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Relations Between Threshold Constants for Yamabe Type Bordism Invariants

Bernd Ammann[1] · **Nadine Große**[2]

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Abstract In the work of Ammann et al. it has turned out that the Yamabe invariant on closed manifolds is a bordism invariant below a certain threshold constant. A similar result holds for a spinorial analogon. These threshold constants are characterized through Yamabe-type equations on products of spheres with rescaled hyperbolic spaces. We give variational characterizations of these threshold constants, and our investigations lead to an explicit positive lower bound for the spinorial threshold constants.

Keywords Dirac operator · Yamabe constant · Yamabe invariant · Conformal Hijazi inequality

Mathematics Subject Classification 53C21 (Primary) 35J60 · 53C27 · 57R65 (Secondary)

1 Introduction

The smooth Yamabe invariant, also called Schoen's σ-constant, of a closed manifold M^m is defined as

✉ Nadine Große
nadine.grosse@math.uni-freiburg.de

Bernd Ammann
bernd.ammann@mathematik.uni-regensburg.de

1 Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

2 Mathematisches Institut, Albert-Ludwigs-Universität, 79104 Freiburg, Germany

$$\sigma^*(M) := \sup \inf \int_M \mathrm{scal}^g \mathrm{dvol}_g \in \Big(-\infty, \sigma^*(\mathbb{S}^m) = m(m-1)\mathrm{vol}(\mathbb{S}^m)^{\frac{2}{m}} \Big],$$

where the supremum runs over all conformal classes $[g_0]$ on M, and the infimum goes over all metrics g in $[g_0]$ such that (M, g) has volume one. The smooth Yamabe invariant is an important geometric quantity. In particular, $\sigma^*(M) > 0$ if and only if M admits a metric of positive scalar curvature. However, the smooth Yamabe invariant is quite mysterious. It is only known for very few examples, e.g., the sphere, cp. Remark 2.4, $\sigma^*(\mathbb{T}^m) = 0$, [20, Corollary 2.5], $\sigma^*(\mathbb{RP}^3) = 2^{-2/3}\sigma^*(\mathbb{S}^3)$, [17, Corollary 2.3], $\sigma^*(\mathbb{C}P^2) = 12\sqrt{2}\pi$, [26,33], and $\sigma^*(\Gamma \setminus \mathbb{H}^3) = -6v_\Gamma^{2/3}$ where v_Γ is the volume of the compact quotient $\Gamma \setminus \mathbb{H}^3$ with respect to the hyperbolic metric, [13], [31, Sect. II.8]. In particular, there is no known example of a manifold of dimension $m \geq 5$ with $\sigma^*(M) \notin \{0, \sigma^*(\mathbb{S}^m)\}$.

In [5] Dahl, Humbert, and the first author proved a surgery formula for the smooth Yamabe invariant, cp. Theorem 2.5. In particular it says that if a manifold N^m is obtained from a closed manifold M^m by a surgery of codimension $m - k \geq 3$ and $\sigma^*(M)$ is below a certain positive threshold constant $\Lambda_{m,k}$, then $\sigma^*(N) \geq \sigma^*(M)$. The threshold constants $\Lambda_{m,k}$ appearing in this result are certain Yamabe-type invariants for special noncompact model spaces $\mathbb{M}_c^{m,k}$ which are products of rescaled hyperbolic spaces and spheres; see Sect. 2.2 for the precise definition. In particular, it follows that the smooth Yamabe invariant is a bordism invariant in the following sense: Suppose that M and N are connected closed smooth spin manifolds of dimension $m \geq 5$ with fundamental group Γ, representing the same element in $\Omega_m^{\mathrm{spin}}(B\Gamma)$, then $0 \leq \sigma^*(M) < \Lambda_m := \min_{k=2,\ldots,m-3} \Lambda_{m,k}$ implies $\sigma^*(M) = \sigma^*(N)$, [5, Sect. 1.4]. Thus, if sufficiently many manifolds with $\sigma^*(M) \in (0, \Lambda_m)$ exist, one obtains a rich and interesting subgroup in the bordism group $\Omega_m^{\mathrm{spin}}(B\Gamma)$ and similar versions hold in oriented bordism classes.

In order to understand the structure of the subgroup, it is essential to get as much knowledge about the surgery constants $\Lambda_{m,k}$ as possible.

If current conjectures about explicit lower bounds for $\Lambda_{m,k}$, see [5, Sect. 1.4], turn out to be true, then the supremum in the smooth Yamabe invariant of $\mathbb{CP}^3$ is not attained by the Fubini–Study metric.

The current article will not give an explicit positive lower bound for $\Lambda_{m,k}$, but it will provide many relations to a spinorial analogue of the problem. The smooth Yamabe invariant $\sigma^*(M)$ has a spinorial analogue $\sigma^*_{\mathrm{spin}}(M)$; cf. Sect. 2.2. For closed manifolds the Hijazi inequality gives $\sigma^*_{\mathrm{spin}}(M) \geq \sigma^*(M)$. As in the Yamabe case there is a surgery formula for σ^*_{spin}, cp. Theorem 2.6, and again a threshold constant $\Lambda^{\mathrm{spin}}_{m,k}$ appears. However, in contrast to the (non-spinorial) Yamabe invariant codimension 2 is allowed as well. This has implications for the smooth Yamabe constant as well: If M and N are arbitrary closed spin manifolds (not necessarily simply connected), and if M is spin-bordant to N, then $\sigma^*_{\mathrm{spin}}(M) < \Lambda^{\mathrm{spin}}_{m,k}$ for $k = 0, \ldots, m-2$ implies $\sigma^*_{\mathrm{spin}}(M) = \sigma^*_{\mathrm{spin}}(N)$. In particular, $\sigma^*_{\mathrm{spin}}(M) \geq \sigma^*(N)$. Finding interesting manifolds with $\sigma^*_{\mathrm{spin}}(M) < \Lambda^{\mathrm{spin}}_{m,k}$ consists of two parts. First, one has to obtain explicit positive lower bounds for $\Lambda^{\mathrm{spin}}_{m,k}$, which is the main subject of the present article, and then one has to find examples for M, which is not covered here.

As the threshold constant are defined as (spinorial) Yamabe-type invariants of noncompact model spaces, one expects in view of the Hijazi inequality that $\Lambda^{\mathrm{spin}}_{m,k} \geq \Lambda_{m,k}$. This question is quite subtle because on noncompact manifolds there are several ways to define Yamabe-type invariants which are sometimes related and sometimes unrelated to each other. One goal of the article is to clarify these relations.

The structure of the article is as follows: In Sect. 2 we fix notation, summarize some preliminaries and recall existing results. In particular, we define the model spaces and several versions of the spinorial and the non-spinorial Yamabe invariant for noncompact manifolds. This allows us to summarize the results of the article in Sect. 3. These results are proved in the remaining sections. In particular, in Sect. 4 we provide a regularity statement for the Euler–Lagrange equation of the spinorial Yamabe functional, which is a nonlinear Dirac eigenvalue equation. For more details, we refer to the end of Sect. 4.

2 Preliminaries

Throughout the article we assume that the reader is familiar with the basic facts about the solution of the Yamabe problem on closed manifolds by Trudinger, Aubin, Schoen and Yau. There are many beautifully written introductions in the literature, e.g., [28,34].

2.1 Notation

In the article a spin manifold always means a manifold admitting a spin structure together with a fixed choice of spin structure.

The notion of a (topological) spin structure can be defined for arbitrary oriented manifolds. A topological spin structure is a principal bundle P for the non-trivial double cover of $\mathrm{GL}_+(n,\mathbb{R})$, together with an equivariant map $\vartheta : P \to P_{\mathrm{GL}_+(n,\mathbb{R})}M$ to the $\mathrm{GL}_+(n,\mathbb{R})$-principal bundle $P_{\mathrm{GL}_+(n,\mathbb{R})}M$ of positively oriented frames. Topological spin structures exist, as soon as an obstruction, the second Stiefel–Whitney class w_2 of its tangent bundle vanishes.

For defining the spinor bundle, we should pass to metric spin structures. By restricting $P_{\mathrm{GL}_+(n,\mathbb{R})}M$ to (positively oriented) orthonormal frames we obtain the $\mathrm{SO}(n)$-principal bundle $P_{\mathrm{SO}(n)}M$ and the $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}(n)}M := \vartheta^{-1}(P_{\mathrm{SO}(n)}M)$. Furthermore, ϑ restricts to an equivariant map $P_{\mathrm{Spin}(n)}(M) \to P_{\mathrm{SO}(n)}M$. A metric spin structure is a $\mathrm{Spin}(n)$-principal bundle together with such an equivariant map. Above we have seen how we obtain a metric spin structure from a topological one, and this construction actually yields a bijection from the set of equivalence classes of topological spin structures to the set of equivalence classes of metric spin structures. Thus one can identify topological and metric spin structures, and they are simply called spin structures. Applying the associated bundle construction for the spinor representation $s : \mathrm{Spin}(n) \to \mathrm{GL}(\$)$ we obtain the spinor bundle $S_M := P_{\mathrm{Spin}(n)}M \times_s \$$. In case the underlying manifold M and its metric are fixed, we write $S = S_M$ for short.

The space of spinors, i.e., sections of S, is denoted by $\Gamma(S)$. The space of smooth compactly supported sections is called $C_c^\infty(M, S)$. The hermitian metric on fibers of

S is written as $\langle .,. \rangle$, the corresponding norm as $|.|$. We write $(.,.)_g$ for the L^2-product of spinors.

We denote by $D\colon C_c^\infty(M,S) \to C_c^\infty(M,S)$ the Dirac operator on (M,g). In case several manifolds or metrics are involved, we sometimes specify its affiliation, i.e., D^g, D^M or $D^{M,g}$. Analogously we proceed for other operators and quantities.

The sphere $\mathbb{S}^1$ carries two spin structures, one of them, the so-called bounding spin structure, is obtained by restricting the unique spin structure on the two-dimensional disk to its boundary. The kernel of the Dirac operator for this spin structure is trivial. The sphere $\mathbb{S}^1$ with the other spin structure represents the non-trivial spin-bordism class in dimension 1. In the article we will always assume that $\mathbb{S}^1$ is equipped with the bounding spin structure, unless stated otherwise.

A Riemannian manifold is of bounded geometry if it is complete, its injectivity radius is bounded from below and the curvature tensor and all derivatives are bounded.

The ball around $x \in M$ with radius ε w.r.t. the metric g on M is written as $B_\varepsilon^{M,g}(x) = B_\varepsilon(x) \subset M$.

In the article we need several Sobolev and Schauder spaces: For $s \in [1,\infty]$ we write $\|.\|_{L^s(g)}$ for the L^s-norm on (M,g). In case the underlying metric is clear from the context we abbreviate by $\|.\|_s$ for short.

Let H_k^s denote both the space of distributions on M and the one of distributional sections in S_M that have finite H_k^snorm given by

$$\|\varphi\|_{H_k^s}^s = \sum_{i=0}^{k} \|(\nabla)^i \varphi\|_{L^s}^s.$$

Here ∇ denotes the covariant derivative on M and S_M, respectively, depending on whether φ is a distribution on M and a distributional section in S_M, respectively. $H_{k,\mathrm{loc}}^s$ means that any restriction of the distribution to a compact subset has to be in H_k^s of that subset.

The space of i-times continuously differentiable functions on M is denoted by $C^i(M)$, and $C^{i,\alpha}$ denotes the corresponding Schauder space for $\alpha \in (0,1]$.

2.2 The Model Spaces $\mathbb{M}_c^{m,k}$

Let $0 \le k \le m-1$ and $c \in [0,1]$. $(\mathbb{M}_c^{m,k} = \mathbb{H}_c^{k+1} \times \mathbb{S}^{m-k-1}, g_c = g_{\mathbb{H}_c^{k+1}} + \sigma^{m-k-1})$ where σ^{m-k-1} denotes the standard metric on $\mathbb{S}^{m-k-1}$ and $(\mathbb{H}_c^{k+1}, g_{\mathbb{H}_c^{k+1}})$ is the rescaled hyperbolic space with scalar curvature $-c^2k(k+1)$ if $c \in (0,1]$ and the Euclidean space if $c = 0$.

We introduce coordinates on $\mathbb{H}_c^{k+1}$ by equipping $\mathbb{R}^{k+1}$ with the metric $g_{\mathbb{H}_c^{k+1}} = \mathrm{d}r^2 + f(r)^2\sigma^k$ where

$$f_c(r) := \sinh_c(r) := \begin{cases} \frac{1}{c}\sinh(cr) & \text{if } c \neq 0 \\ r & \text{if } c = 0. \end{cases}$$

The manifold $(\mathbb{M}_1^{m,k} = \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1}, g_1 = g_{\mathbb{H}^{k+1}} + \sigma^{m-k-1} = \sinh^2 t\, \sigma^k + \mathrm{d}t^2 + \sigma^{m-k-1})$ is conformal to $(\mathbb{S}^m \setminus \mathbb{S}^k, \sigma^m)$, [5, Proposition 3.1],

$$\mathfrak{u}\colon \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1} \to \mathbb{S}^m \setminus \mathbb{S}^k,\ g_1 = f^2 u^* \sigma^m \text{ where } f = f(t) = \cosh^2 t.$$

2.3 Regularity Theory

We recall the standard estimates:

Theorem 2.1 *Let (M^m, g) be a Riemannian spin manifold of bounded geometry. Let $R > 0$ be smaller than the injectivity radius of M, and let $r \in (0, R)$.*

(i) *(Inner L^s-estimate,* [19, Proof of Theorem 8.8]*, spin version* [2, Proof of Theorems 3.2.1 and 3.2.3]*) Let $\varphi \in H^s_{1,\mathrm{loc}}$ be a solution of $D\varphi = \psi$ for $\psi \in H^s_{k,\mathrm{loc}}$. Then, there exists a constant $C = C(s, r, R)$ such that for all $x \in M$*

$$\|\varphi\|_{H^s_{k+1}(B_r(x))} \le C\big(\|\varphi\|_{L^s(B_R(x))} + \|\psi\|_{H^s_k(B_R(x))}\big)$$

(ii) *(Embedding into $C^{0,\gamma}$) Let $m < s$ and $0 \le \gamma \le 1 - \frac{m}{s}$. By the spin version of the proof of* [19, Sect. 7.8 (Theorem 7.26)] *there exists a constant $C = C(s, r)$ such that $H^s_1(B_R(x))$ is continuously embedded in $C^{0,\gamma}(\overline{B_r(x)})$ for all $x \in M$.*

(iii) *(Schauder estimates)* [2, Corollary 3.1.14] *There is a constant $C = C(r, R, k) > 0$ such that for $\alpha > 0$, $\psi \in C^{k,\alpha}$ with $D\varphi = \psi$ weakly it holds for all $x \in M$*

$$\|\varphi\|_{C^{k+1,\alpha}(B_r(x))} \le C\left(\|\varphi\|_{C^k(B_R(x))} + \|\psi\|_{C^{k,\alpha}(B_R(x))}\right).$$

(iv) (Sobolev Embedding into L^p, [19, Theorem 7.26]*) Let $k, \ell \in \mathbb{R}$, $k \ge \ell$ and $s, t \in (1, \infty)$ with $k - (m/s) \ge \ell - (m/t)$, then the restriction map $H^s_k(B_R(x), S) \to H^t_\ell(B_r(x), S)$ is continuous for all $x \in M$ and $r > 0$. For fixed $R > r > 0$ the operator norm of these restriction maps can be chosen uniformly in x.*

2.4 L^s-Invertibility of Dirac Operators

For a complete manifold M, we define the norm $\|\varphi\|_{\tilde{H}^s_1} := \|\varphi\|_s + \|D\varphi\|_s$ for $1 \le s < \infty$. For $1 \le s < \infty$, let $\tilde{H}^s_1 = \tilde{H}^s_1(M, S)$ be the completion of $C^\infty_c(M, S)$ w.r.t. the norm $\|\varphi\|_{\tilde{H}^s_1}$. Then $D_s\colon \tilde{H}^s_1 := \mathrm{dom}\, D_s \subset L^s \to L^s$ is a closed extension of the Dirac operator. By [10, Lemma B.2] we have $(D_s)^* = D_{s^*}$ for $1 < s < \infty$ and $s^{-1} + (s^*)^{-1} = 1$.

Note that on manifolds of bounded geometry and $1 < s < \infty$ the H^s_1-norm and the graph norms $\tilde{H}^s_1$ are equivalent, [10, Lemma A.2].

General properties of the L^s-spectrum of Dirac operators can be found in [10, Appendix]. Here, we only cite the result on L^s-invertibility of our model spaces.

Proposition 2.2 [10, Theorem 1.1] *Let $1 \le s \le \infty$. The Dirac operator $D\colon L^s \to L^s$ on $\mathbb{M}^{m,k}_c$ is L^s-invertible if $\lambda_1 = \frac{m-k-1}{2} > ck\,|1/s - 1/2|$.*

Table 1 Some constants in the article for m-dimensional manifolds

a	p	p^*	q	q^*
$4\frac{m-1}{m-2}$	$\frac{2m}{m-2}$	$\frac{2m}{m+2}$	$\frac{2m}{m-1}$	$\frac{2m}{m+1}$

2.5 Yamabe Type Constants and Yamabe Type Invariants

Let (M^m, g) be a complete m-dimensional Riemannian spin manifold. By Δ^g we denote the Laplacian on (M, g) and by scal^g its scalar curvature. Let $L^g = a\Delta^g + \mathrm{scal}^g$ be the conformal Laplacian where $a = 4\frac{m-1}{m-2}$, see also Table 1. We recall the following definitions:

Definition 2.3 Functionals

$$\mathcal{F}(v) := \frac{\int_M vL^g v \,\mathrm{dvol}_g}{\|v\|^2_{L^{\frac{2m}{m-2}}(g)}}, \quad \mathcal{F}^{\mathrm{spin}}(\varphi) := \frac{\|D^g\varphi\|^2_{L^{\frac{2m}{m+1}}(g)}}{(D^g\varphi, \varphi)_g}$$

For further use we define for the rest of the paper $p := \frac{2m}{m-2}$ and $q := \frac{2m}{m-1}$. The corresponding Euler–Lagrange equations for normalized solutions are the so-called Yamabe equation [34]

$$L^g v = \mu v^{\frac{m+2}{m-2}}, \qquad \|v\|_{L^{p=\frac{2m}{m-2}}} = 1$$

and the spinorial brother [1]

$$D^g\varphi = \lambda|\varphi|^{\frac{2}{m-1}}\varphi, \qquad \|\varphi\|_{L^{q=\frac{2m}{m-1}}} = 1.$$

Yamabe type constants defined by compactly supported test functions

$$Q^*(M, g) := \inf\left\{\mathcal{F}(v) \;\middle|\; v \in C_c^\infty(M, S) \setminus \{0\}\right\},$$
$$\lambda^{+,*}_{\min}(M, g) := \inf\left\{\mathcal{F}^{\mathrm{spin}}(\varphi) \;\middle|\; \varphi \in C_c^\infty(M, S), (D^g\varphi, \varphi)_g > 0\right\}$$

Yamabe type constants defined over solutions

$\widetilde{Q}(M, g)$ is the Yamabe invariant "defined over the solutions", i.e.,

$$\widetilde{Q}(M, g) := \inf\{\mu_v \mid v \in \Omega^{(1)}(M, g)\}$$

where $\Omega^{(1)}(M, g)$ is the set of all nonnegative functions $v \in C^2(M) \cap L^\infty(M) \cap L^2(M, g)$ satisfying $L^g v = \mu_v v^{p-1}$ for a real number μ_v and with $\|v\|_{L^p(M,g)} = 1$ ($p = \frac{2m}{m-2}$ as indicated in Table 1).

Analogously, we introduce a quantity corresponding to $\lambda^{+,*}_{\min}(M,g)$ defined using the solutions of the Euler–Lagrange equation of $\mathcal{F}^{\mathrm{spin}}$:

$$\widetilde{\lambda}^{+}_{\min}(M,g) := \inf\Big\{\lambda\in(0,\infty)\mid \exists\varphi\in L^\infty\cap L^2\cap C^1:\\ 0<\|\varphi\|_{L^{\frac{2m}{m-1}}(M,g)}\le 1,\ D^g\varphi=\lambda|\varphi|^{\frac{2}{m-1}}\varphi\Big\}. \tag{1}$$

We will see in the next section why these different quantities are geometrically relevant.

Renormalized spinorial invariants We also introduce renormalized versions of $\lambda^{+,*}_{\min}$ and $\widetilde{\lambda}^{+}_{\min}$:

$$Q^*_{\mathrm{spin}}(M,g)=4\frac{m-1}{m}\lambda^{+,*}_{\min}(M,g)^2,\ \ \widetilde{Q}_{\mathrm{spin}}(M,g)=4\frac{m-1}{m}\widetilde{\lambda}^{+}_{\min}(M,g)^2. \tag{2}$$

This renormalization will make things simpler.

Yamabe type invariants for compact manifolds Now we define the smooth Yamabe invariant $\sigma^*(M)$ as

$$\sigma^*(M):=\sup_g Q^*(M,g)$$

where the supremum runs over all Riemannian metrics on M. Thus, σ^* only depends on the diffeomorphism type of M. Note that the smooth Yamabe invariant is positive if and only if M admits a metric of positive scalar curvature.

A similar spinorial Yamabe invariant $\tau^+(M)$ was introduced in [11,12]. It is

$$\tau^+(M):=\begin{cases}\sup_{g\in R^{\mathrm{inv}}(M)}\lambda^{+,*}_{\min}(M,g) & \text{if } R^{\mathrm{inv}}(M)\neq\varnothing\\ 0 & \text{if } R^{\mathrm{inv}}(M)=\varnothing,\end{cases}$$

where $R^{\mathrm{inv}}(M)$ is the set of Riemannian metrics on M such that D^g is invertible. The definition of τ^+ is slightly different from the original one in [11,12], but obviously equivalent. Note that for connected closed manifolds one knows from [3] that $R^{\mathrm{inv}}(M)\neq\varnothing$ if and only if the index of M^m in KO_m vanishes. Thus, $\tau^+(M)$ is positive if and only if this index vanishes. The invariant $\tau^+(M)$ only depends on both the diffeomorphism type of M and its spin structure.

These Yamabe type invariants will be considered in this article only in the case that M is compact. In this case the solution of the classical Yamabe problem [34] implies $\widetilde{Q}(M,g)=Q^*(M,g)$, and similar results in the spin case [1] imply $\widetilde{\lambda}^{+}_{\min}(M,g)=\lambda^{+,*}_{\min}(M,g)$. Thus, we also see

$$\sigma^*(M)=\sup_g\widetilde{Q}(M,g),\qquad \tau^+(M)=\begin{cases}\sup_{g\in R^{\mathrm{inv}}(M)}\widetilde{\lambda}^{+}_{\min}(M,g) & \text{if } R^{\mathrm{inv}}(M)\neq\varnothing\\ 0 & \text{if } R^{\mathrm{inv}}(M)=\varnothing.\end{cases}$$

Similar to the above we also define (for M compact) a renormalized version

$$\sigma^*_{\mathrm{spin}}(M) := 4\frac{m-1}{m}\tau^+(M)^2.$$

We want to remark that $\sigma^*(M)$ was considered for non-compact manifolds in [25].

The Λ*-invariants* We define

$$\widetilde{\Lambda}_{m,k} := \inf_{c\in[0,1]} \widetilde{Q}(\mathbb{M}_c^{m,k}) \text{ and } \Lambda^*_{m,k} := \inf_{c\in[0,1]} Q^*(\mathbb{M}_c^{m,k}).$$

These invariants are important because of their relation to the invariant $\Lambda_{m,k}$ that contributes to the Surgery Theorem 2.5. We have $\Lambda_{m,k} = \widetilde{\Lambda}_{m,k}$ unless $m = k-3 \geq 7$ or $m = k-2$ [6, Theorem 3.1 and Proof of Corollary 3.2]. The idea behind the notation is that the invariant with $*$ is the infimum of a functional, the invariant with $\sim$ is defined using solutions of the Euler–Lagrange equation and the invariant without such decoration is the invariant in the surgery theorem. We know from [6, Theorem 3.3] that all these invariants are positive for $0 \leq k \leq m-3$.

In the spinorial case we define similarly

$$\widetilde{\Lambda}^{\mathrm{spin}}_{m,k} = \inf_{c\in[0,1]} \widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,k}) \text{ and } \Lambda^{\mathrm{spin},*}_{m,k} := \inf_{c\in[0,1]} Q^*_{\mathrm{spin}}(\mathbb{M}_c^{m,k}).$$

It is known from [4, Theorem 1.1] that $\widetilde{\Lambda}^{\mathrm{spin}}_{m,k} > 0$ for all $0 \leq k \leq m-2$. The invariant $\Lambda^{\mathrm{spin}}_{m,k}$ in the Spinorial Surgery Theorem 2.6 can be chosen to be $\widetilde{\Lambda}^{\mathrm{spin}}_{m,k}$ for all $0 \leq k \leq m-2$, [4, Corollary 1.4]. We introduce the notation $\Lambda^{\mathrm{spin}}_{m,k} := \widetilde{\Lambda}^{\mathrm{spin}}_{m,k}$ to make the presentation analogous to the non-spin case.

One of the main goals of this article is to search for relations between these five possibly different Λ-invariants.

Remark 2.4 The Q-invariants for the spheres play a special role. We collect the main properties: For all manifolds (M^m, g) it holds $Q^*(M^m, g) \leq Q^*(\mathbb{S}^m)$. If M is spin, then $Q^*_{\mathrm{spin}}(M^m, g) \leq Q^*_{\mathrm{spin}}(\mathbb{S}^m)$. The invariant $Q^*(\mathbb{S}^m)$ is attained by a constant test function v such that $\|v\|_{\frac{2m}{m-2}} = 1$. Thus, $Lv = m(m-1)v = Q^*(\mathbb{S}^m)v^{\frac{m+2}{m-2}}$ and $Q^*(\mathbb{S}^m) = m(m-1)\mathrm{vol}(\mathbb{S}^m)^{\frac{2}{m}}$. The invariant $\lambda^{+,*}_{\min}(\mathbb{S}^m)$ is attained by a Killing spinor φ to the Killing constant $-\frac{1}{2}$. Note that the normalization in (2) is chosen such that $Q^*_{\mathrm{spin}}(\mathbb{S}^m) = Q^*(\mathbb{S}^m)$. Since $\mathbb{S}^m$ is closed $Q^*(\mathbb{S}^m) = \widetilde{Q}(\mathbb{S}^m)$ and $Q^*_{\mathrm{spin}}(\mathbb{S}^m) = \widetilde{Q}_{\mathrm{spin}}(\mathbb{S}^m)$.

Moreover, for (M^m, g) not locally conformally flat and $m \geq 6$, Aubin showed, see [14, p. 292], $Q^*(M^m, g) < Q^*(\mathbb{S}^m)$.

2.6 Surgery-Monotonicity for Yamabe Type Invariants Below Thresholds

In order to define the constant $\Lambda_{m,k}$ mentioned above we set

$$Q^{(2)}(M,g) := \inf\left\{\mu_u \mid u \in \Omega^{(2)}(M,g)\right\}$$

where $\Omega^{(2)}(M,g)$ is the set of all nonnegative functions $u \in C^2(M) \cap L^\infty(M)$ satisfying $L^g u = \mu_u u^{p-1}$ for a nonnegative real number μ_u, $\|u\|_{L^p(M,g)} = 1$, where $p = \frac{2m}{m-2}$ as always, and $\mu_u \|u\|_{L^\infty}^{\frac{4}{m-2}} \geq \frac{(m-k-2)^2(m-1)}{8(m-2)}$. Then, we set, cf. [5, Sect. 3], [6, Sect. 2.6],

$$\Lambda_{m,k} = \min\left\{\widetilde{\Lambda}_{m,k}, \inf_{c\in[0,1]} Q^{(2)}(\mathbb{M}_c^{m,k})\right\}.$$

It follows from [6, Theorem 3.1 and below] that in case $k = m-3 \leq 6$ or $k \leq m-4$ we already have $\Lambda_{m,k} = \widetilde{\Lambda}_{m,k}$.

Theorem 2.5 (Surgery-monotonicity for the Yamabe invariant, [5, Corollary 1.4]) *Assume that N^m is a closed Riemannian manifold that is obtained from M^m by a surgery of codimension $m-k \geq 3$. Then*

$$\sigma^*(N) \geq \min\{\sigma^*(M), \Lambda_{m,k}\}.$$

Note that a surgery from M to N is called spin preserving if the spin structures on M and N extend to a spin structure on the corresponding bordism. In particular this implies that the spin structures on M and N coincide outside the region of surgery.

Theorem 2.6 (Surgery-monotonicity for the spinorial Yamabe invariant, [4, Corollary 1.4]) *Assume that N^m is a closed Riemannian spin manifold that is obtained from M^m by a spin-preserving surgery of codimension $m-k \geq 2$. Then*

$$\sigma^*_{\mathrm{spin}}(N) \geq \min\{\sigma^*_{\mathrm{spin}}(M), \Lambda^{\mathrm{spin}}_{m,k}\}.$$

In the case $k = m-2$ the sphere $\mathbb{S}^1$ carries the bounding spin structure, as explained in the Notation 2.1.

Since there is a whole zoo of different Q- and Λ-invariants, we summarize the logic of our notation in Table 2.

3 Overview of the Results

Many of the inequalities established in this article are summarized in Fig. 1. For example, $\Lambda^{\mathrm{spin},*}_{m,k} = \widetilde{\Lambda}^{\mathrm{spin}}_{m,k} \geq \Lambda^*_{m,k}$ for all $k \leq m-2$. Other inequalities hold under additional assumptions, e.g., in the case $k \leq m-4$ and in the case $k \leq m-3 \leq 3$ we have $\Lambda^*_{m,k} \geq \widetilde{\Lambda}_{m,k}$. Thus, together with previously mentioned relations we obtain

Table 2 Overview of the notation

Label	Meaning
No spin label	Associated with the Yamabe problem
Spin	Associated with the spinorial Yamabe problem for the Dirac operator
$*$	Defined as a variational problem
$\sim$	Defined by solutions of the associated Euler–Lagrange equation (as a nonlinear eigenvalue problem)
No $*$ and no $\sim$	Appears in an associated surgery result
Q	For a fixed Riemannian (spin) manifold
Λ	$= \inf_{c\in[0,1]}$ of the Q-invariants (decorated with the same labels as Λ) for the model spaces $\mathbb{M}_c$
σ	The supremum of the Q-invariants (with the same labels) for a (spin) manifold over all conformal classes

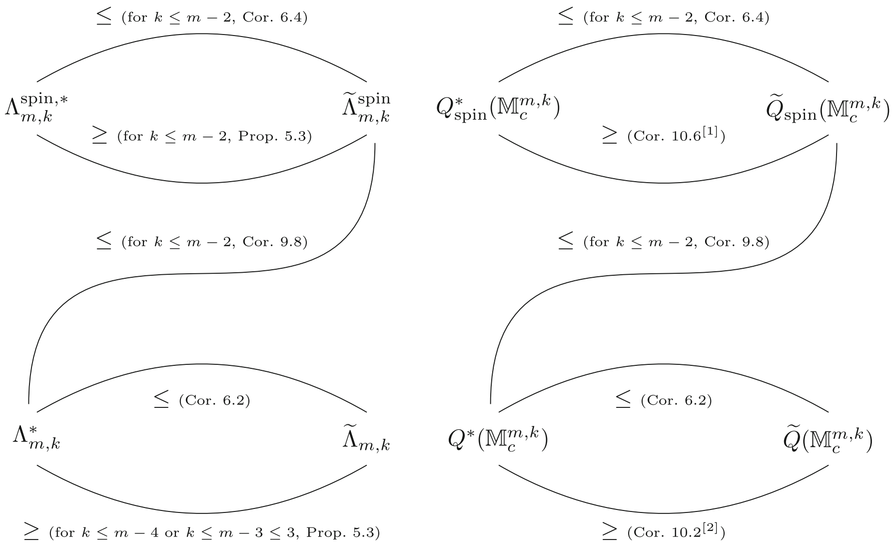

Fig. 1 Summary of the results for the Q-invariants of the model spaces (*right*) and the corresponding Λ-invariants (*left*).
[1]: for $(m-k-1)^2 > c^2k(k+1)$ and $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}^{m,k}_c) < Q^*_{\mathrm{spin}}(\mathbb{S}^m)$
[2]: for $(m-k-1)(m-k-2) > c^2k(k+1)$, $c \in [0,1)$ or for $k \le m-3$, $c = 1$

Theorem 3.1 *In the case $k \le m-4$ and in the case $k \le m-3 \le 3$*

$$\Lambda^{\mathrm{spin},*}_{m,k} = \widetilde{\Lambda}^{\mathrm{spin}}_{m,k} = \Lambda^{\mathrm{spin}}_{m,k} \ge \Lambda^{*}_{m,k} = \widetilde{\Lambda}_{m,k} = \Lambda_{m,k}.$$

Thus, in most of the cases the inequality $\Lambda^{\mathrm{spin}}_{m,k} \ge \Lambda_{m,k}$ conjectured in the introduction holds. Together with the explicit positive lower bounds for $\Lambda_{m,k}$ in [6,7], we then obtain explicit positive lower bounds for $\Lambda^{\mathrm{spin}}_{m,k}$. Theorem 3.1 does not provide for

Table 3 Some explicit lower bounds for $\Lambda_m^{\mathrm{spin}}$: the values are rounded—$Q^*(\mathbb{S}^m)$ is rounded to the nearest multiple of $1/10$ and the lower bounds for $\Lambda_m^{\mathrm{spin}}$ and $Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})$ are always rounded down

m	5	6	7	8	9	10	11	12	13
$Q^*(\mathbb{S}^m)$	80.0	96.3	113.5	130.7	147.9	165.0	182.2	199.3	216.4
$\Lambda_m^{\mathrm{spin}} \geq$	45.1	50.0	65.2	78.7	91.8	104.9	118.1	131.5	145.0
$Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8}) \geq$	–	–	–	121.4	138.5	97.3	135.9	158.7	178.0

Note that $Q^*(\mathbb{H}P^2) = 121.4967\ldots$ is attained by the canonical metric on $\mathbb{H}P^2$, due to Obata's theorem

$\Lambda_{m,m-3}^{\mathrm{spin}}$ for $m > 6$. But nevertheless our techniques also allow us to obtain explicit positive lower bounds for $\Lambda_{m,m-3}^{\mathrm{spin}}$ for $m > 6$; see Sect. 11.

The right-hand side of Fig. 1 gives relations between the Q-invariants of the model spaces. Some of them require additional assumptions which are given as footnotes. The parameter c ranges in the interval $[0, 1]$. However, the case $c = 1$ is very special as then $\mathbb{M}_1^{m,k}$ is conformal to a subset of $\mathbb{S}^m$ which allows much stronger statements. This is summarized in Sect. 7. Another special case is $k = m - 1$. These invariants do not have similar geometric applications. But for the sake of completeness we summarize in Sect. 8.

As already mentioned in the Introduction, the explicit positive lower bounds for $\Lambda_{m,k}^{\mathrm{spin}}$, $k \leq m - 3$ lead to bordism invariant. As we only want to give an overview here, many proofs will be given later, i.e., in Sect. 12.

Let $m \geq 5$. We set

$$\Lambda_m^{\mathrm{spin}} := \min \left\{ \Lambda_{m,2}^{\mathrm{spin}}, \Lambda_{m,3}^{\mathrm{spin}}, \ldots, \Lambda_{m,m-3}^{\mathrm{spin}} \right\}.$$

From Theorem 3.1, Sect. 11, and results in [6] and [7] we obtain explicit positive lower bounds for $\Lambda_m^{\mathrm{spin}}$, summarized in Table 3 for low dimensions.

Using standard techniques from bordism theory (see Sect. 12 for details) one obtains several conclusions:

Proposition 3.2 *Let M be an m-dimensional closed connected, simply connected spin manifold, $\alpha(M) = 0$. If $5 \leq m \leq 7$, then*

$$\sigma_{\mathrm{spin}}^*(M) \geq \Lambda_m^{\mathrm{spin}}.$$

For $m \geq 11$ or $m = 8$ we have

$$\sigma_{\mathrm{spin}}^*(M) \geq \min \left\{ \Lambda_m^{\mathrm{spin}}, Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8}) \right\}.$$

For $m = 9, 10$ we have

$$\sigma_{\mathrm{spin}}^*(M) \geq \min \left\{ \Lambda_{m,1}^{\mathrm{spin}}, \Lambda_m^{\mathrm{spin}}, Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8}) \right\}.$$

Note that by definition $\alpha(M) \neq 0$ implies that there are no invertible Dirac operators, thus by definition $\sigma^*_{\mathrm{spin}}(M) = 0$.

We conjecture $\Lambda^{\mathrm{spin}}_m \leq Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})$ for all $m \geq 11$, which would imply $\sigma^*_{\mathrm{spin}}(M) \geq \Lambda^{\mathrm{spin}}_m$ for all closed simply connected spin manifolds M with dimension $m \geq 5$, $m \neq 9, 10$.

A similar bound also exists for non-simply connected manifolds, namely in this case for $m \neq 9, 10$

$$\sigma^*_{\mathrm{spin}}(M) \geq \min\left\{\Lambda^{\mathrm{spin}}_m, \Lambda^{\mathrm{spin}}_{m,m-2}, Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})\right\} > 0,$$

and for $m = 9, 10$

$$\sigma^*_{\mathrm{spin}}(M) \geq \min\left\{\Lambda^{\mathrm{spin}}_{m,1}, \Lambda^{\mathrm{spin}}_m, \Lambda^{\mathrm{spin}}_{m,m-2}, Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})\right\} > 0.$$

However, this positive lower bound is not explicit as no explicit lower bound for $\Lambda^{\mathrm{spin}}_{m,m-2}$ is currently available. Numerical calculations and some further assumptions indicate that $\Lambda^{\mathrm{spin}}_{m,m-2} < \Lambda^{\mathrm{spin}}_m$.

Proposition 3.3 *Assume that M is an m-dimensional closed connected spin manifold with $m \geq 5$. We consider the bordism groups $\Omega^{\mathrm{spin}}_m(B\Gamma)$, $\Gamma := \pi_1(M)$ where the boundaries and the bordisms are spin manifolds together with maps to $B\Gamma$. Let $c_M \colon M \to B\Gamma$ be a classifying map of the universal covering of M, i.e., the map which induces an isomorphism from $\pi_1(M)$ to $\Gamma = \pi_1(B\Gamma)$. Let $[N, f] = [M, c_M] \in \Omega^{\mathrm{spin}}_m(B\Gamma)$, and let N be connected. Then*

$$\sigma^*_{\mathrm{spin}}(N) \geq \min\left\{\sigma^*_{\mathrm{spin}}(M), \Lambda^{\mathrm{spin}}_m, \Lambda^{\mathrm{spin}}_{m,m-2}\right\}.$$

If N is connected and if f induces an isomorphism from $\pi_1(N)$ to Γ, then

$$\sigma^*_{\mathrm{spin}}(N) \geq \min\left\{\sigma^*_{\mathrm{spin}}(M), \Lambda^{\mathrm{spin}}_m\right\}. \tag{3}$$

Note that every class in $\Omega^{\mathrm{spin}}_m(B\Gamma) \to \mathbb{R}$ can be written as (M, c_M).

By applying (3) twice, it follows from Proposition 3.3 that there is a well-defined map $s^{\mathrm{spin}} \colon \Omega^{\mathrm{spin}}_m \to \mathbb{R}$ such that for all connected, simply connected spin manifolds M

$$s^{\mathrm{spin}}([M]) = \min\left\{\sigma^*_{\mathrm{spin}}(M), \Lambda^{\mathrm{spin}}_m\right\}.$$

Thus if M is a connected spin manifold, we have

$$\sigma^*_{\mathrm{spin}}(M) \geq \min\left\{s^{\mathrm{spin}}([M]), \Lambda^{\mathrm{spin}}_{m,m-2}\right\}.$$

It follows from standard arguments of surgery theory that

$$s^{\mathrm{spin}}([M]+[N]) \geq \min\left\{s^{\mathrm{spin}}([M]), s^{\mathrm{spin}}([N])\right\}.$$

Thus,

$$\Omega_m^{\mathrm{spin},>t} := \left\{[M] \in \Omega_m^{\mathrm{spin}} \mid s^{\mathrm{spin}}([M]) > t\right\}$$

is a subgroup of Ω_m^{spin}. For example, $\Omega_m^{\mathrm{spin},>0}$ is the kernel of the index map $\alpha\colon \Omega_m^{\mathrm{spin}} \to KO_m$.

Similarly as above, for an arbitrary finitely presented group Γ we obtain a well-defined map $s_\Gamma^{\mathrm{spin}}\colon \Omega_m^{\mathrm{spin}}(B\Gamma) \to \mathbb{R}$ as follows: For every $[M, f] \in \Omega_m^{\mathrm{spin}}(B\Gamma)$ where M is connected and f induces an isomorphism from $\pi_1(M)$ to Γ we have

$$s_\Gamma^{\mathrm{spin}}([M, f]) = \min\left\{\sigma^*_{\mathrm{spin}}(M), \Lambda^{\mathrm{spin}}_{m,1}, \Lambda^{\mathrm{spin}}_m\right\}.$$

In this case the minimum includes the constants $\Lambda^{\mathrm{spin}}_{m,1}$ since it is required to show that

$$s_\Gamma^{\mathrm{spin}}([M, f]+[N, g]) \geq \min\left\{s_\Gamma^{\mathrm{spin}}([M, f]), s_\Gamma^{\mathrm{spin}}([N, g])\right\}.$$

Then, analogously as above,

$$\Omega_m^{\mathrm{spin},>t}(B\Gamma) := \{[M, f] \in \Omega_m^{\mathrm{spin}}(B\Gamma) \mid s_\Gamma^{\mathrm{spin}}([M, f]) > t\}$$

is a subgroup of $\Omega_m^{\mathrm{spin}}(B\Gamma)$.

Assume that there is a closed simply connected spin manifold M of dimension $m \geq 5$ with $\sigma^*_{\mathrm{spin}}(M) < \Lambda^{\mathrm{spin}}_m$. For such manifolds one would have: If N is a simply connected closed spin manifold spin-bordant to M, then $\sigma^*_{\mathrm{spin}}(N) = \sigma^*_{\mathrm{spin}}(M)$. An advantage of this bordism result is that we have explicit positive lower bounds for $\Lambda^{\mathrm{spin}}_m$, in contrast to a similar result for the classical Yamabe invariant.

As a consequence, by the Hijazi inequality we have $\sigma^*(N) \leq \sigma^*_{\mathrm{spin}}(M)$, i.e., $\sigma^*_{\mathrm{spin}}(M)$ is an upper bound for the Yamabe invariant for all simply connected manifolds in $[M]$.

This question is related to the open problem of whether there is a manifold in dimension $m \geq 5$ with Yamabe invariant different from 0 and $\sigma^*(\mathbb{S}^m)$. If one finds an M as above, all simply connected manifolds in the spin bordism class of M would have a Yamabe invariant in $(0, \sigma^*(\mathbb{S}^m))$.

Many of the statements on the right-hand side of Fig. 1 are still valid if one replaces the model spaces by arbitrary manifolds of bounded geometry; see Sects. 6 and 9. The inequalities in Sect. 9 are noncompact versions of the Hijazi inequality which is of central importance of our article. The reader should be aware that there are different ways to generalize from the compact to the noncompact setting. We have positive and negative results for the generalization of the Hijazi inequality to the noncompact setting; see Sect. 9. Our investigations also need regularity statements for the Euler–Lagrange equation of the spinorial functional. For this purpose we have included Sect. 4 which might be of independent interest and which goes beyond the requirements of the following sections.

4 Improvements of Regularity for the Dirac Euler–Lagrange Equation

Let (M^m, g) be a Riemannian spin manifold of bounded geometry. In this section, we consider a spinor $\varphi \in L^q$ and $\varphi \in L^s_{\mathrm{loc}}$ for an $s > q$ that fulfills

$$D\varphi = \lambda|\varphi|^{q-2}\varphi \quad \text{weakly,} \tag{4}$$

i.e., in the distributional sense, where as always $q = \frac{2m}{m-1}$. Note that from $\varphi \in L^s_{\mathrm{loc}}$ for an $s > q$ it follows with the methods of [2, Theorem 5.2] that φ is $C^{1,\alpha}$ for all $\alpha \in (0, 1)$. We omit the proof of this local statement since the proof is completely analogous as in [2]. Furthermore, we will only use the fact that φ is continuous which is part of the assumptions in the applications of this subsection.

We want to further examine the regularity of φ. First, we will show that $\varphi \in L^\infty$. For that we need the following auxiliary lemma.

Lemma 4.1 *Fix $\beta, R, \delta > 0$. Let $\varphi \in \Gamma(S_M)$ be continuous with $\|\varphi\|_{L^\infty} = \infty$.*
Then there is a sequence $(x_i)_{i\in\mathbb{N}}$ in M with $|\varphi(x_i)| \geq i$ and

$$|\varphi(x_i)|^{-1}\|\varphi\|_{L^\infty(B_i^R)} \leq 1 + \delta$$

where $B_i^R := B_{R\,|\varphi(x_i)|^{-1/\beta}}(x_i)$.

Proof Let $d(.,.)$ denote the distance in (M, g), and fix $R, \delta > 0$. We prove the claim by contradiction: Assume that there is a constant $C > 0$ such that for all $x \in M$ with $|\varphi(x)| \geq C$ there is y_x with $d(x, y_x) < R\,|\varphi(x)|^{-1/\beta}$ and $|\varphi(y_x)| > (1 + \delta)\,|\varphi(x)|$. Then, we define a sequence x_i recursively by choosing $x_0 \in M$ with $|\varphi(x_0)| \geq C$ and $x_{i+1} = y_{x_i}$ for all $i \geq 0$. Then, $|\varphi(x_i)| \geq (1 + \delta)^i|\varphi(x_0)| \geq (1 + \delta)^i C \to \infty$ as $i \to \infty$. But,

$$\begin{aligned} d(x_i, x_0) &\leq \sum_{j=0}^{i-1} d(x_{j+1}, x_j) \leq \sum_{j=0}^{i-1} R\,|\varphi(x_j)|^{-\frac{1}{\beta}} \leq RC^{-\frac{1}{\beta}} \sum_{j=0}^{i-1}(1+\delta)^{-\frac{j}{\beta}} \\ &\leq \frac{RC^{-\frac{1}{\beta}}}{1-(1+\delta)^{-\frac{1}{\beta}}} < \infty \end{aligned}$$

which then contradicts the continuity of φ. □

Lemma 4.2 *Let (M^m, g) be of bounded geometry. Let $\varphi \in L^q \cap C^0$ be a weak solution of (4). Then $\varphi \in L^\infty$.*

Proof We assume the contrary, i.e., $\|\varphi\|_\infty = \infty$. We fix $\beta := 2(q-2)$, R smaller than the injectivity radius, and some $\delta > 0$. Then applying Lemma 4.1 there is a sequence of points $(x_i)_{i\in\mathbb{N}}$ in M with $|\varphi(x_i)| \geq i$ and $\|\varphi\|_{L^\infty(B_i^R)} \leq (1+\delta)|\varphi(x_i)|$. After passing to a subsequence, every compact subset only contains a finite number of x_i. We thus assume that all B_i^R are pairwise disjoint since this can always be achieved by passing to a further subsequence. We consider the charts for B_i^R given by rescaled exponential maps

$$u_i\colon B_{r_i}(0) \subset \mathbb{R}^m \to B_i^R,\ v \mapsto \exp_{x_i}(\delta_i v)$$

where $m_i := |\varphi(x_i)|$, $\delta_i := m_i^{-\frac{1}{q-2}}$ and $r_i := \delta_i^{-1} R\, |\varphi(x_i)|^{-\frac{1}{\beta}} = R\, |\varphi(x_i)|^{\frac{1}{2(q-2)}}$.

Note that $m_i = |\varphi(x_i)| \geq i \to \infty$ and, hence, $\delta_i \to 0$ and $r_i \to \infty$ as $i \to \infty$. The map u_i induces a map on the frame bundles which lifts to the spinor bundles; for details, see [16]. For simplicity, we denote this lift also by u_i, and set $\psi^i := m_i^{-1} u_i^* \varphi$. Then ψ^i is a spinor on $B_{r_i}(0)$, $|\psi^i(0)| = 1$ and $\|\psi^i\|_{L^\infty(B_{r_i}(0))} \leq 1+\delta$.

Using the comparison of the Dirac operator with the one on the Euclidean space [8, Sects. 3 and 4], we obtain from $D\varphi = \lambda|\varphi|^{q-2}\varphi$ that

$$D^{\mathbb{R}^m}\psi^i + \frac{1}{4}\sum_{\alpha\beta\gamma}\tilde{\Gamma}_{\alpha\beta}^{\gamma} e_\alpha \cdot e_\beta \cdot e_\gamma \cdot \psi^i + \sum_{\alpha\beta}\left(b_\alpha^\beta - \delta_\alpha^\beta\right) e_\alpha \cdot \nabla_{e_\beta}\psi^i = \lambda|\psi^i|^{q-2}\psi^i$$

where δ_α^β denotes the Kronecker symbol, e_α is the standard orthonormal frame on $\mathbb{R}^m$ and

$$\begin{aligned} b_\alpha^\beta &= \delta_\alpha^\beta - \frac{1}{6}\delta_i^2 R^i_{\alpha\lambda\mu\beta} x^\lambda x^\mu + O\left(\delta_i^3|x|^3\right) \to \delta_\alpha^\beta \\ \tilde{\Gamma}_{\alpha\beta}^\gamma &= \partial_\alpha b_\beta^\gamma - \frac{1}{3}\delta_i (R^i_{\alpha\gamma\lambda\beta} + R^i_{\alpha\lambda\gamma\beta}) x^\lambda + O\left(\delta_i^2|x|^2\right) \to 0 \end{aligned}$$

as $\delta_i \to 0$, $i \to \infty$. Here, $R^i_{\alpha\lambda\mu\beta} = g_{x_i}([\nabla_{\partial_\beta}, \nabla_{\partial_\mu}]\partial_\alpha - \nabla_{[\partial_\beta,\partial_\mu]}\partial_\alpha, \partial_\lambda)$ is the Riemannian curvature tensor of g at x_i.

Let K_j, $j = 0,1,2,3$ be compact subsets of $\mathbb{R}^m$ with $K_{j+1} \subset \mathrm{interior}(K_j)$ for $j = 0,1,2$, and let i_0 be big enough such that $K_0 \subset B_{r_{i_0}}(0)$. Since ψ^i is bounded on $B_{r_i}(0)$ for $i \geq i_0$, the inner L^s-estimate in Theorem 2.1 shows that for each s the ψ^i's are uniformly bounded in $H_1^s(K_0)$. Thus, after passing to a subsequence $\psi^i \to \psi$ weakly in $H_1^s(K_0)$. The restriction map $H_1^s(K_0) \to C^{0,\gamma}(K_1)$ is bounded because of Theorem 2.1, hence the ψ^i are uniformly bounded also in $C^{0,\gamma}(K_1)$ for all $\gamma \in (0,1)$. In particular, $|\psi^i|^{q-2}\psi^i$ are uniformly bounded in $C^{0,\gamma}(K_1)$. Thus, by the Schauder estimate (see Theorem 2.1) we obtain $\psi \in C^{1,\gamma}(K_2)$ and, thus, by Arzelà–Ascoli $\psi^i \to \psi$ strongly in C^1 on K_3, after passing to a subsequence. We apply this construction to $K_3 := B_k(0)$ and construct a diagonal subsequence for $k \to \infty$. This subsequence converges locally in C^1 to a spinor ψ on $\mathbb{R}^m$ with $|\psi(0)| = 1$, $\|\psi\|_{L^\infty} \leq 1+\delta$ and $D^{\mathbb{R}^m}\psi = \lambda|\psi|^{q-2}\psi$.

We write $b_i \mathrm{dvol}_{\mathbb{R}^m} = u_i^*\mathrm{dvol}_g$, where $b_i := \sqrt{\det(u_i^* g)} \to 1$ as $i \to \infty$ in C^1 on each compact subset of $\mathbb{R}^m$. As the balls B_i^R are disjoint, $\varphi \in L^q$ implies that

$\int_{B_i^R} |\varphi|^q \, \mathrm{dvol}_g \to 0$ as $i \to \infty$. Thus, we get for all compact $\tilde{K}$ and sufficiently large i

$$\int_{\tilde{K}} |\psi^i|^q \, \mathrm{dvol}_{\mathbb{R}^n} \leq 1.001 \int_{|x|<r_i} |\psi^i|^q b_i \, \mathrm{dvol}_{\mathbb{R}^n}$$
$$= 1.001 \int_{B_i^R} |\varphi|^q \, \mathrm{dvol}_g \to 0 \quad \text{as } i \to \infty.$$

Hence, $\|\psi\|_q = 0$ which contradicts $\psi \in C^1$ and $|\psi(0)| = 1$. Thus, $\varphi \in L^\infty$. □

Lemma 4.3 *Let (M^m, g) be of bounded geometry. Let $\varphi \in L^q \cap C^0$ fulfill weakly (4). Then, $\lim_{x\to\infty} |\varphi| = 0$. Moreover, $\varphi \in C^{1,\gamma}$ for all $\gamma \in (0, 1)$, $\lim_{x\to\infty} \|\varphi\|_{C^{1,\gamma}(B_r(x))} = 0$ for all $r > 0$, and $\|\varphi\|_{C^{1,\gamma}} < \infty$. In particular, φ is uniformly continuous.*

Proof From Lemma 4.2 we have $\varphi \in L^\infty$. Fix $z \in M$ and $\delta > 0$ to be smaller than the injectivity radius. Let $d(.,.)$ denote the distance function on (M, g). We prove the first claim by contradiction: We assume that there is a constant $V > 0$ and a sequence $(x_i)_{i\in\mathbb{N}} \subset M$ with $|\varphi(x_i)| \geq V$, $|x_i| = d(x_i, z) \to \infty$ and $d(x_i, x_j) > 2\delta$.

Let $\varepsilon \in (0, \frac{\delta}{2})$. Since $\varphi \in L^\infty$ and (M, g) has bounded geometry, we obtain by inner L^s-estimates that

$$\|\varphi\|_{H_1^s(B_\varepsilon(x_i))} \leq C_\delta(s)(\|\varphi\|_{L^s(B_{2\varepsilon}(x_i))} + \|\lambda|\varphi|^{q-2}\varphi\|_{L^s(B_{2\varepsilon}(x_i))})$$
$$\leq C C_\delta(s)\mathrm{vol}(B_{2\varepsilon}(x_i))^{\frac{1}{s}} \leq C' \tag{5}$$

where C' does not depend on i.

Fixing $s > m$ and using the Sobolev embedding $H_1^s(B_\varepsilon(x_i)) \hookrightarrow C^{0,\gamma}(B_\varepsilon(x_i))$ we get that $\|\varphi\|_{C^{0,\gamma}(B_\rho(x_i))} \leq C''$ for some $\gamma \in (0, 1)$, $\rho \in (0, \varepsilon)$, and where C'' is independent on i.

With $\varphi \in L^q$ we estimate

$$\|\varphi\|_q^q \geq \sum_i \|\varphi\|^q_{L^q(B_\rho(x_i))} \geq K \sum_i \inf_{x\in B_\rho(x_i)} |\varphi(x)|$$

where $K := \inf_i \mathrm{vol}(B_\rho(x_i))$. Note that $K > 0$ since (M, g) has bounded geometry. Hence, $\inf_{x\in B_\rho(x_i)} |\varphi(x)| \to 0$ as $i \to \infty$. But on the other hand, on each ball $B_\rho(x_i)$ we have for all $x, y \in B_\rho(x_i)$ that $|\varphi(x) - \varphi(y)| \leq C''|x - y|^\gamma \leq C''\rho^\gamma$. Thus

$$V \leq \limsup_{i\to\infty} |\varphi(x_i)| \leq C''\rho^\gamma.$$

By choosing ρ small enough we obtain a contradiction.

Inequality (5) still holds if we replace x_i by an arbitrary $x \in M$. Then, C' does not depend on x. Moreover, choosing s large enough we then have for any $\gamma \in (0, 1)$ that $\|\varphi\|_{C^{0,\gamma}(B_{\rho/2}(x))} < \infty$ for all $x \in M$ and $\lim_{x\to\infty} \|\varphi\|_{C^{0,\gamma}(B_{\rho/2}(x))} = 0$. Thus,

$\varphi \in C^{0,\gamma}$ for any $\gamma \in (0,1)$. Then, by a further bootstrap step we obtain the same statement for $C^{1,\gamma}$ instead of $C^{0,\gamma}$ and for $\rho/3$ instead of $\rho/2$. Thus, for sufficiently large compact subset $\hat{K}$ the norm $\|\varphi\|_{C^{1,\gamma}(M\setminus\hat{K})}$ is arbitrarily close to zero. This implies the lemma. □

Corollary 4.4 *Let (M^m, g) be of bounded geometry. Let $\varphi \in L^q \cap C^0$ be a weak solution of $D\varphi = \lambda|\varphi|^{\frac{2}{m-1}}\varphi$ with $\|\varphi\|_q = 1$. Then, $\varphi \in C^{2,\gamma}$ for all $\gamma \in (0, \frac{2}{m-1}]$ if $m \geq 4$ and all $\gamma \in (0,1)$ otherwise.*

Proof Let $\beta := q - 2 = \frac{2}{m-1}$ and $\psi = |\varphi|^\beta\varphi$. At first we will show that

$$\nabla\psi = |\varphi|^\beta\nabla\varphi + \beta\langle\nabla\varphi, \varphi\rangle|\varphi|^{\beta-2}\varphi \tag{6}$$

is in C^γ for γ as above: By Lemma 4.3 $\varphi \in C^{1,\alpha}$ for all $\alpha \in (0,1)$. Thus, φ is locally Lipschitz and, hence, $|\varphi|^\beta$ is in C^β. Moreover, $\nabla\varphi \in C^\alpha$, thus the first summand in (6) is $C^{\min\{\alpha,\beta\}}$. By [1, Lemma B.1] $|\varphi|^{\beta-2}\varphi\otimes\varphi \in C^\beta$. It follows that $\langle\nabla\varphi, \varphi\rangle|\varphi|^{\beta-2}\varphi$ is C^γ as well. Thus, $\nabla\psi \in C^\gamma$ and $\psi \in C^\alpha$ for all $\alpha \in (0,1)$. Now Schauder estimates, see Theorem 2.1, imply $\varphi \in C^{2,\gamma}$. The corollary then follows. □

Example 4.5 Let us consider Euclidean $\mathbb{R}^m$, $m \geq 2$ with standard basis $(e_i)_{i=1,\ldots,m}$ and with a parallel spinor $\psi_0 \neq 0$. We define

$$\varphi(x^1,\ldots,x^m) := x^1 e_1\cdot\psi_0 - x^2 e_2\cdot\psi_0.$$

Then $\nabla\varphi = dx^1\otimes e_1\cdot\psi_0 - dx^2\otimes e_2\cdot\psi_0$, and thus $D\varphi = e_1\cdot e_1\cdot\psi_0 - e_2\cdot e_2\cdot\psi_0 = -\psi_0 + \psi_0 = 0$. Thus this spinor satisfies (4) with $\lambda = 0$, but is not L^q and many conclusions in this section, particularly the L^∞-bound, do not hold. The example thus shows that the L^q-condition in the above lemmas is necessary.

We know that by Lemma 4.2 φ is in L^∞. However, the following example shows that we cannot derive an upper bound for $\|\varphi\|_{L^\infty}$ which only depends on (M, g), $\|\varphi\|_{L^q}$ and λ.

Example 4.6 Consider again Euclidean $\mathbb{R}^m$. Take a Killing spinor φ on the sphere $(\mathbb{S}^m, \sigma^m)$ normalized such that its $L^{q=\frac{2m}{m-1}}$-norm is one. Then on $\mathbb{S}^m$ we have $D^{\mathbb{S}^m}\varphi = \frac{m}{2}\varphi = \lambda^{+,*}_{\min}(\mathbb{S}^m)|\varphi|^{q-2}\varphi$, cf. Remark 2.4. The stereographic projection h is a conformal map from the sphere with a point removed to the Euclidean space. Now let h^ρ be the composition of φ and the scaling of $\mathbb{R}^m$ by ρ. Then, $g_E = f_\rho^2\,\mathrm{h}^\rho_*(\sigma^m)$ where $f_\rho = \rho^{\frac{1}{2}} f_1$ is the conformal factor. Using the identification of spinor bundles of conformal metrics, cf. [29, Sect. 4], we get a spinor $\tilde{\varphi} = f_\rho^{-\frac{m-1}{2}}\varphi$ fulfilling $D^{\mathbb{R}^m}\tilde{\varphi} = \lambda^{+,*}_{\min}(\mathbb{S}^m)|\tilde{\varphi}|^{q-1}\tilde{\varphi}$ and $\|\tilde{\varphi}\|_{L^q(\mathbb{R}^m)} = 1$. But $\|\tilde{\varphi}\|_{L^\infty(\mathbb{R}^m)} = \rho^{-\frac{m-1}{4}}\|f_1^{-\frac{m-1}{2}}\varphi\|_{L^\infty(\mathbb{R}^m)} \to \infty$ as $\rho \to 0$. We obtain an example where L^∞-norm of solutions cannot be controlled in terms of its L^q-norm, λ and (M, g).

We close this section by some lemmas on removal of singularities for our Euler–Lagrange equations.

Lemma 4.7 *Let (M, g) be an m-dimensional Riemannian spin manifold, and let $S \subset M$ be an embedded submanifold of dimension $\ell \leq m - s^*$ where s^* is the conjugate exponent of s. Assume that φ is a spinor field such that $\|\varphi\|_s < \infty$ for $s \in (1, \infty)$ and $D\varphi = \lambda|\varphi|^{s-2}\varphi$ weakly on $M \setminus S$ for $\lambda \in \mathbb{R}$. Then $D\varphi = \lambda|\varphi|^{s-2}\varphi$ weakly on M.*

Proof We follow the proof for the removal of singularities for weakly harmonic spinors in [3, Lemma 2.4]: Let $U_S(\varepsilon)$ consist of all points of M with distance $\leq \varepsilon$ to S. Let η_δ be a cut-off function with $\eta_\delta = 1$ on $U_S(\delta)$, $\eta_\delta = 0$ on $M \setminus U_S(2\delta)$ and $|\nabla\eta_\delta| \leq 2\delta^{-1}$. Then, we obtain for a smooth and compactly supported spinor ψ on M

$$\begin{aligned}
&\int_M \langle\varphi, D\psi\rangle - \lambda \int_M \left\langle |\varphi|^{s-2}\varphi, \psi \right\rangle \\
&\quad = \int_M \langle\varphi, D(1-\eta_\delta)\psi\rangle - \lambda \int_M \left\langle |\varphi|^{s-2}\varphi, (1-\eta_\delta)\psi \right\rangle \\
&\qquad + \int_M \langle\varphi, \eta_\delta D\psi\rangle + \int_M \langle\varphi, \nabla\eta_\delta \cdot \psi\rangle - \lambda \int_M \left\langle |\varphi|^{s-2}\varphi, \eta_\delta\psi \right\rangle.
\end{aligned}$$

The sum of the first two summands on the right side vanishes since $D\varphi = \lambda|\varphi|^{s-2}\varphi$ weakly on $M \setminus S$. Moreover, $\left|\int_M \langle\varphi, \eta_\delta D\psi\rangle\right| \leq \|\varphi\|_s \|D\psi\|_{L^{s^*}(U_S(2\delta))} \to 0$ and $\left|\int_M \left\langle |\varphi|^{s-2}\varphi, \eta_\delta\psi\right\rangle\right| \leq \|\varphi\|_s^{s/s^*} \|\psi\|_{L^s(U_S(2\delta))} \to 0$ as $\delta \to 0$. The remaining term can be estimated by

$$\begin{aligned}
\left|\int_M \langle\varphi, \nabla\eta_\delta \cdot \psi\rangle\right| &\leq \frac{2}{\delta} \|\varphi\|_{L^s(U_S(2\delta))} \|\psi\|_{L^{s^*}(U_S(2\delta))} \\
&\leq \frac{C}{\delta} \|\varphi\|_{L^s(U_S(2\delta))} \mathrm{vol}(U_S(2\delta) \cap \mathrm{supp}\psi)^{\frac{1}{s^*}} \\
&\leq C' \underbrace{\|\varphi\|_{L^s(U_S(2\delta))}}_{\to 0} \delta^{\frac{m-\ell}{s^*}-1} \to 0.
\end{aligned}$$

□

Lemma 4.8 *Let (M, g) be an m-dimensional Riemannian spin manifold, and let $S \subset M$ be an embedded submanifold of dimension $\ell \leq m - 2s^*$ where s^* is the conjugate exponent of s. Assume that v is a nonnegative function such that $\|v\|_s < \infty$ for $s \in (1, \infty)$ and $Lv = \mu v^{s-1}$ weakly on $M \setminus S$ for $\mu \in \mathbb{R}$. Then $Lv = \mu v^{s-1}$ weakly on M.*

Proof The proof is similar to the one of Lemma 4.7, and we use the notation therein. The cut-off function η_δ is chosen such it fulfills additionally $|\Delta\eta_\delta| \leq 4\delta^{-2}$. Then, the estimates are done analogously. □

5 Gromov–Hausdorff Convergences

Let (M_i, g_i, x_i), $i \in \mathbb{N}$, and $(M_\infty, g_\infty, x_\infty)$ be pointed complete connected Riemannian manifolds. We say that (M_i, g_i, x_i) converges to $(M_\infty, g_\infty, x_\infty)$ in the

C^k-topology of pointed Riemannian manifolds if for every $R > 0$ and every $i \geq i_0(R)$ there is an injective immersion $\varphi_i^R \colon B_{R+1}^{M_\infty, g_\infty}(x_\infty) \to B_{R+1}^{M_i, g_i}(x_i)$ such that $(\varphi_i^R)^* g_i$ converges to g_∞ on $B_R^{M_\infty, g_\infty}(x_\infty)$ in the C^k-topology. If all manifolds above carry spin structures, then we say that they converge in the C^k-topology of pointed Riemannian spin manifolds if additionally the maps φ_i^R preserve the chosen spin structures.

Lemma 5.1 *If (M_i, g_i, x_i) converges to $(M_\infty, g_\infty, x_\infty)$ in the C^2-topology of pointed Riemannian manifolds, then*

$$\limsup_{i\to\infty} Q^*(M_i, g_i) \leq Q^*(M_\infty, g_\infty).$$

If (M_i, g_i, x_i) converges to $(M_\infty, g_\infty, x_\infty)$ in the C^1-topology of pointed Riemannian spin manifolds, then

$$\limsup_{i\to\infty} Q^*_{\mathrm{spin}}(M_i, g_i) \leq Q^*_{\mathrm{spin}}(M_\infty, g_\infty).$$

Proof For a given $\varepsilon > 0$ we take $v \in C_c^\infty(M_\infty)$ with $\mathcal{F}^{g_\infty}(v) < Q^*(M_\infty, g_\infty) + \varepsilon$. Choose $R > 0$ such that the support of v is contained in $B_R^{M_\infty, g_\infty}(x_\infty)$. For sufficiently large i we then have

$$Q^*(M_i, g_i) \leq \mathcal{F}^{g_i}(v \circ (\varphi_i^R)^{-1}) = \mathcal{F}^{(\varphi_i^R)^* g_i}(v) \leq \mathcal{F}^{g_\infty}(v) + \varepsilon < Q^*(M_\infty, g_\infty) + 2\varepsilon$$

where the second inequality uses that $\mathcal{F}^g$ depends only on derivatives of g up to order 2. The first part of the lemma follows in the limit $\varepsilon \to 0$.

The spinorial statement is proven completely analogously. Here, convergence in C^1 is enough since the Dirac operator is of first order. □

In the articles [4] and [5] the following situation was considered. Assume that N^m is obtained from M^m by a surgery of dimension k. Then for any metric g on M a family of special metrics g_ϑ, $\vartheta > 0$, was constructed. It was proved in [5] in combination with estimates given in [6] that for all $k \leq m-4$ and all $k = m-3 \leq 3$ we have

$$\lim_{\vartheta\to 0} Q^*(N, g_\vartheta) \geq \min\left\{Q^*(M, g), \widetilde{\Lambda}_{m,k}\right\}.$$

Similarly it was proven in [4] for $k \leq m-2$ that

$$\lim_{\vartheta\to 0} Q^*_{\mathrm{spin}}(N, g_\vartheta) \geq \min\left\{Q^*_{\mathrm{spin}}(M, g), \widetilde{\Lambda}^{\mathrm{spin}}_{m,k}\right\}.$$

We apply this construction to $M = \mathbb{S}^m$ equipped with the standard metric $g = \sigma^m$. Then $N = S^{k+1} \times S^{m-k-1}$. Thus we obtain a family of metrics g_ϑ on $N = S^{k+1} \times S^{m-k-1}$ with

$$\lim_{\vartheta\to 0} Q^*\left(S^{k+1} \times S^{m-k-1}, g_\vartheta\right) \geq \widetilde{\Lambda}_{m,k} \qquad \text{if } k \leq m-4 \text{ or if } k = m-3 \leq 3,$$

and

$$\lim_{\vartheta\to 0} Q^*_{\mathrm{spin}}\left(S^{k+1}\times S^{m-k-1}, g_\vartheta\right) \geq \widetilde{\Lambda}^{\mathrm{spin}}_{m,k} \qquad \text{if } k \leq m-2.$$

The following lemma is proven with exactly the same methods as in Sect. 6.3 of [5].

Lemma 5.2 *For any $c \in [0,1]$, there are points $x_\vartheta \in S^{k+1}\times S^{m-k-1}$, $\vartheta \in (0,1)$, such that $(S^{k+1}\times S^{m-k-1}, g_\vartheta, x_\vartheta)$ converges in the C^∞-topology of pointed Riemannian manifolds to $(\mathbb{M}^{m,k}_c, x_0)$ where x_0 is an arbitrary base point.*

Lemmas 5.1 and 5.2 imply

$$\lim_{\vartheta\to 0} Q^*\left(S^{k+1}\times S^{m-k-1}, g_\vartheta\right) \leq Q^*(\mathbb{M}^{m,k}_c)$$

and

$$\lim_{\vartheta\to 0} Q^*_{\mathrm{spin}}\left(S^{k+1}\times S^{m-k-1}, g_\vartheta\right) \leq Q^*_{\mathrm{spin}}(\mathbb{M}^{m,k}_c)$$

for all $c \in [0,1]$ with the same restrictions on k as above. Hence, we immediately obtain

Proposition 5.3

$$\widetilde{\Lambda}_{m,k} \leq \Lambda^*_{m,k} \qquad \textit{for } k \leq m-4 \textit{ and for } k = m-3 \leq 3,$$

$$\widetilde{\Lambda}^{\mathrm{spin}}_{m,k} \leq \Lambda^{\mathrm{spin},*}_{m,k} \qquad \textit{for } k \leq m-2.$$

Note that in this proposition we do not get any statement about the invariants for $\mathbb{M}^{m,k}_c$ for a fixed c; compare to Corollary 10.6.

6 Cut-Off Arguments

In this section we use cut-off functions to compare the $*$-invariants (which are defined as the infimum of a functional) with their $\sim$-counterparts (which are defined as the infimum of nonlinear eigenvalues).

Lemma 6.1 *Let (M^m, g) be a complete connected m-dimensional Riemannian manifold. Then, $Q^*(M,g) \leq \widetilde{Q}(M,g)$.*

Proof (cp. [5, Lemma 3.5]) Let $v \in C^2(M)\cap L^\infty(M)\cap L^2(M)$, $v \geq 0$, satisfying $L^g v = \mu_v v^{p-1}$ with $\mu_v \in \mathbb{R}_{\geq 0}$ and $\|v\|_{L^p} = 1$ where $p = \frac{2m}{m-2}$. We fix $z \in M$. Let η_r be a smooth cut-off function with values in $[0,1]$, $\eta_r = 0$ on $M \setminus B_{2r}(z)$, $\eta_r = 1$ on $B_r(z)$, and $|d\eta_r| \leq 2r^{-1}$. Then,

$$Q^*(M,g) \le \frac{\int_M \eta_r v L^g(\eta_r v)\mathrm{dvol}_g}{\|\eta_r v\|^2_{L^p(g)}} = \frac{\int_M \eta_r^2 v L^g v + a|d\eta_r|^2 v^2 \mathrm{dvol}_g}{\|\eta_r v\|^2_{L^p(g)}}$$
$$\le \frac{\int_M \mu_v \eta_r^2 v^p + a4r^{-2}v^2\mathrm{dvol}_g}{\|\eta_r v\|^2_{L^p(g)}} \to \mu_v \text{ as } r \to \infty.$$

□

Corollary 6.2 *$Q^*(\mathbb{M}_c^{m,k}) \le \widetilde{Q}(\mathbb{M}_c^{m,k})$ and $\Lambda^*_{m,k} \le \widetilde{\Lambda}_{m,k}$ for all m, k.*

Lemma 6.3 *Let (M^m, g) be a complete connected m-dimensional Riemannian spin manifold. Assume that D is $L^{q^*=\frac{2m}{m+1}}$-invertible. Then $Q^*_{\mathrm{spin}}(M,g) \le \widetilde{Q}_{\mathrm{spin}}(M,g)$.*

Proof Let $\lambda = \widetilde{\lambda}^+_{\min}(M,g)$. By the definition of $\widetilde{\lambda}^+_{\min}$, cf. (1), there is a $\varphi \in L^2 \cap L^\infty \cap C^1$ with $D\varphi = \lambda|\varphi|^{q-2}\varphi$ and $\|\varphi\|_q = 1$ where $q = \frac{2m}{m-1}$. Then, $D\varphi \in L^{q^*}$, and by the L^{q^*}-invertibility of D we get that $\varphi \in L^{q^*}$. Hence, $\lambda > 0$ since otherwise φ would be a nonzero L^{q^*}-harmonic spinor which contradicts the L^{q^*}-invertibility.

We fix $z \in M$. Let η_r be a smooth cut-off function with values in $[0, 1]$, $\eta_r = 0$ on $M \setminus B_{2r}(z)$, $\eta_r = 1$ on $B_r(z)$, and $|\mathrm{d}\eta_r| \le 2r^{-1}$. Then

$$\underbrace{(D(\eta_r\varphi), \eta_r\varphi)}_{\in\mathbb{R}} = (\mathrm{d}\eta_r \cdot \varphi, \varphi) + \underbrace{(\eta_r^2 D\varphi, \varphi)}_{\in\mathbb{R}} = \lambda \int_M \eta_r^2 |\varphi|^q \mathrm{dvol}_g > 0$$

where we used that the summand including $\mathrm{d}\eta_r$ vanishes due to $\langle \mathrm{d}\eta_r \cdot \varphi, \varphi\rangle_x \in \mathrm{i}\mathbb{R}$. Thus,

$$\lambda^{+,*}_{\min}(M,g,\chi) \le \frac{\|D(\eta_r\varphi)\|^2_{q^*}}{(D(\eta_r\varphi), \eta_r\varphi)} \le \frac{\left(\|\mathrm{d}\eta_r \cdot \varphi\|_{q^*} + \|\eta_r D\varphi\|_{q^*}\right)^2}{\lambda \int_M \eta_r^2|\varphi|^q \mathrm{dvol}_g}$$
$$\le \frac{\left(\frac{2}{r}\|\varphi\|_{q^*} + \lambda\left(\int_M \eta_r^{q^*}|\varphi|^q\mathrm{dvol}_g\right)^{1/q^*}\right)^2}{\lambda\int_M \eta_r^2|\varphi|^q \mathrm{dvol}_g}$$
$$\to \lambda\, \|\varphi\|_q^{q(2-q^*)/q^*} = \lambda\, \|\varphi\|_q^{\frac{2}{m-1}} = \lambda$$

as $r \to \infty$. Note that the summand $\frac{1}{r}\|\varphi\|_{q^*} \to 0$ since $\varphi \in L^{q^*}$ as shown above. Hence, $Q^*_{\mathrm{spin}} \le \widetilde{Q}_{\mathrm{spin}}$. □

Corollary 6.4 *For all $c \in [0,1]$ and $k \le m-1$, we have $Q^*_{\mathrm{spin}}(\mathbb{M}_c^{m,k}, g_c) \le \widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,k}, g_c)$. In particular, $\Lambda^{\mathrm{spin},*}_{m,k} \le \widetilde{\Lambda}^{\mathrm{spin}}_{m,k}$.*

Proof We start with $k \le m-2$. Lemma 6.3 and Proposition 2.2 imply $Q^*_{\mathrm{spin}}(\mathbb{M}_c^{m,k}, g_c) \le \widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,k}, g_c)$ for all $\frac{m-k-1}{2} > ck(\frac{m+1}{2m} - \frac{1}{2}) = \frac{ck}{2m}$, i.e., for all $k \le m-2$ and $c \in [0,1]$.

The remaining case $k = m-1$ follows directly from Lemma 7.4. □

7 The Model Space $\mathbb{M}_1^{m,k}$

For $c = 1$ the model spaces $\mathbb{M}_c^{m,k}$ are very special: The manifold

$$\left(\mathbb{M}_1^{m,k} = \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1}, g_1 = g_{\mathbb{H}^{k+1}} + \sigma^{m-k-1} = \sinh^2 t\, \sigma^k + dt^2 + \sigma^{m-k-1}\right)$$

is conformal to $(\mathbb{S}^m \setminus \mathbb{S}^k, \sigma^m)$, [5, Proposition 3.1],

$$\mathfrak{u} : \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1} \to \mathbb{S}^m \setminus \mathbb{S}^k,\ g_1 = f^2 \mathfrak{u}^* \sigma^m \text{ where } f = f(t) = \cosh t \qquad (7)$$

where $\cosh t = (\sin r)^{-1}$ with $r = \mathrm{dist}(., \mathbb{S}^k)$.

Using this conformal map, we will immediately obtain some of the Q-invariants of $\mathbb{M}_1^{m,k}$.

Lemma 7.1 $Q^*(\mathbb{M}_1^{m,k}) = Q^*(\mathbb{S}^m) = Q^*_{\mathrm{spin}}(\mathbb{M}_1^{m,k})$.

Proof By conformal invariance $Q^*(\mathbb{M}_1^{m,k}, g_1) = Q^*(\mathbb{S}^m \setminus \mathbb{S}^k, \sigma^m)$. Since Q^* is defined over test functions, we have $Q^*(\mathbb{S}^m \setminus \mathbb{S}^k) \geq Q^*(\mathbb{S}^m)$. On the other hand $Q^*(\mathbb{S}^m)$ is the highest possible value for Q^*, see Remark 2.4, and thus $Q^*(\mathbb{M}_1^{m,k}) = Q^*(\mathbb{S}^m)$. With analogous arguments one gets $Q^*_{\mathrm{spin}}(\mathbb{M}_1^{m,k}) = Q^*_{\mathrm{spin}}(\mathbb{S}^m)$. Together with $Q^*(\mathbb{S}^m) = Q^*_{\mathrm{spin}}(\mathbb{S}^m)$ the lemma follows. □

In order to examine $\widetilde{Q}(\mathbb{M}_1^{m,k})$ and $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_1^{m,k})$ we will need modifications of the removal of singularities results in Lemmas 4.7 and 4.8.

Lemma 7.2 *Let (M, g) be an m-dimensional Riemannian spin manifold, and let $S \subset M$ be an embedded submanifold of dimension $\ell \leq m-1$. Assume that φ is a spinor field such that $\int_{U_\varepsilon(S)} \frac{1}{\rho}|\varphi|^2 < \infty$ where $U_\varepsilon(S)$ consists of all points of M with distance $\rho \leq \varepsilon$ to S. Moreover, let $D\varphi = \lambda|\varphi|^{q-2}\varphi$ weakly on $M \setminus S$ for $\lambda > 0$. Then $D\varphi = \lambda|\varphi|^{q-2}\varphi$ weakly on M.*

Proof We adapt the proof of Lemma 4.7: Let $\tilde{\eta}_\delta$ be the function on M defined by

$$\tilde{\eta}_\delta(x) = \begin{cases} 0 & \text{for } \rho := \mathrm{dist}(x, S) \geq \rho_0 := \delta \\ \delta \log(\rho_0/\rho) & \text{for } \rho_0 \geq \rho \geq \rho_1 := \rho_0 e^{-1/\delta} \\ 1 & \text{for } \rho_1 \geq \rho. \end{cases}$$

We smooth out $\tilde{\eta}_\delta$ in such a way that the resulting function η_δ still fulfills $\eta_\delta(x) = 1$ for $\rho \geq \rho_0$, $\eta_\delta(x) = 0$ for $\rho \leq \rho_1$, and $|\nabla\eta_\delta| \leq \frac{2\delta}{\rho}$.

Then, for a smooth and compactly supported spinor ψ on M we obtain

$$\begin{aligned} &\int_M \langle\varphi, D\psi\rangle - \lambda \int_M \left\langle|\varphi|^{q-2}\varphi, \psi\right\rangle \\ &\quad = \int_M \langle\varphi, D(1-\eta_\delta)\psi\rangle - \lambda \int_M \left\langle|\varphi|^{q-2}\varphi, (1-\eta_\delta)\psi\right\rangle \\ &\qquad + \int_M \langle\varphi, \eta_\delta D\psi\rangle + \int_M \langle\varphi, \nabla\eta_\delta \cdot \psi\rangle - \lambda \int_M \left\langle|\varphi|^{q-2}\varphi, \eta_\delta\psi\right\rangle. \end{aligned}$$

The sum of the first two summands on the right side vanishes because the equation holds on $M \setminus S$. The terms $\int_M \langle \varphi, \eta_\delta D\psi\rangle$ and $\int_M \langle |\varphi|^{q-2}\varphi, \eta_\delta \psi\rangle$ vanish for the same reason as in the proof of Lemma 4.7. The remaining term is now estimated by

$$\left|\int_M \langle \varphi, \nabla\eta_\delta \cdot \psi\rangle\right| \le C\delta \int_{U_\delta(S)\cap \mathrm{supp}\psi} \frac{1}{\rho}|\varphi| \le C\delta \left(\int_{U_\delta(S)} \frac{1}{\rho}|\varphi|^2\right)^{\frac{1}{2}} \left(\int_{U_\delta(S)\cap \mathrm{supp}\psi} \frac{1}{\rho}\right)^{\frac{1}{2}}$$

$$\le C'\delta \underbrace{\left(\int_{U_\delta(S)} \frac{1}{\rho}|\varphi|^2\right)^{\frac{1}{2}}}_{\to 0 \text{ as } \delta\to 0} \quad \underbrace{\left(\int_{\delta e^{-1/\delta}}^{\delta} \frac{1}{\rho}\rho^{m-\ell-1}\,\mathrm{d}\rho\right)^{\frac{1}{2}}}_{\text{is } \delta^{-1/2} \text{ for } m=\ell+1 \text{ and } \le \hat{C}\delta^{(m-\ell-1)/2} \text{ else}} \to 0$$

as $\delta \to 0$ which concludes the proof. □

Lemma 7.3 *Let (M, g) be an m-dimensional Riemannian spin manifold, and let $S \subset M$ be an embedded submanifold of dimension $\ell \le m-2$. Assume that v be a nonnegative function such that $\int_{U_\varepsilon(S)} \frac{1}{\rho^2}v^2 < \infty$ where $U_\varepsilon(S)$ consists of all points of M with distance $\rho \le \varepsilon$ to S. Moreover, let $Lv = \mu v^{p-1}$ weakly on $M \setminus S$. Then $Lv = \mu v^{p-1}$ weakly on M.*

Proof We use an analogous argumentation as in the proof above. Now, we smooth out $\tilde{\eta}_\delta$ in such a way that the resulting η_δ fulfills additionally $|\Delta\eta_\delta| \le \frac{4\delta}{\rho^2}$. Then, for $h \in C_c^\infty(M)$ we estimate $\int_M vLh - \int_M v^{p-1}h$ in a similar way—only $\Delta\eta_\delta$ gives rise to a new term:

$$\left|\int_M vh\Delta\eta_\delta\right| \le C\delta \int_{U_\delta(S)\setminus U_{\delta e^{-1/\delta}}(S)\cap \mathrm{supp} h} \frac{1}{\rho^2}v$$

$$\le C\delta \left(\int_{U_\delta(S)} \frac{1}{\rho^2}v^2\right)^{\frac{1}{2}} \left(\int_{U_\delta(S)\setminus U_{\delta e^{-1/\delta}}(S)\cap \mathrm{supp} h} \frac{1}{\rho^2}\right)^{\frac{1}{2}}$$

$$\le C'\delta \underbrace{\left(\int_{U_\delta(S)} \frac{1}{\rho^2}v^2\right)^{\frac{1}{2}}}_{\to 0 \text{ as } \delta\to 0} \quad \underbrace{\left(\int_{\delta e^{-1/\delta}}^{\delta} \frac{1}{\rho^2}\rho^{m-\ell-1}\,d\rho\right)^{\frac{1}{2}}}_{\text{is } \delta^{-1/2} \text{ for } m=\ell+2 \text{ and } \le \hat{C}\delta^{(m-\ell-1)/2} \text{ else}} \to 0 \text{ as } \delta \to 0.$$

□

Lemma 7.4 *For $m \ge 2$*

$$\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_1^{m,k}) = \begin{cases} Q^*(\mathbb{S}^m) & \text{for } k \le m-2 \\ \widetilde{Q}_{\mathrm{spin}}(\mathbb{H}^m) = \infty & \text{for } k = m-1. \end{cases}$$

Proof Let $\varphi \in L^\infty \cap L^2 \cap C^1$ be a solution of $D\varphi = \lambda|\varphi|^{q-2}\varphi$ on $\mathbb{M}_1^{m,k}$ with $0 < \|\varphi\|_{L^q} \le 1$. Using the conformal map $\mathfrak{u}$ in (7) we obtain a C^1-solution $\tilde{\varphi} = f^{\frac{m-1}{2}}\varphi$ of $D^{\sigma^m}\tilde{\varphi} = \lambda|\tilde{\varphi}|^{q-2}\tilde{\varphi}$ on $\mathbb{S}^m \setminus \mathbb{S}^k$ with $0 < \|\tilde{\varphi}\|_{L^q} \le 1$. Moreover, since φ is L^2 we get $\infty > \|\varphi\|_{L^2}^2 = \int_{\mathbb{S}^m\setminus\mathbb{S}^k} f|\tilde{\varphi}|^2\,\mathrm{dvol}_{\sigma^m} = \int_{\mathbb{S}^m\setminus\mathbb{S}^k} (\sin\rho)^{-1}|\tilde{\varphi}|^2\,\mathrm{dvol}_{\sigma^m}$.

In particular, $\tilde{\varphi} \in L^2$. Moreover, $\frac{1}{\rho} - \frac{1}{\sin\rho}$ is bounded by $\mathcal{O}(\varepsilon)$ for $\rho \in (0, \varepsilon)$. Thus, $\int_{U_\varepsilon(\mathbb{S}^k)} \frac{1}{\rho}|\tilde{\varphi}|^2 \,\mathrm{dvol}_{\sigma^m} < \infty$ as well. Because of Lemma 7.2 $\tilde{\varphi}$ solves $D^{\sigma^m}\tilde{\varphi} = \lambda|\tilde{\varphi}|^{q-2}\tilde{\varphi}$ weakly on all of $\mathbb{S}^m$. By regularity theory on compact manifolds $\tilde{\varphi} \in L^q$ implies $\tilde{\varphi} \in H_1^q \subset H_1^{q^*}$. Hence, $\tilde{\varphi}$ can serve as a test function for $\mathcal{F}^{\mathrm{spin}}$ on $\mathbb{S}^m$ which implies $\lambda \geq \widetilde{Q}_{\mathrm{spin}}(\mathbb{S}^m) = Q^*(\mathbb{S}^m)$. Thus, $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_1^{m,k}) \geq Q^*(\mathbb{S}^m)$.

Now let $\tilde{\varphi}$ be a Killing spinor on $\mathbb{S}^m$ with Killing constant $-\frac{1}{2}$ and $\|\tilde{\varphi}\|_{L^q(\mathbb{S}^m)} = 1$. Then $D\tilde{\varphi} = Q^*_{\mathrm{spin}}(\mathbb{S}^m)|\tilde{\varphi}|^{\frac{4}{m-1}}\tilde{\varphi}$. Then using the identification of spinor bundles to conformal metrics as in Example 4.6 the spinor $\varphi = f^{-\frac{m-1}{2}}\tilde{\varphi}$ fulfills the Euler–Lagrange equation for D on $\mathbb{M}_1^{m,k}$ and is in $L^\infty \cap L^q$. Moreover, if $m - k \geq 2$, then

$$\|\varphi\|_{L^2}^2 = C_2 \int_0^\infty \cosh^{1-m} t \sinh^k t \,\mathrm{d}t \leq C_3 + C_4 \int_1^\infty e^{(1-m+k)t} \,\mathrm{d}t < \infty.$$

Thus, for $k \leq m-2$ we obtained $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_1^{m,k}) = Q^*(\mathbb{S}^m)$.

Now let $k = m-1$. Then, $\mathbb{M}_1^{m,m-1}$ corresponds to two copies of the hyperbolic space. Thus, $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_1^{m,m-1}) = \widetilde{Q}_{\mathrm{spin}}(\mathbb{H}^m)$. Now let φ be a solution as above on $\mathbb{H}^m$. By a conformal map we get as above a solution $\tilde{\varphi}$ on the lower hemisphere of $\mathbb{S}^m$. Extending $\tilde{\varphi}$ by zero to all of $\mathbb{S}^m$, we obtain a weak solution to our nonlinear Dirac eigenvalue equation on $\mathbb{S}^m \setminus \mathbb{S}^{m-1}$. Again using Lemma 7.2 we see that $\tilde{\varphi}$ is already a nontrivial weak solution on all of $\mathbb{S}^m$. But since $\tilde{\varphi}$ vanishes on an open subset, this contradicts the unique continuation principle, [15]. Thus, such a solution φ we started with cannot exist. Hence, $\widetilde{Q}_{\mathrm{spin}}(\mathbb{H}^m) = \widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_1^{m,m-1}) = \infty$. ◻

Lemma 7.5 *For $m \geq 3$*

$$\widetilde{Q}(\mathbb{M}_1^{m,k}) = \begin{cases} Q^*(\mathbb{S}^m) & \text{for } k \leq m-3 \\ \infty & \text{for } k = m-2 \\ \widetilde{Q}(\mathbb{H}^m) = \infty & \text{for } k = m-1. \end{cases}$$

Proof We start analogously as in the spin case from above with a nonnegative solution $v \in L^\infty \cap L^2 \cap C^2$ of $Lv = \mu v^{p-1}$ on $\mathbb{M}_1^{m,k}$ and use the conformal map $\mathfrak{u}$ in (7) to obtain $\tilde{v}$ on $\mathbb{S}^m \setminus \mathbb{S}^k$. Analogous as in the proof of Lemma 7.4 we see that $\int_{U_\varepsilon(\mathbb{S}^k)} \frac{1}{\rho^2}\tilde{v}^2 \,\mathrm{dvol}_{\sigma^m} < \infty$ which allows us to use Lemma 7.3 for $k \leq m-2$. Thus, we get as in the last lemma that $\widetilde{Q}(\mathbb{M}_1^{m,k}) \geq Q^*(\mathbb{S}^m)$ for $k \leq m-2$. On the other hand, $\tilde{v} =$ const such that $\|\tilde{v}\|_{L^p(\mathbb{S}^m)} = 1$ is a solution of the Euler–Lagrange equation on $\mathbb{S}^m$. Set $v = f^{-\frac{m-2}{2}}\mathfrak{u}^*\tilde{v} = C\cosh^{-\frac{m-2}{2}} t$ where C is an appropriate constant. Then by conformal invariance, v fulfills the Euler–Lagrange equation on $\mathbb{M}_1^{m,k}$ and is in L^p. Moreover, if $m - k \geq 3$, $v \in L^2$ as can be seen by $\|v\|_{L^2}^2 = C\int_0^\infty \cosh^{2-m} t \sinh^k t \,\mathrm{d}t \leq C_1 + C_2\int_1^\infty e^{(2-m+k)t}\,\mathrm{d}t < \infty$. Hence, for $k \leq m-3$ we have $\widetilde{Q}(\mathbb{M}_1^{m,k}) = Q^*(\mathbb{S}^m)$.

For $m - k \leq 2$ we obtained up to now that each nonnegative solution v on $\mathbb{M}_1^{m,k}$ gives rise to a nonnegative solution $\tilde{v}$ on $\mathbb{S}^m$. By [35, Theorem 5] $\tilde{v}$ is continuous and everywhere positive. For $m-k=2$ and using that $\tilde v$ is continuous and positive we can estimate

$$\int_{U_\varepsilon(\mathbb{S}^{m-2})}\frac{1}{\rho^2}\tilde v^2 \mathrm{dvol}_{\sigma^m}\geq C\int_0^\varepsilon \frac{1}{\rho^2}\rho\,\mathrm{d}\rho.$$

Thus, the left integral is not finite which gives a contradiction. Thus, $\widetilde{Q}(\mathbb{M}_1^{m,m-2})=\infty$.

For $k=m-1$ let $v\in L^\infty\cap L^2\cap C^2$ be a positive solution of $Lv=\mu v^{p-1}$ on $\mathbb{M}_1^{m,m-1}$. Thus, we have two solutions of the same equation on the hyperbolic space. We will show that a nontrivial solution of $Lv=\mu|v|^{p-2}v$ on the hyperbolic space cannot exist in L^2. From a solution on the hyperbolic space we can use the conformal map $\mathfrak{u}$ to obtain a solution $\tilde v$ on the lower hemisphere $\mathbb{S}^m$. We extend $\tilde v$ to the upper hemisphere by reflection and changing its sign on the upper hemisphere. Thus, $\tilde v$ solves $L\tilde v=\mu|\tilde v|^{p-2}\tilde v$ on $\mathbb{S}^m\setminus\mathbb{S}^{m-1}$. Next we show that $\tilde v$ solves this equation weakly on all of $\mathbb{S}^m$. Since $\tilde v$ is an odd function with respect to reflection at the equator, it suffices to test with odd functions $h\in C^\infty(\mathbb{S}^m)$. Thus, there is a constant $C>0$ such that $|h(x)|\leq C\mathrm{dist}(x,\mathbb{S}^{m-1})=C\rho$. Following the arguments in Lemma 7.3 the estimates are done analogously, and it remains to estimate

$$\begin{aligned}\left|\int_M vh\Delta\eta_\delta\right| &\leq C\delta\int_{U_\delta(\mathbb{S}^{m-1})\setminus U_{\delta e^{-1/\delta}}(\mathbb{S}^{m-1})\cap\mathrm{supp}h}\frac{1}{\rho^2}vh\\ &\leq C'\delta\int_{U_\delta(\mathbb{S}^{m-1})\setminus U_{\delta e^{-1/\delta}}(\mathbb{S}^{m-1})\cap\mathrm{supp}h}\frac{1}{\rho}\\ &\leq C''\delta\underbrace{\left(\int_{\delta e^{-1/\delta}}^{\delta}\frac{1}{\rho}\,d\rho\right)^{\frac12}}_{=\delta^{-1/2}}\to 0 \text{ as } \delta\to 0.\end{aligned}$$

Thus, $\tilde v$ solves $L\tilde v=\mu|\tilde v|^{p-2}\tilde v$ weakly on $\mathbb{S}^m$. Then, regularity theory implies that $\tilde v\in C^2$ and thus $\tilde v|_{\mathbb{S}^{m-1}}=0$. Using a conformal transformation from the lower hemisphere to the disk D in $\mathbb{R}^m$, we obtain a solution $\hat v$ of $L\hat v=\mu|\hat v|^{p-2}\tilde v$ on D which is somewhere nonzero in the interior of D and zero on the boundary. This is a contradiction to [36], [39, Theorem III.1.3]. Thus the solution we started with cannot exist, and hence $\widetilde{Q}(\mathbb{H}^m)=\infty$. □

8 The Invariants for $k=m-1$

The constants $\Lambda^*_{m,m-1}$ and $\Lambda^{\mathrm{spin},*}_{m,m-1}$ are easy to determine.

Lemma 8.1 *We have $\Lambda^{\mathrm{spin},*}_{m,m-1}=Q^*(\mathbb{S}^m)$ for all $m\geq 3$ and $\Lambda^*_{m,m-1}=Q^*(\mathbb{S}^m)$ for all $m\geq 2$.*

Proof We show

$$Q^*_{\mathrm{spin}}(\mathbb{M}_c^{m,m-1})=Q^*(\mathbb{M}_c^{m,m-1})=Q^*(\mathbb{S}^m).$$

For $c \neq 0$, our model space $\mathbb{M}_c^{m,m-1}$ is isometric to two copies of the rescaled hyperbolic space $\mathbb{H}_c^m$, and for $c = 0$ it is isometric to two copies of the Euclidean $\mathbb{R}^m$. Thus, $Q^*(\mathbb{M}_c^{m,m-1}) = Q^*(\mathbb{H}_c^m) = Q^*(\mathbb{H}^m) = Q^*(\mathbb{R}^m) = Q^*(\mathbb{S}^m)$, cf. Remark 2.4, and the first equality follows from [32, Lemma 1.10]. Using [21, Lemma 2.0.5] the analogous equations hold for Q^*_{spin} which finishes the proof. □

For the $\widetilde{Q}$-invariants we have by scaling and Lemma 7.4 that $\widetilde{Q}_{\mathrm{spin}}(\mathbb{H}_c^m) = \widetilde{Q}_{\mathrm{spin}}(\mathbb{H}^m) = \infty$ for $c \in (0, 1]$ and $m \geq 2$, and $\widetilde{Q}(\mathbb{H}_c^m) = \widetilde{Q}(\mathbb{H}^m) = \infty$ for $c \in (0, 1]$ and $m \geq 3$. It remains to consider the Euclidean space.

Lemma 8.2 *We have $\widetilde{Q}(\mathbb{R}^m) = \infty$ for $m = 3, 4$, $\widetilde{Q}(\mathbb{R}^m) = Q^*(\mathbb{S}^m)$ for all $m \geq 5$, $\widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^m) = Q^*(\mathbb{S}^m)$ for all $m \geq 3$ and $\widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^2) \geq Q^*(\mathbb{S}^2)$.*

Proof We start examining $\widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^m)$. Let $\varphi \in L^2 \cap L^\infty \cap C^1$ be a solution on $\mathbb{R}^m$ of $D\varphi = \lambda|\varphi|^{q-2}\varphi$ with $0 < \|\varphi\|_{L^q} \leq 1$ for some $\lambda > 0$. By stereographic projection, we have $\sigma^m = \frac{4}{(1+r^2)^2} g_E$ where r is the radial function in $\mathbb{R}^m$. Using the conformal invariance of the nonlinear Dirac eigenvalue equation above, we get for $\tilde{\varphi} = \left(\frac{1+r^2}{2}\right)^{(m-1)/2} \varphi$ that $D^{\sigma^m}\tilde{\varphi} = \lambda|\tilde{\varphi}|^{q-2}\tilde{\varphi}$ and $0 \leq \|\tilde{\varphi}\|_{L^q} \leq 1$ on $\mathbb{S}^m \setminus \{N\}$. Moreover, $\int_{\mathbb{S}^m\setminus\{N\}} \frac{1+r^2}{2}|\tilde{\varphi}|^2 \,\mathrm{dvol}_{\sigma^m} = \int_{\mathbb{R}^m} |\varphi|^2 \,\mathrm{dvol}_E < \infty$ and $1+r^2 = \frac{2}{\sin^2\rho}$ where ρ is the distance to the north pole N. In particular, it now follows similarly to the proof of Lemma 7.4 that $\int_{U_\varepsilon(N)} \frac{1}{\rho^2}|\tilde{\varphi}|^2 \,\mathrm{dvol}_{\sigma^m}$ is finite. In particular, $\int_{U_\varepsilon(N)} \frac{1}{\rho}|\tilde{\varphi}|^2 \,\mathrm{dvol}_{\sigma^m}$ is finite as well. Thus, we can apply Lemma 7.2 and see the nonlinear Dirac eigenvalue equation from above is valid on all of $\mathbb{S}^m$. Thus, we can conclude as in the proof of Lemma 7.4 that $\widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^m) \geq Q^*(\mathbb{S}^m)$. Let $\tilde{\varphi}$ be a Killing spinor to the Killing constant $-\frac{1}{2}$ normalized such that $\|\tilde{\varphi}\|_{L^q} = 1$. Then, $D\tilde{\varphi} = \lambda^{+,*}_{\min}(\mathbb{S}^m)|\tilde{\varphi}|^{q-2}\tilde{\varphi}$. Using stereographic projection we obtain a smooth spinor $\varphi = \left(\frac{1+r^2}{2}\right)^{\frac{-m+1}{2}} \tilde{\varphi}$ on $\mathbb{R}^m$ with L^q-norm one and which satisfies $D\varphi = \lambda^{+,*}_{\min}(\mathbb{S}^m)|\varphi|^{q-2}\varphi$. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^m} |\varphi|^2 \,\mathrm{dvol}_E &= C \int_{\mathbb{R}^m} \left(\frac{1+r^2}{2}\right)^{-m+1} |\tilde{\varphi}|^2 \,\mathrm{dvol}_E \\ &\leq C' \int_0^\infty \left(\frac{1+r^2}{2}\right)^{-m+1} r^{m-1}\,\mathrm{d}r.\end{aligned}$$

Thus, $\varphi \in L^2(\mathbb{R}^m)$ for $m \geq 3$. Hence, $\widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^m) = Q^*(\mathbb{S}^m)$ for $m \geq 3$.

An analogous argumentation for nonnegative solution $v \in C^2 \cap L^\infty \cap L^2$ on $\mathbb{R}^m$ satisfying $Lv = \mu v^{p-1}$ and $\|v\|_{L^p} = 1$ for a $\mu > 0$ gives a nonnegative solution $\tilde{v}$ of the corresponding nonlinear eigenvalue equation on the sphere with $\int_{U_\varepsilon(N)} \frac{1}{\rho^2}\tilde{v}^2 \,\mathrm{dvol}_{\sigma^m} < \infty$. By regularity $\tilde{v} \in L^p \cap H_2^{p^*}$ with $p^* = \frac{2m}{m+2}$. Thus, by the Sobolev embedding theorem $\tilde{v} \in H_1^2$. Thus, similar as in Lemma 7.4 we see that $\widetilde{Q}(\mathbb{R}^m) \geq Q^*(\mathbb{S}^m)$. Moreover, by [35, Theorem 5] $\tilde{v}$ is continuous and everywhere positive. Hence, we can estimate

$$\int_{\mathbb{R}^m} v^2\,\mathrm{dvol}_E = \int_{\mathbb{R}^m}\left(\frac{1+r^2}{2}\right)^{-m+2}\tilde{v}^2\,\mathrm{dvol}_E \geq C\int_0^\infty\left(\frac{1+r^2}{2}\right)^{-m+2} r^{m-1}\,\mathrm{d}r$$
$$\geq C\int_1^\infty r^{-m+3}\,\mathrm{d}r + C'.$$

Thus, for $m=3,4$ the solution v was not in L^2 which contradicts the assumption. Hence, $\widetilde{Q}(\mathbb{R}^m)=\infty$ for $m=3,4$. For $m\geq 5$, we see with an analogous calculation that taking the constant solution v of $Lv = Q^*(\mathbb{S}^m)v^{p-1}$, $\|v\|_{L^p}=1$ on the sphere, we obtain via stereographic projection a solution $\tilde{v}$ on $\mathbb{R}^m$ which is even in $L^2(\mathbb{R}^m)$. Thus, $\widetilde{Q}(\mathbb{R}^m) = Q^*(\mathbb{S}^m)$ for $m\geq 5$. □

Example 8.3 Let φ be a Killing spinor on $\mathbb{S}^2$ with $L^{q=4}$-norm one. Then, $D\varphi = Q^*(\mathbb{S}^2)|\varphi|^2\varphi$. We consider the three-branched covering $h\colon \mathbb{S}^2\to\mathbb{S}^2$, $z\mapsto z^3$. This map preserves the spin structure. Thus, we can pullback φ via h and obtain a spinor $\tilde{\varphi}$ on $\mathbb{S}^2$, cp. [1, Sect. 4] fulfilling $D\tilde{\varphi} = Q^*(\mathbb{S}^2)|\tilde{\varphi}|^2\tilde{\varphi}$ and $\|\tilde{\varphi}\|^4_{L^4}=3$. In particular, $\tilde{\varphi}$ has zeros on the north and the south pole of $\mathbb{S}^2$. Setting $\widehat{\varphi} = \left(\frac{1}{3}\right)^{1/4}\tilde{\varphi}$ we obtain $D\widehat{\varphi} = 3^{1/2}Q^*(\mathbb{S}^2)|\widehat{\varphi}|^2\widehat{\varphi}$ and $\|\widehat{\varphi}\|_{L^4}=1$. Using stereographic projection we obtain a spinor $\psi = (\frac{2}{1+r^2})^{1/2}\widehat{\varphi}$ on $\mathbb{R}^2$ (r being the radial coordinate in $\mathbb{R}^2$) with $D\psi = 3^{1/2}Q^*(\mathbb{S}^2)|\psi|^2\psi$ and $\|\psi\|_{L^4}=1$. Moreover, since $\widehat{\varphi}$ vanishes at the north pole N, $|\widehat{\varphi}(x)|\leq C\rho$ on $U_\varepsilon(N)$ where $\rho = \mathrm{dist}(.,N)$. by the estimate

$$\int_{\mathbb{R}^2\setminus B_r(0)}|\psi|^2\mathrm{dvol}_E = \int_{U_{\varepsilon(r)}(N)}\frac{1+r^2}{2}|\widehat{\varphi}|^2\mathrm{dvol}_{\sigma^2}$$
$$\leq C'\int_0^\varepsilon \rho^2\frac{1+r^2}{2}\rho\,d\rho = C'\int_0^\varepsilon\frac{\rho^3}{\sin^2\rho}\,\mathrm{d}\rho.$$

Thus, $\psi\in L^2(\mathbb{R}^2)$ and $\widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^2)\leq 3^{1/2}Q^*(\mathbb{S}^2)$.

Summarizing we obtained for the spinorial invariants

Corollary 8.4 *We have*

$$\widetilde{\Lambda}^{\mathrm{spin}}_{m,m-1} = \Lambda^{\mathrm{spin}}_{m,m-1} = Q^*(\mathbb{S}^m)\quad \textit{for all } m\geq 3$$

and $3^{1/2}Q^*(\mathbb{S}^2)\geq\widetilde{\Lambda}^{\mathrm{spin}}_{2,1} = \widetilde{Q}_{\mathrm{spin}}(\mathbb{R}^2)\geq Q^*(\mathbb{S}^2) = \Lambda^{\mathrm{spin}}_{2,1}$. *Moreover,*

$$\widetilde{\Lambda}_{m,m-1} = \Lambda_{m,m-1} = Q^*(\mathbb{S}^m)\quad \textit{for all } m\geq 5,$$
$$\infty = \widetilde{\Lambda}_{m,m-1} > \Lambda_{m,m-1} = Q^*(\mathbb{S}^m)\quad \textit{for } m=3,4.$$

9 Hijazi Inequalities

On a closed spin manifold (M^m, g), the Hijazi inequality provides a lower bound of the lowest eigenvalue $\lambda_0^2(g)$ of the square of the Dirac operator by the lowest eigenvalue of the conformal Laplacian $\mu(g)$, [29, Theorem A],

$$\lambda_0^2(g) \geq \frac{m}{4(m-1)}\mu(g). \tag{8}$$

Taking the infimum over all metrics conformal to g with constant volume, one obtains the conformal Hijazi inequality [22]

$$Q^*_{\rm spin}(M) \geq Q^*(M). \tag{9}$$

We call (8) the metric Hijazi inequality and (9) the conformal Hijazi inequality. In this section, we want to discuss whether similar inequalities also hold on noncompact manifolds. In this context one should replace the lowest eigenvalues in (8) by the infimum of the corresponding spectra whereas (9) remains unchanged.

In [22, Theorems 1.1 and 1.2] the metric Hijazi inequality was shown by the second author for complete spin manifold of finite volume fulfilling one of the following conditions:

(1) The infimum of the spectrum of the squared Dirac operator is an eigenvalue.
(2) The infimum of the spectrum of the squared Dirac operator is in the essential spectrum, $m \geq 5$ and the scalar curvature is bounded from below.

In particular, this already implies the conformal Hijazi inequality for manifolds which admit a conformal metric $\bar{g}$ that is complete and of finite volume and where zero is not in the essential spectrum of the Dirac operator for $\bar{g}$ or where the second condition from above is fulfilled, cf. [22, Theorem 1.3].

There are also examples of manifolds of bounded geometry where the metric Hijazi inequality does not hold. The simplest example is the hyperbolic space $\mathbb{H}^m$ where 0 is in the spectrum of the Dirac operator and the spectrum of the conformal Laplacian is $[\mu, \infty)$ with $\mu = 4\frac{m-1}{m-2}\frac{(m-1)^2}{4} - m(m-1) = \frac{m-1}{m-2} > 0$.

On the other hand, the hyperbolic space is conformal to a subset of the standard sphere. Thus, $Q^*_{\rm spin}(\mathbb{H}^m) = Q^*_{\rm spin}(\mathbb{H}^m) = Q^*_{\rm spin}(\mathbb{S}^m) = Q^*(\mathbb{S}^m)$; see Lemma 7.1 for details. Unfortunately it is still unclear whether the conformal Hijazi inequality (9) holds for all complete Riemannian spin manifolds.

In this section we prove slightly modified conformal Hijazi inequalities. Some inequalities are proven only for the model spaces, some on more general manifolds, e.g., for manifolds of bounded geometry with uniformly positive scalar curvature.

Proposition 9.1 *Let (M^m, g) be of bounded geometry with $m \geq 3$. Let $\varphi \in L^q \cap C^0$ and $\lambda \in \mathbb{R}$ with $D\varphi = \lambda|\varphi|^{\frac{2}{m-1}}\varphi$ weakly and $\|\varphi\|_q = 1$ where $q = \frac{2m}{m-1}$. Then $u := |\varphi|^{\frac{m-2}{m-1}}$ satisfies*

$$Lu \leq 4\frac{m-1}{m}\lambda^2 u^{\frac{m+2}{m-2}} \tag{10}$$

in the sense of distributions. Moreover, the equation holds classically outside the zero-set of u.

Proof By Corollary 4.4 and Lemma 4.2 $\varphi \in L^\infty \cap C^2$. We use the idea of Christian Bär and Andrei Moroianu written down in [18, Proposition 3.4]. Let $\alpha := \frac{m-2}{m-1}$ and

$f := \lambda|\varphi|^{q-2}$. We define the Friedrich connection ∇^f as $\nabla^f_X\psi = \nabla_X\psi + \frac{f}{m}X\cdot\psi$. Then for all points where $\varphi \neq 0$ we estimate

$$\begin{aligned}
\mathrm{d}^*\mathrm{d}|\varphi|^\alpha &= \frac{\alpha}{2}|\varphi|^{\alpha-2}\mathrm{d}^*\mathrm{d}|\varphi|^2 - \alpha(\alpha-2)|\varphi|^{\alpha-2}|\mathrm{d}|\varphi||^2\\
&= \alpha|\varphi|^{\alpha-2}\left(\langle\Delta\varphi,\varphi\rangle - |\nabla\varphi|^2 - (\alpha-2)|\mathrm{d}|\varphi||^2\right)\\
&= \alpha|\varphi|^{\alpha-2}\left(\langle\Delta\varphi,\varphi\rangle - |\nabla^f\varphi|^2 - 2\frac{f}{m}\langle D\varphi,\varphi\rangle + \frac{f^2}{m}|\varphi|^2 - (\alpha-2)|\mathrm{d}|\varphi||^2\right)\\
&= \alpha|\varphi|^{\alpha-2}\left(\langle D^2\varphi,\varphi\rangle - \frac{\mathrm{scal}_M}{4}|\varphi|^2 - |\nabla^f\varphi|^2 - 2\frac{f}{m}\langle D\varphi,\varphi\rangle\right.\\
&\qquad\left. + \frac{f^2}{m}|\varphi|^2 - (\alpha-2)|\mathrm{d}|\varphi||^2\right)\\
&= \alpha|\varphi|^{\alpha-2}\left(\langle (D-f)^2\varphi,\varphi\rangle - \frac{\mathrm{scal}_M}{4}|\varphi|^2 - |\nabla^f\varphi|^2\right.\\
&\qquad\left. + 2\frac{m-1}{m}f\langle (D-f)\varphi,\varphi\rangle + \frac{m-1}{m}f^2|\varphi|^2 + \frac{m}{m-1}|\mathrm{d}|\varphi||^2\right)
\end{aligned}$$

where we used in the last step that $\langle [D,f]\varphi,\varphi\rangle = \langle \mathrm{d}f\cdot\varphi,\varphi\rangle$ has to vanish since $\langle \mathrm{d}f\cdot\varphi,\varphi\rangle_x \in \mathrm{i}\mathbb{R}$ but all the other terms are real. By Corollary 4.4, φ is in C^2 and all equations above hold in the classical sense. In particular $(D-f)\varphi = 0$. As the spinor φ is in the kernel of the operator $D-f$ we can use the refined Kato inequality $|\nabla^f\varphi|^2 \geq \frac{m}{m-1}|\mathrm{d}|\varphi||^2$; see [18, (3.9)]. Thus, we get for the u defined in the proposition

$$\mathrm{d}^*\mathrm{d}u \leq -\alpha\frac{\mathrm{scal}_M}{4}u + \alpha\frac{m-1}{m}f^2u.$$

Thus, using $a = \frac{4}{\alpha}$ this means that

$$Lu = a\Delta u + \mathrm{scal}_M u \leq 4\frac{m-1}{m}f^2u = 4\frac{m-1}{m}\lambda^2u^{\frac{m+2}{m-2}}.$$

As remarked above this all holds outside of the zero set of $u = |\varphi|^{\frac{m-2}{m-1}}$. From Corollary 12.2 we see that inequality (10) holds distributionally since u is a nonnegative function. □

Proposition 9.2 *We assume the conditions of Proposition 9.1. Additionally we assume that* $\mathrm{scal}_M \geq s_0 > 0$ *or* $\varphi \in L^{2\frac{m-2}{m-1}}$*, then*

$$Q^*(M,g) \leq 4\frac{m-1}{m}\lambda^2. \tag{11}$$

Proof Let u be defined as in the previous proposition. For any regular value $\varepsilon > 0$ of u, we consider $V_\varepsilon := \{u \geq \varepsilon\}$. Note that by Lemma 4.3 $\lim_{x\to\infty} u(x) = 0$, and hence

$(V_\varepsilon)_\varepsilon$ exhausts M as $\varepsilon \to 0$. Let ν be the exterior unit normal field of the boundary of V_ε. Then, $\partial_\nu u \leq 0$. Thus, integration over V_ε of Inequality (10) multiplied by u gives

$$\begin{aligned} 4\frac{m-1}{m}\lambda^2 \underbrace{\int_{V_\varepsilon} u^{\frac{2m}{m-2}} \mathrm{dvol}_g}_{\to 1 \text{ as } \varepsilon\to 0} &\geq \int_{V_\varepsilon} \left(au\, \mathrm{d}^*\mathrm{d}u + \mathrm{scal}_M u^2\right) \mathrm{dvol}_g \\ &= \int_{V_\varepsilon} \left(a|\mathrm{d}u|^2 + \mathrm{scal}_M u^2\right) \mathrm{dvol}_g - \int_{\partial V_\varepsilon} au\partial_\nu u \,\mathrm{dvol}_g \\ &\geq \int_{V_\varepsilon} \left(a|\mathrm{d}u|^2 + \mathrm{scal}_M u^2\right) \mathrm{dvol}_g. \end{aligned} \tag{12}$$

In the case that $\mathrm{scal}_M \geq s_0 > 0$ this implies $u \in H_1^2(M)$.

For the remaining case that $\varphi \in L^{2\frac{m-2}{m-1}}$ we have $u \in L^2$. Thus, we obtain

$$\lim_{\varepsilon\to 0} \int_{V_\varepsilon} |\mathrm{d}u|^2 \mathrm{dvol}_g \leq 4\frac{m-1}{ma}\lambda^2 + \frac{\sup_M |\mathrm{scal}_M|}{a} \|u\|_{L^2}^2,$$

and this as well implies $u \in H_1^2$.

Sard's theorem tells us that the set of regular ε is dense. In the limit $\varepsilon \to 0$ we then get (11) from (12). □

Example 9.3 (Spherical cap solution on hyperbolic space) Let $B_r \subset \mathbb{S}^m$ be a ball in the standard sphere of radius r. Let φ be a Killing spinor on the sphere with Killing constant $-\frac{1}{2}$ normalized as $|\varphi| = \mathrm{vol}(B_r)^{-\frac{1}{q}}$ for $q = \frac{2m}{m-1}$. Then $\|\varphi\|_{L^p(B_r)} = 1$ and $D^{\mathbb{S}^m}\varphi = \frac{m}{2}\varphi = \frac{m}{2}\mathrm{vol}(B_r)^{\frac{q-2}{q}} |\varphi|^{q-2}\varphi$. Let $u\colon \mathbb{H}^m \to B_r$ be a conformal map from the hyperbolic space to the spherical cap such that $g_{\mathbb{H}^m} = f^2 u^*\sigma^m$. Then, using identification of the spinor bundles, as in Example 4.6 and setting $\tilde{\varphi} := f^{-\frac{m-1}{2}}\varphi$ we get by conformal invariance that

$$D^{\mathbb{H}^m}\tilde{\varphi} = \underbrace{\frac{m}{2}\mathrm{vol}(B_r)^{\frac{q-2}{q}}}_{=:\lambda_r} |\tilde{\varphi}|^{q-2}\tilde{\varphi} \text{ and } \|\tilde{\varphi}\|_{L^q(\mathbb{H}^m)} = 1 \text{ and } \|\tilde{\varphi}\|_{L^\infty(\mathbb{H}^m)} < \infty.$$

Then $\lambda_r \to 0$ as $r \to 0$. Nevertheless, $Q^*(\mathbb{H}^m) = Q^*(\mathbb{S}^m)$. Thus, Proposition 9.2 does not hold without the assumption $\mathrm{scal}_M \geq s_0 > 0$ or $\varphi \in L^{2\frac{m-2}{m-1}}$. Hence, the conformal Hijazi inequality $\widetilde{Q}_{\mathrm{spin}}(\mathbb{H}^m) \geq Q^*(\mathbb{H}^m)$ which trivially follows from Lemma 7.4 is no longer true if we remove the L^2-condition in the definition of $\widetilde{Q}_{\mathrm{spin}}$.

We now want to use these inequalities to prove Hijazi inequalities for the model spaces $\mathbb{M}_c^{m,k}$. In this goal we will examine whether for certain m and k there is a spinor $\varphi \in L^{2\frac{m-2}{m-1}}$ satisfying the assumptions of Proposition 9.1.

Proposition 9.4 *Let $m \geq 3$. Let $0 \leq k < m-2$, $c \in [0,1]$ or $k = m-2$ and $c \in [0,1)$. On the manifold $\mathbb{M}_c^{m,k}$, we consider a spinor field $\varphi \in L^q \cap C^0$ solving*

$$D\varphi = \lambda |\varphi|^{\frac{2}{m-1}} \varphi, \qquad \|\varphi\|_q = 1$$

for $\lambda \in \mathbb{R}$ and $q = \frac{2m}{m-1}$. Then, $\varphi \in L^{2\frac{m-2}{m-1}}$.

We also know that $\varphi \in L^\infty$ by Lemma 4.2.

Lemma 9.5 *Under the assumptions of the proposition we have $(m-2)(m-k-1) > ck$, unless $k = m-2$ and $c = 1$.*

Proof of the lemma The condition $(m-2)(m-k-1) > ck$ is equivalent to $(m-1)(m-k-2) > -(1-c)k$.

Proof of Proposition 9.4 By Proposition 2.2 D is L^r-invertible if

$$\frac{m-k-1}{2} > ck\left|\frac{1}{r} - \frac{1}{2}\right|. \tag{13}$$

By assumption we have $D\varphi = \lambda|\varphi|^{2/(m-1)}\varphi \in L^{\frac{2m}{m+1}}$. The condition (13) for $r := 2m/(m+1)$ is equivalent to $m(m-k-1) > ck$ which is fulfilled by assumption. Thus, we obtain $\varphi \in L^{\frac{2m}{m+1}}$. Hence,

$$D\varphi = \lambda|\varphi|^{q-2}\varphi \in L^{\frac{2m}{(m+1)(q-1)} = \frac{2m(m-1)}{(m+1)^2}}.$$

Note that for $m \geq 3$ we have $\frac{2m(m-1)}{(m+1)^2} \leq 2\frac{m-2}{m-1} =: s$. Hence, using φ in L^∞ we get $D\varphi \in L^s$. Moreover, D is L^s-invertible as condition (13) for $r := s$ is equivalent to $(m-2)(m-k-1) > ck$ which is provided by assumption and Lemma 9.5. Thus $\varphi \in L^s$.

Example 9.6 In the exceptional case $k = m-2$ and $c = 1$ the conclusion of Proposition 9.4 is not correct. To see this, we construct the following example. We consider a Killing spinor on $\mathbb{S}^m$ and transport it conformally to $\mathbb{M}_1^{m,m-2}$, similar to the proof of Lemma 7.4. The spinor falls off as $e^{-(m-1)r/2}$ where r is the distance to a fixed point on $\mathbb{H}^{m-1}$ as introduced in Sect. 2.2. Then the $L^{2\frac{m-2}{m-1}}$-norm of φ is infinite. A similar example is provided by Example 9.3 in the case $k = m-1$ and $c > 0$.

Corollary 9.7 *Consider $\mathbb{M}_c^{m,k}$ with $m \geq 3$. Let $0 \leq k < m-2$, $c \in [0,1]$ or $k = m-2$ and $c \in [0,1)$. Let $\varphi \in L^q \cap C^0$ be a solution of $D\varphi = \lambda|\varphi|^{\frac{2}{m-1}}\varphi$ on $\mathbb{M}_c^{m,k}$ with $\lambda \in \mathbb{R}$ and $\|\varphi\|_q = 1$ where $q = \frac{2m}{m-1}$. Then,*

$$\lambda^2 \geq \frac{m}{4(m-1)} Q^*(\mathbb{M}_c^{m,k}).$$

Proof We set $u = |\varphi|^{\frac{m-2}{m-1}}$. By Proposition 9.4 $u \in L^2$. Then, Proposition 9.2 gives the corollary. □

Corollary 9.8 (Conformal Hijazi inequality for the model spaces) *For* $0 \le k \le m-2$ *and* $c \in [0, 1]$ *we have*

$$\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,k}) \ge Q^*(\mathbb{M}_c^{m,k}).$$

In particular, $\widetilde{\Lambda}_{m,k}^{\mathrm{spin}} \ge \Lambda_{m,k}^*$ *for* $0 \le k \le m-2$.

Proof For $k < m-2$ or $k \le m-2$ and $c < 1$ this follows immediately from Corollary 9.7 and the definition of $\widetilde{Q}_{\mathrm{spin}}$. The remaining case, $k = m-2$ and $c = 1$, was treated in Lemma 7.4. □

Remark 9.9 In the case $k = m-2$ we obtain together with [5, Lemma 3.8] that $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,m-2}) \ge Q^*(\mathbb{M}_c^{m,m-2}) \ge c^{\frac{2}{m}} Q^*(\mathbb{S}^m)$. For a test function $v \in C^\infty(\mathbb{M}_c^{m,m-2})$ that is constant along S^1 one can calculate (since the scalar curvature of S^1 is zero) that

$$\mathcal{F}^{\mathbb{M}_c^{m,m-2}}(v) = c^{\frac{2}{m}} \mathcal{F}^{\mathbb{M}_1^{m,m-2}}(v).$$

Since $Q^*(\mathbb{M}_1^{m,m-2}) = Q^*(\mathbb{S}^m)$ is minimized by a v that is constant along S^1, we have $Q^*(\mathbb{M}_c^{m,m-2}) = c^{\frac{2}{m}} Q^*(\mathbb{S}^m)$. Thus, together we obtain

$$\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,m-2}) \ge Q^*(\mathbb{M}_c^{m,m-2}) = c^{\frac{2}{m}} Q^*(\mathbb{S}^m).$$

In particular, $\Lambda_{m,m-2}^* = 0$.

10 Minimizer of the Variational Problems

The Euler–Lagrange equations of the constants Q^* and $\lambda_{\min}^{+,*}$ defined via functionals read as

$$Lu = Q^* u^{p-1} \text{ with } \|u\|_{p=\frac{2m}{m-2}} = 1$$

and

$$D\varphi = \lambda_{\min}^{+,*} |\varphi|^{q-2}\varphi \text{ with } \|\varphi\|_{q=\frac{2m}{m-1}} = 1.$$

Now assume such minimizing solutions $u \in H_1^2 \cap L^\infty$ and $\varphi \in H_1^{\frac{2m}{m+1}} \cap L^\infty$ exist. Then, we also have $u \in C^2$ and $\varphi \in C^1$. Then, $\widetilde{Q} \le Q^*$. Moreover, by interpolation $\varphi \in L^2$ and, thus, $\widetilde{Q}_{\mathrm{spin}} \le Q_{\mathrm{spin}}^*$.

We now recall some theorems for the existence of such solutions on *almost homogeneous* manifolds (M, g), i.e., on Riemannian manifolds on which there is a relatively compact set $U \subset\subset M$ such that for all $x \in M$ there is an isometry $f: M \to M$ with $f(x) \in U$. Note that a manifold is almost homogeneous if and only if the isometry group $G = \mathrm{Isom}(M, g)$ acts cocompactly on M. This follows since the distance between the orbits on M defines a metric on M/G, and the induced topology is the quotient topology of $\pi: M \to M/G$.

Theorem 10.1 [24, Theorem 13] *Let (M^m, g) be a Riemannian manifold of bounded geometry with $\mathrm{scal}_M \geq \mathrm{const} > 0$ for a constant c and $Q^*(M, g) < Q^*(\mathbb{S}^m)$. Let (M, g) be almost homogeneous. Then, there is a positive smooth solution $u \in H_1^2 \cap L^\infty \cap C^2$ of the Euler–Lagrange equation $Lu = Q^*(M)u^{\frac{m+2}{m-2}}$ and $\|u\|_{\frac{2m}{m-2}} = 1$.*

In the reference, $u \in C^2$ was not explicitly stated, but follows from standard elliptic regularity theory.

Corollary 10.2 *Let $(m-k-1)(m-k-2) > c^2(k+1)k$, $0 \leq c < 1$ or let $k \leq m-3$ and $c = 1$. Then,*

$$\widetilde{Q}(\mathbb{M}_c^{m,k}) \leq Q^*(\mathbb{M}_c^{m,k}).$$

Proof For, $k \leq m-3$ and $c = 1$ we even have equality by Lemmas 7.1 and 7.5. For $k = 0$, $\mathbb{M}_c^{m,0}$ is diffeomorphic to $\mathbb{M}_1^{m,0}$. Thus, by Lemma 7.5 we even have equality. Now let $k > 0$. Then the condition $c < 1$ implies that $\mathbb{M}_c^{m,k}$ is not conformally flat. In the case $m \geq 6$ Aubin's inequality, cp. Remark 2.4, provides $Q^*(\mathbb{M}_c^{m,k}) < Q^*(\mathbb{S}^m)$. For $m < 6$ we have $Q^*(\mathbb{M}_c^{m,k}) < Q^*(\mathbb{S}^m)$ by the following theorem. Thus Theorem 10.1 implies the existence of a solution u as above, which directly implies the corollary. □

Theorem 10.3 [9, Corollary 1] *Let $m = n + k + 1$, $m \geq 3$, $k > 0$, and $c \in [0, 1)$. Then*

$$Q^*(\mathbb{S}^n \times \mathbb{H}_c^{k+1}, \sigma^n + g_c) < Q^*(\mathbb{S}^m, \sigma^m).$$

Theorem 10.4 [23, Theorem 16] *Let (M^m, g) be a Riemannian spin manifold of bounded geometry with $\mathrm{scal}_M \geq C > 0$ for a constant C and $\lambda_{\min}^{+,*}(M, g) < \lambda_{\min}^{+,*}(\mathbb{S}^m)$. Let (M, g) be almost homogeneous. Moreover, assume that the Dirac operator D on M is invertible as an operator from L^s to L^s for $s = \frac{2m}{m+1}$. Then, there is a positive smooth solution $\varphi \in H_1^{\frac{2m}{m+1}} \cap L^\infty \cap C^1$ of the Euler–Lagrange equation $D\varphi = \lambda_{\min}^{+,*}|\varphi|^{\frac{2}{m-1}}\varphi$ and $\|\varphi\|_{\frac{2m}{m-1}} = 1$.*

In the reference, $\varphi \in L^\infty$ was not stated explicitly, but can be seen directly from the proof in [23] or alternatively by Lemma 4.2. Moreover, in the original version of Theorem 10.4 it was requested that D is invertible for all $s \in [\frac{2m}{m+1}, \frac{2m}{m+1} + \varepsilon]$ for some $\varepsilon > 0$. But if D is invertible for $s = \frac{2m}{m+1}$, then it is also invertible for the conjugate exponent $\frac{2m}{m-1}$ and by interpolation for all s in between, cp. [10, Appendix].

For the special case of manifolds that are product spaces $M = N_1 \times N_2$ of an almost homogeneous manifold N_1 and a closed manifold N_2 one can relax the assumption on the positive scalar curvature in Theorem 10.4:

Theorem 10.5 *Let $(M^m = N_1 \times N_2, g)$ be a Riemannian spin manifold that is a product manifold of an almost homogeneous manifold N_1 and a closed manifold N_2. Let $\lambda_{N_2}^2$ be the lowest eigenvalue of the square $(D^{N_2})^2$ of the Dirac operator on N_2, and let $\lambda_{N_2}^2 + \frac{1}{4}\inf \mathrm{scal}_{N_1} =: c > 0$. Moreover, let $\lambda_{\min}^{+,*}(M, g) < \lambda_{\min}^{+,*}(\mathbb{S}^m)$, and assume that the Dirac operator D on M is invertible as an operator from L^{q^*} to L^{q^*} for $q^* = \frac{2m}{m+1}$. Then, there is a positive smooth solution $\varphi \in H_1^{\frac{2m}{m+1}} \cap L^\infty \cap C^1$ of the Euler–Lagrange equation $D\varphi = \lambda_{\min}^{+,*}|\varphi|^{\frac{4}{m-1}}$ and $\|\varphi\|_{\frac{2m}{m-1}} = 1$.*

Proof We start as in the proof of the general result in [23]—here we briefly recall the steps that remain the same: For that let ρ be a radial admissible weight, see [23, Sect. A.1], $\rho \leq 1$. Then, by [23, Lemma 13] we obtain for each $s \in [2, q = \frac{2m}{m-1})$ a sequence $\varphi_s \in H_1^{s^*} \cap C^1$ with $D\varphi_s = \lambda_s^\alpha \rho^{\alpha s}|\varphi_s|^{s-2}\varphi_s$ and $\|\rho^\alpha \varphi_s\|_{L^s} = 1$ where s and s^* are conjugate, $\alpha = \alpha(s) \to 0$ as $s \to q$ and $\mu := \limsup_{s\to q} \lambda_s^\alpha \leq \lambda_{\min}^{+,*}$. Then by [23, Lemmas 14 and 15] a subsequence $\varphi_s = \varphi_{\alpha(s),s}$ converges to a function $\varphi \in H_1^{s^*}$ in C^1-topology on each compact subset, and we have $D\varphi = \mu|\varphi|^{q-2}\varphi$. It remains to show that $\|\varphi\|_{L^q} = 1$, i.e., in particular that φ is nonzero. Then the arguments that φ is a solution as desired just follow again the lines of [23, Lemma 15].

We prove the remaining point by contradiction, i.e., we assume that $\varphi = 0$: Note that by [23, Lemma 33] for each s we have $\lim_{|x|\to\infty}|\varphi_s| = 0$. Thus, we can fix $x_s \in N_1$ such that $\int_{\{x_s\}\times N_2}|\varphi_s|^2$ attains its maximum. By the almost-homogeneity of N_1 we can assume that all x_s are contained in a compact subset K of M. Moreover, then

$$
\begin{aligned}
0 \leq \Delta_{N_1}\int_{\{x_s\}\times N_2}|\varphi_s|^2 &= \int_{\{x_s\}\times N_2}\Delta_{N_1}|\varphi_s|^2\\
&= 2\mathrm{Re}\int_{\{x_s\}\times N_2}\langle\nabla_{N_1}^*\nabla_{N_1}\varphi_s, \varphi_s\rangle - 2\int_{\{x_s\}\times N_2}|\nabla^{N_1}\varphi_s|^2\\
&\leq 2\mathrm{Re}\int_{\{x_s\}\times N_2}\langle\nabla_{N_1}^*\nabla_{N_1}\varphi_s, \varphi_s\rangle.
\end{aligned}
$$

Using $\rho \leq 1$, $D\varphi_s = \lambda_s^\alpha \rho^{\alpha s}|\varphi_s|^{s-2}\varphi_s$ and $\langle \mathrm{d}(\text{function}) \cdot \psi, \psi\rangle_x \in \mathrm{i}\mathbb{R}$ we get

$$
\mathrm{Re}\int_{\{x_s\}\times N_2}\langle D^2\varphi_s, \varphi_s\rangle \leq (\lambda_s^\alpha)^2\int_{\{x_s\}\times N_2}|\varphi_s|^{2(s-1)}.
$$

The square of the Dirac operator decomposes as $D^2 = (D^{N_1})^2 + (\tilde{D}^{N_2})^2$ where $\tilde{D}^{N_2} = \mathrm{diag}(D^{N_2}, -D^{N_2})$ in case that both manifolds are odd dimensional and $\tilde{D}^{N_2} = D^{N_2}$ otherwise; see e.g., [10, Sect. 2.5]. The operators $(\tilde{D}^{N_2})^2$ and $(D^{N_2})^2$ have the same spectrum. Then, together with the Schrödinger–Lichnerowicz formula we obtain

$$\int_{\{x_s\}\times N_2} |\tilde{D}^{N_2}\varphi_s|^2 + \mathrm{Re}\int_{\{x_s\}\times N_2} \langle \nabla^*_{N_1}\nabla_{N_1}\varphi_s, \varphi_s\rangle + \frac{\mathrm{scal}_{N_1}}{4}\int_{\{x_s\}\times N_2} |\varphi_s|^2$$
$$\leq (\lambda_s^\alpha)^2 \int_{\{x_s\}\times N_2} |\varphi_s|^{2(s-1)}$$

and, thus,

$$\left(\lambda_{N_2}^2 + \frac{\mathrm{scal}_{N_1}}{4}\right)\int_{\{x_s\}\times N_2} |\varphi_s|^2 \leq (\lambda_s^\alpha)^2 \max_{\{x_s\}\times N_2} |\varphi_s|^{2(s-2)} \int_{\{x_s\}\times N_2} |\varphi_s|^2.$$

Hence,

$$\left(\lambda_{N_2}^2 + \frac{\mathrm{scal}_{N_1}}{4}\right)(\lambda_s^\alpha)^{-2} \leq \max_{\{x_s\}\times N_2} |\varphi_s|^{2(s-2)}$$
$$c(\lambda_{\min}^{+,*})^{-2} \leq \liminf_{s\to q}\left(\lambda_{N_2}^2 + \frac{\mathrm{scal}_{N_1}}{4}\right)(\lambda_s^\alpha)^{-2} \leq \liminf_{s\to q} \max_{\{x_s\}\times N_2} |\varphi_s|^{2(s-2)}.$$

Note that all x_s are contained in a compact set. Thus, a subsequence of x_s converges to some $x \in M$. But, $\varphi = 0$ means that then the right-hand side is zero which gives the desired contradiction. □

Corollary 10.6 *Let $(m-k-1)^2 > c^2(k+1)k$ and $Q^*_{\mathrm{spin}}(\mathbb{M}_c^{m,k}) < Q^*_{\mathrm{spin}}(\mathbb{S}^m)$. Then, $\widetilde{Q}_{\mathrm{spin}}(\mathbb{M}_c^{m,k}) \leq Q^*_{\mathrm{spin}}(\mathbb{M}_c^{m,k})$.*

Proof This corollary follows from Theorem 10.5. Set $N_1 = \mathbb{H}_c^{k+1}$ and $N_2 = \mathbb{S}^{m-k-1}$. Then $\lambda_{N_2}^2 = \frac{(m-k-1)^2}{4}$ and $\mathrm{scal}_{N_1} = -c^2k(k+1)$. □

Remark 10.7 Note that using Theorem 10.4 would lead to the condition $(m-k-1)(m-k-2) > c^2k(k+1)$. Thus, here Theorem 10.5 gives a better result.

11 Λ-Invariants for $k = m - 3$

For applying the surgery monotonicity formulas, cf. Theorems 2.5 and 2.6, one needs explicit positive lower bounds for $\Lambda_{m,k}$ in dimension $k \leq m-3$ and for $\Lambda^{\mathrm{spin}}_{m,k}$ in dimension $k \leq m-2$. In the case $k \leq m-4$ and in the case $k+3 = m \leq 6$ explicit positive lower bounds for $\Lambda_{m,k}$ were provided in [7]. Using $\Lambda^{\mathrm{spin}}_{m,k} \geq \Lambda_{m,k}$, cp. Corollary 6.4, Proposition 5.3 and [6, Theorem 3.1 and Corollary 3.2], this yields explicit positive lower bounds for $\Lambda^{\mathrm{spin}}_{m,k}$ in these cases. However, the techniques we have developed in the previous sections also provide explicit positive lower bounds for $\Lambda^*_{m,m-3}$ for any $m > 6$ which is the subject of the present section.

Note that by Corollaries 9.8 and 6.4 and by Proposition 6.2 we have $\widetilde{\Lambda}^{\mathrm{spin}}_{m,m-3} = \Lambda^{\mathrm{spin},*}_{m,m-3} \geq \Lambda^*_{m,m-3}$ and $\widetilde{\Lambda}_{m,m-3} \geq \Lambda^*_{m,m-3}$. Thus, any lower bound for $\Lambda^*_{m,m-3}$

Table 4 Some explicit values for $L_{m,m-3} := \inf_{c\in[0,1]} L_m(c^2)$—a lower bound for $\Lambda^*_{m,m-3}$, $\widetilde{\Lambda}_{m,m-3}$ and $\Lambda^{\text{spin}}_{m,m-3}$

m	7	8	9	10	11	12	13	14	15
$Q^*(\mathbb{S}^m)$	113.5	130.7	147.88	165.0	182.2	199.3	216.4	233.5	250.6
$L_{m,m-3}$	65.2	78.7	91.8	104.9	118.1	131.5	145.0	158.6	172.4

The values are rounded—$Q^*(\mathbb{S}^m)$ is rounded to the nearest multiple of $1/10$ and $L_{m,m-3}$ is always rounded down

is also a lower bound for the other three invariants. In Sect. 2.6 we pointed out that $\Lambda^{\text{spin}}_{m,m-3} = \Lambda^{\text{spin},*}_{m,m-3}$.

As a first step we estimate Q_c: It was derived in [7, Theorem 4.1] for all $c \in (0, 1)$ that

$$Q_c \geq \left(\frac{Q_0}{Q_1} - \frac{c^2(k+1)k}{(1-c^2)(m-k-1)(m-k-2)+c^2(k+1)k}\left(\frac{Q_0}{Q_1} - c^{\frac{2(m-k-1)}{m}}\right)\right) Q_1 \tag{14}$$

where we defined $Q_c := Q^*(\mathbb{M}_c^{m,k})$. On the other hand it follows for $m \geq 6$ and $k = m-3$ from [7, Sect. 4.1 (iv)] that

$$Q_0 \geq \frac{ma}{24^{\frac{3}{m}}((m-3)a_{m-3})^{\frac{m-3}{m}}} Q^*(\mathbb{S}^{m-3})^{\frac{m-3}{m}} Q^*(\mathbb{S}^3)^{\frac{3}{m}} =: \widehat{Q}_0.$$

Inequality (14) reads for $k = m-3 \geq 3$ as

$$\begin{aligned} Q_c &\geq \left(\frac{Q_0}{Q_1} - \frac{c^2(m-2)(m-3)}{(1-c^2)2+c^2(m-2)(m-3)}\left(\frac{Q_0}{Q_1} - c^{4/m}\right)\right) Q_1 \\ &\geq \frac{(1-c^2)2\widehat{Q}_0 + c^{2+4/m}(m-2)(m-3)Q_1}{(1-c^2)2+c^2(m-2)(m-3)} =: L_m(c^2). \end{aligned}$$

Since $\widehat{Q}_0$ and $Q_1 = Q^*(\mathbb{S}^m)$ are known explicitly one can compute the infimum of $L_m(c^2)$ numerically for fixed m; see Table 4 for some explicit values.

For general m we can still estimate the infimum of $L_m(c^2)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} L_m(s) &= \frac{-2\widehat{Q}_0 + (1+2/m)s^{2/m}(m-2)(m-3)Q_1}{2+s((m-2)(m-3)-2)} \\ &\quad - \frac{((1-s)2\widehat{Q}_0 + s^{1+2/m}(m-2)(m-3)Q_1)\,[(m-2)(m-3)-2]}{(2+s((m-2)(m-3)-2))^2}. \end{aligned}$$

The condition $\frac{\mathrm{d}}{\mathrm{d}s}L(s) = 0$ is equivalent to

$$f(s) := \frac{(2+s((m-2)(m-3)-2))^2}{(m-2)(m-3)}\frac{\mathrm{d}}{\mathrm{d}s}L(s) = s^{\frac{2}{m}+1}A_0 + s^{\frac{2}{m}}A_1 + A_2 = 0$$

where $A_0 = \frac{2}{m}Q_1((m-2)(m-3)-2)$, $A_1 = 2(\frac{2}{m}+1)Q_1$, and $A_2 = -2\widehat{Q}_0$. There is at least one zero in the interval $(0, 1)$ since $f(0) < 0$ and $f(1) > 0$ (since $A_0 > 0$ and $Q_1 \geq \widehat{Q}_0$). Let $F(u) = f(u^m) = u^{m+2}A_0 + u^2A_1 + A_2$, and let $u_0 \in (0, 1)$ be a zero of F. Then

$$\frac{F(u)}{u-u_0} = u^{m+1}A_0 + u^m A_0 u_0 + \cdots + uA_0u_0^m + uA_1 + A_0u_0^{m+1} + A_1u_0.$$

But for positive u the last polynomial is always positive. Thus, there is only one zero of f in the interval $(0, 1)$.

Moreover, $f(s) > s^{\frac{2}{m}}A_1 + A_2$. Thus, $f(c_2^2) > 0$ where $c_2 := \left(\frac{-A_2}{A_1}\right)^{\frac{m}{2}} = \left(\frac{m\widehat{Q}_0}{(m+2)Q_1}\right)^{\frac{m}{2}}$. Hence,

$$\begin{aligned}\Lambda^*_{m,m-3} &= \inf_{c\in[0,1]} Q_c \geq \inf_{c\in[0,c_2]} L_m(c^2)\\ &\geq \inf_{c\in[0,c_2]} \frac{(1-c^2)2\widehat{Q}_0 + c^{4/m+2}(m-2)(m-3)Q_1}{2+c_2^2((m-2)(m-3)-2)}.\end{aligned}$$

Since $(1-c^2)2\widehat{Q}_0 + c^{4/m+2}(m-2)(m-3)Q_1$ attains its minimum for $c_3^{\frac{4}{m}} = \frac{2m\widehat{Q}_0}{(m+2)(m-2)(m-3)Q_1}$ we obtain

$$\Lambda^*_{m,m-3} \geq \frac{(1-c_3^2)2\widehat{Q}_0 + c_3^{4/m+2}(m-2)(m-3)Q_1}{2+c_2^2((m-2)(m-3)-2)}.$$

Together with the explicit positive lower bounds $\Lambda^*_{3,0} = Q^*(\mathbb{S}^3) = 6\cdot 2^{2/3}\pi^{4/3} = 43.823233\ldots$, $\Lambda^*_{4,1} > 38.9$ [6, Example 4.7], and $\Lambda^*_{5,2} > 45.1$ [6, Example 4.10] we obtain explicit positive lower bounds for all $\Lambda^*_{m,m-3}$.

These methods obviously also yield explicit positive lower bounds for all $\widetilde{Q}_{\mathrm{spin}}(\mathbb{H}^{m-2} \times c\mathbb{S}^2)$.

12 Bordism Arguments

Proof of Proposition 3.2 By a theorem of Stolz, [37, Theorem B], M is spin-bordant to the total space M_0 of an $\mathbb{H}P^2$-bundle over a base Q for which the structure group is $PSp(3)$. In particular, in dimension $m = 5, 6, 7$ this implies that $Q = \varnothing$. Each manifold M in dimension $m = 5, 6, 7$ is a spin boundary.

Moreover, by the extended Stolz theorem [6, Proposition 6.5] we can assume that Q is connected if $m \geq 9$ and that Q is simply connected if $m \geq 11$. Thus, in dimension $m \geq 11$ also M_0 can be chosen to be connected and simply connected. In case that $Q = \varnothing$, set $M_1 := \mathbb{S}^m$ otherwise $M_1 := M_0$.

First, let $m = 5, 6, 7$ or $m \geq 11$. As M_1 is simply connected M can be obtained from M_1 by a sequence of surgeries of dimensions ℓ where $2 \leq \ell \leq m-3$; see [7, Proposition 5.1]. Using Theorem 2.5 this implies that $\sigma^*_{\text{spin}}(M) \geq \min\{\sigma^*_{\text{spin}}(M_1), \Lambda^{\text{spin}}_m\}$.

In the case $m = 5, 6, 7$ and in the case $m \geq 8$ and $Q = \varnothing$ we use $\Lambda^{\text{spin}}_{m,k} = \Lambda^{\text{spin},*}_{m,k} \leq \sigma^*_{\text{spin}}(\mathbb{S}^m)$ for $k \leq m-2$, see Theorem 3.1, and obtain $\sigma^*_{\text{spin}}(M) \geq \Lambda^{\text{spin}}_m$. For $m \geq 11$ and $Q \neq \varnothing$, i.e., $M_0 = M_1$, we use the conformal Hijazi inequality for M_1 and $\sigma^*(M_1) \geq Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})$ for $m \geq 8$, see [38], where $\mathbb{H}P^2 \times \mathbb{R}^{m-8}$ carries the product metric of the standard metrics of both factors. Then we obtain

$$\sigma^*_{\text{spin}}(M) \geq \min\left\{\Lambda^{\text{spin}}_m, Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})\right\}.$$

In dimension $m = 8$ and $Q \neq \varnothing$, M_1 is a disjoint sum of copies of $\mathbb{H}P^2$, possibly with reversed orientation. Using 0-dimensional surgeries M_1 is spin bordant to the connected and simply connected manifold $M_2 := \mathbb{H}P^2 \# \ldots \# \mathbb{H}P^2$, possibly with reversed orientation. Thus, $\sigma^*_{\text{spin}}(M_2) \geq \sigma^*_{\text{spin}}(M_1) = \sigma^*_{\text{spin}}(\mathbb{H}P^2) \geq Q^*(\mathbb{H}P^2)$. As above M can then be obtained from M_2 by surgeries of dimensions ℓ where $2 \leq \ell \leq m-3$. Again using Theorem 2.5 this implies that $\sigma^*_{\text{spin}}(M) \geq \min\{\Lambda^{\text{spin}}_m, \sigma^*_{\text{spin}}(M_2)\} \geq \min\{\Lambda^{\text{spin}}_m, Q^*(\mathbb{H}P^2)\}$.

Now let $m = 9, 10$ and $Q \neq \varnothing$. Then, M can be obtained from M_1 by surgeries of dimensions ℓ where $1 \leq \ell \leq m-3$. Using Theorem 2.5 this implies that $\sigma^*_{\text{spin}}(M) \geq \min\{\Lambda^{\text{spin}}_{m,1}, \Lambda^{\text{spin}}_m, \sigma^*_{\text{spin}}(M_1)\}$. Similar to above we get

$$\sigma^*_{\text{spin}}(M) \geq \min\left\{\Lambda^{\text{spin}}_{m,1}, \Lambda^{\text{spin}}_m, Q^*(\mathbb{H}P^2 \times \mathbb{R}^{m-8})\right\}.$$

The analogous statement for non-simply connected manifolds mentioned after Proposition 3.2 is proven analogously.

Proof of Proposition 3.3 Let (W, F) be the spin bordism from (M, c_M) to (N, f). First, we use 0-dimensional surgery in order to make W connected. We will abuse the notation and also denote the bordism after surgery (W, F). Note that Γ is always finitely presented. Then, again by 0-dimensional surgery, we change W and F such that F induces a surjection on π_1. Next, we use 1-dimensional surgery such that the resulting F induces an injection on π_1.

As a consequence, the resulting map F induces a bijection on π_1. Next, we use 2-dimensional surgeries to kill $\pi_2(W, M)$, cp. [27, Proof of Proposition 2.1.1]. This can always be achieved as every element of $\pi_2(W, M)$ then comes from an element in $\pi_2(W)$ and thus can be represented by an embedded S^2. The condition that W is spin implies that this embedded S^2 has trivial normal bundle. Then, the embedding $M \hookrightarrow W$ is 2-connected and $N \hookrightarrow W$ is 1-connected. Thus, we can obtain N from M

by attaching handles of dimensions ℓ with $2 \leq \ell \leq m-2$. Together with Theorem 2.5 we then obtain the claim.

Appendix: Weak Partial Differential Inequalities

In this Appendix, we recall the connection between viscosity solutions and distributional solutions of weak partial differential inequalities. All functions in this Appendix are real-valued.

Let $\Delta = \mathrm{d}^*\mathrm{d}$ be the geometric Laplacian on functions on a Riemannian manifold (M, g). Assume that P is an operator of the form $P = a\Delta + V$ where a is a positive smooth function and V is a continuous function. Let f be a continuous function.

We say that

$$Pu \leq f$$

holds in the distributional sense if for all compactly supported smooth nonnegative functions v on M

$$\int_M uPv \,\mathrm{dvol}_g \leq \int_M fv \,\mathrm{dvol}_g.$$

We say that

$$Pu \leq f$$

holds in the viscosity sense if u is continuous and for every $p \in M$ and $\varepsilon > 0$ there is a neighborhood U_ε of p and a C^2-function $h_\varepsilon : U_\varepsilon \to \mathbb{R}$ such that $h_\varepsilon(p) = u(p)$, $h_\varepsilon \leq u$ in U_ε and $Ph_\varepsilon(p) \leq f(p)$.

Theorem 12.1 [30, Theorems 1 and 2] *A continuous function u fulfills $Pu \leq f$ in the distributional sense if and only if it also fulfills the inequality in the viscosity sense.*

Actually, in the reference [30, Theorems 1 and 2] the statement is proven for a wide class of second-order operators on $\mathbb{R}^n$. But since this class includes the representation of $a\Delta_M$ in a chart of geodesic coordinates the above theorem follows.

Corollary 12.2 *Let f be a nonnegative continuous function. Let $u \geq 0$ be a continuous function such that $Pu \leq f$ in the classical sense whenever u is positive. Then, u fulfills $Pu \leq f$ in the distributional sense.*

Proof We see that $Pu \leq f$ in the viscosity sense by taking $h_\varepsilon = u$ whenever u is positive and $h_\varepsilon = 0$ otherwise. Then, Theorem 12.1 implies the corollary. □

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