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On Convergence Properties of Tensor Products of Some Operator Sequences

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Abstract We consider sequences of compact bounded linear operators $U_n : L^p(0,1) \to L^p(0,1)$ with certain convergence properties. Several divergence theorems for multiple sequences of tensor products of these operators are proved. These theorems in particular imply that $L\log^{d-1} L$ is the optimal Orlicz space guaranteeing almost everywhere summability of rectangular partial sums of multiple Fourier series in general orthogonal systems.

Keywords Almost everywhere summability · Multiple Fourier series · Operators in functional spaces

Mathematics Subject Classification 42B08 · 42B35 · 47H04 · 47B38

1 Introduction

It is well known that $(C,1)$ means of the rectangular partial sums of d-dimensional Fourier series of the functions from the class $L\log^{d-1} L(\mathbb{T}^d)$ converge almost everywhere and it is the optimal Orlicz space with this property ([20], Ch. 17). The arguments of [20] also imply the optimality of the same class for the convergence of (C,α)-means with $\alpha > 0$. Such properties of Fourier series are based on two fundamental theorems in the theory of differentiation of integrals: If $f \in L\log^{d-1} L(\mathbb{R}^d)$, then

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$$\lim_{\operatorname{diam}(R)\to 0, x\in R} \frac{1}{|R|}\int_R f = f(x) \text{ a.e.}, \tag{1.1}$$

where R denotes a d-dimensional interval with the diameter $\operatorname{diam}(R)$ (Jessen–Marcinkiewicz–Zygmund [9]) and conversely, in each Orlicz space larger than $L\log^{d-1}L(\mathbb{R}^d)$ there exists a function $f(x)$ such that (1.1) fails for any $x \in \mathbb{R}^d$ (Saks [17]).

For two positive quantities a and b the relation $a \lesssim b$ (or $a \gtrsim b$) stands for $a \le c\cdot b$ (or $a \ge c\cdot b$), where $c > 0$ is either an absolute constant or a constant that depends on the dimension d. The notation $\mathbb{I}_E$ denotes the indicator function of a set E. Let $K_n^\alpha(x)$ be the kernel of (C,α)-means of the one-dimensional Fourier series. The following estimate is well known:

$$n\mathbb{I}_{(-1/n,1/n)}(x) \lesssim K_n^\alpha(x) \lesssim \sum_{i=1}^{m(n)} \alpha_i \mathbb{I}_{(-x_i,x_i)}(x), \tag{1.2}$$

where the numbers $\alpha_i > 0$, $0 < x_1 < \cdots < x_{m(n)} \le \pi$ depend on n and satisfy the inequality

$$\sum_{i=1}^{m(n)} x_i\alpha_i \le 1$$

(see [2], Ch. 1, Theorem 4.2, and [20], Ch. 17, Theorem 2.14). The kernel of the (C,α)-means of the rectangular partial sums of d-dimensional Fourier series has the form

$$K_{\mathbf{n}}^\alpha(\mathbf{x}) = K_{n_1}^\alpha(x_1)K_{n_2}^\alpha(x_2)\ldots K_{n_d}^\alpha(x_d),$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{T}^d$. So for the (C,α)-means we have the formula

$$\sigma_{\mathbf{n}}^\alpha(\mathbf{x}, f) = \frac{1}{\pi^d}\int_{-\pi}^{\pi}\ldots\int_{-\pi}^{\pi} f(\mathbf{x}-\mathbf{t})K_{\mathbf{n}}^\alpha(\mathbf{t})dt_1,\ldots,dt_d. \tag{1.3}$$

The relation (1.2) is the basic argument which makes it possible to use integral differentiation theory in the summability problems of the multiple Fourier series. More precisely, using (1.2), one can get the estimate

$$Mf(\mathbf{x}) \lesssim \sup_{\mathbf{n}\in\mathbb{N}^d} |\sigma_{\mathbf{n}}^\alpha(\mathbf{x}, f)| \lesssim Mf(\mathbf{x}) \tag{1.4}$$

where $Mf(\mathbf{x})$ is the ordinary strong maximal function. The right inequality in (1.4) holds for arbitrary $f \in L^1$ while the left one holds for positive functions. Then the optimality of the class $L\log^{d-1}L(\mathbb{T}^d)$ for a.e. convergence of $\sigma_{\mathbf{n}}^\alpha(\mathbf{x}, f)$ can be obtained from the theorems of Jessen–Marcinkiewicz–Zygmund and Saks by using standard arguments.

The right inequality in (1.2) is common for many kernels of summation, while the left one fails for some of them. An example of such a method of summation are the well known logarithmic means

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}, \quad l_n=\sum_{k=1}^{n-1}\frac{1}{k},$$

where $S_k(f)$ denotes the partial sum of the Fourier series of a function $f\in L^1(\mathbb{T})$. It is known that the convergence of Cesàro means of a sequence implies the convergence of the logarithmic means ([19], Ch. 3.9) and the kernel $K_n(x)$ of the logarithmic means of Fourier series has the estimate

$$0\le K_n(x)\lesssim \min\left\{\frac{1}{x\log n},\frac{n}{\log n}\right\},\quad 0<|x|<\pi.$$

One can observe that it satisfies the right inequality of (1.2) and the left estimate is not satisfied. Thus d-dimensional logarithmic means of the functions from $L\log^{d-1}L$ converge almost everywhere. The question of the optimality of $L\log^{d-1}L$ for this convergence property was open. The main result of this paper solves this question positively. Moreover, we establish general divergence theorems for some sequences of compact bounded operators in $L^1(0,1)^d$. These theorems imply that there is no summation method giving a larger a.e. convergence class than $L\log^{d-1}L$ for the rectangular partial sums of the multiple Fourier series in general orthogonal systems.

Let $Q_d=(0,1)^d$ be the unit d-dimensional cube. For a given increasing continuous function

$$\Phi(t):[0,\infty)\to[0,\infty) \tag{1.5}$$

we denote by $\Phi(L)(Q_d)$ the class of functions $f(\mathbf{x})$ defined on Q_d satisfying the inequality

$$\int_{Q_d}\Phi\left(|f(\mathbf{x})|\right)d\mathbf{x}<\infty.$$

If

$$U:L^1(0,1)\to L^1(0,1) \tag{1.6}$$

is a bounded linear operator, then we denote by $(U)_k$ operators

$$(U)_k:L^1(Q_d)\to L^1(Q_d),\quad 1\le k\le d,$$

defined by

$$(U)_k f(x_1,\ldots,x_d)=Uf(x_1,\ldots,x_{k-1},\cdot,x_{k+1},\ldots,x_d). \tag{1.7}$$

In the right side of (1.7) $f(x_1,\ldots,x_{k-1},\cdot,x_{k+1},\ldots,x_d)$ is considered as a function in the variable x_k (the other variables are fixed). Obviously (1.7) is defined for almost all $\mathbf{x} = (x_1, \ldots, x_d)$ and each $(U)_k$ is a bounded linear operator on $L^1(Q_d)$. For a given sequence of bounded linear operators

$$U_n : L^1(0,1) \to L^1(0,1), \quad n = 1, 2, \ldots, \tag{1.8}$$

we define the multiple sequence of operators

$$\mathcal{U}_{\mathbf{n}} = (U_{n_1})_1 \circ (U_{n_2})_2 \circ \cdots \circ (U_{n_d})_d, \quad \mathbf{n} = (n_1, n_2, \ldots, n_d), \tag{1.9}$$

in $L^1(Q_d)$ generated from the tensor products of (1.8).

We will consider operator sequences U_n with the properties

(A) each U_n is a compact linear operator,
(B) if $f \in L^\infty(0,1)$, then $U_n f(x)$ converges to $f(x)$ in measure.

Recall that if U is a compact linear operator on $L^1(0,1)$, then for any sequence of functions $g_n \in L^1(0,1)$, $n = 1, 2, \ldots$, satisfying the condition

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) dx = 0$$

for any $f \in L^\infty(0,1)$, we have $\|U(g_n)\|_1 \to 0$ as $n \to \infty$.

One of the main results of this paper is the following.

Theorem 1 *Let U_n be a sequence of bounded linear operators* (1.8) *with the properties (A) and (B). Then for any function* (1.5) *satisfying*

$$\lim_{t\to\infty} \frac{\Phi(t)}{t \log^{d-1} t} = 0 \tag{1.10}$$

there exists a function $g \in \Phi(L)(Q_d)$, $g(\boldsymbol{x}) \geq 0$, such that

$$\limsup_{\min\{n_k\}\to\infty} |\mathcal{U}_{\boldsymbol{n}} g(\boldsymbol{x})| = \infty$$

at almost every point $\boldsymbol{x} \in Q_d$.

Let $\varphi = \{\varphi_n(x)\}_{n=1}^\infty \subset L^\infty(0,1)$ be an orthonormal system. Denote by $S_n f(x)$ the partial sums of the Fourier series of a function $f \in L^1(0,1)$ in this system. Suppose the matrix $A = \{a_{nk},\ 1 \leq k \leq n,\ n = 1, 2, \ldots\}$ determines a regular method of summation, that is,

$$\lim_{n\to\infty} a_{nk} = 0,$$

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^n |a_{nk}| < \infty,$$

$$\lim_{n\to\infty} \sum_{k=1}^n a_{nk} = 1.$$

The sequence of operators

$$\sigma_n^{\varphi,A} f(x) = \sum_{k=1}^{n} a_{nk} S_k f(x) \tag{1.11}$$

defines A-means of the partial sums of Fourier series of a function $f \in L^1(0,1)$ with respect to the orthonormal system φ. The tensor products of the operators (1.11) defined by

$$\sigma_{\mathbf{n}}^{\varphi,A} = (\sigma_{n_1}^{\varphi,A})_1 \circ (\sigma_{n_2}^{\varphi,A})_2 \circ \cdots \circ (\sigma_{n_d}^{\varphi,A})_d$$

generate A-means of multiple Fourier series with respect to system φ. Observe that the sequence (1.11) satisfies the conditions (A) and (B). So the following theorem is an immediate consequence of Theorem 1.

Theorem 2 *Let $A = \{a_{nk},\ 1 \le k \le n,\ n = 1, 2, \ldots\}$ be a regular method of summation and $\{\varphi_n(x)\}_{n=1}^{\infty} \subset L^{\infty}(0,1)$ be a complete orthonormal system. Then under the condition (1.10) there exists a function $f \in \Phi(L)(Q_d)$, whose Fourier series in the system $\{\varphi_n(x)\}$ is almost everywhere A-divergent, i.e.,*

$$\limsup_{\min\{n_k\}\to\infty} \left|\sigma_{\boldsymbol{n}}^{\varphi,A} f(\boldsymbol{x})\right| = \infty \ a.e.$$

Particular cases of this theorem for double Fourier series were considered in the papers [10] and [5].

Theorem A (Karagulyan, 1989) *If $\{\varphi_n(x)\}_{n=1}^{\infty} \subset L^{\infty}(0,1)$ is a complete orthonormal system and Φ satisfies the condition (1.10), then there exists a function $f \in \Phi(L)(0,1)^2$ with double Fourier series*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\varphi_n(x)\varphi_m(y) \tag{1.12}$$

satisfying the relation

$$\limsup_{\min\{N,M\}\to\infty} \left|\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\varphi_n(x)\varphi_m(y)\right| = \infty \tag{1.13}$$

almost everywhere on $(0,1)^2$.

Theorem B (Getsadze, 2007) *Let $\{\varphi_n(x)\}_{n=1}^{\infty} \subset L^{\infty}(0,1)$ be a complete orthonormal system and Φ satisfies the condition (1.10). Then for any Lebesgue measurable set $E \subset (0,1)^2$ with $mE > 0$ there exists a function $f \in \Phi(L)(0,1)^2$ and a set $E' \subset E$, $mE' > 0$, such that the sequence of rectangular $(C,1)$ means of double Fourier series are unbounded on E'.*

Analogous problems for Walsh systems were considered before by Gàt [3], Nagy [14], Mořicz et al. [13]. It is proved that $L\log L(0,1)^2$ is the maximal Orlicz space for a.e. $(C,1)$ summability of double Fourier series in Walsh–Paley [3] and Walsh–Kaczmarz [14] systems.

In the proofs of the theorems of the present paper we essentially use the method of Haar type systems. This method was first used by Olevskii [15,16] in his work on divergence problems of orthogonal series.

2 Haar Type Systems

Recall the definition of Haar type systems ([12], Ch. 3.1). We say a family of sets $\epsilon = \{E_n : n = 1, 2, \dots\}$ is a dyadic partition of $[0, 1)$ if

$$E_1 = [0,1),\ E_n = E_k^i \subset [0,1), \quad i = 1,2,\dots,2^k, \quad k = 0,1,\dots, \tag{2.1}$$

where $n \geq 2$ has the representation

$$n = 2^k + i,\ 1 \leq i \leq 2^k,\ k = 0, 1, 2, \dots, \tag{2.2}$$

and we have

$$\begin{aligned} &m(E_k^i) = 2^{-k},\ 1 \leq i \leq 2^k, \\ &E_k^i = E_{k+1}^{2i-1} \cup E_{k+1}^{2i}, \\ &E_k^i \cap E_k^j = \varnothing \quad \text{if} \quad i \neq j. \end{aligned} \tag{2.3}$$

Any dyadic partition uniquely defines a Haar type system $\xi = \{\xi_n(x),\ n = 1, 2, \dots\}$ on $[0, 1)$ as follows:

$$\xi_1(x) \equiv 1,$$
$$\xi_n(x) = \begin{cases} 2^{k/2} & \text{if } x \in E_{k+1}^{2i-1}, \\ -2^{k/2} & \text{if } x \in E_{k+1}^{2i}, \\ 0 & \text{if } x \notin E_k^i. \end{cases}$$

If

$$E_n = \Delta_n = \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right), \quad i = 1, 2, \dots, 2^k, \quad k = 0, 1, \dots,$$

then we get the ordinary Haar system, which will be denoted by $\chi = \{\chi_n(x)\}$. It is known (see [12], Ch. 3.9) that for any Haar type system $\xi_n(x)$ there exists a measure preserving transformation $u(x) : [0, 1) \to [0, 1)$ such that

$$\xi_n(x) = \chi_n(u(x)) \text{ a.e.} \tag{2.4}$$

Consequences of this is the basic property that will be used in different situations below.

Examples of dyadic partitions of [0, 1) may be given using the Rademacher system

$$r_n(x) = (-1)^{[2^n x]}, \quad x \in [0, 1), \quad n = 1, 2, \dots.$$

For a given integer $n \geq 2$ of the form (2.2), we define

$$\bar{n} = 2^{k-1} + \left[\frac{i+1}{2}\right].$$

Take an arbitrary sequence of integers $1 \leq p_2 < p_3 < \cdots < p_n \cdots$. The following recurrence formula

$$E_1 = E_2 = [0, 1),\ E_n = \left\{x \in E_{\bar{n}} : (-1)^{n+1} r_{p_{\bar{n}}}(x) > 0\right\},\ n > 2, \tag{2.5}$$

defines a partition of [0, 1). This family of sets uniquely determines a Haar type system as follows:

$$\xi_n(x) = \frac{r_{p_n}(x)\mathbb{I}_{E_n}(x)}{\sqrt{|E_n|}}, \quad n \geq 2. \tag{2.6}$$

We consider the tensor products of the Haar and Haar type systems

$$\chi_{\mathbf{n}}(\mathbf{x}) = \chi_{n_1}(x_1), \dots, \chi_{n_d}(x_n),$$
$$\xi_{\mathbf{n}}(\mathbf{x}) = \xi_{n_1}(x_1), \dots, \xi_{n_d}(x_n),$$

where $\mathbf{x} = (x_1, \dots, x_d) \in Q_d$, $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$. For a given function $f(\mathbf{x}) \in L^1(Q_d)$ let

$$a_{\mathbf{n}} = \int_{Q_d} f(\mathbf{x})\chi_{\mathbf{n}}(\mathbf{x})d\mathbf{x},\ \mathbf{n} = (n_1, n_2, \dots, n_d), \tag{2.7}$$

be the Fourier–Haar coefficients of f. We denote

$$\mathcal{S}^{\xi} f(\mathbf{x}) = \sum_{\mathbf{k}=\mathbf{1}}^{\infty} a_{\mathbf{k}}\xi_{\mathbf{k}}(\mathbf{x}) = \sum_{k_1=1}^{\infty} \dots \sum_{k_1=1}^{\infty} a_{k_1,\dots,k_d}\xi_{k_1}(x_1), \dots, \xi_{k_d}(x_d). \tag{2.8}$$

This series is said to be convergent (a.e., in L^p norm) if its rectangular partial sums

$$\mathcal{S}^{\xi}_{\mathbf{n}} f(\mathbf{x}) = \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} a_{\mathbf{k}}\xi_{\mathbf{n}}(\mathbf{x}) = \sum_{k_1=1}^{n_1} \dots \sum_{k_1=1}^{n_d} a_{k_1,\dots,k_d}\xi_{k_1}(x_1), \dots, \xi_{k_d}(x_d)$$

converges as $\min\{n_i\} \to \infty$. It is well known that the series (2.8) converges in L^1 norm. Besides, we have $\mathcal{S}^{\xi} f(\mathbf{x}) = f(\mathbf{x})$ whenever ξ coincides with the ordinary Haar system. If ξ coincides with the Haar system, then instead of $\mathcal{S}_{\mathbf{n}}^{\xi}$ the notation $\mathcal{S}_{\mathbf{n}}$ will be used. In the one-dimensional case ($d = 1$) the operators $\mathcal{S}^{\xi}$, $\mathcal{S}_{\mathbf{n}}^{\xi}$ and $\mathcal{S}_{\mathbf{n}}$ will be denoted by S^{ξ}, S_n^{ξ} and S_n respectively. Observe that

$$\mathcal{S}^{\xi} = \otimes_{k=1}^{d} \left(S^{\xi}\right)_k = \left(S^{\xi}\right)_1 \circ \cdots \circ \left(S^{\xi}\right)_d , \tag{2.9}$$

$$\mathcal{S}_{\mathbf{n}}^{\xi} = \otimes_{k=1}^{d} \left(S_{n_k}^{\xi}\right)_k = \left(S_{n_1}^{\xi}\right)_1 \circ \cdots \circ \left(S_{n_d}^{\xi}\right)_d . \tag{2.10}$$

Recall that the strong maximal function is defined by

$$Mf(\mathbf{x}) = \sup_{R:\,\mathbf{x}\in R} \frac{1}{|R|} \int_R f(\mathbf{t})d\mathbf{t},$$

where sup is taken over all d-dimensional intervals $R = (a_1, b_1) \times \cdots \times (a_d, b_d) \subset Q_d$ containing the point $\mathbf{x} \in Q_d$. It is well known that

$$\sup_{\mathbf{n}} \left| \sum_{\mathbf{k}=1}^{\mathbf{n}} a_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{x}) \right| \le Mf(\mathbf{x})$$

for any $f \in L^1(Q_d)$ with the Fourier–Haar coefficients (2.7). Thus, using the weak type inequality

$$m\left\{\mathbf{x} \in Q_d : Mf(\mathbf{x}) > \lambda\right\} \le c_d \int_{Q_d} \frac{|f|}{\lambda} \log^{d-1}\left(1 + \frac{|f|}{\lambda}\right),\ \lambda > 0, \tag{2.11}$$

(Fava [1] or Guzman [6], Ch. 2.3) and the relation (2.4), we conclude

$$\begin{aligned} m&\left\{\mathbf{x} \in Q_d : \sup_{\mathbf{n}} \left| \sum_{\mathbf{k}=1}^{\mathbf{n}} a_{\mathbf{k}} \xi_{\mathbf{k}}(\mathbf{x}) \right| > \lambda \right\} \\ &= m\left\{\mathbf{x} \in Q_d : \sup_{\mathbf{n}} \left| \sum_{\mathbf{k}=1}^{\mathbf{n}} a_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{x}) \right| > \lambda \right\} \\ &\le c_d \int_{Q_d} \frac{|f|}{\lambda} \log^{d-1}\left(1 + \frac{|f|}{\lambda}\right),\ \lambda \ge 0, \end{aligned} \tag{2.12}$$

where the equality in (2.12) follows from the definition of the Haar type system.

3 Almost Everywhere Convergence Classes of Functions

The following theorem is the main result of this section.

Theorem 3 *If a sequence of bounded linear operators $U_n : L^1(0,1) \to L^1(0,1)$ satisfies the conditions (A), (B) and $\mathcal{U}_{\mathbf{n}}$ is the multiple sequence of operators* (1.7) *generated by* U_n*, then there exist a Haar type system* $\xi = \{\xi_n(x)\}$ *and a sequence of integers* $0 < \nu(1) < \nu(2) < \cdots < \nu(k) < \cdots$ *such that for any function*

$$f \in \begin{cases} L\log^{d-2}(Q_d) & \text{if } d \geq 2, \\ L^1(0,1) & \text{if } d = 1, \end{cases}$$

we have

$$\lim_{\min\{n_k\}\to\infty} \left(\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^\xi\right) f(\boldsymbol{x}) - \mathcal{S}^\xi_{\boldsymbol{n}} f(\boldsymbol{x})\right) = 0 \tag{3.1}$$

at almost every $\boldsymbol{x} \in Q_d$.

An analogous theorem for martingale operator sequences was proved in the paper [11]. That is, if U_n is an arbitrary sequence of martingale operators, then there exists a sequence of sets $G_{\mathbf{n}} \subset Q_d$ with $m(G_{\mathbf{n}}) \to 1$ as $\min\{n_i\} \to \infty$ such that the relation

$$\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^\xi\right) f(\mathbf{x}) = \mathcal{S}^\xi_{\mathbf{n}} f(\mathbf{x}), \quad x \in G_{\mathbf{n}}, \quad \mathbf{n} \in \mathbb{N}^d,$$

holds for any $f \in L^1(Q_d)$. Some problems related to this martingale theorem were considered before in the papers by Hare and Stokolos [8], Hagelstein [7] and Stokolos [18].

Lemma 1 *If* $\varepsilon_{ni} > 0$*,* $n, i = 1, 2, \ldots$*, then for any sequence of bounded linear operators* (1.8) *satisfying (A) and (B), there exist a sequence of integers* $0 < \nu(1) < \nu(2) < \cdots < \nu(k) < \cdots$ *and a Haar type system* $\xi = \{\xi_n(x)\}$ *such that*

$$m\left\{x \in (0,1) : |U_{\nu(n)}\xi_i(x) - \xi_i(x)| > \varepsilon_{ni}\right\} < \varepsilon_{ni},\ 1 \leq i \leq n, \tag{3.2}$$

$$m\left\{x \in (0,1) : |U_{\nu(n)}\xi_i(x)| > \varepsilon_{ni}\right\} < \varepsilon_{ni},\ i > n. \tag{3.3}$$

Proof We use induction. The system ξ will be found in the form (2.6). Define $\xi_1(x) \equiv 1$ and $p_1 = 1$. Using the property (B) we have $U_\nu \xi_1(x) \to \xi_1(x)$ in measure as $\nu \to \infty$, and so we may take a number $\nu(1)$ satisfying (3.2) for $n = 1$. Then suppose we have already chosen the numbers $\nu(1) < \nu(2) < \cdots < \nu(k-1)$ and the first $k-1$ functions of the system ξ satisfying the relations (2.6), (3.2) and (3.3) for $n, i = 1, 2, \ldots, k-1$. We define the set E_k satisfying (2.5), i.e.,

$$E_k = \left\{x \in E_{\bar{k}} : (-1)^{k+1} r_{p_{\bar{k}}}(x) > 0\right\}.$$

Using the compactness of the operators $U_{\nu(n)}$, $n = 1, 2, \ldots, k-1$, we have

$$\lim_{m\to\infty} \left\| U_{\nu(n)}\left(r_m(x)\mathbb{I}_{E_k}(x)\right)\right\|_1 = 0, \quad n = 1, 2, \ldots, k-1.$$

Thus we can choose a number $m = p_k > p_{k-1}$ such that

$$\left\| U_{\nu(n)}\left(\frac{r_{p_k}(x)\mathbb{I}_{E_k}(x)}{\sqrt{|E_k|}}\right)\right\|_1 < (\varepsilon_{ki})^2, \quad n = 1, 2, \ldots, k-1.$$

Defining $\xi_k = \frac{r_{p_k}\mathbb{I}_{E_k}}{\sqrt{|E_k|}}$ and using Chebyshev's inequality, we get

$$m\left\{x \in (0,1) : U_{\nu(n)}\xi_k(x)\right) > \varepsilon_{ki}\right\} < \varepsilon_{ki}, \quad n = 1, 2, \ldots, k-1. \tag{3.4}$$

Then, using the convergence in measure $U_\nu \xi_i(x) \to \xi_i(x)$ as $\nu \to \infty$, for $i = 1, 2, \ldots, k$, we may choose $\nu(k) > \nu(k-1)$ such that

$$m\left\{x \in (0,1) : |U_{\nu(k)}\xi_i(x) - \xi_i(x)| > \varepsilon_{ki}\right\} < \varepsilon_{ki},\ 1 \le i \le k. \tag{3.5}$$

Combining (3.4) and (3.5) we get (3.2) and (3.3) for $n, i = 1, 2, \ldots, k$. This completes by induction the proof of the lemma. □

Let the function $f \in L\log^{d-1} L(Q_d)$ have Fourier–Haar coefficients $a_{\mathbf{k}}$ defined by (2.7). Suppose $1 \le s < d$ and denote

$$\begin{aligned}\delta_{k_1,\ldots,k_s}&(x_{s+1},\ldots,x_d)\\ &= \sup_{n_{s+1}\ge 1,\ldots,n_d\ge 1}\left|\sum_{k_{s+1}=1}^{n_{s+1}}\cdots\sum_{k_d=1}^{n_d} a_{\mathbf{k}}\prod_{i=s+1}^{d}\xi_{k_i}(x_i)\right|.\end{aligned} \tag{3.6}$$

Lemma 2 *If $f \in L\log^{d-s-1}(Q_d)$, $1 \le s < d$, then*

$$\begin{aligned}&m\left\{(x_1,\ldots,x_d) \in Q_d : \delta_{k_1,\ldots,k_s}(x_{s+1},\ldots,x_d) > \lambda\right\}\\ &\quad\le c_d\,(k_1,\ldots,k_s)^d\int_{Q_d}\frac{|f|}{\lambda}\log^{d-s-1}\left(1+\frac{|f|}{\lambda}\right),\end{aligned} \tag{3.7}$$

for any $\lambda \ge 1$.

Proof Observe that, if the integers $k_1, \ldots, k_s$ are fixed, then the multiple series

$$\sum_{k_{s+1}=1}^{\infty}\cdots\sum_{k_d=1}^{\infty} a_{\mathbf{k}}\prod_{i=s+1}^{d}\chi_{k_i}(x_i)$$

is the Fourier–Haar series of the function

$$\begin{aligned}g(x_{s+1},\ldots,x_d) &= g_{k_1,\ldots,k_s}(x_{s+1},\ldots,x_d)\\ &= \int_{Q_s} f(t_1,\ldots,t_s,x_{s+1},\ldots,x_d)\prod_{i=1}^{s}\chi_{k_i}(t_i)dt_1\ldots dt_s.\end{aligned}$$

Thus, using the notation (3.6) and the inequality (2.12) in the $(d-s)$-dimensional case, we obtain

$$\begin{aligned}&m\left\{(x_{s+1},\ldots,x_d) \in Q_{d-s} : \delta_{k_1,\ldots,k_s}(x_{s+1},\ldots,x_d) > \lambda\right\}\\ &\quad\le c_{d-s}\int_{Q_{d-s}}\Phi\left(\frac{|g(x_{s+1},\ldots,x_d)|}{\lambda}\right)dx_{s+1},\ldots,dx_d,\end{aligned} \tag{3.8}$$

where $\Phi(t) = t\log^{d-s-1}(1+t)$ and $\lambda > 1$. Since $|\chi_n(x)| \le \sqrt{n}$, we get

$$|g(x_{s+1},\dots,x_d)| \le \prod_{i=1}^{s}\sqrt{k_i}\int_{Q_s}|f(t_1,\dots,t_s,x_{s+1},\dots,x_d)|dt_1\dots dt_s. \tag{3.9}$$

It is easy to check that $\Phi(t)$ is a convex function and

$$\Phi(kx) \le k^{s+1}\Phi(x),\ x > 0,\ k \ge 1.$$

Thus, using (3.9) and Jensen's inequality, we obtain

$$\begin{aligned}\Phi&\left(\frac{|g(x_{s+1},\dots,x_d)|}{\lambda}\right)\\ &\le (k_1,\dots,k_s)^{\frac{s+1}{2}}\ \Phi\left(\int_{Q_s}\frac{|f(t_1,\dots,t_s,x_{s+1},\dots,x_d)|}{\lambda}dt_1\dots dt_s\right)\\ &\le (k_1,\dots,k_s)^{\frac{s+1}{2}}\int_{Q_s}\Phi\left(\frac{|f(t_1,\dots,t_s,x_{s+1},\dots,x_d)|}{\lambda}\right)dt_1\dots dt_s.\end{aligned}$$

Integration with respect to variables $x_{s+1},\dots,x_d$ implies

$$\begin{aligned}&\int_{Q_{d-s}}\Phi\left(\frac{|g(x_{s+1},\dots,x_d)|}{\lambda}\right)dx_{s+1},\dots,dx_d\\ &\quad\le (k_1,\dots,k_s)^d\int_{Q_d}\Phi\left(\frac{|f(t_1,\dots,t_d)|}{\lambda}\right)dt_1,\dots,dt_d.\end{aligned}$$

Combining this inequality with (3.8), we get

$$\begin{aligned}&m\{(x_1,\dots,x_d)\in Q_d : \delta_{k_1,\dots,k_s}(x_{s+1},\dots,x_d) > \lambda\}\\ &\quad= m\left\{(x_{s+1},\dots,x_d)\in Q_{d-s} : \delta_{k_1,\dots,k_s}(x_{s+1},\dots,x_d) > \lambda\right\}\\ &\quad\le c_d\left(\prod_{i=1}^{s}k_i\right)^d\int_{Q_d}\Phi\left(\frac{|f(t_1,\dots,t_d)|}{\lambda}\right)dt_1,\dots,dt_d.\end{aligned}$$

□

Proof of Theorem 3 Applying Lemma 1, we fix a Haar type system $\{\xi_n(x)\}$ and a sequence $\nu(n)$ satisfying the conditions (3.2), (3.3) with

$$\varepsilon_{nk} = 4^{-n-k}. \tag{3.10}$$

Then we denote

$$\alpha_k^{(n)}(x) = \begin{cases} U_{\nu(n)}\xi_k(x) - \xi_k(x), & \text{if } 1 \le k \le n,\\ U_{\nu(n)}\xi_k(x), & \text{if } \quad k > n.\end{cases} \tag{3.11}$$

The boundedness of the operators $\mathcal{U}_{\mathbf{n}}$ and the L^1-convergence of the series (2.8) imply

$$\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}\right) f(\mathbf{x}) = \sum_{\mathbf{k}=1}^{\infty} a_{\mathbf{k}} U_{\nu(n_1)}\xi_{k_1}(x_1), \ldots, U_{\nu(n_d)}\xi_{k_d}(x_d). \tag{3.12}$$

Substituting

$$U_{\nu(n_i)}\xi_{k_i}(x_i) = \begin{cases} \alpha_{k_i}^{(n_i)}(x_i), & \text{if} \quad k_i > n_i, \\ \xi_{k_i}(x_i) + \alpha_{k_i}^{(n_i)}(x_i), & \text{if} \ 1 \le k_i \le n_i, \end{cases}$$

in (3.12), we may easily observe that

$$\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}\right) f(\mathbf{x}) = \sum_{I \subset \{1,\ldots,d\}} \ \sum_{\mathbf{k}:\, 1 \le k_i \le n_i,\, i \in I^c} a_{\mathbf{k}} \prod_{i \in I} \alpha_{k_i}^{(n_i)}(x_i) \prod_{i \in I^c} \xi_{k_i}(x_i),$$

where the first sum is taken over all the subsets I of the set $\{1, \ldots, d\}$. If $I = \varnothing$, then we have

$$\begin{aligned} &\sum_{1 \le k_i \le n_i,\, i \in I^c} a_{\mathbf{k}} \prod_{i \in I} \alpha_{k_i}^{(n_i)}(x_i) \prod_{i \in I^c} \xi_{k_i}(x_i) \\ &= \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} a_{\mathbf{k}} \prod_{i=1}^{d} \xi_{k_i}(x_i) = \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} a_{\mathbf{k}} \xi_{\mathbf{k}}(\mathbf{x}) = \left(\mathcal{S}^{\xi} \circ \mathcal{S}_{\mathbf{n}}\right) f(\mathbf{x}). \end{aligned}$$

Thus we get

$$\begin{aligned} &\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}\right) f(\mathbf{x}) - \left(\mathcal{S}^{\xi} \circ \mathcal{S}_{\mathbf{n}}\right) f(\mathbf{x}) \\ &= \sum_{I \neq \varnothing} \ \sum_{\mathbf{k}:\, 1 \le k_i \le n_i,\, i \in I^c} a_{\mathbf{k}} \prod_{i \in I} \alpha_{k_i}^{(n_i)}(x_i) \prod_{i \in I^c} \xi_{k_i}(x_i). \end{aligned} \tag{3.13}$$

Hence, in order to prove the theorem, it is enough to show

$$\lim_{\min\{n_i\} \to \infty} \sum_{\mathbf{k}:\, 1 \le k_i \le n_i,\, i \in I^c} a_{\mathbf{k}} \prod_{i \in I} \alpha_{k_i}^{(n_i)}(x_i) \prod_{i \in I^c} \xi_{k_i}(x_i) = 0 \text{ a.e.} \tag{3.14}$$

whenever $I \neq \varnothing$. Without loss of generality we may suppose that $I = \{1, \ldots, s\}$, $1 \le s \le d$. So we must prove

$$\lim_{\min\{n_i\} \to \infty} \sum_{\mathbf{k}:\, 1 \le k_i \le n_i,\, i > s} a_{\mathbf{k}} \prod_{i=1}^{s} \alpha_{k_i}^{(n_i)}(x_i) \prod_{i=s+1}^{d} \xi_{k_i}(x_i) = 0 \text{ a.e.,} \tag{3.15}$$

where in the case $s = d$ the last product is not considered. Using (3.2), (3.3), (3.10) and (3.11), for the set

$$C_k^{(n)} = \left\{ x \in (0, 1) : \left| \alpha_k^{(n)}(x) \right| < 4^{-(n+k)} \right\}$$

we get

$$m\left(C_k^{(n)}\right) > 1 - 4^{-(n+k)}.$$

Denote

$$C^{(n)} = \bigcap_{k=1}^{\infty} C_k^{(n)} \subset (0,1),$$
$$C = \bigcup_{m\geq 1}\bigcap_{n\geq m} C^{(n)} \subset (0,1),$$
$$A = \{\mathbf{x} = (x_1, \dots, x_d) \in Q_d : x_k \in C\} \subset Q_d.$$

We have

$$m\left(C^{(n)}\right) > 1 - \sum_{k=1}^{\infty} 4^{-(n+k)} > 1 - 4^{-n}.$$

Thus we get $m(C) = 1$ and therefore $m(A) = 1$. Besides, for any $\mathbf{x} \in A$ there exists $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), \dots, n_d(\mathbf{x}))$ such that

$$\left|\alpha_{k_i}^{(n)}(x_i)\right| < 4^{-(n_i+k_i)},\ i = 1, 2, \dots, d,\ k = 1, 2, \dots,$$
$$\text{for any } \mathbf{n} > \mathbf{n}(\mathbf{x}), \quad \mathbf{x} \in A. \tag{3.16}$$

If $s = d$, then (3.15) is immediate. Indeed, we have $|a_{\mathbf{k}}| \leq \|f\|_1\sqrt{k_1, \dots, k_d}$ and so for any $\mathbf{x} \in A$ and $\mathbf{n} > \mathbf{n}(\mathbf{x})$ we get

$$\left|\sum_{\mathbf{k}} a_{\mathbf{k}} \prod_{i=1}^{d} \alpha_{k_i}^{(n_i)}(x_i)\right| \leq \|f\|_1 \sum_{\mathbf{k}} \prod_{i=1}^{d} \sqrt{k_i}, \cdot, 4^{-(n_i+k_i)} \leq \frac{c\|f\|_1}{4^{n_1+\cdots+n_d}}$$

which implies (3.15). At this moment the proof of the theorem in the case $d = 1$ is complete and we can suppose $d \geq 2$.

Now consider the case $1 \leq s < d$. Denote

$$B_{k_1,\dots,k_s}^{n_1,\dots,n_s} = \left\{\mathbf{x} \in Q_d : \delta_{k_1,\dots,k_s}(x_{s+1}, \dots, x_d) < (k_1, \dots, k_s)^d, \cdot, 2^{n_1+k_1+\cdots+n_s+k_s}\right\},$$
$$B^{n_1,\dots,n_s} = \bigcap_{k_1,\dots,k_s=1}^{\infty} B_{k_1,\dots,k_s}^{n_1,\dots,n_s},$$

where $\delta_{k_1,\dots,k_s}$ is the function defined in (3.6). Using Lemma 2, we get

$$m(B_{k_1,\dots,k_s}^{n_1,\dots,n_s}) > 1 - C_f 2^{-(n_1+k_1+\cdots+n_s+k_s)},$$
$$m\left(B^{n_1,\dots,n_s}\right) > 1 - C_f \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} 2^{-(n_1+k_1+\cdots+n_s+k_s)} = 1 - C_f \cdot 2^{-(n_1+,\cdots,+n_s)},$$

where

$$C_f = c_d \int_{Q_d} |f| \log^{d-s-1}(1+|f|).$$

Since by the hypothesis of the theorem $f \in L\log^{d-2} L$ and we have $s \geq 1$, C_f is bounded. Hence for the sets

$$B = \bigcup_{m_i \geq 1:\, i=1,\ldots,s} \; \bigcap_{n_i \geq m_i:\, i=1,\ldots,s} B^{n_1,\ldots,n_s} \subset Q_d$$

we have $m(B) = 1$. Observe that if $\mathbf{x} = (x_1, \ldots, x_d) \in B$, then there exists a vector $\mathbf{m}(\mathbf{x}) = (m_1(x), \ldots, m_d(x))$ such that for any $\mathbf{n} > \mathbf{m}(\mathbf{x})$ we have

$$\delta_{k_1,\ldots,k_s}(x_{s+1}, \ldots, x_d) < (k_1 \ldots k_s)^d \cdot 2^{n_1+k_1+\cdots+n_s+k_s},\ k_i \in \mathbb{N},$$
$$\mathbf{n} > \mathbf{m}(\mathbf{x}), \quad \mathbf{x} = (x_1, \ldots, x_d) \in B. \tag{3.17}$$

Note that the coordinates $m_{s+1}(x), \ldots, m_d(x)$ can be chosen arbitrarily. Combining (3.16) and (3.17), for any $\mathbf{x} \in G = A \cap B$ and $\mathbf{n} > \max\{\mathbf{n}(\mathbf{x}), \mathbf{m}(\mathbf{x})\}$ we get

$$\left| \sum_{\mathbf{k}:\, 1 \leq k_i \leq n_i,\, i > s} a_{\mathbf{k}} \prod_{i=1}^{s} \alpha_{k_i}^{(n_i)}(x_i) \prod_{i=s+1}^{d} \xi_{k_i}(x_i) \right|$$
$$\leq \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} \prod_{i=1}^{s} \left|\alpha_{k_i}^{(n_i)}(x_i)\right| \left| \sum_{k_{s+1}=1}^{n_{s+1}} \cdots \sum_{k_d=1}^{n_d} a_{\mathbf{k}} \prod_{i=s+1}^{d} \xi_{k_i}(x_i) \right|$$
$$\leq \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} \prod_{i=1}^{s} \left|\alpha_{k_i}^{(n_i)}(x_i)\right| \cdot \delta_{k_1,\ldots,k_s}(x_{s+1}, \ldots, x_d)$$
$$\leq \sum_{k_1=1}^{\infty} \cdots \sum_{k_s=1}^{\infty} 4^{-(n_1+k_1+\cdots+n_s+k_s)} (k_1 \ldots k_s)^d \cdot 2^{n_1+k_1+\cdots+n_s+k_s}$$
$$< C_d \cdot 2^{-(n_1+\cdots+n_s)} \tag{3.18}$$

where $C_d > 0$ is a constant. Since $m(G) = 1$, (3.18) completes the proof of Theorem 3. □

The functions $f(\mathbf{x}),\ g(\mathbf{x}) \in L^1(Q_d)$ are said to be equivalent ($f \sim g$), if they have the same distribution function, that is,

$$m\{\mathbf{x} \in Q_d : f(\mathbf{x}) > \lambda\} = m\{\mathbf{x} \in Q_d : g(\mathbf{x}) > \lambda\}, \quad \lambda \in \mathbb{R}.$$

Theorem 3 immediately implies

Theorem 4 *Let U_n be the operator sequence (1.8) satisfying the conditions (A) and (B). If the Fourier–Haar series*

$$\sum_{\boldsymbol{n}=1}^{\infty} a_{\boldsymbol{n}} \chi_{\boldsymbol{n}}(\boldsymbol{x}) \tag{3.19}$$

of a function $f \in L^1(Q_d)$ diverges almost everywhere, then there exists a function $g \in L^1(Q_d)$ such that $g \sim f$ and

$$\mathcal{U}_{\boldsymbol{n}} g(\boldsymbol{x}) \text{ diverges a.e. as } \min\{n_i\} \to \infty. \tag{3.20}$$

Proof Since the series (3.19) diverges a.e., the same also holds for the series

$$\sum_{\mathbf{n}=1}^{\infty} a_{\mathbf{n}} \xi_{\mathbf{n}}(\mathbf{x}), \tag{3.21}$$

where $\xi = \{\xi_n\}$ is the Haar type system obtained by Theorem 3. On the other hand (3.21) converges in L^1 norm to a function

$$g = \mathcal{S}^{\xi} f \in L^1(Q_d).$$

We have $g \sim f$ and

$$\mathcal{U}_{\nu(\mathbf{n})} g(\mathbf{x}) = (\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}) f(\mathbf{x}).$$

Thus, according to (3.1), we get

$$\lim_{\min\{n_i\}\to 0} \mathcal{U}_{\nu(\mathbf{n})} g(\mathbf{x}) - \mathcal{S}^{\xi}_{\mathbf{n}} f(\mathbf{x}) = 0 \text{ a.e.}$$

and then the a.e. divergence of the partial sums $\mathcal{S}^{\xi}_{\mathbf{n}} f(\mathbf{x})$ of the series (3.21) yields the divergence of $\mathcal{U}_{\nu(\mathbf{n})} g(\mathbf{x})$, which completes the proof. □

Proof of Theorem 1 If Φ satisfies the condition (1.10), then there exists a function $f \in \Phi(L)(Q_d)$, $f(\mathbf{x}) \geq 0$, whose Fourier–Haar series (3.19) diverges a.e. We will also have $g \in \Phi(L)(Q_d)$, where $g \sim f$ is the function obtained by Theorem 4. Then the relation (3.20) completes the proof of Theorem 1. □

So we consider the sequence of convolution operators

$$U_n f(x) = \int_0^1 K_n(x-t) f(t) dt, \tag{3.22}$$

where the kernels $K_n \in L^{\infty}[0, 1)$ are 1-periodic functions and form an approximation of identity. That is

1. $\int_0^1 K_n(t)dt \to 1$ as $n \to \infty$,
2. $K_n^*(x) = \sup_{|x|\le|t|\le 1/2} |K_n(t)| \to 0$ as $n \to \infty, 0 < |x| < 1/2$,
3. $\sup_n \int_0^1 K_n^*(x) < \infty$.

It is well known that such an operator sequence U_n satisfies the conditions (A) and (B). Moreover, $U_n f(x)$ converges in L^p for any $f \in L^p$, $1 \le p < \infty$, and the convergence is uniform while f is a continuous 1-periodic function. Let (1.9) be the multiple operator sequence generated from (3.22). It can be written in the form

$$\mathcal{U}_{\mathbf{n}} f(\mathbf{x}) = \int_{Q_d} K_{n_1}(t_1), \dots, K_{n_d}(t_d) f(\mathbf{x}-\mathbf{t}) dt_1, \dots, dt_d. \tag{3.23}$$

The following theorem determines the exact Orlicz class of functions guaranteeing a.e. convergence for the sequence of operators (1.9). The first part of the theorem is based on a standard argument (see, for example, [2] Theorem 4.2) and immediately follows from the weak estimate of the strong maximal function.

Theorem 5 *Let $\mathcal{U}_{\boldsymbol{n}}$ be the sequence of operators* (1.9) *generated by* (3.22). *Then*
(1) *if $f \in L\log^{d-1} L(Q_d)$, then $\mathcal{U}_{\boldsymbol{n}} f(\boldsymbol{x}) \to f(\boldsymbol{x})$ a.e. as $\min\{n_i\} \to \infty$,*
(2) *if the function Φ satisfies the condition*

$$\lim_{t\to\infty} \frac{\Phi(t)}{t\log^{d-1} t} = 0,$$

then there exists a function $f \in \Phi(L)(Q_d)$, $f(x) \ge 0$, such that

$$\limsup_{\min\{n_i\}\to\infty} |\mathcal{U}_{\boldsymbol{n}} f(\boldsymbol{x})| = \infty \text{ a.e. on } Q_d. \tag{3.24}$$

If in addition $K_n(x) \ge 0$, then (3.24) *holds everywhere.*

Proof We may suppose that all the functions are 1-periodic in each variable. Since $K_n^*(x)$ is even and decreasing on [0, 1/2], we may find a step function of the form

$$\varphi_n(x) = \sum_{i=1}^{m(n)} a_i^{(n)} \mathbb{I}_{(-x_i^{(n)}, x_i^{(n)})}(x),\; a_i^{(n)} \ge 0,\; x_i^{(n)} \ge 0,$$

such that $K_n^*(x) \le \varphi_n(x)$ and

$$\int_0^1 \varphi_n(x)dx = \sum_{i=1}^{m} 2x_i^{(n)} a_i^{(n)} < 2\int_0^1 K_n^*(x)dx < B.$$

This implies that

$$\begin{aligned}
|\mathcal{U}_{\mathbf{n}} f(\mathbf{x})| &= \left|\int_{Q_d} K_{n_1}(t_1), \dots, K_{n_d}(t_d) f(\mathbf{x}-\mathbf{t}) dt_1, \dots, dt_d\right| \\
&\le \int_{Q_d} K_{n_1}^*(t_1), \dots, K_{n_d}^*(t_d) |f(\mathbf{x}-\mathbf{t})| dt_1, \dots, dt_d \\
&\le \int_{Q_d} \varphi_{n_1}(t_1), \dots, \varphi_{n_d}(t_d) |f(\mathbf{x}-\mathbf{t})| dt_1, \dots, dt_d
\end{aligned}$$

$$= \sum_{i=1}^{m(n_1)} \cdots \sum_{i=1}^{m(n_d)} \prod_{k=1}^{d} (2x_i^{(n_k)} a_i^{(n_k)})$$
$$\times \frac{1}{2^d x_i^{(n_1)}, \ldots, x_i^{(n_d)}} \int_{-x_i^{(n_1)}}^{x_i^{(n_1)}} \cdots \int_{-x_i^{(n_d)}}^{x_i^{(n_d)}} |f(\mathbf{x}-\mathbf{t})| dt_1, \ldots, dt_d$$
$$\leq Mf(\mathbf{x}) \sum_{i=1}^{m(n_1)} 2x_i^{(n_1)} a_i^{(n_1)} \ldots \sum_{i=1}^{m(n_d)} 2x_i^{(n_d)} a_i^{(n_d)} \leq B^d Mf(\mathbf{x}). \quad (3.25)$$

Hence, according to (2.11), we have

$$m\{\mathbf{x} \in Q_d : \sup_{\mathbf{n}} |\mathfrak{U}_{\mathbf{n}} f(\mathbf{x})| > \lambda\} \leq c_d \int_{Q_d} \frac{|f|}{\lambda} \log^{d-1}\left(1 + \frac{|f|}{\lambda}\right). \quad (3.26)$$

Now take a function $f \in L \log^{d-1} L(Q_d)$. Let $\lambda > 0$ be an arbitrary number. Observe that for any $\varepsilon > 0$ we can write f in the form $f = g + h$ where g is continuous and

$$\int_{Q_d} \frac{2|h|}{\lambda} < \varepsilon, \quad \int_{Q_d} \frac{2|h|}{\lambda} \log^{d-1}\left(1 + \frac{2|h|}{\lambda}\right) < \varepsilon.$$

From the continuity of g we have $\mathfrak{U}_{\mathbf{n}} g(\mathbf{x})$ uniformly converges to $g(\mathbf{x})$. Thus, applying (3.26) and Chebyshev's inequality, we get

$$m\left\{\mathbf{x} \in Q_d : \limsup_{\min\{n_i\}\to\infty} |\mathfrak{U}_{\mathbf{n}} f(\mathbf{x}) - f(\mathbf{x})| > \lambda\right\}$$
$$= m\left\{\mathbf{x} \in Q_d : \limsup_{\min\{n_i\}\to\infty} |\mathfrak{U}_{\mathbf{n}} h(\mathbf{x}) - h(\mathbf{x})| > \lambda\right\}$$
$$\leq m\left\{\mathbf{x} \in Q_d : \sup_{\mathbf{n}} |\mathfrak{U}_{\mathbf{n}} h(\mathbf{x})| > \lambda/2\right\} + \{\mathbf{x} \in Q_d : |h(\mathbf{x})| > \lambda/2\}$$
$$\leq c_d \int_{Q_d} \frac{2|h|}{\lambda} \log^{d-1}\left(1 + \frac{2|h|}{\lambda}\right) + \int_{Q_d} \frac{2|h|}{\lambda} < (c_d + 1)\varepsilon.$$

Since $\varepsilon > 0$ can be small enough, we obtain

$$m\left\{\mathbf{x} \in Q_d : \limsup_{\min\{n_i\}\to\infty} |\mathfrak{U}_{\mathbf{n}} f(\mathbf{x}) - f(\mathbf{x})| > \lambda\right\} = 0$$

for any $\lambda > 0$. This implies the first part of the theorem.

To prove the second part, we apply Theorem 1. Then we find a function $f \in \Phi(L)(Q_d)$, $f(\mathbf{x}) \geq 0$, satisfying (3.24) almost everywhere. To get everywhere divergence in the case $K_n(x) \geq 0$, we modify the function $f(\mathbf{x})$ as follows. Suppose

$E \subset Q_d$ is the set where (3.24) doesn't hold. We have $mE = 0$. Define a sequence of open sets $G_n \subset Q_d$, $E \subset G_n \subset G_{n-1}$, such that

$$m(G_n) < 2^{-n}.$$

Then we consider the function

$$\tilde{f}(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}), \quad g(\mathbf{x}) = \sum_{n=1}^{\infty} n \cdot \mathbb{I}_{G_n}(\mathbf{x}).$$

It is easy to check that g and so $\tilde{f}$ is from $\Phi(L)$ and

$$\lim_{\min\{n_i\}\to\infty} \mathcal{U}_{\mathbf{n}} g(\mathbf{x}) = +\infty, \quad \mathbf{x} \in E.$$

The using the positivity of the operators $\mathcal{U}_{\mathbf{n}}$, one can easily get the divergence of $\mathcal{U}_{\mathbf{n}} \tilde{f}(\mathbf{x})$ at any $\mathbf{x} \in Q_d$. □

4 Estimates of L^p-Norms

In this section we suppose $p \geq 1$ is fixed and consider a sequence of operators U_n satisfying (A) and a stronger condition

(B_p) if $f \in L^p(0, 1)$, then $\|U_n f - f\|_{L^p(0,1)} \to 0$ ($p \geq 1$),

instead of (B). Note that, according to the Banach–Steinhaus theorem, condition (B_p) implies

$$1 \leq M = \sup_{n\geq 1} \|U_n\|_{L^p\to L^p} < \infty. \tag{4.1}$$

The following theorem is the main result of this section.

Theorem 6 *If $1 \leq p < \infty$, $\delta_n \searrow 0$ and the sequence of bounded linear operators U_n in $L^1(0, 1)$ satisfies the conditions (A) and (B_p), then there exist a Haar type system $\xi = \{\xi_n(x)\}$ and a sequence of integers $0 < \nu(1) < \nu(2) < \cdots < \nu(k) < \cdots$ such that*

$$\left\| \left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}\right) - \mathcal{S}^{\xi}_{\boldsymbol{n}} \right\|_p < \delta_m, \quad \min\{n_k\} \geq m. \tag{4.2}$$

The proof of the next lemma is similar to Lemma 1. So it will be stated briefly.

Lemma 3 *Let $p \geq 1$, $\varepsilon_i \searrow 0$ and the sequence of bounded linear operators (1.8) satisfies the conditions (A) and (B_p). Then there exist a sequence of integers $0 < \nu(1) < \nu(2) < \cdots < \nu(k) < \cdots$ and a Haar type system $\xi = \{\xi_n(x)\}$ such that*

$$\left\| U_{\nu(n)}\xi_i(x) - \xi_i(x) \right\|_p < \varepsilon_n, \quad i = 1, 2, \ldots, n, \tag{4.3}$$

$$\|U_{\nu(n)}\xi_i(x)\|_p < \varepsilon_i, \quad i > n. \tag{4.4}$$

for any $n = 1, 2, \dots$.

Proof We will use induction. Define $\xi_1(x) \equiv 1$. Using the property (B_p), we may find a number $\nu(1)$, satisfying (4.3) for $n = 1$. Then suppose we have already chosen the numbers $\nu(1) < \nu(2) < \cdots < \nu(k)$ and the first k functions of the system $\xi = \{\xi_n(x)\}$, satisfying the relations (4.3) and (4.4) for $n = 1, 2, \dots, k$. From the compactness of the operators follows the existence of a number $p_{k+1} > p_k$ such that

$$\left\| U_{\nu(i)}\left(r_{p_{k+1}}(x)\mathbb{I}_{E_k}(x)\right)\right\|_p < \varepsilon_{k+1}, \quad i = 1, 2, \dots, k.$$

Defining $\xi_{k+1} = r_{p_{k+1}}\mathbb{I}_{E_k}$ we will have (4.4) for $i = k+1$ and for each $1 \le n \le k$. Then using property (B_p), we may chose $\nu(k+1)$ satisfying (4.3) for $n = k+1$ and for each $1 \le i \le k+1$. This completes the induction and the proof of Lemma 3. □

The following lemma was proved in [11].

Lemma 4 ([11]) *If U and V are bounded linear operators on $L^1[0, 1)$, then*

$$(V)_n \circ (U)_m = (U)_m \circ (V)_n, \quad n \neq m, \quad 1 \le n, m \le d.$$

Proof (Proof of Theorem 6) One-dimensional case To prove (4.2) in the one-dimensional case, we must construct a Haar type system ξ and a sequence of integers $\nu(n)$ such that

$$\left\| U_{\nu(n)} \circ S^{\xi} - S_n^{\xi}\right\|_p < \delta_n, \quad n = 1, 2, \dots. \tag{4.5}$$

Using Lemma 3, we find ξ with the relations (4.3) and (4.4), where the sequence $\varepsilon_n \searrow 0$ satisfies the inequality

$$\varepsilon_n < \delta_n/4^n, \quad n = 1, 2, \dots.$$

Take an arbitrary function

$$f(x) = \sum_{n=1}^{\infty} a_n \chi_n(x) \in L^p.$$

We have

$$S^{\xi} f(x) = \sum_{k=1}^{\infty} a_k \xi_k(x),$$

$$S_n^{\xi} f(x) = \sum_{k=1}^{n} a_k \xi_k(x),$$

$$\left(U_{\nu(n)} \circ S^{\xi}\right) f(x) = \sum_{k=1}^{\infty} a_k U_{\nu(n)} \xi_k(x).$$

Thus, using the bound $|a_k| \le \sqrt{k}\|f\|_p$ and conditions (4.3), (4.4), we get

$$\begin{aligned}
&\left\|\left(U_{\nu(n)} \circ S^\xi - S_n^\xi\right) f(x)\right\|_p \\
&\quad = \left\|\sum_{k=1}^{n} a_k \left(U_{\nu(n)}\xi_k(x) - \xi_k(x)\right) + \sum_{k=n+1}^{\infty} a_k U_{\nu(n)}\xi_k(x)\right\|_p \\
&\quad \le \varepsilon_n \sum_{k=1}^{n} |a_k| + \sum_{k=n+1}^{\infty} |a_k|\varepsilon_k \\
&\quad \le \|f\|_p \left(n\sqrt{n}\varepsilon_n + \sum_{k=n+1}^{\infty} \sqrt{k}\varepsilon_k\right) < \delta_n \|f\|_p
\end{aligned}$$

which implies (4.5).

The general case Applying the one-dimensional case of the theorem, we may find a Haar type system with

$$\left\|U_{\nu(n)} \circ S^\xi - S_n^\xi\right\|_p < \gamma_n, \quad n = 1, 2, \ldots, \tag{4.6}$$

where

$$\gamma_n \searrow 0, \quad \gamma_n \le \delta_n / M^d,$$

and M is the constant defined in (4.1). We claim that

$$\left\|\otimes_{k=1}^{\mu} \left(U_{\nu(n_k)} \circ S^\xi\right)_k - \otimes_{k=1}^{\mu} \left(S_{n_k}^\xi\right)_k\right\|_p < \gamma_{\min\{n_1,\ldots,n_\mu\}} \cdot M^\mu \tag{4.7}$$

The proof of (4.7) is by induction on the dimension $\mu = 1, 2, \ldots, d$. The case $\mu = 1$ is just (4.6), since by (4.1) we have $M \ge 1$. Writing (4.6) with respect to each coordinate, we get

$$\left\|(\mathcal{U}_{\nu(n)} \circ S^\xi)_k - (S_n^\xi)_k\right\|_p < \gamma_n. \tag{4.8}$$

Suppose the case of dimension $\mu - 1$ is already proved, that is,

$$\left\|\otimes_{k=1}^{\mu-1} \left(U_{\nu(n_k)} \circ S^\xi\right)_k - \otimes_{k=1}^{\mu-1} \left(S_{n_k}^\xi\right)_k\right\|_p \le \gamma_{\min\{n_1,\ldots,n_{\mu-1}\}} M^{\mu-1}. \tag{4.9}$$

Let us prove the case of dimension μ. Observe that

$$\begin{aligned}
&\otimes_{k=1}^{\mu} \left(U_{\nu(n_k)} \circ S^\xi\right)_k - \otimes_{k=1}^{\mu} \left(S_{n_k}^\xi\right)_k \\
&\quad = \left[\otimes_{k=1}^{\mu-1} \left(U_{\nu(n_k)} \circ S^\xi\right)_k\right] \circ \left[\left(U_{\nu(n_\mu)} \circ S^\xi\right)_\mu - \left(S_{n_\mu}^\xi\right)_\mu\right] \\
&\qquad + \left[\otimes_{k=1}^{\mu-1} \left(U_{\nu(n_k)} \circ S^\xi\right)_k - \otimes_{k=1}^{\mu-1} \left(S_{n_k}^\xi\right)_k\right] \circ \left(S_{n_\mu}^\xi\right)_\mu.
\end{aligned} \tag{4.10}$$

Besides, we have

$$\left\| \left(S^{\xi}_{n_\mu} \right)_\mu \right\|_p \le 1,$$

$$\left\| \otimes_{k=1}^{\mu-1} \left(U_{\nu(n_k)} \circ S^{\xi} \right)_k \right\|_p \le \prod_{k=1}^{\mu-1} \left\| U_{\nu(n_k)} \right\|_p \left\| \left(S^{\xi} \right)_k \right\|_p \le M^{\mu-1},$$

and therefore, also using (4.8), (4.9) and (4.10), we get the estimate

$$\left\| \otimes_{k=1}^{\mu} \left(U_{\nu(n_k)} \circ S^{\xi} \right)_k - \otimes_{k=1}^{\mu} \left(S^{\xi}_{n_k} \right)_k \right\|_p$$
$$\le \gamma_{n_\mu} M^{\mu-1} + \gamma_{\min\{n_1,\dots,n_{\mu-1}\}} \cdot M^{\mu-1} \le \gamma_{\min\{n_1,\dots,n_\mu\}} M^{\mu},$$

which completes the induction and the proof of (4.7). Then, applying Lemma 4 several times, we obtain

$$\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi} = \otimes_{k=1}^{d} \left(U_{\nu(n_k)} \right)_k \circ \otimes_{k=1}^{d} \left(S^{\xi} \right)_k = \otimes_{k=1}^{d} \left(U_{\nu(n_k)} \circ S^{\xi} \right)_k ,$$

and therefore we get

$$\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi} \right) - \mathcal{S}^{\xi}_{\mathbf{n}} = \otimes_{k=1}^{d} \left(U_{\nu(n_k)} \circ S^{\xi} \right)_k - \otimes_{k=1}^{d} \left(S^{\xi}_{n_k} \right)_k$$

which means that in the case $\mu = d$ the inequality (4.2) coincides with (4.7). Theorem 6 is proved. □

If $a \lesssim b$ and $a \gtrsim b$ are satisfied at the same time, then we write $a \sim b$.

For the operator sequence $\mathcal{U}_{\mathbf{n}}$ generated by (1.8) we consider the maximal operator

$$\mathcal{U}^* f(\mathbf{x}) = \sup_{\mathbf{n}} \left| \mathcal{U}_{\mathbf{n}} f(\mathbf{x}) \right| .$$

The norm of this operator is defined by

$$\|\mathcal{U}^*\|_p = \sup_{\|f\|_p \le 1} \|\mathcal{U}^* f(\mathbf{x})\|_p.$$

This quantity describes the least constant $c > 0$ for which the inequality

$$\left\| \mathcal{U}^* f(\mathbf{x}) \right\|_p \le c \|f\|_p$$

holds for any $f \in L^p(Q_d)$. The similar operator for the partial sums of Fourier–Haar series is denoted by

$$\mathcal{S}^* f(\mathbf{x}) = \sup_{\mathbf{n}} \left| \mathcal{S}_{\mathbf{n}} f(\mathbf{x}) \right| .$$

We will consider also the maximal operator generated by a Haar type system defined by

$$(\mathcal{S}^{\xi})^* f(\mathbf{x}) = \sup_{\mathbf{n}} \left|\mathcal{S}^{\xi}_{\mathbf{n}} f(\mathbf{x})\right|$$

The following estimate is well known:

$$\|Mf(\mathbf{x})\|_p \sim \left(\frac{p}{p-1}\right)^d \|f\|_p, \quad 1 < p < \infty, \tag{4.11}$$

(see, for example, [4]), which also implies

$$\|(\mathcal{S}^{\xi})^*\|_p = \|\mathcal{S}^*\|_p \sim \left(\frac{p}{p-1}\right)^d. \tag{4.12}$$

We prove the following

Theorem 7 *If* $1 < p < \infty$ *and the sequence of bounded linear operators* (1.8) *satisfies conditions (A) and* (B_p) *and* $\mathcal{U}_{\boldsymbol{n}}$ *is generated by* (1.8)*, then*

$$\|\mathcal{U}^*\|_p \geq \|\mathcal{S}^*\|_p. \tag{4.13}$$

Proof Let $\varepsilon > 0$ be arbitrary. Using (4.12) we may choose a function $f \in L^p(Q_d)$ with $\|f\|_p = 1$ such that

$$\|\mathcal{S}^* f(\mathbf{x})\|_p > \|\mathcal{S}^*\|_p - \varepsilon.$$

Obviously we can fix an integer m such that

$$\left\|\sup_{\mathbf{n}:\, n_i \leq m} |\mathcal{S}_{\mathbf{n}} f(\mathbf{x})|\right\|_p \geq \|\mathcal{S}^*\|_p - 2\varepsilon. \tag{4.14}$$

We take an arbitrary sequence $\delta_n \searrow 0$ such that $\delta_k = \varepsilon/m^d$, $k = 1, 2, \ldots, m$. Applying Theorem 6 with this sequence, we determine a Haar type system ξ and a sequence of integers $\nu(n)$ satisfying (4.2). Denote $g(\mathbf{x}) = \mathcal{S}^{\xi} f(\mathbf{x})$. We have $\|g\|_p = \|f\|_p = 1$, and from (4.2), (4.14) it follows that

$$\begin{aligned}
\|\mathcal{U}^* g(\mathbf{x})\|_p &\geq \left\|\sup_{\mathbf{n}} \left|\mathcal{U}_{\nu(\mathbf{n})} g(\mathbf{x})\right|\right\|_p = \left\|\sup_{\mathbf{n}} \left|\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}\right) f(\mathbf{x})\right|\right\|_p \\
&\geq \left\|\sup_{\mathbf{n}:\, n_i \leq m} \left|\left(\mathcal{U}_{\nu(\mathbf{n})} \circ \mathcal{S}^{\xi}\right) f(\mathbf{x})\right|\right\|_p \\
&\geq \left\|\sup_{\mathbf{n}:\, n_i \leq m} \left|\mathcal{S}^{\xi}_{\mathbf{n}} f(\mathbf{x})\right|\right\|_p - m^d \cdot \frac{\varepsilon}{m^d} = \left\|\sup_{\mathbf{n}:\, n_i \leq m} |\mathcal{S}_{\mathbf{n}} f(\mathbf{x})|\right\|_p - \varepsilon \\
&> \|\mathcal{S}^*\|_p - 3\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain (4.13). □

Theorem 8 *Let $1 < p < \infty$ and the kernels $K_n(x)$ form an approximation of identity. Then the multiple operator sequence $\mathcal{U}_{\boldsymbol{n}}$ defined in* (3.23) *satisfies the relation*

$$\|\mathcal{U}^*\|_p \sim \left(\frac{p}{p-1}\right)^d.$$

Proof The lower bound

$$\|\mathcal{U}^*\|_p \gtrsim \left(\frac{p}{p-1}\right)^d$$

immediately follows from (4.12) and Theorem 7. To prove the upper bound we use the estimate (3.25). So we have

$$|\mathcal{U}^* f(\mathbf{x})| \le c \cdot Mf(\mathbf{x}) \tag{4.15}$$

where $Mf(\mathbf{x})$ is the strong maximal function. From (4.15) and (4.11) we conclude

$$\|\mathcal{U}^* f(\mathbf{x})\|_p \lesssim \left(\frac{p}{p-1}\right)^d \|f\|_p$$

and therefore we get $\|\mathcal{U}^*\|_p \lesssim \left(\frac{p}{p-1}\right)^d$, which completes the proof of the theorem. □

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