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Cohomologies of Generalized Complex Manifolds and Nilmanifolds

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Abstract We study generalized complex cohomologies of generalized complex structures constructed from certain symplectic fiber bundles over complex manifolds. We apply our results in the case of left-invariant generalized complex structures on nilmanifolds and to their space of small deformations.

Keywords Cohomology · Generalized complex · Nilmanifold · Deformation

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Introduction

Generalized complex geometry, in the sense of Hitchin, Gualtieri, and Cavalcanti, [5,10,12], unifies symplectic and complex geometries in a unitary framework. In such

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a way, it clarifies the parallelism between results for (non-Kähler) complex manifolds and for (non-Kähler) symplectic manifolds.

We recall that a generalized complex structure on a differentiable manifold M is an endomorphism $\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$ such that $\mathcal{J}^2 = -1$ and the i-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ is involutive with respect to the Courant bracket (1). If $\omega \in \wedge^2 M$ (viewed as an isomorphism $TM \to T^*M$) is a symplectic structure, respectively $J \in \mathrm{End}(TM)$ is a complex structure on M, then

$$\mathcal{J}_\omega := \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array}\right), \qquad \text{respectively} \qquad \mathcal{J}_J := \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array}\right)$$

are generalized complex structures on M. In view of the generalized Darboux theorem [10, Theorem 3.6] proved by Gualtieri, these examples constitute the basic models of generalized complex structures near regular points.

A generalized complex structure $\mathcal{J}$ on M of dimension $2n$ yields a decomposition of complex differential forms $\wedge^\bullet T^*M \otimes \mathbb{C} = \bigoplus_{j=-n}^{n} U^j$, whence the bi-differential $\mathbb{Z}$-graded complex

$$\left(\mathcal{U}^\bullet,\ \partial,\ \overline{\partial}\right).$$

In this note, we are interested in the generalized Dolbeault cohomologies

$$GH^\bullet_\partial(M) := \frac{\ker \partial}{\mathrm{im}\, \partial} \qquad \text{and} \qquad GH^\bullet_{\overline{\partial}}(M) := \frac{\ker \overline{\partial}}{\mathrm{im}\, \overline{\partial}},$$

and in the generalized Bott–Chern and Aeppli cohomologies

$$GH^\bullet_{BC}(M) := \frac{\ker \partial \cap \ker \overline{\partial}}{\mathrm{im}\, \partial\overline{\partial}} \qquad \text{and} \qquad GH^\bullet_{A}(M) := \frac{\ker \partial\overline{\partial}}{\mathrm{im}\, \partial + \mathrm{im}\, \overline{\partial}}.$$

(Note that, in the complex case, the generalized ∂ and $\overline{\partial}$ operators coincide with the complex operators, and so, up to a change of graduation, the above cohomologies are exactly the Dolbeault and the Bott–Chern cohomologies. In the symplectic case, the generalized Dolbeault cohomology is isomorphic to the de Rham cohomology, and the generalized Bott–Chern cohomology has been studied by Tseng and Yau; see [18–21].)

More precisely, look at the i-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ of $\mathcal{J} \in \mathrm{End}((TM \oplus T^*M) \otimes \mathbb{C})$ with the Lie algebroid structure given by the Courant bracket and the projection $\pi \colon L \to TM \otimes \mathbb{C}$. Take a generalized holomorphic bundle, that is, a complex vector bundle E with a Lie algebroid connection

$$\overline{\partial} \colon \mathcal{C}^\infty(\wedge^k L^* \otimes E) \to \mathcal{C}^\infty(\wedge^{k+1} L^* \otimes E)$$

satisfying $\overline{\partial} \circ \overline{\partial} = 0$. Consider

$$GH^{n-\bullet}_{\overline{\partial}}(M, E) := H^\bullet(L, E) := \frac{\ker\left(\overline{\partial} \colon \mathcal{C}^\infty(\wedge^\bullet L^* \otimes E) \to \mathcal{C}^\infty(\wedge^{\bullet+1} L^* \otimes E)\right)}{\mathrm{im}\left(\overline{\partial} \colon \mathcal{C}^\infty(\wedge^{\bullet-1} L^* \otimes E) \to \mathcal{C}^\infty(\wedge^\bullet L^* \otimes E)\right)}.$$

One way to construct generalized complex structures on manifolds is the following. Let $p\colon X \to B$ be a symplectic fiber bundle with a generic fiber (F, σ). Assume that the base B is a compact complex manifold and that there is a closed form ω on the total space X which restricts to the symplectic form σ on the generic F. Then we can construct a non-degenerate pure form, and then a generalized complex structure on X.

We construct the following Leray spectral sequence for computing the generalized cohomology of such an X.

Corollary 2.2 *Let $p\colon X \to B$ be a symplectic fiber bundle with a generic fiber (F, σ) of dimension 2ℓ such that:*

- *B is a compact complex manifold of complex dimension k;*
- *we have a closed form ω on the total space X which restricts to the symplectic form σ on the generic F.*

Consider the generalized complex structure $\mathcal{J}$ on X defined by ω and the complex structure of B and the i-eigenbundle L of $\mathcal{J}$. Let W be a complex vector bundle over X such that $W = p^*W'$ for a holomorphic vector bundle W' over the complex manifold B. We regard W as a generalized holomorphic bundle. Consider the flat vector bundle $\mathbf{H}(F) = \bigcup_{x \in B} H^\bullet(F_b)$ over B.

Then there exists a spectral sequence $\left\{E_r^{\bullet,\bullet}\right\}_r$ which converges to $GH_{\overline{\partial}}^{k+\ell-\bullet}(X)$ such that

$$E_2^{p,q} \cong GH_{\overline{\partial}}^{k-p}(B, \mathbf{H}^{\ell-q}(F)).$$

As an application of the above results, we investigate generalized cohomologies of nilmanifolds $M = \Gamma\backslash G$, that is, compact quotients of connected simply connected nilpotent Lie groups G. We consider left-invariant generalized complex structures on M, equivalently, linear generalized complex structures on the Lie algebra $\mathfrak{g}$ of G. Note that left-invariant generalized complex structures on nilmanifolds are generalized Calabi–Yau, that is, the canonical line bundle K is trivial; whence $GH_{\overline{\partial}}^{n-\bullet}(M) = H^\bullet(L)$.

In this context, we have a generalized complex decomposition also at the level of the Lie algebra, namely, $\wedge^\bullet \mathfrak{g}^* = \bigoplus_j \mathfrak{U}^j$, and a (finite dimensional) bi-differential $\mathbb{Z}$-graded sub-complex

$$\left(\mathfrak{U}^\bullet,\, \partial,\, \overline{\partial}\right) \to \left(\mathcal{U}^\bullet,\, \partial,\, \overline{\partial}\right).$$

It induces the map $GH_{\overline{\partial}}^\bullet(\mathfrak{g}) \to GH_{\overline{\partial}}^\bullet(M)$ in cohomology, which is in fact always injective.

Corollary 5.4 *Let G be a connected simply connected nilpotent Lie group and $\mathfrak{g}$ the Lie algebra of G. We suppose that G admits a lattice Γ and consider the $\mathbb{Q}$-structure $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ induced by Γ. We assume that there exists an ideal $\mathfrak{h} \subset \mathfrak{g}$ so that:*

(i) *$\mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{h}$ is a $\mathbb{Q}$-structure of $\mathfrak{h}$;*
(ii) *$\mathfrak{g}/\mathfrak{h}$ admits a complex structure J;*
(iii) *we have a closed 2-form $\omega \in \wedge^2\mathfrak{g}^*$ yielding $\omega \in \wedge^2\mathfrak{h}^*$ non-degenerate form on $\mathfrak{h}$;*

(iv) $\iota\colon \wedge^{\bullet,\bullet}(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{C} \to \wedge^{\bullet,\bullet}\Gamma H\backslash G$ *induces an isomorphism on the Dolbeault cohomology.*

Then the inclusion $\iota\colon (\mathfrak{U}^\bullet, \partial, \overline{\partial}) \to (\mathcal{U}^\bullet, \partial, \overline{\partial})$ *induces isomorphisms* $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$*, and* $GH_{\partial}(\mathfrak{g}) \cong GH_{\partial}(\Gamma\backslash G)$*, and* $GH_{BC}(\mathfrak{g}) \cong GH_{BC}(\Gamma\backslash G)$.

As regards the fourth assumption, we note that it holds, e.g., when J is either bi-invariant, or holomorphically parallelizable, or Abelian, or rational, or nilpotent; see [9] and the references therein.

As an explicit example, we study a generalized complex structure on the Kodaira–Thurston manifold in Sect. 7. Another application of the previous result can be found in Angella et al. [2].

The above invariance result for generalized cohomologies is stable under small deformations.

Theorem 6.1 *Let* $\Gamma\backslash G$ *be a nilmanifold with a left-invariant generalized complex structure* $\mathcal{J}$*; denote by* $\mathfrak{g}$ *the Lie algebra of* G*. If the isomorphism* $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$ *holds on the original generalized complex structure* $\mathcal{J}$*, then the same isomorphism holds on the deformed generalized complex structure* $\mathcal{J}_{\epsilon(t)}$ *for sufficiently small* t.

For complex case, theorems of this type are found in [1,3,8].

Finally, we apply the above result on nilmanifolds to study their space of small deformations. In particular, we prove that any small deformation of a generalized complex structure on a nilmanifold with invariant generalized cohomology is (equivalent to) a left-invariant structure.

Theorem 6.2 *Let* $\Gamma\backslash G$ *be a nilmanifold with a left-invariant generalized complex structure* $\mathcal{J}$*; denote by* $\mathfrak{g}$ *the Lie algebra of* G*. If the isomorphism* $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$ *holds on the original generalized complex structure* $\mathcal{J}$*, then any sufficiently small deformation of generalized complex structure is equivalent to a left-invariant generalized complex structure* $\mathcal{J}_\epsilon$ *with* $\epsilon \in \wedge^2\mathfrak{L}^*$ *satisfying the Maurer–Cartan equation.*

This result is a generalization of [17, Theorem 2.6].

1 Generalized Complex Structures

Let M be a compact differentiable manifold of dimension $2n$. Consider the vector bundle $TM \oplus T^*M$, endowed with the natural symmetric pairing

$$\left\langle X + \xi \,\middle|\, Y + \eta \right\rangle := \frac{1}{2}\,(\xi(Y) + \eta(X)).$$

We define the action of $TM \oplus T^*M$ on $\wedge^\bullet T^*M$ so that

$$(X + \xi) \cdot \rho = i_X\rho + \xi \wedge \rho$$

We define the *Courant bracket* on the space $\mathcal{C}^\infty(TX \oplus T^*X)$ such that

$$[X+\xi,\ Y+\eta] := [X,\ Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\,\mathrm{d}\,(\iota_X\eta - \iota_Y\xi)\,. \tag{1}$$

A *generalized complex structure* on M is an endomorphism $\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$ such that $\mathcal{J}^2 = -1$ and the i-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ involutive with respect to the Courant bracket.

A form ρ in $\wedge^\bullet T^*M \otimes \mathbb{C}$ is called *pure* if it can be written as

$$\rho = e^{B+i\omega}\Omega$$

where $B, \omega \in \wedge^2 T^*M$ and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ with $\theta_1, \ldots, \theta_k \in T^*M \otimes \mathbb{C}$. A pure form $\rho \in \wedge^\bullet T^*M \otimes \mathbb{C}$ is non-degenerate if

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0.$$

For a generalized complex structure $\mathcal{J}$ with the i-eigenbundle L, we have the *canonical line bundle* $K \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ such that

$$L = \mathrm{Ann}(K) = \left\{ v \in (TM \oplus T^*M) \otimes \mathbb{C} \;\middle|\; v \cdot K = 0 \right\}.$$

Any $\rho \in K$ is a non-degenerate pure form and any $\phi \in \mathcal{C}^\infty(K)$ is integrable, i.e., there exists $v \in \mathcal{C}^\infty(TX \oplus T^*X)$ satisfying

$$d\phi = v \cdot \phi.$$

Conversely, if we have a line bundle $K \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ so that any $\rho \in K$ is a non-degenerate pure form and any $\phi \in \mathcal{C}^\infty(K)$ is integrable, then we have a generalized complex structure whose i-eigenbundle is $L = \mathrm{Ann}(K)$.

For a generalized complex manifold $(M, \mathcal{J})$ with the i-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ and the canonical line bundle $K \subset \wedge^\bullet T^*M \otimes \mathbb{C}$, for $j \in \mathbb{Z}$, we define

$$U^j := \wedge^{n-j}\bar{L} \cdot K \;\subseteq\; \wedge^\bullet X \otimes \mathbb{C}.$$

Then we have

$$\wedge^\bullet T^*M \otimes \mathbb{C} = \bigoplus_{j=-n}^{n} U^j.$$

Denote $\mathcal{U}^j = \mathcal{C}^\infty(U^j)$. Then, by the integrability, we have $d\mathcal{U}^j \subset \mathcal{U}^{j-1} \oplus \mathcal{U}^{j+1}$. We consider the decomposition $d = \partial + \overline{\partial}$ such that

$$\partial\colon \mathcal{U}^j \to \mathcal{U}^{j+1} \qquad \text{and} \qquad \overline{\partial}\colon \mathcal{U}^j \to \mathcal{U}^{j-1}.$$

Hence we have the bi-differential $\mathbb{Z}$-graded complexes $(\mathcal{U}^\bullet, \partial, \overline{\partial})$.

We define the generalized Dolbeault cohomologies

$$GH^{\bullet}_{\partial}(M) := \frac{\ker\left(\partial\colon \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet+1}\right)}{\operatorname{im}\left(\partial\colon \mathcal{U}^{\bullet-1} \to \mathcal{U}^{\bullet}\right)},$$
$$GH^{\bullet}_{\overline{\partial}}(M) := \frac{\ker\left(\overline{\partial}\colon \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet-1}\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathcal{U}^{\bullet+1} \to \mathcal{U}^{\bullet}\right)}.$$

Define also the generalized Bott–Chern and Aeppli cohomologies

$$GH^{\bullet}_{BC}(M) := \frac{\ker\left(\partial\colon \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet+1}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet-1}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet}\right)},$$
$$GH^{\bullet}_{A}(M) := \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet}\right)}{\operatorname{im}\left(\partial\colon \mathcal{U}^{\bullet-1} \to \mathcal{U}^{\bullet}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{U}^{\bullet+1} \to \mathcal{U}^{\bullet}\right)}.$$

A *generalized Hermitian metric* on a generalized complex manifold $(M, \mathcal{J})$ is a self-adjoint orthogonal transformation $\mathcal{G} \in \operatorname{End}(TM \oplus TM^*)$ such that $\langle \mathcal{G}v, v\rangle > 0$ for $v \neq 0$ and $\mathcal{J}\mathcal{G} = \mathcal{G}\mathcal{J}$. For a generalized Hermitian metric $\mathcal{G}$, we can define the generalized Hodge star operator $\star : \mathcal{U}^{\bullet} \to \mathcal{U}^{\bullet}$ (see [6, Section 3]) and its conjugation $\bar{\star}$. Define $\overline{\partial}^* = -\bar{\star}\overline{\partial}\bar{\star}$ and $\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Then $\Delta_{\overline{\partial}}$ is an elliptic operator and every cohomology class $\alpha \in GH^{\bullet}_{\overline{\partial}}(M)$ admits a unique representative $a \in \ker \Delta_{\overline{\partial}}$.

It is known that the vector bundle L with the Courant bracket and the projection $\pi : L \to TM \otimes \mathbb{C}$ is a Lie algebroid. By this, we have the differential graded algebra structure on $\mathcal{C}^{\infty}(\wedge^{\bullet} L^*)$ with the differential $d_L : \mathcal{C}^{\infty}(\wedge^k L^*) \to \mathcal{C}^{\infty}(\wedge^{k+1} L^*)$ such that

$$d_L\omega(v_1, \dots, v_{k+1}) = \sum_{i<j} (-1)^{i+j-1} \omega([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{k+1})$$
$$+ \sum_{i=1}^{k+1} (-1)^i \pi(v_i) \left(\omega(v_1, \dots, \hat{v}_i, \dots, v_{k+1})\right).$$

It is known that $(\mathcal{C}^{\infty}(\wedge^{\bullet} L^*), d_L)$ is an elliptic complex (see [10, Proposition 3.12]). A *generalized holomorphic bundle* is a complex vector bundle E with a Lie algebroid connection

$$\overline{\partial}\colon \mathcal{C}^{\infty}(\wedge^k L^* \otimes E) \to \mathcal{C}^{\infty}(\wedge^{k+1} L^* \otimes E)$$

satisfying $\overline{\partial} \circ \overline{\partial} = 0$. For a generalized holomorphic bundle $(E, \overline{\partial})$, we define the Lie algebroid cohomology

$$H^{\bullet}(L, E) = \frac{\ker\left(\overline{\partial}\colon \mathcal{C}^{\infty}(\wedge^{\bullet} L^* \otimes E) \to \mathcal{C}^{\infty}(\wedge^{\bullet+1} L^* \otimes E)\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathcal{C}^{\infty}(\wedge^{\bullet-1} L^* \otimes E) \to \mathcal{C}^{\infty}(\wedge^{\bullet} L^* \otimes E)\right)}.$$

Identifying $L^* = \overline{L}$ by the pairing, $\overline{\partial} : \mathcal{U}^{n-k} \to \mathcal{U}^{n-k-1}$ can be viewed as a Lie algebroid connection

$$\overline{\partial} : \mathcal{C}^\infty(\wedge^k L^* \otimes K) \to \mathcal{C}^\infty(\wedge^{k+1} L^* \otimes K)$$

such that

$$\overline{\partial}(\omega \otimes s) = d_L \omega \otimes s + (-1)^k \omega \otimes ds.$$

Hence the canonical line bundle K is generalized holomorphic and we have $GH^{n-\bullet}_{\overline{\partial}}(M) = H^\bullet(L, K)$. For a generalized holomorphic bundle $(E, \overline{\partial})$, we denote $GH^{n-\bullet}_{\overline{\partial}}(M, E) = H^\bullet(L, K \otimes E)$.

If there exists a nowhere-vanishing closed section $\rho \in \mathcal{C}^\infty(K)$, we call $\mathcal{J}$ a *generalized Calabi–Yau structure*. In this case, we have $GH^{n-\bullet}_{\overline{\partial}}(M) = H^\bullet(L)$.

By the identification $L^* = \overline{L}$, we can define the Schouten bracket [-, =] on $\mathcal{C}^\infty(\wedge^\bullet L^*)$. For sufficiently small $\epsilon \in \mathcal{C}^\infty(\wedge^2 L^*)$, we obtain the small deformation of the isotropic subspace

$$L_\epsilon = (1 + \epsilon)L \subset (TM \oplus T^*M) \otimes \mathbb{C}.$$

Consider the endomorphism $\mathcal{J}_\epsilon \in \mathrm{End}(TM \oplus T^*M)$ whose i-eigenbundle and $-i$-eigenbundle are L_ϵ and $\overline{L_\epsilon}$ respectively. Then $\mathcal{J}_\epsilon$ is a generalized complex structure if and only if ϵ satisfies the Maurer–Cartan equation:

$$d_L \epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0.$$

As similar to Complex Geometry, we can apply the Kuranishi theory. Choose a Hermitian metric on L. Consider the adjoint operator d_L^*, the Laplacian operator $\Delta_L = d_L d_L^* + d_L^* d_L$, the projection $H : \mathcal{C}^\infty(\wedge^\bullet L^*) \to \ker \Delta_L$ and the Green operator G (i.e., the operator on $\mathcal{C}^\infty(\wedge^\bullet L^*)$ so that $G\Delta_L + H = \mathrm{id}$). Let $\epsilon_1 \in \ker \Delta_L \cap \mathcal{C}^2(\wedge^\bullet L^*)$. We consider the formal power series $\epsilon(\epsilon_1)$ with values in $\mathcal{C}^\infty(\wedge^\bullet L^*)$ given inductively by

$$\epsilon_r(\epsilon_1) = \frac{1}{2} \sum_{s=1}^{r-1} d_L^* G[\epsilon_s(\epsilon_1), \epsilon_{r-s}(\epsilon_1)].$$

Then, for sufficiently small ϵ_1, the formal power series $\epsilon(\epsilon_1)$ converges.

Theorem 1.1 ([10, Theorem 5.5]) *Any sufficiently small deformation of the generalized complex structure $\mathcal{J}$ is equivalent to a generalized complex structure $\mathcal{J}_{\epsilon(\epsilon_1)}$ for some $\epsilon_1 \in \ker \Delta_L \cap \mathcal{C}^2(\wedge^\bullet L^*)$ such that $\epsilon(\epsilon_1)$ satisfies the Maurer–Cartan equation.*

Example 1.2 Let M be a compact $2n$-dimensional manifold endowed with a symplectic structure $\omega \in \wedge^2 M$. Consider the induced isomorphism $\omega \colon TM \to T^*M$. The symplectic structure gives rise to the generalized complex structure

$$\mathcal{J}_\omega := \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array}\right).$$

In this case, we obtain the i-eigenbundle

$$L = \{X - i\omega(X) : X \in TM \otimes \mathbb{C}\},$$

the canonical line bundle $K = \langle e^{i\omega} \rangle$ and

$$U^{n-\bullet} = \Phi\left(\wedge^\bullet X \otimes \mathbb{C}\right),$$

where

$$\Phi(\alpha) := \exp(\mathrm{i}\,\omega)\left(\exp\left(\frac{\Lambda}{2\,\mathrm{i}}\right)\alpha\right),$$

and $\Lambda := -\iota_{\omega^{-1}}$. In particular, we have the Lie algebroid isomorphism $TM \otimes \mathbb{C} \cong L$, $\mathcal{J}$ is generalized Calabi–Yau and hence we have an isomorphism $H^*(M) \cong H^*(L) \cong GH^{n-\bullet}_{\overline{\partial}}(M)$. Moreover, we have [6, Corollary 1],

$$\Phi\,\mathrm{d} = \overline{\partial}\Phi \quad \text{and} \quad \Phi\,\mathrm{d}^\Lambda = 2\,\mathrm{i}\,\partial\Phi,$$

where $\mathrm{d}^\Lambda := [\mathrm{d}, \Lambda]$ and this implies that $GH^k_{BC}(X)$ and $GH^k_A(X)$ are isomorphic to the symplectic Bott–Chern and Aeppli cohomologies introduced and studied by Tseng and Yau; see [18–21].

Example 1.3 Let M be a compact $2n$-dimensional manifold endowed with a complex structure $J \in \mathrm{End}(TM)$. The complex structure induces the generalized complex structure

$$\mathcal{J}_J := \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array}\right),$$

where $J^* \in \mathrm{End}(T^*M)$ denotes the dual endomorphism of $J \in \mathrm{End}(TX)$. In this case, we obtain the i-eigenbundle $L = T^{0,1}M \oplus T^{*1,0}M$, the canonical line bundle $K = \wedge^n T^{*1,0}M$ and

$$\mathcal{U}^\bullet = \bigoplus_{p-q=\bullet} \wedge^{p,q} X,$$

with the differentials

$$\partial = \partial_J \qquad \text{and} \qquad \overline{\partial} = \overline{\partial}_J,$$

where ∂_J and $\overline{\partial}_J$ are the usual Dolbeault operators on a complex manifold. The Lie algebroid complex $\mathcal{C}^\infty(\wedge^\bullet L^*)$ is $\mathcal{C}^\infty(\wedge^\bullet(T^{1,0}M \oplus T^{*0,1}M))$ with the differential d_L which is the usual Dolbeault operator.

2 Fibrations and Spectral Sequences

A *symplectic fiber bundle* is a smooth fiber bundle $p\colon X \to B$ so that the fiber F is a compact symplectic manifold and the structural group is the group of symplectomorphisms. Let $p\colon X \to B$ be a symplectic fiber bundle with a generic fiber (F, σ) such that:

- B is a compact complex manifold of complex dimension k;
- we have a closed form ω on the total space X which restricts to the symplectic form σ on the generic F.

Taking a local trivialization $U \times F \subset X$, for a local holomorphic coordinates set $(z_1, \ldots, z_k)$ in U we obtain a non-degenerate pure form

$$\rho = e^{i\omega} dz_1 \wedge \cdots \wedge dz_k$$

and it gives a generalized complex structure on X whose i-eigenbundle L is given by

$$L_{|U} = T^{0,1}U \oplus T^{*1,0}U \oplus \{X - i\omega(X) : X \in TF \otimes \mathbb{C}\}.$$

We consider the sub-bundle S so that $S_{|U} = \{X - i\omega(X) : X \in TF \otimes \mathbb{C}\} \subset L_{|U}$. Then, S is involutive with respect to the Courant bracket.

For $b \in B$ and $F_b = p^{-1}(b)$, denoting by $H^\bullet(F_b)$ the $\mathbb{C}$-valued de Rham cohomology of F_b, we consider the vector bundle $\mathbf{H}(F) = \bigcup_{x \in B} H^\bullet(F_b)$. Then $\mathbf{H}(F)$ is a flat vector bundle over B. Hence, in this case, $\mathbf{H}(F)$ is a holomorphic vector bundle over the complex manifold B.

Consider the bundle $\mathcal{F} = TF_b \otimes \mathbb{C}$ of the vectors tangent to the fibers. Then $\mathcal{F}$ is a Lie algebroid. Consider the Lie algebroid cohomology $H^*(\mathcal{F})$; then we have an isomorphism

$$H^*(\mathcal{F}) \cong \mathcal{C}^\infty(\mathbf{H}(F));$$

see [11, Chapter I. 2.4].

Let $(W, \overline{\partial})$ be a generalized holomorphic bundle over X. Define the subspace $F^p\mathcal{C}^\infty(\wedge^\bullet L^* \otimes W) \subset \mathcal{C}^\infty(\wedge^\bullet L^* \otimes W)$ so that

$$\begin{aligned} &F^p\mathcal{C}^\infty(\wedge^{p+q} L^* \otimes W) \\ &\quad = \left\{\phi \in \mathcal{C}^\infty(\wedge^{p+q} L^* \otimes W) \;\middle|\; \phi(X_1, \ldots, X_{p+q}) = 0 \text{ for } X_{\ell_1}, \ldots, X_{\ell_{q+1}} \in S\right\}. \end{aligned}$$

Then $F^p\mathcal{C}^\infty(\wedge^\bullet L^* \otimes W)$ is a decreasing bounded filtration of $(\mathcal{C}^\infty(\wedge^\bullet L^* \otimes W), \overline{\partial})$. Hence we obtain the spectral sequence $\left\{E_r^{\bullet,\bullet}\right\}_r$ for this filtration.

We suppose that $W = p^*W'$ for a holomorphic vector bundle W' over the complex manifold B. For a local holomorphic coordinates set $(z_1, \ldots, z_k)$ of B, locally we have

$$E_0^{p,q} \cong \wedge^p \left\langle d\bar{z}_1, \ldots, d\bar{z}_k, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k} \right\rangle \otimes_{\mathcal{C}^\infty(B)} W' \otimes_{\mathcal{C}^\infty(B)} \mathcal{C}^\infty(\wedge^q S^*)$$

with the differential

$$d_0 = \mathrm{id} \otimes d_S$$

where d_S is the differential on the Lie algebroid complex $\mathcal{C}^\infty(\wedge^q S^*)$. By using the ω, we have a Lie algebroid isomorphism $\mathcal{F} \ni X \mapsto X - i\omega(X) \in S$. Hence we obtain

$$E_1^{p,q} \cong \wedge^p \left\langle d\bar{z}_1, \dots, d\bar{z}_k, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k} \right\rangle \otimes_{\mathcal{C}^\infty(B)} W' \otimes_{\mathcal{C}^\infty(B)} \mathbf{H}^\bullet(F)$$

with the differential

$$d_1 = \overline{\partial}_B$$

where $\overline{\partial}_B$ is the usual Dolbeault operator on the complex manifold B. Thus, globally, we obtain

$$E_1^{p,q} \cong \mathcal{C}^\infty(\wedge^p L_B^* \otimes W' \otimes \mathbf{H}^q(F))$$

with the differential $d_1 = \overline{\partial}$ which is the Lie algebroid connection on the holomorphic bundle $W' \otimes \mathbf{H}(F)$ where $L_B = T^{0,1}B \oplus T^{*1,0}B$. Hence we have

$$E_2^{p,q} \cong H^p(L_B, W' \otimes \mathbf{H}^q(F)).$$

We have shown the following result.

Theorem 2.1 *Let $p\colon X \to B$ be a symplectic fiber bundle with a generic fiber (F, σ) such that:*

- *B is a compact complex manifold of complex dimension k;*
- *we have a closed form ω on the total space X which restricts to the symplectic form σ on the generic F.*

Consider the generalized complex structure $\mathcal{J}$ on X defined by ω and the complex structure of B and the i-eigenbundle L of $\mathcal{J}$. Let W be a complex vector bundle over X such that $W = p^*W'$ for a holomorphic vector bundle W' over the complex manifold B. We regard W as a generalized holomorphic bundle.

Then there exists a spectral sequence $\{E_r^{\bullet,\bullet}\}_r$ which converges to $H^\bullet(L, W)$ such that

$$E_2^{p,q} \cong H^p(L_B, W' \otimes \mathbf{H}^q(F)).$$

Set $W = K$ which is the canonical line bundle of $(X, \mathcal{J})$. Then as a bundle, we have $p^*K_B \cong K$ where K_B is the canonical line bundle of the complex manifold B. Hence we have:

Corollary 2.2 *Consider the same setting in Theorem 2.1. Suppose* $\dim B = 2k$, $\dim F = 2\ell$.

Then there exists a spectral sequence $\left\{E_r^{\bullet,\bullet}\right\}_r$ *which converges to* $GH_{\overline{\partial}}^{k+\ell-\bullet}(X)$ *such that*

$$E_2^{p,q} \cong GH_{\overline{\partial}}^{k-p}(B, \mathbf{H}^{\ell-q}(F)).$$

3 Generalized Complex Structures on Lie Algebras

Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra. We consider the Lie algebra $\mathbb{D}\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ with the bracket

$$[X+\zeta, Y+\eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\zeta$$

for $X, Y \in \mathfrak{g}$ and $\zeta, \eta \in \mathfrak{g}^*$. A generalized complex structure on $\mathfrak{g}$ is a complex structure on $\mathbb{D}\mathfrak{g}$ which is orthogonal with respect to the pairing

$$\langle X+\zeta, Y+\eta\rangle = \frac{1}{2}(\zeta(Y) + \eta(X)).$$

Consider the complex $\wedge^\bullet \mathfrak{g}_{\mathbb{C}}^*$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. A form $\rho \in \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^*$ is a pure form of type k if it can be written as

$$\rho = e^{B+i\omega}\Omega$$

where $B, \omega \in \wedge^2\mathfrak{g}^*$ and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ with $\theta_1, \ldots, \theta_k \in \wedge^1\mathfrak{g}_{\mathbb{C}}^*$. A pure form $\rho \in \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^*$ of type k is non-degenerate if

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0.$$

A pure form $\rho \in \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^*$ of type k is integrable if there exists $X + \zeta \in \mathbb{D}\mathfrak{g}$ such that

$$d\rho = (X+\zeta)\cdot\rho.$$

Theorem 3.1 ([7]) *If* $\mathfrak{g}$ *is nilpotent, then any non-degenerate integrable pure form is closed.*

For a non-degenerate integrable pure form $\rho \in \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^*$ of type k, we have the sub Lie algebra $\mathfrak{L} \subset \mathbb{D}\mathfrak{g}_{\mathbb{C}}$ such that

$$\mathfrak{L} = \mathrm{Ann}(\rho) = \{X+\zeta \in \mathbb{D}\mathfrak{g}_{\mathbb{C}} | (X+\zeta)\cdot\rho = 0\}.$$

We have the decomposition $\mathbb{D}\mathfrak{g}_{\mathbb{C}} = \mathfrak{L} \oplus \overline{\mathfrak{L}}$ and this gives a generalized complex structure on $\mathfrak{g}$.

Define $\mathfrak{U}^\bullet \subset \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^*$ such that $\mathfrak{U}^n = \langle\rho\rangle$ and $\mathfrak{U}^{n-r} = \wedge^r\overline{\mathfrak{L}} \cdot \mathfrak{U}^n$. Then, by the integrability, we have $d\mathfrak{U}^j \subset \mathfrak{U}^{j-1} \oplus \mathfrak{U}^{j+1}$. We consider the decomposition $d = \partial + \overline{\partial}$

such that $\partial\colon \mathfrak{U}^j \to \mathfrak{U}^{j+1}$ and $\overline{\partial}\colon \mathfrak{U}^j \to \mathfrak{U}^{j-1}$. Hence we have the bi-differential $\mathbb{Z}$-graded complex $(\mathfrak{U}^\bullet, \partial, \overline{\partial})$. We define

$$GH^\bullet_{\partial}(\mathfrak{g}) := \frac{\ker\left(\partial\colon \mathfrak{U}^\bullet \to \mathfrak{U}^{\bullet+1}\right)}{\operatorname{im}\left(\partial\colon \mathfrak{U}^{\bullet-1} \to \mathfrak{U}^\bullet\right)},$$

$$GH^\bullet_{\overline{\partial}}(\mathfrak{g}) := \frac{\ker\left(\overline{\partial}\colon \mathfrak{U}^\bullet \to \mathfrak{U}^{\bullet-1}\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathfrak{U}^{\bullet+1} \to \mathfrak{U}^\bullet\right)},$$

$$GH^\bullet_{BC}(\mathfrak{g}) := \frac{\ker\left(\partial\colon \mathcal{U}^\bullet \to \mathfrak{U}^{\bullet+1}\right) \cap \ker\left(\overline{\partial}\colon \mathfrak{U}^\bullet \to \mathfrak{U}^{\bullet-1}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathfrak{U}^\bullet \to \mathfrak{U}^\bullet\right)},$$

$$GH^\bullet_{A}(\mathfrak{g}) := \frac{\ker\left(\partial\overline{\partial}\colon \mathfrak{U}^\bullet \to \mathfrak{U}^\bullet\right)}{\operatorname{im}\left(\partial\colon \mathfrak{U}^{\bullet-1} \to \mathfrak{U}^\bullet\right) + \operatorname{im}\left(\overline{\partial}\colon \mathfrak{U}^{\bullet+1} \to \mathfrak{U}^\bullet\right)}.$$

By the integrability $d\rho = (X + \zeta)\rho$ and from the identification of $\mathfrak{L}^* = \overline{\mathfrak{L}}$ by the pairing, we can consider $\langle\rho\rangle$ as an $\mathfrak{L}$-module and we can identify $(\mathfrak{U}^{n-\bullet}, \overline{\partial})$ with $\wedge^\bullet \mathfrak{L}^* \otimes \langle\rho\rangle$ as a cochain complex of the Lie algebra $\mathfrak{L}$ with values in the module $\langle\rho\rangle$ (cf. [10, p. 98]). In particular, if $d\rho = 0$, then we have $\mathfrak{U}^{n-\bullet} \cong \wedge^\bullet \mathfrak{L}^*$.

We consider the following special case for using techniques of spectral sequences.

Example 3.2 Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an ideal of $\mathfrak{g}$. Consider the differential graded algebra extension

$$\wedge^\bullet \mathfrak{g}^* = \wedge^\bullet (\mathfrak{g}/\mathfrak{h})^* \otimes \wedge^\bullet \mathfrak{h}^*$$

dualizing the Lie algebra extension

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0.$$

We assume that:

- $\mathfrak{g}/\mathfrak{h}$ admits a complex structure J;
- we have a closed 2-form $\omega \in \wedge^2 \mathfrak{g}^*$ yielding $\omega \in \wedge^2 \mathfrak{h}^*$ non-degenerate form on $\mathfrak{h}$.

Consider the $\pm i$-eigenspace decomposition

$$(\mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C} = (\mathfrak{g}/\mathfrak{h})^{1,0} \oplus (\mathfrak{g}/\mathfrak{h})^{0,1}.$$

Take a basis $Z_1, \ldots, Z_k$ of $(\mathfrak{g}/\mathfrak{h})^{1,0}$ and the dual basis $\theta_1, \ldots, \theta_k$ of $(\mathfrak{g}/\mathfrak{h})^{*1,0}$. Then we have the non-degenerate integrable pure form

$$\rho = e^{i\omega}\theta_1 \wedge \cdots \wedge \theta_k.$$

We have

$$\mathfrak{L} = \langle\theta_1, \ldots, \theta_k, \overline{Z_1}, \ldots, \overline{Z_k}\rangle \oplus \{X - i\omega(X) : X \in \mathfrak{h} \otimes \mathbb{C}\}$$

and consider the subspace $\mathfrak{S} = \{X - i\omega(X) \mid X \in \mathfrak{h} \otimes \mathbb{C}\}$. By $d\omega = 0$ in $\wedge^\bullet \mathfrak{g}^*$ and $\omega \in \wedge^2 \mathfrak{h}^*$, $\mathfrak{S}$ is an ideal of $\mathfrak{L}$. We have $\mathfrak{L}/\mathfrak{S} \cong \mathfrak{L}_J = (\mathfrak{g}/\mathfrak{h})^{0,1} \oplus (\mathfrak{g}/\mathfrak{h})^{*1,0}$. We have the isomorphism $\mathfrak{h} \otimes \mathbb{C} \ni X \mapsto X - i\omega(X) \in \mathfrak{S}$.

By the Hochschild–Serre spectral sequence, we have the spectral sequence $\{E_r^{\bullet,\bullet}\}_r$ which converges to $H^\bullet(\mathfrak{L})$ such that

$$E_2^{p,q} = H^p(\mathfrak{L}/\mathfrak{S}, H^q(\mathfrak{S})).$$

4 De Rham and Dolbeault Cohomology of Nilmanifolds

Let G be a connected simply connected nilpotent Lie group and $\mathfrak{g}$ the Lie algebra of G. A $\mathbb{Q}$-structure of $\mathfrak{g}$ is a $\mathbb{Q}$-subalgebra $\mathfrak{g}_\mathbb{Q} \subset \mathfrak{g}$ such that $\mathfrak{g}_\mathbb{Q} \otimes \mathbb{R} = \mathfrak{g}$. It is known that $\mathfrak{g}$ admits a $\mathbb{Q}$-structure if and only if G admits a lattice (namely, a cocompact discrete subgroup); see, e.g., [16]. More precisely, considering the exponential map $\exp\colon \mathfrak{g} \to G$ which is an diffeomorphism, we can say that:

- for a $\mathbb{Q}$-structure $\mathfrak{g}_\mathbb{Q} \subset \mathfrak{g}$, taking a basis $X_1, \ldots, X_n$ of $\mathfrak{g}_\mathbb{Q}$, the group generated by $\exp(\mathbb{Z}\langle X_1, \ldots, X_n\rangle)$ is a lattice in G;
- for a lattice $\Gamma \subset G$, the $\mathbb{Q}$-span of $\exp^{-1}(\Gamma)$ is a $\mathbb{Q}$-structure of $\mathfrak{g}$.

If G admits a lattice Γ, we call $\Gamma\backslash G$ a nilmanifold.

We suppose that G admits a lattice Γ and consider the $\mathbb{Q}$-structure $\mathfrak{g}_\mathbb{Q} \subset \mathfrak{g}$ induced by Γ as above. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra and $H = \exp(\mathfrak{h})$. We suppose that $\mathfrak{g}_\mathbb{Q} \cap \mathfrak{h}$ is a $\mathbb{Q}$-structure of $\mathfrak{h}$. Then $H \cap \Gamma$ is a lattice of H; see [16, Remark 2.16]. If $\mathfrak{h}$ is an ideal, then H is normal and we obtain the fiber bundle $\Gamma\backslash G \to \Gamma H\backslash G$ with the fiber $\Gamma \cap H\backslash H$.

For a nilmanifold $\Gamma\backslash G$, regarding the cochain complex $\wedge^\bullet \mathfrak{g}^*$ as the space of left-invariant differential forms on $\Gamma\backslash G$, we have the inclusion

$$\iota : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \Gamma\backslash G.$$

Theorem 4.1 ([15]) *The inclusion* $\iota\colon \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \Gamma\backslash G$ *induces a cohomology isomorphism*

$$H^\bullet(\mathfrak{g}) \cong H^\bullet(\Gamma\backslash G).$$

Suppose that $\mathfrak{g}$ admits a complex structure J. Then we can define the Dolbeault complex $\wedge^{\bullet,\bullet}\mathfrak{g}^* \otimes \mathbb{C}$ of $(\mathfrak{g}, J)$. Consider the left-invariant complex structure on the nilmanifold $\Gamma\backslash G$ induced by J and the Dolbeault complex $\wedge^{\bullet,\bullet}\Gamma\backslash G$. Then we have the inclusion $\iota\colon \wedge^{\bullet,\bullet} \mathfrak{g}^* \otimes \mathbb{C} \to \wedge^{\bullet,\bullet}\Gamma\backslash G$.

Let $\mathfrak{L}_J = \mathfrak{g}^{0,1} \oplus \mathfrak{g}^{*1,0}$. We consider the Lie algebroid $L_{\Gamma\backslash G} = T^{0,1}\Gamma\backslash G \oplus T^{*1,0}\Gamma\backslash G$ for the generalized complex structure associated with the complex structure on $\Gamma\backslash G$. Then we have $\mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}) = \mathcal{C}^\infty(\Gamma\backslash G) \otimes \wedge^\bullet \mathfrak{L}_J^*$ and we have the inclusion

$$\kappa\colon \wedge^\bullet \mathfrak{L}_J^* \to \mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}).$$

Proposition 4.2 ([13]) *If the inclusion* $\iota\colon \wedge^{\bullet,\bullet}\mathfrak{g}^*\otimes\mathbb{C}\to\wedge^{\bullet,\bullet}\Gamma\backslash G$ *induces an isomorphism on the Dolbeault cohomology, then the inclusion* $\kappa\colon \wedge^{\bullet}\mathfrak{L}_J^*\to\mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G})$ *induces a cohomology isomorphism.*

Let W be a complex-valued $\mathfrak{g}$-module. We regard W as a $\mathfrak{g}^{0,1}$-module and so an $\mathfrak{L}_J$-module. We consider the cochain complex $\wedge^\bullet\mathfrak{L}_J^*\otimes W$ of the Lie algebra with values in the module W. Consider the flat complex vector bundle $\mathbf{W}$ over $\Gamma\backslash G$ given by W. We regard $\mathbf{W}$ as a holomorphic bundle over $\Gamma\backslash G$ and so a generalized holomorphic bundle on $\Gamma\backslash G$. We have $\mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}\otimes\mathbf{W})=\mathcal{C}^\infty(\Gamma\backslash G)\otimes\wedge^\bullet\mathfrak{L}_J^*\otimes W$ and we have the inclusion

$$\kappa:\wedge^\bullet\mathfrak{L}_J^*\otimes W\to\mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}\otimes\mathbf{W}).$$

Proposition 4.3 *We suppose that the inclusion* $\iota:\wedge^{\bullet,\bullet}\mathfrak{g}^*\otimes\mathbb{C}\to\wedge^{\bullet,\bullet}\Gamma\backslash G$ *induces an isomorphism on the Dolbeault cohomology and* W *is a nilpotent* $\mathfrak{g}$*-module. Then the inclusion*

$$\kappa:\wedge^\bullet\mathfrak{L}_J^*\otimes W\to\mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}\otimes\mathbf{W})$$

induces a cohomology isomorphism.

Proof The proof is by induction on the dimension of W.

Suppose first $\dim W=1$. Then W is the trivial $\mathfrak{g}$-module and hence the statement follows from Proposition 4.2.

In case $\dim W=n>1$, by Engel's theorem, we have an $(n-1)$-dimensional $\mathfrak{g}$-submodule $\tilde{W}\subset W$ such that the quotient $W/\tilde{W}$ is the trivial submodule. The exact sequence

$$0\longrightarrow\tilde{W}\longrightarrow W\longrightarrow W/\tilde{W}\longrightarrow 0$$

gives the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge^\bullet\mathfrak{L}_J^*\otimes\tilde{W} & \longrightarrow & \wedge^\bullet\mathfrak{L}_J^*\otimes W & \longrightarrow & \wedge^\bullet\mathfrak{L}_J^*\otimes W/\tilde{W} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}\otimes\tilde{\mathbf{W}}) & \longrightarrow & \mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}\otimes\mathbf{W}) & \longrightarrow & \mathcal{C}^\infty(\wedge^\bullet L^*_{\Gamma\backslash G}\otimes\mathbf{W}/\tilde{\mathbf{W}}) & \longrightarrow & 0
\end{array}$$

such that the horizontal sequences are exact. Considering the long exact sequence of cohomologies, by the Five Lemma (see, e.g., [14]), the proposition follows inductively. □

5 Left-Invariant Generalized Complex Structures on Nilmanifolds

Let G be a connected simply connected nilpotent Lie group and $\mathfrak{g}$ the Lie algebra of G. We assume that G admits a lattice Γ. We consider the nilmanifold $\Gamma\backslash G$.

We assume that $\mathfrak{g}$ admits a generalized complex structure associated with a non-degenerate integrable pure form $\rho \in \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}}$ of type k. Then we have the left-invariant generalized complex structure $\mathcal{J}$ of type k on the nilmanifold $\Gamma\backslash G$.

Consider the bi-differential $\mathbb{Z}$-graded complexes $(\mathfrak{U}^\bullet, \partial, \overline{\partial})$ associated with $(\mathfrak{g}, \rho)$ and $(\mathcal{U}^\bullet, \partial, \overline{\partial})$ associated with $(\Gamma\backslash G, \mathcal{J})$. Then the inclusion $\iota : \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \to \wedge^\bullet \Gamma\backslash G \otimes \mathbb{C}$ can be considered as a homomorphism $(\mathfrak{U}^\bullet, \partial, \overline{\partial}) \to (\mathcal{U}^\bullet, \partial, \overline{\partial})$ of bi-differential $\mathbb{Z}$-graded complexes.

Proposition 5.1 *There exists a homomorphism* $\mu\colon (\mathcal{U}^\bullet, \partial, \overline{\partial}) \to (\mathfrak{U}^\bullet, \partial, \overline{\partial})$ *such that* $\mu \circ \iota = \mathrm{id}$. *Hence the induced map* $\iota\colon GH_{\overline{\partial}}(\mathfrak{g}) \to GH_{\overline{\partial}}(\Gamma\backslash G)$ *is injective.*

Proof Let $d\nu$ be a bi-invariant volume form such that $\int_{\Gamma\backslash G} d\nu = 1$. We define the map $\mu\colon \wedge^\bullet \Gamma\backslash G \to \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}}$ as follows: for $\alpha \in \wedge^\bullet \Gamma\backslash G$, the left-invariant form $\mu(\alpha)$ is defined by

$$\mu(\alpha)(X_1, \ldots, X_p) = \int_{\Gamma\backslash G} \alpha(\tilde{X}_1, \ldots, \tilde{X}_p) d\nu,$$

where $\tilde{X}_1, \ldots, \tilde{X}_p$ are vector fields on $\Gamma\backslash G$ induced by $X_1, \ldots, X_p \in \mathfrak{g}$. Then we have $d \circ \mu = \mu \circ d$ and $\mu \circ \iota = \mathrm{id}$. We have $\mu(\mathcal{U}^\bullet) \subset \mathfrak{U}^\bullet$. We consider $\mathcal{J}$ as an operator on $\mathcal{U}^\bullet$ such that

$$\mathcal{J}(\alpha) = ip\alpha$$

for $\alpha \in \mathcal{U}^p$. Then we have $d\mathcal{J} - \mathcal{J}d = -i(\partial - \overline{\partial})$; see [5,6]. By $\mu \circ \mathcal{J} = \mathcal{J} \circ \mu$, we have $\mu \circ \partial = \partial \circ \mu$ and $\mu \circ \overline{\partial} = \overline{\partial} \circ \mu$. Hence the proposition follows.

Corollary 5.2 *If the induced map* $\iota\colon GH_{\overline{\partial}}(\mathfrak{g}) \to GH_{\overline{\partial}}(\Gamma\backslash G)$ *is an isomorphism, then the induced maps* $\iota\colon GH_{\partial}(\mathfrak{g}) \to GH_{\partial}(\Gamma\backslash G)$ *and* $\iota\colon GH_{BC}(\mathfrak{g}) \to GH_{BC}(\Gamma\backslash G)$ *are also isomorphisms.*

Proof By using the complex conjugation, we can easily prove that $\iota\colon GH_{\partial}(\mathfrak{g}) \to GH_{\partial}(\Gamma\backslash G)$ is an isomorphism if $\iota : GH_{\overline{\partial}}(\mathfrak{g}) \to GH_{\overline{\partial}}(\Gamma\backslash G)$ is an isomorphism.

Now, [4, Corollary 1.2] implies that if $\iota\colon GH_{\overline{\partial}}(\mathfrak{g}) \to GH_{\overline{\partial}}(\Gamma\backslash G)$ and $\iota\colon GH_{\partial}(\mathfrak{g}) \to GH_{\partial}(\Gamma\backslash G)$ are isomorphisms; then $\iota\colon GH_{BC}(\mathfrak{g}) \to GH_{BC}(\Gamma\backslash G)$ is an isomorphism.

By Theorem 3.1, in our settings, we have isomorphisms $GH^{n-\bullet}_{\overline{\partial}}(\mathfrak{g}) \cong H^\bullet(\mathfrak{L})$ and $GH^{n-\bullet}_{\overline{\partial}}(\Gamma\backslash G) \cong H^\bullet(L)$. Thus, $H^\bullet(\mathfrak{L}) \cong H^\bullet(L)$ if and only if $GH^{n-\bullet}_{\overline{\partial}}(\mathfrak{g}) \cong GH^{n-\bullet}_{\overline{\partial}}(\Gamma\backslash G)$.

Let G be a connected simply connected nilpotent Lie group and $\mathfrak{g}$ the Lie algebra of G. We suppose that G admits a lattice Γ and consider the $\mathbb{Q}$-structure $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ induced by Γ. We assume that there exists an ideal $\mathfrak{h} \subset \mathfrak{g}$ so that:

(i) $\mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{h}$ is a $\mathbb{Q}$-structure of $\mathfrak{h}$;
(ii) $\mathfrak{g}/\mathfrak{h}$ admits a complex structure J;
(iii) we have a closed 2-form $\omega \in \wedge^2 \mathfrak{g}^*$ yielding $\omega \in \wedge^2 \mathfrak{h}^*$ non-degenerate form on $\mathfrak{h}$.

Then, as in Example 3.2, we obtain the non-degenerate integrable pure form $\rho \in \wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}$ and the Lie algebra $\mathfrak{L}$ and its ideal $\mathfrak{S}$. We obtain the symplectic fiber bundle $\Gamma\backslash G \to \Gamma H\backslash G$ over the complex base $\Gamma H\backslash G$ with the symplectic fiber $\Gamma \cap H\backslash H$ as in Sect. 2. The left-invariant generalized complex structure given by ρ is the generalized complex structure constructed in Sect. 2. Consider the Lie algebroids L and S as in Sect. 2. Then $\mathfrak{L}$ and $\mathfrak{S}$ give the global frame of L and S respectively.

Consider the cochain complex $\wedge^{\bullet}\mathcal{L}^{*}$ and $\mathcal{C}^{\infty}(\wedge^{\bullet}L^{*})$. Then we have $\mathcal{C}^{\infty}(\wedge^{\bullet}L^{*}) = \mathcal{C}^{\infty}(\Gamma\backslash G) \otimes \wedge^{\bullet}\mathcal{L}^{*}$ and we have the inclusion

$$\wedge^{\bullet}\mathcal{L}^{*} \to \mathcal{C}^{\infty}(\wedge^{\bullet}L^{*}).$$

For the ideal $\mathfrak{S}$, we consider the filtration

$$F^{p} \wedge^{p+q} \mathfrak{L}^{*} = \left\{\phi \in \wedge^{p+q}\mathfrak{L}^{*} \;\middle|\; \omega(X_{1}, \ldots, X_{p+q}) = 0 \text{ for } X_{\ell_1}, \ldots, X_{\ell_{q+1}} \in \mathfrak{S}\right\}.$$

This filtration gives the spectral sequence $\left\{{}'E_{r}^{\bullet,\bullet}\right\}_{r}$ which converges to $H^{\bullet}(\mathfrak{L})$ such that

$${}'E_{2}^{p,q} = H^{p}(\mathfrak{L}/\mathfrak{S}, H^{q}(\mathfrak{S})).$$

By the identifications $\mathfrak{L}/\mathfrak{S} \cong \mathfrak{L}_{J}$ and $\mathfrak{S} = \mathfrak{h} \otimes \mathbb{C}$, we have

$${}'E_{2}^{p,q} = H^{p}(\mathfrak{L}_{J}, H^{q}(\mathfrak{h} \otimes \mathbb{C})).$$

The filtration $F^{p} \wedge^{\bullet} \mathfrak{L}^{*}$ can be extended to the filtration of $\mathcal{C}^{\infty}(\wedge^{\bullet}L^{*})$ constructed in Sect. 2. Hence the inclusion $\wedge^{\bullet}\mathcal{L}^{*} \to \mathcal{C}^{\infty}(\wedge^{\bullet}L^{*})$ induces the spectral sequence homomorphism ${}'E_{\bullet}^{\bullet,\bullet} \to E_{\bullet}^{\bullet,\bullet}$ such that the homomorphism ${}'E_{2}^{\bullet,\bullet} \to E_{2}^{\bullet,\bullet}$ is identified with the map

$$H^{p}(\mathfrak{L}_{J}, H^{q}(\mathfrak{h} \otimes \mathbb{C})) \to H^{p}(L_{\Gamma H\backslash G}, \mathbf{H}^{q}(\Gamma \cap H\backslash H)).$$

By Theorem 4.1, the flat bundle $\mathbf{H}^{q}(\Gamma \cap H\backslash H)$ over $\Gamma H\backslash G$ is derived from the $\mathfrak{g}/\mathfrak{h}$-module $H^{q}(\mathfrak{h} \otimes \mathbb{C})$. The $\mathfrak{g}/\mathfrak{h}$-module $H^{q}(\mathfrak{h} \otimes \mathbb{C})$ being induced by the adjoint representation on the nilpotent Lie algebra $\mathfrak{g}$, it is a nilpotent $\mathfrak{g}/\mathfrak{h}$-module. If $\iota : \wedge^{\bullet,\bullet}(\mathfrak{g}/\mathfrak{h})^{*}\otimes\mathbb{C} \to \wedge^{\bullet,\bullet}\Gamma H\backslash G$ induces an isomorphism on the Dolbeault cohomology, then the homomorphism ${}'E_{2}^{\bullet,\bullet} \to E_{2}^{\bullet,\bullet}$ is an isomorphism.

Hence, by Proposition 4.3, we obtain the following result.

Theorem 5.3 *Let G be a connected simply connected nilpotent Lie group and $\mathfrak{g}$ the Lie algebra of G. We suppose that G admits a lattice Γ and consider the $\mathbb{Q}$-structure $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ induced by Γ. We assume that there exists an ideal $\mathfrak{h} \subset \mathfrak{g}$ so that:*

- (i) $\mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{h}$ *is a $\mathbb{Q}$-structure of $\mathfrak{h}$;*
- (ii) $\mathfrak{g}/\mathfrak{h}$ *admits a complex structure J;*
- (iii) *we have a closed 2-form $\omega \in \wedge^{2}\mathfrak{g}^{*}$ yielding $\omega \in \wedge^{2}\mathfrak{h}^{*}$ non-degenerate form on $\mathfrak{h}$;*

(iv) $\iota\colon \wedge^{\bullet,\bullet}(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{C} \to \wedge^{\bullet,\bullet}\Gamma H\backslash G$ *induces an isomorphism on the Dolbeault cohomology (e.g., J is bi-invariant, Abelian, or rational, i.e., $J(\mathfrak{g}_{\mathbb{Q}}/\mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}) \subset \mathfrak{g}_{\mathbb{Q}}/\mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$).*

Then the inclusion

$$\wedge^{\bullet}\mathcal{L}^* \to \mathcal{C}^{\infty}(\wedge^{\bullet}L^*)$$

induces a cohomology isomorphism.

Corollary 5.4 *In the same assumptions of Theorem 5.3, the inclusion $\iota\colon (\mathfrak{U}^{\bullet}, \partial, \overline{\partial}) \to (\mathcal{U}^{\bullet}, \partial, \overline{\partial})$ induces isomorphisms $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$, and $GH_{\partial}(\mathfrak{g}) \cong GH_{\partial}(\Gamma\backslash G)$, and $GH_{BC}(\mathfrak{g}) \cong GH_{BC}(\Gamma\backslash G)$.*

6 Deformation and Cohomology

We consider a nilmanifold $\Gamma\backslash G$ with a left-invariant generalized complex structure $\mathcal{J}$. We consider the Lie algebra $\mathfrak{L} \subset (\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathbb{C}$ and the cochain complex $\wedge^{\bullet}\mathfrak{L}^*$. By the identification $\overline{\mathfrak{L}} = \mathfrak{L}^*$, we have the bracket on $\mathfrak{L}^*$. Consider the Schouten bracket on $\wedge^{\bullet}\mathfrak{L}^*$. Then, for the inclusion $\wedge^{\bullet}\mathcal{L}^* \subset \mathcal{C}^{\infty}(\wedge^{\bullet}L^*)$, the Schouten bracket on $\wedge^{\bullet}\mathfrak{L}^*$ can be extended to the Schouten bracket on $\mathcal{C}^{\infty}(\wedge^{\bullet}L^*)$.

We assume that we have a smooth family $\epsilon(t) \in \wedge^2 L^*$ which satisfies the Maurer–Cartan equation

$$d_{\mathfrak{L}}\epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0$$

such that $\epsilon(0) = 0$. Then we have deformations $\mathcal{J}_{\epsilon(t)}$ of $\mathcal{J}$.

Theorem 6.1 *Let $\Gamma\backslash G$ be a nilmanifold with a left-invariant generalized complex structure $\mathcal{J}$; denote by $\mathfrak{g}$ the Lie algebra of G. If the isomorphism $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$ holds on the original generalized complex structure $\mathcal{J}$, then the same isomorphism holds on the deformed generalized complex structure $\mathcal{J}_{\epsilon(t)}$ for sufficiently small t.*

Proof Take a smooth family of generalized Hermitian metrics for the generalized complex structures $\mathcal{J}_{\epsilon(t)}$. We obtain the smooth family $\Delta_{\bar{\partial}}(t)$ of elliptic operators on $\wedge^{\bullet}\Gamma\backslash G \otimes \mathbb{C}$ such that $\Delta_{\bar{\partial}}(t)(\wedge^{\bullet}\mathfrak{g}^* \otimes \mathbb{C}) \subset \wedge^{\bullet}\mathfrak{g}^* \otimes \mathbb{C}$.

Take a Hermitian metric on $\mathfrak{g} \otimes \mathbb{C}$ and extend it to $T\Gamma\backslash G \otimes \mathbb{C}$. Consider the completion $W^0(\wedge^{\bullet}\Gamma\backslash G \otimes \mathbb{C})$ with respect to the L^2-norm. Consider the orthogonal complement $(\ker \Delta_{\bar{\partial}}(t))^{\perp}$ in $W^0(\wedge^{\bullet}\Gamma\backslash G \otimes \mathbb{C})$. It is known that for sufficiently small t, we have $(\ker \Delta_{\bar{\partial}}(0))^{\perp} \cap \ker \Delta_{\bar{\partial}}(t) = 0$.

We can easily show that any cohomology class in $GH_{\overline{\partial}}(\mathfrak{g})$ admits a unique representative in $\ker \Delta_{\bar{\partial}}(0)$. Hence, by the isomorphism $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$, we have $\ker \Delta_{\bar{\partial}}(0) \subset \wedge^{\bullet}\mathfrak{g}^* \otimes \mathbb{C}$. This implies that $(\wedge^{\bullet}\mathfrak{g}^* \otimes \mathbb{C})^{\perp} \subset (\ker \Delta_{\bar{\partial}}(0))^{\perp}$. By $(\ker \Delta_{\bar{\partial}}(0))^{\perp} \cap \ker \Delta_{\bar{\partial}}(t) = 0$, we have $\ker \Delta_{\bar{\partial}}(t) \subset \wedge^{\bullet}\mathfrak{g}^* \otimes \mathbb{C}$. Hence, on the deformed generalized complex structure $\mathcal{J}_{\epsilon(t)}$, any cohomology class in $GH_{\overline{\partial}}(\Gamma\backslash G)$ admits a representative in $\wedge^{\bullet}\mathfrak{g}^* \otimes \mathbb{C}$ and the theorem follows. □

Theorem 6.2 *Let $\Gamma\backslash G$ be a nilmanifold with a left-invariant generalized complex structure $\mathcal{J}$; denote by $\mathfrak{g}$ the Lie algebra of G. If the isomorphism $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$ holds on the original generalized complex structure $\mathcal{J}$, then any sufficiently small deformation of generalized complex structure is equivalent to a left-invariant generalized complex structure $\mathcal{J}_\epsilon$ with $\epsilon \in \wedge^2\mathfrak{L}^*$ satisfying the Maurer–Cartan equation.*

Proof By the isomorphism $GH_{\overline{\partial}}(\mathfrak{g}) \cong GH_{\overline{\partial}}(\Gamma\backslash G)$, we have the isomorphism $H^*(\mathfrak{L}) \cong H^*(L)$.

Take a Hermitian metric on $\mathfrak{L}$. Since $\mathfrak{L}$ gives the global frame of L, it gives a Hermitian metric on L. Consider the adjoint operator d_L^*, the Laplacian operator $\Delta_L = d_L d_L^* + d_L^* d_L$, the projection $H : \mathcal{C}^\infty(\wedge^\bullet L^*) \to \ker\Delta_L$ and the Green operator G. Obviously, these operators can be extended to $\wedge^\bullet\mathfrak{L}^*$. Since $\wedge^\bullet\mathfrak{L}^*$ is finite dimensional, we can easily prove that any cohomology class in $H^*(\mathfrak{L})$ admits a unique representative in $\ker\Delta_L$. Hence, combining with the Hodge theory on the elliptic complex $(\mathcal{C}^\infty(\wedge^\bullet L^*), d_L)$, we have $\ker\Delta_L \subset \wedge^\bullet\mathfrak{L}^*$. Hence, for $\epsilon_1 \in \ker\Delta_L$, the formal power series $\epsilon(\epsilon_1)$ as in Theorem 1.1 is valued in $\wedge^\bullet\mathfrak{L}^*$. Thus the theorem follows from Theorem 1.1.

7 Example: The Kodaira–Thurston Manifold

We consider the real Heisenberg group $H_3(\mathbb{R})$ which is the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. Then $H_3(\mathbb{R})$ admits the lattice $H_3(\mathbb{Z}) = GL_3(\mathbb{Z}) \cap H_3(\mathbb{R})$. We consider the Lie group $H_3(\mathbb{R}) \times \mathbb{R}$ with the lattice $H_3(\mathbb{Z}) \times \mathbb{Z}$.

Let $\mathfrak{g} = \langle X_1, X_2, X_3, X_4\rangle$ such that $[X_1, X_2] = X_3$ and other brackets are 0. Then $\mathfrak{g}$ is the Lie algebra of $H_3(\mathbb{R}) \times \mathbb{R}$ and the basis X_1, X_2, X_3, X_4 gives the $\mathbb{Q}$-structure associated with the lattice $H_3(\mathbb{Z}) \times \mathbb{Z}$. Consider the ideal $\mathfrak{h} = \langle X_2, X_3\rangle$. In this case, the assumptions in Theorem 5.3 hold.

Take the dual basis $\{x_1, x_2, x_3, x_4\}$ of $\{X_1, X_2, X_3, X_4\}$ and consider $\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}} = \wedge^\bullet\langle x_1, x_2, x_3, x_4\rangle$. Consider the non-degenerate integrable pure form

$$\rho = e^{ix_2\wedge x_3} \wedge (x_1 + ix_4)$$

of type 1. We have

$$\mathfrak{L} = \langle X_1 + iX_4, x_1 + ix_4, X_2 - ix_3, X_3 + ix_2\rangle.$$

In this case, we have $\mathfrak{S} = \langle X_2 - ix_3, X_3 + ix_2\rangle$ and $\mathfrak{S}$ is an ideal. We obtain

$$\mathfrak{U}^2 = \langle\rho\rangle$$

$$
\begin{aligned}
\mathfrak{U}^{1} &= \langle e^{i x_2 \wedge x_3},\ e^{i x_2 \wedge x_3} \wedge (x_1 + i x_4) \wedge (x_1 - i x_4),\ (x_1 + i x_4) \wedge x_3, \\
&\quad (x_1 + i x_4) \wedge x_2 \rangle \\
\mathfrak{U}^{0} &= \langle e^{i x_2 \wedge x_3} \wedge (x_1 - i x_4),\ x_3,\ x_2,\ x_3 \wedge (x_1 + i x_4) \wedge (x_1 - i x_4), \\
&\quad x_2 \wedge (x_1 + i x_4) \wedge (x_1 - i x_4),\ e^{-i x_2 \wedge x_3} \wedge (x_1 + i x_4) \rangle \\
\mathfrak{U}^{-1} &= \langle e^{-i x_2 \wedge x_3},\ e^{-i x_2 \wedge x_3} \wedge (x_1 + i x_4) \wedge (x_1 - i x_4),\ (x_1 - i x_4) \wedge x_3, \\
&\quad (x_1 - i x_4) \wedge x_2 \rangle \\
\mathfrak{U}^{-2} &= \langle \bar{\rho} \rangle .
\end{aligned}
$$

We have that the only non-trivial differentials are

$$
\begin{aligned}
d\left((x_1 + i x_4) \wedge x_3\right) = \overline{\partial}\left((x_1 + i x_4) \wedge x_3\right) &= i x_1 \wedge x_2 \wedge x_4, \\
d\left(x_3\right) &= -\frac{1}{2}(x_1 + i x_4) \wedge x_2 - \frac{1}{2}(x_1 - i x_4) \wedge x_2, \\
d\left((x_1 - i x_4) \wedge x_3\right) = \partial\left((x_1 + i x_4) \wedge x_3\right) &= i x_1 \wedge x_2 \wedge x_4 .
\end{aligned}
$$

Define the Kodaira–Thurston manifold as the compact quotient

$$
M := \left(H_3(\mathbb{Z}) \times \mathbb{Z}\right) \Big\backslash \left(H_3(\mathbb{R}) \times \mathbb{R}\right).
$$

By Corollary 5.4, we get:

$$
\begin{aligned}
GH^{2}_{\overline{\partial}}(M) &= \langle [\rho] \rangle \\
GH^{1}_{\overline{\partial}}(M) &= \langle [e^{i x_2 \wedge x_3}],\ [e^{i x_2 \wedge x_3} \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)] \rangle \\
GH^{0}_{\overline{\partial}}(M) &= \langle [e^{i x_2 \wedge x_3} \wedge (x_1 - i x_4)],\ [x_2],\ [x_3 \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)], \\
&\quad [e^{-i x_2 \wedge x_3} \wedge (x_1 + i x_4)] \rangle \\
GH^{-1}_{\overline{\partial}}(M) &= \langle [e^{-i x_2 \wedge x_3}],\ [e^{-i x_2 \wedge x_3} \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)] \rangle \\
GH^{-2}_{\overline{\partial}}(M) &= \langle [\bar{\rho}] \rangle ,
\end{aligned}
$$

and

$$
\begin{aligned}
GH^{2}_{BC}(M) &= \langle [\rho] \rangle \\
GH^{1}_{BC}(M) &= \langle [e^{i x_2 \wedge x_3}],\ [e^{i x_2 \wedge x_3} \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)], \\
&\quad [(x_1 + i x_4) \wedge x_2] \rangle \\
GH^{0}_{BC}(M) &= \langle [e^{i x_2 \wedge x_3} \wedge (x_1 - i x_4)],\ [x_2],\ [x_3 \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)], \\
&\quad [x_2 \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)],\ [e^{-i x_2 \wedge x_3} \wedge (x_1 + i x_4)] \rangle \\
GH^{-1}_{BC}(M) &= \langle [e^{-i x_2 \wedge x_3}],\ [e^{-i x_2 \wedge x_3} \wedge (x_1 + i x_4) \wedge (x_1 - i x_4)], \\
&\quad [(x_1 - i x_4) \wedge x_2] \rangle \\
GH^{-2}_{BC}(M) &= \langle [\bar{\rho}] \rangle .
\end{aligned}
$$

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