$. Together with (25), we obtain for $i \geq i_\eta$

$$
-T^{(t)}_{k,N'}(\Pi_i|\mathrm{m}_{X_i}) \geq \int \left[a^-(\sigma)\varrho_0(\sigma_0)^{-\frac{1}{N}} + a^+(\sigma)\varrho_1(\sigma_1)^{-\frac{1}{N}}\right]\hat\Pi_i(d\sigma) - M\frac{\epsilon^2}{\lambda^2}. \tag{26}
$$

7. Now, choose $\lambda = \sqrt{\epsilon}$, a sequence $\epsilon_i \downarrow 0$ for $i \to 0$, and pick for every $i \in \mathbb{N}$ a measure $\hat\Pi_i$ as in (26). Since $\mathcal{G}^L(X)$ is compact with respect to d_∞, Prohorov's theorem yields that there is a subsequence of $\hat\Pi_i$ that converges to a dynamic transference plan Π that is supported on $\mathcal{G}^L(X)$. Recall that $(e_j)_\star\hat\Pi_i = \mu_j$ for $j = 0, 1$ and all i. By a modification of Lemma 6.3 (replacing $\tau_{k^{-/+},N}$ by $a^{-/+}$), it follows that

$$
\text{RHS in (26)} \to \int \left[a^-(\sigma)\varrho_0(\sigma_0)^{-\frac{1}{N}} + a^+(\sigma)\varrho_1(\sigma_1)^{-\frac{1}{N}}\right]\Pi(d\sigma) - M\epsilon \quad \text{if } i \to \infty.
$$

We show that $(e_0,e_1)_\star\Pi =: \pi$ is an optimal coupling, and Π is supported on $\mathcal{G}(X)$.

The first claim follows by construction of $\hat\Pi_i$. We have an explicit representation for the coupling $\hat\pi_i$ by (23) that is the same coupling as constructed by Sturm in [34] (more precisely, this is $\bar q^r$ on page 154). It is an almost optimal coupling between μ_0 and μ_1, and the error becomes small if i is large. Therefore, since $\hat\pi_i \to \pi$ weakly and since the total cost of couplings is lower semi-continuous with respect to weak convergence of couplings, π is optimal for μ_0 and μ_1.

For the second claim, we decompose $\hat\Pi_i$ with respect to $X^{\sqrt{\epsilon_i}}$. Recall

$$
\begin{aligned}
\int f(\sigma)\hat\Pi_i(d\sigma) = \int_{X_i^2}\int_{X^2}\int_{\mathcal{G}(X_i)} & f((\Xi_{i,\gamma})_\star(x_0,x_1))\frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(y_0)\varrho_{1,i}(y_1)}\\
&\times Q_i(y_1,dx_1)Q_i(y_0,dx_0)\Pi_{i,y_0,y_1}(d\gamma)\pi_i(dy_0,dy_1).
\end{aligned}
\tag{27}
$$

We set for $f \in C_b(\mathcal{G}^L(X))$

$$L_{i,\sqrt{\epsilon_i}} f := \int_{X_i^2}\int_{X^2}\int_{\mathcal{G}(X_i)} f((\Xi_{i,\gamma})_\star(x_0,x_1)) 1_{(\mathcal{X}^{\sqrt{\epsilon_i}})^2}(x_0,y_0,x_1,y_1)\frac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(y_0)\varrho_{1,i}(y_1)}$$
$$\times Q_i(y_1,dx_1)Q_i(y_0,dx_0)\Pi_{i,y_0,y_1}(d\gamma)\pi_i(dy_0,dy_1).$$

By Riesz' theorem, $L_{i,\sqrt{\epsilon_i}}$ yields a measure $\tilde{\Pi}_i$ such that $L_{i,\sqrt{\epsilon_i}} f = \int f(\sigma)d\tilde{\Pi}_i(\sigma)$ and that is supported on $\mathcal{G}^{2\epsilon_i+2\sqrt{\epsilon_i}}(X)$ by construction of $X^{\sqrt{\epsilon_i}}$.

Note that $\mathcal{G}^{2\epsilon_i+2\sqrt{\epsilon_i}}(X) \subset \mathcal{G}^{2\epsilon_j+2\sqrt{\epsilon_j}}(X)$ for $i \geq j$, and since $\mathcal{G}^{2\epsilon+2\sqrt{\epsilon}}(X)$ is compact, by diagonal argument, we find a subsequence such that $\tilde{\Pi}_i$ converges to a measure $\tilde{\Pi}$ supported on $\mathcal{G}^{2\epsilon_i+2\sqrt{\epsilon_i}}(X)$ for every $i \in \mathbb{N}$. Since $\mathcal{G}(X) = \bigcap_{i\in\mathbb{N}} \mathcal{G}^{2\epsilon_i+2\sqrt{\epsilon_i}}(X)$ by the Arzela–Ascoli theorem, $\tilde{\Pi}$ is supported on $\mathcal{G}(X)$. For every i, we consider the decomposition $\hat{\Pi}_i - \tilde{\Pi}_i =: \bar{\Pi}_i$ and as in (25) we see that $0 \leq \bar{\Pi}_i(\mathcal{G}^L(X)) \leq M\epsilon_i$ for all $i \geq i_\eta$. Hence, $(\Pi - \tilde{\Pi})(\mathcal{G}^L(X)) = 0$, and therefore $\mathcal{G}(X)$ has full Π-measure.

Together with the convergence of $Q^i(\mu_{t,i})$ to μ_t (step **3.**) the curvature-dimension condition on X_i and lower semi-continuity of S_N, we get

$$S_N(\mu_t|\,\mathrm{m}_X) \leq -\int\left[a^-(\sigma)\varrho_0(\sigma_0)^{-\frac{1}{N}} + a^+(\sigma)\varrho_1(\sigma_1)^{-\frac{1}{N}}\right]\Pi(d\sigma). \qquad (28)$$

Since η was arbitrary, application of another compactness argument on X yields the inequality for k instead $k-\eta$.

8. Now, we will check that $(e_t)_\star\Pi = \mu_t$ where $(\mu_t)_{t\in[0,1]}$ is the geodesic that we found in step 3.

Consider again $\tilde{\Pi}_i$. It is a finite measure on $\mathcal{G}^{2\epsilon+2\sqrt{\epsilon}}(X)$ and by normalization, we can make it a probability measure. Recall again (27). If $g \in C_b(X_i \times X)$, then

$$\alpha_i \int g(e_t(\gamma), e_t(\Xi_{i,\gamma}(x_0,x_1)))1_{(\mathcal{X}^{\sqrt{\epsilon}})^2}(x_0,y_0,x_1,y_1)$$
$$\times \tfrac{\varrho_0(x_0)\varrho_1(x_1)}{\varrho_{0,i}(y_0)\varrho_{1,i}(y_1)} Q_i(y_1,dx_1)Q_i(y_0,dx_0)\Pi_i(d\gamma)$$

defines a coupling between $(e_t)_\star\Pi_i = \mu_{t,i}$ from step 3 and $(e_t)_\star(\alpha_i\tilde{\Pi}_i)$ where $\alpha_i > 0$ is a normalization constant with $\alpha_i \to 1$ if $i \to \infty$. Choosing $g = \mathrm{d}_Z^2|_{X_i\sqcup X}$, we obtain by construction of Ξ_i and $X^{\sqrt{\epsilon_i}}$ that $\mathrm{d}_{Z,W}(\mu_{t,i},(e_t)_\star\tilde{\Pi}_i) \leq 2\sqrt{\epsilon_i} + 2\epsilon_i$. Recall $\mu_{t,i} \to \mu_t$ weakly. Moreover, weak convergence of $\alpha_i\tilde{\Pi}_i$ to $\tilde{\Pi}$ implies weak convergence of $(e_t)_\star\alpha_i\tilde{\Pi}_i$ to $(e_t)_\star\tilde{\Pi}$. Consequently, $(e_t)_\star\tilde{\Pi} = \mu_t$ since $\mathrm{d}_{Z,W}(\mu_{t,i},(e_t)_\star\tilde{\Pi}_i) \leq 2\sqrt{\epsilon_i} + 2\epsilon_i \to 0$. Moreover, since $((e_t)_\star\Pi - (e_t)_\star\tilde{\Pi})(X) = 0$, we have $\mu_t = (e_t)_\star\Pi$.

9. In the last step, we want to remove the remaining assumptions, namely continuity of k and boundedness of ϱ_j and $a^{-/+}$.

Consider general probability measures $\mu_0, \mu_1 \in \mathcal{P}^2(\mathrm{m}_X)$ with densities ρ_j for $j = 0, 1$. Fix an arbitrary optimal coupling $\tilde{\pi}$ between μ_0 and μ_1, and set for $r \in (0,\infty)$

$$E_r := \left\{(x_0,x_1)) \in X^2 : \rho_0(x_0) \leq r, \rho_1(x_1) \leq r\right\},\ \alpha_r = \tilde{\pi}(E_r),\ \tilde{\pi}^r := \frac{1}{\alpha_r}\tilde{\pi}|_{E_r}.$$

The coupling $\tilde{\pi}^r$ is an optimal coupling between its marginals μ_0^r and μ_1^r such that

$$\mathrm{d}_{X,W}(\mu_j, \mu_j^r) \le \epsilon \quad \text{for } j = 0, 1 \text{ if } r > 0 \text{ sufficiently large.} \tag{29}$$

Depending on $r > 0$ we can construct μ_t^r and Π^r as before. After successively choosing subsequences - that μ_t^r converges weakly to a probability μ_t for $t \in [0, 1] \cap \mathbb{Q}$. Then, again as in step (vi) of the proof of Theorem 3.1 in [34] μ_t extends to geodesic between μ_0 and μ_1 and $\liminf_{i\to\infty} S_N(\mu_t^r|\mathrm{m}_{X_\infty}) \ge S_N(\mu_t|\mathrm{m}_{X_\infty})$ for $t \in [0, 1]$.

Set $a^-(\sigma)\varrho_0(\sigma_0)^{-\frac{1}{N}} + a^+(\sigma)\varrho_1(\sigma_1)^{-\frac{1}{N}} = \psi(\gamma)$. ψ is integrable w.r.t. Π, since the coefficients $a^{+/-}$ are bounded, since ρ_0 and ρ_1 are probability densities for μ_0 and μ_1, respectively, and since Π is a coupling between μ_0 and μ_1. Therefore, if we set $\Pi^\epsilon = \alpha_r\Pi^r + \Psi_\star\tilde{\pi}|_{X^2\setminus E_r}$, it follows that

$$\lim_{\epsilon\to 0}\left|\int \psi(\gamma)d\Pi^\epsilon(\gamma) - \int \psi(\gamma)d\Pi^r(\gamma)\right| = 0.$$

(see also step (v) in the proof of Theorem 3.1 in [34] for similar argument) Now, by compactness, we can choose subsequence ϵ_i such that Π^{ϵ_i} converges weakly to an optimal coupling Π between μ_0 and μ_1. Since Π^ϵ is a coupling between μ_0 and μ_1 for every $\epsilon > 0$, we can apply again Lemma 6.3. Hence,

$$\begin{aligned} S_N(\mu_t|\mathrm{m}_{X_\infty}) &\le \liminf_{i\to\infty} S_N(\mu_t^{r(\epsilon_i)}|\mathrm{m}_{X_\infty}) \le \limsup_{i\to\infty} -\int \psi(\gamma)d\Pi^{\epsilon_i}(\gamma) \\ &\le -\int \psi(\gamma)d\Pi(\gamma). \end{aligned}$$

If k is lower semi-continuous, we take monotone sequence of continuous functions k_n that approximates k from below. Since we can repeat all the previous steps, for any n, we obtain an optimal dynamic coupling Π^n and a Wasserstein geodesic μ_t^n such that (28) holds with k replaced by k_n. The right-hand side of (28) is monotone with respect to k_n. Therefore, we obtain

$$\begin{aligned} S_N(\mu_t^n|\mathrm{m}_X) &\le -\int \left[a_{k_n}^-(\gamma)\varrho_0(\gamma_0)^{-\frac{1}{N}} + a_{k_n}^+(\gamma)\varrho_1(\gamma_1)^{-\frac{1}{N}}\right]\Pi^n(d\gamma) \\ &\le -\int \left[a_{k_{\hat{n}}}^-(\gamma)\varrho_0(\gamma_0)^{-\frac{1}{N}} + a_{k_{\hat{n}}}^+(\gamma)\varrho_1(\gamma_1)^{-\frac{1}{N}}\right]\Pi^n(d\gamma). \end{aligned}$$

for $n \ge \hat{n}$. Compactness yields a subsequence such that Π_{n_i} and $\mu_t^{n_i}$ converge if $i \to \infty$ and by another application of Lemma 6.3 the limits of Π and μ_t satisfy

$$S_N(\mu_t|\mathrm{m}_X) \le -\int \left[a_{k_{\hat{n}}}^-(\gamma)\varrho_0(\gamma_0)^{-\frac{1}{N}} + a_{k_{\hat{n}}}^+(\gamma)\varrho_1(\gamma_1)^{-\frac{1}{N}}\right]\Pi(d\gamma). \tag{30}$$

We let $\hat{n} \to \infty$. Then the theorem of monotone convergence yields the estimate

$$S_N(\mu_t|\,\mathrm{m}_X) \le -\int \left[a_k^-(\sigma)\varrho_0(\sigma_0)^{-\frac{1}{N}} + a_k^+(\sigma)\varrho_1(\sigma_1)^{-\frac{1}{N}}\right]\Pi(d\sigma). \tag{31}$$

Finally, by a similar reasoning as before, $C \nearrow \infty$ yields $a^{-/+}(\gamma) \nearrow \tau^{(t)}_{k_\gamma^{-/+},N}(|\dot{\gamma}|) \in \mathbb{R}\cup\{\infty\}$ for any $\gamma \in \mathcal{G}(X)$, and again by monotone convergence, the left-hand side in (31) converges to

$$S_N(\mu_t|\,\mathrm{m}_X) \le -\int \left[\tau^{(1-t)}_{k_\gamma^-,N}(|\dot{\gamma}|)\varrho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{k_\gamma^+,N}(|\dot{\gamma}|)\varrho_1(\gamma_1)^{-\frac{1}{N}}\right]\Pi(d\gamma).$$

This finishes the proof. □

Corollary 6.8 *Let $(M_i, g_{M_i})_{i\in\mathbb{N}}$ be a family of compact Riemannian manifolds with*

$$\mathrm{ric}_{M_i} \ge k_i \;\;\&\;\; \dim_{M_i} \le N \;\;\&\;\; \mathrm{diam}_{M_i} \le L$$

where $k_i : M_i \to \mathbb{R}$ are lower semi-continuous functions such that $k_i \ge -C$ for some $C > 0$. Let $\overline{\mathrm{vol}}_{M_i}$ be the normalized Riemannian volume. Then, there exists subsequence of $(M_i, \mathrm{d}_{M_i}, \overline{\mathrm{vol}}_{M_i})$ that converges in measured Gromov–Hausdorff sense to a normalized metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$, there exists a subsequence $(i_n)_{n\in\mathbb{N}}$ and a lower semi-continuous function $k : X \to \mathbb{R}$ such that $\liminf \kappa_{i_n} \ge \kappa$ and $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the condition $CD(k, N)$.

Proof By Gromov's precompactness theorem, there is a converging subsequence $(M_i, \mathrm{d}_{M_i}, \overline{\mathrm{vol}}_{M_i})$ in the measured Gromov–Hausdorff sense, and a normalized limit metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$. $\overline{\mathrm{vol}}$ is the normalized Riemannian volume. Consider a compact metric space (Z, d_Z) where the convergence is realized, and define $\hat{\kappa} : Z \to \mathbb{R}$ with $\hat{\kappa}(x_i) = k_i(x_i)$ if $x_i \in X_i$, and $+\infty$ otherwise. Let $\kappa : Z \to \mathbb{R}$ be the lower semi-continuous envelope of $\hat{\kappa}$ (see, for instance, [12]). We have $\hat{\kappa} > -\infty$ since $k_i \ge -C > -\infty$. We define $\kappa|_X = k : X \to \mathbb{R}$. By compactness, κ is uniformly lower semi-continuous. More precisely, for $\epsilon > 0$, there is $\delta > 0$ such that for all $x, y \in Z$ with $\mathrm{d}_Z(x, y) \le \delta$ we have $f(y) \ge f(x) - \epsilon$. This implies $\liminf \kappa_i \ge \kappa$ in the sense of our definition. Hence, applying the previous theorem yields the statement.

Remark 6.9 Recall the notion of pointed measured Gromov–Hausdorff (GH) and pointed measured Gromov convergence from [17]. These notions generalize the previous concepts of convergence of metric measure spaces to the context of non-compact spaces and measures with infinite mass. From Remark 3.29 in [17], we see that if a sequence of pointed metric measure spaces converges in pointed measured GH sense to a metric measure space that is a length space, then pointed measured GH convergence is equivalent to measured GH convergence of closed R-balls around the center point to closed R-balls around the center point in the limit space. Hence, it is possible to extend the previous stability statement to pointed measured GH convergence.

7 Non-branching Spaces and Tensorization Property

Lemma 7.1 *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a non-branching metric measure space that satisfies $CD(k, N)$. Then, for every $x \in \operatorname{supp} \mathrm{m}_X$, there exists a unique geodesic between x and m_X-a.e. $y \in X$. Consequently, there exists a measurable map $\Psi : X^2 \to \mathcal{G}(X)$ such that $\Psi(x, y)$ is the unique geodesic between x and y $\mathrm{m}_X \otimes \mathrm{m}_X$-a.e. .*

Proof Since k is bounded from below on any ball $B_R(x)$ by Theorem 5.3, one can adapt the proof of Lemma 4.1 in [34]. □

Proposition 7.2 *Let $k : X \to \mathbb{R}$ be admissible, $N \geq 1$ and $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space that is non-branching. Then the following statements are equivalent*

(i) *$(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the curvature-dimension condition $CD(k, N)$.*
(ii) *For each pair, $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$, there exists an optimal dynamic transference plan Π such that*

$$\varrho_t(\gamma_t)^{-\frac{1}{N}} \geq \tau^{(1-t)}_{k^-_\gamma, N'}(|\dot\gamma|)\varrho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{k^+_\gamma, N}(|\dot\gamma|)\varrho_1(\gamma_1)^{-\frac{1}{N}}. \tag{32}$$

for all $t \in [0, 1]$ and Π-a.e. $\gamma \in \mathcal{G}(X)$. Here, ϱ_t is the density of the push-forward of Π under the map $\gamma \mapsto \gamma_t$. That is determined by

$$\int_X u(y)\varrho_t(y)d\,\mathrm{m}_X(y) = \int u(\gamma_t)d\Pi(\gamma).$$

for all bounded measurable functions $u : X \to \mathbb{R}$.

Proof "$\Leftarrow$": Let $N' > N$ and $\varrho_i d\,\mathrm{m}_X = \mu_i \in \mathcal{P}_2(X, \mathrm{m}_X)$ for $i = 0, 1$. Hölder's inequality yields

$$\begin{aligned}\varrho_t(\gamma_t)^{-\frac{1}{N'}} &\geq \left(\tau^{(1-t)}_{k^-_\gamma, N'}(|\dot\gamma|)\varrho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{k^+_\gamma, N}(|\dot\gamma|)\varrho_1(\gamma_1)^{-\frac{1}{N}}\right)^{\frac{N}{N'}} \\ &\geq \tau^{(1-t)}_{k^-_\gamma, N'}(|\dot\gamma|)^{\frac{N}{N'}}(1-t)^{(1-\frac{N}{N'})}\varrho_0(\gamma_0)^{\frac{-1}{N'}} + \tau^{(t)}_{k^+_\gamma, N'}(|\dot\gamma|)^{\frac{N}{N'}}t^{(1-\frac{N}{N'})}\varrho_1(\gamma_1)^{\frac{-1}{N'}}\end{aligned}$$

In addition, Lemma 3.14 yields the estimate

$$\tau^{(1-t)}_{k^-_\gamma, N}(|\dot\gamma|)^{\frac{N}{N'}}(1-t)^{1-\frac{N}{N'}} \geq \tau^{(1-t)}_{k^-_\gamma, N'}(|\dot\gamma|)$$

and similarly for the term involving k^+_γ. Finally, integrating the previous inequality with respect to Π yields the condition $CD(k, N)$.

"$\Rightarrow$": Consider probability measures $\mu_i = \varrho_i d\,\mathrm{m}_X$ for $i = 0, 1$. Let Π be an optimal dynamic coupling. Since for $\mathrm{m}_X \otimes \mathrm{m}_X$-a.e. pair (x, y), there exists a unique geodesic $\gamma_{x,y}$, there exist an optimal coupling π such that Π can be written in the form: $\delta_{\gamma_{x,y}} d\pi(x, y)$. Therefore, the curvature-dimension condition for μ_0 and μ_1 becomes

$$\int_X \int \varrho_t(\gamma_t)^{-\frac{1}{N}} \delta_{\gamma_{x,y}}(d\gamma) d\pi(x,y)$$
$$\geq \int_{X^2} \int \left[\tau^{(1-t)}_{k^-_\gamma, N}(|\dot\gamma|) \varrho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{k^+_\gamma, N}(|\dot\gamma|) \varrho_1(\gamma_1)^{-\frac{1}{N}} \right] \delta_{\gamma_{x,y}}(d\gamma) d\pi(x,y).$$

Now, we can follow exactly the proof of the corresponding result in [34]. □

Proposition 7.3 *Let* $(X, \mathrm{d}_X, \mathrm{m}_X)$ *be a non-branching metric measure space that satisfies* $CD(k, N)$*, let* $k' : X \to \mathbb{R}$ *be lower semi-continuous and let* $V : X \to [0, \infty)$ *be strongly* $k'V$*-convex in the sense of Definition 3.16. Then* $(X, \mathrm{d}_X, V^{N'} \mathrm{m}_X)$ *satisfies the condition* $CD(k + k', N + N')$.

Proof The proof is a straightforward calculation using the characterization of $CD(k, N)$ for non-branching spaces, Corollary 4.2 and Hölder's inequality. □

Theorem 7.4 *Let* $(X_i, \mathrm{d}_{X_i}, \mathrm{m}_{X_i})$ *be non-branching metric measure spaces for* $i = 1, \dots, k$ *satisfying the condition* $CD(k_i, N_i)$ *for admissible functions* $k_i : X_i \to \mathbb{R}$ *and* $N_i \geq 1$*. Then the metric measure space*

$$\left(\Pi^k_{i=1} X_i, \sqrt{\sum\nolimits^k_{i=1} \mathrm{d}^2_{X_i}}, \bigotimes\nolimits^k_{i=1} \mathrm{m}_{X_i} \right) = (Y, \mathrm{d}_Y, \mathrm{m}_Y)$$

satisfies the condition

$$CD\left(\min_{i=1,\dots,k} k_i, \max_{i=1,\dots,k} N \right)$$

where $(\min_{i=1,\dots,k} k_i)(x_1, \dots, x_k) = \min\{k_i(x_i) : i = 1, \dots, k\}$.

Proof It is enough to consider $k = 2$ and measures of μ_0 and μ_1 in $\mathcal{P}_2(Y, \mathrm{m}_Y)$ of the form $\mu_0 = \mu_0^{(1)} \otimes \mu_0^{(2)}$ and $\mu_1 = \mu_1^{(1)} \otimes \mu_1^{(2)}$. Then general case follows in the same way as in [8], for instance. Consider dynamic optimal couplings $\Pi^{(i)}$ for $\mu_0^{(i)}$ and $\mu_0^{(i)}$ such that (32) holds according to our curvature assumption. Let $(e_0, e_1)_\star \Pi^{(i)} = \pi^{(i)}$. The pushforward of $\pi^{(1)} \otimes \pi^{(2)}$ with respect to

$$(x_0^{(1)}, x_1^{(1)}, x_0^{(2)}, x_1^{(2)}) \mapsto (x_0^{(1)}, x_0^{(2)}, x_1^{(1)}, x_1^{(2)})$$

becomes an optimal coupling π between μ_0 and μ_1. There is also a measurable map $(\gamma^{(1)}, \gamma^{(2)}) \in \mathcal{G}(X_1) \times \mathcal{G}(X_2) \mapsto (\gamma^{(1)}, \gamma^{(2)}) \in \mathcal{G}(Z)$. Therefore, we can consider the pushforward Π of $\Pi^{(1)} \times \Pi^{(2)}$ with respect to this map. Since $(e_0, e_1)_\star \Pi = \pi$, Π is an optimal dynamic plan for μ_0 and μ_1.

Claim: For geodesics $\gamma^{(1)} \in \mathcal{G}(X_1)$ *and* $\gamma^{(2)} \in \mathcal{G}(X_2)$ *consider* $\gamma = (\gamma^{(1)}, \gamma^{(2)}) \in \mathcal{G}(Y)$*, then we have*

$$\tau^{(t)}_{k_{1,\gamma}, N_1}(|\dot\gamma^{(1)}|)^{N_1} \cdot \tau^{(t)}_{k_{2,\gamma}, N_2}(|\dot\gamma^{(2)}|)^{N_2} \geq \tau^{(t)}_{k_\gamma, N_1+N_2}(|\dot\gamma|)^{N_1+N_2}$$

The claim follows immediately from Corollary 4.2 combined with the observations that $\tau^{(t)}_{k_\gamma, N}(|\dot\gamma|) = \tau^{(t)}_{k_\gamma |\dot\gamma|^2, N}(1)$, that $|\dot\gamma|^2 = |\dot\gamma^{(1)}|^2 + |\dot\gamma^{(2)}|^2$, and that

$$k_i \circ \bar{\gamma}^{(i)}(t|\dot{\gamma}^{(i)}|) = k_i \circ \gamma^{(i)}(t) \geq \min_{i=1,2} \{k_i \circ \gamma(t)\} = \left(\min_{i=1,2} k_i \circ \bar{\gamma}\right)(t|\dot{\gamma}|).$$

for $i = 1, 2$. The rest of the proof works exactly like the proof of the corresponding result in [13]. □

8 Globalization of the Reduced Curvature-Dimension Condition

Definition 8.1 If we replace in Definition 4.4

$$\tau^{(1-t)/(t)}_{k_\gamma^{-/+},N'}(|\dot{\gamma}|) \text{ by } \sigma^{(1-t)/(t)}_{k_\gamma^{-/+},N'}(|\dot{\gamma}|).$$

we say $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the *reduced curvature-dimension condition* $CD^*(k, N)$. Obviously, we always have that $CD(k, N)$ implies $CD^*(k, N)$.

We say that $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the the curvature-dimension condition *locally* - denoted by $CD_{loc}(k, N)$ - if for any point x there exists a neighborhood U_x such that for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$ with bounded support in U_x, then one can find a geodesic $\mu_t \in \mathcal{P}_2(X, \mathrm{m}_X)$ and an optimal dynamic coupling $\Pi \in \mathcal{P}(\mathcal{G}(X))$ such that (12) holds. Similarly, we define $CD^*_{loc}(k, N)$.

Remark 8.2 All the previous results of this article also hold for the condition $CD^*(k, N)$ although constants and estimates are in general not sharp.

Theorem 8.3 *Let* $(X, \mathrm{d}_X, \mathrm{m}_X)$ *be a non-branching and geodesic metric measure space with* $\operatorname{supp} \mathrm{m}_X = X$. *Let* $k : X \to \mathbb{R}$ *be admissible. Then the curvature-dimension condition* $CD^*(k, N)$ *holds if and only if it holds locally.*

Proof We only have to show the implication $CD^*_{loc}(k, N)$ implies $CD^*(k, N)$. Let us assume the curvature-dimension condition holds locally. Therefore, a Bishop–Gromov volume growth result holds locally, and it implies the space is locally compact. Then the metric Hopf-Rinow theorem implies that X is proper. Hence, we can assume that X is compact. Otherwise, we choose an exhaustion of X with compact balls $\overline{B_R(o)}$ such that the optimal transport between measures supported in $B_R(o)$ does not leave $\overline{B_{2R}(o)}$. For instance, compare with the proof of Theorem 5.1 in [8]. Similar to the proof of Proposition 7.2, one can also see that a measure contraction property holds locally. Then, the result of [10] implies uniqueness of L^2-Wasserstein geodesics.

By compactness of X, there is $\lambda \in (0, \operatorname{diam}_X)$, finitely many disjoints sets $L_1, \ldots, L_k$ that cover X and have non-zero measure, and finitely many open sets $M_1, \ldots, M_k$ such that $B_\lambda(L_i) \subset M_i$ for $i \in \{1, \ldots k\}$ and such that (12) holds in M_i for each i (for instance, see the proof of Theorem 5.1 in [8]).

Let $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$ be arbitrary and let μ_t be the L^2-Wasserstein geodesic between μ_0 and μ_1. Consider $\mu_{\bar{t}}$ and $\mu_{\bar{s}}$ such that $\bar{s} - \bar{t} \leq \lambda / \operatorname{diam}_X$. We define $\nu_\tau = \mu_{(1-\tau)\bar{t}+\tau\bar{s}}$ is a geodesic between $\mu_{\bar{t}}$ and $\mu_{\bar{s}}$, and any transport geodesic has length less than λ. Π denotes the optimal dynamic transference plan that corresponds to ν_t. We decompose ν_0 with respect to $(L_i)_{i=1,\ldots,k}$ as follows

$$\nu_0 = \sum_{i=1}^{k} \frac{1}{\nu_0(L_i)} \nu_0|_{L_i} = \sum_{i=1}^{k} \nu_0^i.$$

Define $\mathcal{L}_i = \{\gamma \in \mathcal{G}(X) : \gamma(0) \in L_i\}$ with $\nu_0(L_i) = \Pi(\mathcal{L}_i)$. The restriction property of optimal transport yields that $\Pi^i = \Pi(\mathcal{L}_i)^{-1}\Pi|_{\mathcal{L}_i}$ are optimal dynamic couplings between ν_0^i and $\nu_1^i = (e_1)_*\Pi^i$ and Π^i induces a geodesic ν_τ^i between ν_0^i and $\nu_1^i = (e_1)_*\Pi^i$. By construction, ν_1^i is supported in M_i. Hence, the condition $CD(k, N)$ implies

$$\varrho_t^i(\gamma(t))^{-\frac{1}{N}} \geq \sigma_{k_\gamma^-,N}^{(1-t)}(|\dot{\gamma}|)\varrho_0^i(\gamma(0))^{-\frac{1}{N}} + \sigma_{k_\gamma^+,N}^{(t)}(|\dot{\gamma}|)\varrho_1^i(\gamma(1))^{-\frac{1}{N}}$$

for Π^i-a.e. $\gamma \in \mathcal{G}(X)$ where $\varrho_t^i d\,\mathrm{m}_X = d\nu_t^i$. In particular, ν_t is absolutely continuous with density $\varrho_t = \sum_{i=1}^k \varrho_t^i$.

The measures ν_0^i are disjoint. Therefore, the measures ν_t^i for $i = 1, \ldots, k$ are disjoint for any $t \in [0, 1)$ (see for instance, Lemma 2.6 in [8]). Since any optimal transport between absolutely continuous probability measures is induced by an optimal map, we can conclude that also ν_1^i are disjoint. Therefore, for any $t \in [0, 1]$,

$$\varrho_t(x)^{-\frac{1}{N}} = \sum_{i=1}^{k} \frac{1}{\Pi(\mathcal{L}_i)} \varrho_t^i(x)^{-\frac{1}{N}} \tag{33}$$

where $\varrho_t^i d\,\mathrm{m}_X = d\nu_t^i$. Hence

$$\varrho_t(\gamma(t))^{-\frac{1}{N}} \geq \sigma_{k_\gamma^-,N}^{(1-t)}(|\dot{\gamma}|)\varrho_0(\gamma(0))^{-\frac{1}{N}} + \sigma_{k_\gamma^+,N}^{(t)}(|\dot{\gamma}|)\varrho_1(\gamma(1))^{-\frac{1}{N}}$$
$$\text{for } \Pi - \text{a.e.}\gamma \in \mathcal{G}(X).$$

In particular, the previous argument holds for each $\bar{s}, \bar{t} \in [0, 1] \cap \mathbb{Q}$. Thus, if μ_t is the unique geodesic between μ_0, μ_1 and Π is the corresponding optimal dynamic plan, we showed that

$$\rho_{\tau(t)}(\gamma(\tau(t)))^{-\frac{1}{N}} \geq \sigma_{k_\gamma^-,N}^{(1-\tau(t))}((s-t)|\dot{\gamma}|)\rho_t(\gamma(t))^{-\frac{1}{N}}$$
$$+ \sigma_{k_\gamma^+,N}^{(\tau(t))}((s-t)|\dot{\gamma}|)\rho_s(\gamma(s))^{-\frac{1}{N}}$$

for Π-a.e. geodesic γ and each $\bar{t}, \bar{s} \in [0, 1] \cap \mathbb{Q}$ where $\tau(t) = (1-t)\bar{t} + t\bar{s}$. If we pick such a geodesic γ, the inequality holds also globally along γ for ρ_t by Corollary 3.13. Then the result follows. □

9 Curvature-Dimension Condition with Variable Dimension Bound

We briefly discuss two possibilities to define a curvature-dimension condition $CD(k, \mathcal{N})$ with variable dimension bound $\mathcal{N} : X \to (0, \infty)$. Following the previous approach for variable lower curvature bounds, it is not obvious how to pose a reasonable definition when $N = \mathcal{N}$ is variable as well. This is because for our definition, we use the N-Reny entropy functional where N is a constant parameter.

However, the problem can be resolved in the following way. Consider the non-branching situation of Sect. 7, and the reduced curvature-dimension condition $CD^*(\kappa, N)$ that is introduced in Sect. 8. A metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD^*(k, N)$ if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}^2(X, \mathrm{m}_X)$ there exists a geodesic $\mu_t \in \mathcal{P}^2(X, \mathrm{m}_X)$ and a dynamic optimal plan Π with $(e_t)_\star \Pi = \mu_t$ and

$$\rho_t(\gamma_t)^{-\frac{1}{N}} \geq \sigma^{(1-t)}_{\kappa^-_\gamma/N}(|\dot\gamma|)\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma^{(t)}_{\kappa^+_\gamma/N}(|\dot\gamma|)\rho_0(\gamma_0)^{-\frac{1}{N}} \tag{34}$$

for Π-a.e. $\gamma \in \mathcal{G}(X)$, where $\mu_t = \rho_t d\,\mathrm{m}_X$. (34) is equivalent to the following differential inequality in the distributional sense:

$$\frac{d^2}{dt^2}\rho_t(\gamma_t)^{-\frac{1}{N}} \leq -\frac{\kappa_\gamma(t)|\dot\gamma|^2}{N}\rho_t(\gamma_t)^{-\frac{1}{N}} \quad \text{on } [0,1]. \tag{35}$$

Note (35) implies that $t \mapsto \rho_t(\gamma_t)$ is absolutely continuous. If $N = \mathcal{N}$ is variable along γ, (35) would involve second derivatives of $\mathcal{N}$ along γ. However, one can fix this problem by another reformulation. If we assume that $\rho_t(\gamma_t)$ is C^2 in t for Π-a.e. γ (this is true, for instance, on Riemannian manifolds), then (35) is equivalent to the Ricatti equation

$$\frac{d^2}{dt^2}\log\rho_t(\gamma_t) + \frac{1}{N}\left(\frac{d}{dt}\log\rho_t(\gamma_t)\right)^2 + \kappa_\gamma(t)\,|\dot\gamma|^2 \leq 0. \tag{36}$$

Here, it is no problem to replace N by an upper semi-continuous function $\mathcal{N} \circ \gamma$. Then, the following definition is meaningful.

Definition 9.1 Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a non-branching metric measure space, and let $\kappa : X \to \mathbb{R}$ be lower semi-continuous and let $\mathcal{N} : X \to (0, \infty)$ be upper semi-continuous. We say $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD^*(\kappa, \mathcal{N})$ if for each pair $\mu_0, \mu_1 \in \mathcal{P}^2(X, \mathrm{m}_X)$ there exists a geodesic $\mu_t \in \mathcal{P}^2(X, \mathrm{m}_X)$ and a dynamic optimal plan Π such that $(e_t)_\star \Pi = \rho_t\,\mathrm{m}_X \in \mathcal{P}^2(\mathrm{m}_X)$, $\rho_t(\gamma_t)$ is absolutely continuous in $t \in [0, 1]$ for Π-a.e. γ and (36) holds in the distributional sense with N replaced by $\mathcal{N} \circ \gamma$.

Similarly, one can define an entropic curvature-dimension condition $CD^e(\kappa, \mathcal{N})$ for κ lower semi-continuous and $\mathcal{N}$ upper semi-continuous following ideas of [14] and [19] where no non-branching assumption is required. However, Definition 9.1 has an important disadvantage. It is not clear to the author how to formulate an equivalent, integrated version that is desirable for proving geometric consequences and stability properties. Moreover, the full curvature-dimension condition $CD(\kappa, \mathcal{N})$ does not make sense since the coefficients $\tau^{(t)}_{\kappa_\gamma}(|\dot\gamma|)$ are not derived from an ODE.

Following a clever suggestion of the referee another possibility to extend our definition to variable upper dimension bounds would be as follows.

Definition 9.2 Consider an admissible function $k : X \to \mathbb{R}$, and a upper semi-continuous function $\mathcal{N} : X \to \mathbb{R}$ bounded from above. A metric measure space

$(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the *curvature-dimension condition* $CD(k, \mathcal{N})$ if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(X, \mathrm{m}_X)$ with bounded support there exists a geodesic $(\nu_t)_{t\in[0,1]} \subset \mathcal{P}_2(X, \mathrm{m}_X)$ and a dynamic optimal coupling $\Pi \in \mathcal{P}(X)$ such that $(e_t)_\star \Pi = \nu_t$ and

$$S_{N'}(\nu_t) \leq -\int \left[\tau^{(1-t)}_{k^-_\gamma, N'}(|\dot\gamma|)\varrho_0\left(e_0(\gamma)\right)^{-\frac{1}{N'}} + \tau^{(t)}_{k^+_\gamma, N'}(|\dot\gamma|)\varrho_1\left(e_1(\gamma)\right)^{-\frac{1}{N'}} \right] d\Pi(\gamma) \tag{37}$$

for all $t \in [0, 1]$ and all $N' \geq \overline{N}$ where $\overline{N} = \overline{N}(\Pi) = \sup_{\gamma \operatorname{supp} \Pi, t\in[0,1]} \mathcal{N}(\gamma(t))$.

The upper bound for $\mathcal{N}$ guarantees that $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD(\kappa, N)$ for some N, and in particular it is locally compact. Therefore $\overline{N}(\Pi)$ is well defined. This definition seems to be better for proving geometric consequences and stability of the condition, and it might be possible that one can extend the results of this article.

Example 9.3 First, it is clear that any $CD(\kappa, N)$-space satisfies a condition $CD(\kappa, \mathcal{N})$ for N constant and $\mathcal{N}$ upper semi-continuous if $\mathcal{N} \geq N$.

Now, we construct a more interesting example. Let $N, N' \in [1, \infty)$ with $N > N'$, and $K \in (0, \infty)$. Consider

$$I = \left[-\pi\sqrt{\frac{K}{N-1}}, 0\right] \cup \left(0, \pi\sqrt{\frac{K}{N'-1}}\right]$$

and define

$$f(x) = \begin{cases} \cos_{K/(N-1)}(x)^{N-1} & \text{if } x \in [-\pi\sqrt{\frac{K}{N-1}}, 0], \\ \cos_{K/(N'-1)}(x)^{N'-1} & \text{if } x \in (0, \pi\sqrt{\frac{K}{N'-1}}]. \end{cases}$$

Then, $(I, |\cdot|_2, f d\mathcal{L}^1)$ satisfies the curvature-dimension condition $CD(K, \mathcal{N})$ in the sense of Definition 9.2 and the condition $CD^*(K, \mathcal{N})$ in the sense of Definition 9.1 where

$$\mathcal{N}(x) = \begin{cases} N & \text{if } x \in [-\pi\sqrt{\frac{K}{N-1}}, 0], \\ N' & \text{if } x \in (0, \pi\sqrt{\frac{K}{N'-1}}]. \end{cases}$$

Note that the space satisfies the condition $CD(K, N)$ already.

Acknowledgements The author is very grateful to Yu Kitabeppu for his keen interest in this project and many fruitful discussions. The author also wants to thank the unknown referee for providing useful comments that helped to improve the present article, and the referee who gave the idea for Definition 9.2 on variable upper dimension bounds. Major parts of this work have been written during the Junior Trimester Programm "Optimal Transport" at the Hausdorff Institute of Mathematics (HIM) in Bonn. The author also wishes to thank the HIM for the excellent working condition and the stimulation and open atmosphere.

References

1. Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. **163**(7), 1405–1490 (2014)
2. Ambrosio, L., Gigli, N., Savaré, G.: Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. Ann. Probab. **43**(1), 339–404 (2015)
3. Ambrose, W.: A theorem of Myers. Duke Math. J. **24**, 345–348 (1957)
4. Anastasiei, M., Peter, I.R.: A compactness theorem in Finsler geometry. Publ. Math. Debr. **84**(1–2), 75–88 (2014)
5. Aubry, E.: Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature. Ann. Sci. École Norm. Sup. (4) **40**(4), 675–695 (2007)
6. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry. Graduate Studies in Mathematics, vol. 33. American Mathematical Society, Providence, RI (2001)
7. Brenier, Y.: Polar factorization and monotone rearrangement of vector-valued functions. Commun. Pure Appl. Math. **44**(4), 375–417 (1991)
8. Bacher, K., Sturm, K.-T.: Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. J. Funct. Anal. **259**(1), 28–56 (2010)
9. Cordero-Erausquin, D., McCann, R.J., Schmuckenschläger, M.: A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. Invent. Math. **146**(2), 219–257 (2001)
10. Cavalletti, F., Huesmann, M.: Existence and uniqueness of optimal transport maps. Ann. Inst. H. Poincaré Anal. Non Linéaire **32**(6), 1367–1377 (2015)
11. Cavalletti, F., Mondino, A.: Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower ricci curvature bounds. arxiv:1502.06465 (2015)
12. Dal Maso, G.: An Introduction to Γ-Convergence, Progress in Nonlinear Differential Equations and their Applications, vol. 8. Birkhäuser Boston Inc, Boston, MA (1993)
13. Deng, Q., Sturm, K.-T.: Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, II. J. Funct. Anal. **260**(12), 3718–3725 (2011)
14. Erbar, M., Kuwada, K., Sturm, K.-T.: On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. Invent. Math. **201**(3), 993–1071 (2015)
15. Gregory, J.: Galloway, compactness criteria for Riemannian manifolds. Proc. Am. Math. Soc. **84**(1), 106–110 (1982)
16. Gigli, N.: An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature. Anal. Geom. Metr. Spaces **2**, 169–213 (2014)
17. Gigli, N., Mondino, A., Savaré, G.: Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. Proc. Lond. Math. Soc. (3) **111**(5), 1071–1129 (2015)
18. Guan, P., Wang, G.: Conformal deformations of the smallest eigenvalue of the Ricci tensor. Am. J. Math. **129**(2), 499–526 (2007)
19. Ketterer, C.: Evolution variational inequality and wasserstein control in variable curvature context. arxiv:1509.02178 (2015)
20. Ketterer, C.: Ricci curvature bounds for warped products. J. Funct. Anal. **265**(2), 266–299 (2013)
21. Ketterer, C.: Cones over metric measure spaces and the maximal diameter theorem. J. Math. Pures Appl. (9) **103**(5), 1228–1275 (2015)
22. Ketterer, C.: Obata's rigidity theorem for metric measure spaces. Anal. Geom. Metr. Spaces **3**, 278–295 (2015)
23. Lott, J., Villani, C.: Weak curvature conditions and functional inequalities. J. Funct. Anal. **245**(1), 311–333 (2007)
24. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) **169**(3), 903–991 (2009)
25. McCann, R.J.: Polar factorization of maps on Riemannian manifolds. Geom. Funct. Anal. **11**(3), 589–608 (2001)
26. Ohta, S.: On the measure contraction property of metric measure spaces. Comment. Math. Helv. **82**(4), 805–828 (2007)
27. Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. Commun. Partial Differ. Equ. **26**(1–2), 101–174 (2001)
28. Petersen, P., Sprouse, C.: Integral curvature bounds, distance estimates and applications. J. Differ. Geom. **50**(2), 269–298 (1998)

29. Petersen, P., Wei, G.: Relative volume comparison with integral curvature bounds. Geom. Funct. Anal. **7**(6), 1031–1045 (1997)
30. Rajala, T.: Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm. J. Funct. Anal. **263**(4), 896–924 (2012)
31. Schneider, R.: Konvexe Flächen mit langsam abnehmender Krümmung. Arch. Math. (Basel) **23**, 650–654 (1972)
32. Simon, B.: Convexity, Cambridge Tracts in Mathematics, vol. 187. Cambridge University Press, Cambridge (2011). An analytic viewpoint
33. Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. **196**(1), 65–131 (2006)
34. Sturm, K.-T.: On the geometry of metric measure spaces. II. Acta Math. **196**(1), 133–177 (2006)
35. Sturm, K.-T.: Metric measure spaces with variable Ricci bounds and couplings of Brownian motions. Festschrift Masatoshi Fukushima (Interdiscip. Math. Sci.) **17**, 553–575 (2015)
36. Veysseire, L.: A harmonic mean bound for the spectral gap of the Laplacian on Riemannian manifolds. C. R. Math. Acad. Sci. Paris **348**(23–24), 1319–1322 (2010)
37. Villani, Cédric: Optimal Transport, Old and New, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338. Springer, Berlin (2009)
38. Zhang, S.: A theorem of Ambrose for Bakry-Emery Ricci tensor. Ann. Glob. Anal. Geom. **45**(3), 233–238 (2014)