J Geom Anal (2017) 27:1274–1285
DOI 10.1007/s12220-016-9718-7

On Compactness of Hankel and the $\overline{\partial}$-Neumann Operators on Hartogs Domains in $\mathbb{C}^2$

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Received: 27 October 2015 / Published online: 23 June 2016

Abstract We prove that on smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$ compactness of the $\overline{\partial}$-Neumann operator is equivalent to compactness of all Hankel operators with symbols smooth on the closure of the domain.

Keywords Hankel operators · $\overline{\partial}$-Neumann problem · Hartogs domains

Mathematics Subject Classification Primary 32W05 · Secondary 47B35

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $L^2_{(0,q)}(\Omega)$ denote the space of square integrable $(0, q)$ forms for $0 \leq q \leq n$. The complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is a densely defined, closed, self-adjoint linear operator on $L^2_{(0,q)}(\Omega)$. Hörmander in [7] showed that when Ω is bounded and pseudoconvex, $\Box$ has a bounded solution operator N_q, called the $\overline{\partial}$-Neumann operator, for all q. Kohn in [9] showed that the Bergman projection, denoted by $\mathbf{B}$ below, is connected to the $\overline{\partial}$-Neumann operator via the following formula

$$\mathbf{B} = \mathbf{I} - \overline{\partial}^* N_1 \overline{\partial}$$

where $\mathbf{I}$ denotes the identity operator. For more information about the $\overline{\partial}$-Neumann problem we refer the reader to two books [4, 15].

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Let $A^2(\Omega)$ denote the space of square integrable holomorphic functions on Ω and $\phi \in L^\infty(\Omega)$. The Hankel operator with symbol ϕ, $H_\phi : A^2(\Omega) \to L^2(\Omega)$ is defined by

$$H_\phi g = [\phi, \mathbf{B}]g = (\mathbf{I} - \mathbf{B})(\phi g).$$

Using Kohn's formula one can immediately see that

$$H_\phi g = \overline{\partial}^* N_1(g\overline{\partial}\phi)$$

for $\phi \in C^1(\overline{\Omega})$. It is clear that H_ϕ is a bounded operator; however, its compactness depends on both the function theoretic properties of the symbol ϕ as well as the geometry of the boundary of the domain Ω (see [6]).

The following observation is relevant to our work here. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\phi \in C(\overline{\Omega})$. If $\overline{\partial}$-Neumann operator N_1 is compact on $L^2_{(0,1)}(\Omega)$ then the Hankel operator H_ϕ is compact (see [15, Proposition 4.1]).

We are interested in the converse of this observation. Namely,

Assume that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and H_ϕ is compact on $A^2(\Omega)$ for all symbols $\phi \in C(\overline{\Omega})$. Then is the $\overline{\partial}$-Neumann operator N_1 compact on $L^2_{(0,1)}(\Omega)$?

This is known as D'Angelo's question and first appeared in [12, Remark 2].

The answer to D'Angelo's question is still open in general but there are some partial results. Fu and Straube in [13] showed that the answer is yes if Ω is convex. Çelik and the first author [2, Corollary 1] observed that if Ω is not pseudoconvex then the answer to D'Angelo's question may be no. Indeed, they constructed an annulus type domain Ω where H_ϕ is compact on $A^2(\Omega)$ for all symbols $\phi \in C(\overline{\Omega})$; yet, the $\overline{\partial}$-Neumann operator N_1 is not compact on $L^2_{(0,1)}(\Omega)$.

Remark 1 One can extend the definition of Hankel operators from holomorphic functions to the $\overline{\partial}$-closed $(0, q)$-forms (denoted by $K^2_{(0,q)}(\Omega)$) and ask the analogous problem at the forms level. In this case, an affirmative answer was obtained in [3]. Namely, for $1 \le q \le n-1$ if $H^q_\phi = [\phi, \mathbf{B}_q]$ is compact on $K^2_{(0,q)}(\Omega)$ for all symbols $\phi \in C^\infty(\overline{\Omega})$ then the $\overline{\partial}$-Neumann operator N_{q+1} is compact on $L^2_{(0,q)}(\Omega)$.

In this paper, we provide an affirmative answer to D'Angelo's question on smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$.

Theorem 1 *Let Ω be a smooth bounded pseudoconvex Hartogs domain in $\mathbb{C}^2$. The $\overline{\partial}$-Neumann operator N_1 is compact on $L^2_{(0,1)}(\Omega)$ if and only if H_ψ is compact on $A^2(\Omega)$ for all $\psi \in C^\infty(\overline{\Omega})$.*

As mentioned above, compactness of N_1 implies that H_ψ is compact on any bounded pseudoconvex domain (see [12,15, Proposition 4.4.1]). The key ingredient of our proof of the converse is the characterization of the compactness of N_1 in terms of ground state energies of certain Schrödinger operators as previously explored in [5,14].

We will need a few lemmas before we prove Theorem 1.

Lemma 1 *Let $A(a,b)=\{z\in\mathbb{C}: a<|z|<b\}$ for $0<a<b<\infty$ and $d_{ab}(w)$ be the distance from w to the boundary of $A(a,b)$. Then there exists $C>0$ such that*

$$\int_{A(a,b)}(d_{ab}(w))^2|w|^{2n}dV(w)\le\frac{C}{n^2}\int_{A(a,b)}|w|^{2n}dV(w)$$

for nonzero integer n.

Proof We will use the fact that $d_{ab}(w)=\min\{b-|w|,|w|-a\}$ with polar coordinates to compute the first integral. One can compute that

$$\int_{A(a,b)}|w|^{2n}dV(w)=\frac{\pi}{n+1}\left(b^{2n+2}-a^{2n+2}\right)$$

for $n\ne -1$. Let $c=\frac{a+b}{2}$. Then

$$\begin{aligned}\int_{A(a,b)}(d_{ab}(w))^2|w|^{2n}dV(w)&=\int_{A(a,c)}(|w|-a)^2|w|^{2n}dV(w)\\&\quad+\int_{A(c,b)}(b-|w|)^2|w|^{2n}dV(w)\\&=2\pi\int_a^c\left(a^2\rho^{2n+1}-2a\rho^{2n+2}+\rho^{2n+3}\right)d\rho\\&\quad+2\pi\int_c^b\left(b^2\rho^{2n+1}-2b\rho^{2n+2}+\rho^{2n+3}\right)d\rho\\&=2\pi\left(b^{2n+4}-a^{2n+4}\right)\left(\frac{1}{2n+2}-\frac{2}{2n+3}+\frac{1}{2n+4}\right)\\&\quad+2\pi(a^2-b^2)\frac{c^{2n+2}}{2n+2}+4\pi(b-a)\frac{c^{2n+3}}{2n+3}\\&=\frac{\pi(b^{2n+4}-a^{2n+4})}{(n+1)(n+2)(2n+3)}-\frac{\pi c^{2n+2}(b^2-a^2)}{(n+1)(2n+3)}.\end{aligned}$$

In the last equality we used the fact that $c=\frac{a+b}{2}$. Then one can show that

$$\lim_{n\to\pm\infty}\frac{n^2\int_{A(a,b)}(d_{ab}(w))^2|w|^{2n}dV(w)}{\int_{A(a,b)}|w|^{2n}dV(w)}=\frac{b^2}{2}.$$

Therefore, there exists $C>0$ such that

$$\int_{A(a,b)}(d_{ab}(w))^2|w|^{2n}dV(w)\le\frac{C}{n^2}\int_{A(a,b)}|w|^{2n}dV(w)$$

for nonzero integer n. □

We note that throughout the paper $\|.\|_{-1}$ denotes the Sobolev -1 norm.

Lemma 2 *Let $\Omega = \{(z, w) \in \mathbb{C}^2 : z \in D$ and $\phi_1(z) < |w| < \phi_2(z)\}$ be a bounded Hartogs domain. Then there exists $C > 0$ such that*

$$\|g(z)w^n\|_{-1} \leq \frac{C}{n}\|g(z)w^n\|$$

for any $g \in L^2(D)$ and nonzero integer n, as long as the right-hand side is finite.

Proof We will denote the distance from (z, w) to the boundary of Ω by $d_\Omega(z, w)$. We note that $W^{-1}(\Omega)$ is the dual of $W_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $W^1(\Omega)$. Furthermore,

$$\|f\|_{-1} = \sup\{|\langle f, \phi\rangle| : \phi \in C_0^\infty(\Omega), \|\phi\|_1 \leq 1\}$$

for $f \in W^{-1}(\Omega)$. Then there exists $C_1 > 0$ such that

$$\|f\|_{-1} \leq \|d_\Omega f\| \sup\{\|\phi/d_\Omega\| : \phi \in C_0^\infty(\Omega), \|\phi\|_1 \leq 1\} \leq C_1\|d_\Omega f\|.$$

In the second inequality above we used the fact that (see [4, Proof of Theorem C.3]) there exists $C_1 > 0$ such that $\|\phi/d_\Omega\| \leq C_1\|\phi\|_1$ for all $\phi \in W_0^1(\Omega)$.

Let $d_z(w)$ denote the distance from w to the boundary of $A(\phi_1(z), \phi_2(z))$. Then there exists $C_1 > 0$ such that

$$\begin{aligned}\|g(z)w^n\|_{-1}^2 \leq& C_1 \int_\Omega (d_\Omega(z, w))^2 |g(z)|^2 |w|^{2n} dV(z, w)\\ \leq& C_1 \int_D |g(z)|^2 \int_{\phi_1(z)<|w|<\phi_2(z)} (d_z(w))^2 |w|^{2n} dV(w).\end{aligned}$$

Lemma 1 and the assumption that Ω is bounded imply that there exists $C_2 > 0$ such that

$$\int_{\phi_1(z)<|w|<\phi_2(z)} (d_z(w))^2 |w|^{2n} dV(w) \leq \frac{C_2}{n^2} \int_{\phi_1(z)<|w|<\phi_2(z)} |w|^{2n} dV(w).$$

Then

$$\begin{aligned}&\int_D |g(z)|^2 \int_{\phi_1(z)<|w|<\phi_2(z)} (d_z(w))^2 |w|^{2n} dV(w)\\ &\leq \frac{C_2}{n^2} \int_D |g(z)|^2 \int_{\phi_1(z)<|w|<\phi_2(z)} |w|^{2n} dV(w)\\ &= \frac{C_2}{n^2}\|g(z)w^n\|^2.\end{aligned}$$

Therefore, for $C = \sqrt{C_1 C_2}$ we have $\|g(z)w^n\|_{-1} \leq \frac{C}{n}\|g(z)w^n\|$ for nonzero integer n. □

Lemma 3 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\psi \in C^1(\overline{\Omega})$. Then H_ψ is compact if and only if for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|H_\psi h\|^2 \leq \varepsilon \|h\overline{\partial}\psi\| \|h\| + C_\varepsilon \|h\overline{\partial}\psi\|_{-1} \|h\| \tag{1}$$

for $h \in A^2(\Omega)$.

Proof First assume that H_ψ is compact. Then

$$\|H_\psi h\|^2 = \langle H_\psi^* H_\psi h, h\rangle \leq \|H_\psi^* H_\psi h\| \|h\|$$

for $h \in A^2(\Omega)$. Compactness of H_ψ implies that H_ψ^* is compact. Now we apply the compactness estimate in [8, Proposition V.2.3] to H_ψ^*. For $\varepsilon > 0$ there exists a compact operator K_ε such that

$$\begin{aligned}\|H_\psi^* H_\psi h\| &\leq \frac{\varepsilon}{2\|\overline{\partial}^* N\|} \|H_\psi h\| + \|K_\varepsilon H_\psi h\| \\ &\leq \frac{\varepsilon}{2} \|h\overline{\partial}\psi\| + \|K_\varepsilon H_\psi h\|.\end{aligned}$$

In the second inequality we used the fact that $H_\psi h = \overline{\partial}^* N(h\overline{\partial}\psi)$. Since Ω is bounded pseudoconvex $\overline{\partial}^* N$ is bounded and hence $K_\varepsilon \overline{\partial}^* N$ is compact. Now we use the fact that $H_\psi h = \overline{\partial}^* N(h\overline{\partial}\psi)$ and [15, Lemma 4.3] for the compact operator $K_\varepsilon \overline{\partial}^* N$ to conclude that there exists $C_\varepsilon > 0$ such that

$$\|K_\varepsilon H_\psi h\| \leq \frac{\varepsilon}{2} \|h\overline{\partial}\psi\| + C_\varepsilon \|h\overline{\partial}\psi\|_{-1}.$$

Therefore, for $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|H_\psi h\|^2 \leq \varepsilon \|h\overline{\partial}\psi\| \|h\| + C_\varepsilon \|h\overline{\partial}\psi\|_{-1} \|h\|$$

for $h \in A^2(\Omega)$.

To prove the converse assume (1) and choose $\{h_j\}$ a sequence in $A^2(\Omega)$ such that $\{h_j\}$ converges to zero weakly. Then the sequence $\{h_j\}$ is bounded and $\|h_j\overline{\partial}\psi\|_{-1}$ converges to 0 (as the imbedding from L^2 into Sobolev -1 is compact). The inequality (1) implies that there exists $C > 0$ such that for every $\varepsilon > 0$ there exists J such that $\|H_\psi h_j\|^2 \leq C\varepsilon$ for $j \geq J$. That is, $\{H_\psi h_j\}$ converges to 0. That is, H_ψ is compact. □

The following lemma is contained in [10, Remark 1]. The superscripts on the Hankel operators are used to emphasize the domains.

Lemma 4 ([10]) *Let Ω_1 be a bounded pseudoconvex domain in $\mathbb{C}^n$ and Ω_2 be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary. Assume that $U = \Omega_1 \cap \Omega_2$ is connected, $\phi \in C^1(\overline{\Omega}_1)$, and $H_\phi^{\Omega_1}$ is compact on $A^2(\Omega_1)$. Then H_ϕ^U is compact on $A^2(U)$.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1 We present the proof of the nontrivial direction. That is, we assume that H_ψ is compact on $A^2(\Omega)$ for all $\psi \in C^\infty(\overline{\Omega})$ and prove that N_1 is compact. Our proof is along the lines of the proof of [5, Theorem 1.1].

Let $\rho(z, w)$ be a smooth defining function for Ω that is invariant under rotations in w. That is, $\rho(z, w) = \rho(z, |w|)$,

$$\Omega = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) < 0\},$$

and $\nabla\rho$ is nonvanishing on $b\Omega$. Let $\Gamma_0 = \{(z, w) \in b\Omega : \rho_{|w|}(z, |w|) = 0\}$ and

$$\Gamma_k = \{(z, w) \in b\Omega : |\rho_{|w|}(z, |w|)| \geq 1/k\}$$

for $k = 1, 2, \ldots$. We will show that Γ_k is B-regular for $k = 0, 1, 2, \ldots$ by establishing the estimates (2) and (3) below and invoking [5, Lemma 10.2]. Then

$$b\Omega = \bigcup_{k=0}^{\infty} \Gamma_k$$

and [11, Proposition 1.9] implies that $b\Omega$ is B-regular (satisfies Property (P) in Catlin's terminology). This will be enough to conclude that N_1 is compact on $L^2_{(0,1)}(\Omega)$

The proof of the fact that Γ_0 is B-regular is essentially contained in [5, Lemma 10.1] together with the following fact: Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. If $H_{\overline{z}}$ and $H_{\overline{w}}$ are compact on $A^2(\Omega)$ then there is no analytic disc in $b\Omega$ (see [6, Corollary 1]).

Now we will prove that Γ_k is B-regular for any fixed $k \geq 1$. Let $(z_0, w_0) \in \Gamma_k$, we argue in two cases. The first case is when $\rho_{|w|}(z_0, |w_0|) < 0$ and the second case is $\rho_{|w|}(z_0, |w_0|) > 0$.

We continue with the first case. Assume that $b\Omega$ near (z_0, w_0) is given by $|w| = e^{-\varphi(z)}$. Let $D(z_0, r)$ denote the disc centered at z_0 with radius r and

$$U_{a,b} = D(z_0, a) \times \{w \in \mathbb{C} : |w_0| - b < |w| < |w_0| + b\}$$

for $a, b > 0$. Then let us choose $a, a_1, b, b_1 > 0$ such that $a_1 > a$, $b_1 > |w_0| + b$, the open sets

$$U = \Omega \cap U_{a,b} = \left\{(z, w) \in \mathbb{C}^2 : z \in D(z_0, a),\ e^{-\varphi(z)} < |w| < |w_0| + b\right\}$$

and $U_1 = \Omega \cap U^{a_1,b_1}$ are connected where

$$U^{a_1,b_1} = \left\{(z, w) \in \mathbb{C}^2 : \frac{|z - z_0|^2}{a_1^2} + \frac{|w|^2}{b_1^2} < 1\right\},$$

and finally $\overline{U} \subset U_1$. Then

$$U_1 = \left\{(z, w) \in \mathbb{C}^2 : z \in V_1, e^{-\varphi(z)} < |w| < e^{-\alpha(z)}\right\}$$

where V_1 is a domain in $\mathbb{C}$ such that $\overline{D(z_0, a)} \subset V_1 \subset D(z_0, a_1)$ and

$$\alpha(z) = \log a_1 - \log b_1 - \frac{1}{2}\log\left(a_1^2 - |z - z_0|^2\right).$$

One can check that α is subharmonic on $D(z_0, a_1)$, while pseudoconvexity of Ω implies that the function φ is superharmonic on $D(z_0, a_1)$. Furthermore, since B-regularity is invariant under holomorphic change of coordinates, by mapping under $(z, w) \to (z, \lambda w)$ for some $\lambda > 1$, we may assume that

$$U_1 \subset D(z_0, a_1) \times \{w \in \mathbb{C} : |w| > 1\}.$$

For any $\beta \in C_0^\infty(D(z_0, a))$ let us choose $\psi \in C^\infty(\overline{V_1})$ such that $\psi_{\overline{z}} = \beta$. Lemma 4 implies that the Hankel operator $H_\psi^{U_1}$ (we use the superscript U_1 to emphasize the domain) is compact on the Bergman space $A^2(U_1)$.

Let

$$\lambda_n(z) = -\log\left(\frac{\pi}{n-1}\left(e^{(2n-2)\varphi(z)} - e^{(2n-2)\alpha(z)}\right)\right)$$

for $n = 2, 3, \ldots$. One can check that since φ is superharmonic and α is subharmonic, the function λ_n is subharmonic. Let $S_{\lambda_n}^{V_1}$ be the canonical solution operator for $\overline{\partial}$ on $L^2(V_1, \lambda_n)$. If $f_n = H_\psi^{U_1} w^{-n}$ then we claim that

$$f_n(z, w) = g_n(z) w^{-n}$$

where $g_n = S_{\lambda_n}^{V_1}(\beta d\overline{z})$ and $n = 2, 3, \ldots$. Clearly $H_\psi^{U_1} w^{-n} = f_n \in L^2(U_1)$ and

$$\overline{\partial} g_n(z) w^{-n} = \beta(z) w^{-n} d\overline{z}.$$

To prove the claim we will just need to show that $g_n(z)w^{-n}$ is orthogonal to $A^2(U_1)$. That is, we need to show that $\langle g_n(z)w^{-n}, h(z)w^m\rangle_{U_1} = 0$ for any $h(z) \in A^2(V_1)$ and $m \in \mathbb{Z}$. Then

$$\begin{aligned}\langle g_n(z)w^{-n}, h(z)w^m\rangle_{U_1} &= \int_{U_1} g_n(z)w^{-n}\overline{h(z)w^m}\,dV(z)dV(w)\\ &= \int_{V_1} g_n(z)\overline{h(z)}\,dV(z)\int_{e^{-\varphi(z)}<|w|<e^{-\alpha(z)}} w^{-n}\overline{w^m}\,dV(w).\end{aligned}$$

Unless $m=-n$ the integral $\int_{e^{-\varphi(z)}<|w|<e^{-\alpha(z)}} w^{-n}\overline{w^m}dV(w)=0$. So let us assume that $m=-n$. In that case we get

$$\int_{V_1} g_n(z)\overline{h(z)}dV(z)\int_{e^{-\varphi(z)}<|w|<e^{-\alpha(z)}} w^{-n}\overline{w^m}dV(w)=\int_{V_1} g_n(z)\overline{h(z)}e^{-\lambda_n(z)}dV(z).$$

The integral on the right-hand side above is zero because g_n is orthogonal to $A^2(V_1,\lambda_n)$. Therefore,

$$g_n(z)w^{-n}=H_{\psi}^{U_1}w^{-n}.$$

The equality above implies that $\frac{\partial g_n}{\partial \overline{z}}=\frac{\partial \psi}{\partial \overline{z}}=\beta$. Then the compactness estimate (1) implies that

$$\begin{aligned}\int_{D(z_0,a)}|g_n(z)|^2e^{-\lambda_n(z)}dV(z) &\le \|g_n(z)w^{-n}\|_{U_1}^2\\ &\le \varepsilon\|\beta(z)w^{-n}\|_{U_1}\|w^{-n}\|_{U_1}\\ &\quad+C_\varepsilon\|\beta(z)w^{-n}\|_{W^{-1}(U_1)}\|w^{-n}\|_{U_1}\\ &=\varepsilon\left(\int_{D(z_0,a)}|\beta(z)|^2e^{-\lambda_n(z)}dV(z)\right)^{1/2}\\ &\quad\times\left(\int_{V_1}e^{-\lambda_n(z)}dV(z)\right)^{1/2}\\ &\quad+C_\varepsilon\|\beta(z)w^{-n}\|_{W^{-1}(U_1)}\left(\int_{V_1}e^{-\lambda_n(z)}\right)^{1/2}.\end{aligned}$$

Then by Lemma 2 there exists $C>0$ such that

$$\|\beta(z)w^{-n}\|_{W^{-1}(U_1)}\le\frac{C}{n}\|\beta(z)w^{-n}\|_{U_1}=\frac{C}{n}\|\beta\|_{L^2(D(z_0,a),\lambda_n)}.$$

We note that to get the equality above we used the fact that β is supported in $D(z_0,a)$. Hence we get

$$\|g_n\|^2_{L^2(D(z_0,a),\lambda_n)}\le\left(\varepsilon+\frac{CC_\varepsilon}{n}\right)\|\beta\|_{L^2(D(z_0,a),\lambda_n)}\|1\|_{L^2(V_1,\lambda_n)}.$$

For any $\varepsilon>0$ there exists an integer n_ε such that

$$\frac{CC_\varepsilon}{n}+\frac{\pi a_1}{\sqrt{n-1}}\le\varepsilon$$

for $n \geq n_\varepsilon$. Then

$$\|g_n\|^2_{L^2(D(z_0,a),\lambda_n)} \leq 2\varepsilon\|\beta\|_{L^2(D(z_0,a),\lambda_n)}\|1\|_{L^2(V_1,\lambda_n)} \leq 2\varepsilon^2\|\beta\|_{L^2(D(z_0,a),\lambda_n)}$$

for $n \geq n_\varepsilon$ because $U \subset D(z_0, a) \times \{w \in \mathbb{C} : |w| > 1\}$ and

$$\begin{aligned}\|1\|_{L^2(V_1,\lambda_n)} &\leq \|1\|_{L^2(D(z_0,a_1),\lambda_n)} \\ &= \left(\int_{D(z_0,a_1)} \frac{\pi}{n-1}\left(e^{(2n-2)\varphi(z)} - e^{(2n-2)\alpha(z)}\right) dV(z)\right)^{1/2} \\ &\leq \left(\int_{D(z_0,a_1)} \frac{\pi}{n-1} dV(z)\right)^{1/2} \\ &= \frac{\pi a_1}{\sqrt{n-1}}.\end{aligned}$$

Let $u \in C_0^\infty(D(z_0, a))$ and $n \geq n_\varepsilon$. Then

$$\begin{aligned}&\int_{D(z_0,a)} |u(z)|^2 e^{\lambda_n(z)} dV(z) \\ &\quad = \sup\left\{|\langle u, \beta\rangle_{D(z_0,a)}|^2 : \beta \in C_0^\infty(D(z_0, a)),\ \|\beta\|^2_{L^2(D(z_0,a),\lambda_n)} \leq 1\right\} \\ &\quad \leq \sup\left\{|\langle u, (g_n)_{\overline{z}}\rangle_{D(z_0,a)}|^2 : \|g_n\|^2_{L^2(D(z_0,a),\lambda_n)} \leq 2\varepsilon^2\right\} \\ &\quad = \sup\left\{|\langle u_z, g_n\rangle_{D(z_0,a)}|^2 : \|g_n\|^2_{L^2(D(z_0,a),\lambda_n)} \leq 2\varepsilon^2\right\} \\ &\quad \leq 2\varepsilon^2 \int_{D(z_0,a)} |u_z(z)|^2 e^{\lambda_n(z)} dV(z).\end{aligned}$$

There exists $0 < c < 1$ such that $e^{-\varphi(z)} < ce^{-\alpha(z)}$ for $z \in D(z_0, a)$. Then

$$\frac{\pi}{n-1} e^{(2n-2)\varphi(z)}\left(1 - c^{2n-2}\right) < e^{-\lambda_n(z)} < \frac{\pi}{n-1} e^{(2n-2)\varphi(z)}.$$

So for large n we have

$$\frac{\pi}{2(n-1)} e^{(2n-2)\varphi(z)} < e^{-\lambda_n(z)} < \frac{\pi}{n-1} e^{(2n-2)\varphi(z)}$$

and

$$\begin{aligned}&\frac{n-1}{\pi}\int_{D(z_0,a)} |u(z)|^2 e^{(2-2n)\varphi(z)} dV(z) \\ &\quad < \int_{D(z_0,a)} |u(z)|^2 e^{\lambda_n(z)} dV(z) \\ &\quad \leq 2\varepsilon^2 \int_{D(z_0,a)} |u_z(z)|^2 e^{\lambda_n(z)} dV(z)\end{aligned}$$

$$\leq \frac{4\varepsilon^2(n-1)}{\pi}\int_{D(z_0,a)}|u_z(z)|^2e^{(2-2n)\varphi(z)}dV(z).$$

That is, for any $\varepsilon > 0$ and $u \in C_0^\infty(D(z_0, a))$

$$\int_{D(z_0,a)}|u(z)|^2e^{(2-2n)\varphi(z)}dV(z) \leq 4\varepsilon^2\int_{D(z_0,a)}|u_z(z)|^2e^{(2-2n)\varphi(z)}dV(z) \qquad (2)$$

for large n.

The estimate in (2) is identical to the one in [5, p. 38, proof of Lemma 10.2]. That is $\lambda^m_{n\varphi}(D(z_0, a)) \to \infty$ as $n \to \infty$ (see [5, Definition 2.3]). Since φ is smooth and subharmonic, [5, Theorem 1.5] implies that $\lambda^e_{n\varphi}(D(z_0, a)) \to \infty$ as $n \to \infty$. We note that [5, Theorem 1.5] implies that if $\lambda^m_{n\varphi}(D(z_0, a)) \to \infty$ as $n \to \infty$ then $\lambda^e_{n\varphi}(D(z_0, a)) \to \infty$ as $n \to \infty$. This is enough to conclude that Γ_k is B-regular. This argument is contained in the proof of Proposition 9.1 converse of (1) in [5, p. 33]. We repeat the argument here for the convenience of the reader.

Let $V = \{z \in D(z_0, a) : \Delta\varphi(z) > 0\}$ and $K_0 = \overline{D(z_0, a/2)} \setminus V$. Then V is open and K_0 is a compact subset of $D(z_0, a)$. Furthermore, $\Delta\varphi = 0$ on K_0. If K_0 has non-trivial fine interior then it supports a nonzero function $f \in W^1(\mathbb{C})$ (see [15, Proposition 4.17]). Then

$$\lambda^e_{n\varphi}(D(z_0, a)) \leq \frac{\|\nabla f\|^2}{\|f\|^2} < \infty \quad \text{for all} \quad n.$$

Which is a contradiction. Hence K_0 has empty fine interior which implies that K_0 satisfies property (P) (see [15, Proposition 4.17] or [11, Proposition 1.11]). Therefore, for $M > 0$ there exists an open neighborhood O_M of K_0 and $b_M \in C_0^\infty(O_M)$ such that $|b_M| \leq 1/2$ on O_M and $\Delta b_M > M$ on K_0. Furthermore, using the assumption that $|w| > 0$ on Γ_k one can choose M_1 such that the function $g_{M_1}(z, w) = M_1(|w|^2e^{\varphi(z)} - 1) + b_M(z)$ has the following properties: $|g_{M_1}| \leq 1$ and the complex Hessian $H_{g_{M_1}}(W) \geq M\|W\|^2$ on $\Gamma_k \cap \overline{D(z_0, a)}$ where W is complex tangential direction. Then [1, Proposition 3.1.7] implies that $\Gamma_k \cap \overline{D(z_0, a/2)}$ satisfies property (P) (hence it is B-regular). Therefore, [15, Corollary 4.13] implies that Γ_k is B-regular.

The computations in the second case (that is $\rho_{|w|}(z_0, |w_0|) > 0$) are very similar. So we will just highlight the differences between the two cases. We define

$$U^{a_1,b_1} = \left\{(z, w) \in \mathbb{C}^2 : |w| > b_1|z - z_0|^2 + a_1\right\}$$

and

$$U_1 = \Omega \cap U^{a_1,b_1} = \left\{(z, w) \in \mathbb{C}^2 : z \in V_1, e^{-\alpha(z)} < |w| < e^{-\varphi(z)}\right\}$$

where V_1 is a domain in $\mathbb{C}$ and where $\alpha(z) = -\log(b_1|z - z_0|^2 + a_1)$ is a strictly superharmonic function. One can show that bU^{a_1,b_1} is strongly pseudoconvex. We choose $a, a_1, b, b_1 > 0$ such that such that $\overline{D(z_0, a)} \subset V_1$ and U is given by

$$U = \Omega \cap U_{a,b} = \left\{(z, w) \in \mathbb{C}^2 : z \in D(z_0, a), e^{-\alpha(z)} < |w| < e^{-\varphi(z)}\right\}$$

where $U_{a,b} = D(z_0, a) \times \{w \in \mathbb{C} : |w_0| - b < |w| < |w_0| + b\}$. Furthermore, we define

$$\lambda_n(z) = -\log\left(\frac{\pi}{n+1}\left(e^{-(2n+2)\varphi(z)} - e^{-(2n+2)\alpha(z)}\right)\right)$$

for $n = 0, 1, 2, \ldots$ and by scaling U_1 in w variable if necessary, we will assume that $U_1 \subset D(z_0, a_1) \times \{w \in \mathbb{C} : |w| < 1\}$ so that $\|1\|_{L^2(D(z_0,a_1),\lambda_n)}$ goes to zero as $n \to \infty$. One can check that λ_n is subharmonic for all n.

We take functions $\beta \in C_0^\infty(D(z_0, a))$ and consider symbols $\psi \in C^\infty(\overline{V_1})$ such that $\psi_{\overline{z}} = \beta$. Then we consider the functions $H_\psi w^n$ for $n = 0, 1, 2, \ldots$. Calculations similar to the ones in the previous case reveal that $g_n(z)w^n = H_\psi w^n$ where $g_n = S_{\lambda_n}^{V_1}(\beta d\overline{z})$. Using similar manipulations and again the compactness estimate (1) we conclude that for any $\varepsilon > 0$ there exists an integer n_ε such that for $u \in C_0^\infty(D(z_0, a))$ and $n \geq n_\varepsilon$ we have

$$\int_{D(z_0,a)} |u(z)|^2 e^{(2n+2)\varphi(z)} dV(z) \leq \varepsilon \int_{D(z_0,a)} |u_z(z)|^2 e^{(2n+2)\varphi(z)} dV(z). \qquad (3)$$

Finally, an argument similar to the one right after (2) implies that Γ_k is B-regular. □

Acknowledgements We would like to thank the anonymous referee for constructive comments. The work of the second author was partially supported by a grant from the Simons Foundation (#353525), and also by a University of Michigan–Dearborn CASL Faculty Summer Research Grant. Both authors would like to thank the American Institute of Mathematics for hosting a workshop during which this project was started.

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