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Calabi's Conjecture of the Kähler–Ricci Soliton Type

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Abstract In this paper, we discuss Calabi's equation of the Kähler–Ricci soliton type on a compact Kähler manifold. This equation was introduced by Zhu as a generalization of Calabi's conjecture. We give necessary and sufficient conditions for the unique existence of a solution for this equation on a compact Kähler manifold with a holomorphic vector field which has a zero point. We also consider the case of a nowhere vanishing holomorphic vector field, and give sufficient conditions for the unique existence of a solution for this equation.

Keywords Kähler–Ricci soliton · Holomorphic vector field · Calabi's conjecture · Geometric flow

Mathematics Subject Classification Primary 53C25; Secondary 53C55 · 58E11

1 Introduction

Let (M, ω) be an m-dimensional compact Kähler manifold. In Kähler geometry, the following theorem is widely known as Calabi's conjecture:

Theorem 1.1 *Let* $\Omega \in 2\pi c_1(M)$ *be a real* $(1, 1)$*-form. Then there exists a unique Kähler form* ω' *in the Kähler class* $[\omega]$ *such that* $\mathrm{Ric}(\omega) = \Omega$.

Yau [8] proved this theorem by the continuity method and Cao [1] also proved it by using some geometric flow. This theorem is deeply related to Kähler–Einstein metrics. For instance, as an immediate corollary, we have

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Corollary 1.2 *If $c_1(M) = 0$, then there exists a unique Ricci-flat Kähler form ω' in Kähler class $[\omega]$.*

As a generalization of Calabi's conjecture, Zhu [9] considered the following problem:

Problem 1.3 (Calabi's conjecture of the Kähler–Ricci soliton type). *Let $\Omega \in 2\pi c_1(M)$ be a real $(1, 1)$-form and X be a holomorphic vector field on M. Then, does there exist a Kähler form ω' in the Kähler class $[\omega]$ such that*

$$\mathrm{Ric}(\omega') - \Omega = \mathcal{L}_X \omega'? \tag{1.1}$$

Here $\mathcal{L}_X$ denotes the Lie derivative along X. We call (1.1) Calabi's equation of the Kähler–Ricci soliton type. One of the motivations for which he introduced Eq. (1.1) was to study Kähler–Ricci solitons. A Kähler form ω' is called a Kähler–Ricci soliton if it satisfies

$$\mathrm{Ric}(\omega') - \omega' = \mathcal{L}_X \omega' \tag{1.2}$$

for some holomorphic vector field X. In particular, if $X = 0$, then a Kähler–Ricci soliton is nothing but a Kähler–Einstein metric. Clearly, a Kähler–Ricci soliton ω' is a solution for (1.1) when $\Omega = \omega'$. In his paper, Zhu [9] showed the following theorem:

Theorem 1.4 [9] *Let (M, ω) be a compact Kähler manifold with $c_1(M) > 0$. Let $\Omega \in 2\pi c_1(M)$ be a positive definite $(1, 1)$-form on M and X be a holomorphic vector field on M. Then Eq. (1.1) has a unique solution ω' in the Kähler class $[\omega]$ if and only if*

(i) *There exists a maximal compact subgroup K of $\mathrm{Aut}_0(M)$ such that it contains the one-parameter family $\{\exp(t\, \mathrm{Im}\, X)\}_{t\in\mathbb{R}}$,*
(ii) *$\mathcal{L}_X \Omega$ is a real $(1, 1)$-form on M.*

Here $\mathrm{Aut}_0(M)$ is the identity component of the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of M.

One of the main purposes of this paper is to remove the assumption that Ω is positive definite and give a partial answer to Problem 1.3. Our first main result is as follows:

Theorem 1.5 *Let (M, ω) be a compact Kähler manifold and $\Omega \in 2\pi c_1(M)$ be a real $(1, 1)$-form on M. Suppose that a holomorphic vector field X has a zero point. Then Eq. (1.1) has a unique solution ω' in the Kähler class $[\omega]$ if and only if*

(i) *There exists a maximal compact subgroup K of $\mathrm{Aut}_0(M)$ such that it contains the one-parameter family $\{\exp(t\, \mathrm{Im}\, X)\}_{t\in\mathbb{R}}$,*
(ii) *$\mathcal{L}_X \Omega$ is a real $(1, 1)$-form on M.*

As a corollary of Theorem 1.5, we have

Corollary 1.6 *Let (M, ω) be a compact Kähler manifold. Let $\Omega \in 2\pi c_1(M)$ be a real $(1, 1)$-form on M and X be a holomorphic vector field on M. Suppose $H^1(M; \mathbb{R}) = 0$. Then Eq. (1.1) has a unique solution ω' in the Kähler class $[\omega]$ if and only if*

(i) *There exists a maximal compact subgroup K of* $\mathrm{Aut}_0(M)$ *such that it contains the one-parameter family* $\{\exp(t\,\mathrm{Im}\,X)\}_{t\in\mathbb{R}}$,
(ii) $\mathcal{L}_X\Omega$ *is a real* $(1,1)$*-form on* M.

In particular, if M is a Fano manifold, i.e., $c_1(M) > 0$, then M satisfies the condition $H^1(M;\mathbb{R}) = 0$. Zhu used the continuity method in the proof of his theorem, but we show Theorem 1.5 by using a geometric flow.

We also consider the case of a nowhere vanishing holomorphic vector field X. This case is more complicated because the harmonic part of $i_X\omega$ does not vanish. Under the condition that X has no zero point, we show the following theorem:

Theorem 1.7 *Let* (M,ω) *be a compact Kähler manifold and* $\Omega \in 2\pi c_1(M)$ *be a real* $(1,1)$*-form on* M. *Let* X *be a holomorphic vector field which has no zero point. Assume that both* $\{\exp(t\,\mathrm{Re}\,X)\}_{t\in\mathbb{R}}$ *and* $\{\exp(t\,\mathrm{Im}\,X)\}_{t\in\mathbb{R}}$ *are periodic. Moreover, suppose that* $\mathcal{L}_X\Omega$ *is a real* $(1,1)$*-form on* M. *Then Eq.* (1.1) *has a unique solution* ω' *in the Kähler class* $[\omega]$.

We organize this paper as follows. In Sect. 2, we review some basic facts in Kähler geometry. In Sect. 3, we show the necessity part of Theorem 1.5 (cf. [9]). In Sects. 4, 5 and 6, we introduce a geometric flow, and prove the long time existence and the convergence of the flow (cf. [1,7]). In Sect. 7, we consider the case of a nowhere vanishing holomorphic vector field.

2 Preliminaries

Let M be an m-dimensional compact Kähler manifold and ω be a Kähler form on M. In local coordinates $(z^1,\ldots,z^m)$, ω has an expression

$$\omega = \sqrt{-1}\sum_{i,j=1}^{m} g_{i\bar{j}}\,dz^i \wedge d\bar{z}^j,$$

where $(g_{i\bar{j}})$ is a positive definite Hermitian matrix. Recall that $g_{i\bar{j}}$ satisfy the Kähler identities

$$\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}, \tag{2.1}$$

where $\partial_i = \partial/\partial z^i$ and $\partial_{\bar{j}} = \partial/\partial\bar{z}^j$. For arbitrary Kähler form ω' in the Kähler class $[\omega]$, there exists a smooth real function φ on M such that

$$\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi.$$

The Ricci form $\mathrm{Ric}(\omega)$ of ω is given by

$$\mathrm{Ric}(\omega) = \sqrt{-1}\sum_{i,j=1}^{m} R_{i\bar{j}}\,dz^i \wedge d\bar{z}^j = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

and it represents $2\pi c_1(M)$.

Let $\eta(M)$ be the space of holomorphic vector fields on M. For each holomorphic vector field X, there exists a unique function $\theta_X(\omega)$ such that

$$\begin{cases} \mathcal{L}_X\omega = \sqrt{-1}\partial\bar{\partial}\theta_X(\omega), \\ \int_M \theta_X(\omega)\,\frac{\omega^m}{m!} = 0. \end{cases} \tag{2.2}$$

Put $\alpha_X := i_X\omega - \sqrt{-1}\bar{\partial}\theta_X(\omega)$. Then α_X is a harmonic (0, 1)-form with respect to ω. The following propositions are widely known, but we give proofs for the reader's convenience.

Proposition 2.1 *Let $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler form on M in the Kähler class $[\omega]$. Then*

$$\theta_X(\omega_\varphi) = \theta_X(\omega) + X(\varphi). \tag{2.3}$$

Proof Let $\omega_s = \omega + s\sqrt{-1}\partial\bar{\partial}\varphi$. From the definition of θ_X, we have

$$\sqrt{-1}\partial\bar{\partial}\theta_X(\omega_s) = \sqrt{-1}\partial\bar{\partial}\theta_X(\omega) + s\sqrt{-1}\partial\bar{\partial}X(\varphi) \tag{2.4}$$

and hence

$$\theta_X(\omega_s) = \theta_X(\omega) + sX(\varphi) + c_s \tag{2.5}$$

for some constants c_s. Clearly, $c_0 = 0$.

We now compute

$$\begin{aligned} 0 &\equiv \frac{d}{ds}\int_M (\theta_X(\omega) + sX(\varphi) + c_s)\,\frac{\omega_s^m}{m!} \\ &= \int_M \left(X(\varphi) + \frac{dc_s}{ds} + (\theta_X(\omega) + sX(\varphi) + c_s)\,\triangle_{\omega_s}\varphi\right)\frac{\omega_s^m}{m!} \\ &= \int_M X(\varphi)\,\frac{\omega_s^m}{m!} + \int_M \varphi\mathcal{L}_X\left(\frac{\omega_s^m}{m!}\right) + \frac{dc_s}{ds}\int_M \frac{\omega_s^m}{m!} \\ &= \frac{dc_s}{ds}\int_M \frac{\omega_s^m}{m!}, \end{aligned} \tag{2.6}$$

where $\triangle_{\omega_s} = g_s^{i\bar{j}}\partial_i\partial_{\bar{j}}$ denotes the complex Laplacian with respect to ω_s. Thus we conclude $c_s \equiv 0$. □

Proposition 2.2 *α_X is independent of the choice of ω' in the Kähler class $[\omega]$.*

Proof From Proposition 2.1, it follows that

$$i_X\omega_\varphi - \sqrt{-1}\bar{\partial}\theta_X(\omega_\varphi) = i_X\omega - \sqrt{-1}\bar{\partial}\theta_X(\omega) + \sqrt{-1}\left(i_X\partial\bar{\partial}\varphi - \bar{\partial}(X(\varphi))\right). \tag{2.7}$$

Since X is holomorphic, we have $i_X\partial\bar{\partial}\varphi - \bar{\partial}(X(\varphi)) = 0$. □

Proposition 2.3 [4] $\alpha_X \equiv 0$ *if and only if X has a zero point.*

Proof Suppose $\alpha_X \equiv 0$. Let $p \in M$ be a point at which $\theta_X(\omega)$ attains its maximum. Then X vanishes at p. Conversely, suppose X vanishes at $q \in M$. Since α_X is harmonic, $\bar\partial^* \alpha_X = 0$ and $\partial \alpha_X = 0$. Thus we have

$$\begin{aligned} 0 \le \int_M |\alpha_X|^2_\omega \frac{\omega^m}{m!} &= \int_M (i_X\omega - \sqrt{-1}\bar\partial\theta_X(\omega), \alpha_X)_\omega \frac{\omega^m}{m!} \\ &= \int_M (i_X\omega, \alpha_X)_\omega \frac{\omega^m}{m!} \\ &= \int_M \bar\alpha_X(X) \frac{\omega^m}{m!}. \end{aligned} \tag{2.8}$$

Furthermore, $\bar\alpha_X(X)$ is a holomorphic function on M. Since M is compact and $X_q = 0$, it follows that $\bar\alpha_X(X) \equiv 0$. Therefore, $\alpha_X \equiv 0$. □

As a corollary of Proposition 2.3, we have

Corollary 2.4 *Suppose $H^1(M; \mathbb{R}) = 0$. Then, for arbitrary holomorphic vector field X, $\alpha_X \equiv 0$.*

3 Calabi's Equation of the Kähler–Ricci Soliton Type

Let $\Omega \in 2\pi c_1(M)$ be a real $(1,1)$-form on M and X be a holomorphic vector field on M. In this section, we assume a Kähler form ω is a solution for Calabi's equation of the Kähler–Ricci soliton type:

$$\mathrm{Ric}(\omega) - \Omega = \mathcal{L}_X\omega. \tag{3.1}$$

The aim of this section is to derive the necessary conditions for the existence of the solution ω, which was obtained by Zhu ([9]).

Since $\mathrm{Ric}(\omega)$ and Ω are real $(1,1)$-forms on M, we can see $\mathcal{L}_X\omega$ is a real $(1,1)$-form. Therefore, $\mathrm{Im}\, X$ is a Killing vector field, that is, $\mathrm{Im}\, X$ generates a one-parameter group of isometries of (M, ω). Thus, there exists a maximal compact subgroup K of $\mathrm{Aut}_0(M)$ such that it contains the one-parameter group $\{\exp(t\, \mathrm{Im}\, X)\}_{t\in\mathbb{R}}$.

Moreover, we can see the following:

Proposition 3.1 [9] *Assume that there exists a solution ω for (3.1). Then $\mathcal{L}_X\Omega$ is a real $(1,1)$-form on M.*

Proof First note that $\theta_X(\omega)$ is a real-valued function. We have

$$\begin{aligned} \mathcal{L}_{\mathrm{Re}\, X} \mathrm{Ric}(\omega) &= \frac{d}{dt}\Big|_{t=0} (\exp(t\, \mathrm{Re}\, X))^* \mathrm{Ric}(\omega) \\ &= -\sqrt{-1}\partial\bar\partial\triangle_\omega\theta_X(\omega), \end{aligned} \tag{3.2}$$

and

$$\mathcal{L}_{\mathrm{Im}\, X} \mathrm{Ric}(\omega) = 0. \tag{3.3}$$

Hence $\mathcal{L}_X \operatorname{Ric}(\omega)$ is a real (1, 1)-form.

Furthermore, we have

$$X(\theta_X(\omega)) = g_{i\bar{j}} X^i \overline{X^j} - \sqrt{-1}\bar{\alpha}_X(X). \tag{3.4}$$

Since X is holomorphic and $\partial \alpha_X = 0$, it follows that $\bar{\partial}(\bar{\alpha}_X(X)) = 0$. Thus we obtain

$$\sqrt{-1}\partial\bar{\partial} X(\theta_X(\omega)) = \sqrt{-1}\partial\bar{\partial}(g_{i\bar{j}} X^i \overline{X^j}). \tag{3.5}$$

Hence $\mathcal{L}_X(\mathcal{L}_X\omega) = \sqrt{-1}\partial\bar{\partial} X(\theta_X(\omega))$ is a real (1, 1)-form, and we conclude $\mathcal{L}_X\Omega$ is a real (1, 1)-form. □

Consequently, we complete the proof of the necessity part of Theorem 1.5.

4 A Geometric Flow of the Kähler–Ricci Soliton Type

In order to show that Calabi's equation of the Kähler–Ricci soliton type has a solution, in this section, we introduce a geometric flow. We also show the short-time existence of the flow.

Let X be a holomorphic vector field on M. We assume that there exists a maximal compact subgroup $K \subset \operatorname{Aut}_0(M)$ such that $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}} \subset K$. By changing ω if necessary, we may assume that ω is a K-invariant Kähler form. Let $\Omega \in 2\pi c_1(M)$ be a real (1, 1)-form such that $\mathcal{L}_X\Omega$ is a real (1, 1)-form. Since $\Omega \in 2\pi c_1(M)$, there exists a real-valued function f on M such that

$$\begin{cases} \operatorname{Ric}(\omega) - \Omega = \sqrt{-1}\partial\bar{\partial} f, \\ \int_M e^f \frac{\omega^m}{m!} = \int_M \frac{\omega^m}{m!}. \end{cases} \tag{4.1}$$

Now we consider the following flow:

$$\begin{cases} \frac{d}{dt}\omega_t = -\operatorname{Ric}(\omega_t) + \Omega + \mathcal{L}_X\omega_t, \\ \omega_0 = \omega. \end{cases} \tag{4.2}$$

By the definition of this flow, we can see the following lemma:

Lemma 4.1 *The flow* (4.2) *preserves its de Rham cohomology class.*

Therefore, the flow (4.2) is equivalent to the following parabolic complex Monge–Ampère equation:

$$\begin{cases} \dot{\varphi}_t = \log \frac{\omega_t^m}{\omega^m} - f + \theta_X(\omega) + X(\varphi_t), \\ \varphi_0 = 0. \end{cases} \tag{4.3}$$

First we consider the short-time existence.

Theorem 4.2 *There exists a positive constant $T > 0$ such that a unique solution φ_t for* (4.3) *exists for $0 \le t < T$.*

Proof Let $\Omega_t = (\exp(-t \operatorname{Re} X))^* \Omega$. We consider the following flow:

$$\begin{cases} \frac{d}{dt}\tilde{\omega}_t = -\operatorname{Ric}(\tilde{\omega}_t) + \Omega_t, \\ \tilde{\omega}_0 = \omega. \end{cases} \tag{4.4}$$

Equation (4.4) has a unique short-time solution. We now fix $s \in \mathbb{R}$. Since $\mathcal{L}_{\operatorname{Im} X}\Omega = 0$ and $[\operatorname{Re} X, \operatorname{Im} X] = 0$, it follows that

$$\begin{aligned} (\exp(s \operatorname{Im} X))^* (\exp(-t \operatorname{Re} X))^* \Omega &= (\exp(-t \operatorname{Re} X))^* (\exp(s \operatorname{Im} X))^* \Omega \\ &= (\exp(-t \operatorname{Re} X))^* \Omega. \end{aligned} \tag{4.5}$$

Moreover, since $\mathcal{L}_{\operatorname{Im} X}\omega = 0$, we have

$$(\exp(s \operatorname{Im} X))^* \tilde{\omega}_0 = \omega. \tag{4.6}$$

Therefore, the uniqueness of the solution for (4.4) implies

$$(\exp(s \operatorname{Im} X))^* \tilde{\omega}_t = \tilde{\omega}_t, \tag{4.7}$$

and hence, $\mathcal{L}_{\operatorname{Im} X}\tilde{\omega}_t = 0$.

Thus

$$\omega_t = (\exp(t \operatorname{Re} X))^* \tilde{\omega}_t \tag{4.8}$$

is the unique short-time solution for (4.2). □

5 A Priori Estimates

In this section, let us assume that X has a zero point. Then, from Proposition 2.3, $\alpha_X \equiv 0$. First, we need the following lemma:

Lemma 5.1 (see [2,9]) *Let (M, ω) be a compact Kähler manifold. Let $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler form. Suppose that $\mathcal{L}_X\omega$ and $\mathcal{L}_X\omega_\varphi$ are real $(1, 1)$-forms. Then $\|\theta_X(\omega)\|_{C^0} = \|\theta_X(\omega_\varphi)\|_{C^0}$.*

Proof First note that $\theta_X(\omega_\varphi)$ and $\theta_X(\omega)$ are real functions. Suppose $\theta_X(\omega_\varphi)$ and $\theta_X(\omega)$ attain their maximum at p and q, respectively. Since $i_X\omega = \sqrt{-1}\bar{\partial}\theta_X(\omega)$ and $i_X\omega_\varphi = \sqrt{-1}\bar{\partial}\theta_X(\omega_\varphi)$, X vanishes at p and q. Thus, from Proposition 2.1, we can see that

$$\theta_X(\omega_\varphi)(p) = \theta_X(\omega)(p) \le \theta_X(\omega)(q), \tag{5.1}$$
$$\theta_X(\omega)(q) = \theta_X(\omega_\varphi)(q) \le \theta_X(\omega_\varphi)(p). \tag{5.2}$$

Hence $\max \theta_X(\omega) = \max \theta_X(\omega_\varphi)$. Similarly, we see $\min \theta_X(\omega) = \min \theta_X(\omega_\varphi)$. □

5.1 Volume Ratio Estimate

Let φ_t be the solution for (4.3). Now we shall prove some estimates for φ_t. Differentiating (4.3), we obtain

$$(\partial_t - \triangle_t - X)\dot{\varphi}_t = 0. \tag{5.3}$$

Then the maximum principle implies the following:

Proposition 5.2 *There exists a positive constant C_1 depending only on f and $\theta_X(\omega)$ such that*

$$|\dot{\varphi}_t| \leq C_1 \tag{5.4}$$

for all $t \geq 0$.

Moreover, by (4.3), Lemma 5.1 and Proposition 5.2, we obtain the following estimate:

Proposition 5.3 *There exists a positive constant C_2 depending only on f and $\theta_X(\omega)$ such that*

$$\left| \log \frac{\omega_{\varphi_t}^m}{\omega^m} \right| \leq C_2. \tag{5.5}$$

5.2 C^2 Estimate

Next let $Y_t := g^{i\bar{j}} g_{t,i\bar{j}} = m + \triangle_\omega \varphi_t$. We shall show an estimate for Y_t.

Proposition 5.4

$$Y_t \leq C_3 \tag{5.6}$$

for some positive constant C_3 independent of t.

Proof Let

$$\psi_t := \varphi_t - \frac{1}{\mathrm{Vol}(M)} \int_M \varphi_t \frac{\omega^m}{m!}, \tag{5.7}$$

where $\mathrm{Vol}(M) = \int_M \omega^m/m!$. Since $\omega_{\varphi_t} = \omega_{\psi_t}$, we consider ψ_t instead of φ_t.

Now we compute in normal coordinates with respect to ω at $p \in M$, i.e., $g_{i\bar{j}}(p) = \delta_{ij}$ and $\partial_k g_{i\bar{j}}(p) = \partial_{\bar{k}} g_{i\bar{j}}(p) = 0$. Furthermore, we may assume that $g_{t,i\bar{j}}(p) = \lambda_i \delta_{ij}$ and then $Y_t(p) = \sum \lambda_\alpha$. First we show the following inequality:

Lemma 5.5 *We have*

$$(\partial_t - \triangle_t - X) \log Y_t(p) \leq \frac{1}{Y_t} \left(-R_{i\bar{i}k\bar{k}}(p) \frac{\lambda_i}{\lambda_k} + \Omega_{i\bar{i}}(p) + \lambda_i \partial_i X^i(p) \right), \tag{5.8}$$

where $R_{i\bar{j}k\bar{l}}$ is the curvature tensor for ω.

Proof of Lemma 5.5 Using (4.3), we have

$$Y_t \partial_t \log Y_t = -R_{t,i\bar{i}} + \Omega_{i\bar{i}} + \partial_i \partial_{\bar{i}} (\theta_X(\omega) + X(\psi_t)). \tag{5.9}$$

A straightforward computation gives

$$\triangle_t Y_t(p) = R_{i\bar{i}k\bar{k}}(p)\frac{\lambda_i}{\lambda_k} - R_{t,i\bar{i}}(p) + \frac{1}{\lambda_i\lambda_k}\partial_i g_{t,k\bar{j}}(p)\partial_{\bar{i}} g_{t,j\bar{k}}(p). \tag{5.10}$$

Furthermore, we have

$$\begin{aligned}
|\partial Y_t|^2_{\omega_t}(p) &= \sum \frac{1}{\lambda_i}\partial_i g_{t,j\bar{j}}\partial_{\bar{i}} g_{t,k\bar{k}} \\
&\leq \sum_{j,l}\left(\sum_i \frac{1}{\lambda_i}|\partial_i g_{t,j\bar{j}}|^2\right)^{\frac{1}{2}}\left(\sum_k \frac{1}{\lambda_k}|\partial_k g_{t,l\bar{l}}|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_j \lambda_j^{\frac{1}{2}}\left(\sum_i \frac{1}{\lambda_j\lambda_i}|\partial_i g_{t,j\bar{j}}|^2\right)^{\frac{1}{2}}\right)^2 \\
&\leq \left(\sum_k \lambda_k\right)\left(\sum_{i,j}\frac{1}{\lambda_j\lambda_i}|\partial_i g_{t,j\bar{j}}|^2\right) \leq Y_t\left(\sum_{i,j,l}\frac{1}{\lambda_j\lambda_i}|\partial_i g_{t,j\bar{l}}|^2\right).
\end{aligned} \tag{5.11}$$

Here the first and second inequalities follow from the Cauchy–Schwarz inequality. Combining (5.10) and (5.11), we obtain

$$\begin{aligned}
-Y_t\triangle_t \log Y_t(p) &= -\triangle_t Y_t(p) + \frac{1}{Y_t}|\partial Y_t|^2_{\omega_t}(p) \\
&\leq -R_{i\bar{i}k\bar{k}}(p)\frac{\lambda_i}{\lambda_k} + R_{t,i\bar{i}}(p).
\end{aligned} \tag{5.12}$$

We also have

$$\begin{aligned}
\partial_i\partial_{\bar{i}}(\theta_X(\omega) + X(\psi_t)) &= \partial_i(g_{t,k\bar{i}}X^k) \\
&= X^k\partial_k g_{i\bar{i}} + g_{t,k\bar{i}}\partial_i X^k \\
&= X(Y_t) + \lambda_i\partial_i X^i.
\end{aligned} \tag{5.13}$$

Here we used the Kähler identities (2.1). Combining (5.9), (5.12) and (5.13), we complete the proof of Lemma 5.5. □

From Lemma 5.5 and $\lambda_i \leq Y_t$, it follows that

$$(\partial_t - \triangle_t - X)\log Y_t \leq C\sum_i \frac{1}{\lambda_i} + C', \tag{5.14}$$

where a positive constant C depends only on Ω and a lower bound of the bisectional curvature for ω, and $C' = \|\nabla^\omega X\|_{C^0(M,\omega)}$. Moreover, from (4.3), Proposition 5.2 and Proposition 5.3, it follows that

$$(\partial_t - \triangle_t - X)\psi_t = \log \frac{\omega_t^m}{\omega^m} - f + \theta_X(\omega) - \triangle_t \psi_t + \frac{1}{\mathrm{Vol}(M)} \int_M \dot{\varphi}_t \frac{\omega^m}{m!}$$
$$\geq \sum \frac{1}{\lambda_i} - C'', \tag{5.15}$$

where $C'' > 0$ depends on $f, \theta_X(\omega)$ and m. Let $w_t := \log Y_t - (C+1)\psi_t$. Then, by (5.14) and (5.15), we obtain

$$(\partial_t - \triangle_t - X)w_t \leq C \sum_i \frac{1}{\lambda_i} + C' - (C+1)\left(\sum \frac{1}{\lambda_i} - C''\right)$$
$$= -\sum_i \frac{1}{\lambda_i} + C_4, \tag{5.16}$$

where $C_4 = C' + C''(C+1)$.

Now we compute $\sum_i \lambda_i^{-1}$. We have

$$Y_t e^{-\log \frac{\omega_t^m}{\omega^m}} = \sum_i \lambda_i \prod_k \frac{1}{\lambda_k}$$
$$= \sum_i \prod_{k \neq i} \frac{1}{\lambda_k}$$
$$\leq \left(\sum \frac{1}{\lambda_i}\right)^{m-1}. \tag{5.17}$$

By using Proposition 5.3, we obtain

$$Y_t \leq e^{C_2} \left(\sum \frac{1}{\lambda_i}\right)^{m-1}. \tag{5.18}$$

We now need the following lemma (see [6]).

Lemma 5.6 *Let f be a positive function on (M, ω). Suppose φ satisfies*

$$\frac{\omega_\varphi^m}{\omega^m} = f. \tag{5.19}$$

Then we have

$$\operatorname*{osc}_M \varphi := \sup_M \varphi - \inf_M \varphi \leq C \tag{5.20}$$

for some positive constant depending only on (M, ω) and $\|f\|_{C^0}$.

From Proposition 5.3, Lemma 5.6 and (5.18), we see that

$$e^{w_t} = e^{-(C+1)\psi_t} Y_t < C_5 \left(\sum \frac{1}{\lambda_i}\right)^{m-1} \tag{5.21}$$

for some positive constant C_5 independent of t. Thus, from (5.16) and (5.21), it follows that

$$(\partial_t - \triangle_t - X)w_t < -\frac{1}{C_5} e^{\frac{w_t}{m-1}} + C_4. \tag{5.22}$$

Note that we can choose positive constants C_4 and C_5 such that

$$w_0 \equiv \log m < (m-1)\log C_4 C_5. \tag{5.23}$$

Now we prove

$$w_t < (m-1)\log C_4 C_5 \tag{5.24}$$

for all $t > 0$ by contradiction. We assume that

$$\max w_{t_0} = w_{t_0}(p_0) = (m-1)\log C_4 C_5, \tag{5.25}$$

$$w_t < (m-1)\log C_4 C_5 \qquad (t < t_0) \tag{5.26}$$

for some $t_0 > 0$. From (5.22), (5.25) and (5.26), we have

$$0 \le \frac{d}{dt} w_{t_0}(p_0) < 0, \tag{5.27}$$

a contradiction. Thus we obtain

$$w_t < (m-1)\log C_4 C_5 \tag{5.28}$$

and hence we complete the proof. □

Propositions 5.3 and 5.4 immediately imply the following proposition:

Proposition 5.7

$$C_6^{-1}\omega \le \omega_t \le C_6\omega \tag{5.29}$$

for some positive constant C_6 independent of t.

Proof We have $\lambda_i < Y_t \le C_3$ from Proposition 5.4. On the other hand, from Propositions 5.3 and 5.4, it follows immediately that

$$\frac{1}{\lambda_i} < \sum_i \frac{1}{\lambda_i} \le \prod_k \frac{1}{\lambda_k} \left(\sum \lambda_i\right)^{m-1} \le C \tag{5.30}$$

for some positive constant C independent of t. □

5.3 C^3 Estimate

In this subsection, we shall show the following proposition:

Proposition 5.8 *There exists a positive constant C_7 independent of t such that*

$$\left|\nabla^0 g_t\right|^2_{\omega_t} \leq C_7, \tag{5.31}$$

where ∇^0 is the Levi-Civita connection for ω_0.

Our proof is a slight modification of the argument in [5] (see also [7]). Let $\sigma_t := \exp(-t \operatorname{Re} X)$. We prove Proposition 5.8 by computing pullbacks under σ_t. More precisely, we put

$$\tilde{\omega}_t := \sigma_t^* \omega_t, \tag{5.32}$$
$$\hat{\omega}_t := \sigma_t^* \omega_0, \tag{5.33}$$

and let $\tilde{\nabla}$, $\hat{\nabla}$ be the Levi-Civita connections for $\tilde{\omega}_t$, $\hat{\omega}_t$, respectively. Then

$$S := \left|\hat{\nabla} \tilde{g}_t\right|^2_{\tilde{\omega}_t} = \sigma_t^* \left|\nabla^0 g_t\right|^2_{\omega_t}. \tag{5.34}$$

Therefore we show the uniform boundedness of S instead of $|\nabla^0 g_t|^2_{\omega_t}$. We define a tensor $\Psi^k_{i,p}$ by

$$\Psi := \tilde{\nabla} - \hat{\nabla}. \tag{5.35}$$

Then we can express Ψ as

$$\Psi^k_{ip} = \tilde{g}^{k\bar{l}} \partial_i \tilde{g}_{p\bar{l}} - \hat{g}^{k\bar{l}} \partial_i \hat{g}_{p\bar{l}} \tag{5.36}$$

and S as

$$S = |\Psi|^2_{\tilde{\omega}_t} = \tilde{g}^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{g}_{p\bar{q}} \Psi^k_{ip} \overline{\Psi^l_{jq}}. \tag{5.37}$$

Now let us prove Proposition 5.8.

Proof of Proposition 5.8 We compute in normal coordinates with respect to $\tilde{\omega}_t$ at $p \in M$. First, a straightforward computation gives

$$\begin{aligned} \tilde{\Delta} S = {} & \tilde{R}_{j\bar{i}} \Psi^k_{ip} \overline{\Psi^k_{jp}} + \tilde{R}_{q\bar{p}} \Psi^k_{ip} \overline{\Psi^k_{iq}} - \tilde{R}_{k\bar{l}} \Psi^k_{ip} \overline{\Psi^l_{ip}} \\ & + 2 \operatorname{Re} \left(\tilde{\nabla}_\alpha \tilde{\nabla}_{\bar{\alpha}} \Psi^k_{ip} \overline{\Psi^k_{ip}} \right) + \left|\overline{\tilde{\nabla}} \Psi\right|^2_{\tilde{\omega}_t} + \left|\tilde{\nabla} \Psi\right|^2_{\tilde{\omega}_t}, \end{aligned} \tag{5.38}$$

where $\tilde{\Delta}$ is the Laplacian with respect to $\tilde{\omega}_t$ and $\tilde{R}_{i\bar{j}}$ is the Ricci tensor of $\tilde{\omega}_t$.

Recall that $\tilde{\omega}_t$ satisfies (4.4). Moreover, we have

$$\frac{d}{dt} \hat{\omega}_t = \mathcal{L}_X \hat{\omega}_t = \sqrt{-1} \partial \bar{\partial} \sigma_t^* \theta_X(\omega_0) =: \sqrt{-1} \partial \bar{\partial} \hat{\theta}_t. \tag{5.39}$$

By using (4.4), (5.36) and (5.39), we obtain

$$\begin{aligned}\partial_t S =& \left(\tilde{R}_{j\bar{i}} - \Omega_{t,j\bar{i}}\right) \Psi^k_{ip}\overline{\Psi^k_{jp}} + \left(\tilde{R}_{q\bar{p}} - \Omega_{t,q\bar{p}}\right) \Psi^k_{ip}\overline{\Psi^k_{iq}} - \left(\tilde{R}_{k\bar{l}} - \Omega_{t,k\bar{l}}\right) \Psi^k_{ip}\overline{\Psi^l_{ip}} \\ &+ \partial_t \Psi^k_{ip}\overline{\Psi^k_{ip}} + \Psi^k_{ip}\overline{\partial_t \Psi^k_{ip}} \\ =& \left(\tilde{R}_{j\bar{i}} - \Omega_{t,j\bar{i}}\right) \Psi^k_{ip}\overline{\Psi^k_{jp}} + \left(\tilde{R}_{q\bar{p}} - \Omega_{t,q\bar{p}}\right) \Psi^k_{ip}\overline{\Psi^k_{iq}} - \left(\tilde{R}_{k\bar{l}} - \Omega_{t,k\bar{l}}\right) \Psi^k_{ip}\overline{\Psi^l_{ip}} \\ &+ 2\,\mathrm{Re}\left(\left(-\tilde{\nabla}_i R_{p\bar{k}} + \tilde{\nabla}_i \Omega_{t,p\bar{k}} + \hat{g}^{k\bar{\delta}}\hat{g}^{\gamma\bar{l}}\tilde{\nabla}_i \hat{g}_{p\bar{l}}\partial_p\partial_{\bar{q}}\hat{\theta}_t + \sqrt{-1}\hat{g}^{k\bar{l}}\tilde{\nabla}_i(\mathcal{L}_X\hat{\omega}_t)_{p\bar{l}}\right)\overline{\Psi^k_{ip}}\right).\end{aligned} \tag{5.40}$$

From (5.38) and (5.40), we obtain

$$\begin{aligned}\left(\partial_t - \tilde{\Delta}\right) S \le& \; CS - 2\,\mathrm{Re}\left(\tilde{\nabla}_\alpha\tilde{\nabla}_{\bar{\alpha}}\Psi^k_{ip}\overline{\Psi^k_{ip}}\right) \\ &+ 2\,\mathrm{Re}\left(\left(-\tilde{\nabla}_i R_{p\bar{k}} + \tilde{\nabla}_i \Omega_{t,p\bar{k}} + \hat{g}^{k\bar{\delta}}\hat{g}^{\gamma\bar{l}}\tilde{\nabla}_i \hat{g}_{p\bar{l}}\partial_p\partial_{\bar{q}}\hat{\theta}_t + \sqrt{-1}\hat{g}^{k\bar{l}}\tilde{\nabla}_i(\mathcal{L}_X\hat{\omega}_t)_{p\bar{l}}\right)\overline{\Psi^k_{ip}}\right)\end{aligned} \tag{5.41}$$

for some positive constant C independent of t.

From (5.36), we have

$$\partial_{\bar{\alpha}}\Psi^k_{ip} = -\tilde{R}^k_{ji\bar{\alpha}} + \hat{R}^k_{ji\bar{\alpha}}, \tag{5.42}$$

$$\begin{aligned}\tilde{\nabla}_\alpha\tilde{\nabla}_{\bar{\alpha}}\Psi^k_{ip} &= -\tilde{\nabla}_\alpha\tilde{R}^k_{ji\bar{\alpha}} + \tilde{\nabla}_\alpha\hat{R}^k_{ji\bar{\alpha}} \\ &= -\tilde{\nabla}_i\tilde{R}_{p\bar{k}} + \tilde{\nabla}_\alpha\hat{R}^k_{ji\bar{\alpha}}.\end{aligned} \tag{5.43}$$

Here we used the Bianchi identities $\tilde{\nabla}_\alpha\tilde{R}^k_{ji\bar{\beta}} = \tilde{\nabla}_i\tilde{R}^k_{j\alpha\bar{\beta}}$. Combining Proposition 5.7, (5.41) and (5.42), we obtain

$$\begin{aligned}(\partial_t - \tilde{\Delta})S \le& \; CS + 2\,\mathrm{Re}\left(\left(-\tilde{\nabla}_\alpha\hat{R}^k_{ji\bar{\alpha}} + \tilde{\nabla}_i\Omega_{t,p\bar{k}}\right.\right. \\ &\left.\left.+ \hat{g}^{k\bar{\delta}}\hat{g}^{\gamma\bar{l}}\tilde{\nabla}_i \hat{g}_{p\bar{l}}\partial_p\partial_{\bar{q}}\hat{\theta}_t + \sqrt{-1}\hat{g}^{k\bar{l}}\tilde{\nabla}_i(\mathcal{L}_X\hat{\omega}_t)_{p\bar{l}}\right)\overline{\Psi^k_{ip}}\right) \\ \le& \; C'\left(S + \sqrt{S}\right) \le 2C'(S+1)\end{aligned} \tag{5.44}$$

for some positive constant C' independent of t.

Furthermore, put $\tilde{Y}_t := \hat{g}^{i\bar{j}}\tilde{g}_{i\bar{j}} = \sigma_t^* Y_t$. Then, from Proposition 5.7, (5.9), (5.10) and (5.13), we have

$$\left(\partial_t - \tilde{\Delta}\right)\tilde{Y}_t \le C'' - \frac{1}{C''}S \tag{5.45}$$

for some positive constants C'' independent of t.

Let $Q := S + C''(2C'+1)\tilde{Y}_t$. Then, by (5.44) and (5.45), we obtain

$$\left(\partial_t - \tilde{\Delta}\right) Q \le -S + C_7 \tag{5.46}$$

for some positive constant C_7 independent of t. Note that we can choose the positive constant C_7 such that

$$mC''(2C'+1) < C_7. \tag{5.47}$$

Then the same argument in Sect. 5.2 implies

$$S \leq C_7, \tag{5.48}$$

and hence, we complete the proof. □

Combining the above estimates and standard Schauder theory, we conclude that a solution for (4.3) exists for a long time.

Theorem 5.9 *A solution φ_t for (4.3) exists for all time $t \in [0, \infty)$.*

Proof Let T be the maximal time. Assume $T < \infty$. From the estimates which we show in this section and standard Schauder theory (see [3]), there exists a positive constant C independent of $t \in [0, T)$ such that $\|\varphi_t\|_{C^{2,\varepsilon}} \leq C$. Now let $(z^1, \dots, z^m)$ be local coordinates of M and $z^i = x^i + \sqrt{-1}x^{m+i}$. Differentiating (4.3) with respect to x^l, we obtain

$$(\partial_t - \triangle_t - v)\frac{\partial \varphi_t}{\partial x^l} = g_t^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial x^l} - g^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial x^l} - \frac{\partial f}{\partial x^l} + \frac{\partial \theta(\omega)}{\partial x^l} + \frac{\partial v^i}{\partial x^l}\frac{\partial \varphi_t}{\partial x^i}, \tag{5.49}$$

where $v = v^i \partial_{x^i} = \mathrm{Re}\, X$. We have a uniform $C^{0,\varepsilon}$ estimate for the right-hand side. Therefore, standard Schauder theory implies that for arbitrary k, there exists a positive constant C_k independent of $t \in [0, T)$ such that $\|\varphi_t\|_{C^k} \leq C_k$. Then there exists a smooth function φ_T and a time sequence $\{t_n\}$ which converges to T such that $\varphi_{t_n} \to \varphi_T$ in C^k. This is a contradiction, and hence φ_t exists for all time $t \in [0, \infty)$. □

6 Convergence of the Flow

In this section, we show the convergence of the flow (4.2). Let ψ_t be the function defined as in (5.7). First note that from the estimates in the previous section, we have uniform C^k estimates for ψ_t $(k = 0, 1, \dots)$. Hence there is a sequence $\{t_n\}$ such that

$$\psi_{t_n} \to \psi_\infty \tag{6.1}$$

in C^∞ for some smooth function ψ_∞ on M.

Now, we prove the following lemma in order to show the convergence of the flow:

Lemma 6.1 *Let M be a compact Riemannian manifold and g_t be Riemannian metrics such that for any nonnegative integers k, l $|\partial_t^k \nabla^l g_t|$ is uniformly bounded. Here ∇ is the Levi-Civita connection for g_0 and $|\cdot|$ is the norm with respect to g_0. For a smooth vector field v on M, we consider the following equation:*

$$(\partial_t - \triangle_t - v) f_t = 0. \tag{6.2}$$

Then there exists a positive constant $0 < \gamma < 1$ *independent of* t *such that*

$$\operatorname{osc} f_{t_2} \leq \gamma \operatorname{osc} f_{t_1} \qquad (t_1 + 1 \leq t_2) \tag{6.3}$$

for arbitrary solution f_t *for* (6.2).

Proof First note that the maximum principle implies that, for arbitrary solution f_t for (6.2), $\sup_M f_t$ is monotonically decreasing and $\inf_M f_t$ is monotonically increasing, and hence, $\operatorname{osc} f_t$ is monotonically decreasing. Therefore, we have only to consider the case $t_2 = t_1 + 1$. We prove this lemma by contradiction. We assume that the statement is not true. Then for arbitrary positive integer n, there exists a solution $f_t^{(n)}$ for (6.2) and $t_1^{(n)}$ such that

$$\operatorname*{osc}_M f^{(n)}_{t_1^{(n)}+1} > \left(1 - \frac{1}{n}\right) \operatorname*{osc}_M f^{(n)}_{t_1^{(n)}}. \tag{6.4}$$

We may assume that

$$\sup_M f^{(n)}_{t_1^{(n)}} = 1, \qquad \inf_M f^{(n)}_{t_1^{(n)}} = 0. \tag{6.5}$$

(6.4) implies

$$\inf_M f^{(n)}_{t_1^{(n)}+1} < \frac{1}{n} \quad \text{or} \quad \sup_M f^{(n)}_{t_1^{(n)}+1} > 1 - \frac{1}{n}. \tag{6.6}$$

Replacing $f^{(n)}$ by $1 - f^{(n)}$, if necessary, we may assume that

$$\inf_M f^{(n)}_{t_1^{(n)}+1} < \frac{1}{n}. \tag{6.7}$$

Put $h_s^{(n)} := f^{(n)}_{t_1^{(n)}+\frac{1}{2}+s}$. Then we have

$$0 \leq h_s^{(n)} \leq 1, \quad \operatorname*{osc}_M h^{(n)}_{\frac{1}{2}} > 1 - \frac{1}{n}, \quad \inf_M h^{(n)}_{\frac{1}{2}} < \frac{1}{n}. \tag{6.8}$$

Moreover, $h_s^{(n)}$ satisfies

$$\left(\partial_s - \triangle_{t_1^{(n)}+\frac{1}{2}+s} - v\right) h_s^{(n)} = 0. \tag{6.9}$$

Since we have uniform C^k estimates ($k = 0, 1, \dots$) for g_t, sequences $\{g_s^{(n)} := g_{t_1^{(n)}+\frac{1}{2}+s}\}$ and $\{h_s^{(n)}\}$ have convergent subsequences with limits $\hat{g}_s$ and $\hat{h}_s$, respectively. Note that $\hat{g}_s$ are smooth Riemannian metrics and $\hat{h}_s$ are functions that are smooth as functions on M and of class C^1 as functions of $s \in [0, \infty)$ such that

$$(\partial_s - \triangle_{\hat{g}_s} - v)\hat{h}_s = 0, \tag{6.10}$$

$$0 \leq \hat{h}_s \leq 1, \quad \operatorname*{osc}_M \hat{h}_{\frac{1}{2}} \geq 1, \quad \inf_M \hat{h}_{\frac{1}{2}} = 0. \tag{6.11}$$

By the maximum principle, we can see that $\hat{h}_s \equiv 0$. This is a contradiction. □

Since $\dot{\varphi}_t$ satisfies (5.3), Lemma 6.1 implies

$$\operatorname{osc} \dot{\varphi}_{t_2} \leq e^{-a} \operatorname{osc} \dot{\varphi}_{t_1} \qquad (t_1 + 1 \leq t_2) \tag{6.12}$$

for some positive constant a independent of t. From the definition of ψ_t, we see that $\operatorname{osc} \dot{\varphi}_t = \operatorname{osc} \dot{\psi}_t$. Thus we have

$$\sup \dot{\psi}_t \leq \operatorname{osc} \dot{\psi}_t \leq Ce^{-at} \tag{6.13}$$

for some positive constant C independent of t. Hence, we obtain the following theorem:

Theorem 6.2 *ψ_t converges in C^∞ to ψ_∞ and $\dot{\varphi}_t$ converges in C^∞ to a constant.*

Next, we prove the uniqueness of the solution for (3.1). Let ω^0, ω^1 be solutions for (3.1) and $\omega^s := (1-s)\omega^0 + s\omega^1$. Then, we have a solution ω_t^s for

$$\begin{cases} \frac{d}{dt}\omega_t^s = -\operatorname{Ric}(\omega_t^s) + \Omega + \mathcal{L}_X \omega_t^s, \\ \omega_0^s = \omega^s. \end{cases} \tag{6.14}$$

Note that $\omega_t^0 \equiv \omega^0$ and $\omega_t^1 \equiv \omega^1$. From Theorem 6.2, ω_t^s converges to some Kähler metric $\omega_\infty^s = \omega + \sqrt{-1}\partial\bar{\partial}\psi_\infty^s$ and ω_∞^s satisfies

$$0 = -\operatorname{Ric}(\omega_\infty^s) + \Omega = \mathcal{L}_X \omega_\infty^s. \tag{6.15}$$

Differentiating (6.15) with respect to s, we obtain

$$\sqrt{-1}\partial\bar{\partial}\left((\triangle_{\omega_\infty^s} + X)\frac{d}{ds}\psi_\infty^s\right) = 0. \tag{6.16}$$

Therefore, the maximum principle implies $\omega_\infty^s = \omega^0 = \omega^1$.
Consequently, we complete the proof of the sufficiency part of Theorem 1.5.

7 More General Cases

In this section, we consider the case of a nowhere vanishing holomorphic vector field X. First, note that in Sect. 5 we use the assumption that X has a zero point only in the proof of Lemma 5.1. Hence, if we show Lemma 5.1 under the condition that X has a zero point, then, by using the argument in Sect. 5, we can show the existence of the solution for (1.3).

The goal of this section is to show the following lemma:

Lemma 7.1 *Let (M, ω) be a compact Kähler manifold and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler form. Let X be a nowhere vanishing holomorphic vector field. Suppose that $\mathcal{L}_X\omega$ and $\mathcal{L}_X\omega_\varphi$ are real $(1,1)$-forms. Assume both $\{\exp(t \operatorname{Re} X)\}_{t\in\mathbb{R}}$ and*

$\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}}$ *are periodic. Then there exists a constant* C *independent of* φ *such that*

$$|X(\varphi)| \le C. \tag{7.1}$$

7.1 The Case $m = 1$

Let us prove Lemma 7.1 when $m = 1$. Hence, we consider the case where M is a 1-complex torus. When M is a 1-complex torus, we can remove the assumption that $\{\exp(t \operatorname{Re} X)\}_{t\in\mathbb{R}}$ and $\{\exp(t \operatorname{Im} X)\}_{t\in\mathbb{R}}$ are periodic. Now we shall show the following:

Lemma 7.2 *Let* M *be a 1-complex torus and* $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ *be a Kähler form. Let* X *be a holomorphic vector field. Suppose that* $\mathcal{L}_X\omega$ *and* $\mathcal{L}_X\omega_\varphi$ *are real* $(1, 1)$*-forms. Then*

$$|X(\varphi)| \le C \tag{7.2}$$

for some constant C *depending only on* M, X *and* ω.

Proof We may assume that

$$M = \mathbb{C}/(\xi_1\mathbb{Z} + \xi_2\mathbb{Z}) \qquad (\xi_1, \xi_2 \in \mathbb{C},\ \operatorname{Re}\xi_1 \ne 0) \tag{7.3}$$

and $X = \partial/\partial z$. Note that for any Kähler form ω on M,

$$k\sqrt{-1}dz \wedge d\bar{z} \in [\omega], \tag{7.4}$$

where $k = \int_M [\omega] / \int_M \sqrt{-1}dz \wedge d\bar{z}$. Therefore, we may assume $\omega = k\sqrt{-1}dz \wedge d\bar{z}$.

Put $z = x + \sqrt{-1}y$. Suppose $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler form and $X(\varphi)$ is a real function. Using the natural projection $\pi : \mathbb{C} \longrightarrow \mathbb{C}/(\xi_1\mathbb{Z} + \xi_2\mathbb{Z})$, we identify φ and ω with their pullbacks. Since ω_φ is a Kähler form, φ satisfies

$$k + \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi > 0. \tag{7.5}$$

From $0 = 2\operatorname{Im} X(\varphi) = \partial_y\varphi$, we see that

$$\varphi(x, y) = \varphi(x). \tag{7.6}$$

Hence, from (7.5), we have

$$k + \frac{1}{4}\frac{\partial^2}{\partial x^2}\varphi > 0. \tag{7.7}$$

Moreover,

$$\varphi(x + \operatorname{Re}\xi_1) = \varphi(x + \operatorname{Re}\xi_1, \operatorname{Im}\xi_1) = \varphi(x), \tag{7.8}$$

i.e., φ is $|\operatorname{Re}\xi_1|$-periodic. Note that $\partial_x\varphi$ and $\partial_x^2\varphi$ are also $|\operatorname{Re}\xi_1|$-periodic.

Let $x_0 \in \mathbb{R}$ be a minimizer of φ. For arbitrary $\zeta \in [0, |\operatorname{Re}\xi_1|]$, we have

$$\begin{aligned}\frac{\partial\varphi}{\partial x}(\zeta + x_0) &= \int_{x_0}^{\zeta+x_0} \frac{\partial^2\varphi}{\partial x^2}\,dx \\ &> -4k\int_{x_0}^{\zeta+x_0} dx > -4k|\operatorname{Re}\xi_1|. \end{aligned} \tag{7.9}$$

Hence we obtain $2\operatorname{Re} X(\varphi) > -4k|\operatorname{Re}\xi_1|$.

On the other hand, let $x_1 \in \mathbb{R}$ be a maximizer of $\partial_x\varphi$. We may assume $\partial_x\varphi(x_1) > 0$. Then there exists $\zeta' \in (0, |\operatorname{Re}\xi_1|)$ such that $\zeta' + x_1$ is a minimizer of φ. Thus we have

$$\begin{aligned}-\frac{\partial\varphi}{\partial x}(x_1) &= \int_{x_1}^{\zeta'+x_1} \frac{\partial^2\varphi}{\partial x^2}\,dx \\ &> -4k\int_{x_1}^{\zeta'+x_1} du > -4k|\operatorname{Re}\xi_1|. \end{aligned} \tag{7.10}$$

Hence we conclude

$$|X(\varphi)| = |\operatorname{Re} X(\varphi)| \leq 2k|\operatorname{Re}\xi_1|. \tag{7.11}$$

□

7.2 The Case $m \geq 2$

Now we prove Lemma 7.1 when $m \geq 2$.

Proof of Lemma 7.1 The proof is similar to the proof of Corollary 5.3 in [9]. Since $[\operatorname{Re} X, \operatorname{Im} X] = 0$, $\{\operatorname{Re} X, \operatorname{Im} X\}$ defines a holomorphic foliation $\mathcal{F}_X$ on M. From the assumption that both $\{\exp(t\operatorname{Re} X)\}_{t\in\mathbb{R}}$ and $\{\exp(t\operatorname{Im} X)\}_{t\in\mathbb{R}}$ are periodic, we see that every leaf of $\mathcal{F}_X$ is a compact Riemann surface and the leaf space $M/\mathcal{F}_X$ is compact. The condition $X_p \neq 0$ for arbitrary $p \in M$ implies that every leaf of $\mathcal{F}_X$ is a 1-complex torus. Therefore, applying Lemma 7.2 to each leaf of $\mathcal{F}_X$, we obtain the desired uniform estimate. □

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