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Proper Modifications of Generalized p-Kähler Manifolds

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Abstract In this paper, we consider a proper modification $f : \tilde{M} \to M$ between complex manifolds, and study when a generalized p-Kähler property goes back from M to $\tilde{M}$. When f is the blow-up at a point, every generalized p-Kähler property is conserved, while when f is the blow-up along a submanifold, the same is true for $p = 1$. For $p = n - 1$, we prove that the class of compact generalized balanced manifolds is closed with respect to modifications, and we show that the fundamental forms can be chosen in the expected cohomology class. We also get some partial results in the non-compact case; finally, we end the paper with some examples of generalized p-Kähler manifolds.

Keywords Kähler manifold · p-Kähler manifold · SKT manifold · Blow-up · Modification · Balanced manifold

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1 Introduction

Let M be a complex manifold of dimension $n \geq 2$, let p be an integer, $1 \leq p \leq n-1$. We shall consider three families of maps, namely:

$\pi_O : \tilde{M} \to M$, which is the blow-up of M at a point O;

$\pi : \tilde{M} \to M$, which is the blow-up of M along a submanifold Y;

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$f : \tilde{M} \to M$, which is a proper modification of M with center Y and exceptional set E.

We will study, in this context, when a generalized p-Kähler property (indicated as "p-Kähler" property, see Definition 2.3) goes back from M to $\tilde{M}$, or what kind of weaker properties can be obtained. So we unify and generalize analogous results on hermitian symplectic, SKT, balanced, strongly Gauduchon manifolds.

The obvious way is to start from a "p-Kähler" form Ω on M, and consider the pull-back $f^*\Omega$ on $\tilde{M}$, which is "closed" because f is holomorphic; for the same reason, we get $f^*\Omega \geq 0$, and $f^*\Omega > 0$ on $\tilde{M} - E$, since $f|_{\tilde{M}-E} : \tilde{M} - E \to M - Y$ is biholomorphic.

Nevertheless, strict positivity is not preserved in general; for instance, if $y \in Y$, and $F := f^{-1}(y)$ is a fiber of dimension k, and $\omega > 0$ is the (1, 1)-form of a Kähler metric on M, it holds that $\int_F (f^*\omega)^k = 0$.

The case when M is Kähler is well known: while π (and π_O) preserve the Kähler property, so that $\tilde{M}$ is Kähler too, the famous example of Hironaka (a compact threefold X which is given by a modification of $\mathbb{P}_3$) shows that the Kähler property is not preserved by modifications.

Hironaka's example X is a Moishezon manifold: this proves that it is "2-Kähler" (i.e., 2K, 2WK, 2S, 2PL; see Sect. 2) because it is balanced [5]; X is not 1K (nor 1WK, 1S, 1PL) since it contains a curve that bounds.

But in general, when we perform a modification of a "1-Kähler" manifold M, it is not guaranteed that $\tilde{M}$ is regular (in the sense of Varouchas, see Subsection 7.4, i.e., a manifold satisfying the $\partial\overline{\partial}$-Lemma). Moreover, it is well known that if $\tilde{M}$ satisfies the $\partial\overline{\partial}$-Lemma, so does M (see [11]); but it is not known yet if a modification of a regular manifold is regular too: this fact sheds further light on the context of the question we stated above.

The first result we get (Theorem 3.1) extends the very classical statement: *The blow-up at a point of a Kähler manifold is a Kähler manifold too.* We prove, with a unified proof, that the same also holds for hermitian symplectic, pluriclosed, SKT, balanced, strongly Gauduchon, …manifolds: in general, for "p-Kähler" manifolds. This result allows one to construct new examples of "p-Kähler" manifolds.

Next, in Theorem 3.2, we extend another classical result, that is: *If M is a Kähler manifold, and $\tilde{M}$ is obtained from M blowing up a submanifold, then $\tilde{M}$ is Kähler too.* We give a very short proof in the general case of "1-Kähler" manifolds, which includes also pluriclosed (i.e., SKT) and hermitian symplectic manifolds. The analogous result cannot hold in the generic "p-Kähler" case, as we prove by a suitable example.

As for compact "$(n-1)$-Kähler" manifolds, we complete the study of the invariance of the property of being "balanced" with respect to modifications, initiated in [8] in the classical case, and due to [25] in the sG case: in Theorem 4.1, we prove that a modification $\tilde{M}$ of a compact "$(n-1)$-Kähler" manifold M, is "$(n-1)$-Kähler", and in Theorem 4.3 we prove that, when $\tilde{M}$ is "$(n-1)$-Kähler", then M is "$(n-1)$-Kähler" too. Next we give a partial result in the "p-Kähler" case.

Here the compactness hypothesis is needed to use the characterization of "p-Kähler" manifolds by means of positive currents (see Theorem 2.4). But, owing to the use of currents, we lose the link between metrics on M and $\tilde{M}$: we recapture the link (that is, $f_*\tilde{\omega}^{n-1}$ is cohomologous to ω^{n-1}) in Proposition 5.1; this result is proved in a more general setting in Theorem 5.2.

We look also for another kind of generalization of our main result in [7], i.e., *A proper modification $\tilde{M}$ of a compact balanced manifold M is balanced.* Indeed, we consider non-compact manifolds, but suppose that the center Y is compact. In this case, we can consider the Bott–Chern and the Aeppli cohomologies with compact supports, and use a modified version of the characterization theorem by positive currents; we can prove (see Theorem 6.2) that, under mild cohomological hypotheses, if M is *locally balanced with respect to* Y, then $\tilde{M}$ is *locally balanced with respect to* E.

We end the paper in Sect. 7 with some examples and some remarks on the "exactness" of the "p-Kähler" form.

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2 Preliminaries

Let X be a complex manifold of dimension $n \geq 2$, let p be an integer, $1 \leq p \leq n-1$; we refer to [20] (see also [1]) as regards notation and terminology. To define positivity for forms and currents, let us start from a complex n-dimensional (Euclidean) vector space E, its associated (Euclidean) vector spaces of (p, q)-forms $\Lambda^{p,q}(E^*)$, and a (orthonormal) basis $\{\varphi_1, \dots, \varphi_n\}$ for E^*.

Let us denote $\varphi_I := \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}$, where $I = (i_1, \dots, i_p)$, $\sigma_p := i^{p^2} 2^{-p}$ and $\Lambda^{p,p}_{\mathbb{R}}(E^*) := \{\psi \in \Lambda^{p,p}(E^*) / \psi = \overline{\psi}\}$. Let $p + k = n$.

We obviously get that $\{\sigma_p \varphi_I \wedge \overline{\varphi_I}, |I| = p\}$ is a (orthonormal) basis for $\Lambda^{p,p}_{\mathbb{R}}(E^*)$, and

$$dV = \left(\frac{i}{2}\varphi_1 \wedge \overline{\varphi_1}\right) \wedge \cdots \wedge \left(\frac{i}{2}\varphi_n \wedge \overline{\varphi_n}\right) = \sigma_n \varphi_I \wedge \overline{\varphi_I}, \; I = (1, \dots, n)$$

is a volume form.

Definition 2.1 (1) An (n, n)-form τ is called *positive* (*strictly positive*) if $\tau = c\, dV$ with $c \geq 0$ ($c > 0$). We shall write $\tau \geq 0$ ($\tau > 0$).

(2) $\eta \in \Lambda^{p,0}(E^*)$ is called *simple* (or decomposable) if and only if there are $\{\psi_1, \dots, \psi_p\} \in E^*$ such that $\eta = \psi_1 \wedge \cdots \wedge \psi_p$.

(3) $\Omega \in \Lambda^{p,p}_{\mathbb{R}}(E^*)$ is called *strongly positive* ($\Omega \in SP^p$) if and only if $\Omega = \sigma_p \sum_j \eta_j \wedge \overline{\eta_j}$, with η_j simple.

(4) $\Omega \in \Lambda^{p,p}_{\mathbb{R}}(E^*)$ is called *weakly positive* ($\Omega \in WP^p$) if and only if for all $\psi_j \in E^*$, and for all $I = (i_1, \dots, i_k)$ with $k + p = n$, $\Omega \wedge \sigma_k \psi_I \wedge \overline{\psi_I}$ is a positive (n, n)-form. It is called *transverse* when it is strictly weakly positive, i.e., when $\Omega \wedge \sigma_k \psi_I \wedge \overline{\psi_I}$ is a strictly positive (n, n)-form for $\sigma_k \psi_I \wedge \overline{\psi_I} \neq 0$ (i.e., $\psi_{i_1}, \dots, \psi_{i_k}$ linearly independent).

Remark a) There is also an intermediate "natural" definition of positivity, given in terms of eigenvalues, or as follows: "$\Omega \in \Lambda^{p,p}_{\mathbb{R}}(E^*)$ is positive ($\Omega \in P^p$) if and only if for every $\eta \in \Lambda^{k,0}(E^*)$, $(k+p=n)$, $\Omega \wedge \sigma_k \eta \wedge \overline{\eta}$ is a positive (n, n)-form." (see [20], Theorem 1.2).

b) Positive forms (as in (a)) are not considered either by Lelong [22] or by Demailly [12]; both of them call positive forms (this is the "classical sense") what we call weakly positive forms. The strongly positive forms are called *decomposable* by Lelong.
c) The sets P^p, SP^p, WP^p and their interior parts are indeed convex cones; moreover, there are obvious inclusions:

$$SP^p \subseteq P^p \subseteq WP^p \subseteq \Lambda_{\mathbb{R}}^{p,p}, \quad (SP^p)^{int} \subseteq (P^p)^{int} \subseteq (WP^p)^{int}.$$

d) When $p = 1$ or $p = n - 1$, the three cones coincide, since every $(1, 0)$-form is simple (and hence also every $(n - 1, 0)$-form is simple).
e) In the intermediate cases, $1 < p < n - 1$, the inclusions are strict: indeed, if $\{\varphi_1, \dots, \varphi_4\}$ is a basis for $\Lambda^{1,0}(\mathbb{C}^4)$, then it is easy to prove that $\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4$ is not a simple $(2, 0)$-form; moreover, in $\mathbb{C}^n$, $(\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4) \wedge \varphi_5 \wedge \cdots \wedge \varphi_{p+2}$ is not a simple $(p, 0)$-form, for $p > 2$.
By Proposition 1.5 in [20], this implies that $(\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4) \wedge (\overline{\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4})$ is a positive $(2, 2)$-form which is not strongly positive.
Moreover, the authors exhibit a (p, p)-form which is in the interior of the cone WP^p, but has a negative eigenvalue, so it does not belong to the cone P^p.
f) Duality. Using the volume form dV, for $p + k = n$ we get the pairing

$$f : \Lambda^{p,p}(E^*) \times \Lambda^{k,k}(E^*) \to \mathbb{C}$$

given by $f(\Omega, \Psi)dV = \Omega \wedge \Psi$. So it is not hard to prove that:

$$\Omega \in SP^p \iff \forall\, \Psi \in WP^k, \Omega \wedge \Psi \geq 0,$$
$$\Omega \in P^p \iff \forall\, \Psi \in P^k, \Omega \wedge \Psi \geq 0.$$

g) Consider $\Lambda_{p,q}(E)$, the space of (p, q)-vectors: as before, $X \in \Lambda_{p,0}(E)$ is called a simple vector if $X = v_1 \wedge \cdots \wedge v_p$ for some $v_j \in E$; in this case, when $X \neq 0$, $\sigma_p^{-1} X \wedge \overline{X}$ is called a *strictly strongly positive* (p, p)-vector.

Claim *$\Omega \in \Lambda_{\mathbb{R}}^{p,p}(E^*)$ is transverse if and only if $\Omega(\sigma_p^{-1} X \wedge \overline{X}) > 0$ for every $X \in \Lambda_{p,0}(E)$, $X \neq 0$ and simple.*

Proof of the claim Using the pairing described above, we get an isomorphism $g : \Lambda_{p,p}(E) \to \Lambda^{k,k}(E^*)$ given as $f(\Omega, g(A)) = \Omega(A)$, i.e.,

$$f(\Omega, g(A))dV = \Omega \wedge g(A) := \Omega(A)dV, \quad \forall A \in \Lambda_{p,p}(E), \forall \Omega \in \Lambda^{p,p}(E^*).$$

If $\{e_1, \dots, e_n\}$ denotes the dual basis of $\{\varphi_1, \dots, \varphi_n\}$, it is not hard to check that for all $I = (i_1, \dots, i_p)$, $g(\sigma_p^{-1} e_I \wedge \overline{e_I}) = \sigma_k \varphi_J \wedge \overline{\varphi_J}$ with $J = \{1, \dots, n\} - I$.

Thus the isomorphism g transforms (p, p)-vectors of the form $\sigma_p^{-1} X \wedge \overline{X}$, with X simple (i.e., strongly positive vectors), into strongly positive (k, k)-forms (of the form $\sigma_k \eta_j \wedge \overline{\eta_j}$, with η_j simple). Hence we get

$$\Omega\left(\sigma_p^{-1}X\wedge\overline{X}\right)dV=\Omega\wedge g\left(\sigma_p^{-1}X\wedge\overline{X}\right)=\Omega\wedge\sigma_k\eta\wedge\overline{\eta}$$

and the claim follows.

Let us go back to manifolds: we denote by $\mathcal{D}^{p,p}(X)_{\mathbb{R}}$ the space of compactly supported real (p,p)-forms on X and by $\mathcal{E}^{p,p}(X)_{\mathbb{R}}$ the space of real (p,p)-forms on X.

Their dual spaces are: $\mathcal{D}'_{p,p}(X)_{\mathbb{R}}$ (also denoted by $\mathcal{D}'^{k,k}(X)_{\mathbb{R}}$, where $p+k=n$), the space of real currents of bidimension (p,p) or bidegree (k,k), which we call (k,k)-currents, and $\mathcal{E}'_{p,p}(X)_{\mathbb{R}}$ (also denoted by $\mathcal{E}'^{k,k}(X)_{\mathbb{R}}$), the space of compactly supported real (k,k)-currents on X.

We shall denote by $[Y]$ the current given by the integration on the irreducible analytic subset Y.

We shall define weakly positive, positive, strongly positive currents (see, for instance, [20]). For simplicity, let N be a *compact* n-dimensional manifold, and $1\le p\le n-1$. □

Definition 2.2 (1) $\Omega\in\mathcal{E}^{p,p}(N)_{\mathbb{R}}$ is called strongly positive (resp., positive, weakly positive, transverse or strictly weakly positive) if $\forall\, x\in N,\ \Omega_x\in SP^p(T'_xN^*)$ (resp., $P^p(T'_xN^*),\ WP^p(T'_xN^*),\ (WP^p(T'_xN^*))^{int}$).
These spaces of forms are denoted by $SP^p(N),\ P^p(N), WP^p(N), (WP^p(N))^{int}$.

(2) Let $T\in\mathcal{D}'_{p,p}(N)_{\mathbb{R}}$ be a current of bidimension (p,p) on N. Then we have:
weakly positive currents: $T\in WP_p(N)\iff T(\Omega)\ge 0\ \ \forall\,\Omega\in SP^p(N)$.
positive currents: $T\in P_p(N)\iff T(\Omega)\ge 0\ \ \forall\,\Omega\in P^p(N)$.
strongly positive currents: $T\in SP_p(N)\iff T(\Omega)\ge 0\ \ \forall\,\Omega\in WP^p(N)$.

Notation $\Omega\ge 0$ denotes that Ω is weakly positive; $\Omega>0$ denotes that Ω is transverse; $T\ge 0$ means that T is strongly positive. Thus:

Claim $\Omega>0$ *if and only if* $T(\Omega)>0$ *for every* $T\ge 0, T\ne 0$.

Remark There are obvious inclusions between the previous cones of currents, that is, $SP_p(N)\subseteq P_p(N)\subseteq WP_p(N)$. Demailly ([12], Definition III.1.13) does not consider $P_p(N)$, and indicates $WP_p(N)$ as the cone of positive currents; there is no uniformity of notation in the papers of Alessandrini and Bassanelli.

Moreover, let us recall that, if f is a holomorphic map, and $T\ge 0$, then $f_*T\ge 0$.

We shall need the De Rham cohomology, and also the Bott–Chern and the Aeppli cohomologies (the notation is not standard, so that we recall them below): they can be described using forms or currents of the same bidegree:

$$H^{k,k}_{dR}(X,\mathbb{R}):=\frac{\{\varphi\in\mathcal{E}^{k,k}(X)_{\mathbb{R}};\,d\varphi=0\}}{\{d\psi;\,\psi\in\mathcal{E}^{2k-1}(X)_{\mathbb{R}}\}}\simeq\frac{\{T\in\mathcal{D}'^{k,k}(X)_{\mathbb{R}};\,dT=0\}}{\{dS;\,S\in\mathcal{D}'^{2k-1}(X)_{\mathbb{R}}\}}$$

$$H^{k,k}_{\partial\overline{\partial}}(X,\mathbb{R})=\Lambda^{k,k}_{\mathbb{R}}(X)=H^{k,k}_{BC}(X,\mathbb{R}):=\frac{\{\varphi\in\mathcal{E}^{k,k}(X)_{\mathbb{R}};\,d\varphi=0\}}{\{i\partial\overline{\partial}\psi;\,\psi\in\mathcal{E}^{k-1,k-1}(X)_{\mathbb{R}}\}}\simeq$$

$$\simeq\frac{\{T\in\mathcal{D}'^{k,k}(X)_{\mathbb{R}};\,dT=0\}}{\{i\partial\overline{\partial}A;\,A\in\mathcal{D}'^{k-1,k-1}(X)_{\mathbb{R}}\}}$$

$$H^{k,k}_{\partial+\overline{\partial}}(X,\mathbb{R}) = V^{k,k}_{\mathbb{R}}(X) = H^{k,k}_{A}(X,\mathbb{R}) := \frac{\{\varphi \in \mathcal{E}^{k,k}(X)_{\mathbb{R}};\, i\partial\overline{\partial}\varphi = 0\}}{\{\varphi = \partial\overline{\eta} + \overline{\partial}\eta;\, \eta \in \mathcal{E}^{k,k-1}(X)\}} \simeq$$

$$\simeq \frac{\{T \in \mathcal{D}'^{k,k}(X)_{\mathbb{R}};\, i\partial\overline{\partial}T = 0\}}{\{\partial\overline{S} + \overline{\partial}S;\, S \in \mathcal{D}'^{k,k-1}(X)\}}.$$

In general, when the class of a current vanishes in one of the previous cohomology groups, we say that the current "bounds" or is "exact".

We collect what we called in the Introduction "**p-Kähler**" **properties** in the following definition (see [1], and also the next Remarks).

Definition 2.3 Let X be a complex manifold of dimension $n \geq 2$, let p be an integer, $1 \leq p \leq n-1$.

(1) X is a p-Kähler (**pK**) manifold if it has a closed transverse (p, p)-form Ω.
(2) X is a weakly p-Kähler (**pWK**) manifold if it has a transverse (p, p)-form Ω with $\partial\Omega = \partial\overline{\partial}\alpha$ for some form α.
(3) X is a p-symplectic (**pS**) manifold if it has a closed transverse real $2p$-form Ψ; that is, $d\Psi = 0$ and $\Omega := \Psi^{p,p}$ (the (p, p)-component of Ψ) is transverse.
(4) X is a p-pluriclosed (**pPL**) manifold if it has a transverse (p, p)-form Ω with $\partial\overline{\partial}\Omega = 0$.

Notice that: $pK \Longrightarrow pWK \Longrightarrow pS \Longrightarrow pPL$; as regards examples and differences under these classes of manifolds, see [1].

When X satisfies one of these definitions, in the rest of the paper we will call it generically a "**p-Kähler" manifold**; the form Ω, called a "**p-Kähler" form**, is said to be "closed". This may be a little bit worrying to read, but the benefit is that we do not write a lot of similar proofs.

Remark For $p = 1$, a transverse form is the fundamental form of a hermitian metric, so that we can consider 1-Kähler (i.e., Kähler), weakly 1-Kähler, 1-symplectic, 1-pluriclosed *metrics*. 1-symplectic manifolds are also called *hermitian symplectic* [27].

In [13], pluriclosed (i.e., 1-pluriclosed) metrics are defined (see also [27]), while in [14] a 1PL metric (manifold) is called a *strong Kähler metric (manifold) with torsion* (SKT).

For $p = n-1$, we get a hermitian metric too, because every transverse $(n-1, n-1)$-form Ω is in fact given by $\Omega = \omega^{n-1}$, where ω is a transverse $(1, 1)$-form (see, for instance, [23], p. 279). This case was studied by Michelsohn in [23], where $(n-1)$-Kähler manifolds are called *balanced* manifolds.

Moreover, $(n-1)$-symplectic manifolds are called *strongly Gauduchon manifolds (sG)* by Popovici (compare Definition 2.3 (3) and Theorem 2.4 (3) with [24], Definition 4.1 and Propositions 4.2 and 4.3; see also [25]), while $(n-1)$-pluriclosed metrics are called *standard* or *Gauduchon metrics*. Recently, weakly $(n-1)$-Kähler manifolds have been called *superstrong Gauduchon (super sG)* [26].

In the case of a *compact* manifold N, we got the following characterization (see [1], Theorems 2.1, 2.2, 2.3, 2.4)

Theorem 2.4 *(1) Characterization of compact p-Kähler (pK) manifolds.*
N has a strictly weakly positive (i.e., transverse) (p, p)-form Ω with $\partial\Omega = 0$, if and only if N has no strongly positive currents $T \neq 0$, of bidimension (p, p), such that $T = \partial\overline{S} + \overline{\partial}S$ for some current S of bidimension $(p, p+1)$ (i.e., T "bounds" in $H^{k,k}_{\partial+\overline{\partial}}(N)$, i.e., T is the (p, p)-component of a boundary).

(2) Characterization of compact weakly p-Kähler (pWK) manifolds.
N has a strictly weakly positive (p, p)-form Ω with $\partial\Omega = \partial\overline{\partial}\alpha$ for some form α, if and only if N has no strongly positive currents $T \neq 0$, of bidimension (p, p), such that $T = \partial\overline{S}+\overline{\partial}S$ for some current S of bidimension $(p, p+1)$ with $\partial\overline{\partial}S = 0$ (i.e., T is closed and "bounds" in $H^{k,k}_{\partial+\overline{\partial}}(N)$).

(3) Characterization of compact p-symplectic (pS) manifolds.
N has a real $2p$-form $\Psi = \sum_{a+b=2p} \Psi^{a,b}$, such that $d\Psi = 0$ and the (p, p)-form $\Omega := \Psi^{p,p}$ is strictly weakly positive, if and only if N has no strongly positive currents $T \neq 0$, of bidimension (p, p), such that $T = dS$ for some current S (i.e., T is a boundary with respect to the De Rham cohomology).

(4) Characterization of compact p-pluriclosed (pPL) manifolds.
N has a strictly weakly positive (p, p)-form Ω with $\partial\overline{\partial}\Omega = 0$, if and only if N has no strongly positive currents $T \neq 0$, of bidimension (p, p), such that $T = i\partial\overline{\partial}A$ for some current A of bidimension $(p+1, p+1)$ (i.e., T "bounds" in $H^{k,k}_{\partial\overline{\partial}}(N)$).

Remark Every compact complex manifold supports Gauduchon metrics, that is, is $(n-1)$PL: in fact, by Theorem 2.4 (4), if T is a strongly positive $(1, 1)$-current, such that $T = i\partial\overline{\partial}A$, A turns out to be a plurisubharmonic function; but N is compact, so that A is constant, and $T = 0$.

Lastly, let us recall a Support Theorem, which we shall frequently use for $p = n-1$.

Theorem 2.5 (see [6],Theorem 1.5) *Let X be an n-dimensional complex manifold, E a compact analytic subset of X; call $\{E_j\}$ the irreducible components of E of dimension p. Let T be a weakly positive $\partial\overline{\partial}$-closed current of bidimension (p, p) on X such that supp $T \subseteq E$. Then there exist $c_j \geq 0$ such that $S := T - \sum_j c_j[E_j]$ is a weakly positive $\partial\overline{\partial}$-closed current of bidimension (p, p) on X, supported on the union of the irreducible components of E of dimension bigger than p.*

3 Blow-up of Manifolds

Let M be a connected complex manifold, with $n = dim M \geq 2$, let p be an integer, $1 \leq p \leq n-1$. As stated in the Introduction, we shall consider three kinds of proper modifications:

$\pi_O : \tilde{M} \to M$, which is the blow-up of M at a point O;
$\pi : \tilde{M} \to M$, which is the blow-up of M along a compact submanifold Y;
and an arbitrary proper modification of M with compact center Y, $f : \tilde{M} \to M$.

Recall that a complex manifold $\tilde{M}$ together with a proper holomorphic map $f : \tilde{M} \to M$ is called a (smooth) proper modification of M if there is a thin set Y in M such that $f^{-1}(Y)$ is thin in $\tilde{M}$, and the restricted map f from $\tilde{M} - f^{-1}(Y)$ to $M - Y$ is biholomorphic.

Grauert and Remmert (see [17], pp. 214–215) proved among other things that Y can be chosen as an analytic subset of codimension ≥ 2 such that $E := f^{-1}(Y)$ is an analytic subset of pure codimension one in $\tilde{M}$, called the exceptional set of the modification.

We will study, in this context, when a "p-Kähler" property goes back from M to $\tilde{M}$. The problem is completely solved for π_O by Theorem 3.1, for which we give a proof by direct computation, which unifies all "p-Kähler" cases, some of which are well known when $p = 1$.

Theorem 3.1 *Let $\pi_O : \tilde{M} \to M$ be the blow-up of M at a point O; for every $p, 1 \leq p \leq n-1$, whenever M is "p-Kähler", $\tilde{M}$ is also "p-Kähler".*

Proof First of all, let us recall the classical proof for Kähler manifolds. Let us choose coordinates $\{z_j\}$ around $O \in M$, such that on $U_{2\epsilon} := \{||z|| < 2\epsilon\}$ and $\tilde{U}_{2\epsilon} := \pi_O^{-1}(U_{2\epsilon})$, π_O is nothing but the blow-up of $\mathbb{C}^n$ at 0, with exceptional set $E := \pi_O^{-1}(O) \simeq \mathbb{P}_{n-1}$.

With obvious notation, consider a cut-off function $\chi \in C_0^\infty(U_{2\epsilon})$, $\chi = 1$ on U_ϵ, and put, for $x \in \tilde{U}_{2\epsilon}$,

$$\tilde{\theta}_x := i\partial\overline{\partial}(\chi(\pi_O(x))log||x||^2),$$

where $i\partial\overline{\partial}(log||x||^2)$ is just the pull-back of the Fubini–Study $(1, 1)$-form on $\mathbb{P}_{n-1}$ under the map $j : \tilde{U}_{2\epsilon} \to \mathbb{P}_{n-1}$ which is the identity on $\mathbb{P}_{n-1}$ and maps every $x \in \tilde{U}_{2\epsilon} - \mathbb{P}_{n-1}$ to the line $[x] \in \mathbb{P}_{n-1}$ that passes through x (see [18], p. 186).

The form $\tilde{\theta}$ turns out to be a global closed real $(1, 1)$-form, with supp $\tilde{\theta} \subset \tilde{U}_{2\epsilon}$; moreover, $\tilde{\theta} \geq 0$ on $\tilde{U}_\epsilon$. For $x \in E$, $\tilde{\theta}_x(\sigma_1^{-1} v \wedge \overline{v}) > 0$ only on vectors $v \in T'_x E$: that is, $\tilde{\theta}$ is not strictly positive on E.

Nevertheless, starting from a Kähler form Ω on M, we can consider $\pi_O^*\Omega$ which is a closed real $(1, 1)$-form on $\tilde{M}$, with $\pi_O^*\Omega \geq 0$ and $(\pi_O^*\Omega)_x(\sigma_1^{-1} v \wedge \overline{v}) > 0$ when $x \in E$ and $v \in T'_x\tilde{M}$ is orthogonal to $T'_x E$.

Moreover, for x in the closure of $\tilde{U}_{2\epsilon} - \tilde{U}_\epsilon$, the values of $(\pi_O^*\Omega)_x$ on strictly strongly positive $(1, 1)$-vectors $\sigma_1^{-1} v \wedge \overline{v}$ have a positive lower bound. Hence there is a $c > 0$ such that $\tilde{\Omega} := \pi_O^*\Omega + c\tilde{\theta}$ is a Kähler form for $\tilde{M}$.

Notice that this proof (the classical one) also works for "1-Kähler" manifolds; indeed, the summand $c\tilde{\theta}$ is d-closed, and hence also "closed" (see Definition 2.3).

On the contrary, in the generic "p-Kähler" case, starting from a "p-Kähler" form Ω on M, $\tilde{\Omega} := \pi_O^*\Omega + c\tilde{\theta}^p$ is not strictly weakly positive on E, because when $p > 1$, in a p-vector $X = v_1 \wedge \cdots \wedge v_p$ it is possible to have, for instance, $v_1 \in (T'_x E)^\perp$, $v_2, \ldots, v_p \in T'_x E$, so that both summands vanish on the strictly strongly positive (p, p)-vector $\sigma_p^{-1} X \wedge \overline{X}$.

When $p > 1$, we can argue as follows. Let us consider the standard Kähler form on $U_{2\epsilon}$, i.e.,

$$\omega = i\partial\overline{\partial}||z||^2 = \frac{i}{2}\sum_j dz_j \wedge d\overline{z}_j.$$

Put $\Theta := \pi_O^*\omega \wedge \tilde{\theta}^{p-1} + \tilde{\theta}^p$.

Claim *The form Θ is a closed real (p, p)-form on $\tilde{M}$, with supp $\Theta \subset \tilde{U}_{2\epsilon}$; moreover, $\Theta \geq 0$ on $\tilde{U}_\epsilon$ and $\Theta > 0$ on E.*

This is a local construction, based only on the geometry of the blow-up π_O.
Now, let Ω be a "p-Kähler" form for M; the following claim is clear.

Claim *$\pi_O^*\Omega$ is a "closed" real (p, p)-form on $\tilde{M}$, with $\pi_O^*\Omega \geq 0$; moreover, for x in the closure of $\tilde{U}_{2\epsilon} - \tilde{U}_\epsilon$, the values of $(\pi_O^*\Omega)_x$ on strictly strongly positive (p, p)-vectors have a positive lower bound.*

Hence, there is a $c > 0$ such that $\tilde{\Omega} := \pi_O^*\Omega + c\Theta$ is a "closed" transverse (p, p)-form on $\tilde{M}$, that is, a "p-Kähler" form for $\tilde{M}$. □

The case 1PL (where $\tilde{\Omega}$ is simply $\pi_O^*\Omega + c\tilde{\theta}$) was proved in [14], 3.1. The authors also proved, using a similar technique, the persistence of the 1PL property for a blow-up π along a submanifold, as in the classical 1K case (3.2 ibid.). Let us give here a simpler proof, which includes all "1-Kähler" cases, by using the fact that π is a projective morphism.

Recall that a blow-up is a projective morphism, hence it is a *Kähler morphism* (in the sense of [16], Definition 4.1; recall also [31], pp. 23–24); this means that there is an open covering $\{U_j\}$ of $\tilde{M}$, and, for every j, smooth functions $p_j : U_j \to \mathbb{C}$ such that:

$\forall y \in M$, the restriction of p_j to $U_j \cap \pi^{-1}(y)$ is strictly plurisubharmonic, and
$p_j - p_k$ is pluriharmonic on $U_j \cap U_k$.

This gives a *relative Kähler form* $\tilde{\beta}$ for π, that is, $\tilde{\beta} := i\partial\overline{\partial}p_j$ on U_j gives a globally defined real closed $(1, 1)$-form, strictly positive on the fibers (but notice that the $(1, 1)$-form $\tilde{\beta}$ may not be ≥ 0 in all directions).

Theorem 3.2 *Let $\pi : \tilde{M} \to M$ be the blow-up of M along a compact submanifold $Y \subset M$; if M is "1-Kähler", then $\tilde{M}$ is "1-Kähler" too.*

Proof Following [16], Lemma 4.4, choose a "1-Kähler" form ω for M; since Y is compact, there is a constant $C > 0$ such that $\tilde{\omega} := \tilde{\beta} + C\pi^*\omega > 0$; since $\tilde{\beta}$ is d-closed, $\tilde{\omega}$ turns out to be "closed". □

Remark Example 7.3 in Sect. 7 proves that Theorem 3.2 cannot hold for a generic $p > 1$; the case $p = n - 1$ is discussed in the next section, for compact manifolds, and in Sect. 6 for non-compact manifolds.

4 Modifications of Compact Manifolds

While we cannot use the previous proof in the case $p > 1$, nor one similar to that of Theorem 3.1, on *compact* manifolds we can also solve the case $p = n-1$ for arbitrary modifications, as done in [6], Theorem 2.4 in case K and in [25] in case S. In fact, following Theorem 2.4, we will not construct "closed" transverse forms, but we will prove that "exact" strongly positive currents must vanish.

Theorem 4.1 *Let M, $\tilde{M}$ be compact n-dimensional manifolds, let $f : \tilde{M} \to M$ be a modification with center Y (an analytic subset of codimension ≥ 2) and exceptional set E (whose $(n-1)$-dimensional irreducible components are $\{E_j\}$). If M is "$(n-1)$-Kähler", then $\tilde{M}$ is "$(n-1)$-Kähler" too.*

Proof Notice that every compact complex n-dimensional manifold is $(n-1)PL$, as we pointed out in Sect. 2.

Let $T \geq 0$ be an "exact" $(1, 1)$-current on $\tilde{M}$, as stated in the Characterization Theorem 2.4. Since f_*T has the same properties on M, we get $f_*T = 0$, which implies that *supp* $T \subseteq E$, because $f|_{\tilde{M}-E}$ is a biholomorphism. More precisely, $T = \sum c_j[E_j]$, $c_j \geq 0$, by the Support Theorem 2.5. Therefore $T = 0$ by the following proposition (which is more general, since the current is not supposed to be positive and M, $\tilde{M}$ are not compact). □

Proposition 4.2 *Let M, $\tilde{M}$ be n-dimensional manifolds, let $f : \tilde{M} \to M$ be a proper modification with compact center $Y \subset M$ (an analytic subset of codimension ≥ 2) and exceptional set E (whose $(n-1)$-dimensional irreducible components are $\{E_j\}$). Let $R = \sum c_j[E_j]$, $c_j \in \mathbb{R}$; R is a closed real $(1, 1)$-current on $\tilde{M}$. The following statements are equivalent:*

(1) R is the component of a boundary, i.e., its class vanishes in $H^{1,1}_{\partial+\overline{\partial}}(\tilde{M})$;
(2) R is a boundary, i.e., its class vanishes in $H^{1,1}_{dR}(\tilde{M}, \mathbb{R})$;
(3) R is $\partial\overline{\partial}$-exact, i.e., its class vanishes in $H^{1,1}_{\partial\overline{\partial}}(\tilde{M})$;
(4) $c_j = 0\ \forall j$, i.e., $R = 0$.

Proof The implications (4) ⇒ (3) ⇒ (2) ⇒ (1) are obvious.

(1) ⇒ (2): see Lemma 8 in [4], where the hypothesis is: R is a closed $(1, 1)$-current on $\tilde{M}$ such that $f_*R = 0$ (notice that if $R = \sum c_j[E_j]$, $c_j \in \mathbb{R}$, then $f_*R = 0$, because codim$Y \geq 2$). We recall here the proof.

Let $R = \partial\overline{S} + \overline{\partial}S$ for some $(1, 0)$-current S; since R is closed, we get $\partial\overline{\partial}S = 0$. Consider ∂S: it is a $\overline{\partial}$-closed $(2, 0)$-current, hence it is a holomorphic 2-form on $\tilde{M}$; the same holds for $\partial(f_*S)$ on M.

Since $\partial(f_*S)$ is smooth and ∂-exact, we can find a $(1, 0)$-form φ and a distribution $t = a + ib$ on M such that $f_*S = \varphi + \partial t = \varphi + \partial a + i\partial b$.

The explanation is the following (see Sect. 2): consider the isomorphism j (induced by the identity) between smooth and non-smooth (i.e., involving currents) cohomology: for instance, j maps the class $[\gamma]$ of a smooth ∂-closed $(2, 0)$-form γ in the cohomology space $H^{2,0}_{\partial}(M)$, to the class $\{\gamma\}$, in the cohomology space of currents (denoted for the moment by $K^{2,0}_{\partial}(M)$). Call α the holomorphic 2-form $\partial(f_*S)$ on M. Since $\partial\alpha = 0$, $[\alpha] \in H^{2,0}_{\partial}(M)$; but by definition, $\{\alpha\} = 0 \in K^{2,0}_{\partial}(M)$, thus $0 = [\alpha] \in H^{2,0}_{\partial}(M)$, so that $\alpha = \partial\mu$ for some smooth 1-form μ. Therefore $\partial(f_*S - \mu) = 0$, hence $\{f_*S - \mu\} \in K^{1,0}_{\partial}(M) \simeq H^{1,0}_{\partial}(M)$, that is, there is a smooth form ν such that $\{f_*S - \mu\} = \{\nu\}$, i.e., there is a distribution t such that $f_*S - \mu = \nu + \partial t$, as stated.

Now we use $f_*R = 0$ as follows:

$$0 = f_*(\partial\overline{S} + \overline{\partial}S) = \partial(\overline{\varphi} + \overline{\partial a} - i\overline{\partial b}) + \overline{\partial}(\varphi + \partial a + i\partial b) = \partial\overline{\varphi} + \overline{\partial}\varphi - 2i\partial\overline{\partial}b.$$

Thus $\partial\overline{\partial}b$ is smooth, hence also b is smooth, and we can pull it back to $\tilde{M}$.

Define $s := S - f^*(\varphi + i\partial b)$; we get

$$\begin{aligned}&\overline{\partial}s + \partial\overline{s} = \overline{\partial}(S - f^*(\varphi + i\partial b)) + \partial(\overline{S} - f^*(\overline{\varphi} - i\overline{\partial}b)) =\\&\overline{\partial}S + \partial\overline{S} - f^*(\overline{\partial}\varphi - i\partial\overline{\partial}b) - f^*(\partial\overline{\varphi} - i\partial\overline{\partial}b)) =\\&R - f^*(\overline{\partial}\varphi + \partial\overline{\varphi} - 2i\partial\overline{\partial}b) = R;\end{aligned}$$

moreover,

$$\partial s = \partial S - \partial f^*\varphi = \partial S - f^*(\partial(f_*S));$$

both summands are holomorphic 2-forms on $\tilde{M}$, and they coincide outside the exceptional set E: therefore they coincide, hence $\partial s = 0$. Thus $R = d(s + \overline{s})$ is a boundary.

(2) $\Rightarrow$ (3): Let $R = dQ = \partial\overline{S} + \overline{\partial}S$, for a real 1-current $Q = Q^{0,1} + Q^{1,0} = \overline{S} + S$, where S is a ∂-closed $(1,0)$-current. As before, $0 = f_*R = d(f_*Q)$, so that we can choose a smooth representative of the cohomology class of the d-closed 1-current f_*Q on M; that is, $f_*Q = \varphi + da$, where φ is a smooth closed 1-form and a is a distribution on M.

Let $q := Q - f^*\varphi$; it holds that

$$dq = dQ - f^*d\varphi = dQ = R$$

and, as regards the (0, 1)-part,

$$f_*q^{0,1} = f_*\left(Q^{0,1} - f^*\varphi^{0,1}\right) = f_*Q^{0,1} - \varphi^{0,1} = \overline{\partial}a.$$

Since R is a (1, 1)-current, $\overline{\partial}q^{0,1} = 0$, so it represents a class in $H^{0,1}_{\overline{\partial}}(\tilde{M}) \simeq H^{0,1}_{\overline{\partial}}(M)$ (a classical result), but this class vanishes in M, because $f_*q^{0,1} = \overline{\partial}a$; thus it vanishes in $\tilde{M}$, i.e., $q^{0,1} = \overline{\partial}b$.

Hence

$$R = dq = \partial q^{0,1} + \overline{\partial}\overline{q^{0,1}} = \partial\overline{\partial}(b - \overline{b}).$$

(3) $\Rightarrow$ (4): Suppose $R = i\partial\overline{\partial}a$: since $f_*R = 0$, f_*a is pluriharmonic on M (there is a smooth pluriharmonic function h such that $f_*a = h$ a.e.). Hence $f^*h = h \circ f$ is pluriharmonic on $\tilde{M}$, so that $R = i\partial\overline{\partial}(a - f^*h)$, where the distribution $a - f^*h$ is supported on E, because $f|_{\tilde{M}-E}$ is a biholomorphism.

Let x be a smooth point, $x \in E$ (as a matter of fact, $x \in E_k$ for some k); choose a neighborhood U of x with coordinates $\{w_j\}$ such that, in U, $R = c_k[E_k] = ic_k\pi^{-1}\partial\overline{\partial}log||w_n||$; thus in U the distribution

$$ic_k\pi^{-1}log||w_n|| - (a - f^*h)$$

is pluriharmonic, hence smooth. This implies that $a - f^*h$, which is a distribution supported on E, vanishes in U. We conclude in this manner that $R = \sum c_j[E_j] = i\partial\overline{\partial}(a - f^*h) = 0$. □

More than that, we can prove that the class of compact "$(n-1)$-Kähler" manifolds is closed with respect to modifications.

Theorem 4.3 *Let M, $\tilde{M}$ be compact n-dimensional manifolds, let $f : \tilde{M} \to M$ be a modification. If $\tilde{M}$ is "$(n-1)$-Kähler", then M is "$(n-1)$-Kähler" too.*

Proof The case $(n-1)PL$ is obvious. The case $(n-1)K$ is proved in [8], the case $(n-1)S$ is proved by Popovici in [25]; the proofs are similar, nevertheless, as the author says in the Introduction, the arguments are considerably simplified by the fact that one can handle the "pull-back" of d-closed positive $(1,1)$-currents by their local potentials. Let us consider here the WK-case, to complete the proof of the theorem.

Take a $(1,1)$-current $T \geq 0$ on M, such that $dT = 0$ and $T = \partial\overline{S} + \overline{\partial}S$. Consider the following result:

Theorem 4.4 (Theorem 3 in [8]) *Let M, $\tilde{M}$ be complex manifolds, and let $f : \tilde{M} \to M$ be a proper modification. Let T be a positive $\partial\overline{\partial}$-closed $(1,1)$-current on M. Then there is a unique positive $\partial\overline{\partial}$-closed $(1,1)$-current $\tilde{T}$ on $\tilde{M}$ such that $f_*\tilde{T} = T$ and $\tilde{T} \in f^*\{T\} \in H^{1,1}_{\partial+\overline{\partial}}(\tilde{M}, \mathbb{R})$.*

Looking carefully through the details of the proof (see also Theorem 3.9 and Proposition 3.10 in [7]), it is not hard to notice that, when T is d-closed, $\tilde{T}$ becomes d-closed too (in the estimates, this is the "classical case").

Thus, in our situation, $\tilde{T}$ is a closed positive $(1,1)$-current on $\tilde{M}$ such that $\tilde{T} \in f^*\{T\} = 0 \in H^{1,1}_{\partial+\overline{\partial}}(\tilde{M}, \mathbb{R})$: this means that $\tilde{T} = \partial\overline{s} + \overline{\partial}s = 0$, since $\tilde{M}$ is "$(n-1)$-Kähler".

Therefore $T = f_*\tilde{T} = 0$. □

Example 7.3 shows that similar results cannot hold for a generic p, also when the exceptional set $E \subset \tilde{M}$ is supposed to be pK as the manifold M, and the modification is simply a blow-up. Hence, to study when a generalized p-Kähler property goes back from M to $\tilde{M}$, we must add some hypothesis on E, as in the following result.

Since we shall use only here forms and currents on a (singular) analytic subset (that is, the exceptional set E), we refer to [10], pp. 575–577 for definitions and details; here, for a (p,p)-form $\tilde{\Omega}$ on $\tilde{M}$, we indicate by $i_E^*\tilde{\Omega} > 0$ the fact that, for every strongly positive current $t \neq 0$ on E, it holds that $((i_E)_*t, \tilde{\Omega}) > 0$.

Proposition 4.5 *Let M, $\tilde{M}$ be compact n-dimensional manifolds, let $f : \tilde{M} \to M$ be a modification with center $Y \subset M$ and exceptional set E (call $i_E : E \to \tilde{M}$ the inclusion); let $1 \leq p < n-1$ and suppose M is "p-Kähler". If there is a (p,p)-form $\tilde{\Omega}$ on $\tilde{M}$ such that $i_E^*\tilde{\Omega} > 0$ and $f_*(d\tilde{\Omega})$ (or $f_*(i\partial\overline{\partial}\tilde{\Omega})$ in case PL) is a smooth form, then $\tilde{M}$ is "p-Kähler" too.*

Proof Let $T \geq 0$, $T \neq 0$, be an "exact" current of bidimension (p, p) on $\tilde{M}$. Since f_*T has the same properties on M, we get $f_*T = 0$, which implies that supp $T \subseteq E$. By Theorem 1.24 in [10], there is a strongly positive current t on E such that $T = (i_E)_*t$; thus $(T, \tilde{\Omega}) = ((i_E)_*t, \tilde{\Omega}) > 0$, when $t \neq 0$.

Arguing as in Proposition 4.2, since $f_*(d\tilde{\Omega})$ is smooth and exact, we have a (p, p)-form Ψ on M such that $f_*(d\tilde{\Omega}) = d\Psi$; moreover, $f^*(d\Psi) = f^*(f_*(d\tilde{\Omega})) = d\tilde{\Omega}$, since they are smooth forms, which coincide on $\tilde{M} - E$. Therefore, when $T = dS$, we get a contradiction:

$$(T, \tilde{\Omega}) = (dS, \tilde{\Omega}) = (S, d\tilde{\Omega}) = (S, f^*(d\Psi)) = (dS, f^*\Psi) = (f_*T, \Psi) = 0.$$

When $T = \partial\overline{S} + \overline{\partial}S$, the proof is similar, since by dimensional reasons, $(\partial\overline{S} + \overline{\partial}S, \tilde{\Omega}) = (d(S + \overline{S}), \tilde{\Omega})$.

In the pPL case, we have only to replace the operator d by the operator $\partial\overline{\partial}$. □

5 Link Between "p-Kähler" Forms on M and $\tilde{M}$

Notice that, using currents, in Theorem 4.1 we lose the connection between metrics on M and $\tilde{M}$: nevertheless, we can prove the following link:

Proposition 5.1 *Let M, $\tilde{M}$ be compact n-dimensional manifolds, let $f : \tilde{M} \to M$ be a modification. For every "$(n-1)$-Kähler" metric h with form ω on M, there is an "$(n-1)$-Kähler" metric $\tilde{h}$ with form $\tilde{\omega}$ on $\tilde{M}$ such that ω^{n-1} and $f_*\tilde{\omega}^{n-1}$ are in the same (relevant) cohomology class.*

In the case K, this is Corollary 4.9 in [7]; we consider here a more general context, namely, that of "p-Kähler" manifolds with $p > \dim Y$, not necessarily compact.

Theorem 5.2 *Let $f : \tilde{M} \to M$ be a proper modification with a compact center $Y \subset M$ and exceptional set E. Suppose $\tilde{M}$ and M are "p-Kähler" manifolds, with $p > dim Y$, having "p-Kähler" forms $\tilde{\Omega}$ and Ω. Then there is a "p-Kähler" form Γ on $\tilde{M}$ such that $f_*\Gamma$ is "cohomologous" to Ω.*

Here $f_*\Gamma$ is "cohomologous" to Ω means: $\{f_*\Gamma\} = \{\Omega\} \in H^{p,p}_{\partial\overline{\partial}}(M)$ in the case K, $\{f_*\Gamma\} = \{\Omega\} \in H^{p,p}_{\partial+\overline{\partial}}(M)$ in the cases WK, S, PL. The case pK, with M and $\tilde{M}$ compact manifolds, is proved in [7], Theorem 4.8; as regards the case $(n-1)S$, see Theorem 1.2 in [32]; we will prove here the general case.

Proof Our goal is to get, as in Theorem 3.1, a positive constant c such that $\Gamma := f^*\Omega + c\Theta$ is the required form, where Θ is null-cohomologous and is obtained by changing $f_*\tilde{\Omega}$.

Let us recall the following classical result (see, for instance, [29] p. 251):

Remark Let Y be an s-dimensional compact analytic subset of M; Y has a fundamental system of neighborhoods $\{U\}$ such that $H^q_{dR}(U, \mathbb{R}) = 0$ for $q > 2s$, and, for every coherent sheaf $\mathcal{F}$, $H^q(U, \mathcal{F}) = 0$ for $q > s$.

In [9] we studied the case of 1-convex manifolds, where the cohomology groups $H^q(U, \mathcal{F})$ are finite dimensional when $q > 0$. We proved there the following result:

Theorem 5.3 ([9], Theorem 2.4) *Let M be a complex manifold, and let $\mathcal{O}^k$ be the sheaf of germs of holomorphic k-forms on M. Suppose $dim H^j(M, \mathcal{O}^k) < \infty\ \forall k \geq 0,\ \forall j \geq s$. Then the cohomology groups $H^{p,p}_{\partial\overline{\partial}}(M)$ and $H^{p,p}_{\partial+\overline{\partial}}(M)$ are Hausdorff topological vector spaces for every $p \geq s$.*

Adapting its proof, which is based on an accurate analysis of exact sequences of sheaves and cohomology groups, we get in our situation (where the cohomology groups vanish):

Claim *Let Y be an s-dimensional compact analytic subset of M; Y has a fundamental system of neighborhoods $\{U\}$ such that $H^q_{dR}(U, \mathbb{R}) = 0$ for $q > 2s$, and, for every coherent sheaf $\mathcal{F}$, $H^q(U, \mathcal{F}) = 0$ for $q > s$. Thus Y has a fundamental system of neighborhoods $\{U\}$ such that $H^{p,p}_{\partial\overline{\partial}}(U) = 0,\ H^{p,p}_{\partial+\overline{\partial}}(U) = 0$ for $p > s$.*

To give a hint of the first step ($s = 0$, $p = 1$) of the proof of this Claim, let us consider $\mathcal{H}$, the sheaf of germs of real pluriharmonic functions, with the following well-known exact sequences of sheaves (see [9], p. 260):

$$0 \to \mathbb{R} \stackrel{i}{\to} \mathcal{O} \stackrel{Re}{\to} \mathcal{H} \to 0$$

where $i(c) = ic,\ c \in \mathbb{R}$, and $Ref(z) = f(z) + \overline{f(z)}$; and

$$0 \to \mathcal{H} \stackrel{j}{\to} \mathcal{E}^{0,0}_{\mathbb{R}} \stackrel{i\partial\overline{\partial}}{\to} \mathcal{E}^{1,1}_{\mathbb{R}} \stackrel{\partial+\overline{\partial}}{\to} (\mathcal{E}^{2,1} \oplus \mathcal{E}^{1,2})_{\mathbb{R}} \to \cdots$$

where j is the standard inclusion.

From the second one we can compute $H^{1,1}_{\partial\overline{\partial}}(U)$, so that $H^{1,1}_{\partial\overline{\partial}}(U) \simeq H^1(U, \mathcal{H})$. Indeed, we get a short exact sequence

$$0 \to \mathcal{H} \to \mathcal{E}^{0,0}_{\mathbb{R}} \to Ker(\partial + \overline{\partial}) \to 0$$

and the associated exact sequence

$$0 \to H^0(U, \mathcal{H}) \to H^0\left(U, \mathcal{E}^{0,0}_{\mathbb{R}}\right) \to H^0\left(U, Ker(\partial + \overline{\partial})\right) \to H^1(U, \mathcal{H}) \to 0.$$

Thus

$$H^1(U, \mathcal{H}) \simeq \frac{H^0(U, Ker(\partial + \overline{\partial}))}{H^0(U, Im(i\partial\overline{\partial}))} = H^{1,1}_{\partial\overline{\partial}}(U).$$

From the first exact sequence, we get

$$\cdots \to H^1(U, \mathcal{O}) \to H^1(U, \mathcal{H}) \to H^2(U, \mathbb{R}) \to \cdots.$$

Thus, by the previous Remark, we get $H^{1,1}_{\partial\overline{\partial}}(U) \simeq H^1(U, \mathcal{H}) = 0$.

Let us turn back to the proof of Theorem 5.2; choose U as in the previous Claim, and consider $f_*\tilde{\Omega}$. While in the case pK, $f_*\tilde{\Omega}$ is closed, so that by $H^{p,p}_{\partial\overline{\partial}}(U) = 0$ we get $f_*\tilde{\Omega} = i\partial\overline{\partial}R$ on U, in the other cases it holds that $\partial\overline{\partial}f_*\tilde{\Omega} = 0$, so that thanks to $H^{p,p}_{\partial+\overline{\partial}}(U) = 0$ we get $f_*\tilde{\Omega} = \partial\overline{S} + \overline{\partial}S$, for some $(p, p-1)$-current S on U.

Recall that cohomology classes can be represented by currents or by forms (see also the proof of Proposition 4.2): thus, since $f_*\tilde{\Omega}$ is smooth on $M - Y$, we get on $U - Y$:

(a) $f_*\tilde{\Omega} = i\partial\overline{\partial}\alpha$ for some real $(p-1, p-1)$-form α on $U - Y$ in the case pK, and
(b) $f_*\tilde{\Omega} = \partial\overline{\beta} + \overline{\partial}\beta$ for some $(p, p-1)$-form β on $U - Y$ in the cases pWK, pS and pPL.

Claim *In the previous notation, on $U - Y$ we get, respectively:*

(a) $i\partial\overline{\partial}(R-\alpha) = 0$, thus $R - \alpha = \gamma + \partial\overline{C} + \overline{\partial}C$, where γ is a real $\partial\overline{\partial}$-closed form and C is a $(p-1, p-2)$-current; when $p = 1$, $R - \alpha = \gamma$ are smooth functions;
(b) $\partial\overline{(S-\beta)} + \overline{\partial}(S-\beta) = 0$, thus $S - \beta = \gamma + \partial A + \overline{\partial}B$, where γ is a $(p, p-1)$-form such that $\partial\overline{\gamma} + \overline{\partial}\gamma = 0$, A is a real $(p-1, p-1)$-current, B is a $(p, p-2)$-current.
(c) when $p = 1$, $\partial\overline{(S-\beta)} + \overline{\partial}(S-\beta) = 0$, thus $S - \beta = \gamma + \alpha + \partial h$, where γ is a $(1,0)$-form such that $\partial\overline{\gamma} + \overline{\partial}\gamma = 0$, α is a holomorphic 1-form, h is a real distribution.

Proof of the Claim In the case (a), γ is a smooth representative of the class $\{R - \alpha\} \in H^{p-1,p-1}_{\partial+\overline{\partial}}(U - Y)$; when $p = 1$, $R - \alpha$ itself is smooth.

In the case (b), for $p > 1$, the proof is more involved: we can use exact sequences of sheaves and their cohomology groups as done in [9] (see the proof of Proposition 2.2 there). In particular, let us consider

$$\dots(\mathcal{E}^{p,p-2} \oplus \mathcal{E}^{p-1,p-1} \oplus \mathcal{E}^{p-2,p})_{\mathbb{R}} \overset{\sigma_{2p-2}}{\to} (\mathcal{E}^{p,p-1} \oplus \mathcal{E}^{p-1,p})_{\mathbb{R}} \overset{\sigma_{2p-1}}{\to} \mathcal{E}^{p,p}_{\mathbb{R}} \overset{\sigma_{2p}}{\to} \mathcal{E}^{p+1,p+1}_{\mathbb{R}} \dots$$

where the maps are, respectively,

$\sigma_{2p-2}(\zeta, \eta, \overline{\zeta}) = (\overline{\partial}\zeta + \partial\eta, \overline{\partial}\eta + \partial\overline{\zeta})$, $\sigma_{2p-1}(\varphi, \overline{\varphi}) = (\overline{\partial}\varphi + \partial\overline{\varphi})$, $\sigma_{2p} = i\partial\overline{\partial}$.

Notice that $H^{p-1,p-1}_{\partial+\overline{\partial}}(U - Y)$ is given by $\frac{Ker(\sigma_{2p})}{Im(\sigma_{2p-1})}$ on $U - Y$, but here we need $\frac{Ker(\sigma_{2p-1})}{Im(\sigma_{2p-2})}$ on $U - Y$.

Since $\partial\overline{(S-\beta)} + \overline{\partial}(S-\beta) = 0$, i.e., $\sigma_{2p-1}(S-\beta, \overline{S-\beta}) = 0$, it represents a class in $\frac{Ker(\sigma_{2p-1})}{Im(\sigma_{2p-2})}$ on $U - Y$. Choose a smooth representative of this class: this means precisely $S - \beta = \gamma + \partial A + \overline{\partial}B$, as stated in the Claim.

In the case c), when $p = 1$, the exact sequence of sheaves is the following

$$\dots\left(\mathcal{O}^1 \oplus \mathcal{E}^{0,0} \oplus \overline{\mathcal{O}}^1\right)_{\mathbb{R}} \overset{\sigma_0}{\to} \left(\mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}\right)_{\mathbb{R}} \overset{\sigma_1}{\to} \mathcal{E}^{1,1}_{\mathbb{R}} \overset{\sigma_2}{\to} \mathcal{E}^{2,2}_{\mathbb{R}} \dots$$

where the maps are, respectively,

$\sigma_0(\alpha, h, \overline{\alpha}) = (\alpha + \partial h, \overline{\partial}h + \overline{\alpha})$, $\sigma_1(\varphi, \overline{\varphi}) = (\overline{\partial}\varphi + \partial\overline{\varphi})$, $\sigma_2 = i\partial\overline{\partial}$.

Since $\partial\overline{(S-\beta)} + \overline{\partial}(S-\beta) = 0$, i.e., $\sigma_1(S-\beta, \overline{S-\beta}) = 0$, it represents a class in $\frac{Ker(\sigma_1)}{Im(\sigma_0)}$ on $U - Y$. Choose a smooth representative of this class: this means precisely $S - \beta = \gamma + \alpha + \partial h$, as stated in the Claim.

Going back to the proof of Theorem 5.2, choose a neighborhood $W \subset\subset U$ of Y, and take a cut-off function $\chi \in C_0^\infty(U)$, $\chi = 1$ on W. Define:

$D := \chi(\alpha + \gamma) + \partial(\chi \overline{C}) + \overline{\partial}(\chi C)$,

$F := \chi(\beta + \gamma) + \partial(\chi A) + \overline{\partial}(\chi B)$ when $p > 1$, and

$F := \chi(\beta + \gamma + \alpha) + \partial(\chi h)$ when $p = 1$.

D and F are currents on $M - Y$; moreover, it is easy to check that $i\partial\overline{\partial}D$ and $\partial\overline{F} + \overline{\partial}F$ are smooth on $M - Y$, so that we can pull them back to $\tilde{M} - E$ (let us denote by g the restriction of f to $\tilde{M} - E$). Thus $\Theta := g^*(i\partial\overline{\partial}D)$ and $\Theta' := g^*(\partial\overline{F} + \overline{\partial}F)$ are, respectively, (p, p)-forms on $\tilde{M} - E$, which coincide with $\tilde{\Omega}$ on $f^{-1}(W) - E$.

So they extend to the whole of $\tilde{M}$: note that they are supported on $f^{-1}(U)$ and transverse on $f^{-1}(W)$.

Thus we can pick $c > 0$ such that $\Gamma := f^*\Omega + c\Theta$ or $\Gamma' := f^*\Omega + c\Theta'$ are transverse forms on $\tilde{M}$. Γ and Γ' are "closed" because Ω is "closed", and

(a) Θ is $i\partial\overline{\partial}$-exact on $\tilde{M} - E$ and coincides with $\tilde{\Omega}$ (which is "closed") near E;

(b) Θ' is $(\partial + \overline{\partial})$-exact on $\tilde{M} - E$ and coincides with $\tilde{\Omega}$ (which is "closed") near E.

Moreover, in the first case $f_*\Gamma - \Omega = i\partial\overline{\partial}(cD)$, and in the other case $f_*\Gamma' - \Omega' = \partial(\overline{cF}) + \overline{\partial}(cF)$. □

6 Currents in the Non-compact Case

In the non-compact case, we cannot use the classical characterization of Kähler manifolds by currents, which has been introduced by Sullivan [28] and by Harvey and Lawson [21]; hence we have no results about a generic modification. Nevertheless, in [9] we studied 1-convex manifolds (which are not compact but have a specific compact "soul") by positive currents. This technique can be used to get a partial result on proper modifications. Thus we consider the following definition, where M is a complex n-dimensional manifold.

Definition 6.1 Let Y be a compact analytic subset of M; M is said to be *locally* "p-Kähler"*with respect to* Y if every neighborhood U of Y, $U \subset\subset M$, is "p-Kähler", in the sense that there is a real closed (p, p)-form Ω on M such that $\Omega > 0$ on the compact set $\overline{U}$.

Our aim is to prove:

Theorem 6.2 *Let M, $\tilde{M}$ be n-dimensional manifolds, let $f : \tilde{M} \to M$ be a proper modification with compact center Y and exceptional set E (whose $(n-1)$-dimensional irreducible components are $\{E_j\}$). Suppose $\dim H^j(\tilde{M}, \mathcal{O}^r) < \infty\ \forall r \geq 0,\ \forall j \geq 1$. If M is locally $(n-1)K$ with respect to Y, then $\tilde{M}$ is locally $(n-1)K$ with respect to E.*

Let us recall the notation and some results from [9], which we shall use in the proof. For every n-dimensional manifold X, $SP_p(X)_c$ denotes the closed convex cone of strongly positive currents of bidimension (p, p) (or bidegree (k, k), $p + k = n$) and compact support, while $B_p(X)_c$ is the space of currents of bidimension (p, p) and compact support, which are (p, p)-components of a compactly supported boundary current, that is, the class of the current vanishes in

$$H^{k,k}_{\partial+\overline{\partial}}(X,\mathbb{R})_c = \frac{\{T \in \mathcal{E}'_{p,p}(X)_{\mathbb{R}}; i\partial\overline{\partial}T = 0\}}{\{\partial\overline{S} + \overline{\partial}S;\ S \in \mathcal{E}'_{p,p+1}(X)\}}.$$

In the non-compact case, it is not guaranteed that the operators we need are topological homomorphisms (so that the orthogonal space to the Kernel coincides with the Image), but to get this fact it suffices to require a mild cohomological condition, as stated in the next result:

Proposition 6.3 (Corollary 2.5 in [9]) *Let X be a complex manifold such that $dim H^j(X,\mathcal{O}^r) < \infty\ \forall r \geq 0,\ \forall j \geq 1$. Then $d_p := d : \mathcal{E}^{p,p}_{\mathbb{R}}(X) \to (\mathcal{E}^{p+1,p} \oplus \mathcal{E}^{p,p+1})_{\mathbb{R}}(X)$ and $\partial\overline{\partial}_p := \partial\overline{\partial} : \mathcal{E}^{p-1,p-1}_{\mathbb{R}}(X) \to \mathcal{E}^{p,p}_{\mathbb{R}}(X)$ are topological homomorphisms for every $p \geq 1$.*

So we got:

Theorem 6.4 (see Theorem 3.2 in [9]) *Let X be a complex manifold, $dim X = n$, and K a compact subset of X; let $1 \leq p \leq n-1$ and suppose $d_p := d : \mathcal{E}^{p,p}_{\mathbb{R}}(X) \to (\mathcal{E}^{p+1,p} \oplus \mathcal{E}^{p,p+1})_{\mathbb{R}}(X)$ is a topological homomorphism. Then:*

there is no current $T \neq 0$, $T \in SP_p(X)_c \cap B_p(X)_c$, $suppT \subseteq K \iff$ there is a real closed (p,p)-form Ω on M such that $\Omega > 0$ on K.

Now we can prove Theorem 6.2.

Proof Fix a neighborhood U of E in $\tilde{M}$, $U \subset\subset \tilde{M}$, and let T be a *bad* current, i.e., $T \in SP_{n-1}(\tilde{M})_c \cap B_{n-1}(\tilde{M})_c$, supp $T \subseteq K := \overline{U}$. Thus f_*T is a *bad* current on M supported in $f(K)$, which is a compact neighborhood of Y: since M is locally $(n-1)$-Kähler with respect to Y, we get $f_*T = 0$, so that supp $T \subseteq E$.

By the Support Theorem 2.5, T is closed (in fact, $T = \sum c_j[E_j],\ c_j \geq 0$), and moreover $T = \partial\overline{S} + \overline{\partial}S$ for some compactly supported $(1,0)$-current S, so that we get $\partial\overline{\partial}S = 0$.
Consider ∂S: it is a $\overline{\partial}$-closed $(2,0)$-current, hence it is a holomorphic 2-form with compact support: therefore, $\partial S = 0$ and $T = d(S + \overline{S})$ is d-exact. But no $(n-1)$-dimensional component of E is null-homologous, by the structure of the homology of $\tilde{M}$: hence $T = \sum c_j[E_j] = 0$. □

The other "p-Kähler" cases are not known.

Notice that when $i\partial\overline{\partial}_{p+1} : \mathcal{E}^{p,p}_{\mathbb{R}} \to \mathcal{E}^{p+1,p+1}_{\mathbb{R}}$ is a topological homomorphism (which is true in our hypothesis by Proposition 6.3), then we have a similar characterization theorem (see [9], Remark 3.4):

Proposition 6.5 *Let X be a complex manifold, $dim X = n$, and K a compact subset of X; let $1 \leq p \leq n-1$ and suppose $i\partial\overline{\partial}_{p+1} : \mathcal{E}^{p,p}_{\mathbb{R}} \to \mathcal{E}^{p+1,p+1}_{\mathbb{R}}$ is a topological homomorphism. Then:*

there is no current $T \neq 0$, $T \in SP_p(X)_c \cap (Im(i\partial\overline{\partial}_{p+1}))_c$, $suppT \subseteq K \iff$ there is a real (p,p)-form Ω on M such that $i\partial\overline{\partial}\Omega = 0$ and $\Omega > 0$ on K.

But when $p = n-1$, $T \in SP_{n-1}(X)_c \cap (Im(i\partial\overline{\partial}_n))_c$ means that $T = i\partial\overline{\partial}g$, with g a plurisubharmonic function with compact support: so g is a constant, and $T = 0$. This means that every n-dimensional complex manifold is $(n-1)$PL with respect to its compact subsets (as expected).

7 Examples and Remarks

7.1 Hironaka's manifold X (see [19], p. 444 or [5]) is given by a modification $f : X \to \mathbb{P}_3$, where the center Y is a plane curve with a node. It is a Moishezon manifold, containing a null-homologous curve. Thus it is not "1-Kähler". X is a balanced manifold (see [5] or [8]) so that it is "$(n-1)$-Kähler".

We can also consider a modification given as follows: take $\pi_O : \tilde{\mathbb{P}}_3 \to \mathbb{P}_3$, the blow-up of $\mathbb{P}_3$ at a point O, and take its exceptional divisor $E \simeq \mathbb{P}_2$. In this $\mathbb{P}_2 \subset \tilde{\mathbb{P}}_3$, take a plane curve Y with a node, for instance, that given in coordinates by the equation $z_2^2 = z_1^2 + z_1^3$, where $z_3 = 0$ is the local equation of $\mathbb{P}_2 \subset \tilde{\mathbb{P}}_3$.

Take the modification $f : \tilde{M} \to \tilde{\mathbb{P}}_3$ of center Y like that of Hironaka's example, i.e., in a little ball near the origin ($z_1 = z_2 = z_3 = 0$) blow up first one branch of Y, then the other; outside the origin, just blow up Y. Then glue together, to obtain the modification $f : \tilde{M} \to \tilde{\mathbb{P}}_3$. As with Hironaka's manifold, $\tilde{M}$ is not Kähler.

Finally, consider $\pi_O \circ f : \tilde{M} \to \mathbb{P}_3$. It is a modification of a projective manifold, whose center is a point: but the resulting compact threefold is not "1-Kähler".

7.2 In [6] we build an example to show that, even in the case of modifications, we can sometime pull-back "p-Kähler" properties for $p > 1$. Indeed, we consider a smooth modification $\tilde{X}$ of $\mathbb{P}_5$, where the center Y is a surface with a singularity; the singular fiber has two irreducible components, one of which is biholomorphic to $\mathbb{P}_2$ and the other is a holomorphic fiber bundle over $\mathbb{P}_1$ with $\mathbb{P}_2$ as fiber. We show that $\tilde{X}$ is not Kähler, because it contains a copy of Hironaka's manifold, but it is "p-Kähler" for every $p > 1$.

Recall that Hironaka's threefold X, and also the compact manifold $\tilde{X}$ just described, are "p-Kähler" for every $p > 1$ and belong to Fujiki's class $\mathcal{C}$.

But this cannot be the general case: for instance, $M := X \times \mathbb{P}_{n-3} \in \mathcal{C}$, but it cannot be "$(n-2)$-Kähler", otherwise, using the projection p_X onto the first factor, X would be "1-Kähler". In fact, the projection p_X is a holomorphic submersion, so that the following result applies (see, for instance, [2], Proposition 3.1):

Proposition 7.1 *Let M be a compact m-dimensional manifold, and let $\pi : M \to N$ be a proper holomorphic submersion with p-dimensional fibers. If M is q-Kähler for $m > q > p$, then N is $(q-p)$-Kähler.*

7.3 On the contrary, we give here an example that shows that we cannot *always* pull-back "p-Kähler" properties by blowing up, even in the compact case. We use the class of manifolds we constructed in [3], namely, the compact nilmanifolds $\eta\beta_{2n+1}$, $n \geq 1$: let us recall the definition.

Let G be the following subgroup of $GL(n+2, \mathbb{C})$:

$$G := \{A \in GL(n+2, \mathbb{C})/A = \begin{pmatrix} 1 & X & z \\ 0 & I_n & Y \\ 0 & 0 & 1 \end{pmatrix}, z \in \mathbb{C}, X, Y \in \mathbb{C}^n\},$$

and let Γ be the subgroup of G given by matrices with entries in $\mathbb{Z}[i]$. Γ is a discrete subgroup and the homogeneous manifold $\eta\beta_{2n+1} := G/\Gamma$ becomes a holomorphically parallelizable compact connected complex nilmanifold of dimension $2n+1$ (for $n = 1$, $\eta\beta_3$ is nothing but the Iwasawa manifold I_3). The standard basis for holomorphic 1-forms on $\eta\beta_{2n+1}$ is $\{\varphi_1, \dots, \varphi_{2n+1}\}$; the φ_j are all closed, except $d\varphi_{2n+1} = -\varphi_1 \wedge \varphi_2 - \cdots - \varphi_{2n-1} \wedge \varphi_{2n}$.

Recall the following results:

Theorem 7.2 (see Theorem 3.2 and Theorem 4.2 in [3])

(1) For $\eta\beta_{2n+1}$, as for all holomorphically parallelizable manifolds, for a fixed p, all "p-Kähler" conditions are equivalent.
(2) The manifold $\eta\beta_{2n+1}$ is not pK for $1 \le p \le n$ and is pK for $n+1 \le p \le 2n$.

To build our example, let us consider $M = \eta\beta_7$, $Y = \eta\beta_3 = I_3$ as a submanifold of M (in an obvious way; see, for instance, (4.4) in [3]). In particular, M is 4K, Y is 2K but not Kähler. Consider $\pi : \tilde{M} \to M$, the blow-up of M along Y; if $\tilde{M}$ were 4K too, then the exceptional set E would also be 4K, but by definition π induces a holomorphic submersion from E to Y with 3-dimensional fibers. Thus by Proposition 7.1, Y would be Kähler.

7.4 Taking into account these examples, let us collect what we have got until now for the case of a modification $f : \tilde{M} \to M$ of a *compact* "1-Kähler" manifold M:

a) $\tilde{M}$ is obviously $(n-1)$PL.
b) If M is Kähler (i.e., 1K), then it is regular (in the sense of Varouchas, that is, it satisfies the $\partial\overline{\partial}$-Lemma; see [11,30]), so that also $\tilde{M}$ is regular, which implies that it is $(n-1)$WK and $(n-1)$S by the following result stated in [1]: *On a regular manifold,* $\forall\, p$, $pWK = pS = pPL$. *Thus, every regular manifold is* $(n-1)WK$, *since* $(n-1)WK = (n-1)PL$.
   As stated in Theorem 4.1, when M is 1K, $\tilde{M}$ is also $(n-1)$K (a direct proof was given in [6]). Nevertheless, $\tilde{M}$ may not be "1-Kähler", as examples in Subsection 7.1 show, also when the center is only a point. But Example 7.2 shows that $\tilde{M}$ can be "p-Kähler" for every $p > 1$.
c) If M is "1-Kähler", then in case K and S, M is also "$(n-1)$-Kähler", so that $\tilde{M}$ is $(n-1)$K or, respectively, $(n-1)$S (see, for instance, [2]). We do not know in general if, when M is 1WK, then $\tilde{M}$ is $(n-1)$WK; this is true for a wide class of manifolds, for instance when $H^{2,0}(M) = 0$, because in this case $(n-1)$WK = $(n-1)$S (see [1]).

7.5 We recall here an example proposed by Yachou [33].

Take $G = SL(2, \mathbb{C})$, $\Gamma = SL(2, \mathbb{Z})$, and consider the holomorphic 1-forms η, α, β on $M := G/\Gamma$ induced by the standard basis for $\mathfrak{g}^*$: it holds that

$$d\alpha = -2\eta \wedge \alpha, \quad d\beta = 2\eta \wedge \beta, \quad d\eta = \alpha \wedge \beta.$$

The standard fundamental form, given by $\omega = \frac{i}{2}(\alpha \wedge \overline{\alpha} + \beta \wedge \overline{\beta} + \eta \wedge \overline{\eta})$, satisfies $d\omega^2 = 0$, so that ω^2 is a balanced form: but it is exact, since

$$\omega^2 = d\left(\frac{1}{16}\alpha \wedge d\overline{\alpha} + \frac{1}{16}\beta \wedge d\overline{\beta} + \frac{1}{4}\eta \wedge d\overline{\eta}\right).$$

7.6 Let us end with a particular question, related to Example 7.5, i.e., the fact that, on compact Kähler manifolds, $\int_M \omega^n = \text{vol } M > 0$, so that ω is not "exact", while a 2K form can be exact, as seen in Subsection 7.5 (see also the Introduction of [15]). Notice that, when M is not compact, a "p-Kähler" form can be exact: this is the case, for instance, of p-complete manifolds (see [9], Proposition 4.4).

Suppose M, $\tilde{M}$ are complex manifolds and $f : \tilde{M} \to M$ is a proper modification with compact center Y; suppose moreover that $\tilde{M}$ is "p-Kähler": can the class of its "p-Kähler" form $\tilde{\Omega}$ vanish, in one of the following cohomology groups: $H^{p,p}_{\partial\overline{\partial}}(\tilde{M})$, $H^{p,p}_{\partial+\overline{\partial}}(\tilde{M})$, $H^{p,p}_{dR}(\tilde{M})$?

In most cases, the answer is no: suppose $\dim Y = s, 0 \leq s \leq n-2$: the cases $p = n-1$ and $p = n-1-s$ are completely solved by the existence of compact analytic subvarieties of the right dimension in $\tilde{M}$, namely, the maximal irreducible components of E and a fiber $f^{-1}(y)$. These are closed currents, which vanish when applied to an "exact" form, but they must give a positive number when applied to transverse forms, since they have positive volume. For the same reason, the answer is the same, for every p, on blow-ups with center at a point O: they have enough compact subvarieties on E.

In some other cases, when M and $\tilde{M}$ are compact, we can use the pull-back of a suitable p-Kähler form on M: in particular, this holds when M is balanced, as follows:

Proposition 7.3 *Let $f : \tilde{M} \to M$ be a modification, M a compact balanced manifold with form ω, $\tilde{M}$ a compact "1-Kähler" manifold with "1-Kähler" form $\tilde{\omega}$. Then $\tilde{\omega}$ is never "exact".*

Proof The case in $H^{p,p}_{\partial\overline{\partial}}(\tilde{M})$ is obvious, because $\tilde{\omega} = i\partial\overline{\partial}g > 0$ implies g is constant.

In the case 1S, if $\tilde{\omega} = \psi^{1,1}$ with $d\psi = 0$, we ask for the possibility $\psi = d\alpha$. Notice that $f^*\omega^{n-1} \geq 0$ and $f^*\omega^{n-1} > 0$ outside E; thus we get

$$0 < \int_{\tilde{M}} \tilde{\omega} \wedge f^*\omega^{n-1} = \int_{\tilde{M}} \psi \wedge f^*\omega^{n-1} = \int_{\tilde{M}} d\alpha \wedge f^*\omega^{n-1} = -\int_{\tilde{M}} \alpha \wedge d(f^*\omega^{n-1}),$$

which vanishes since ω is balanced.

In the cases 1WK and 1PL, starting by $\partial\tilde{\omega} = \partial\overline{\partial}\alpha$ or $\partial\overline{\partial}\tilde{\omega} = 0$, we ask for the possibility $\tilde{\omega} = \partial\overline{\mu} + \overline{\partial}\mu$; this can be solved as above.

In case the 1K, starting by $d\tilde{\omega} = 0$, we can ask if $\tilde{\omega} = d\beta$. As above, the answer is negative, also when M is only $(n-1)$WK. □

Claim *Arguing as in the previous proposition, if M is compact Kähler and $\tilde{M}$ is "p-Kähler", its form cannot be exact.*

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