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On a Classification of 4-d Gradient Ricci Solitons with Harmonic Weyl Curvature

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Abstract We study a characterization of 4-dimensional (not necessarily complete) gradient Ricci solitons (M, g, f) which have harmonic Weyl curvature, i.e., $\delta W = 0$. Roughly speaking, we prove that the soliton metric g is locally isometric to one of the following four types: an Einstein metric, the product $\mathbb{R}^2 \times N_\lambda$ of the Euclidean metric and a 2-d Riemannian manifold of constant curvature $\lambda \neq 0$, a certain singular metric and a locally conformally flat metric. The method here is motivated by Cao–Chen's works (in Trans Am Math Soc 364:2377–2391, 2012; Duke Math J 162:1003–1204, 2013) and Derdziński's study on Codazzi tensors (in Math Z 172:273–280, 1980). Combined with the previous results on locally conformally flat solitons, our characterization yields a new classification of 4-d complete steady solitons with $\delta W = 0$. For the shrinking case, it re-proves the rigidity result (Fernández-López and García-Río in Math Z 269:461–466, 2011; Munteanu and Sesum in J. Geom Anal 23:539–561, 2013) in 4-d. It also helps to understand the expanding case; we now understand all 4-d non-conformally flat ones with $\delta W = 0$. We also characterize *locally* 4-d (not necessarily complete) gradient Ricci solitons with harmonic curvature.

Keywords Gradient Ricci soliton · Harmonic Weyl tensor · Codazzi tensor

Mathematics Subject Classification 53C21 · 53C25

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1 Introduction

A gradient Ricci soliton consists of a Riemannian manifold (M, g) and a smooth function f satisfying $\nabla df = -Rc + \lambda g$, where Rc denotes the Ricci tensor of g and λ is a constant. Gradient Ricci solitons are essential in Hamilton's Ricci flow theory as singularity models of the flow. So it is important to understand their geometry and classify them. A gradient Ricci soliton is said to be shrinking, steady or expanding if λ is positive, zero or negative, respectively.

Two-dimensional gradient Ricci solitons are well understood; see [2] and references therein. Any 3-d complete noncompact non-flat shrinker (shrinking Ricci soliton) is proved to be a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$ in [8]; see also [22,25,26]. For the 3-d gradient steadiers (steady Ricci solitons), one may refer to [3,4] and references therein.

In higher dimension, there are numerous rigidity and classification results under various geometric conditions. For the relevance to the current work, we shall focus on locally conformally flat solitons and their generalizations.

Complete locally conformally flat gradient shrinkers are classified to be a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{S}^{n-1} \times \mathbb{R}$, $n \geq 4$, in [9,29,32]; see also [18,25]. Complete locally conformally flat gradient steadiers are classified to be either flat or isometric to the Bryant soliton [6,10]. The 4-d half conformally flat steadiers and shrinkers are studied in [14]. More generally, Bach-flat shrinkers are classified in [7] and Bach-flat steadiers with positive Ricci curvature in [5].

A gradient soliton is said to be rigid if it is isometric to a quotient of $N \times \mathbb{R}^k$ where N is an Einstein manifold and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor. Fernández-López and García-Río [19] showed that an n-dimensional compact Ricci soliton (M, g) is rigid if and only if it has harmonic Weyl tensor W. Then Munteanu and Sesum [24] proved that any n-dimensional complete gradient shrinker with harmonic Weyl tensor is rigid. In [31], Wu, Wu and Wylie showed that a 4-d complete gradient shrinker with $\delta W^+ = 0$ is either Einstein, or a finite quotient of $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$.

The purpose of this article is to study 4-dimensional gradient Ricci solitons (M, g, f) which have harmonic Weyl curvature. This work is most related to the above-mentioned works on locally conformally flat solitons and to [24] on shrinking solitons with $\delta W = 0$. The latter needs control on geometric decay of curvature and volume from the shrinker condition, while the former resorts to the nonnegative curvedness of metrics for locally conformally flat shrinking or steady solitons, which is proved in [12,32].

As our study includes steady and expanding solitons with $\delta W = 0$, we can use neither geometric decay nor nonnegative curvedness. This work takes a different approach and is inspired by Cao and Chen's works [6,7] and Derdziński's [17]. Note that the harmonicity of the Weyl tensor provides a Codazzi tensor $Rc - \frac{R}{6}g$. Riemannian metrics with a Codazzi tensor which have more than two distinct eigenvalue functions of Ricci tensor have been little understood; see Chap. 16 of [1]. In this article, combining with the soliton condition, we managed to analyze in detail the Codazzi tensor with three and four distinct eigenvalues.

Our argument is mostly local and produces a *local* description of soliton metrics and potential functions. So far, we worked out only in four dimensions, but we hope that our perspective might provide some way to understand the higher-dimensional case.

The main theorem of this paper is as follows.

Theorem 1.1 *Any four-dimensional (not necessarily complete) connected gradient Ricci soliton (M, g, f) with harmonic Weyl curvature is one of the following four types.*

(i) *g is an Einstein metric with f a constant function.*
(ii) *For each point $p \in M$, there exists a neighborhood V of p such that (V, g) is isometric to a domain in the product $\mathbb{R}^2 \times N_\lambda$ where $\mathbb{R}^2$ has the Euclidean metric and N_λ is a 2-dimensional Riemannian manifold of constant curvature $\lambda \neq 0$. And $f = \frac{\lambda}{2}|x|^2$ modulo a constant on the Euclidean factor.*
(iii) *For each point $p \in M$, there exists a neighborhood V of p with coordinates (s, t, x_3, x_4) such that (V, g) is isometric to a domain in $\mathbb{R}^4 \backslash \{s = 0\}$ with the Riemannian metric $ds^2 + s^{\frac{2}{3}} dt^2 + s^{\frac{4}{3}} \tilde{g}$, where $\tilde{g}$ is the Euclidean metric on the (x_3, x_4)-plane. Also, $\lambda = 0$ and $f = \frac{2}{3} \ln(s)$ modulo a constant.*
(iv) *For each point p in an open dense subset of M, there exists a neighborhood V of p with coordinates (s, t, x_3, x_4) such that (V, g) is isometric to a domain in $\mathbb{R} \times W^3$ with the warped product metric $ds^2 + h(s)^2 \tilde{g}$, where $\tilde{g}$ is a constant curvature metric on a 3-manifold W^3 and f is not constant. And g is locally conformally flat.*

For the 4-d *complete* shrinking soliton case, we re-prove the rigidity result in [19,24] by a distinct method. For the 4-d complete steady case, with the result of [6,10] on locally conformally flat solitons, we obtain the following classification.

Theorem 1.2 *A 4-dimensional complete steady gradient Ricci soliton with $\delta W = 0$ is either Ricci flat, or isometric to the Bryant soliton.*

The expanding solitons are much less rigid, and many works have been done recently, e.g., [13,15,28,30] and references therein. We prove:

Theorem 1.3 *A 4-dimensional complete expanding gradient Ricci soliton with harmonic Weyl curvature is one of the following:*

(i) *g is an Einstein metric with f a constant function.*
(ii) *g is isometric to a finite quotient of $\mathbb{R}^2 \times N_\lambda$ where $\mathbb{R}^2$ has the Euclidean metric and N_λ is a 2-dimensional Riemannian manifold of constant curvature $\lambda < 0$. And $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor.*
(iii) *g is locally conformally flat.*

In [28] Petersen and Wylie proved that any complete gradient Ricci soliton with harmonic curvature is rigid. But it is not clear if their argument extends to work for a local soliton. The classification of any (not necessarily complete) gradient Ricci soliton with harmonic curvature comes from Theorem 1.1; we demonstrated it as Corollary 8.3 in the final section.

To prove Theorem 1.1, from the harmonic Weyl curvature condition on gradient Ricci solitons, we observe by the arguments of [7,19] that $\frac{\nabla f}{|\nabla f|}$ is a Ricci-eigenvector field with its eigenvalue λ_1, there is a local function s with $\nabla s = \frac{\nabla f}{|\nabla f|}$, and λ_1 and R are functions of s only. Next we obtain important geometric information on (Ricci-)eigenvalues, eigenvectors and eigenspaces from the Codazzi tensor $Rc - \frac{R}{6}g$ through Derdziński's Lemma 2.4 and its extension Lemma 2.8.

Based on all the above, we show in Lemma 2.7 that the Ricci-eigenvalues λ_i, $i = 1, \ldots, 4$, locally depend only on the variable s; this key lemma is crucial in the later argument. Then we divide the proof of Theorem 1.1 into several cases, depending on the distinctiveness of $\lambda_2, \lambda_3, \lambda_4$. There arise two subtle cases: when these three are pairwise distinct and when exactly two of them are equal. In the latter case we reduce the analysis to ordinary differential equations in Lemma 6.1 and resolve them to get the types (ii) and (iii). In the former we compute on the soliton equation using Codazzi tensor property, which eliminates the case, in Proposition 3.4.

The last case $\lambda_2 = \lambda_3 = \lambda_4$ is relatively simpler and produces the types (i) and (iv). Theorem 1.2, 1.3 and Corollary 8.3 on the harmonic curvature case can be easily deduced from Theorem 1.1.

This paper is organized as follows. In Sect. 2 we develop properties common to any gradient Ricci solitons with harmonic Weyl curvature and nonconstant f; in particular we prove that λ_i's, $i = 1, \ldots, 4$, depend only on s. In Section 3 we study the case where the three λ_i's, $i = 2, 3, 4$, are pairwise distinct. In Sections 4, 5 and 6, we analyze the case where two of the three λ_i's, $i = 2, 3, 4$, are equal. In Sect. 7, we treat the remaining case where $\lambda_2 = \lambda_3 = \lambda_4$. In the final Sect. 8, we summarize and prove theorems.

2 Gradient Ricci Solitons with Harmonic Weyl Curvature

We shall begin by recalling some properties of a gradient Ricci soliton with harmonic Weyl curvature in a few lemmas.

Lemma 2.1 *For any gradient Ricci soliton (M, g, f), we have:*

(i) $\frac{1}{2}dR = R(\nabla f, \cdot)$, *where R in the left-hand side denotes the scalar curvature, and $R(\cdot, \cdot)$ is a Ricci tensor.*
(ii) $R + |\nabla f|^2 - 2\lambda f = constant$.

Our notational convention is as follows: for orthonormal vector fields E_i, $i = 1, \ldots, n$, on an n-dimensional Riemannian manifold, the curvature components are

$$R_{ijkl} := R(E_i, E_j, E_k, E_l) = \langle \nabla_{E_i}\nabla_{E_j}E_k - \nabla_{E_j}\nabla_{E_i}E_k - \nabla_{[E_i,E_j]}E_k, E_l\rangle.$$

We recall the formula (2.1) in [19]:

Lemma 2.2 *For a gradient Ricci soliton (M^n, g, f) with harmonic Weyl curvature on an n-dimensional manifold M^n, we have:*

$$\begin{aligned} R(X, Y, Z, \nabla f) &= \frac{1}{n-1}R(X, \nabla f)g(Y, Z) - \frac{1}{n-1}R(Y, \nabla f)g(X, Z) \\ &= \frac{1}{2(n-1)}dR(X)g(Y, Z) - \frac{1}{2(n-1)}dR(Y)g(X, Z). \end{aligned}$$

One may mimic arguments in [7] and get the next lemma.

Lemma 2.3 *Let (M^n, g, f) be a gradient Ricci soliton with harmonic Weyl curvature. Let c be a regular value of f and $\Sigma_c = \{x | f(x) = c\}$ be the level surface of f. Then the following hold:*

(i) *Where $\nabla f \neq 0$, $E_1 := \frac{\nabla f}{|\nabla f|}$ is an eigenvector field of Rc.*
(ii) *R and $|\nabla f|^2$ are constant on a connected component of Σ_c.*
(iii) *There is a function s locally defined with $s(x) = \int \frac{df}{|\nabla f|}$, so that $ds = \frac{df}{|\nabla f|}$ and $E_1 = \nabla s$.*
(iv) *$R(E_1, E_1)$ is constant on a connected component of Σ_c.*
(v) *Near a point in Σ_c, the metric g can be written as $g = ds^2 + \sum_{i,j>1} g_{ij}(s, x_2, \ldots, x_n) dx_i \otimes dx_j$, where $x_2, \ldots, x_n$ is a local coordinate system on Σ_c.*
(vi) *$\nabla_{E_1} E_1 = 0$.*

Proof Lemma 2.2 gives $R(\nabla f, X) = 0$ for $X \perp \nabla f$, hence $E_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector of Rc.

As $dR = 2R(\nabla f, \cdot)$ from Lemma 2.1, $dR(X) = 0$ for $X \perp \nabla f$. Also, $\frac{1}{2}\nabla_X |\nabla f|^2 = -R(\nabla f, X) + \lambda g(\nabla f, X) = 0$ for $X \perp \nabla f$. We proved (ii). $d(\frac{df}{|\nabla f|}) = -\frac{1}{2|\nabla f|^{\frac{3}{2}}} d|\nabla f|^2 \wedge df = 0$ as $\nabla_X(|\nabla f|^2) = 0$ for $X \perp \nabla f$. So, (iii) is proved.

Locally, R may be considered as a function of the local variable s only. We can express $dR(E_1) = \frac{dR}{ds} ds(E_1) = \frac{dR}{ds} g(\nabla s, \nabla s) = \frac{dR}{ds}$. By Lemma 2.1, we have $dR(E_1) = 2R(E_1, E_1)|\nabla f|$, so $R(E_1, E_1)$ is constant on a connected component of Σ_c.

As ∇f and the level surfaces of f are perpendicular, one gets (v).

For (vi), one follows the proof of Proposition 5.1 in [7]; with the local coordinates $s, x_2, \ldots, x_n$ in (v), one readily gets $\nabla s = \frac{\partial}{\partial s}$ so that $[\frac{\partial}{\partial x_i}, \nabla s] = 0$. Then $\langle \nabla s, \nabla s \rangle = 1$ and $\langle \frac{\partial}{\partial x_i}, \nabla s \rangle = 0$ yield (vi). □

A Codazzi tensor on a Riemannian manifold M is a symmetric tensor A of covariant order 2 such that $d^\nabla A = 0$, which can be written in local coordinates as $\nabla_k A_{ij} = \nabla_i A_{kj}$. Derdziński [17] described the following: for a Codazzi tensor A and a point x in M, let $E_A(x)$ be the number of distinct eigenvalues of A_x, and set $M_A = \{x \in M \mid E_A$ is constant in a neighborhood of $x\}$, so that M_A is an open dense subset of M and that in each connected component of M_A, the eigenvalues are well-defined and differentiable functions. The next lemma is from Sect. 2 of [17].

Lemma 2.4 *For a Codazzi tensor A on a Riemannian manifold M, in each connected component of M_A,*

(i) *Given distinct eigenfunctions λ, μ of A and local vector fields v, u such that $Av = \lambda v$, $Au = \mu u$ with $|u| = 1$, it holds that $v(\mu) = (\mu - \lambda)\langle \nabla_u u, v \rangle$.*
(ii) *For each eigenfunction λ, the λ-eigenspace distribution is integrable, and its leaves are totally umbilic submanifolds of M.*
(iii) *Eigenspaces of A form mutually orthogonal differentiable distributions.*

When a Riemannian manifold M of dimension $n \geq 4$ has harmonic Weyl curvature, i.e., $\delta W = 0$, it is equivalent to $d^{\nabla}(Rc - \frac{R}{2n-2}g) = 0$. So, $\mathcal{A} := Rc - \frac{R}{2n-2}g$ is a Codazzi tensor. By Lemma 2.4, each eigenspace distribution of $\mathcal{A}$ is integrable in the open dense subset $M_{\mathcal{A}}$ of M. The leaves are totally umbilic submanifolds of M. Let $D_1, \ldots, D_k$ be all the eigenspace distributions of $\mathcal{A}$ in a connected component of $M_{\mathcal{A}}$. Then, the Ricci tensor also has $D_1, \ldots, D_k$ as its eigenspace distributions. Let the dimension of D_l be d_l for $l = 1, \ldots, k$. Then in a neighborhood of each point of the connected component of $M_{\mathcal{A}}$, there exist orthonormal Ricci-eigenvector fields E_i, $i = 1, \ldots, n$, with corresponding eigenfunctions λ_i such that $E_1, \ldots, E_{d_1} \in D_1$, $E_{d_1+1}, \ldots, E_{d_1+d_2} \in D_2, \ldots,$ and $E_{d_1+\cdots+d_{k-1}+1}, \ldots, E_n \in D_k$.

Let (M^n, g, f) be a gradient Ricci soliton with harmonic Weyl curvature. As a gradient Ricci soliton, (M, g, f) is real analytic in harmonic coordinates; see [21] or argue as in [20, Prop. 2.4]. Then if f is not constant, $\{\nabla f \neq 0\}$ is open and dense in M. As in the above paragraph, we consider orthonormal Ricci-eigenvector fields E_i in a neighborhood of each point in $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$. By just requiring $E_1 = \frac{\nabla f}{|\nabla f|}$ to be in D_1 and using Lemma 2.3, we obtain:

Lemma 2.5 *Let (M^n, g, f) be an n-dimensional gradient Ricci soliton with harmonic Weyl curvature and non-constant f. For any point p in the open dense subset $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ of M^n, there is a neighborhood U of p where there exist orthonormal Ricci-eigenvector fields E_i, $i = 1, \ldots, n$ such that for all the eigenspace distributions $D_1, \ldots, D_k$ of $\mathcal{A}$ in U,*

(i) *$E_1 = \frac{\nabla f}{|\nabla f|}$ is in D_1,*
(ii) *for $i > 1$, E_i is tangent to smooth level hypersurfaces of f,*
(iii) *let d_l be the dimension of D_l for $l = 1, \ldots, k$, then $E_1, \ldots, E_{d_1} \in D_1$, $E_{d_1+1}, \ldots, E_{d_1+d_2} \in D_2, \ldots,$ and $E_{d_1+\cdots+d_{k-1}+1}, \ldots, E_n \in D_k$.*

These local orthonormal Ricci-eigenvector fields E_i of Lemma 2.5 shall be called an *adapted frame field* of (M, g, f).

For an adapted frame field E_i, $i = 1, \ldots, n$, with $R_{ij} := R(E_i, E_j) = \lambda_i \delta_{ij}$, from Lemma 2.2, for $j \in \{2, \ldots, n\}$ we get

$$R(E_1, E_j, E_j, \nabla f) = \frac{1}{n-1} Ric(E_1, \nabla f) = \frac{1}{2(n-1)} dR(E_1). \tag{1}$$

Due to Lemma 2.3, in a neighborhood of a point $p \in M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, f and R may be considered as functions of the variable s only, and we write the derivative in s by a prime: $f' = \frac{df}{ds}$ and $R' = \frac{dR}{ds}$, etc. We recall $dR(E_1) = R' ds(E_1) = R' g(\nabla s, \nabla s) = R'$ and similarly $df(E_1) = f'$. Also, $df(E_1) = g(\nabla f, \frac{\nabla f}{|\nabla f|}) = |\nabla f|$. So, $|\nabla f| = f'$. Then (1) becomes:

$$R_{1jj1} |\nabla f| = \frac{1}{n-1} R_{11} |\nabla f| = \frac{1}{2(n-1)} R'. \tag{2}$$

Lemma 2.6 *For a gradient Ricci soliton (M, g, f) with harmonic Weyl curvature, and for a local adapted frame field $\{E_i\}$ in $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, setting $\zeta_i = -\langle \nabla_{E_i} E_i, E_1 \rangle$, for $i > 1$, we have:*

$$\nabla_{E_1} E_1 = 0, \quad \text{and} \quad \nabla_{E_i} E_1 = \frac{1}{|\nabla f|}(\lambda - \lambda_i) E_i. \tag{3}$$

$$\zeta_i = \frac{1}{|\nabla f|}(\lambda - \lambda_i). \tag{4}$$

Proof From Lemma 2.3 we get $\nabla_{E_1} E_1 = 0$. From the gradient Ricci soliton equation, for $i > 1$, $\nabla_{E_i} E_1 = \nabla_{E_i}(\frac{\nabla f}{|\nabla f|}) = \frac{\nabla_{E_i} \nabla f}{|\nabla f|} = \frac{-R(E_i,\cdot)+\lambda g(E_i,\cdot)}{|\nabla f|} = -\frac{1}{|\nabla f|}(\lambda_i - \lambda) E_i$. Then, $\zeta_i = -\langle \nabla_{E_i} E_i, E_1 \rangle = \langle E_i, \nabla_{E_i} E_1 \rangle = \frac{1}{|\nabla f|}(\lambda - \lambda_i)$. □

Lemma 2.7 *For a 4-dimensional gradient Ricci soliton (M, g, f) with harmonic Weyl curvature, and for a local adapted frame field $\{E_i\}$ in $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, the Ricci-eigenfunctions λ_i, $i = 1, \ldots, 4$, are constant on a connected component of a regular level hypersurface Σ_c of f, and so depend on the local variable s only. And ζ_i, $i = 2, 3, 4$, in Lemma 2.6 also depend on s only. In particular, we have $E_i(\lambda_j) = E_i(\zeta_k) = 0$ for $i, k > 1$ and any j.*

Proof We write $R_{ij} := R(E_i, E_j)$. Recall that $\lambda_i = R_{ii}$. We set $Rc^1 = Rc$ and for $k \geq 2$, $Rc^k_{ij} = \sum_{s_1,s_2,\ldots,s_{k-1}=1}^{4} R_{is_1} R_{s_1 s_2} \cdots R_{s_{k-1} j}$ with its trace $\mathrm{tr}(Rc^k) = \sum_{i=1}^{4} (\lambda_i)^k$. We will show $\mathrm{tr}(Rc^k)$, $k = 1, 2, 3$, depend on s only.

First, $R = \mathrm{tr}(Rc^1)$ and $\lambda_1 = R_{11}$ depend on s only by Lemma 2.3. Next, for $k \geq 1$, writing the Hessian $\nabla_j \nabla_i R := \nabla_{E_j} \nabla_{E_i} R$, by Lemma 2.6 we compute the following:

$$\begin{aligned}
&\sum_{j,s_1,s_2,\ldots,s_{k-1}=1}^{4} (\nabla_j \nabla_{s_1} R) R_{s_1 s_2} \cdots R_{s_{k-1} j} \\
&= \sum_{j=1}^{4} (\nabla_j \nabla_j R)(R_{jj})^{k-1} \\
&= (\nabla_1 \nabla_1 R)\lambda_1^{k-1} + \sum_{i>1} (\nabla_i \nabla_i R)\lambda_i^{k-1} \\
&= (R'')\lambda_1^{k-1} + \sum_{i>1} \{E_i E_i(R) - (\nabla_{E_i} E_i) R\}\lambda_i^{k-1} \\
&= (R'')\lambda_1^{k-1} - \sum_{i>1} \frac{R'}{|\nabla f|}\left(\lambda_i^k - \lambda \cdot \lambda_i^{k-1}\right).
\end{aligned} \tag{5}$$

In particular, for $k = 1$, (5) shows that

$$\sum_{j=1}^{4} \nabla_j \nabla_j R = R'' - \sum_{i>1} \frac{R'}{|\nabla f|}(\lambda_i - \lambda) = R'' - \frac{R'}{|\nabla f|}(R - \lambda_1 - 3\lambda),$$

which depends only on s. We drop summation symbols using the Einstein summation convention below.

$$\sum_{j=1}^{4} \frac{1}{2}\nabla_j \nabla_j R = \nabla_j (f_i R_{ij}) = f_{ij} R_{ij} + f_i \nabla_j R_{ij} = -(R_{ij} - \lambda g_{ij}) R_{ij} + \frac{1}{2} f_i R_i$$
$$= -R_{ij} R_{ij} + \lambda R + \frac{1}{2} f' R'.$$

So, $\mathrm{tr}(Rc^2) = R_{ij} R_{ij}$ depends only on s.

We shall use the Codazzi equation $\nabla_k R_{ij} = \nabla_i R_{kj} - \frac{R_i}{6} g_{kj} + \frac{R_k}{6} g_{ij}$.

$$\begin{aligned}\nabla_k (f_i R_{ij} R_{jk}) &= f_{ik} R_{ij} R_{jk} + f_i (\nabla_k R_{ij}) R_{jk} + f_i R_{ij} \nabla_k R_{jk} \\ &= -(R_{ik} - \lambda g_{ik}) R_{ij} R_{jk} + f_i (\nabla_i R_{kj} - \frac{R_i}{6} g_{kj} + \frac{R_k}{6} g_{ij}) R_{jk} \\ &\quad + \frac{1}{2} f_i R_{ij} R_j \\ &= -\mathrm{tr}(Rc^3) + \lambda R_{ij} R_{ij} + \frac{1}{2} f_i \nabla_i (R_{jk} R_{jk}) - f_i \frac{R_i}{6} R + \frac{f_i R_k}{6} R_{ik} \\ &\quad + \frac{1}{2} f_i R_{ij} R_j.\end{aligned} \tag{6}$$

All terms except $\mathrm{tr}(Rc^3)$ in the right-hand side of (6) depend on s only. From (5) we also get

$$\begin{aligned}2\nabla_k (f_i R_{ij} R_{jk}) &= \nabla_k (R_j R_{jk}) = (\nabla_k R_j) R_{jk} + \frac{1}{2} R_j R_j \\ &= R'' R_{11} - \sum_{i>1} \frac{R'}{|\nabla f|} (R_{ii}^2 - \lambda R_{ii}) + \frac{1}{2} R_j R_j,\end{aligned}$$

which depends only on s. So, we compare this with (6) to see that $\mathrm{tr}(Rc^3)$ depends only on s. Now λ_1 and $\sum_{i=1}^{4} (\lambda_i)^k$, $k = 1, \ldots, 3$, depend only on s. This implies that each λ_i, $i = 1, \ldots, 4$, is a constant depending only on s. By (4), ζ_i, $i = 2, 3, 4$ depend on s only. □

We now extend Lemma 2.4 (i):

Lemma 2.8 *For a Riemannian metric g of dimension $n \geq 4$ with harmonic Weyl curvature, consider orthonormal vector fields E_i, $i = 1, \ldots, n$, such that $Rc(E_i, \cdot) = \lambda_i g(E_i, \cdot)$. Then the following hold:*

(i) $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k \rangle + \nabla_{E_i} \langle E_k, \mathcal{A} E_j \rangle = (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k \rangle + \nabla_{E_j} \langle E_k, \mathcal{A} E_i \rangle$, *for any $i, j, k = 1, \ldots, n$.*

(ii) *If $k \neq i$ and $k \neq j$,* $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k \rangle = (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k \rangle$.

Proof The tensor $\mathcal{A} = Rc - \frac{R}{2n-2} g$ is a Codazzi tensor with eigenfunctions $\lambda_i - \frac{R}{2n-2}$. We have

$$\begin{aligned}\langle(\nabla_{E_i}\mathcal{A})E_j, E_k\rangle &= -\langle\nabla_{E_i}E_j, \mathcal{A}E_k\rangle - \langle\nabla_{E_i}E_k, \mathcal{A}E_j\rangle + \nabla_{E_i}\langle E_k, \mathcal{A}E_j\rangle \\ &= -(\lambda_k - \frac{R}{2n-2})\langle\nabla_{E_i}E_j, E_k\rangle - (\lambda_j - \frac{R}{2n-2})\langle\nabla_{E_i}E_k, E_j\rangle + \nabla_{E_i}\langle E_k, \mathcal{A}E_j\rangle \\ &= (\lambda_j - \lambda_k)\langle\nabla_{E_i}E_j, E_k\rangle + \nabla_{E_i}\langle E_k, \mathcal{A}E_j\rangle.\end{aligned}$$

As $\mathcal{A}$ is a Codazzi tensor, $\langle(\nabla_{E_i}\mathcal{A})E_j, E_k\rangle = \langle(\nabla_{E_j}\mathcal{A})E_i, E_k\rangle$. So, we get (i). Then (ii) holds since $\nabla_{E_i}\langle E_k, \mathcal{A}E_j\rangle = \nabla_{E_j}\langle E_k, \mathcal{A}E_i\rangle = 0$. □

Lemma 2.9 *For a gradient Ricci soliton (M, g, f) with harmonic Weyl curvature, and for a local adapted frame field $\{E_i\}$ in $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, the following holds.*

For $i, j, k > 1$, with $k \neq i$ and $k \neq j$, setting $\Gamma_{ij}^k := \langle\nabla_{E_i}E_j, E_k\rangle$, $(\zeta_k - \zeta_j)\Gamma_{ij}^k = (\zeta_k - \zeta_i)\Gamma_{ji}^k$, $(\zeta_k - \zeta_j)\Gamma_{ij}^k = (\zeta_i - \zeta_j)\Gamma_{kj}^i$ and $\Gamma_{ij}^k = -\Gamma_{ik}^j$.

Proof From (4) and Lemma 2.8, $(\zeta_k - \zeta_j)\Gamma_{ij}^k = (\zeta_k - \zeta_i)\Gamma_{ji}^k$. Others hold readily. □

3 4-Dimensional Solitons with Distinct $\lambda_2, \lambda_3, \lambda_4$

Let (M, g, f) be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature and non-constant f. In a neighborhood of any point in the open dense subset $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ of M, there exists an adapted frame field E_j, $j = 1, 2, 3, 4$, of Lemma 2.5 with its eigenfunction λ_j

We may only consider three cases depending on the distinctiveness of $\lambda_2, \lambda_3, \lambda_4$: the first case is when $\lambda_i, i = 2, 3, 4$, are all equal (on an open subset), and the second is when exactly two of the three are equal. And the last is when the three $\lambda_i, i = 2, 3, 4$, are mutually different.

In this section we shall study the last case.

Lemma 3.1 *Let (M, g, f) be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature and non-constant f. Suppose that for an adapted frame field E_j, $j = 1, 2, 3, 4$, in an open subset W of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, the eigenfunctions $\lambda_2, \lambda_3, \lambda_4$ are distinct from each other. Then the following hold in W:*
for $i, j > 1$, $i \neq j$,

$$\nabla_{E_1}E_1 = 0, \quad \nabla_{E_i}E_1 = \zeta_i E_i, \quad \nabla_{E_i}E_i = -\zeta_i E_1, \quad \nabla_{E_1}E_i = 0.$$
$$\nabla_{E_i}E_j = \Gamma_{ij}^k E_k \text{ where } k \neq 1, i, j.$$

Proof From Lemma 2.6 we have $\nabla_{E_1}E_1 = 0$ and $\nabla_{E_i}E_1 = \zeta_i E_i$. From Lemma 2.4 (i) and Lemma 2.7, $\langle\nabla_{E_i}E_i, E_j\rangle = 0$. And $\langle\nabla_{E_i}E_i, E_1\rangle = -\langle E_i, \nabla_{E_i}E_1\rangle = -\zeta_i$. So, we get $\nabla_{E_i}E_i = -\zeta_i E_1$. Now, $-\langle\nabla_{E_i}E_j, E_i\rangle = 0$, $\langle\nabla_{E_i}E_j, E_j\rangle = 0$. And $\langle\nabla_{E_i}E_j, E_1\rangle = -\langle\nabla_{E_i}E_1, E_j\rangle = 0$. So, $\nabla_{E_i}E_j = \Gamma_{ij}^k E_k$ where $k \neq 1, i, j$. Clearly $\Gamma_{ij}^k = -\Gamma_{ik}^j$.

From Lemma 2.8 (ii), $(\lambda_i - \lambda_j)\langle\nabla_{E_1}E_i, E_j\rangle = (\lambda_1 - \lambda_j)\langle\nabla_{E_i}E_1, E_j\rangle$. As $\langle\nabla_{E_i}E_1, E_j\rangle = 0$, $\langle\nabla_{E_1}E_i, E_j\rangle = 0$. This gives $\nabla_{E_1}E_i = 0$. □

From the above lemma, we may write

$$[E_2, E_3] = \alpha E_4, \quad [E_3, E_4] = \beta E_2, \quad [E_4, E_2] = \gamma E_3. \tag{7}$$

Lemma 3.2 *Under the hypothesis of Lemma 3.1, we have the following relation on ζ_i's and the coefficients of (7).*

$$E_1(\alpha) = \alpha(\zeta_4 - \zeta_2 - \zeta_3), \quad E_1(\beta) = \beta(\zeta_2 - \zeta_3 - \zeta_4), \quad E_1(\gamma) = \gamma(\zeta_3 - \zeta_2 - \zeta_4)$$
$$\beta = \frac{(\zeta_3 - \zeta_4)^2}{(\zeta_2 - \zeta_3)^2}\alpha, \quad \gamma = \frac{(\zeta_2 - \zeta_4)^2}{(\zeta_2 - \zeta_3)^2}\alpha.$$

Proof From Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ applied to $(X, Y, Z) = (E_1, E_2, E_3)$ gives $E_1(\alpha) = \alpha(\zeta_4 - \zeta_2 - \zeta_3)$. Apply it to E_1, E_2, E_4 and E_1, E_3, E_4, we get the next two.

Using $2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle$ for vector fields X, Y, Z, from Lemma 2.9 we get: $\frac{-\alpha-\gamma+\beta}{2} = \Gamma^3_{24} = \frac{(\zeta_2-\zeta_4)}{\zeta_3-\zeta_4}\Gamma^2_{34} = \frac{(\zeta_2-\zeta_4)}{\zeta_3-\zeta_4}\frac{\alpha-\gamma+\beta}{2}$. So, $-\alpha - \gamma + \beta = \frac{(\zeta_2-\zeta_4)}{\zeta_3-\zeta_4}(\alpha - \gamma + \beta)$. By symmetry, we have $-\beta - \alpha + \gamma = \frac{(\zeta_3-\zeta_2)}{\zeta_4-\zeta_2}(\beta - \alpha + \gamma)$ and $-\gamma - \beta + \alpha = \frac{(\zeta_4-\zeta_3)}{\zeta_2-\zeta_3}(\gamma - \beta + \alpha)$. From these, we can get the other formulas. □

Lemma 3.3 *Let a four-dimensional gradient Ricci soliton (M, g, f) with harmonic Weyl curvature satisfy the hypothesis of Lemma 3.1. Then the following hold in W: For distinct $i, j, k > 1$, $R_{1ii1} = -\zeta_i' - \zeta_i^2 = R_{1jj1}$, where $\zeta_i' = \frac{d\zeta_i}{ds}$, $R_{1ij1} = 0$.*

$$R_{11} = -3\zeta_2' - 3\zeta_2^2.$$
$$R_{22} = -\zeta_2' - \zeta_2^2 - \zeta_2\zeta_3 - \zeta_2\zeta_4 - 2\Gamma^2_{34}\Gamma^2_{43}.$$
$$R_{33} = -\zeta_3' - \zeta_3^2 - \zeta_3\zeta_2 - \zeta_3\zeta_4 + 2\frac{(\zeta_2 - \zeta_4)}{\zeta_3 - \zeta_4}\Gamma^2_{34}\Gamma^2_{43}.$$
$$R_{44} = -\zeta_4' - \zeta_4^2 - \zeta_4\zeta_2 - \zeta_4\zeta_3 + 2\frac{(\zeta_2 - \zeta_3)}{\zeta_4 - \zeta_3}\Gamma^2_{34}\Gamma^2_{43}.$$
$$R_{1i} = 0, \quad R_{ij} = E_k(\Gamma^k_{ij}).$$

Proof One uses Lemma 3.1 and Lemma 2.7. Recall $R_{1ii1} = R_{1jj1}$ from (2). By direct computation we get $R_{1ii1} = -\zeta_i' - \zeta_i^2$, $R_{jiij} = -\zeta_j\zeta_i - \Gamma^k_{ji}\Gamma^j_{ik} - \Gamma^k_{ji}\Gamma^j_{ki} + \Gamma^k_{ij}\Gamma^j_{ki}$ and $R_{kijk} = E_k(\Gamma^k_{ij})$. Use Lemma 2.9 to express R_{33} and R_{44}. □

Here we set $a := \zeta_2$, $b := \zeta_3$ and $c := \zeta_4$. From the soliton equation $\lambda - \zeta_i f' = R_{ii}$, $i > 1$ and Lemma 3.3,

$-(a - b)f' = R_{22} - R_{33} = (b - a)c - 2\{1 + \frac{(a-c)}{b-c}\}\Gamma^2_{34}\Gamma^2_{43}$. So,

$$f' = c + 2\frac{(a + b - 2c)}{(a - b)(b - c)}\Gamma^2_{34}\Gamma^2_{43}. \tag{8}$$

Similarly, $-(a - c)f' = (c - a)b - 2\{1 + \frac{(a-b)}{c-b}\}\Gamma^2_{34}\Gamma^2_{43}$. So,

$$f' = b + 2\frac{(a+c-2b)}{(a-c)(c-b)}\Gamma^2_{34}\Gamma^2_{43}. \tag{9}$$

From (8) and (9), we get

$$4\Gamma^2_{34}\Gamma^2_{43} = \frac{(a-b)(a-c)(b-c)^2}{(a^2+b^2+c^2-ab-bc-ac)}, \tag{10}$$

$$f' = \frac{a^2b+a^2c+ab^2+ac^2+b^2c+c^2b-6abc}{2(a^2+b^2+c^2-ab-bc-ac)}. \tag{11}$$

We are now ready to prove the following.

Proposition 3.4 *Let* (M, g, f) *be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature and non-constant* f*. For any adapted frame field* E_j*,* $j = 1, 2, 3, 4$*, in an open dense subset* $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ *of* M*, the three eigenfunctions* $\lambda_2, \lambda_3, \lambda_4$ *cannot be pairwise distinct, i.e., at least two of the three coincide.*

Proof Suppose that $\lambda_2, \lambda_3, \lambda_4$ are pairwise distinct. We shall prove then that g should be an Einstein metric, so a contradiction.
In this proof again we set $a := \zeta_2$, $b := \zeta_3$ and $c := \zeta_4$. From (10) and Lemma 2.9,

$$(\alpha-\gamma+\beta)^2 = 4(\Gamma^2_{34})^2 = 4\Gamma^2_{34}\Gamma^2_{43}\frac{(a-b)}{(a-c)} = \frac{(a-b)^2(b-c)^2}{(a^2+b^2+c^2-ab-bc-ac)}.$$

For convenience set $P := a^2+b^2+c^2-ab-bc-ac$. From Lemma 3.2,

$$(\alpha-\gamma+\beta)^2 = \alpha^2\left\{1-\frac{(a-c)^2}{(a-b)^2}+\frac{(b-c)^2}{(a-b)^2}\right\}^2 = \frac{4\alpha^2(b-c)^2}{(a-b)^2}.$$

So, $\alpha^2 = \frac{(a-b)^4}{4P}$. Since a, b, c are all functions of s only, so is α.
Differentiating this in s and using $b'-a' = a^2-b^2$ and $c'-a' = a^2-c^2$, we get

$$\begin{aligned}
2\alpha\alpha' &= \frac{(a-b)^3(a'-b')}{P} \\
&\quad -\frac{(a-b)^4(2aa'+2bb'+2cc'-ab'-ba'-ac'-ca'-cb'-bc')}{4P^2} \\
&= \frac{-(a-b)^3(a^2-b^2)}{P} \\
&\quad -\frac{(a-b)^4\{(a-b)(a'-b')+(a-c)(a'-c')+(b-c)(b'-c')\}}{4P^2} \\
&= \frac{-(a-b)^4(a+b)}{P} \\
&\quad +\frac{(a-b)^4\{(a-b)(a^2-b^2)+(a-c)(a^2-c^2)+(b-c)(b^2-c^2)\}}{4P^2}
\end{aligned}$$

$$= -\frac{(a-b)^4}{P}\left[(a+b) - \frac{\{2(a^3+b^3+c^3-3abc)+6abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2\}}{4P}\right]$$

$$= -\frac{(a-b)^4}{P}\left[(a+b) - \frac{(a+b+c)}{2} - \frac{\{6abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2\}}{4P}\right]$$

$$= -\frac{(a-b)^4}{P}\left[\frac{(a+b-c)}{2} - \frac{\{6abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2\}}{4P}\right].$$

Meanwhile, from Lemma 3.2 and $\alpha^2 = \frac{(a-b)^4}{4P}$,

$$2\alpha\alpha' = 2\alpha E_1(\alpha) = -2\alpha^2(a+b-c) = -\frac{(a-b)^4}{2P}(a+b-c).$$

Equating these two expressions for $2\alpha\alpha'$, we get: $6abc = a^2b + b^2a + a^2c + c^2a + b^2c + c^2b$. From (11), $f' = 0$. So, g is an Einstein metric. □

4 4-Dimensional Soliton with $\lambda_2 \neq \lambda_3 = \lambda_4$

In this section we begin to study the case when exactly two of the three eigenvalues $\lambda_2, \lambda_3, \lambda_4$ are equal. We may well assume that $\lambda_2 \neq \lambda_3 = \lambda_4$.

Lemma 4.1 *Let (M, g, f) be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field E_j, $j = 1, 2, 3, 4$, on an open subset of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$. Then the following hold on the open subset:*

$\nabla_{E_1} E_1 = 0$.
$\nabla_{E_i} E_1 = \zeta_i(s) E_i$ *for* $i = 2, 3, 4$, *with* $\zeta_i(s) = \frac{1}{|\nabla f|}(\lambda - \lambda_i)$.
$\nabla_{E_2} E_2 = -\zeta_2(s) E_1$.
$\nabla_{E_3} E_3 = -\zeta_3 E_1 - \beta_3 E_4$, $\nabla_{E_4} E_4 = -\zeta_4 E_1 + \beta_4 E_3$, *for some functions* β_3 *and* β_4.
$\nabla_{E_1} E_2 = 0$, $\nabla_{E_1} E_3 = \rho E_4$ *and* $\nabla_{E_1} E_4 = -\rho E_3$ *for some function* ρ.
$\nabla_{E_2} E_3 = q E_4$ *and* $\nabla_{E_2} E_4 = -q E_3$ *for some function* q.
$\nabla_{E_3} E_2 = 0$ *and* $\nabla_{E_4} E_2 = 0$.
$\nabla_{E_3} E_4 = \beta_3 E_3$ *and* $\nabla_{E_4} E_3 = -\beta_4 E_4$.
$[E_1, E_2] = -\zeta_2 E_2$ *and* $[E_3, E_4] = \beta_3 E_3 + \beta_4 E_4$.

In particular, the distribution spanned by E_1 and E_2 is integrable. So is that spanned by E_3 and E_4.

Proof The formula for $\nabla_{E_i} E_1$, $i \geq 1$, comes from (3).

Then from Lemma 2.7 and Lemma 2.4 (i): $(\lambda_2 - \lambda_i)\langle \nabla_{E_2} E_2, E_i\rangle = E_i(\lambda_2) = 0$ for $i = 3, 4$ and $\langle \nabla_{E_2} E_2, E_1\rangle = -\langle E_2, \nabla_{E_2} E_1\rangle = -\zeta_2(s)$. So, $\nabla_{E_2} E_2 = -\zeta_2(s)E_1$. By similar argument, $\nabla_{E_3} E_3 = -\zeta_3 E_1 - \beta_3 E_4$, $\nabla_{E_4} E_4 = -\zeta_4 E_1 + \beta_4 E_3$, for some functions β_3 and β_4.

From Lemma 2.8 (ii), $(\lambda_2 - \lambda_i)\langle \nabla_{E_1} E_2, E_i\rangle = (\lambda_1 - \lambda_i)\langle \nabla_{E_2} E_1, E_i\rangle = (\lambda_1 - \lambda_i)\langle \zeta_2 E_2, E_i\rangle = 0$, for $i = 3, 4$. So, $\langle \nabla_{E_1} E_2, E_i\rangle = 0$, for $i = 3, 4$. As $\langle \nabla_{E_1} E_2, E_1\rangle = -\langle E_2, \nabla_{E_1} E_1\rangle = 0$, we have $\nabla_{E_1} E_2 = 0$.

As $\langle \nabla_{E_1} E_3, E_2\rangle = -\langle E_3, \nabla_{E_1} E_2\rangle = 0$, one can readily get $\nabla_{E_1} E_3 = \rho E_4$ for some function ρ and $\nabla_{E_1} E_4 = -\rho E_3$. And $\nabla_{E_2} E_3 = q E_4$ for some function q and $\nabla_{E_2} E_4 = -q E_3$.

From Lemma 2.8 (ii), $(\lambda_2 - \lambda_4)\langle \nabla_{E_3} E_2, E_4\rangle = (\lambda_3 - \lambda_4)\langle \nabla_{E_2} E_3, E_4\rangle = 0$. So, $\langle \nabla_{E_3} E_2, E_4\rangle = 0$. As we have $\langle \nabla_{E_3} E_2, E_a\rangle = 0$ for $i = 1, 3$ from above, we get $\nabla_{E_3} E_2 = 0$. Similarly, $\nabla_{E_4} E_2 = 0$.

One can easily compute $\nabla_{E_3} E_4 = \beta_3 E_3$ and $\nabla_{E_4} E_3 = -\beta_4 E_4$. From above we get $[E_1, E_2] = -\zeta_2 E_2$ and $[E_3, E_4] = \beta_3 E_3 + \beta_4 E_4$. □

Lemma 4.2 *Let D^1 and D^2 be both two-dimensional smooth integrable distributions on a domain Ω of a four-dimensional manifold that span the tangent space $T_p\Omega$ for each $p \in \Omega$. Let p_0 be a point in Ω. Then there is a coordinate neighborhood (x_1, x_2, x_3, x_4) near p_0 so that D^1 is tangent to the 2-dimensional level sets* $\{(x_1, x_2, x_3, x_4)|\ x_3, x_4$ constants$\}$ *and D^2 is tangent to the level sets* $\{(x_1, x_2, x_3, x_4)|\ x_1, x_2$ constants$\}$.

Proof By Frobenius's theorem, there is a coordinate neighborhood $\mathbf{x} := (x, y, z, w)$ near p_0 so that D^2 is tangent to the sets $\{(x, y, z, w)|\ x, y$ constants$\}$. We may assume that $(x(p_0), y(p_0), w(p_0), z(p_0)) = (0, 0, 0, 0)$.

Then there are two vector fields $v_1 = (a_1, b_1, c_1, d_1) := a_1\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y} + c_1\frac{\partial}{\partial z} + d_1\frac{\partial}{\partial w}$ and $v_2 = (a_2, b_2, c_2, d_2)$ for points p near p_0 in D^1, with $(a_1(p), b_1(p))$ and $(a_2(p), b_2(p))$ being linearly independent as two-dimensional vectors; if not, D^1_p and D^2_p will not span $T_p\Omega$.

By considering $X_1 := \alpha_1 v_1 + \beta_1 v_2$ and $X_2 := \alpha_2 v_1 + \beta_2 v_2$ for smooth functions α_i, β_i, we have smooth vector fields $X_1, X_2 \in D^1$, of the form $X_1(p) = (1, 0, a_1(p), a_2(p))$ and $X_2 = (0, 1, b_1(p), b_2(p))$ for p near p_0 with smooth functions $a_i, b_i, i = 1, 2$.

Consider the one-parameter subgroup ϕ_t of X_1 and ψ_s of X_2: $\frac{d}{dt}\phi_t(p) = (1, 0, a_1(\phi_t(p)), a_2(\phi_t(p)))_{\phi_t(p)}$ and $\frac{d}{ds}\psi_s(p) = (0, 1, b_1(\psi_s(p)), b_2(\psi_s(p)))$.

Define a map Φ on a neighborhood of the origin in $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ into $\mathbb{R}^4 = \{(x, y, z, w)\}$ by $\Phi(x_1, x_2, x_3, x_4) := \phi_{x_1}\psi_{x_2}(0, 0, x_3, x_4)$. This Φ gives a local coordinate system near p_0. From $\frac{d}{ds}\psi_s(p) = (0, 1, b_1(\psi_s(p)), b_2(\psi_s(p)))$, we get $\psi_{x_2}(0, 0, x_3, x_4) = (0, x_2, *, *)$ and similarly $\phi_{x_1}\psi_{x_2}(0, 0, x_3, x_4) = \phi_{x_1}(0, x_2, *, *) = (x_1, x_2, *, *)$.

So, $\Phi(x_1, x_2, x_3, x_4) = (x_1, x_2, *, *)$. Then we get $\Phi_*(\frac{\partial}{\partial x_3}), \Phi_*(\frac{\partial}{\partial x_4}) \in$ span $(\frac{\partial}{\partial z}, \frac{\partial}{\partial w}) = D^2$. So, D^2 is spanned by $\Phi_*(\frac{\partial}{\partial x_3})$ and $\Phi_*(\frac{\partial}{\partial x_4})$.

As D^1 is integrable, in a neighborhood of each point $q_0 := (0, 0, c, d)$ near the origin, there is a unique surface S_{q_0} containing q_0 which is tangent to the distribution

D^1 at each point of S_{q_0}. As X_1 and X_2 are vector fields on S_{q_0}, at each point $q \in S_{q_0}$ we have $\{\psi_{x_2}(q) \mid x_2 \in (-\epsilon, \epsilon)\} \subset S_{q_0}$ and $\{\phi_{x_1}(q) \mid x_1 \in (-\epsilon, \epsilon)\} \subset S_{q_0}$ for small ϵ. Therefore, the set $\{\phi_{x_1}\psi_{x_2}(0, 0, c, d) \mid x_1, x_2 \in (-\varepsilon, \varepsilon)\}$, for small ε, coincides with S_{q_0} near q_0. So, we get $\Phi_*(\frac{\partial}{\partial x_1}), \Phi_*(\frac{\partial}{\partial x_2}) \in D^1$, and D^1 is spanned by $\Phi_*(\frac{\partial}{\partial x_1}), \Phi_*(\frac{\partial}{\partial x_2})$. Now we have obtained a new coordinate system $\Phi^{-1} \circ \mathbf{x}$ with the desired property. This proves the lemma. □

Using Lemma 4.1 and Lemma 4.2, we can express the metric g in the following lemma.

Lemma 4.3 *Let (M, g, f) be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field E_j, $j = 1, 2, 3, 4$, on an open subset U of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$.*

Then for each point p_0 in U, there exists a neighborhood V of p_0 in U with coordinates (s, t, x_3, x_4) such that $\nabla s = \frac{\nabla f}{|\nabla f|}$ and g can be written on V as

$$g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}, \tag{12}$$

where $p := p(s)$ and $h := h(s)$ are smooth functions and $\tilde{g}$ is (a pull-back of) a Riemannian metric on a 2-dimensional domain with x_3, x_4 coordinates.

We get $E_1 = \frac{\partial}{\partial s}$ and $E_2 = \frac{1}{p}\frac{\partial}{\partial t}$.

Proof Let D^1 be the 2-dimensional distribution spanned by $E_1 = \nabla s$ and E_2. Also let D^2 be the one spanned by E_3 and E_4. Then D^1 and D^2 are both integrable by Lemma 4.1. We may consider the coordinates (x_1, x_2, x_3, x_4) from Lemma 4.2, so that D^1 is tangent to the 2-dimensional level sets $\{(x_1, x_2, x_3, x_4) | \, x_3, x_4 \text{ constants}\}$ and D^2 is tangent to the level sets $\{(x_1, x_2, x_3, x_4) | \, x_1, x_2 \text{ constants}\}$. As D^1 and D^2 are g-orthogonal, we can get the metric description for g as follows:

$g = g_{11}dx_1^2 + g_{12}dx_1 \odot dx_2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{34}dx_3 \odot dx_4 + g_{44}dx_4^2$, where $\odot$ is the symmetric tensor product and g_{ij} are functions of (x_1, x_2, x_3, x_4).

As $E_1 = \nabla s \in D^1$, we have $ds = g(E_1, \cdot)$. We define a 1-form $\omega_2(\cdot) := g(E_2, \cdot)$. One can readily see that $ds^2 + \omega_2^2 = g_{11}dx_1^2 + g_{12}dx_1 \odot dx_2 + g_{22}dx_2^2$. In fact, one may feed (E_i, E_j) to both sides and use the fact that each of E_1 and E_2 is of the form $a\partial_1 + b\partial_2$ as they are tangent to the sets $\{(x_1, x_2, x_3, x_4) | \, x_3, x_4 \text{ constants}\}$, while each of E_3 and E_4 is of the form $c\partial_3 + d\partial_4$ for a similar reason; here we have set $\partial_i := \frac{\partial}{\partial x_i}$.

Recalling $[E_1, E_2] = -\zeta_2(s)E_2$, we define a function $p(s) = e^{\int_{s_0}^{s} \zeta_2(u)du}$ for a constant s_0 so that $\zeta_2 = \frac{p'}{p}$. Then, the 2-form $d(\frac{\omega_2}{p})$ satisfies $d(\frac{\omega_2}{p})(E_1, E_2) = -\frac{dp\wedge\omega_2}{p^2}(E_1, E_2) + \frac{1}{p}d\omega_2(E_1, E_2) = -\frac{p'}{p^2} + \frac{p'}{p^2} = 0$. And for $i \in \{3, 4\}$ and for any $j \in \{1, 2, 3, 4\}$, $d(\frac{\omega_2}{p})(E_i, E_j) = -\frac{dp\wedge\omega_2}{p^2}(E_i, E_j) + \frac{1}{p}d\omega_2(E_i, E_j) = \frac{1}{p}d\omega_2(E_i, E_j) = -\frac{1}{p}\omega_2([E_i, E_j]) = 0$ by Lemma 4.1.

So, $d(\frac{\omega_2}{p}) = 0$ and $\frac{\omega_2}{p} = dt$ for some function t modulo a constant in a neighborhood of p_0. The metric g can now be written as

$$g = ds^2 + p(s)^2 dt^2 + g_{33}dx_3^2 + g_{34}dx_3 \odot dx_4 + g_{44}dx_4^2, \tag{13}$$

where g_{ij} are functions of (x_1, x_2, x_3, x_4). In the coordinate system (s, t, x_3, x_4), one easily gets $E_1 = \frac{\partial}{\partial s}$ and $E_2 = \frac{1}{p}\frac{\partial}{\partial t}$.

Now we use new coordinates (s, t, x_3, x_4) in computations below, so that $\partial_1 = \frac{\partial}{\partial s}$ and $\partial_2 = \frac{\partial}{\partial t}$, etc. From Lemma 4.1, we have $\langle \nabla_{E_i} E_j, E_2\rangle = 0$ for $i, j \in \{3, 4\}$. As ∂_3 and ∂_4 are both of the form $\gamma E_3 + \delta E_4$, we have that $\langle \nabla_{\partial_i} \partial_j, \partial_2\rangle = 0$ for $i, j \in \{3, 4\}$.

We set $g_{ij} = g(\partial_i, \partial_j)$. Due to (13), for $i, j \in \{3, 4\}$:

$$
\begin{aligned}
0 =&\langle \nabla_{\partial_i} \partial_j, \partial_2\rangle = \sum_{k=1}^{4} \langle \Gamma_{ij}^k \partial_k, \partial_2\rangle \\
&= \sum_{k,l=1}^{4} \langle \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})\partial_k, \partial_2\rangle \\
&= -\sum_{k,l=1}^{4} \frac{1}{2} g^{kl} \partial_l g_{ij} \langle \partial_k, \partial_2\rangle = -\frac{1}{2}\partial_2 g_{ij}.
\end{aligned} \tag{14}
$$

We have shown:

$$
\frac{\partial g_{33}}{\partial t} = \frac{\partial g_{34}}{\partial t} = \frac{\partial g_{44}}{\partial t} = 0. \tag{15}
$$

We consider the second fundamental form of a leaf for D^2 with respect to E_1: $H^{E_1}(u, u) = -\langle \nabla_u u, E_1\rangle$. As D^2 is totally umbilic by Lemma 2.4 (ii), $H^{E_1}(u, u) = \zeta \cdot g(u, u)$ for some function ζ and any u tangent to D^2. Then, $H^{E_1}(E_3, E_3) = -\langle \nabla_{E_3} E_3, E_1\rangle = \zeta_3$ So, $\zeta = \zeta_3$, which is a function of s only by Lemma 2.7.

For $i, j \in \{3, 4\}$, we compute similarly as in (14),

$$
\begin{aligned}
\zeta_3 g_{ij} &= H^{E_1}(\partial_i, \partial_j) = -\left\langle \nabla_{\partial_i} \partial_j, \frac{\partial}{\partial s}\right\rangle = -\left\langle \sum_k \Gamma_{ij}^k \partial_k, \frac{\partial}{\partial s}\right\rangle \\
&= -\sum_k \left\langle \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})\partial_k, \frac{\partial}{\partial s}\right\rangle = \frac{1}{2}\frac{\partial}{\partial s} g_{ij}.
\end{aligned}
$$

So, $\frac{1}{2}\frac{\partial}{\partial s} g_{ij} = \zeta_3 g_{ij}$. Integrating it, for $i, j \in \{3, 4\}$, we get $g_{ij} = e^{C_{ij}} h(s)^2$. Here the function $h(s) > 0$ is independent of i, j, and each function C_{ij} depends only on x_3, x_4 by (15).

Now g can be written as $g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}$, where $\tilde{g}$ can be viewed as a Riemannian metric in a domain of the (x_3, x_4)-plane. □

5 Analysis of the Metric When $\lambda_2 \neq \lambda_3 = \lambda_4$

We shall study more about the metric $g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}$ of (12) obtained in Lemma 4.3.

Lemma 5.1 *Let (M, g, f) be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature which satisfies the hypothesis of Lemma 4.3. For the metric $g =$*

$ds^2+p(s)^2dt^2+h(s)^2\tilde{g}$ of (12), the two-dimensional metric $\tilde{g}$ has constant curvature, say k.

Proof In local coordinates $(x_1 := s,\ x_2 := t,\ x_3, x_4)$ of Lemma 4.3, we write some Christoffel symbols Γ_{ij}^k and Ricci curvature of g. In this proof, for any $(0,2)$-tensor P, $P(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ shall be denoted by P_{ij}. We let $\tilde{\nabla}$, $\tilde{\Gamma}_{ij}^k$ and $R_{ij}^{\tilde{g}}$ be the Levi-Civita connection, Christoffel symbols and Ricci curvature of $\tilde{g}$, respectively. For $i, j, k \in \{3, 4\}$, we get:

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k$$
$$R_{ij} = -\tilde{g}_{ij}\{hh'' + \frac{p'}{p}hh' + h'^2\} + R_{ij}^{\tilde{g}}. \tag{16}$$

From (16), for $i, j, k \in \{3, 4\}$, we have $\nabla_k \tilde{g}_{ij} = \tilde{\nabla}_k \tilde{g}_{ij} = 0$ and $\nabla_k R_{ij}^{\tilde{g}} = \tilde{\nabla}_k R_{ij}^{\tilde{g}}$ so that $\nabla_k R_{ij} = \tilde{\nabla}_k R_{ij}^{\tilde{g}}$. The condition $\delta W = 0$ gives $\nabla_k R_{ij} - \nabla_j R_{ik} = -\frac{R_j}{6} g_{ki} + \frac{R_k}{6} g_{ij}$. For $i, j, k \in \{3, 4\}$, $R_j = R_k = 0$, so $\nabla_k R_{ij} = \nabla_j R_{ik}$.

Then, we get $\tilde{\nabla}_k R_{ij}^{\tilde{g}} = \tilde{\nabla}_j R_{ik}^{\tilde{g}}$. By the contracted second Bianchi identity the 2-dimensional metric $\tilde{g}$ then has constant curvature. □

The metric $\tilde{g}$ of Lemma 5.1 is locally isometric to the Riemannian metric $g_0 = dr^2 + u(r)^2 d\theta^2$ on a domain in $\mathbb{R}^2$ with polar coordinates (r, θ), where $u(r) = r$ when $k = 0$, $u(r) = \sin(\sqrt{k} \cdot r)$ when $k > 0$ or $u(r) = \sinh(\sqrt{-k} \cdot r)$ when $k < 0$. We may identify $\tilde{g}$ with g_0 locally and set $e_3 = \frac{\partial}{\partial r}$ and $e_4 = \frac{1}{u(r)} \frac{\partial}{\partial \theta}$, which then form an orthonormal basis of $\tilde{g}$.

Lemma 5.2 *For the local soliton metric $g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}$ of (12) obtained in Lemma 4.3 with the metric $\tilde{g}$ of constant curvature k, if we set $E_1 = \frac{\partial}{\partial s}$, $E_2 = \frac{1}{p(s)} \frac{\partial}{\partial t}$, $E_3 = \frac{1}{h(s)} e_3$ and $E_4 = \frac{1}{h(s)} e_4$, where e_3 and e_4 are as in the above paragraph, then the connection form, Ricci and scalar curvature of g are as below. Here $R_{ij} = R(E_i, E_j)$ and $R_{ijkl} = R(E_i, E_j, E_k, E_l)$.*

$\nabla_{E_1} E_i = 0$, *for* $i = 1, 2, 3, 4$.
$\nabla_{E_i} E_1 = \zeta_i E_i$, *for* $i = 2, 3, 4$ *with* $\zeta_2 = \frac{p'}{p}$, $\zeta_3 = \zeta_4 = \frac{h'}{h}$.
$\nabla_{E_2} E_2 = -\zeta_2 E_1$, $\quad \nabla_{E_3} E_3 = -\zeta_3 E_1$, $\quad \nabla_{E_4} E_4 = -\zeta_4 E_1 + \beta_4 E_3$.
$\nabla_{E_2} E_3 = \nabla_{E_3} E_2 = \nabla_{E_4} E_2 = \nabla_{E_2} E_4 = 0$.
$\nabla_{E_3} E_4 = 0$, $\quad \nabla_{E_4} E_3 = -\beta_4 E_4$, *where* $\beta_4 = \frac{u'(r)}{h(s)u(r)}$.

$$R_{1221} = -\frac{p''}{p} = -\zeta_2' - \zeta_2^2 \ = \ R_{1ii1} = -\zeta_i' - \zeta_i^2 = -\frac{h''}{h}, \quad \text{for } i \geq 3.$$

$$R_{11} = -3\zeta_2' - 3\zeta_2^2 = -3\frac{h''}{h}.$$

$$R_{22} = -\zeta_2' - \zeta_2^2 - 2\zeta_2\zeta_3 = -\frac{h''}{h} - 2\frac{p'}{p}\frac{h'}{h}.$$

$$R_{33} = R_{44} = -\zeta_3' - \zeta_3^2 - \zeta_3\zeta_2 - (\zeta_3)^2 + \frac{k}{h^2} = -\frac{h''}{h} - \frac{p'}{p}\frac{h'}{h} - \frac{(h')^2}{h^2} + \frac{k}{h^2}.$$

$$R_{ij} = 0, \quad \text{if } i \neq j.$$
$$R = -6\zeta_3' - 6\zeta_3^2 - 4\zeta_3\zeta_2 - 2(\zeta_3)^2 + 2\frac{k}{h^2} = -6\frac{h''}{h} - 4\frac{p'}{p}\frac{h'}{h} - 2\frac{(h')^2}{h^2} + 2\frac{k}{h^2}.$$

Proof One may verify all the formulas by direct computation. In particular, $\zeta_2 = \frac{p'}{p}$ and $\zeta_3 = \zeta_4 = \frac{h'}{h}$. We get $\frac{p''}{p} = \frac{h''}{h}$ from (2). □

What emerges from the above discussions can be highlighted as the following soliton on an open set, which results from Lemmas 4.3 and 5.1:

A four-dimensional gradient Ricci soliton (M, g, f) with harmonic Weyl curvature has a connected coordinate neighborhood $(V, (s, t, x_3, x_4)) \subset M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, in which

$$g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g} \quad \text{on } V, \tag{17}$$

where $\tilde{g}$ is a 2-dimensional Riemannian metric of constant curvature k on an (x_3, x_4)-domain. We have the adapted frame field

$$E_1 = \frac{\nabla f}{|\nabla f|} = \frac{\partial}{\partial s}, \quad E_2 = \frac{1}{p}\frac{\partial}{\partial t}, \quad E_3 = \frac{1}{h}e_3, \quad E_4 = \frac{1}{h}e_4 \text{ on } V,$$
$$\text{and } \lambda_2 \neq \lambda_3 = \lambda_4, \tag{18}$$

where e_3 and e_4 are an orthonormal frame field of $\tilde{g}$ as in Lemma 5.2.

Remark 5.3 As mentioned in Sect. 2, g and f are real analytic (in harmonic coordinates), so is $|\nabla f|$ where $\nabla f \neq 0$. The Ricci eigenvalues λ_i are real analytic in $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$. So are $\zeta_i(s) = \frac{1}{|\nabla f|}(\lambda - \lambda_i)$.

Also $R' = dR(E_1)$ is real analytic since it equals $dR(\frac{\nabla f}{|\nabla f|})$. From (2) $R(E_1, E_2, E_2, E_1)$ is real analytic. As $-\zeta_2' - \zeta_2^2 = -\zeta_3' - \zeta_3^2 = R(E_1, E_2, E_2, E_1)$, ζ_2' as well as ζ_3' are real analytic.

To exploit the real analyticity, we shall use the following simple fact: if $P \cdot Q$ equals zero (identically) on an open connected set W for two real analytic functions P and Q, then either P equals zero on W or Q equals zero on W.

For the rest of this section we denote $a := \zeta_2$ and $b := \zeta_3$ for convenience.

In the adapted frame field $\{E_i\}$ of (18), we can write components of the soliton equation $\nabla df(E_i, E_i) = -(Rc - \lambda g)(E_i, E_i)$, $i = 1, 2, 3$ as follows:

$$f'' = 3a' + 3a^2 + \lambda. \tag{19}$$
$$f'a = a' + a^2 + 2ba + \lambda. \tag{20}$$
$$f'b = b' + b^2 + ba + b^2 - \frac{k}{h^2} + \lambda. \tag{21}$$

In the next section we are going to deduce several linear or quadratic equations in a and b from (19)–(21) and $\delta W = 0$. But before we get to it, in the next three lemmas we shall understand three linear cases (when $a = 0$, $b = 0$ and $a + b = 0$ on a domain).

Lemma 5.4 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), the function a cannot vanish on V.*

Proof If $a = 0$, then $b' + b^2 = a' + a^2 = 0$. Integrate for $b = \frac{h'}{h}$ to get $\frac{h'}{h} = \frac{1}{s-c}$ for a constant c, as $b \neq a = 0$. So, $h = c_h(s - c)$, for a constant $c_h \neq 0$. From (20), $\lambda = 0$. From (19), $f'' = 0$ and f' is constant. From (21) we get $f' = \frac{1}{s-c}(1 - \frac{k}{c_h^2})$. Then, $c_h^2 = k > 0$ and $f' = 0$. So, g is Einstein, a contradiction to the hypothesis $\lambda_2 \neq \lambda_3$. □

Lemma 5.5 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), assume that $b = 0$ on V. Then g is locally isometric to a domain in $\mathbb{R}^2 \times (N, \tilde{g})$ with $g = ds^2 + s^2dt^2 + \tilde{g}$, where $\tilde{g}$ is a Riemannian metric of constant curvature $\lambda \neq 0$ on a two-dimensional manifold N. And $f = \frac{\lambda}{2}s^2 + C_1$, for a nonzero constant C_1.*

Proof If $b = 0$, then $a' + a^2 = 0$. Integrate for $a = \frac{p'}{p}$ to get $\frac{p'}{p} = \frac{1}{s-c_1}$ for a constant c_1, as $a \neq b = 0$. So, $p = c_p(s - c_1)$, for a constant $c_p \neq 0$. As h is constant, we set $h = h_0 > 0$.

From (20), $f' = \lambda(s - c_1)$. We get $f(s) = \frac{1}{2}\lambda(s - c_1)^2 + C_1$. If $\lambda = 0$, then f is constant and g is Einstein, which violates the $\lambda_2 \neq \lambda_3$ hypothesis. So, $\lambda \neq 0$. From (21), we have $\frac{k}{h_0^2} = \lambda$. And by absorbing a constant to the variable t, we can write the metric $g = ds^2 + (s - c_1)^2dt^2 + h_0^2\tilde{g}$, where $h_0^2\tilde{g}$ is a Riemannian metric of constant curvature $\frac{k}{h_0^2} = \lambda$. The metric g is isometric to $ds^2 + s^2dt^2 + h_0^2\tilde{g}$. This proves the lemma. □

Lemma 5.6 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), the function $a + b$ cannot vanish on V.*

Proof Suppose $a + b = 0$ on V. Then $a' - b' = b^2 - a^2 = 0$. So, $a - b = C$, a constant. Then $a = \frac{p'}{p} = \frac{C}{2}$, $b = \frac{h'}{h} = -\frac{C}{2}$. As $a \neq b$, $C \neq 0$. Then $h = c_h e^{-\frac{C}{2}s}$ for a constant $c_h > 0$. Put it into (20) and (21), and we have $k = \lambda = 0$ and f' is a constant. Then (19) gives $C^2 = 0$, which is a contradiction. □

6 Characterization of the Metric When $\lambda_2 \neq \lambda_3 = \lambda_4$

In this section we shall characterize the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18).

From (20) and (21),

$$(a - b)f' = b(a - b) + \frac{k}{h^2}. \tag{22}$$

Differentiating, $(a - b)'f' + (a - b)f'' = b'(a - b) + b(a - b)' - 2\frac{kh'}{h^3}$.
Meanwhile, from (19), (22) and $a' - b' = -a^2 + b^2$,

$$\begin{aligned}(a - b)'f' + (a - b)f'' &= -(a^2 - b^2)f' + (a - b)(-\lambda_1 + \lambda)\\ &= (a + b)\Big\{ - b(a - b) - \frac{k}{h^2}\Big\} + (a - b)(-\lambda_1 + \lambda).\end{aligned}$$

So, we get $b'(a-b)+b(b^2-a^2)-2\frac{kh'}{h^3} = (a+b)\{-b(a-b)-\frac{k}{h^2}\}+(a-b)(-\lambda_1+\lambda)$. Then, as $b = \frac{h'}{h}$,

$$\begin{aligned} b'(a-b) &= (a+b)\Big\{-\frac{k}{h^2}\Big\} + 2\frac{kh'}{h^3} + (a-b)(-\lambda_1+\lambda) \\ &= (a-b)\Big\{-\frac{k}{h^2}\Big\} + (a-b)(-\lambda_1+\lambda). \end{aligned}$$

As $\lambda_2 \neq \lambda_3$, we have $a-b \neq 0$. We then have:

$$2(b'+b^2)+b^2-\frac{k}{h^2}+\lambda = 0. \tag{23}$$

From (20), (21) and $b' = a'+a^2-b^2$, we have

$$b(a'+a^2+2ba+\lambda) = a\Big(b'+b^2+ba+b^2-\frac{k}{h^2}+\lambda\Big),$$

and so

$$-(a-b)a'-a^3+ab^2+\lambda(b-a) = -a\frac{k}{h^2}. \tag{24}$$

Next, we shall exploit the harmonic Weyl curvature condition. In $\{E_i\}$, we have $\nabla_k R_{ij} - \nabla_j R_{ik} = -\frac{R_j}{6}g_{ki} + \frac{R_k}{6}g_{ij}$. Then as $\nabla_{E_1}E_2 = \nabla_{E_1}E_3 = 0$,

$$\begin{aligned} 0 &= \nabla_1 R_{22} - \nabla_2 R_{12} - \frac{R'}{6} \\ &= \nabla_1(R_{22}) + R(\nabla_{E_2}E_1, E_2) + R(\nabla_{E_2}E_2, E_1) - \frac{R'}{6} \\ &= (R_{22})' + aR_{22} - aR_{11} - \frac{R'}{6}. \end{aligned} \tag{25}$$

$$\begin{aligned} 0 &= \nabla_1 R_{33} - \nabla_3 R_{13} - \frac{R'}{6} \\ &= \nabla_1(R_{33}) + R(\nabla_{E_3}E_1, E_3) + R(\nabla_{E_3}E_3, E_1) - \frac{R'}{6} \\ &= (R_{33})' + bR_{33} - bR_{11} - \frac{R'}{6}. \end{aligned} \tag{26}$$

Subtracting (26) from (25), with Lemma 5.2 we get $(-ab+b^2-\frac{k}{h^2})'+a(-a'-a^2-2ab)-(a-b)(-3a'-3a^2)-b(-b'-b^2-ba-b^2+\frac{k}{h^2}) = 0$, from which we obtain

$$-(a-b)a'-a^3+b^3+2a^2b-2ab^2 = b\frac{k}{h^2}. \tag{27}$$

Subtracting (24) from (27),

$$(a-b)(2ab-b^2+\lambda) = (a+b)\frac{k}{h^2}. \tag{28}$$

Lemma 6.1 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), assume that $k \neq 0$. Then the following holds:*

$$b(\lambda + 3ab)(\lambda - 2a^2 + ab) = 0. \tag{29}$$

Proof We start from (28). Our hypothesis $k \neq 0$ and Lemma 5.6 implies that $2ab - b^2 + \lambda$ does not vanish. So, we may take the natural log of (28) and differentiate it:

$$-a - b + \frac{2a'b + 2ab' - 2bb'}{2ab - b^2 + \lambda} = \frac{a' + b'}{a + b} - 2b.$$

Then put $b' = a' + a^2 - b^2$ into it:

$$\frac{aa' + (a - b)(a^2 - b^2)}{2ab - b^2 + \lambda} = \frac{a' + a^2 - b^2}{a + b}.$$

Arranging terms, we obtain:

$$-a'(a^2 + b^2 - ab - \lambda) = (a^2 - b^2)(a^2 - 2ab - \lambda). \tag{30}$$

Meanwhile, using that $a - b \neq 0$, from $b \times (24) + a \times (27) = 0$ we have

$$-(a + b)a' = a^3 + 2ab^2 + \lambda b. \tag{31}$$

Removing a' in (30) and (31) and simplifying, we can get:

$$b(\lambda + 3ab)(\lambda - 2a^2 + ab) = 0.$$

□

We need to characterize the two equalities appearing in (29): $\lambda + 3ab = 0$ and $\lambda - 2a^2 + ab = 0$.

Lemma 6.2 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), assume that $k \neq 0$ and that h is not constant. Then $\lambda + 3ab$ does not vanish on V.*

Proof If $\lambda + 3ab = 0$ vanishes, we have $0 = a'b + ab' = (b' + b^2 - a^2)b + ab' = (a + b)(b' + b^2 - ab)$. By Lemma 5.6, we have $b' + b^2 = ab = -\frac{\lambda}{3}$. Due to (23), $b^2 - \frac{k}{h^2} = -\frac{\lambda}{3}$. From (21), $f'b = -\frac{\lambda}{3} - \frac{\lambda}{3} - \frac{\lambda}{3} + \lambda = 0$. As h is not constant, we have $f' = 0$, a contradiction. □

We study the equation $\lambda - 2a^2 + ab = 0$:

Lemma 6.3 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), assume that $k \neq 0$ and that h is not constant. Then $\lambda - 2a^2 + ab$ does not vanish on V.*

Proof If $\lambda - 2a^2 + ab = 0$ on V, put $\lambda = 2a^2 - ab$ into (31) to get: $-a' = \frac{a^3+2a^2b+ab^2}{a+b} = a(a+b)$. So, $a' + a^2 + ab = 0$, i.e., $p''h + p'h' = 0$. Integrating this, we get $p'h = c_1$ for a constant c_1. As $\frac{h''}{h} = \frac{p''}{p}$, we have $h''p + p'h' = 0$, which integrates to $h'p = c_2$ for a constant c_2. As a does not vanish by Lemma 5.5 and $b \neq 0$ from the hypothesis, c_1c_2 is not zero. So $\frac{h'}{h} = \frac{c_2}{c_1}\frac{p'}{p}$, i.e., $b = ca$, for $c \neq 0$. So, $0 = \lambda - 2a^2 + ab = \lambda + (c-2)a^2$.

If $c \neq 2$, then a is a nonzero constant. $a' + a^2 + ab = 0$ yields $a + b = 0$, which is not possible by Lemma 5.6.

If $c = 2$, then $\lambda = 0$ and $2a = b$. Put these and $a' + a^2 + ab = 0$ into (20) to get $f' = 2a$. Then from (21), we get $k = 0$, a contradiction. □

Lemma 6.4 *For the soliton metric g of (17) with harmonic Weyl curvature and with the adapted frame field (18), assume that $k = 0$.*

Then g is locally isometric to the metric $ds^2 + s^{\frac{2}{3}}dt^2 + s^{\frac{4}{3}}\tilde{g}$ on a domain of $\mathbb{R}^4$, where $\tilde{g}$ is flat. Also, $\lambda = 0$ and $f = \frac{2}{3}\ln s + C_2$, for a constant C_2.

Furthermore, the Ricci curvature components and scalar curvature of g are as follows: $R_{11} = \frac{2}{3s^2}$, $R_{22} = -\frac{2}{9s^2}$, $R_{33} = R_{44} = -\frac{4}{9s^2}$, $R_{ij} = 0$, $i \neq j$, and $R = -\frac{4}{9s^2}$. And the Weyl curvature of g is not zero.

Proof As $k = 0$ and $a \neq b$, $2ab - b^2 + \lambda = 0$ from (28). From the computation in Lemma 5.2, we get $R = -6(a' + a^2) - 8ab - 2\lambda$. (25) becomes:

$$\begin{aligned}
0 &= -\{a' + a^2 + 2ab\}' - a\{a' + a^2 + 2ab\} + 3a(a' + a^2) \\
&\quad - \frac{1}{6}\{-6(a' + a^2) - 8ab - 2\lambda\}' \\
&= -\frac{2}{3}(ab)' + 2a(a' + a^2 - ab) \\
&= -\frac{2}{3}\{a'b + a(a' + a^2 - b^2)\} + 2a(a' + a^2 - ab) \\
&= -\frac{2}{3}a'b + \frac{4}{3}aa' + \frac{4}{3}a^3 + \frac{2}{3}ab^2 - 2a^2b.
\end{aligned}$$

We get:

$$(2a - b)(a' + a^2 - ab) = 0.$$

If $a' + a^2 - ab = 0$, we get $p'' = \frac{p'h'}{h}$. Then $\frac{p'}{h} = c_1$, a constant. From $\frac{h''}{h} = \frac{p''}{p} = \frac{p'h'}{ph}$, we also get $\frac{h'}{p} = c_2$, a constant. So, $ab = \frac{p'h'}{ph} = c_1c_2$. And $2ab - b^2 + \lambda = 0$ tells that b is a constant. If $b = 0$, then $\lambda = k = 0$ and from (20) $f'a = 0$. So, $f' = 0$ and g is Einstein, a contradiction to the hypothesis. Now b is a nonzero constant. Then $b' + b^2 = a' + a^2 = ab$ gives $a = b$, a contradiction to the hypothesis.

If $2a = b$, then $0 = 2ab - b^2 + \lambda = \lambda$. From $a' + a^2 = b' + b^2 = 2a' + 4a^2$, we get $a' + 3a^2 = 0$. Integrating it to get $a = \frac{p'}{p} = \frac{1}{3s - c_2}$ for a constant c_2. (20) gives

$f'a = 2a^2$, so that $f' = 2a = \frac{2}{3s-c_2}$. As $2\frac{p'}{p} = \frac{h'}{h}$, we have $p^2 = e^c h$ for a constant c. We get $p = e^{c_3}(3s - c_2)^{\frac{1}{3}}$ and $h = e^{c_4}(3s - c_2)^{\frac{2}{3}}$.

So, g is locally isometric to the metric $ds^2 + s^{\frac{2}{3}}dt^2 + s^{\frac{4}{3}}\tilde{g}$ on a domain of $\mathbb{R}^4$, where $\tilde{g}$ is flat. And $f = \frac{2}{3}\ln s + C_2$, for a constant C_2.

One can check that the above (g, f) satisfy the soliton equation including (19), (20), (21) and the harmonicity of the Weyl curvature, and so is a steady Ricci soliton. One can easily compute the curvature components of g. □

Based on the real analyticity of a, b, a' and b' from Remark 5.3, we combine the previous lemmas to obtain the next proposition.

Proposition 6.5 *Let (M, g, f) be a four-dimensional gradient Ricci soliton with harmonic Weyl curvature. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field E_j, $j = 1, 2, 3, 4$, in an open subset U of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$.*

Then for each point p_0 in U, there exists a neighborhood V of p_0 in U with coordinates (s, t, x_3, x_4) in which (V, g, f) can be one of the following:

- (i) *(V, g) is isometric to a domain in $\mathbb{R}^2 \times N$ with $g = ds^2 + s^2dt^2 + \tilde{g}$, where $(N, \tilde{g})$ is a Riemannian manifold of constant curvature $\lambda \neq 0$. And $f = \frac{\lambda}{2}s^2 + C_1$, for a constant C_1.*
- (ii) *(V, g) is isometric to a domain in $\mathbb{R}^4$ with the Riemannian metric $ds^2 + s^{\frac{2}{3}}dt^2 + s^{\frac{4}{3}}\tilde{g}$, where $\tilde{g}$ is flat. Also, $\lambda = 0$ and $f = \frac{2}{3}\ln s + C_2$, for a constant C_2. The metric g is not locally conformally flat.*

Proof We exploit the real analyticity. Lemma 6.4 settles the $k = 0$ case. Lemma 6.1 divides the $k \neq 0$ case into three subcases $b = 0$, $\lambda + 3ab = 0$ and $\lambda - 2a^2 + ab = 0$ which are treated in Lemmas 5.5, 6.2 and 6.3, respectively. □

7 4-Dimensional Soliton with $\lambda_2 = \lambda_3 = \lambda_4$.

In this section we treat the remaining case of $\lambda_2 = \lambda_3 = \lambda_4$ for an adapted frame field.

Proposition 7.1 *Suppose that (M, g, f) is a four-dimensional gradient Ricci soliton with harmonic Weyl curvature and non-constant f and that $\lambda_2 = \lambda_3 = \lambda_4 \neq \lambda_1$ for an adapted frame field in an open subset U of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$.*

Then for each point p_0 in U, there exists a neighborhood V of p_0 in U where g is a warped product:

$$g = ds^2 + h(s)^2\tilde{g}, \tag{32}$$

for a positive function h, where the Riemannian metric $\tilde{g}$ has constant curvature, say k. In particular, g is locally conformally flat.

Proof Near p_0 in U, we use a local coordinate system $(x_1 := s,\ x_2, x_3, x_4)$ from Lemma 2.3 (v) in which the metric $g = ds^2 + \sum_{i,j\geq 2}^{4} g_{ij}dx_i dx_j$ with $g_{ij} = g_{ij}(x_1, \ldots, x_4)$.

By Lemma 2.7, near p_0, each λ_i, $i = 1, 2, 3, 4$ is a function of s only. We consider the second fundamental form of the level hypersurfaces Σ_c of f with respect to E_1:

$H^{E_1}(u,u) = -\langle \nabla_u u, E_1\rangle$. As Σ_c is totally umbilic by Lemma 2.4 (ii), $H^{E_1}(u,u) = G \cdot g(u,u)$ for any u tangent to Σ_c and some function G. Then, by Lemma 2.4 (i) $\langle \nabla_{E_2} E_2, E_1\rangle = \frac{\lambda_2' - \frac{1}{6}R'}{\lambda_2 - \lambda_1}$ So, $G = -\frac{\lambda_2' - \frac{1}{6}R'}{\lambda_2 - \lambda_1}$ is a function of s only.

For $i, j \in \{2,3,4\}$, setting $\partial_i := \frac{\partial}{\partial x_i}$, we compute,

$$G(s)\cdot g_{ij} = H^{E_1}(\partial_i, \partial_j) = -\left\langle \nabla_{\partial_i}\partial_j, \frac{\partial}{\partial s}\right\rangle = -\left\langle \sum_{k=1}^{4} \Gamma_{ij}^k \partial_k, \frac{\partial}{\partial s}\right\rangle$$
$$= -\sum_k \left\langle \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})\partial_k, \frac{\partial}{\partial s}\right\rangle = \frac{1}{2}\frac{\partial}{\partial s} g_{ij}.$$

So, $\frac{1}{2}\frac{\partial}{\partial s} g_{ij} = G(s) g_{ij}$. Integrating it, we get $g_{ij} = e^{C_{ij}} w(s)$. Here the function $w(s)$ is independent of i, j and each C_{ij} depends only on x_2, x_3, x_4.

Now g can be written as $g = ds^2 + h(s)^2 \tilde{g}$, where $\tilde{g}$ can be viewed as a Riemannian metric in a domain of the (x_2, x_3, x_4)-plane.

To prove that $\tilde{g}$ has constant curvature, we modify the proof of Derdziński's Lemma 4 in [17], which is stated for the harmonic curvature case.

For $i, j \in \{2,3,4\}$, we compute the Christoffel symbols and Ricci curvature of g:

$$\Gamma_{ij}^1 = -hh'\tilde{g}_{ij}, \quad \Gamma_{1j}^i = \frac{h'}{h}\delta_{ij},$$
$$R_{1i} = 0, \quad R_{11} = -3\frac{h''}{h}, \quad R_{ij} = -\tilde{g}_{ij}(hh'' + 2h'^2) + R_{ij}^{\tilde{g}}. \tag{33}$$

The condition $\delta W = 0$ gives $\nabla_k R_{ij} - \nabla_j R_{ik} = -\frac{R_j}{6} g_{ki} + \frac{R_k}{6} g_{ij}$. In particular, for $i, j \in \{2,3,4\}$, $\nabla_1 R_{ij} - \nabla_j R_{i1} = \frac{R_1}{6} g_{ij}$. From (33),

$$\begin{aligned}\frac{\partial_1 R}{6} h^2 \tilde{g}_{ij} &= \frac{\partial_1 R}{6} g_{ij} = \nabla_1 R_{ij} - \nabla_i R_{1j}\\ &= \partial_1 R_{ij} - R(\nabla_{\partial_1}\partial_j, \partial_i) + R(\nabla_{\partial_i}\partial_j, \partial_1)\\ &= \partial_1 R_{ij} - \frac{h'}{h} R(\partial_j, \partial_i) - hh' R(\partial_1, \partial_1)\tilde{g}(\partial_i, \partial_j)\\ &= -\tilde{g}_{ij}\partial_1(h''h + 2h'^2) - \frac{h'}{h}\left[-\tilde{g}_{ij}\left(hh'' + 2h'^2\right) + R_{ij}^{\tilde{g}}\right] - hh' R_{11}\tilde{g}_{ij}.\end{aligned}$$

As R depends only on s, so does $\partial_1 R = \frac{\partial R}{\partial s}$. Therefore, we get $R_{ij}^{\tilde{g}} = H(s)\cdot \tilde{g}_{ij}$ for a function $H(s)$ of s only. So, $\tilde{g}$ is a 3-dimensional Einstein metric. □

For the metric in (32), h and f satisfy the following equations from $\nabla\nabla f + Rc = \lambda g$:

$$f'' - 3\frac{h''}{h} = \lambda, \tag{34}$$
$$\frac{h'}{h} f' + \frac{2k}{h^2} - \frac{h''}{h} - 2\frac{(h')^2}{h^2} = \lambda. \tag{35}$$

Remark 7.2 If all λ_i's, $i = 1, \dots, 4$, are equal, then the metric is Einstein. And if f is not constant, then the conclusion of Proposition 7.1 still holds. In fact, from Sect. 1 of [11], the Einstein metric g becomes locally of the form $g = ds^2 + (f'(s))^2 \tilde{g}$ where $\tilde{g}$ has constant curvature. Then, the soliton can be seen to be either Gaussian or a flat metric with $\nabla df = 0$; see also Proposition 2 of [28].

8 Classification of Gradient Ricci Solitons with Harmonic Weyl Curvature

We are going to combine Propositions 3.4, 6.5 and 7.1 to prove Theorem 1.1 after we settle the next lemma:

Lemma 8.1 *No two of the local four types of solitons* (i)–(iv) *in the statement of Theorem* 1.1 *can exist on a connected soliton.*

Proof When the real analytic function f is constant in an open subset, then it is constant on M as M is connected. So, if a soliton is the type (i) in an open subset, it will be so on M.

If g is a locally conformally flat metric on an open subset U with non-constant f, then $|W|^2 = 0$ on U and the real analytic function $|W|^2 = 0$ everywhere on M. So, g is locally conformally flat on M and f is nowhere constant on M. The types (ii) and (iii) do not satisfy $|W|^2 = 0$.

If g is isometric, on an open subset V, to a domain in $\mathbb{R}^2 \times N_\lambda$, then $R = 2\lambda$ on V and by real analyticity $R = 2\lambda$ on M. But if g is isometric, on another open subset W, to the metric $ds^2 + s^{\frac{2}{3}} dt^2 + s^{\frac{4}{3}} \tilde{g}$, then the scalar curvature $R = -\frac{4}{9s^2}$ is not locally constant. This proves the lemma. □

Proof of Theorem 1.1 Due to Lemma 8.1 we may consider only one type on M. When f is constant, it corresponds to the type (i).

So, suppose that f is not constant. Note that the statement (iv) holds by Proposition 7.1 and Remark 7.2. We denote the open dense subset $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ by K. If $K = M$, then the statements for (ii) and (iii) also hold from Proposition 6.5..

For the rest of proof we assume that there is a point $p_0 \in M \backslash K$.

When (K, g) is of the type (ii), (K, g) is locally isometric to $\mathbb{R}^2 \times N_\lambda$, where the Ricci tensor is parallel. As K is dense in M, the Ricci tensor is parallel near p_0 with eigenvalues λ and 0 of both multiplicity two by continuity. We can decompose the tangent bundle over a neighborhood of p_0: $TM = \eta_1 \oplus \eta_2$, where η_1, η_2 are 2-dimensional parallel distributions with $Rc_{|\eta_1} = \lambda \cdot \mathrm{Id}$ and $Rc_{|\eta_2} = 0 \cdot \mathrm{Id}$. By de Rham's decomposition theorem [27, Sect. 8.3.1], p_0 has an open ball $B \subset M$ with p_0 as the center, where B is isometric to (to be identified with) a ball in $\mathbb{R}^2 \times N_\lambda$. Now we can just solve for f from the gradient soliton equation $\nabla df = -Rc + \lambda g$ to get: $f = \frac{\lambda s^2}{2} + C$ where $s(\cdot) := d_{\mathbb{R}^2}(p_0, \cdot)$ is the Euclidean distance function from p_0. So, a neighborhood of p_0 is of type (ii).

Suppose that (K, g) is of the type (iii). Let $\gamma_1 : [0, 1] \to M$ be a smooth path with $\gamma_1(0) = p_0$ and $\gamma_1(1) \in K$. Let $c \in [0, 1)$ be the largest element in $\{t \in [0, 1) \mid \gamma_1(t) \in M \backslash K\}$. Define γ to be the restriction of γ_1 on $[c, 1]$. Set $p := \gamma(c)$ which is in $M \backslash K$.

Then $\gamma((c, 1]) \subset K$. Near any point $q \in \gamma((c, 1])$, by Proposition 6.5 we have local coordinates neighborhood $B_q \subset K$ with (s_q, t, x_3, x_4) in which $f = \frac{2}{3}\ln(s_q) + C_q$ with the function s_q and constant C_q depending on q. In a neighborhood $B_r \subset K$ of another point $r \in \gamma((c, 1])$, we have a similar expression of $f = \frac{2}{3}\ln(s_r) + C_r$. On a possible overlap region $B_q \cap B_r$, $\frac{2}{3}\ln(s_q) + C_q = \frac{2}{3}\ln(s_r) + C_r$. By taking its gradient, we have $\frac{\nabla s_q}{s_q} = \frac{\nabla s_r}{s_r}$. As $\nabla s_q = \frac{\nabla f}{|\nabla f|} = \nabla s_r$, we get $s_q = s_r$ and then $C_q = C_r$.

We may set $s := s_q$ and $C := C_q$ which are independent of q and $f = \frac{2}{3}\ln(s) + C$ near $\gamma((c, 1])$. As $|\nabla s| \equiv 1$, the oscillation of s along γ is less than or equal to the length of γ, which is finite. So, $|\nabla f| = \frac{2}{3s}$ cannot be zero at p. From Lemma 6.3, the Ricci-eigenfunctions of g are $\lambda_1 = \frac{2}{3s^2}$, $\lambda_2 = -\frac{2}{9s^2}$, $\lambda_3 = \lambda_4 = -\frac{4}{9s^2}$. So, p shall stay in $M_{\mathcal{A}}$ by definition. Then $p \in K$. This contradiction implies that $M \backslash K$ is an empty set.

Proposition 3.4 shows that there are no other types than (i)–(iv). This proves the theorem. □

We remark that the incomplete steady gradient soliton in Theorem 1.1 (iii) has negative scalar curvature, in contrast to the fact that complete steady gradient solitons should have nonnegative scalar curvature.

As a Corollary to Theorem 1.1, we state a classification of 4-dimensional complete gradient Ricci solitons with harmonic Weyl curvature. The case Theorem 1.1 (iii) can only yield an incomplete soliton. And for case (ii), when g is complete and locally isometric to $\mathbb{R}^2 \times N_\lambda$, its universal cover is isometric to $\mathbb{R}^2 \times N_\lambda$.

Theorem 8.2 *Let (M, g, f) be a complete four-dimensional gradient Ricci soliton $\nabla df = -Rc + \lambda g$ with harmonic Weyl curvature. Then it is one of the following:*

(i) *g is an Einstein metric with f a constant function.*
(ii) *g is isometric to a finite quotient of $\mathbb{R}^2 \times N_\lambda$ where $\mathbb{R}^2$ has the Euclidean metric and N_λ is a 2-dimensional Riemannian manifold of constant curvature $\lambda \neq 0$. And $f = \frac{\lambda}{2}|x|^2$ modulo a constant on the Euclidean factor.*
(iii) *g is locally conformally flat.*

Complete locally conformally flat *steady* gradient Ricci solitons are classified to be either flat or isometric to the Bryant soliton, in [6,10]. This result and Theorem 8.2 yield Theorem 1.2. We also understand better complete expanding gradient Ricci solitons with harmonic Weyl curvature as in Theorem 1.3.

As mentioned in the Introduction, we can show the local classification of gradient Ricci soliton with *harmonic curvature* as a corollary of Theorem 1.1.

Corollary 8.3 *Let (M, g, f) be a (not necessarily complete) four-dimensional gradient Ricci soliton satisfying $\nabla df = -Rc + \lambda g$ with harmonic curvature. Then it is locally one of the three types* (i)–(iii) *below; for each point p, there exists a neighborhood V of p such that (V, g, f) can be one of the following:*

(i) *g is an Einstein metric and f is constant.*
(ii) *g is isometric to a domain in $\mathbb{R}^2 \times N_\lambda$ where $\mathbb{R}^2$ has the Euclidean metric and N_λ is a 2-dimensional Riemannian manifold of constant curvature $\lambda \neq 0$. And $f = \frac{\lambda}{2}|x|^2$ modulo a constant on the Euclidean factor.*

(iii) *g is isometric either to a domain in the Gaussian soliton or to a domain in* $\mathbb{R} \times M_\lambda$ *with the product metric, where* M_λ *is a 3-dimensional Riemannian manifold of constant curvature* $\frac{\lambda}{2} \neq 0$, *and* $f = \frac{\lambda}{2}|x|^2$ *modulo a constant on the Euclidean factor.*

Proof In this proof we do not rely on Theorem 1.2 of [28] as it works for a complete soliton.

The soliton metric $ds^2 + s^{\frac{2}{3}}dt^2 + s^{\frac{4}{3}}\tilde{g}$ in Theorem 1.1 (iii) does not have constant scalar curvature, so does not have harmonic curvature.

Note that the above (iii) should come from Theorem 1.1 (iv), in which the metric is of the form $g = ds^2 + h(s)^2\tilde{g}$, where $\tilde{g}$ has constant curvature. Lemma 2.1 (ii) gives $R + |\nabla f|^2 - 2\lambda f =$ constant. We differentiate with the local variable s where $|\nabla f| \neq 0$, and get $2f'f'' = 2\lambda f'$ since R is constant. So, $f'' = \lambda$. From (34), $h'' = 0$. Either $h = a$ or $h = bs$ for constants $a, b \neq 0$ after shifting s by a constant.

When $h = a$, from (35) we get $\frac{k}{a^2} = \frac{\lambda}{2}$. We have $g = ds^2 + \tilde{g}$ where $\tilde{g}$ has constant curvature $\frac{\lambda}{2}$. And we may set $f = \frac{\lambda}{2}s^2 + C$ by shifting s. As f is not constant, $\lambda \neq 0$.

When $h = bs$, using (35) and $f'' = \lambda$ we obtain that $f' = \lambda s$ and $k = b^2$. We get $f = \frac{1}{2}\lambda s^2 + C$ so that $\lambda \neq 0$. And $g = ds^2 + s^2\tilde{g}$, where $\tilde{g}$ has constant curvature $+1$. This yields the Gaussian soliton.

(As an alternative to settle (iii), Sect. 2.2 of [10] may be cited. But that section is based on the existence of a self-similar solution, which exists if the soliton metric is complete [33]. Here the metric may be incomplete.) □

Remark 8.4 In Theorem 1.1 (iii) we got a four-dimensional incomplete soliton. One may ask if there exist *complete* non-conformally flat gradient Ricci solitons of dimension ≥ 5 with harmonic Weyl curvature and $\lambda \leq 0$.

There are a number of objects to study by extending our method; it would be interesting to characterize the higher-dimensional gradient Ricci solitons with harmonic Weyl curvature as well as other Ricci solitons. Of course, other geometric structures than solitons can also be approached by the method here.

Remark 8.5 There is much literature on *orbifolds* in the theory of Ricci flow, for instance, [16,23]. As our result is a local description, it is possible to state an orbifold version of Theorem 8.2.

Remark 8.6 B.L. Chen proved a local version of a Hamilton–Ivey type estimate for three dimensions in [12], which has been extended to the $W = 0$ case by Zhang [32]. From Theorem 1.1, one may ask if such a local version still holds when $\delta W = 0$.

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