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On Nonuniqueness of Geodesics in Asymptotic Teichmüller Space

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Abstract In an infinite-dimensional Teichmüller space, it is known that the geodesic connecting two points can be unique or not. In this paper, we study the situation on the geodesic in the universal asymptotic Teichmüller space $AT(\Delta)$. We introduce the notions of substantial point and non-substantial point in $AT(\Delta)$. The set of all non-substantial points is open and dense in $AT(\Delta)$. It is shown that there are infinitely many geodesics joining a non-substantial point to the basepoint. Although we have difficulty in dealing with the substantial points, we give an example to show that there are infinitely many geodesics connecting a certain substantial point and the basepoint. It is also shown that there are always infinitely many straight lines containing two points in $AT(\Delta)$.

Keywords Teichmüller space · Asymptotic Teichmüller space · Geodesic · Substantial boundary point · Substantial point

Mathematics Subject Classification Primary 30C75, 30C62

1 Introduction

Let S be a hyperbolic Riemann surface, that is, it is covered by a holomorphic map: $\varpi : \Delta \to S$, where $\Delta = \{|z| < 1\}$ is the open unit disk. Let $T(S)$ be the Teichmüller space of S. A quotient space of the Teichmüller space $T(S)$, called the asymptotic Teichmüller space and denoted by $AT(S)$, was introduced by Gardiner and Sullivan

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(see [11] for $S = \Delta$ and by Earle, Gardiner and Lakic for arbitrary hyperbolic S [2,4,10]).

$AT(S)$ is interesting only when $T(S)$ is infinite dimensional, which occurs when S has border or when S has infinite topological type; otherwise, $AT(S)$ consists of just one point. In recent years, the asymptotic spaces have been extensively studied; for example, one can refer to [2–4,8,18,19,25].

We shall use some geometric terminology adapted from [1] by Busemann. Let X and Y be metric spaces. An isometry of X into Y is a distance preserving map. A straight line in Y is a (necessarily closed) subset L that is an isometric image of the real line $\mathbb{R}$. A geodesic in Y is an isometric image of a non-trivial compact interval of $\mathbb{R}$. Its endpoints are the images of the endpoints of the interval, and we say that the geodesic joins its endpoints.

Geodesics play an important role in the theory of Teichmüller spaces. In a finite-dimensional Teichmüller space $T(S)$, there is always a unique geodesic connecting two points. The situation is substantially different in an infinite-dimensional Teichmüller space (see [5,12,15–17,23,24]). Generally, for a Strebel point, there is a unique geodesic connecting it and the basepoint. A natural question is whether the geodesic connecting two points in $AT(S)$ is unique. In [6], by a lengthy computation Fan tried to give certain examples to show the nonuniqueness of geodesics in $AT(\Delta)$ or in more general $AT(S)$. Unfortunately, there is a gap in his proof.

The motivation of this paper is to investigate the nonuniqueness of geodesics in the universal asymptotic Teichmüller space $AT(\Delta)$. We introduce the notions of substantial point and non-substantial point in $AT(\Delta)$. The set of all non-substantial points is open and dense in $AT(\Delta)$. The first main result is the following theorem.

Theorem 1 *For every non-substantial point in $AT(\Delta)$, there are infinitely many geodesics joining it with the basepoint.*

We have some difficulty in dealing with the substantial points. Nevertheless, we give an example to show that there are infinitely many geodesics connecting a certain substantial point and the basepoint which might support the conjecture that there are always infinitely many geodesics connecting two points in $AT(\Delta)$.

In a finite-dimensional Teichmüller space, there is a unique straight line passing through two points [13]. In an infinite-dimensional Teichmüller space, the work in [5] shows that if and only if τ is a Strebel point, there is a unique straight line passing through τ and the basepoint in $T(S)$. The second main result of this paper characterizes the nonuniqueness of straight lines containing two points in $AT(\Delta)$.

Theorem 2 *For any two points in $AT(\Delta)$, there are infinitely many straight lines containing them.*

This paper is organized as follows. In Sect. 2, we introduce some basic notions in the Teichmüller space theory. Theorems 1 and 2 are proved in Sects. 3 and 4 separately. We investigate the relationship on the substantial boundary points for points along a geodesic in Sect. 5. In Sect. 6, an example is given to show the nonuniqueness of geodesics joining a certain substantial point with the basepoint. Some parallel results in the infinitesimal setting are obtained in the last section.

The method used here can also be used to deal with some more general cases. However, there are some difficulties in solving the problem in all cases.

2 Some Preliminaries

2.1 Teichmüller Space and Asymptotic Teichmüller Space

Let S be a Riemann surface of topological type. The Teichmüller space $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable Riemann surface $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are equivalent if there is a conformal map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every $t \in [0, 1]$ and every p in the ideal boundary of S. Denote by $[f]$ the Teichmüller equivalence class of f; also sometimes denote the equivalence class by $[\mu]$ where μ is the Beltrami differential of f.

The asymptotic Teichmüller space is the space of a larger equivalence classes. The definition of the new equivalence classes is exactly the same as the previous definition with one exception; the word *conformal* is replaced by *asymptotically conformal*. A quasiconformal map f is asymptotically conformal if for every $\epsilon > 0$, there is a compact subset E of S, such that the dilatation of f outside of E is less than $1 + \epsilon$. Accordingly, denote by $[[f]]$ or $[[\mu]]$ the asymptotic equivalence class of f.

Denote by $Bel(S)$ the Banach space of Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on S with finite L^∞-norm and by $M(S)$ the open unit ball in $Bel(S)$.

For $\mu \in M(S)$, define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

Define $h^*(\mu)$ to be the infimum over all compact subsets E contained in S of the essential supremum norm of the Beltrami differential $\mu(z)$ as z varies over $S\backslash E$ and $h([\mu])$ to be the infimum of $h^*(\nu)$ taken over all representatives ν of the class $[\mu]$. It is obvious that $h([\mu]) \leq k_0([\mu])$. Following [5], $[\mu]$ is called a Strebel point if $h([\mu]) < k_0(\tau)$; otherwise, τ is called a non-Strebel point.

Put

$$h([[\mu]]) = \inf\{h^*(\nu) : \nu \in [[\mu]]\}.$$

We say that μ is extremal in $[\mu]$ if $\|\mu\|_\infty = k_0([\mu])$ and μ is asymptotically extremal if $h^*(\mu) = h([[\mu]])$. The relation $h([\mu]) = h([[\mu]])$ is due to the definition.

The Teichmüller metric d_T between two points $\tau, \sigma \in T(S)$ is defined as follows:

$$d_T(\tau, \sigma) = \frac{1}{2} \inf_{\mu\in\tau,\ \nu\in\sigma} \log \frac{1 + \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}{1 - \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}.$$

The asymptotic Teichmüller metric d_{AT} between two points $\widetilde{\tau}, \widetilde{\sigma} \in AT(S)$ is defined by

$$
\begin{aligned}
d_{AT}(\widetilde{\tau}, \widetilde{\sigma}) &= \frac{1}{2} \inf_{\mu \in \widetilde{\tau},\ \nu \in \widetilde{\sigma}} \log \frac{1 + \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}{1 - \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty} \\
&= \frac{1}{2} \inf_{\mu \in \widetilde{\tau},\ \nu \in \widetilde{\sigma}} \log \frac{1 + h^*((\mu - \nu)/(1 - \bar{\nu}\mu))}{1 - h^*((\mu - \nu)/(1 - \bar{\nu}\mu))}.
\end{aligned} \tag{2.1}
$$

In particular, the distance between $[[\mu]]$ and the basepoint $[[0]]$ is

$$
d_{AT}([[\mu]], [[0]]) = \frac{1}{2} \log H([[\mu]]), \text{ where } H([[\mu]]) = \frac{1 + h([[\mu]])}{1 - h([[\mu]])}. \tag{2.2}
$$

2.2 Tangent Spaces to Teichmüller Space and Asymptotic Teichmüller Space

The cotangent space to $T(S)$ at the basepoint is the Banach space $Q(S)$ of integrable holomorphic quadratic differentials φ on S with L^1-norm

$$
\|\varphi\| = \iint_S |\varphi(z)|\, dxdy < \infty.
$$

In what follows, let $Q^1(S)$ denote the unit sphere of $Q(S)$. Moreover, let $Q^1_d(S)$ denote the set of all degenerating sequence $\{\varphi_n\} \subset Q^1(S)$. By definition, a sequence $\{\varphi_n\}$ is called degenerating if it converges to 0 uniformly on compact subsets of S.

Two Beltrami differentials μ and ν in $Bel(S)$ are said to be infinitesimally equivalent if

$$
\iint_S (\mu - \nu)\varphi\, dxdy = 0, \text{ for any } \varphi \in Q(S).
$$

The tangent space $Z(S)$ of $T(S)$ at the basepoint is defined as the set of the quotient space of $Bel(S)$ under the equivalence relations. Denote by $[\mu]_Z$ the equivalence class of μ in $Z(S)$.

$Z(S)$ is a Banach space and actually [10] its standard sup-norm satisfies

$$
\|[\mu]_Z\| := \sup_{\varphi \in Q^1(S)} \left| \iint_S \mu\varphi\, dxdy \right| = \inf\{\|\nu\|_\infty : \nu \in [\mu]_Z\}.
$$

Two Beltrami differentials μ and ν in $Bel(S)$ are said to be infinitesimally asymptotically equivalent if

$$
\sup_{Q^1_d(S)} \limsup_{n \to \infty} \left| \iint_S (\mu - \nu)\varphi_n\, dxdy \right| = 0,
$$

where the first *supremum* is taken when $\{\varphi_n\}$ varies over $Q^1_d(S)$.

The tangent space $AZ(S)$ of $AT(S)$ at the basepoint is defined as the set of the quotient space of $Bel(S)$ under the asymptotic equivalence relation. Denote by $[[\mu]]_{AZ}$ the equivalence class of μ in $AZ(S)$.

Define $b([\mu]_Z)$ to be the infimum over all elements in the equivalence class $[\mu]_Z$ of the quantity $b^*(\nu)$. Here $b^*(\nu)$ is the infimum over all compact subsets E contained in

S of the essential supremum of the the Beltrami differential ν as z varies over $S - E$. It is obvious that $b^*(\mu) \leq \|[\mu]_Z\|$. $[\mu]_Z$ is called an infinitesimal Strebel point if $b([\mu]_Z) < \|[\mu]_Z\|$.

Put

$$b([[\mu]]_{AZ}) = \inf\{b^*(\nu) : \nu \in [[\mu]]_{AZ}\}.$$

We say that μ is (infinitesimally) extremal if $\|\mu\|_\infty = \|[\mu]_Z\|$ and μ is (infinitesimally) asymptotically extremal if $b^*(\mu) = b([[\mu]]_{AZ})$. We also have $b([\mu]_Z) = b([[\mu]]_{AZ})$ [10].

$AZ(S)$ is a Banach space and its standard infinitesimal asymptotic norm satisfies (see [10])

$$\|[[\mu]]_{AZ}\| := \sup_{Q_d^1(S)} \limsup_{n\to\infty} \left| \iint_S \mu\varphi_n \, dxdy \right| = \inf\{\|\nu\|_\infty : \nu \in [[\mu]]_{AZ}\} = b([[\mu]]_{AZ}).$$

2.3 Substantial Boundary Points and Hamilton Sequence

Now we define the notion of boundary dilatation of a quasiconformal mapping at a boundary point. For a Riemann surface, the meaning of what is a boundary point can be problematic. However, if S can be embedded into a larger surface $\widetilde{S}$ such that the closure of S in $\widetilde{S}$ is compact, then it is possible to define the boundary dilatation. In this section, we assume that S is such a surface.

Let p be a point on ∂S and let $\mu \in Bel(S)$. Define

$$h_p^*(\mu) = \inf\{\operatorname{esssup}_{z\in U\bigcap S} |\mu(z)| : \ U \text{ is an open neighborhood in } \widetilde{S} \text{ containing } p\}$$

to be the boundary dilatation of μ at p. If $\mu \in M(S)$, define

$$h_p([\mu]) = \inf\{h_p^*(\nu) : \ \nu \in [\mu]\}$$

to be the boundary dilatation $[\mu]$ at p. For a general $\mu \in Bel(S)$, define

$$b_p([\mu]_Z) = \inf\{h_p^*(\nu) : \ \nu \in [\mu]_Z\}$$

to be the boundary dilatation of $[\mu]_Z$ at p. If we define the quantities

$$h_p([[\mu]]) = \inf\{h_p^*(\nu) : \ \nu \in [[\mu]]\}, \quad b_p([[\mu]]_{AZ}) = \inf\{h_p^*(\nu) : \ \nu \in [[\mu]]_{AZ}\},$$

then $h_p([\mu]) = h_p([[\mu]])$ and $b_p([\mu]_Z) = b_p([[\mu]]_{AZ})$. In particular, Lakic [14] proved that when S is a plane domain,

$$h([[\mu]]) = \max_{p\in\partial S} h_p([[\mu]]), \quad b([[\mu]]_{AZ}) = \max_{p\in\partial S} b_p([[\mu]]_{AZ}).$$

As is well known, μ is extremal if and only if it has a so-called Hamilton sequence, namely, a sequence $\{\psi_n\} \subset Q^1(S)$, such that

$$\lim_{n\to\infty} \iint_S \mu\psi_n(z)dxdy = \|\mu\|_\infty. \tag{2.3}$$

Similarly, by Theorem 8 on p. 281 in [10], μ is asymptotically extremal if and only if it has an asymptotic Hamilton sequence, namely, a degenerating sequence $\{\psi_n\} \subset Q^1(S)$, such that

$$\lim_{n\to\infty} \iint_S \mu\psi_n(z)dxdy = h^*(\mu). \tag{2.4}$$

Now, we assume that S is a plane domain with two or more boundary points. Then, the following lemma derives from Theorem 6 on p. 333 in [10]:

Lemma 2.1 *The following three conditions are equivalent for every boundary point p of S and every asymptotically or infinitesimal asymptotically extremal representative μ:*

(1) $h([\mu]) = h_p([\mu])$ *(equivalently,* $h([[\mu]]) = h_p([[\mu]])$*),*

(2) $b([\mu]) = b_p([\mu])$ *(equivalently,* $b([[\mu]]_{AZ}) = b_p([[\mu]]_{AZ})$*),*

(3) *there exists an asymptotic Hamilton sequence for μ degenerating towards p, i.e., a sequence $\{\psi_n\} \subset Q^1(S)$ converging uniformly to 0 on compact subsets of $S\backslash\{p\}$, such that*

$$\lim_{n\to\infty} \iint_S \mu\psi_n(z)dxdy = h_p^*(\mu). \tag{2.5}$$

If one of three conditions in the lemma holds at some $p \in \partial S$, we call p a substantial boundary point for $[[\mu]]$ (or $[\mu]$) and $[[\mu]]_{AZ}$ (or $[\mu]_Z$), respectively.

3 Geodesics Joining Non-substantial Points with the Basepoint

$[[\mu]]$ (or $[[\mu]]_{AZ}$) is called a substantial point in $AT(\Delta)$ (or $AZ(\Delta)$), if every $p \in \partial\Delta$ is a substantial boundary point for $[[\mu]]$ (or $[[\mu]]_{AZ}$); otherwise, $[[\mu]]$ (or $[[\mu]]_{AZ}$) is called a non-substantial point.

Let SP and ISP denote the collection of all (infinitesimal) substantial points in $AT(\Delta)$ and $AZ(\Delta)$, respectively. Since every substantial point can be approximated by a sequence of non-substantial points, it is clear that $AT(\Delta)\backslash SP$ and $AZ(\Delta)\backslash ISP$ are open and dense in $AT(\Delta)$ and $AZ(\Delta)$, respectively.

Let $d_H(z_1, z_2)$ denote the hyperbolic distance between two points z_1, z_2 in Δ, i.e.,

$$d_H(z_1, z_2) = \frac{1}{2}\log\frac{1 + |\frac{z_1 - z_2}{1-\bar{z}_1 z_2}|}{1 - |\frac{z_1 - z_2}{1-\bar{z}_1 z_2}|}.$$

To prove Theorem 1, we need a series of lemmas.

Lemma 3.1 *Let t_1, t_2 be two complex numbers and k_1, k_2 be two real numbers. Then we have*

$$\left|\frac{(t_1-t_2)k_1}{1-\overline{t_2}t_1k_1^2}\right| \le \left|\frac{(t_1-t_2)k_2}{1-\overline{t_2}t_1k_2^2}\right|, \quad \textit{if } 0<k_1\le k_2 \textit{ and } k_2^2|t_1t_2|<1. \tag{3.1}$$

Proof Without any loss of generality, we may assume that $t_1t_2 \neq 0$. Let k be a real variable and put

$$F(k) = \left|\frac{(t_1-t_2)k}{1-\overline{t_2}t_1k^2}\right|^2 = \frac{|t_1-t_2|^2k^2}{1+|t_1t_2|^2k^4-2k^2Re(\overline{t_2}t_1)}.$$

It is easy to verify that $F'(k)\ge 0$ as $k\in(0,1/\sqrt{|t_1t_2|})$. Therefore $F(k)$ is an increasing function on $(0,1/\sqrt{|t_1t_2|})$ and hence (3.1) holds. □

Lemma 3.2 *Let $\mu\in Bel(\Delta)$ and $p\in\partial\Delta$. Then, for any given $\epsilon>0$,*
(1) if $\mu\in M(\Delta)$, then there exists a Beltrami differential $\nu\in[\mu]$ such that ν is an asymptotical extremal and $h_p^*(\nu)<h_p([\mu])+\epsilon$;
(2) there exists a Beltrami differential $\nu\in[\mu]_Z$ such that ν is an asymptotical extremal and $b_p^*(\nu)<b_p([\mu]_Z)+\epsilon$.

Proof We only show the first part for the second part follows from a similar argument.

Case 1. $h_p([\mu])=h([\mu])$.

By Theorem 2 on p. 296 of [10], there exists a Beltrami differential $\nu\in[\mu]$ such that ν is an asymptotical extremal representative, that is, $h^*(\nu)=h([\mu])$. It obviously yields $h_p^*(\nu)<h_p([\mu])+\epsilon$.

Case 2. $h_p([\mu])<h([\mu]):=h$.

By the definition of boundary dilatation, there exists a Beltrami differential $\chi(z)\in[\mu]$ such that $h_p^*(\chi)<\min\{h_p([\mu])+\epsilon,h\}$. Let $B(p)=\{z\in\Delta:\ |z-p|<r\}$ for small $r>0$. Then, when r is sufficiently small, $|\chi(z)|<\min\{h_p([\mu])+\epsilon,h\}$ in $B(p)$ almost everywhere.

Restrict χ on $\Delta\backslash B(p)$ and regard $[\chi]$ as a point in the Teichmüller space $T(\Delta\backslash B(p))$. Then $h([\chi])=h$ (if necessary, let $B(p)$ be smaller). By Theorem 2 on p. 296 of [10] again, we can choose an asymptotical extremal in $[\chi]$, say $\nu_1(z)$. Define

$$\nu(z)=\begin{cases}\nu_1(z), & z\in\Delta\backslash B(p),\\ \chi(z), & z\in B(p).\end{cases}$$

Then, ν is the desired asymptotical extremal in $[\mu]$. □

In [6], Fan obtained a sufficient condition to determine two different geodesics. That is the following theorem.

Theorem A *Let μ and ν be two asymptotically extremal Beltrami differentials in $[[\mu]] \in AT(\Delta)$. If*

$$\sup_{Q_d^1(\Delta)} \limsup_{n\to\infty} \left| \iint_\Delta (\mu - \nu)\phi_n \, dxdy \right| > 0,$$

then the two geodesics $[[t\mu]]$ and $[[t\nu]]$ ($0 \le t \le 1$) are different, where the first supremum is taken when $\{\phi_n\}$ varies over $Q_d^1(\Delta)$.

The following corollary is direct.

Corollary 3.1 *Suppose μ and ν be two asymptotically extremal Beltrami differentials in their classes in $AT(\Delta)$ respectively. If $h([[\mu]]) = h([[\nu]])$ and*

$$\sup_{Q_d^1(\Delta)} \limsup_{n\to\infty} \left| \iint_\Delta (\mu - \nu)\phi_n \, dxdy \right| > 0,$$

then the two geodesics $[[t\mu]]$ and $[[t\nu]]$ ($0 \le t \le 1$) are different.

Proof of Theorem 1 Suppose $[[\mu]]$ is not a substantial point in $AT(\Delta)$. Let $h = h([[\mu]])$. There is a non-substantial boundary point $q \in \partial\Delta$ such that $h_q([[\mu]]) < h$. By Lemma 3.2, it is convenient to assume that μ is an asymptotical extremal representative in $[[\mu]]$ satisfying $h_q^*(\mu) < h$.

By the definition of boundary dilatation, we can find a small neighborhood $B(q)$ of q in Δ such that $|\mu(z)| \le \rho < h$ for some $\rho > 0$ in $B(q)$ almost everywhere. Therefore for any $\zeta \in \partial\Delta \cap \partial B(q)$, $h_\zeta^*(\mu) \le \rho$.

Choose $\delta(z) \in M(\Delta)$ such that $\|\delta\|_\infty \le \beta < h - \rho$ and $\delta(z) = 0$ when $z \in \Delta \backslash B(q)$.

Let Σ be the collection of the real-valued functions $\sigma(t)$ defined on $[0, h]$ with the following conditions:

(A) σ is continuous with $\sigma(0) = 0$ and $\sigma(h) = 0$,

(B) $\frac{|s-t|\rho/h + |\sigma(t)-\sigma(s)|\beta}{1-(s\rho/h+|\sigma(s)|\beta)(t\rho/h+|\sigma(t)|\beta)} \le \frac{|s-t|}{1-st}$, $s, t \in [0, h]$.

We claim that Σ contains uncountably many elements. At first, let σ be a Lipschitz continuous function on $[0, h]$ with the following conditions,

(i) for some small $\alpha > 0$, $|\sigma(s) - \sigma(t)| < \alpha|s - t|$, t, $s \in [0, h]$,

(ii) $\sigma(0) = 0$ and $\sigma(h) = 0$,

(iii) for some small t_0 in $(0, h)$, $\sigma(t) \equiv 0$ when $t \ge t_0$.

Secondly, we show that when t_0 and α are sufficiently small, σ belongs to Σ, for which it suffices to show that σ satisfies the condition (B). Let t, $s \in [0, h]$. It is no harm to assume that $t \le s$.

Case 1. $h \ge t \ge t_0$.

Since $\sigma(s) = \sigma(t) = 0$, by Lemma 3.1, we have

$$\frac{|s-t|\rho/h + |\sigma(t)-\sigma(s)|\beta}{1-(s\rho/h+|\sigma(s)|\beta)(t\rho/h+|\sigma(t)|\beta)} = \frac{|s-t|\rho/h}{1-st(\rho/h)^2} \\ \leq \left|\frac{s-t}{1-st}\right|.$$

Case 2. $0 \leq t < t_0$.

Put $\gamma = \rho/h + \alpha\beta$ and choose small $\alpha > 0$ such that $\gamma < 1$. On the one hand, since $|\sigma(t)| \leq \alpha t$ and $|\sigma(s)| \leq \alpha s$, it holds that

$$\frac{|s-t|\rho/h + |\sigma(t)-\sigma(s)|\beta}{1-(s\rho/h+|\sigma(s)|\beta)(t\rho/h+|\sigma(t)|\beta)} \leq \left|\frac{(s-t)(\rho/h+\alpha\beta)}{1-(s\rho/h+|\sigma(s)|\beta)(t\rho/h+|\sigma(t)|\beta)}\right| \\ \leq \left|\frac{(s-t)(\rho/h+\alpha\beta)}{1-[\rho/h+\alpha\beta][t_0(\rho/h+\alpha\beta)]}\right| = \gamma\left|\frac{s-t}{1-t_0\gamma^2}\right|.$$

On the other hand, we have

$$\left|\frac{s-t}{1-st}\right| \geq \left|\frac{s-t}{1+t_0}\right|.$$

When t_0 is sufficiently small, we can get

$$\left|\frac{s-t}{1+t_0}\right| \geq \gamma\left|\frac{s-t}{1-t_0\gamma^2}\right|.$$

Therefore, when t_0 and α are sufficiently small, σ satisfies the condition (B).

For a given $\sigma \in \Sigma$, define for $t \in [0, h]$,

$$\mu_t(z) = \begin{cases} t\mu(z)/h, & z \in \Delta\backslash B(q), \\ t\mu(z)/h + \sigma(t)\delta(z), & z \in B(q). \end{cases} \tag{3.2}$$

Step 1 We prove that $\{[[\mu_t]] : t \in [0, h]\}$ is a geodesic connecting $[[0]]$ and $[[\mu]]$. It is sufficient to verify that whenever $t, s \in [0, h]$,

$$d_{AT}([[\mu_t]], [[\mu_s]]) = d_H(t, s) = \frac{1}{2}\log\frac{1+|s-t|/(1-st)}{1-|s-t|/(1-st)}. \tag{3.3}$$

Let $f_s : \Delta \to \Delta$ and $f_t : \Delta \to \Delta$ be quasiconformal mappings with Beltrami differentials μ_s and μ_t respectively. It is convenient to assume that $t \neq 0$ and $t \neq s$. Set $F_{s,t} = f_s \circ f_t^{-1}$ and assume that the Beltrami differential of $F_{s,t}$ is $\nu_{s,t}$. Then a simple computation shows,

$$\nu_{s,t} \circ f_t(z) = \frac{1}{\tau}\frac{\mu_s(z)-\mu_t(z)}{1-\overline{\mu_t(z)}\mu_s(z)},$$

where $z = f_t^{-1}(w)$ for $w \in \Delta$ and $\tau = \overline{\partial f_t}/\partial f_t$. We have

$$\nu_{s,t} \circ f_t(z) = \begin{cases} \frac{1}{\tau} \frac{s-t}{1-st|\mu(z)|^2/h^2} \frac{\mu(z)}{h}, & z \in \Delta \backslash B(q), \\ \frac{1}{\tau} \frac{(s-t)\mu(z)/h+[\sigma(s)-\sigma(t)]\delta(z)}{1-[s\mu(z)/h+\sigma(s)\delta(z)]\overline{t\mu(z)/h+\sigma(t)\delta(z)}}, & z \in B(q). \end{cases} \quad (3.4)$$

Since $\sigma(t) \in \Sigma$, due to condition (B) we see that restricted on $f_t(B(q))$,

$$\|\nu_{s,t}\|_\infty \le \frac{|s-t|}{1-st}. \quad (3.5)$$

Suppose $p \in \partial\Delta \cap \partial(\Delta \backslash B(q))$ is a substantial boundary point for $[[\mu]]$. By Lemma 2.1 there is a degenerating Hamilton sequence $\{\psi_n\} \subset Q^1(\Delta)$ towards p such that

$$h = \lim_{n\to\infty} \iint_\Delta \mu(z)\psi_n(z)dxdy.$$

Then we have

$$t = \lim_{n\to\infty} \iint_\Delta \mu_t(z)\psi_n(z)dxdy.$$

On the other hand, it is easy to see that $h([[\mu_t]]) = h^*(\mu_t) = t$ and hence μ_t is an asymptotical extremal. Therefore, the Beltrami differential $\widetilde{\mu}_t$ of f_t^{-1} is also an asymptotical extremal where

$$\widetilde{\mu}_t = -\mu_t(f_t^{-1})\overline{\partial f_t^{-1}}/\partial f_t^{-1} = -\frac{1}{\tau}\frac{t\mu(z)}{h}$$

Thus $f_t(p) \in \partial\Delta \cap \partial(\Delta \backslash f_t(B(q)))$ is a substantial boundary point for $[[\widetilde{\mu}_t]]$ and there is a degenerating Hamilton sequence $\{\widetilde{\psi}_n\} \subset Q^1(\Delta)$ towards $f_t(p)$ such that

$$\lim_{n\to\infty} \iint_\Delta \widetilde{\mu}_t \widetilde{\psi}_n(w)dudv = \lim_{n\to\infty} \iint_{\Delta\backslash f_t(B(q))} \widetilde{\mu}_t \widetilde{\psi}_n(w)dudv = h([[\widetilde{\mu}_t]]) = t.$$

Note that when $w = f_t(z) \in \Delta \backslash f_t(B(q))$,

$$\nu_{s,t}(w) = \frac{1}{\tau}\frac{s-t}{1-st|\mu(z)|^2/h^2}\frac{\mu(z)}{h} = \frac{t-s}{1-st|\mu(f_t^{-1}(w))|^2/h^2}\frac{\widetilde{\mu}_t(w)}{t}.$$

By a simple analysis, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\iint_{\Delta}\nu_{s,t}(w)\widetilde{\psi}_n(w)dudv\\
&=\lim_{n\to\infty}\iint_{\Delta}\frac{t-s}{1-st|\mu(f_t^{-1}(w))|^2/h^2}\frac{\widetilde{\mu}_t(w)}{t}\widetilde{\psi}_n(w)dudv\\
&=\lim_{n\to\infty}\iint_{\Delta\backslash f_t(B(q))}\frac{t-s}{1-st|\mu(f_t^{-1}(w))|^2/h^2}\frac{\widetilde{\mu}_t(w)}{t}\widetilde{\psi}_n(w)dudv\\
&=\lim_{n\to\infty}\iint_{\Delta\backslash f_t(B(q))}\frac{t-s}{1-st}\frac{\widetilde{\mu}_t}{t}\widetilde{\psi}_n(w)dudv=\frac{t-s}{1-st}.
\end{aligned} \tag{3.6}$$

In terms of (3.4) and Lemma 3.1, it is not hard to prove that $h^*_\zeta(\nu_{s,t}) \le \frac{|s-t|}{1-st}$ when $\zeta \in \partial\Delta \cap \partial(\Delta\backslash f_t(B(q)))$. Thus, by (3.5), (3.6) and Lemma 2.1, it follows that $h([[\nu_{s,t}]]) = \frac{|s-t|}{1-st}$, $\nu_{s,t}$ is asymptotically extremal and the equality (3.3) holds.

Step 2. We show that, when $\sigma(t)$ varies over Σ and $\delta(z)$ varies over $M(\Delta)$ suitably, respectively, we can get infinitely many different geodesics.

Firstly, choose $\delta(z)$ in $M(\Delta)$ such that

$$\sup_{Q_d^1(\Delta)}\ \limsup_{n\to\infty}\left|\iint_{\Delta}\delta\varphi_n\,dxdy\right| = c > 0, \tag{3.7}$$

where the *supremum* is over all sequences $\{\varphi_n\}$ in $Q_d^1(\Delta)$ degenerating towards q.

Secondly, we choose small t_0 in $(0, h)$, small $\alpha > 0$ and $\sigma \in \Sigma$ such that $\sigma(t) \equiv 0$ when $t \in [t_0, h]$ and $\sigma(t) = \alpha t$ when $t \in [0, t_0/2]$.

Claim *When α varies in a small range, the geodesics $[[\mu_t]]$ ($t \in [0, h]$) are mutually different.*

Let α_1 and α_2 be two small different positive numbers and $\sigma_j(t) = \alpha_j t$ when $t \in [0, t_0]$ ($j = 1, 2$), respectively. Now, the corresponding expression of equation (3.2) is

$$\mu_t^j(z) = \begin{cases} t\mu(z)/h, & z \in \Delta\backslash B(q),\\ t\mu(z)/h + \sigma_j(t)\delta(z), & z \in B(q),\ \ j = 1, 2.\end{cases}$$

They correspond to geodesics $G_j = \{[[\mu_t^j]] : t \in [0, h]\}$ ($j = 1, 2$), respectively. Note that when $t \in [0, t_0/2]$,

$$\mu_t^j(z) = \begin{cases} t\mu(z)/h, & z \in \Delta\backslash B(q),\\ t\mu(z)/h + t\alpha_j\delta(z), & z \in B(q),\ \ j = 1, 2.\end{cases}$$

Define

$$\mu^j(z) = \begin{cases} \mu(z)/h, & z \in \Delta\backslash B(q),\\ \mu(z)/h + \alpha_j\delta(z), & z \in B(q),\ \ j = 1, 2.\end{cases}$$

Since

$$\sup_{Q_d^1(\Delta)} \limsup_{n\to\infty} \left| \iint_\Delta (\mu^1 - \mu^2)\varphi_n \, dxdy \right| = \sup_{Q_d^1(\Delta)} \limsup_{n\to\infty} \left| \iint_\Delta (\alpha_1 - \alpha_2)\delta\varphi_n \, dxdy \right|$$
$$\geq |\alpha_1 - \alpha_2| c > 0,$$

by Corollary 3.1, the geodesics G_1 and G_2 are different.

Fixing small $\alpha > 0$ and letting δ vary suitably in $M(\Delta)$, we can also get infinitely many geodesics as desired. The proof of Theorem 1 is completed.

We say that μ is a non-Strebel extremal if it is an extremal representative in the non-Strebel point $[\mu]$ (or $[\mu]_Z$). Suppose μ ($\neq 0$) is a non-Strebel extremal. Then $[t\mu]$ and $[[t\mu]]$ ($t \in [0, 1]$) are the geodesics in $T(\Delta)$ and $AT(\Delta)$ respectively. If μ is uniquely extremal in $[\mu]$ with constant modulus, then the geodesic joining $[\mu]$ with $[0]$ is unique in $T(\Delta)$ [5]. Suppose $[[\mu]]$ is a non-substantial point in $AT(\Delta)$ in addition. Then by Theorem 1 there are infinitely many geodesics joining $[[\mu]]$ with $[[0]]$ in $AT(\Delta)$.

In [6], the uniquely extremal Beltrami differential k constructed by Fan for his Theorem 3.1 actually has the property: except that two points at infinity are two substantial boundary points, the boundary dilatation of k at other boundary points is identically zero. Fan's proof of Theorem 3.1 is lengthy and complicated. Unfortunately, there is a serious gap in his proof. In his argument, he should have used the definition of the asymptotic Teichmüller metric by (2.1) carefully. To be precise, he should prove that restricted on $f(S)$, $\widetilde{\nu}(w) = \frac{1}{\tau}\frac{\delta k - \mu_\delta}{1 - \delta k \overline{\mu_\delta}} \circ f^{-1}(w)$ satisfies $h([[\widetilde{\nu}(w)]]) > 0$ if it is possible, where f is the quasiconformal map from S onto $f(S)$ with Beltrami differential μ_δ and $\tau = \overline{f_z}/f_z$. But Fan directly used the computation $h([[\frac{\delta k - \mu_\delta}{1 - \delta k \overline{\mu_\delta}}]]) > 0$ on S to deduce that $d_{AT}(\widetilde{\gamma}(\delta), \gamma(\delta)) > 0$, which is problematic. There is a similar problem with the proof of his other main result Theorem 4.1. In our argument, the construction of Σ-class just helps overcome these difficulty in computation.

For completeness, here we give a new example.

Example Let $\phi(z)$ be holomorphic on $\overline{\Delta}$ except that it has poles of at most order 2 on $\partial\Delta$. Assume that $\phi(z)$ has a second-order pole at $z = 1$. Then by Reich's result [22], $\mu = k\frac{\overline{\phi}}{|\phi|}$ ($k \in (0, 1)$) is uniquely extremal and $[\mu]$ is a non-Strebel point in $T(\Delta)$. In addition, it is easy to check that $h_\zeta^*(\mu) = 0$ if $\zeta \in \partial\Delta$ is neither a pole nor a zero of $\phi(z)$. Therefore, $[[\mu]]$ is not a substantial point in $AT(\Delta)$.

4 Straight Lines Containing Two Points

The following lemma says that a non-Strebel extremal as an asymptotical extremal representative always exists in a class $[[\mu]]$.

Lemma 4.1 *Let $\mu \in Bel(S)$. Then,*

(1) *if $\mu \in M(S)$, then there exists a Beltrami differential $\nu \in [[\mu]]$ such that ν is a non-Strebel extremal;*

(2) *there exists a Beltrami differential* $\nu \in [[\mu]]_{AZ}$ *such that* ν *is a non-Strebel extremal.*

Proof We only show the first part (1).

By Theorem 2 on p. 296 of [10], there is an asymptotical extremal representative in $[[\mu]]$, say μ, such that $h([[\mu]]) = h^*(\mu)$. If $h^*(\mu) = 0$, let ν be identically zero. If $h^*(\mu) > 0$, put

$$\nu(z) = \begin{cases} \mu(z), & |\mu(z)| \le h^*(\mu), \\ h^*(\mu)\mu(z)/|\mu(z)|, & |\mu(z)| > h^*(\mu). \end{cases}$$

In either case, it is easy to verify that $\nu \in [[\mu]]$ and is a non-Strebel extremal. □

Let μ ($\neq 0$) be a non-Strebel extremal. Then there are infinitely many straight lines containing $[0]$ and $[\mu]$ in $T(\Delta)$ [5]. However, it cannot be directly inferred that there are infinitely many straight lines containing $[[0]]$ and $[[\mu]]$ in $AT(\Delta)$ since the topologies induced by metrics in $T(\Delta)$ and $AT(\Delta)$ are essentially different.

Proof of Theorem 2 Up to an isometry of $AT(\Delta)$, it suffices to prove that for any $[[\mu]]$ ($\neq [[0]]$) in $AT(\Delta)$, there are infinitely many straight lines passing through $[[\mu]]$ and $[[0]]$. By Lemma 4.1, we choose a non-Strebel extremal representative in $[[\mu]]$, say μ. Then $k_0([\mu]) = h([[\mu]]) = h^*(\mu) := h$.

Case 1. $[[\mu]]$ is a substantial point.

Fix a boundary point $p \in \partial\Delta$. Let $B(p) = \{z \in \Delta : |z - p| < r\}$ for small $r > 0$ and $E = \Delta \backslash B(p)$. Define for $t \in (-1, 1)$,

$$\mu_t(z) := \begin{cases} t\mu(z)/h, & z \in \Delta,\ |t| \le h, \\ t\mu(z)/h, & z \in \Delta\backslash E,\ |t| > h, \\ sgn(t)\mu(z), & z \in E,\ |t| > h. \end{cases} \tag{4.1}$$

We prove that $G_E = \{[[\mu_t]] : t \in (-1, 1)\}$ is a straight line passing through $[[0]]$ and $[[\mu]]$. Note that G_E differs from the straight line $G[\mu] = \{[[t\mu/h]] : t \in (-1, 1)\}$ only when $|t| > h$. It is sufficient to show the following two points: for any given $\rho \in (h, 1)$,

(i) $d_{AT}([[\mu_{-\rho}]], [[\mu_\rho]]) = d_H(-\rho, \rho)$;
(ii) $\{[[\mu_t]] : t \in [0, \rho]\}$ and $\{[[\mu_t]] : t \in [-\rho, 0])\}$ are two geodesics

and

$$d_{AT}([[\mu_{-\rho}]], [[0]]) = d_{AT}([[0]], [[\mu_\rho]]) = \frac{1}{2} d_H(-\rho, \rho). \tag{4.2}$$

(i) is relatively clear since on E, $|\rho\mu(z)|/\rho < \rho|\mu(z)|/h$ for $\rho > h$, so is (4.2). Due to symmetry, for (ii), it suffices to show that $\{[[\mu_t]] : t \in [0, \rho]\}$ is a geodesic. This is reduced to proving that $\{[[\mu_t]] : t \in [0, h]\}$ and $\{[[\mu_t]] : t \in [h, \rho])\}$ are two geodesics, and

$$d_{AT}([[0]], [[\mu_h]]) + d_{AT}([[\mu_h]], [[\mu_\rho]]) = d_H(0, \rho).$$

In such a case, we only need to check that $\{[[\mu_t]] : t \in [h, \rho])\}$ is a geodesic with length $d_H(h, \rho)$. In fact, when $\rho \le t < s \le h$, using the previous notation, we have

$$\nu_{s,t} \circ f_t(z) = \begin{cases} \frac{1}{\tau} \frac{s-t}{1-st|\mu(z)|^2/h^2} \frac{\mu(z)}{h}, & z \in \Delta \backslash E, \\ 0, & z \in E. \end{cases} \tag{4.3}$$

Now, it is evident that

$$d_{AT}([[\mu_t]], [[\mu_s]]) = \frac{1}{2} \log \frac{1 + h([[\nu_{s,t}]])}{1 - h([[\nu_{s,t}]])} = d_H(t, s)\ t, s \in [h, \rho].$$

Comparing $[[\mu_t]]$ with $[[t\mu/h]]$ as $|t| > h$, we find that $[[\mu_t]]$ is no longer a substantial point since the boundary points in the interior of $\partial\Delta \cap \partial E$ are no longer substantial ones. Therefore, when the boundary point p or the neighborhood $B(p)$ varies, we get infinitely many different straight lines.

Case 2. $[[\mu]]$ is not a substantial point.

By Theorem 1, there are infinitely many geodesics connecting $[[0]]$ and $[[\mu]]$. We can then extend these geodesics to straight lines by uniformly defining, $[[\mu_t]] = [[t\mu/h]]$ for $t \in (-1, 0) \cup (h, 1)$. The verification is similar to Case 1 and is omitted here. This completes the proof of Theorem 2.

5 Relationship on Substantial Boundary Points for Points Along a Geodesic

In this section, we investigate the relationship on substantial boundary points for the points along a geodesic. We have the following result.

Theorem 3 *Suppose* $h([[\mu]]) = h \in (0, 1)$ *and* $\{[[\mu_t]] : t \in (0, h)\}$ *is a geodesic connecting* $[[0]]$ *and* $[[\mu]]$ *such that* $d_{AT}([[0]], [[\mu_t]]) = d_H(0, t)$ *for* $t \in (0, h)$. *If* $p \in \partial\Delta$ *is a substantial boundary point for* $[[\mu]]$, *then* p *is a substantial boundary point for all* $[[\mu_t]]$, $t \in (0, h)$.

Proof Let $h = h([[\mu]])$. Given $t \in (0, h)$, let $\nu(z) \in [[\mu_t]]$ be an asymptotic extremal representative with $h^*(\nu) = t$. We need to show that $h_p^*(\nu) = t$.

Let $f : \Delta \to \Delta$ and $g : \Delta \to \Delta$ be the quasiconformal mappings with Beltrami differentials μ and ν, respectively. Let Λ be an asymptotical extremal quasiconformal mapping in the asymptotic equivalence class $[[f \circ g^{-1}]]$. Assume that $\lambda(w)$ is the Beltrami differential of Λ where $w = g(z)$, and $h([[\lambda]]) = h^*(\lambda) = \alpha$. Put $F = \Lambda \circ g$. Then F and f is asymptotically equivalent and

$$\mu_F = \frac{\nu + \lambda \circ g \cdot \tau}{1 + \overline{\nu} \cdot \lambda \circ g \cdot \tau}, \tag{5.1}$$

where $\tau = \overline{\partial_z g}/\partial_z g$. Since

$$d_{AT}([[0]], [[\mu]]) = d_{AT}([[0]], [[\mu_t]]) + d_{AT}([[\mu_t]], [[\mu]]),$$

we have
$$\frac{1}{2}\log\frac{1+h}{1-h}=\frac{1}{2}\log\frac{1+t}{1-t}+\frac{1}{2}\log\frac{1+\alpha}{1-\alpha},$$
equivalently,
$$\frac{1+h}{1-h}=\frac{1+t}{1-t}\cdot\frac{1+\alpha}{1-\alpha}.$$
This leads to
$$h=\frac{t+\alpha}{1+t\alpha}. \tag{5.2}$$

On the other hand, by (5.1) we have

$$|\mu_F|=\left|\frac{\nu+\lambda\circ g\cdot\tau}{1+\overline{\nu}\cdot\lambda\circ g\cdot\tau}\right|\le\frac{|\nu|+|\lambda\circ g|}{1+|\nu||\lambda\circ g|}.$$

Therefore, by the definition of boundary dilatation, we get

$$h=h_p([[\mu]])\le h_p^*(\mu_F)\le\frac{h_p^*(\nu)+h_{g(p)}^*(\lambda)}{1+h_p^*(\nu)h_{g(p)}^*(\lambda)}\le\frac{t+\alpha}{1+t\alpha}. \tag{5.3}$$

Notice that $h_p^*(\nu)\le t$ and $h_{g(p)}^*(\lambda)\le\alpha$. Combining (5.2) and (5.3), we must have $h_p^*(\nu)=t$. This concludes the proof. □

The following corollary follows immediately.

Corollary 5.1 *If* $[[\mu]]$ *is a substantial point in* $AT(\Delta)$*, then every point in a geodesic connecting* $[[0]]$ *and* $[[\mu]]$ *is a substantial point.*

There is a natural projection from $T(\Delta)$ onto $AT(\Delta)$,

$$\pi: T(\Delta)\to AT(\Delta),$$
$$[\mu]\to[[\mu]].$$

If μ is a non-Strebel extremal, then the projection of any geodesic connecting $[0]$ and $[\mu]$ in $T(\Delta)$ under π is a geodesic connecting $[[0]]$ and $[[\mu]]$ in $AT(\Delta)$. Therefore, we have the following corollary.

Corollary 5.2 *If* μ *is a non-Strebel extremal and* $p\in\partial\Delta$ *is a substantial boundary point for* $[\mu]$*, then* p *is a substantial boundary point for all points in a geodesic connecting* $[0]$ *and* $[\mu]$ *in* $T(\Delta)$.

Remark 1 If $p\in\Delta$ is not a substantial boundary point for $[[\mu]]$, it is possible that p is a substantial boundary point for some point (hence for infinitely many points) in the geodesic connecting $[[0]]$ and $[[\mu]]$ (see Case 1 in the proof of Theorem 2 in Sect. 4).

6 An Example for Geodesics Joining a Substantial Point with the Basepoint

The situation on the geodesics joining a substantial point with the basepoint is not clear. The difficulty can be seen from Theorem 3, for which the method used in the proof of Theorem 1 does not apply for a substantial point. However, one must not expect that the geodesic passing through a substantial point and the basepoint is necessarily unique. We now construct a certain counterexample to show how it is.

We divide the construction of the example into three steps.

Step 1 At first, we introduce an example, which we describe below, every point $\zeta \in \partial\Delta$ is a substantial boundary point. The example was shown in [7] and was said to be due to Reich by an oral communication. For the sake of clarity and completeness, here we include the detail for construction which was demonstrated in [7].

Example Let ϕ_n be the sequence defined by

$$\phi_n(z) := \frac{(n+2)z^n}{2\pi}.$$

For a fixed number k, $0 < k < 1$, we define, for every $n \in \mathbb{N}$,

$$\kappa_n(z) := k\frac{\bar{z}^n}{|z|^n}.$$

Then for $0 \le \rho_1 \le \rho_2 \le 1$

$$\iint_{\rho_1<|z|<\rho_2} \kappa_n\phi_n dxdy = \frac{(n+2)k}{2\pi}\int_0^{2\pi}\int_{\rho 1}^{\rho_2} r^{n+1}drd\theta = k(\rho_2^{n+2} - \rho_1^{n+2})$$

and

$$\iint_{\rho_1<|z|<\rho_2} |\phi_n|dxdy = \frac{n+2}{2\pi}\int_0^{2\pi}\int_{\rho 1}^{\rho_2} r^{n+1}drd\theta = \rho_2^{n+2} - \rho_1^{n+2}.$$

Choose a number $n_1 \in \mathbb{N}$. Then there is a number r_1, $0 < r_1 < 1$, with

$$r_1^{n_1+2} > 1 - \frac{1}{2}$$

and we compute

$$\iint_{0<|z|<r_1} \kappa_{n_1}\phi_{n_1} dxdy = kr_1^{n_1+2} > k(1-\frac{1}{2})$$

and

$$\iint_{r_1<|z|<1} |\phi_{n_1}|dxdy = 1 - r_1^{n_1+2} < \frac{1}{2}.$$

Next we choose $n_2 > n_1$ such that

$$r_1^{n_2+2} < \frac{1}{2^2}.$$

Then there is a number r_2, $r_1 < r_2 < 1$, such that

$$r_2^{n_2+2} > 1 - \frac{1}{2^2}$$

and we may also have $r_2 > r_1 + \frac{1-r_1}{2}$. We compute

$$\iint_{|z|<r_1} |\phi_{n_2}| dxdy = r_1^{n_2+2} < \frac{1}{2^2},$$

$$\iint_{r_1<|z|<r_2} \kappa_{n_2}\phi_{n_2} dxdy = k(r_2^{n_2+2} - r_1^{n_2+2}) > k(1 - \frac{1}{2^2} - \frac{1}{2^2}) = k(1 - \frac{1}{2})$$

and

$$\iint_{r_2<|z|<1} |\phi_{n_2}| dxdy = 1 - r_2^{n_2+2} < \frac{1}{2^2},$$

Proceeding with this construction, we get a sequence n_j ($n_j \to \infty$) and a sequence r_j, $r_1 < r_2 < \cdots < 1$, $r_j \to 1$ ($j \to \infty$). Furthermore, because of $r_{j-1}^{n_j+2} < 1/2^j$, $r_j^{n_j+2} > 1 - 1/2^j$, we have for $j \geq 2$

$$\iint_{|z|<r_{j-1}} |\phi_{n_j}| dxdy = r_{j-1}^{n_j+2} < \frac{1}{2^j}, \tag{6.1}$$

$$\iint_{r_{j-1}<|z|<r_j} \kappa_{n_j}\phi_{n_j} dxdy = k(r_j^{n_j+2} - r_{j-1}^{n_j+2}) > k(1 - \frac{1}{2^j} - \frac{1}{2^j}) = k(1 - \frac{1}{2^{j-1}}), \tag{6.2}$$

$$\iint_{r_j<|z|<1} |\phi_{n_j}| dxdy = 1 - r_j^{n_j+2} < \frac{1}{2^j}. \tag{6.3}$$

Clearly, $\{\phi_n\} \subset Q^1(\Delta)$ is a degenerating sequence in Δ. Set $E_j = \{z : r_{j-1} \leq |z| < r_j\}$ for $j \geq 1$ where we let $r_0 = 0$. Define

$$\kappa(z) := \begin{cases} \kappa_{n_1}(z), & z \in E_1, \\ \kappa_{n_2}(z), & z \in E_2, \\ \vdots & \\ \kappa_{n_j}(z), & z \in E_j, \\ \vdots & \end{cases} \tag{6.4}$$

Then $\kappa(z)$ has constant modulus k. Regard $\kappa(z)$ as the complex dilatation of a quasiconformal self-mapping f of Δ. By (6.1), (6.2) and (6.3) we have

$$Re \iint_{\Delta} \kappa\phi_{n_j} dxdy \geq Re \iint_{r_{j-1}<|z|<r_j} \kappa_{n_j}\phi_{n_j} dxdy - \iint_{|z|\leq r_{j-1} \text{ or } r_j\leq|z|<1} k|\phi_{n_j}|dxdy \geq k(1-\frac{1}{2^{j-1}}) - \frac{k}{2^{j-1}}. \tag{6.5}$$

Thus, we have

$$\lim_{j\to\infty} Re \iint_{\Delta} \kappa\phi_{n_j} dxdy = k$$

and hence ϕ_{n_j} is a Hamilton sequence for the extremal complex dilatation κ. Moreover, as Fehlmann and Sakan noted in their paper, by Theorem 1.1 in [7], every $\zeta \in \partial\Delta$ is a substantial boundary point for $[\kappa]$.

Step 2. With some modification on κ, we define a new complex dilatation as follows,

$$\mu(z) := \begin{cases} \alpha\kappa(z), & z \in E_{2m-1}, \\ \beta\kappa(z), & z \in E_{2m}, \end{cases}$$

where $m \geq 1$ and the constants $\alpha, \beta \in [0, 1/k)$.

Claim *μ is extremal and* $k_0([\mu]) = \max\{\alpha k, \beta k\}$. *Moreover, every $\zeta \in \partial\Delta$ is a substantial boundary point for* $[\mu]$.

Proof If $\alpha = \beta$, then $\mu = \alpha\kappa$ and the claim is a fortiori.

Let $\alpha < \beta$ first. By the reasoning deriving (6.5), we have

$$Re \iint_{\Delta} \mu\phi_{n_{2m}} dxdy \geq \beta Re \iint_{E_{2m}} \kappa_{2m}\phi_{n_{2m}} dxdy - \beta \iint_{\Delta\setminus E_{2m}} k|\phi_{n_{2m}}|dxdy \geq \beta k(1-\frac{1}{2^{2m-1}}) - \frac{\beta k}{2^{2m-1}}.$$

Thus, we get

$$\lim_{m\to\infty} Re \iint_{\Delta} \mu\phi_{n_{2m}} dxdy = \beta k.$$

Hence μ is extremal with $\|\mu\|_\infty = \beta k$ and $\phi_{n_{2m}}$ is a degenerating Hamilton sequence. Similarly, if $\alpha > \beta$, then μ is extremal with $\|\mu\|_\infty = \alpha k$ and $\phi_{n_{2m-1}}$ is a degenerating Hamilton sequence. Anyway, the aforementioned reason implies that every $\zeta \in \partial\Delta$ is a substantial boundary point for $[\mu]$ or $[[\mu]]$. This claim is proved. □

Step 3. Fix $\alpha \in (0, 1)$ and $\beta = 1$. Then μ is extremal with $\|\mu\|_\infty = k$ and $[[\mu]]$ is a substantial point in $AT(\Delta)$. We construct infinitely many geodesics connecting $[[\mu]]$ and the basepoint.

Let Σ' be the collection of the real-valued functions $\sigma(t)$ defined on $[0, k]$ with the following conditions:

(A) σ is continuous with $\sigma(0) = 0$ and $\sigma(k) = k$,

(B) $\frac{|\sigma(s)-\sigma(t)|\alpha}{|1-\sigma(t)\sigma(s)\alpha^2|} \leq \frac{|s-t|}{1-st}$, $t, s \in [0, k]$.
Since $0 < \alpha < 1$, it is easy to verify that Σ' contains uncountably many elements.

Given $\sigma \in \Sigma'$, define for $t \in [0, k]$,

$$\mu_t(z) := \begin{cases} \sigma(t)\mu(z)/k, & z \in E_{2m-1},\ m \geq 1, \\ t\mu(z)/k, & z \in E_{2m},\ m \geq 1. \end{cases} \tag{6.6}$$

One easily proves that $\{[[\mu_t]] : t \in [0, k]\}$ is a geodesic connecting $[[0]]$ and $[[\mu]]$.

Fix some t_0 in $(0, k)$. Choose $\sigma(t) \in \Sigma'$ such that $\sigma(t) = \lambda t$ when $t \in [0, t_0]$ where $\lambda \in (0, 1)$ is sufficiently small. We show that for different λ, these geodesics are mutually different.

Let $\lambda_1, \lambda_2 \in (0, 1)$ ($\lambda_1 > \lambda_2$) be small and $\sigma_j(t) = \lambda_j t$ when $t \in [0, t_0]$ ($j = 1, 2$), respectively. Now, on $[0, t_0]$ the corresponding expression of Eq. (6.6) is

$$\mu_t^j(z) := \begin{cases} \lambda_j t\mu(z)/k, & z \in E_{2m-1},\ m \geq 1, \\ t\mu(z)/k, & z \in E_{2m},\ m \geq 1. \end{cases} \tag{6.7}$$

They correspond to geodesic segments $G_j = \{[[\mu_t^j]] : t \in [0, t_0]\}$ ($j = 1, 2$), respectively.

Define

$$\mu^j(z) := \begin{cases} \lambda_j\mu(z)/k, & z \in E_{2m-1},\ m \geq 1. \\ \mu(z)/k, & z \in E_{2m},\ m \geq 1. \end{cases}$$

Then

$$\mu^1 - \mu^2 = \begin{cases} (\lambda_1 - \lambda_2)\mu(z)/k, & z \in E_{2m-1},\ m \geq 1, \\ 0, & z \in E_{2m},\ m \geq 1. \end{cases}$$

Since

$$\lim_{m\to\infty} \iint_\Delta (\mu^1 - \mu^2)\phi_{n_{2m-1}}\, dxdy = \frac{1}{k}(\lambda_1 - \lambda_2) \lim_{m\to\infty} \iint_\Delta \mu\phi_{n_{2m-1}} dxdy$$
$$= \lambda_1 - \lambda_2 > 0,$$

by Corollary 3.1, the geodesic segments G_1 and G_2 are different.

The example serves to give infinitely many geodesics connecting the infinitesimal substantial point $[[\mu]]_{AZ}$ and the basepoint in $AZ(\Delta)$ as well.

In an infinite-dimensional Teichmüller space, there always exist closed geodesics and the spheres are not convex due to Li's work [17] (also see [5]). Here a closed geodesic means to be locally shortest. As a byproduct of the example, the following result in the asymptotic Teichmüller space is fairly direct.

Theorem 4 *There exist closed geodesics in the universal asymptotic Teichmüller space $AT(\Delta)$ and hence the spheres in $AT(\Delta)$ are not convex.*

Proof Define

$$\eta_1(z) := \begin{cases} \kappa(z), & z \in E_{2m-1},\ m \geq 1, \\ 0, & z \in E_{2m},\ m \geq 1, \end{cases}$$

$$\eta_2(z) := \begin{cases} 0, & z \in E_{2m-1},\ m \geq 1, \\ \kappa(z), & z \in E_{2m},\ m \geq 1, \end{cases}$$

$\eta_3(z) = -\eta_1(z)$ and $\eta_4(z) = -\eta_2(z)$. Let $R = \frac{1}{2}\log\frac{1+k}{1-k}$. It is easy to derive that

$$d_{AT}[[0]], [[\eta_j]] = R,\ \ j = 1, 2, 3, 4,\ \ 2R = d_{AT}([[\eta_1]], [[\eta_3]]) = d_{AT}([[\eta_2]], [[\eta_4]])$$

and

$$\begin{aligned} R &= d_{AT}([[\eta_1]], [[\eta_2]]) = d_{AT}([[\eta_2]], [[\eta_3]]) = d_{AT}([[\eta_3]], [[\eta_4]]) \\ &= d_{AT}([[\eta_4]], [[\eta_1]]). \end{aligned}$$

Define for $t \in [0, k]$

$$\mu_t(z) := \begin{cases} \sigma(t)\kappa(z)/k, & z \in E_{2m-1},\ m \geq 1, \\ t\kappa(z)/k, & z \in E_{2m},\ m \geq 1, \end{cases}$$

where $\sigma(t) = \frac{k-t}{1-tk}$ as $t \in [0, k]$. Using the same notation as in the proof of Theorem 1, we have

$$\nu_{s,t} \circ f_t(z) = \begin{cases} \frac{1}{\tau}\frac{\sigma(s)-\sigma(t)}{1-\sigma(s)\sigma(t)}\frac{\kappa(z)}{k}, & z \in E_{2m-1},\ m \geq 1, \\ \frac{1}{\tau}\frac{s-t}{1-st}\frac{\kappa(z)}{k}, & z \in E_{2m},\ m \geq 1. \end{cases} \tag{6.8}$$

Observe that

$$\left|\frac{\sigma(s)-\sigma(t)}{1-\sigma(s)\sigma(t)}\frac{\kappa(z)}{k}\right| = \left|\frac{\sigma(s)-\sigma(t)}{1-\sigma(s)\sigma(t)}\right| = \left|\frac{s-t}{1-st}\right| = \left|\frac{s-t}{1-st}\frac{\mu(z)}{k}\right|,\ t,\ s \in [0, k].$$

It is not hard to prove that that whenever $t,\ s \in [0, k]$,

$$d_T([\mu_t], [\mu_s]) = d_{AT}([[\mu_t]], [[\mu_s]]) = d_H(t, s).$$

Hence, $\{[\mu_t] :\ t \in [0, k]\}$ is a geodesic connecting $[\eta_1]$ and $[\eta_2]$ in the universal Teichmüller space $T(\Delta)$ as well as $\{[[\mu_t]] :\ t \in [0, k]\}$ is a geodesic connecting $[[\eta_1]]$ and $[[\eta_2]]$ in $AT(\Delta)$. Similarly, one can construct the geodesic connecting η_2 and η_3, and so on. Thus, we construct closed geodesics in $T(\Delta)$ and $AT(\Delta)$ simultaneously. In particular, the latter is the image of the former under the natural projection π. Moreover, all points in the closed geodesic are substantial ones.

Consider the sphere centered at $[[\eta_1]]$ and with radius R in $AT(\Delta)$. By the construction, there are two geodesics connecting $[[\eta_2]]$ and $[[\eta_4]]$. One is $[[\eta_2]] \to [[\eta_1]] \to [[\eta_4]]$ which is located inside the sphere; the other is $[[\eta_2]] \to [[\eta_3]] \to [[\eta_4]]$ which is located outside the sphere. Now it is clear that the sphere is not convex. □

One can check that the geodesic joining $[\kappa]$ (defined by (6.4)) with $[0]$ in $T(\Delta)$ is not unique. However, it is not clear up to present whether the geodesic connecting the substantial point $[[\kappa]]$ and $[[0]]$ in $AT(\Delta)$ is unique.

7 Geodesics and Straight Lines in the Tangent Space

The following theorem is the counterpart of Theorem 1 in $AZ(\Delta)$.

Theorem 5 *Suppose* $[[\mu]]_{AZ}$ *is not a substantial point in* $AZ(\Delta)$*, i.e.,* $[[\mu]]_{AZ} \in AZ(\Delta)\backslash ISP$*. Then there are infinitely many geodesics connecting* $[[\mu]]_{AZ}$ *and the basepoint* $[[0]]_{AZ}$*.*

Proof Let $b = b([[\mu]]_{AZ})$. Since $[[\mu]]_{AZ}$ is not substantial point, there is a point $q \in \partial\Delta$ which is not a substantial boundary point for $[[\mu]]_{AZ}$. By Lemma 3.2, we may assume that μ is an asymptotical extremal representative in $[[\mu]]_{AZ}$ such that $b_q^*(\mu) < b$.

By the definition of boundary dilatation, we can find a small neighborhood $B(q)$ of q in Δ such that $|\mu(z)| \le \rho < b$ for some $\rho > 0$ in $B(q)$ almost everywhere. Therefore for any $\zeta \in \partial\Delta \cap \partial B(q)$, $b_\zeta^*(\mu) \le \rho$.

Choose $\delta(z) \in Bel(\Delta)$ such that $\|\delta\|_\infty \le \beta < b - \rho$ and $\delta(z) = 0$ when $z \in \Delta\backslash B(q)$.

Let Σ'' be the collection of the real-valued functions $\sigma(t)$ defined on $[0, b]$ with the following conditions:

(A) σ is continuous with $\sigma(0) = 0$ and $\sigma(b) = 0$,

(B) $|s - t|\rho/b + |\sigma(t) - \sigma(s)|\beta \le |s - t|, t,\ s \in [0, b]$.

Since $\rho < b$ and $\beta < b - \rho$, Σ'' contains uncountably many elements. In fact, if σ is a Lipschitz continuous function on $[0, b]$ with the following conditions,

(i) for some small $\alpha > 0$, $|\sigma(s) - \sigma(t)| < \alpha|s - t|, t,\ s \in [0, b]$,

(ii) $\sigma(0) = 0$ and $\sigma(b) = 0$,

(iii) $\rho/b + \alpha\beta < 1$,

then $\sigma \in \Sigma''$.

Given $\sigma \in \Sigma''$, define for $t \in [0, b]$,

$$\mu_t(z) = \begin{cases} t\mu(z)/b, & z \in \Delta\backslash B(q), \\ t\mu(z)/b + \sigma(t)\delta(z), & z \in B(q). \end{cases} \tag{7.1}$$

We show that $[[\mu_t]]_{AZ}$ $(t \in [0, b])$ is a geodesic. It is sufficient to verify that

$$\|[[\mu_s - \mu_t]]_{AZ}\| = |s - t|,\ t,\ s \in [0, b]. \tag{7.2}$$

At first, it is obvious that

$$\|\mu_s - \mu_t\|_\infty = |s - t|.$$

Suppose $p \in \partial\Delta$ is a substantial boundary point for $[[\mu]]_{AZ}$. By Lemma 2.1 there is a degenerating Hamilton sequence $\{\psi_n\} \subset Q^1(\Delta)$ towards p such that

$$b = \lim_{n\to\infty} \iint_\Delta \mu(z)\psi_n(z)dxdy.$$

Therefore, we have

$$s - t = \lim_{n\to\infty} \iint_\Delta [\mu_s(z) - \mu_t(z)]\psi_n(z)dxdy,$$

which implies the equality (7.2).

It remains to show that there are infinitely many geodesics passing through $[[\mu]]_{AZ}$ and $[[0]]_{AZ}$ when σ varies over Σ'' and $\delta(z)$ varies over $Bel(\Delta)$ suitably, respectively.

Choose $\delta(z)$ in $Bel(\Delta)$ such that (3.7) holds. Fix a small t_0 in $(0, b)$. Choose $\sigma \in \Sigma''$ such that $\sigma(t) \equiv 0$ when $t \geq t_0$ and $\sigma(t) = \alpha t$ when $t \in [0, t_0/2]$ where $\alpha > 0$ satisfying $\rho/b + \alpha\beta < 1$. Note that when $t \in [0, t_0/2]$,

$$\mu_t(z) = \begin{cases} t\mu(z)/b, & z \in \Delta\backslash B(q), \\ t\mu(z)/b + t\alpha\delta(z), & z \in B(q). \end{cases}$$

Due to the equality (3.7), the geodesics $G_\alpha = \{[[\mu_t]]_{AZ} : t \in [0, b]\}$ are mutually different when α varies in a small range.

Fixing small $\alpha > 0$ and letting δ vary suitably in $Bel(\Delta)$, we can also get infinitely many geodesics as required. □

The counterpart of Theorem 2 in the infinitesimal setting follows from an almost identical argument.

Theorem 6 *For any two points in $AZ(\Delta)$, there are infinitely many straight lines containing them.*

The following is the infinitesimal version of Theorem 3.

Theorem 7 *Suppose $b([[\mu]]_{AZ}) = b \in (0, +\infty)$ and $\{[[\mu_t]]_{AZ} : t \in (0, b)\}$ is a geodesic connecting $[[0]]_{AZ}$ and $[[\mu]]_{AZ}$ such that $d_{AZ}([[0]]_{AZ}, [[\mu_t]]_{AZ}) = t$ for $t \in (0, b)$. If $p \in \partial\Delta$ is a substantial boundary point for $[[\mu]]_{AZ}$, then p is a substantial boundary point for all $[[\mu_t]]_{AZ}$, $t \in (0, b)$.*

At last, we end the paper with the infinitesimal version of Theorem 4.

Theorem 8 *There exist closed geodesics in the tangent space $AZ(\Delta)$ and hence the spheres in $AZ(\Delta)$ are not convex.*

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