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Explicit Hodge-Type Decomposition on Projective Complete Intersections

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Abstract We construct an explicit homotopy formula for the $\bar{\partial}$-complex on a reduced complete intersection subvariety $V \subset \mathbb{CP}^n$. This formula can be interpreted as an explicit Hodge-type decomposition for residual currents on V. As a first application of this formula we obtained the explicit Hodge decomposition on arbitrary Riemann surfaces.

Keywords $\bar{\partial}$-Operator · Complete intersection · Hodge decomposition

Mathematics Subject Classification Primary: 14C30 · 32S35 · 32C30

1 Introduction

The goal of the present article is to construct an explicit Hodge-type decomposition for the $\bar{\partial}$-operator on complete intersection subvarieties of $\mathbb{CP}^n$ and to obtain for those varieties a constructive version of the classical theorem of Hodge [26,27,42]:

To Carlos Berenstein on occasion of his 70-th birthday.

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Hodge Theorem *Let $V \subset \mathbb{CP}^n$ be an algebraic manifold. Let $Z^{(p,q)}(V)$ be the space of smooth $\bar\partial$-closed (p,q)-forms on V, and $B^{(p,q)}(V)$—the space of smooth $\bar\partial$-exact (p,q)- forms on V. Then*

(i) *there exist a finite-dimensional projection operator $L: Z^{(p,q)}(V) \to H^{(p,q)}(V)$ into the subspace of real analytic $\bar\partial$-closed forms on V and for $q > 0$ a linear operator $I: Z^{(p,q)}(V) \to C^{(p,q-1)}(V)$ such that for an arbitrary $\phi \in Z^{(p,q)}(V)$ the following equality is satisfied*

$$\phi = \bar\partial I[\phi] + L[\phi], \tag{1.1}$$

(ii) *a form $\phi \in Z^{(p,q)}(V)$ is $\bar\partial$-exact iff $L[\phi] = 0$.*

Theorems of this type have many applications, especially in algebraic geometry. However, for some important applications there are at least two difficulties. The first difficulty is caused by the non-constructiveness of the following remarkable Hodge's statement: V has to be equipped with an hermitian metric, and then projection operator L can be chosen to be orthogonal onto the subspace of harmonic $\bar\partial$-closed forms on V (see [8,15,26]). The second difficulty is caused by too abstract formulations of necessary results for applications to varieties with singularities (see [11,16,17,38]).

The first difficulty has been overcome (rather recently) only for special cases ($\mathbb{CP}^n$ and some flag manifolds) in [7,13,14,21,40]. An analytic technique for overcoming the second difficulty was initiated in [22] using an important theory of residual currents of Coleff and Herrera [10], based on resolution of singularities of Hironaka [25]. In the present article we further develop our homotopy formulas for the $\bar\partial$-operator from [21] and combine them with the theory of residual currents to obtain a constructive version of a Hodge-type decomposition for residual $\bar\partial$-cohomologies on complete intersection subvarieties of $\mathbb{CP}^n$.

The main result of the article is formulated in Theorem 1 below. We notice that the decomposition obtained in this theorem, which explicitly depends only on polynomials defining V, is new even in the case of a nonsingular curve in $\mathbb{CP}^2$.

Before formulating this result we have to recap some of the definitions from [23]. Let V be a complete intersection subvariety

$$V = \left\{z \in \mathbb{CP}^n : \; P_1(z) = \cdots = P_m(z) = 0\right\} \tag{1.2}$$

of dimension $n-m$ in $\mathbb{CP}^n$ defined by a collection $\{P_k\}_{k=1}^m$ of homogeneous polynomials. Let

$$\left\{U_\alpha = \left\{z \in \mathbb{CP}^n : \; z_\alpha \neq 0\right\}\right\}_{\alpha=0}^n$$

be the standard covering of $\mathbb{CP}^n$, and let

$$\mathbf{F}^{(\alpha)}(z) = \begin{bmatrix} F_1^{(\alpha)}(z) \\ \vdots \\ F_m^{(\alpha)}(z) \end{bmatrix} = \begin{bmatrix} P_1(z)/z_\alpha^{\deg P_1} \\ \vdots \\ P_m(z)/z_\alpha^{\deg P_m} \end{bmatrix}$$

be collections of nonhomogeneous polynomials satisfying

$$\mathbf{F}^{(\alpha)}(z) = A_{\alpha\beta}(z)\cdot\mathbf{F}^{(\beta)}(z) = \begin{bmatrix} \left(z_\beta/z_\alpha\right)^{\deg P_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(z_\beta/z_\alpha\right)^{\deg P_m} \end{bmatrix}\cdot\mathbf{F}^{(\beta)}(z)$$

on $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Following [16] and [17] we consider a line bundle $\mathcal{L}$ on V with transition functions

$$l_{\alpha\beta}(z) = \det A_{\alpha\beta} = \left(\frac{z_\beta}{z_\alpha}\right)^{\sum_{k=1}^m \deg P_k}$$

on $U_{\alpha\beta}$ and the *dualizing bundle* on a complete intersection subvariety V

$$\omega_V^\circ = \omega_{\mathbb{CP}^n}\otimes\mathcal{L}, \tag{1.3}$$

where $\omega_{\mathbb{CP}^n}$ is the canonical bundle on $\mathbb{CP}^n$.

For $q = 1,\ldots,n-m$ we denote by $\mathcal{E}^{(n,n-m-q)}(V,\mathcal{L}) = \mathcal{E}^{(0,n-m-q)}\left(V,\omega_V^\circ\right)$ the space of C^∞ differential forms of bidegree $(n, n-m-q)$ with coefficients in $\mathcal{L}$, i.e., the space of collections of forms

$$\left\{\gamma_\alpha \in \mathcal{E}^{(n,n-m-q)}(U_\alpha)\right\}_{\alpha=0}^n$$

satisfying

$$\gamma_\alpha = l_{\alpha\beta}\cdot\gamma_\beta + \sum_{k=1}^m F_k^{(\alpha)}\cdot\gamma_k^{\alpha\beta} \text{ on } U_\alpha\cap U_\beta. \tag{1.4}$$

Then following [10,23,35] we define residual currents and $\bar\partial$-closed residual currents on V. By a residual current of homogeneity zero $\phi \in C_R^{(0,q)}(V)$ we call a collection $\left\{\Phi_\alpha^{(0,q)}\right\}_{\alpha=0}^n$ of C^∞ differential forms satisfying equalities

$$\Phi_\alpha = \Phi_\beta + \sum_{k=1}^m F_k^{(\alpha)}\cdot\Omega_k^{(\alpha\beta)} \text{ on } U_\alpha\cap U_\beta, \tag{1.5}$$

acting on $\gamma \in \mathcal{E}^{(n,n-m-q)}(V,\mathcal{L})$ by the formula

$$\langle\phi,\gamma\rangle = \sum_\alpha \int_{U_\alpha}\vartheta_\alpha\gamma_\alpha\wedge\Phi_\alpha\bigwedge_{k=1}^m\bar\partial\frac{1}{F_k^{(\alpha)}} \stackrel{\text{def}}{=} \lim_{t\to 0}\sum_\alpha\int_{T_\alpha^{\epsilon(t)}}\vartheta_\alpha\frac{\gamma_\alpha\wedge\Phi_\alpha}{\prod_{k=1}^m F_k^{(\alpha)}}, \tag{1.6}$$

where $\{\vartheta_\alpha\}_{\alpha=0}^n$ is a partition of unity subordinate to the covering $\{U_\alpha\}_{\alpha=0}^n$, and the limit in the right-hand side of (1.6) is taken along an admissible path in the sense of

Coleff–Herrera [10], i.e., an analytic map $\epsilon : [0, 1] \to \mathbb{R}^m$ satisfying conditions

$$\begin{cases} \lim_{t\to 0} \epsilon_m(t) = 0, \\ \lim\limits_{t\to 0} \dfrac{\epsilon_j(t)}{\epsilon_{j+1}^l(t)} = 0, \text{ for any } l \in \mathbb{N} \text{ and } j = 1, \dots, m-1, \end{cases} \tag{1.7}$$

and

$$T_\alpha^{\epsilon(t)} = \left\{ z \in U_\alpha : \left| F_k^{(\alpha)}(z) \right| = \epsilon_k(t) \text{ for } k = 1, \dots, m \right\}. \tag{1.8}$$

Condition (1.7), though looking technical, is essential for the existence of the limit in the right-hand side of (1.6), and cannot be replaced by a simpler condition $\epsilon_j(t) \to 0$, $t \to 0$, $j = 1, \dots, m$, as was shown by Passare and Tsikh in [34].

A residual current ϕ we call $\bar\partial$-closed $\left(\text{denoted } \phi \in Z_R^{(0,q)}(V)\right)$, if there exists a representation $\left\{ \Phi_\alpha^{(0,q)} \right\}_{\alpha=0}^n$ of this current satisfying the following condition

$$\bar\partial \Phi_\alpha = \sum_{k=1}^m F_k^{(\alpha)} \cdot \Omega_k^{(\alpha)} \text{ on } U_\alpha. \tag{1.9}$$

It is easy to see that because of the classical Grothendieck–Dolbeault lemma the definition above coincides with the standard definition of smooth $\bar\partial$-closed differential forms on complex manifolds.

In Theorem 1 below we prove the existence of an explicit Hodge-type representation formula for $\bar\partial$-closed residual currents and its main properties. For simplification of formulation and of the exposition below we assume existence of holomorphic functions $g_\alpha \in H(U_\alpha)$ for $\alpha \in (0, \dots, n)$ satisfying

$$\begin{aligned} (a) \quad & V'_\alpha = \left\{ z \in U_\alpha : F_1^{(\alpha)}(z) = \cdots = F_m^{(\alpha)}(z) = g_\alpha(z) = 0 \right\} \\ & \qquad \text{is a complete intersection in } U_\alpha, \\ (b) \quad & (V \cap U_\alpha) \setminus V'_\alpha \text{ is a submanifold in } U_\alpha. \end{aligned} \tag{1.10}$$

Existence of such functions is a corollary of the local description of analytic sets (see [39]).

Theorem 1 *Let $V \subset \mathbb{CP}^n$ be a reduced complete intersection subvariety as in* (1.2). *Then*

(i) *there exist an explicit finite-dimensional projection operator (see formula* (3.24) *below)*

$$L_{n-m} : Z_R^{(0,n-m)}(V) \to Z_R^{(0,n-m)}(V)$$

into the subspace of real analytic $\bar\partial$-closed residual currents and explicit linear operators (see formula (4.22) *below)*

$$I_q : Z_R^{(0,q)}(V) \to C^{(0,q-1)}(V)$$

into the spaces of currents on V for $q = 1, \ldots, n-m$, so that the following equality is satisfied for an arbitrary $\phi \in Z_R^{(0,q)}(V)$:

$$\phi = \bar{\partial} I_q[\phi] + L_q[\phi], \tag{1.11}$$

(ii) *for $q = 1, \ldots, n-m-1$ we have $L_q = 0$, and therefore $I_q[\phi]$ is a current-solution of equation $\bar{\partial}\psi = \phi$, which is a residual current on $V \setminus \bigcup_\alpha V'_\alpha$ defined by the forms in $C^\infty\left(U_\alpha \setminus V'_\alpha\right)$,*

(iii) *a $\bar{\partial}$-closed residual current $\phi \in Z_R^{(0,n-m)}(V)$ of homogeneity zero is $\bar{\partial}$-exact, i.e., there exists a current $\psi \in C^{(0,n-m-1)}(V)$ such that $\phi = \bar{\partial}\psi$, iff*

$$L_{n-m}[\phi] = 0. \tag{1.12}$$

Remark 1 We interpret formula (1.11) as equality of currents, which are principal values of the residues of Coleff and Herrera taken along admissible paths. Precise definitions and explanations are given in the end of Sect. 2 and in Sect. 5. Such interpretation with application to explicit solvability of $\bar{\partial}$-equation on Stein reduced complete intersections in pseudoconvex domains was introduced in [22], motivated by the works of Coleff, Herrera, and Lieberman [10,20]. In [1,2] such interpretation was used to obtain similar solvability of $\bar{\partial}$-equation on reduced pure-dimensional Stein spaces. In the present article we use this interpretation in the problem of constructing an explicit Hodge-type decomposition of $\bar{\partial}$-closed residual currents on reduced, compact, complete intersection subvarieties in $\mathbb{CP}^n$ with nontrivial $\bar{\partial}$-cohomologies of highest degree. An important feature of the obtained decomposition is condition (1.12), which is similar to condition (ii) in the Hodge Theorem, but with explicit integral operator L_{n-m}. Another important feature of decomposition (1.11) is the real analyticity of the form $L_{n-m}[\phi]$ in some neighborhood of V even for the case of singular reduced complete intersections.

Remark 2 Works of Passare [35,36], and of Berenstein et al. [4], based on fundamental results of Atiyah [3], Bernstein and Gelfand [5], and Bernstein [6] lead to the following simplified version of the original Coleff–Herrera–Lieberman residue formula

$$\langle \Phi, \gamma \rangle = \lim_{\substack{\lambda_1, \ldots, \lambda_m \to 0 \\ \mathrm{Re}\lambda_j > 0,\ j=1,\ldots,m}} \sum_\alpha \vartheta_\alpha \frac{\gamma_\alpha \wedge \bar{\partial}\left|F_1^{(\alpha)}\right|^{2\lambda_1} \wedge \cdots \wedge \bar{\partial}\left|F_m^{(\alpha)}\right|^{2\lambda_m} \wedge \Phi_\alpha}{\prod_{j=1}^m F_j^{(\alpha)}}.$$

It was used in [4] for the following division and interpolation problem:

for given holomorphic functions $f_1, \ldots, f_p$ and an arbitrary holomorphic function f on a Stein variety V find an explicit representation

$$f = f_1 g_1 + \cdots + f_p g_p + h$$

with g_j being holomorphic on V, so that the remainder h vanishes on V iff f belongs to the ideal generated by $f_1, \ldots, f_p$.

Remark 3 In the original version of this article at http://arxiv.org/abs/1405.7411 we mentioned two interrelated simple applications of Theorem 1 that we were working on.

(i) The first of those applications was the construction of an explicit Hodge decomposition on complex curves in $\mathbb{CP}^3$. We have completed the above mentioned construction in "archived" article http://arxiv.org/abs/1507.03272, where we proved the following: *Explicit Formula in Hodge decomposition* Let $\mathcal{X}$ be a smooth algebraic curve, and let $\mathcal{X} \xrightarrow{\varrho} \mathbb{CP}^2$ be an immersion of $\mathcal{X}$ into $\mathbb{CP}^2$, such that $\mathcal{C} = \varrho(\mathcal{X})$ is an algebraic curve with r nodal points. Let points $p_1^{(i)}, p_2^{(i)} \in \mathcal{X}$ be such that $\varrho(p_1^{(i)}) = \varrho(p_2^{(i)}) = p^{(i)} \in \mathcal{C}$. Let $\{\gamma_i\}_{i=1}^r$ in $\mathcal{X}$ be a collection of paths such that $\gamma_i(0) = p_1^{(i)}$ and $\gamma_i(1) = p_2^{(i)}$, and $\{f_i\}_{i=1}^r \in \mathcal{E}(\mathcal{X})$ be a collection of functions with supports in some neighborhoods $U_i \supset \gamma_i$ such that

$$\begin{cases} f_i(p_1^{(i)}) = 0, \ f_i(p_2^{(i)}) = 1, \\ \partial f_i = 0 \text{ in } V_i \Subset U_i. \end{cases} \tag{1.13}$$

Let L and I be the operators from Theorem 1 for the complete intersection curve $\mathcal{C}$. We define operators $\mathcal{L}$ and $\mathcal{I}$ as follows

$$\mathcal{L}[\phi] = (L[\phi_*])^* - \sum_{i=1}^{r} a_i[\phi](L[(\bar{\partial} f_i)_*])^*, \tag{1.14}$$

and

$$\mathcal{I}[\phi] = \left(I\left[\phi_* - \sum_{i=1}^{r} a_i[\phi](\bar{\partial} f_i)_*\right]\right)^* + \sum_{i=1}^{r} a_i[\phi] f_i, \tag{1.15}$$

where $_*$ and * denote respectively direct and inverse images under the immersion ϱ and

$$a_i[\phi] = \int_{\mathcal{X}} \phi \wedge \overline{(L[(\bar{\partial} f_i)_*])^*} \quad \text{for } \phi \in \mathcal{E}^{(0,1)}(\mathcal{X}). \tag{1.16}$$

Let $\left\{\omega_j\right\}_{j=1}^{g}$ be an orthonormal basis of holomorphic $(1, 0)$-forms on $\mathcal{X}$, i.e.,

$$\int_{\mathcal{X}} \omega_j \wedge \bar{\omega}_k = \delta_{jk}, \ j, k = 1, \ldots, g.$$

Then Hodge operators H_1 and R_1 in decomposition (1.1) admit the following representations

$$\begin{aligned} H_1[\phi] &= \sum_{j=1}^{g} \left(\int_{\mathcal{C}} \phi \wedge \omega_j\right) \bar{\omega}_j, \\ R_1[\phi] &= \mathcal{I}\big[\phi + (\mathcal{L} - H_1)[\phi]\big] + \text{const.} \end{aligned} \tag{1.17}$$

(ii) The second application is the construction of an explicit Green's function for solutions of inverse conductivity problem on bordered surfaces in $\mathbb{R}^3$. In this direction we have the following result:

Let $\widetilde{V}$ be a smooth algebraic curve in $\mathbb{CP}^3$ with homogeneous coordinates w_0, w_1, w_2, w_3, and let $\mathbb{CP}^2_\infty = \left\{w \in \mathbb{CP}^3 : w_0 = 0\right\}$. Then $\mathbb{C}^3 = \mathbb{CP}^3 \setminus \mathbb{CP}^2_\infty$ is a complex affine space with coordinates $\{z_k = w_k/w_0,\ k = 1, 2, 3\}$.

Let $V = \widetilde{V} \setminus (\widetilde{V} \cap \mathbb{CP}^2_\infty)$ be a connected algebraic curve in $\mathbb{C}^3$. We denote by

$L^p_{(1,1)}(V)$ – the space of differential forms of type (1, 1) on V with coefficients in $L^p(V)$,

and by

$$\widetilde{W}^{1,p}_{(1,0)}(V) = \left\{f \in L^\infty_{(1,0)}(V) :\ \bar{\partial} f \in L^p_{(1,1)}(V),\ \operatorname{supp} \bar{\partial} f \Subset V_0 \subset V\right\} \text{ for } p > 2.$$

Using operators R and H from (1.17) we define

$$\begin{aligned}
&\widehat{R}_\theta[\phi] = R\left(\left((dz_1 + \theta dz_2) \lrcorner \phi\right) \wedge (dz_1 + \theta dz_2)\right),\\
&R_{\lambda,\theta}[f] = e_{-\lambda,\theta}\overline{R\left[\overline{e_{\lambda,\theta} f}\right]},\ \text{ where } e_{\lambda,\theta}(z) = e^{\lambda(z_1+\theta z_2)-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)} \text{ for } \lambda, \theta \in \mathbb{C},\\
&H_{\lambda,\theta}[f] = e_{-\lambda,\theta}\overline{H\left[\overline{e_{\lambda,\theta} f}\right]}.
\end{aligned}$$

Explicit Faddeev Type Green Function The kernel $g_{\lambda,\theta}(\zeta, z)$ on $V \times V$ with $\lambda, \theta \in \mathbb{C}$ of the integral operator $R_{\lambda,\theta} \circ \widehat{R}_\theta$ is called the Faddeev type Green function for the operator $\bar{\partial}\,(\partial + \lambda dz)$ and the explicit Hodge formula from (1.17) implies the following explicit version of the Hodge type result presented already in [19]:

If $\phi \in L^\infty_{(1,1)}$, then the function $u = R_{\lambda,\theta} \circ \widehat{R}_\theta[\phi]$ satisfies the following equation

$$\bar{\partial}\,(\partial + \lambda(dz_1 + \theta dz_2))\, u = \phi + \bar{\lambda}(d\bar{z}_1 + \bar{\theta} d\bar{z}_2) \wedge \left(H_{\lambda,\theta} \circ \widehat{R}_\theta[\phi]\right)$$

on V.

Function $g_{\lambda,\theta}(\zeta, z)$ is the main tool in the construction of an explicit solution of the Inverse Conductivity Problem on bordered Riemann surfaces, which we are currently working on.

Remark 4 In the future we still plan to extend the result of Theorem 1 to the case of locally complete intersections in $\mathbb{CP}^n$ with $n \geq 3$, which might be considered as a natural level of generality for explicit formulas, as implied by Hartshorne [17]. In http://arxiv.org/abs/1507.03272 we made a progress in this direction by constructing explicit Hodge decomposition on arbitrary Riemann surfaces, which are embeddable into $\mathbb{CP}^3$ (see [15,17]), but not necessarily embeddable into $\mathbb{CP}^2$ as complete intersections.

2 Integral Formulas on Domains in Projective Spaces

In this section we construct a Cauchy–Weil–Leray type integral formula for differential forms on a domain U in $\mathbb{CP}^n$. We start with the Koppelman-type formula from [21] (Proposition 1.2) and [18] (Theorem 3.2) going back to Moisil [32], Fueter [12], Bochner [9], Martinelli [31]. This formula is a modification for the case of $\mathbb{CP}^n$

of the original Koppelman formula announced by Koppelman in [28] (1967). The first complete proof of Koppelman's formula was given in the Polyakov's paper [37] (06.1970), where it was used to obtain a Weil-type integral formula [41] for differential forms on analytic polyhedra, while in the papers of Lieb [30] (07.1970) and Øvrelid [33] (11.1970) Koppelman's formula was used to obtain an integral formula of Leray-type [29] for differential forms on strongly pseudoconvex domains. In the present article we use formulas of both types: Weil-type formula for a tubular neighborhood of a subvariety in $\mathbb{CP}^n$ and Leray-type formula for the unit sphere $\mathbb{S}^{2n+1}(1) \subset \mathbb{C}^{n+1}$.

In [21] we identified the forms on $\mathbb{CP}^n$ with their lifts to $\mathbb{S}^{2n+1}(1)$ satisfying appropriate homogeneity conditions and constructed integral formulas for the lifted forms. The proposition below is a reformulation of Proposition 1.2 from [21].

Proposition 2.1 *Let $\{P_k\}_1^m$ be homogeneous polynomials defining the variety V as in (1.2), let $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$, and let $\Phi^{(0,q)}$ be a form of homogeneity zero on the domain*

$$U^{\epsilon} = \left\{ z \in \mathbb{S}^{2n+1}(1) : \; |P_k(z)| < \epsilon_k \quad \text{for } k = 1, \ldots, m \right\}. \tag{2.1}$$

Then the following equality is satisfied for $z \in U^{\epsilon}$

$$\Phi^{(0,q)}(z) = \bar{\partial}_z J_q^{\epsilon}[\Phi](z) + J_{q+1}^{\epsilon}\left[\bar{\partial}\Phi\right](z) + K_q^{\epsilon}[\Phi](z), \tag{2.2}$$

with

$$J_q^{\epsilon}[\Psi](z) = -\frac{n!}{(2\pi i)^{n+1}} \int_{U^{\epsilon}\times[0,1]} \Psi(\zeta) \wedge \omega'_{q-1}\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)} + \lambda \frac{\bar{\zeta}}{B(\zeta,z)} \right) \wedge \omega(\zeta),$$

and

$$K_q^{\epsilon}[\Psi](z) = -\frac{n!}{(2\pi i)^{n+1}} \int_{bU^{\epsilon}\times[0,1]} \Psi(\zeta) \wedge \omega'_{q}\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)} + \lambda \frac{\bar{\zeta}}{B(\zeta,z)} \right) \wedge \omega(\zeta),$$

where

$$B^*(\zeta,z) = \sum_{j=0}^{n} \bar{z}_j \cdot \left(\zeta_j - z_j\right), \qquad B(\zeta,z) = \sum_{j=0}^{n} \bar{\zeta}_j \cdot \left(\zeta_j - z_j\right),$$

$$\omega(\zeta) = d\zeta_0 \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n, \qquad \omega'(\eta) = \sum_{k=0}^{n} (-1)^{k-1} \eta_k \bigwedge_{j\neq k} d\eta_j$$

and ω'_q is the $(0,q)$-component with respect to z of the form ω'.

We will transform the right-hand side of equality (2.2) into a Cauchy–Weil–Leray type formula. For this transformation we need the following Weil-type lemma.

Lemma 2.2 *Let $P(\zeta)$ be a homogeneous polynomial of variables $\zeta_0, \dots, \zeta_n$ of degree d. Then there exist polynomials $\left\{Q^i(\zeta, z)\right\}_{i=0}^n$ satisfying:*

$$\begin{cases} P(\zeta) - P(z) = \sum_{i=0}^n Q^i(\zeta, z) \cdot (\zeta_i - z_i)\,, \\ Q^i(\lambda\zeta, \lambda z) = \lambda^{d-1} \cdot Q^i(\zeta, z) \; for \; \lambda \in \mathbb{C}. \end{cases} \tag{2.3}$$

Proof We notice that it suffices to prove the lemma for homogeneous monomials. We prove the lemma for homogeneous monomials by induction with respect to the number of variables. Using the one-variable equality

$$\zeta^d - z^d = (\zeta - z) \cdot \left(\sum_{j=0}^{d-1} \zeta^{d-1-j} \cdot z^j \right) \tag{2.4}$$

we obtain the statement of the lemma for an arbitrary monomial depending only on one variable.

To prove the step of induction we consider a monomial $\zeta_0^{d_0} \cdots \zeta_k^{d_k}$ with $k \geq 1$ and $\sum_{j=0}^k d_j = d$. Then we obtain the following equality

$$\zeta_0^{d_0} \cdots \zeta_k^{d_k} - z_0^{d_0} \cdots z_k^{d_k} = \left(\zeta_0^{d_0} - z_0^{d_0}\right) \cdot \zeta_1^{d_1} \cdots \zeta_k^{d_k} + z_0^{d_0} \cdot \left(\zeta_1^{d_1} \cdots \zeta_k^{d_k} - z_1^{d_1} \cdots z_k^{d_k}\right).$$

Using equality (2.4) for the first term of the right-hand side of equality above we obtain

$$Q^0(\zeta, z) = \left(\sum_{j=0}^{d_0-1} \zeta_0^{d_0-1-j} \cdot z_0^j \right) \cdot \zeta_1^{d_1} \cdots \zeta_k^{d_k}.$$

Using then the inductive assumption for the polynomial

$$\zeta_1^{d_1} \cdots \zeta_k^{d_k} - z_1^{d_1} \cdots z_k^{d_k}$$

we obtain the existence of polynomials $\left\{q^i(\zeta, z)\right\}_{i=1}^n$ satisfying conditions (2.3). Therefore, defining for $i = 1, \dots, n$

$$Q^i(\zeta, z) = z_0^{d_0} \cdot q^i(\zeta, z)$$

we obtain the necessary coefficients for a monomial in $k + 1$ variables. □

The integrals in the sought formula will be taken over a special chain

$$\mathcal{C}^\epsilon = \sum_{|J| \geq 1} \Gamma_J^\epsilon \times \Delta_J, \tag{2.5}$$

where $J = \left(j_1, \ldots, j_p\right)$ is a multiindex with $|J| = p \leq m$,

$$\Gamma_J^\epsilon = \left\{\zeta \in \mathbb{S}^{2n+1}(1) : \; |P_j(\zeta)| = \epsilon_j \text{ for } j \in J, \; |P_k(\zeta)| < \epsilon_k \text{ for } k \notin J\right\},$$
$$\Delta_J = \left\{\lambda, \mu_{j_1}, \ldots, \mu_{j_p} \in \mathbb{R}^{p+1} : \; \lambda + \sum_{i=1}^{p} \mu_{j_i} \leq 1\right\}.$$

The boundary of chain $\mathcal{C}^\epsilon$ is the chain

$$\mathcal{B}^\epsilon = -\sum_{j=1}^{m} \Gamma_j^\epsilon \times \Lambda + \sum_{|J|\geq 1} \left((-1)^{|J|-1}\Gamma_J^\epsilon \times \Delta_J' + \Gamma_J^\epsilon \times \Lambda_J\right),$$

where

$$\Lambda = [0, 1],$$
$$\Delta_J' = \left\{\mu_{j_1}, \ldots, \mu_{j_p} \in \mathbb{R}^p : \; \sum_{i=1}^{p} \mu_{j_i} \leq 1\right\},$$
$$\Lambda_J = \left\{\lambda, \mu_{j_1}, \ldots, \mu_{j_p} \in \mathbb{R}^{p+1} : \; \lambda + \sum_{i=1}^{p} \mu_{j_i} = 1\right\}.$$

In the following proposition we construct a Cauchy–Weil–Leray type formula on ϵ-neighborhoods of complete intersection subvarieties in $\mathbb{CP}^n$.

Proposition 2.3 *Let*

$$V = \left\{z \in \mathbb{CP}^n : \; P_1(z) = \cdots = P_m(z) = 0\right\}$$

be a complete intersection subvariety in $\mathbb{CP}^n$ *of dimension* $n - m$*, and let* $\Phi^{(0,q)}$ *be a differential form on an open neighborhood* $U \supset V$.

Then for U^ϵ *as in* (2.1) *and arbitrary* $z \in U^\epsilon$ *the following equality holds*

$$\Phi(z) = \bar{\partial}_z I_q^\epsilon [\Phi](z) + I_{q+1}^\epsilon \left[\bar{\partial}\Phi\right](z) + L_q^\epsilon [\Phi](z), \tag{2.6}$$

where

$$I_q^\epsilon [\Phi](z) = -\frac{n!}{(2\pi i)^{n+1}} \int_{U^\epsilon \times [0,1]} \Phi(\zeta)$$
$$\wedge\, \omega_{q-1}'\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta, z)} + \lambda\frac{\bar{\zeta}}{B(\zeta, z)}\right) \wedge \omega(\zeta)$$

$$+\frac{n!}{(2\pi i)^{n+1}}\sum_{|J|\geq 1}\int_{\Gamma^{\epsilon}_J\times\Delta_J}\Phi(\zeta)\wedge\omega'_{q-1}\Bigg((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}$$

$$+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta), \tag{2.7}$$

and

$$L^{\epsilon}_q[\Phi](z)=\sum_{|J|=n-q}(-1)^{|J|-1}\frac{n!}{(2\pi i)^{n+1}}\int_{\Gamma^{\epsilon}_J\times\Delta'_J}\Phi(\zeta)$$

$$\wedge\,\omega'_q\Bigg((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta), \tag{2.8}$$

with coefficients $\left\{Q^i_k\right\}^{i=0,\dots,n}_{k=1,\dots,m}$ *satisfying conditions* (2.3) *from Lemma* 2.2.

The forms defined by (2.7) *and* (2.8) *on* U^{ϵ} *admit the descent onto a neighborhood of* V *in* $\mathbb{CP}^n$.

Proof Applying Stokes's formula to the form

$$\Phi(\zeta)\wedge\omega'_q\Bigg((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta)$$

we obtain equality

$$\int_{bU^{\epsilon}\times[0,1]}\Phi(\zeta)\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)$$

$$=\sum_{j=1}^{m}\int_{\Gamma^{\epsilon}_j\times\Lambda}\Phi(\zeta)\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)$$

$$=\sum_{|J|\geq 1}(-1)^{|J|-1}\int_{\Gamma^{\epsilon}_J\times\Delta'_J}\Phi(\zeta)$$

$$\wedge\,\omega'_q\Bigg((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta)$$

$$+\sum_{|J|\geq 1}\int_{\Gamma^{\epsilon}_J\times\Lambda_J}\Phi(\zeta)\wedge\omega'_q\Bigg((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta)$$

$$-\sum_{|J|\geq 1}\int_{\Gamma^{\epsilon}_J\times\Delta_J}\bar{\partial}\Phi(\zeta)\wedge\omega'_q\Bigg((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}$$

$$+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta)$$

$$+(-1)^{q+1}\sum_{|J|\geq 1}\int_{\Gamma_J^{\epsilon}\times\Delta_J}\Phi(\zeta)\wedge d_{\zeta,\lambda,\mu}\omega_q'\left((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}\right.$$

$$\left.+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right)\wedge\omega(\zeta).$$

Then using equality

$$d_{\zeta,\lambda,\mu}\omega_r'(\eta)\wedge\omega(\zeta)+\bar{\partial}_z\omega_{r-1}'(\eta)\wedge\omega(\zeta)=0 \qquad (r=1,\ldots,n) \tag{2.9}$$

for

$$\left\{\eta_j=(1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}_j}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}_j}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right\}_{j=0}^{n}$$

we transform the equality above for $n\geq 2$ into

$$\begin{aligned}
&\int_{bU^{\epsilon}\times[0,1]}\Phi(\zeta)\wedge\omega_{q-1}'\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)\\
&=\sum_{|J|\geq 1}(-1)^{|J|-1}\int_{\Gamma_J^{\epsilon}\times\Delta_J'}\Phi(\zeta)\\
&\quad\wedge\omega_q'\left((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right)\wedge\omega(\zeta)\\
&\quad+\sum_{|J|\geq 1}\int_{\Gamma_J^{\epsilon}\times\Lambda_J}\Phi(\zeta)\\
&\quad\wedge\omega_q'\left((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right)\wedge\omega(\zeta)\\
&\quad-\sum_{|J|\geq 1}\int_{\Gamma_J^{\epsilon}\times\Delta_J}\bar{\partial}\Phi(\zeta)\wedge\omega_q'\left((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}\right.\\
&\quad\left.+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right)\wedge\omega(\zeta)\\
&\quad+\sum_{|J|\geq 1}\int_{\Gamma_J^{\epsilon}\times\Delta_J}\Phi(\zeta)\wedge\bar{\partial}_z\omega_{q-1}'\left((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}\right.\\
&\quad\left.+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right)\wedge\omega(\zeta),
\end{aligned} \tag{2.10}$$

and finally obtain from (2.2) equality (2.6) with

$$
\begin{aligned}
L_q^{\epsilon}[\Phi](z) = \sum_{|J|\geq 1} (-1)^{|J|-1} \frac{n!}{(2\pi i)^{n+1}} \int_{\Gamma_J^{\epsilon}\times\Delta_J'} \Phi(\zeta)\wedge\omega_q'\Bigg((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)} \\
+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta) \\
+\sum_{|J|\geq 1}\frac{n!}{(2\pi i)^{n+1}}\int_{\Gamma_J^{\epsilon}\times\Lambda_J}\Phi(\zeta)\wedge\omega_q'\Bigg((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{\zeta}}{B(\zeta,z)} \\
+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)\wedge\omega(\zeta).
\end{aligned}
$$

Then we notice that because of the holomorphic dependence on z we have for $q\geq 1$ the equality

$$
\omega_q'\Bigg((1-\sum_{k=1}^{m}\mu_k)\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\Bigg)=0.
$$

Since the dimension of Γ_J^{ϵ} is equal to $2n+1-|J|$ and the form Φ has q differentials of the form $d\bar{\zeta}$, we conclude that the only terms in the first sum of the right-hand side of the formula for $L_q^{\epsilon}[\Phi]$ that have a nonzero contribution are the terms with

$$
|J| = n-q.
$$

From the last two observations we obtain formula (2.8) for $L_q^{\epsilon}[\Phi]$.

The fact that the forms $L_q^{\epsilon}[\Phi]$ and $I_q^{\epsilon}[\Phi]$ have homogeneity zero, as the form Φ, follows from the homogeneity properties of the functions $B(\zeta,z)$ and $B^*(\zeta,z)$ and from the homogeneity property (2.3) of the coefficients $Q_s^i(\zeta,z)$. □

We interpret formula (2.6) as a formula for residual currents

$$
\langle\phi,\gamma\rangle = \pm\left\langle I_q^{\epsilon}[\phi],\bar{\partial}\gamma\right\rangle+\left\langle I_{q+1}^{\epsilon}\left[\bar{\partial}\phi\right],\gamma\right\rangle+\left\langle L_q^{\epsilon}[\phi],\gamma\right\rangle, \tag{2.11}
$$

where all terms in the right-hand side are understood as residual currents, i.e., for example for an arbitrary $\gamma\in\mathcal{E}_c^{(n,n-m-q)}(V,\mathcal{L})$ with support in U_α we mean

$$
\left\langle L_q^{\epsilon}[\phi],\gamma\right\rangle = \lim_{\tau\to 0}\int_{T_\alpha^{\delta(\tau)}}\gamma(z)\wedge\frac{L_q^{\epsilon}[\phi](z)}{\prod_{k=1}^{m}F_k^{(\alpha)}(z)}, \tag{2.12}
$$

where we denote by $L_q^{\epsilon}[\phi](z)$ the descent of this form onto $\mathbb{CP}^n$.

Formula (2.11) is a preliminary form of the Hodge-type decomposition formula for $\bar{\partial}$-closed residual currents on V. In what follows we will consider the limits of the terms in the right-hand side of (2.6) as $\epsilon \to 0$, and interpret the limit of operator $I_q^{\epsilon}[\Phi]$ as a solution operator on V and the limit of $L_q^{\epsilon}[\Phi]$ as a Hodge-type projection operator.

3 Hodge-Type Projection

In this section we transform formula (2.8) into a residual form by considering the limit of L_q^{ϵ} as $\epsilon \to 0$. We perform this transformation in several steps. First we observe that the only nonzero terms in this formula are those that have $q = n - |J|$. But for subvariety V we have $|J| \leq m$, and therefore operator L_q^{ϵ} contains nonzero integrals only for $q \geq n-m$. On the other hand, since we are considering only the cohomologies of degree less or equal to $n-m$, where $n-m$ is the dimension of V, in formula (2.8) we have the exact equalities $q = n-m$, $|J| = m$, and therefore $J = (1, \ldots, m)$.

First we transform formula (2.8) for $L_q^{\epsilon}[\Phi](z)$ with $z \in U^{\epsilon}$ by integrating with respect to variables $\mu_k \in \Delta'_J$, and obtain

$$\begin{aligned} L_q^{\epsilon}[\Phi](z) &= (-1)^{n-q-1}\frac{n!}{(2\pi i)^{n+1}}\int_{\Gamma_J^{\epsilon}\times\Delta'_J}\Phi(\zeta) \\ &\quad \wedge\omega'_q\left((1-\mu)\frac{\bar{z}}{B^*(\zeta,z)}+\sum_{k=1}^{m}\mu_k\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}\right)\wedge\omega(\zeta) \\ &= C(n,m,d)\int_{\Gamma_J^{\epsilon}}\Phi(\zeta)\wedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q=n-m}\right]\wedge\omega(\zeta). \end{aligned} \tag{3.1}$$

Then, using expression

$$B^*(\zeta,z) = \sum_{j=0}^{n}\bar{z}_j\cdot\left(\zeta_j - z_j\right) = -1 + \sum_{j=0}^{n}\bar{z}_j\cdot\zeta_j$$

and its corollary

$$\left(B^*(\zeta,z)\right)^{-q-1} = (-1)^{q+1}\left(1-\sum_{j=0}^{n}\bar{z}_j\cdot\zeta_j\right)^{-q-1}$$

in the integral from the right-hand side of (3.1) we obtain

$$\int_{\Gamma_J^{\epsilon}} \Phi(\zeta) \wedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q}\right] \wedge \omega(\zeta)$$

$$= (-1)^{q+1} \lim_{\substack{\eta\to 1\\ \eta<1}} \int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k\}_{k=1}^m\}} \left(1-\eta\sum_{j=0}^{n}\bar{z}_j\cdot\zeta_j\right)^{-q-1}$$

$$\times \Phi(\zeta) \wedge \det\left[\bar{z}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{m}\ \overbrace{d\bar{z}}^{q}\right] \wedge \omega(\zeta)$$

$$= (-1)^{q+1} \lim_{\substack{\eta\to 1\\ \eta<1}} \sum_{r=0}^{\infty} c_r\eta^r \cdot \int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k\}_{k=1}^m\}} \langle\bar{z}\cdot\zeta\rangle^r$$

$$\times \Phi(\zeta) \wedge \det\left[\bar{z}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{m}\ \overbrace{d\bar{z}}^{q}\right] \wedge \omega(\zeta), \tag{3.2}$$

where we denoted $\langle\bar{z}\cdot\zeta\rangle = \sum_{j=0}^{n}\bar{z}_j\zeta_j$.

For $\zeta, z \in \mathbb{S}^{2n+1}(1)$ such that $\{|P_k(\zeta)| = \epsilon_k\}_{k=1}^m$ and $\{|P_k(z)| < \epsilon_k\}_{k=1}^m$ we use in the differential form

$$\det\left[\bar{z}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{m}\ \overbrace{d\bar{z}}^{q}\right]$$

the following representation with absolutely converging series

$$\frac{Q_k^s(\zeta,z)}{P_k(\zeta)-P_k(z)} = \frac{Q_k^s(\zeta,z)}{P_k(\zeta)}\cdot\left(1-\frac{P_k(z)}{P_k(\zeta)}\right)^{-1} = \frac{Q_k^s(\zeta,z)}{P_k(\zeta)}\cdot\left(1+\sum_{l=1}^{\infty}\left(\frac{P_k(z)}{P_k(\zeta)}\right)^l\right) \tag{3.3}$$

and obtain the equality

$$\int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k\}_{k=1}^m\}} \langle \bar{z}\cdot\zeta\rangle^r \Phi(\zeta) \times \det\left[\bar{z}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\omega(\zeta)$$

$$= \sum_{|A|\geq 0} C(A)\int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k\}_{k=1}^m\}} \langle \bar{z}\cdot\zeta\rangle^r \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\cdot\frac{P^A(z)}{P^A(\zeta)}$$

$$\wedge \det\left[\bar{z}\ \overbrace{Q_k(\zeta,z)}^{n-q}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\omega(\zeta), \tag{3.4}$$

where $A=(a_1,\ldots,a_m)$ is a multiindex,

$$P^A(\zeta)=P_1^{a_1}(\zeta)\cdots P_m^{a_m}(\zeta),$$

and $|A|=a_1+\cdots+a_m$.

Using Theorem 1.7.6(2) from [10] (see also [23] Prop. 2.3) we obtain that the residual currents defined by the terms in the right-hand side of (3.4) with $|A|\geq 1$ are zero-currents from the point of view of (2.12), and therefore we can simplify the expression for $L_q^{\epsilon}[\Phi]$ as follows

$$L_q^{\epsilon}[\Phi](z) = C(n,m,d)\lim_{\substack{\eta\to 1\\ \eta<1}}\sum_{r=0}^{\infty} c_r\eta^r \cdot \int_{\{|\zeta|=1,\{|P_k(\zeta)=\epsilon_k\}_{k=1}^m\}} \langle\bar{z}\cdot\zeta\rangle^r$$

$$\times\Phi(\zeta)\wedge\det\left[\bar{z}\ \overbrace{\frac{Q_k(\zeta,z)}{P_k(\zeta)-P_k(z)}}^{n-q}\ \overbrace{d\bar{z}}^{q}\right]\wedge\omega(\zeta)$$

$$= C(n,m,d)\lim_{\substack{\eta\to 1\\ \eta<1}}\sum_{r=0}^{\infty}\int_{\{|\zeta|=1,\{|P_k(\zeta)=\epsilon_k\}_{k=1}^m\}} c_r\eta^r\langle\bar{z}\cdot\zeta\rangle^r\frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$

$$\bigwedge\det\left[\bar{z}\ \overbrace{Q_k(\zeta,z)}^{n-q}\ \overbrace{d\bar{z}}^{q}\right]\wedge\omega(\zeta). \tag{3.5}$$

Before continuing with further transformation of (3.5) we prove a lemma, in which we slightly modify the result of Coleff and Herrera from [10] to obtain the existence of residual limits over deformed admissible tubes for reduced complete intersections.

Lemma 3.1 *Let* $\{F_1,\ldots,F_m\}$ *be polynomials on* $\mathbb{C}^n$*, let*

$$V=\left\{\zeta\in\mathbb{C}^n : F_1(\zeta)=\cdots=F_m(\zeta)=0\right\} \tag{3.6}$$

be a reduced complete intersection subvariety, and let g *be a holomorphic function* g *satisfying:*

(i) $V' = \{\zeta : F_1(\zeta) = \cdots = F_m(\zeta) = g(\zeta) = 0\}$ *is a complete intersection,*
(ii) *for any* $z \in V \setminus V'$ *there exists a neighborhood* W_z*, such that* $(V \cap W_z) \setminus V'$ *is a submanifold in* W_z.

Then for an arbitrary differential form $\Phi(\zeta, u) \in \mathcal{E}_c^{(n,n-m)}(\mathbb{C}^n)$ *real analytic with respect to parameters* $u_1, \ldots, u_s$*, and a collection of real-valued functions* $\{\chi_k(\zeta)\}_{k=1}^m \in \mathcal{E}_c(\mathbb{C}^n)$*, such that* $\chi_k(\zeta) \geq 1$ *for* $|\zeta| < 1$*, the limit along an admissible path* $\{\epsilon_k(t)\}_{k=1}^m$ *defined in* (1.7)

$$\begin{aligned}
&\lim_{t\to 0}\int_{\{|F_k(\zeta)|\cdot\chi_k(\zeta)=\epsilon_k(t)\}_{k=1}^m} \frac{\Phi(\zeta,u)}{\prod_{k=1}^m F_k(\zeta)}\\
&\stackrel{\text{def}}{=} \lim_{\eta\to 0}\lim_{t\to 0}\int_{\{|g(\zeta)|>\eta,\ \{|F_k(\zeta)|\cdot\chi_k(\zeta)=\epsilon_k(t)\}_{k=1}^m\}} \frac{\Phi(\zeta,u)}{\prod_{k=1}^m F_k(\zeta)}\\
&= \lim_{\eta\to 0}\lim_{t\to 0}\int_{\{|g(\zeta)|>\eta,\ \{|F_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\Phi(\zeta,u)}{\prod_{k=1}^m F_k(\zeta)}
\end{aligned}\tag{3.7}$$

exists and is real analytic with respect to parameters $u_1, \ldots, u_s$.

Proof We assume that the analytic set V is a subset of a polydisk $\mathcal{P}^n = \{|\zeta_i| < 1,\ i = 1, \ldots, n\}$, such that the restriction of the projection

$$\pi : \mathcal{P}^n \to \mathcal{P}^{n-m},$$

defined by the formula $\pi(\zeta_1, \ldots, \zeta_n) = (\zeta_{m+1}, \ldots, \zeta_n)$, to $V \cap \mathcal{P}$ is a finite proper covering, and the holomorphic function g on $\mathcal{P}^n$ is such that $\dim\left\{V \cap \{g(\zeta) = 0\}\right\} = n - m - 1$.

For a point $z \in V$, such that $|g(z)| > \eta$ we consider the nonholomorphic complex coordinates

$$w_1(\zeta) = F_1(\zeta)\cdot\chi_1(\zeta), \ldots, w_m(\zeta) = F_m(\zeta)\cdot\chi_m(\zeta), \zeta_{m+1}, \ldots, \zeta_n$$

in a small enough neighborhood of the point z. Then for the $(m, 0)$-form

$$\Phi(\zeta,z,u) = \phi(\zeta,z,u)\bigwedge_{j=1}^m d\zeta_j = \left(\bigwedge_{j=m+1}^n d\zeta_j \bigwedge_{j=m+1}^n d\bar{\zeta}_j\right) \rfloor\ \Phi(\zeta,u)\Big|_{\{\zeta_j=\zeta_j(z)\}_{j=m+1}^n}$$

we have

$$\begin{aligned}
&\lim_{t\to 0}\int_{\{|F_k(\zeta)|\cdot\chi_k(\zeta)=\epsilon_k(t)\}_{k=1}^m\{\zeta_j=\zeta_j(z)\}_{j=m+1}^n} \frac{\Phi(\zeta,z,u)}{\prod_{k=1}^m F_k(\zeta)}\\
&= \lim_{t\to 0}\int_{\{|w_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\{\zeta_j=\zeta_j(z)\}_{j=m+1}^n} \frac{\Psi(\zeta,z,u)}{\prod_{k=1}^m w_k(\zeta)},
\end{aligned}$$

where

$$\Psi(\zeta, z, u) = \psi(\zeta, z, u) \bigwedge_{j=1}^{m} d\zeta_j = \Phi(\zeta, z, u) \cdot \prod_{k=1}^{m} \chi_k(\zeta).$$

Using equalities

$$\begin{aligned}
&\Psi(\zeta, z, u)\Big|_{\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \\
&= \phi(\zeta, z, u) \cdot \prod_{k=1}^{m} \chi_k(\zeta) \cdot \det{}^{-1}\left[\frac{\partial F_k}{\partial \zeta_l}\right] \bigwedge_{k=1}^{m} dF_k, \\
&\frac{\partial w_j}{\partial \zeta_l}\Big|_{\{F_k(\zeta)=0\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \\
&= \frac{\partial}{\partial \zeta_l}\left[F_j \cdot \chi_j(\zeta)\right]\Big|_{\{F_k(\zeta)=0\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \\
&= \frac{\partial F_j}{\partial \zeta_l}(\zeta) \cdot \chi_j(\zeta)\Big|_{\{F_k(\zeta)=0\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \quad \text{for } j = 1, \ldots, m,
\end{aligned}$$

and the corollary of the second one

$$\begin{aligned}
&\prod_{k=1}^{m} \chi_k(\zeta) \bigwedge_{k=1}^{m} dF_k(\zeta)\Big|_{\{F_k(\zeta)=0\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \\
&= \bigwedge_{k=1}^{m} dw_k(\zeta)\Big|_{\{F_k(\zeta)=0\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}},
\end{aligned}$$

we obtain for z with $|g(z)| > \delta$ the equality

$$\begin{aligned}
&\lim_{t\to 0} \int_{\{|w_k(\zeta)|=\epsilon_k(t)\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \frac{\Psi(\zeta, z, u)}{\prod_{k=1}^{m} w_k(\zeta)} \\
&= \lim_{t\to 0} \int_{\{|w_k(\zeta)|=\epsilon_k(t)\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \phi(\zeta, z, u) \cdot \prod_{k=1}^{m} \chi_k(\zeta) \\
&\quad \cdot \det{}^{-1}\left[\frac{\partial F_k}{\partial \zeta_l}\right] \frac{\bigwedge_{k=1}^{m} dF_k(\zeta)}{\prod_{k=1}^{m} w_k(\zeta)} \\
&= \lim_{t\to 0} \int_{\{|w_k(\zeta)|=\epsilon_k(t)\}_{k=1}^{m}\{\zeta_j=\zeta_j(z)\}_{j=m+1}^{n}} \phi(\zeta, z, u) \\
&\quad \cdot \det{}^{-1}\left[\frac{\partial F_k}{\partial \zeta_l}\right] \frac{\bigwedge_{k=1}^{m} dw_k(\zeta)}{\prod_{k=1}^{m} w_k(\zeta)}
\end{aligned}$$

$$= (2\pi i)^k \phi(\zeta(z), z, u) \cdot \det{}^{-1}\left[\frac{\partial F_k}{\partial \zeta_l}\right](\zeta(z))$$
$$= \lim_{t\to 0} \int_{\{|F_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m \{\zeta_j=\zeta_j(z)\}_{j=m+1}^n} \Phi(\zeta, z, u) \cdot \det{}^{-1}\left[\frac{\partial F_k}{\partial \zeta_l}\right] \frac{\bigwedge_{k=1}^m dF_k(\zeta)}{\prod_{k=1}^m F_k(\zeta)}.$$

From the last equality we obtain the equality

$$\lim_{t\to 0} \int_{\{|F_k(\zeta)|\cdot \chi_k(\zeta)=\epsilon_k(t)\}_{k=1}^m \{\zeta_j=\zeta_j(z)\}_{j=m+1}^n} \frac{\Phi(\zeta, z, u)}{\prod_{k=1}^m F_k(\zeta)} = \mathrm{res}_{\{\mathbf{F},\pi\}}(\Phi, z),$$

which in combination with equality

$$\lim_{t\to 0} \int_{\{|F_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m} \frac{\Phi(\zeta, u)}{\prod_{k=1}^m F_k(\zeta)} = \lim_{\eta\to 0} \int_{V\cap\{|g(z)|>\eta\}} \mathrm{res}_{\{\mathbf{F},\pi\}}(\Phi, z), \tag{3.8}$$

from Theorem 1.8.3 in [10] (see also [24] Prop. 2.2) and existence of the limit in the left-hand side of (3.8), following from Theorem 1.7.2 in [10], implies the existence of the limit in the right-hand side of (3.7).

To prove the real analyticity of the limit in the right-hand side of (3.7) with respect to real variables $u_1, \ldots, u_s$ we represent those variables in terms of complex variables

$$u_r = 1/2\,(w_r + \bar{w}_r).$$

Then the resulting form can be considered as a restriction of a form analytically depending on $2s$ complex variables $\{w_1, \ldots, w_s, v_1, \ldots, v_s\}$ obtained after substitution $v_r = \bar{w}_r$. Then from Lemma 2.4 in [24] we obtain an analytic dependence of the residual integral on $\{w_1, \ldots, w_s, v_1, \ldots, v_s\}$, and, as a corollary, its real analytic dependence on the original parameters $u_1, \ldots, u_s$. □

In the next lemma using Lemma 3.1 we prove the existence of residual limits for the integrals on a sphere in $\mathbb{C}^{n+1}$, which are present in formula (3.5).

Lemma 3.2 *Let $V \subset \mathbb{CP}^n$ be a reduced complete intersection subvariety as in (1.2) satisfying (1.10), let $U \supset V$ be an open neighborhood of V in $\mathbb{CP}^n$, and let $\Phi \in \mathcal{E}_c^{(0,n-m)}(U \cap U_\alpha)$ be a differential form of homogeneity zero on $U \cap U_\alpha$ for some $\alpha \in (0, \ldots, n)$.*

Then formula

$$\lim_{t\to 0} \int_{\{|\zeta|=\tau, \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{z} \cdot \zeta \rangle^r \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta, z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \omega(\zeta), \tag{3.9}$$

where $\{\epsilon_k(t)\}_{k=1}^m$ is an admissible path, defines a differential form of homogeneity zero on U, real analytic with respect to z.

If $\Phi(\zeta) = \sum_{k=1}^m F_k^{(\alpha)}(\zeta)\Omega_k(\zeta)$ with $\Omega_k \in \mathcal{E}_c^{(0,n-m)}(U \cap U_\alpha)$, then the limit above is equal to zero.

Proof Without loss of generality we may assume that $\alpha = 0$ in (3.9). We transform the integral in this formula as follows

$$\int_{\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{z}\cdot\zeta\rangle^r \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \omega(\zeta)$$
$$= \int_{\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\langle \bar{z}\cdot\zeta\rangle^r}{\tau^2} \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]$$
$$\wedge\left(\sum_{i=0}^n \bar{\zeta}_i d\zeta_i\right) \wedge \omega'(\zeta),$$

where $\omega'(\zeta) = \sum_{i=0}^n (-1)^i \zeta_i d\zeta_0 \wedge \overset{\overset{i}{\vee}}{\cdots} \wedge d\zeta_n$.

Then, using the nonhomogeneous coordinates

$$\zeta_0,\ w_1 = \zeta_1/\zeta_0, \ldots, w_n = \zeta_n/\zeta_0 \tag{3.10}$$

and equality

$$1 + \sum_{i=1}^n w_i \cdot \bar{w}_i = \frac{\tau^2}{\zeta_0\cdot\bar{\zeta}_0} \tag{3.11}$$

on the sphere $\mathbb{S}^{2n+1}(\tau)$ of radius τ in $\mathbb{C}^{n+1}$ we represent the form $\sum_{i=0}^n \bar{\zeta}_i d\zeta_i$ in

$$\widetilde{U}_0(\tau) = \left\{\zeta\in\mathbb{C}^{n+1} : |\zeta| = \tau, \zeta_0 \neq 0\right\}$$

as

$$\sum_{i=0}^n \bar{\zeta}_i d\zeta_i = \bar{\zeta}_0 d\zeta_0 + \sum_{i=1}^n \bar{\zeta}_0\cdot\bar{w}_i\,(\zeta_0 dw_i + w_i d\zeta_0) = \bar{\zeta}_0\left(1 + \sum_{i=1}^n \bar{w}_i\cdot w_i\right) d\zeta_0$$
$$+ \zeta_0\cdot\bar{\zeta}_0\left(\sum_{i=1}^n \bar{w}_i dw_i\right) = \frac{\tau^2}{\zeta_0} d\zeta_0 + \zeta_0\cdot\bar{\zeta}_0\left(\sum_{i=1}^n \bar{w}_i dw_i\right). \tag{3.12}$$

For the form $\omega'(\zeta)$ using equalities $d\zeta_i = \zeta_0 dw_i + w_i d\zeta_0$ for $i = 1, \ldots, n$ we obtain

$$\omega'(\zeta) = \sum_{i=0}^n (-1)^i \zeta_i d\zeta_0 \wedge \overset{\overset{i}{\vee}}{\cdots} \wedge d\zeta_n = \zeta_0 \bigwedge_{j=1}^n \left(\zeta_0 dw_j + w_j d\zeta_0\right)$$
$$- \zeta_0 w_1 d\zeta_0 \wedge (\zeta_0 dw_2 + w_2 d\zeta_0) \wedge \cdots \wedge (\zeta_0 dw_n + w_n d\zeta_0)$$
$$+ \zeta_0 w_2 d\zeta_0 \wedge (\zeta_0 dw_1 + w_1 d\zeta_0) \wedge \overset{\overset{2}{\vee}}{\cdots} \wedge (\zeta_0 dw_n + w_n d\zeta_0)$$
$$+ \cdots$$

$$
\begin{aligned}
&\quad + (-1)^n \zeta_0 w_n d\zeta_0 \wedge (\zeta_0 dw_1 + w_1 d\zeta_0) \wedge \cdots \wedge (\zeta_0 dw_{n-1} + w_{n-1} d\zeta_0)\\
&= \zeta_0 \bigwedge_{j=1}^{n} \left(\zeta_0 dw_j + w_j d\zeta_0\right) + \zeta_0^n \sum_{j=1}^{n} (-1)^j w_j d\zeta_0 \wedge dw_1 \wedge \overset{j}{\overset{\vee}{\cdots}} \wedge dw_n\\
&= \zeta_0^{n+1} dw_1 \wedge \cdots \wedge dw_n.
\end{aligned} \tag{3.13}
$$

Using formulas (3.12) and (3.13), we obtain the equality

$$
\begin{aligned}
&\int_{\{|\zeta|=\tau, \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\langle \bar{z}\cdot\zeta\rangle^r}{\tau^2} \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\\
&\quad \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \left(\sum_{i=0}^{n} \bar{\zeta}_i d\zeta_i\right) \wedge \omega'(\zeta)\\
&= \int_{\{|\zeta|=\tau, \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\langle \bar{z}\cdot\zeta\rangle^r}{\tau^2} \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\\
&\quad \wedge \left(\frac{\tau^2}{\zeta_0} d\zeta_0 + \zeta_0 \cdot \bar{\zeta}_0 \left(\sum_{i=1}^{n} \bar{w}_i dw_i\right)\right) \wedge \zeta_0^{n+1} \bigwedge_{j=1}^{n} dw_j\\
&= \int_{\{|\zeta|=\tau, \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0 + \sum_{j=1}^{n} \bar{z}_j \cdot w_j\right)^r \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\\
&\quad \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \left(\zeta_0^{n+r} d\zeta_0\right) \bigwedge_{j=1}^{n} dw_j.
\end{aligned}
$$

Then, using the nonhomogeneous polynomials

$$
F_k(w) = F_k^{(0)}(w) = P_k(\zeta)/\zeta_0^{\deg P_k} \tag{3.14}
$$

and denoting $\chi(w) = \left(1 + \sum_{i=1}^n w_i \bar{w}_i\right)^{-\frac{1}{2}}$, and $\chi_k(w) = \chi(w)^{\deg P_k}$, so that

$$
|P_k(\zeta)| = |F_k(w)| \cdot |\zeta_0|^{\deg P_k} = |F_k(w)| \cdot \chi_k(w) \text{ on } \mathbb{S}^{2n+1}(1)
$$

we obtain the equality

$$
\begin{aligned}
&\int_{\{|\zeta|=\tau, \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0 + \sum_{j=1}^{n} \bar{z}_j \cdot w_j\right)^r \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\\
&\quad \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \left(\zeta_0^{n+r} d\zeta_0\right) \bigwedge_{j=1}^{n} dw_j
\end{aligned}
$$

$$= \int_{\{|\zeta|=\tau, \{|F_k(w)|\cdot\chi_k(w)=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0 + \sum_{j=1}^n \bar{z}_j \cdot w_j\right)^r \left(\zeta_0^{n+r-\sum_{k=1}^m \deg P_k} d\zeta_0\right)$$

$$\times \frac{\Phi(\zeta)}{\prod_{k=1}^m F_k(w)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \bigwedge_{j=1}^n dw_j$$

$$= \int_{\{|\zeta|=\tau, \{|F_k(w)|\cdot\chi_k(w)=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0 + \sum_{j=1}^n \bar{z}_j \cdot w_j\right)^r \left(\zeta_0^{n+r-\sum_{k=1}^m \deg P_k} d\zeta_0\right)$$

$$\times \frac{\Phi(w)}{\prod_{k=1}^m F_k(w)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \bigwedge_{j=1}^n dw_j$$

$$= i \int_0^{2\pi} e^{i\left(n+r-\sum_{k=1}^m \deg P_k+1\right)\phi_0} d\phi_0$$

$$\int_{\{w\in U_0, \{|F_k(w)|\cdot\chi_k(w)=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0 + \sum_{j=1}^n \bar{z}_j \cdot w_j\right)^r$$

$$\times \rho_0(w)^{n+r-\sum_{k=1}^m \deg P_k+1} \cdot \frac{\Phi(w)}{\prod_{k=1}^m F_k(w)}$$

$$\wedge \det\left[\bar{z}\ \overbrace{Q(e^{i\phi_0}, w, z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \bigwedge_{j=1}^n dw_j, \tag{3.15}$$

where we used the notation $\zeta_0 = \rho_0(w) \cdot e^{i\phi_0}$ with

$$\rho_0(w) = \frac{\tau}{\sqrt{1 + \sum_{i=1}^n |w_i|^2}}$$

depending on $\{w_i, \bar{w}_i\}_{i=1}^n$ on the sphere $\mathbb{S}^{2n+1}(\tau)$ according to formula (3.11).

Applying Lemma 3.1 to the interior integral in the right-hand side of (3.15) we obtain the existence of the limit in (3.9) and its real analytic dependence on z. Applying Theorem 1.7.6(2) from [10] (see also [23] Prop. 2.3) to the interior integral in the right-hand side of (3.15) we obtain that the limit in (3.9) is equal to zero if $\Phi(\zeta) = \sum_{k=1}^m F_k^{(\alpha)}(\zeta)\Omega_k(\zeta)$ with $\Omega_k \in \mathcal{E}_c^{(0,n-m)}(U \cap U_\alpha)$. □

We further simplify the right-hand side of (3.5) using the following lemma.

Lemma 3.3 *Let ϕ be a $\bar{\partial}$-closed residual current defined by a collection of forms $\left\{\Phi_\alpha^{(0,n-m)}\right\}_{\alpha=0}^n$ of homogeneity zero on a neighborhood U of the reduced subvariety*

$$V = \left\{\zeta \in \mathbb{CP}^n : P_1(\zeta) = \cdots = P_m(\zeta) = 0\right\}$$

satisfying (1.5) *and* (1.9)*, and let* $\Phi(\zeta) = \sum_{\alpha=0}^{n} \vartheta_\alpha(\zeta)\Phi_\alpha(\zeta)$ *be a differential form of homogeneity zero on* U.

Then for an arbitrary $\gamma \in \mathcal{E}_c^{(n,0)}(V, \mathcal{L})$ *the equality*

$$\lim_{\tau\to 0}\int_{T_\beta^{\delta(\tau)}} \wedge \frac{\gamma(z)}{\prod_{k=1}^m F_k^{(\beta)}(z)} \bigwedge \left(\lim_{t\to 0}\int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{z}\cdot\zeta\rangle^r \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \omega(\zeta)\right) = 0 \tag{3.16}$$

holds unless

$$r \leq \sum_{k=1}^{m} \deg P_k - n - 1. \tag{3.17}$$

Proof We notice that for all values of $a > 0$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ the sets

$$S(a) = \left\{\zeta \in \mathbb{S}^{2n+1}(a) : \left\{|P_k(\zeta)| = \epsilon_k \cdot a^{\deg P_k}\right\}_{k=1}^m\right\}$$

are real analytic subvarieties of $\mathbb{S}^{2n+1}(a)$ of real dimension $2n+1-m$ satisfying

$$c\cdot a^{2n+1-m}\cdot \text{Volume}\,(S(1)) < \text{Volume}_{2n+1-m}\,(S(a)) < C\cdot a^{2n+1-m}\cdot \text{Volume}\,(S(1)). \tag{3.18}$$

We denote

$$\Phi_\alpha(\zeta,z) = \langle\bar{z}\cdot\zeta\rangle^r \cdot \frac{\Phi_\alpha(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \omega(\zeta),$$

and apply Stokes's formula to the differential form

$$\beta(\zeta,z) = \sum_{\alpha=1}^{N} \vartheta_\alpha(\zeta)\Phi_\alpha(\zeta,z) = \sum_{\alpha=1}^{N} \langle\bar{z}\cdot\zeta\rangle^r \cdot \frac{\vartheta_\alpha(\zeta)\Phi_\alpha(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \omega(\zeta) \tag{3.19}$$

on the variety

$$\left\{\zeta \in \mathbb{C}^{n+1} : \left\{|P_k(\zeta)| = \epsilon_k \cdot |\zeta|^{\deg P_k}\right\}_{k=1}^m,\ a < |\zeta| < 1\right\}$$

with the boundary

$$\left\{\zeta : |\zeta| = a,\ \left\{|P_k(\zeta)| = \epsilon_k \cdot a^{\deg P_k}\right\}_{k=1}^m\right\} \bigcup \left\{\zeta : |\zeta| = 1,\ \{|P_k(\zeta)| = \epsilon_k\}_{k=1}^m\right\}.$$

Then using equality (1.9) we obtain the equality

$$\int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \beta(\zeta,z) - \int_{\left\{|\zeta|=a,\ \left\{|P_k(\zeta)|=\epsilon_k(t)\cdot a^{\deg P_k}\right\}_{k=1}^m\right\}} \beta(\zeta,z)$$
$$= \sum_{\alpha=1}^{N}\sum_{j=1}^{m}\int_a^1 d\tau \int_{\left\{|\zeta|=\tau,\left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}} F_j^{(\alpha)}(\zeta)\cdot\beta_j^{(\alpha)}(\zeta,z)$$
$$+\sum_{\alpha=1}^{N}\int_a^1 d\tau \int_{\left\{|\zeta|=\tau,\left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}} \bar{\partial}\vartheta_\alpha(\zeta)\wedge\Big(d|\zeta|\rfloor\ \Phi_\alpha(\zeta,z)\Big) \tag{3.20}$$

for arbitrary t and $0 < a < 1$.

Using estimate (3.18) and the homogeneity property

$$\Phi_\beta(t\cdot\zeta) = \bar{t}^{-(n-m)}\cdot\Phi_\beta(\zeta) \tag{3.21}$$

of the coefficients of $\Phi^{(0,n-m)}$ from Proposition 1.1 in [21] we obtain that if

$$r+n+1-\sum_{k=1}^{m}\deg P_k > 0, \tag{3.22}$$

then

$$\left|\int_{\left\{|\zeta|=a,\ \left\{|P_k(\zeta)|=\epsilon_k(t)\cdot a^{\deg P_k}\right\}_{k=1}^m\right\}} \beta(\zeta,z)\right|$$
$$< C'(\epsilon)\cdot a^{r+2n+1-m-(n-m)-\sum_{k=1}^m \deg P_k} \longrightarrow 0$$

as t is fixed and $a\to 0$.

For the first sum of integrals in the right-hand side of (3.20) we have

$$\left|\int_a^1 d\tau \int_{\left\{|\zeta|=\tau,\ \left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}} F_j^{(\alpha)}(\zeta)\cdot\beta_j^{(\alpha)}(\zeta,z)\right|$$
$$< C\int_a^1 d\tau\cdot\tau^{r+2n+1-m-(n-m)-\sum_{k=1}^m \deg P_k}$$
$$\times\int_{\left\{|\zeta|=\tau,\ \left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}} F_j^{(\alpha)}(\zeta)\cdot\Big(d|\zeta|\rfloor\,\beta_j^{(\alpha)}(\zeta,z)\Big)$$
$$< C'(\epsilon)\frac{\Big(1-a^{r+n+2-\sum_{k=1}^m \deg P_k}\Big)}{r+n+2-\sum_{k=1}^m \deg P_k}\longrightarrow 0 \tag{3.23}$$

as $t\to 0$, since $a<1$, condition (3.22) is satisfied, and $C'(\epsilon)\to 0$ as $t\to 0$ by Lemma 3.2.

For the second sum of integrals in the right-hand side of (3.20) using equality $\sum_\alpha \bar\partial \vartheta_\alpha = 0$, equality (1.5) for residual currents of homogeneity zero, and Lemma 3.2 we obtain as in (3.23) that the limit of this sum is also zero as $t \to 0$.

This completes the proof of the lemma. □

Combining the results of Lemmas 3.2 and 3.3 with formula (3.1) we obtain the following

Proposition 3.4 *Let $V \subset \mathbb{CP}^n$ be a reduced complete intersection subvariety as in (1.2) satisfying conditions (1.10), let ϕ be a $\bar\partial$-closed residual current defined by a collection of forms $\left\{\Phi_\alpha^{(0,n-m)}\right\}_{\alpha=0}^n$ of homogeneity zero on a neighborhood U of V, and let $\Phi(\zeta) = \sum_{\alpha=0}^n \vartheta_\alpha(\zeta)\Phi_\alpha(\zeta)$.*

Then for an admissible path $\epsilon(t)$ and operator $L_{n-m}^{\epsilon(t)}$ the following equality is satisfied

$$\begin{aligned}
\lim_{t\to 0} L_{n-m}^{\epsilon(t)}[\Phi](z) &= \sum_{0\le r\le d-n-1} C(n,m,d,r) \lim_{t\to 0} \int_{\{|\zeta|=1,\ \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar z \cdot \zeta\rangle^r \\
&\quad \times \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \det\left[\bar z\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar z}^{n-m}\right] \wedge \omega(\zeta) \\
&= \sum_{0\le r\le d-n-1} C(n,m,d,r)\,(2\pi)^m\, i^{m+1} \sum_{\alpha=0}^n \int_0^{2\pi} d\phi_\alpha \\
&\quad \times Res_V \left\{ \left\langle \bar z \cdot w^{(\alpha)}\right\rangle^r \Phi_\alpha(\zeta) \wedge \det\left[\bar z\ \overbrace{Q(e^{i\phi_\alpha}, w^{(\alpha)}, z)}^{m}\ \overbrace{d\bar z}^{n-m}\right] \bigwedge_{j=1}^n dw_j^{(\alpha)} \right\},
\end{aligned} \tag{3.24}$$

where $d = \sum_{k=1}^m \deg P_k$, $\langle \bar z \cdot w^{(\alpha)}\rangle = \bar z_0 + \sum_{j=1}^n \bar z_j w_j^\alpha$, and Res_V is the residue of Coleff–Herrera [10].

4 Estimates for the Solution Operator

In this section we analyze the solution operators I_q^ϵ, specifically estimates for limits of those operators as $\epsilon \to 0$. In the estimates below we slightly abuse the notation by using the same letter C in all estimates for constants that do not depend on ϵ, τ and η.

In the next lemma we simplify expression (2.7) for I_q^ϵ by eliminating the first integral in its right-hand side.

Lemma 4.1 *Let $V \subset \mathbb{CP}^n$ be a reduced subvariety as in (1.2), and let g_α be an analytic function on $U_\alpha \subset \mathbb{CP}^n$ as in Theorem 1.*

Then for a fixed $\eta > 0$ and an arbitrary $z \in U_\alpha$, such that $|g_\alpha(z)| > \eta$, we have the following equality

$$\lim_{t\to 0}\int_{U^{\epsilon(t)}\times[0,1]}\vartheta_\beta(\zeta)\Phi(\zeta)\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)=0, \quad (4.1)$$

where $\epsilon(t)$ is an admissible path, and $\beta \in (0,\dots,n)$.

Proof For a fixed z with $|g_\alpha(z)| > \eta$ we choose $\tau > 0$ so that for $|\zeta - z| < \tau$ we have $|g_\alpha(\zeta)| > \eta/2$. Then we represent the integral in (4.1) as

$$\begin{aligned}
&\int_{U^\epsilon\times[0,1]}\vartheta_\beta(\zeta)\Phi(\zeta)\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)\\
&=\int_{(U^\epsilon\cap\{|\zeta-z|<\tau\})\times[0,1]}\vartheta_\beta(\zeta)\Phi(\zeta)\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)\\
&\quad+\int_{(U^\epsilon\cap\{|\zeta-z|>\tau\})\times[0,1]}\vartheta_\beta(\zeta)\Phi(\zeta)\\
&\quad\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta). \qquad (4.2)
\end{aligned}$$

To estimate the first integral in the right-hand side of (4.2) we introduce the coordinates

$$\begin{cases} t=\mathrm{Im}B(\zeta,z)=\mathrm{Im}B^*(\zeta,z),\\ \rho_k=|P_k(\zeta)| \ \text{for}\ k=1\dots,m, \end{cases} \qquad (4.3)$$

and obtain the following estimate

$$\begin{aligned}
&\left|\int_{(U^\epsilon\cap\{|\zeta-z|<\tau\})\times[0,1]}\vartheta_\beta(\zeta)\Phi(\zeta)\wedge\omega'_{q-1}\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)\right|\\
&\le C\cdot\int_0^\tau dt\int_0^\epsilon\rho_1 d\rho_1\cdots\int_0^\epsilon\rho_m d\rho_m\int_0^\tau\frac{r^{2(n-m)}dr}{\left(t+\sum_{k=1}^m\rho_k^2+r^2\right)^{n+1}}\\
&\le C\cdot\int_0^\tau dt\int_0^\epsilon\rho_1 d\rho_1\cdots\int_0^\epsilon\rho_m d\rho_m\int_0^\tau\frac{dr}{\left(t+\sum_{k=1}^m\rho_k^2+r^2\right)^{m+1}}\\
&\le C\cdot\int_0^\epsilon\rho_1 d\rho_1\int_0^\tau dt\int_0^\tau\frac{dr}{\left(t+\rho_1^2+r^2\right)^2}\\
&\le C\cdot\int_0^\epsilon d\rho_1\int_0^\tau\frac{\rho_1 dr}{\rho_1^2+r^2}\le C\cdot\int_0^\epsilon d\rho_1\int_0^\infty\frac{du}{1+u^2}\le C\cdot\epsilon\ \to\ 0 \qquad (4.4)
\end{aligned}$$

as $\epsilon \to 0$.

For the second integral in the right-hand side of (4.2) we have

$$\left|\int_{(U^{\epsilon}\cap\{|\zeta-z|>\tau\})\times[0,1]} \vartheta_{\beta}(\zeta)\Phi(\zeta)\wedge\omega'_q\left((1-\lambda)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}\right)\wedge\omega(\zeta)\right|\to 0 \tag{4.5}$$

as $\epsilon\to 0$ because the integrand in (4.5) is uniformly bounded for $\{\zeta : |\zeta-z|>\tau\}$, and the volume of U^{ϵ} goes to zero as $\epsilon\to 0$.

Combining estimates (4.4) and (4.5) we obtain the statement of the lemma. □

To estimate the rest of the integrals in (2.7) we transform those integrals by integrating the kernels with respect to variables $\lambda,\ \mu_j\in\Delta_J$ for $j\in J$ and obtain

$$\begin{aligned}
&\sum_{|J|\geq 1}\int_{\Gamma^{\epsilon}_J\times\Delta_J}\Phi(\zeta)\\
&\quad\wedge\omega'_{q-1}\left((1-\lambda-\sum_{k=1}^{m}\mu_k)\frac{\bar{z}}{B^*(\zeta,z)}+\lambda\frac{\bar{\zeta}}{B(\zeta,z)}+\sum_{j\in J}\mu_j\frac{Q_j(\zeta,z)}{P_j(\zeta)-P_j(z)}\right)\wedge\omega(\zeta)\\
&\quad=\sum_{|J|\geq 1}C(n,q,|J|)\int_{\Gamma^{\epsilon}_J}\Phi(\zeta)\\
&\quad\bigwedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{\frac{Q_j(\zeta,z)}{P_j(\zeta)-P_j(z)}}^{|J|}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q}\right]\wedge\omega(\zeta).
\end{aligned}\tag{4.6}$$

Then we further transform the integrals in the right-hand side of (4.6) using series representation (3.3) and obtain

$$\begin{aligned}
&\int_{\Gamma^{\epsilon}_J}\Phi(\zeta)\wedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{\frac{Q_j(\zeta,z)}{P_j(\zeta)-P_j(z)}}^{|J|}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q}\right]\wedge\omega(\zeta)\\
&\quad=\sum_{|A|\geq 0}\int_{\Gamma^{\epsilon}_J}\frac{\Phi(\zeta)}{\prod_{j\in J}P_j(\zeta)}\cdot\frac{P^A(z)}{P^A(\zeta)}\\
&\quad\wedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{|J|}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q}\right]\wedge\omega(\zeta),
\end{aligned}\tag{4.7}$$

where we assume that $J=(j_1,\ldots,j_p)$, denote by $A=(a_1,\ldots,a_p)$ a multiindex, by

$$P^A(\zeta) = P_{j_1}^{a_1}(\zeta)\cdots P_{j_p}^{a_p}(\zeta),$$

and by $|A| = a_1 + \cdots + a_p$.

As before in (3.4), using Theorem 1.7.6(2) from [10] (see also [23] Prop. 2.3) we obtain that the residual currents defined by the terms in the right-hand side of (4.7) with $|A| \geq 1$ are zero-currents from the point of view of (2.12), and therefore we can simplify formula (2.7) for $I_q^{\epsilon(t)}[\Phi]$ as follows

$$I_q^{\epsilon(t)}[\Phi](z) = \sum_{|J|\geq 1} C(n,q,|J|)\int_{\Gamma_J^{\epsilon(t)}} \frac{\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)} \bigwedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{|J|}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q}\right]\wedge\omega(\zeta). \quad (4.8)$$

In the next lemma we further simplify formula (4.8) for $I_q^{\epsilon(t)}$.

Lemma 4.2 *Let $V \subset \mathbb{CP}^n$ be a reduced subvariety as in (1.2), and let g_α be an analytic function on $U_\alpha \subset \mathbb{CP}^n$ as in Theorem 1.*

Then for a fixed $\eta > 0$, an arbitrary $z \in U_\alpha$, such that $|g_\alpha(z)| > \eta$, and J, such that $|J| = p < m$ we have

$$\lim_{t\to 0}\int_{\Gamma_J^{\epsilon(t)}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)} \bigwedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{|J|}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q}\right]\wedge\omega(\zeta) = 0, \quad (4.9)$$

for an admissible path $\epsilon(t)$, and $\beta \in (0,\ldots,n)$.

Proof For a fixed z with $|g_\alpha(z)| > \eta$ we choose $\tau > 0$ as in Lemma 4.1, so that for $|\zeta - z| < \tau$ we have $|g_\alpha(\zeta)| > \eta/2$. Then we represent the integral in (4.9) as

$$\int_{\Gamma_J^{\epsilon(t)}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)} \bigwedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{|J|}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q}\right]\wedge\omega(\zeta)$$

$$= \int_{\Gamma_J^{\epsilon(t)}\cap\{|\zeta-z|<\tau\}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)}$$

$$\bigwedge \det \left[\frac{\bar{z}}{B^*(\zeta,z)} \ \frac{\bar{\zeta}}{B(\zeta,z)} \ \overbrace{Q_j(\zeta,z)}^{|J|} \ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1} \ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q} \right] \wedge \omega(\zeta)$$

$$+ \int_{\Gamma_J^{\epsilon(t)} \cap \{|\zeta - z| > \tau\}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)}$$

$$\bigwedge \det \left[\frac{\bar{z}}{B^*(\zeta,z)} \ \frac{\bar{\zeta}}{B(\zeta,z)} \ \overbrace{Q_j(\zeta,z)}^{|J|} \ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1} \ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q} \right] \wedge \omega(\zeta). \quad (4.10)$$

For the first integral in the right-hand side of (4.10) using the coordinates from (4.3) and estimate

$$\left|\bar{z} \wedge \bar{\zeta}\right| \leq C \cdot |\zeta - z|$$

we obtain

$$\Bigg| \int_{\Gamma_J^{\epsilon(t)} \cap \{|\zeta - z| < \tau\}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)}$$

$$\bigwedge \det \left[\frac{\bar{z}}{B^*(\zeta,z)} \ \frac{\bar{\zeta}}{B(\zeta,z)} \ \overbrace{Q_j(\zeta,z)}^{|J|} \ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1} \ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q} \right] \wedge \omega(\zeta) \Bigg|$$

$$\leq C \cdot \int_0^\tau dt \int_0^{2\pi} d\phi_{j_1} \cdots \int_0^{2\pi} d\phi_{j_p} \int_0^\epsilon \rho_1 d\rho_1 \cdots \int_0^\epsilon \rho_{m-p} d\rho_{m-p}$$

$$\times \int_0^\tau \frac{r^{2(n-m)} dr}{\left(t + \sum_{k=1}^{m-p} \rho_k^2 + r^2\right)^{n-p+1}}$$

$$\leq C \cdot \int_0^\tau dt \int_0^\epsilon \rho_1 d\rho_1 \cdots \int_0^\epsilon \rho_{m-p} d\rho_{m-p} \int_0^\tau \frac{dr}{\left(t + \sum_{k=1}^{m-p} \rho_k^2 + r^2\right)^{m-p+1}}$$

$$\leq C \cdot \int_0^\tau dt \int_0^\epsilon \rho_1 d\rho_1 \int_0^\tau \frac{dr}{\left(t + \rho_1^2 + r^2\right)^2}$$

$$\leq C \cdot \int_0^\epsilon \rho_1 d\rho_1 \int_0^\tau \frac{dr}{\left(\rho_1^2 + r^2\right)} \leq C \cdot \int_0^\epsilon \rho_1 d\rho_1 \int_0^\tau \frac{dr}{\rho_1^2 \left(1 + (r/\rho_1)^2\right)}$$

$$\leq C \cdot \int_0^\epsilon d\rho_1 \int_0^\infty \frac{du}{1+u^2} \leq C \cdot \epsilon \to 0, \quad (4.11)$$

as $\epsilon \to 0$.

For the second integral in (4.10) we may assume without loss of generality that $\beta = 0$ and Φ is a smooth differential form with support in $\{|\zeta - z| > \tau\}$, and rewrite this integral using polynomials $\{F_k\}_1^m$ from (3.14) as

$$\int_{\Gamma_J^{\epsilon(t)} \cap \{|\zeta - z| > \tau\}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{j\in J} P_j(\zeta)}$$
$$\bigwedge \det \left[\frac{\bar{z}}{B^*(\zeta,z)} \; \frac{\bar{\zeta}}{B(\zeta,z)} \; \overbrace{Q_j(\zeta,z)}^{|J|} \; \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1} \; \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-|J|-q} \right] \wedge \omega(\zeta)$$
$$= \lim_{t\to 0} \int_{\left\{ \begin{array}{l} |F_j(\zeta)| \cdot \chi_j(\zeta) = \epsilon_j(t) \text{ for } j \in J, \\ |F_k(\zeta)| \cdot \chi_k(\zeta) < \epsilon_k(t) \text{ for } k \notin J \end{array} \right\}} \frac{\Psi(\zeta,z)}{\prod_{j\in J} F_j(\zeta)}$$

with a smooth form $\Psi(\zeta, z)$ analytically depending on $(z, \bar{z})$ for $z \in \{U_\alpha : |g_\alpha(z)| > \eta\}$ and compact support in $\{|\zeta - z| > \tau\}$.

Then as in Lemma 3.1 we obtain equality

$$\lim_{t\to 0} \int_{\left\{ \begin{array}{l} |F_j(\zeta)| \cdot \chi_j(\zeta) = \epsilon_j(t) \text{ for } j \in J, \\ |F_k(\zeta)| \cdot \chi_k(\zeta) < \epsilon_k(t) \text{ for } k \notin J \end{array} \right\}} \frac{\Psi(\zeta,z)}{\prod_{j\in J} F_j(\zeta)}$$
$$= \lim_{\eta\to 0} \lim_{t\to 0} \int_{\left\{ |g_\beta(\zeta)| > \eta, \begin{array}{l} |F_j(\zeta)| \cdot \chi_j(\zeta) = \epsilon_j(t) \text{ for } j \in J, \\ |F_k(\zeta)| \cdot \chi_k(\zeta) < \epsilon_k(t) \text{ for } k \notin J \end{array} \right\}} \frac{\Psi(\zeta,z)}{\prod_{j\in J} F_j(\zeta)}$$

reducing the proof of (4.9) to the proof of equality

$$\lim_{t\to 0} \int_{\left\{ |g_\beta(\zeta)| > \eta, \begin{array}{l} |F_j(\zeta)| \cdot \chi_j(\zeta) = \epsilon_j(t) \text{ for } j \in J, \\ |F_k(\zeta)| \cdot \chi_k(\zeta) < \epsilon_k(t) \text{ for } k \notin J \end{array} \right\}} \frac{\Psi(\zeta,z)}{\prod_{j\in J} F_j(\zeta)} = 0. \tag{4.12}$$

In the proof of (4.12) we use the method used in Lemma 2.3 in [24]. Namely, localizing the problem we assume that the set $V \cap \{|g_\beta(z)| > \eta\}$ is a submanifold in a polydisk $\mathcal{P}^n$ of the form

$$S = \{u \in \mathcal{P}^n : u_1 = \cdots = u_m = 0\},$$

and the integral in (4.12) can be represented as

$$\lim_{t\to 0} \int_{\left\{ \begin{array}{l} |u_j| \cdot \chi_j(u) = \epsilon_j(t) \text{ for } j \in J, \\ |u_k| \cdot \chi_k(u) < \epsilon_k(t) \text{ for } k \notin J \end{array} \right\}} \frac{f(u,z)}{\prod_{j\in J} u_j} = 0.$$

□

In the next lemma we obtain an explicit form of a solution operator for $\bar{\partial}$-equation on residual currents.

Lemma 4.3 *Let $V \subset \mathbb{CP}^n$ be a reduced subvariety as in (1.2), and let g_α be an analytic function on $U_\alpha \subset \mathbb{CP}^n$ as in Theorem 1.*

Then for a fixed $\eta > 0$, $J = (1, \dots, m)$, an admissible path $\epsilon(t)$, and $\beta \in (0, \dots, n)$ we have:

(i) *the limits of integrals*

$$\lim_{t\to 0}\int_{\Gamma_J^{\epsilon(t)}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \bigwedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-m-q}\right] \wedge \omega(\zeta) \quad (4.13)$$

are well-defined continuous functions on $\{U_\alpha : |g_\alpha(z)| > \eta\}$,

(ii) *if*

$$\Phi(\zeta)\Big|_{U_\beta} = \sum_{k=1}^m F_k^{(\beta)}(\zeta)\cdot\Psi_k(\zeta), \quad (4.14)$$

where $\left\{F_k^{(\beta)}\right\}_1^m$ are the polynomials from (3.14), then the limit in (4.13) is equal to zero.

Proof For a fixed z with $|g_\alpha(z)| > \eta$ we choose $\tau > 0$ so that for $|\zeta - z| < \tau$ we have $|g_\alpha(\zeta)| > \eta/2$. Then for $\epsilon < \tau$ we represent the integral in (4.13) as

$$\int_{\Gamma_J^\epsilon} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1} P_k(\zeta)} \bigwedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-m-q}\right] \wedge \omega(\zeta)$$

$$= \int_{\Gamma_J^\epsilon \cap\{|\zeta-z|<\sqrt{\epsilon}\}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \bigwedge \det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-m-q}\right] \wedge \omega(\zeta)$$

$$+\int_{\Gamma_J^\epsilon\cap\{|\zeta-z|>\sqrt{\epsilon}\}}\frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$

$$\bigwedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q_j(\zeta,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta). \quad (4.15)$$

For the first integral in the right-hand side of (4.15) we have

$$\left|\int_{\Gamma_J^\epsilon\cap\{|\zeta-z|<\sqrt{\epsilon}\}}\frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\right.$$

$$\left.\bigwedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta)\right|$$

$$\leq C\cdot\int_0^{\sqrt{\epsilon}}dt\int_0^{2\pi}d\phi_1\cdots\int_0^{2\pi}d\phi_m\int_0^{\sqrt{\epsilon}}\frac{r^{2(n-m)}dr}{\left(t+r^2\right)^{n-m+1}}$$

$$\leq C\cdot\int_0^{\sqrt{\epsilon}}dt\int_0^{\sqrt{\epsilon}}\frac{dr}{t+r^2}\leq C\cdot\int_0^{\sqrt{\epsilon}}\frac{dt}{\sqrt{t}}\int_0^{\infty}\frac{du}{1+u^2}\leq C\cdot\sqrt[4]{\epsilon}\to 0, \quad (4.16)$$

as $\epsilon\to 0$.

For the second integral in (4.15) we denote

$$K(\zeta,z) = \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$

$$\bigwedge\det\left[\frac{\bar{z}}{B^*(\zeta,z)}\ \frac{\bar{\zeta}}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar{\zeta}}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta)$$

and consider $z^{(1)}, z^{(2)}$ such that $|z^{(1)}-z^{(2)}|<\epsilon/4$.

Then using relations

$$|\zeta-z|>\sqrt{\epsilon}\Longrightarrow|B(\zeta,z)|>\epsilon/2,$$
$$|z^{(1)}-z^{(2)}|<\epsilon/4,\quad\left|B(\zeta,z^{(1)})\right|>\epsilon/2\Longrightarrow\left|B(\zeta,z^{(2)})\right|>\epsilon/4,$$

we obtain

$$
\begin{aligned}
&\left|\int_{\Gamma_J^\epsilon\cap\{|\zeta-z^{(1)}|>\sqrt{\epsilon}\}}\left(K(\zeta,z^{(1)})-K(\zeta,z^{(2)})\right)\right|\\
&\quad\leq\left|\int_{\Gamma_J^\epsilon\cap\{|B(\zeta,z^{(1)})|>\epsilon/2\}}\left(K(\zeta,z^{(1)})-K(\zeta,z^{(2)})\right)\right|\\
&\quad\leq C\epsilon\cdot\int_0^A dt\int_0^{2\pi}d\phi_1\cdots\int_0^{2\pi}d\phi_m\int_0^A\frac{r^{2(n-m)}dr}{\left(\epsilon+t+r^2\right)^{n-m+2}}\\
&\quad\leq C\epsilon\cdot\int_0^A dt\int_0^A\frac{dr}{\left(\epsilon+t+r^2\right)^2}\leq C\epsilon\cdot\int_0^A\frac{dr}{\left(\epsilon+r^2\right)}\\
&\quad\leq C\epsilon\cdot\int_0^A\frac{dr}{\epsilon\left(1+\left(r/\sqrt{\epsilon}\right)^2\right)}\leq C\sqrt{\epsilon}\cdot\int_0^\infty\frac{du}{1+u^2}\leq C\sqrt{\epsilon}.
\end{aligned}
\tag{4.17}
$$

From estimates (4.16) and (4.17) we obtain claim (i) of the lemma.

Claim (ii) of the lemma for the first integral in (4.15) follows from estimate (4.16), and for the second integral in (4.15) it follows from Lemma 3.1 with additional application of Theorem 1.7.6(2) from [10] (see also [23] Prop. 2.3). □

In the proposition below we prove smoothness of limits of integrals in (4.13) under assumption of smoothness of the forms defining current ϕ.

Proposition 4.4 *Let $V\subset\mathbb{CP}^n$ be a reduced subvariety as in (1.2), let g_α be an analytic function on $U_\alpha\subset\mathbb{CP}^n$ as in Theorem 1, and let the $\bar\partial$-closed current ϕ be defined by a C^∞ form Φ.*

Then for a fixed $\eta>0$, $J=(1,\ldots,m)$, an admissible path $\epsilon(t)$, the limits of integrals in (4.13) represent C^∞ forms on $\{U_\alpha:\ |g_\alpha(z)|>\eta\}$.

Proof For a fixed z with $|g_\alpha(z)|>\eta$ we choose $\tau>0$ as in Lemma 4.1, so that for $|\zeta-z|<\tau$ we have $|g_\alpha(\zeta)|>\eta/2$. Then, as in Lemma 4.2, we represent the integral in (4.9) as

$$
\int_{\Gamma_J^{\epsilon(t)}}\frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\bigwedge\det\left[\frac{\bar z}{B^*(\zeta,z)}\ \frac{\bar\zeta}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar z}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar\zeta}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta)
$$

$$
=\int_{\Gamma_J^{\epsilon(t)}\cap\{|\zeta-z|<\tau\}}\frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}
$$

$$
\bigwedge\det\left[\frac{\bar z}{B^*(\zeta,z)}\ \frac{\bar\zeta}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar z}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar\zeta}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta)
$$

$$+\int_{\Gamma_J^{\epsilon(t)}\cap\{|\zeta-z|>\tau\}}\frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$

$$\bigwedge\det\left[\frac{\bar z}{B^*(\zeta,z)}\ \frac{\bar\zeta}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar z}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar\zeta}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta). \tag{4.18}$$

From Theorem 1.7.2 in [10] (see also [24] Prop. 2.2) we obtain that the second integral in the right-hand side of (4.18) represents a C^∞ form with respect to z, since the functions $|B(\zeta,z)|$, $|B^*(\zeta,z)|$ are separated from zero uniformly with respect to z for $|\zeta-z|>\tau$.

To prove the statement of the proposition for the first integral in the right-hand side of (4.18) we use the following lemma.

Lemma 4.5 *Let $V\subset\mathbb{CP}^n$ be a reduced subvariety as in (1.2), let $z\in V$ be a nonsingular point, and let Φ be a C^{l+1} form with compact support in a neighborhood $U_z\ni z$, such that $U_z\cap \mathrm{sing}V=\emptyset$.*

Then for $J=(1,\dots,m)$ and an admissible path $\epsilon(t)$ the limit

$$\lim_{t\to0}\int_{\Gamma_J^{\epsilon(t)}\cap U}\frac{\Phi(\zeta)\cdot\overbrace{Q(\zeta,z)}^{m}}{\prod_{k=1}^m P_k(\zeta)}\wedge\frac{\overbrace{d\bar z}^{q-1}\wedge\overbrace{d\bar\zeta}^{n-m-q}}{(B^*(\zeta,z))^q\,(B(\zeta,z))^{n-m-q+1}}\wedge\omega(\zeta), \tag{4.19}$$

defines a C^l-form on some neighborhood of V.

Proof Using equalities

$$\frac{\partial}{\partial\bar z_k}\left[\frac{1}{(B^*(\zeta,z))^p}\right]=\frac{\partial}{\partial\bar z_k}\left[\frac{1}{\left(-1+\sum_{j=0}^n\bar z_j\zeta_j\right)^p}\right]=-p\left[\frac{\zeta_k}{(B^*(\zeta,z))^{p+1}}\right],$$

$$\frac{\partial}{\partial\bar z_k}\left[\frac{1}{(B(\zeta,z))^p}\right]=\frac{\partial}{\partial\bar z_k}\left[\frac{1}{\left(1-\sum_{j=0}^n\bar\zeta_j z_j\right)^p}\right]=0, \tag{4.20}$$

we obtain

$$\frac{\partial^l}{\partial\bar z_k^l}\int_{\Gamma_J^{\epsilon(t)}\cap U}\frac{\Phi(\zeta)\cdot\overbrace{Q(\zeta,z)}^{m}}{\prod_{k=1}^m P_k(\zeta)}\wedge\frac{\overbrace{d\bar z}^{q-1}\wedge\overbrace{d\bar\zeta}^{n-m-q}}{(B^*(\zeta,z))^q\,(B(\zeta,z))^{n-m-q+1}}\wedge\omega(\zeta)$$

$$=\int_{\Gamma_J^{\epsilon(t)}\cap U}\frac{\Phi(\zeta)\cdot\overbrace{Q(\zeta,z)}^{m}}{\prod_{k=1}^m P_k(\zeta)}\cdot\frac{\partial^l}{\partial\bar z_k^l}\left[\frac{1}{(B^*(\zeta,z))^q}\right]\wedge\frac{\overbrace{d\bar z}^{q-1}\wedge\overbrace{d\bar\zeta}^{n-m-q}}{(B(\zeta,z))^{n-m-q+1}}\wedge\omega(\zeta)$$

$$= (-1)^l q \cdots (q+l-1) \int_{\Gamma_J^{\epsilon(t)} \cap U} \frac{\Phi(\zeta) \cdot \overbrace{Q(\zeta, z)}^{m}}{\prod_{k=1}^m P_k(\zeta)}$$

$$\wedge \frac{\zeta_k^l \cdot \overbrace{d\bar{z}}^{q-1} \wedge \overbrace{d\bar{\zeta}}^{n-m-q}}{(B^*(\zeta, z))^{q+l} (B(\zeta, z))^{n-m-q+1}} \wedge \omega(\zeta).$$

To estimate the right-hand side of the equality above we assume without loss of generality that for some $i \in (0, \ldots, n)$ we have $z_i \neq 0$ in U_z. Then using equalities

$$\frac{\partial}{\partial \zeta_k} \left[\frac{1}{(B^*(\zeta, z))^p} \right] = \frac{\partial}{\partial \zeta_k} \left[\frac{1}{\left(-1 + \sum_{j=0}^n \bar{z}_j \zeta_j\right)^p} \right] = -p \left[\frac{\bar{z}_k}{(B^*(\zeta, z))^{p+1}} \right],$$

$$\frac{\partial}{\partial \zeta_k} \left[\frac{1}{(B(\zeta, z))^p} \right] = \frac{\partial}{\partial \zeta_k} \left[\frac{1}{\left(1 - \sum_{j=0}^n \bar{\zeta}_j z_j\right)^p} \right] = 0, \tag{4.21}$$

we obtain for $q > 1$

$$(-1)^l q \cdots (q+l-1) \bar{z}_i^{l+1} \int_{\Gamma_J^{\epsilon(t)} \cap U} \frac{\Phi(\zeta) \cdot \overbrace{Q(\zeta, z)}^{m}}{\prod_{k=1}^m P_k(\zeta)}$$

$$\wedge \frac{\zeta_k^l \cdot \overbrace{d\bar{z}}^{q-1} \wedge \overbrace{d\bar{\zeta}}^{n-m-q}}{(B^*(\zeta, z))^{q+l} (B(\zeta, z))^{n-m-q+1}} \wedge \omega(\zeta)$$

$$= -\frac{1}{q-1} \int_{\Gamma_J^{\epsilon(t)} \cap U} \frac{\Phi(\zeta) \cdot \overbrace{Q(\zeta, z)}^{m}}{\prod_{k=1}^m P_k(\zeta)} \frac{\partial^{l+1}}{\partial \zeta_i^{l+1}} \left[\frac{1}{(B^*(\zeta, z))^{q-1}} \right]$$

$$\wedge \frac{\zeta_k^l \cdot \overbrace{d\bar{z}}^{q-1} \wedge \overbrace{d\bar{\zeta}}^{n-m-q}}{(B(\zeta, z))^{n-m-q+1}} \wedge \omega(\zeta)$$

$$= \frac{(-1)^l}{q-1} \int_{\Gamma_J^{\epsilon(t)} \cap U} \frac{\partial^{l+1}}{\partial \zeta_i^{l+1}} \left[\frac{\zeta_k^l \Phi(\zeta) \cdot \overbrace{Q(\zeta, z)}^{m}}{\prod_{k=1}^m P_k(\zeta)} \right]$$

$$\wedge \frac{\overbrace{d\bar{z}}^{q-1} \wedge \overbrace{d\bar{\zeta}}^{n-m-q}}{(B^*(\zeta, z))^{q-1} (B(\zeta, z))^{n-m-q+1}} \wedge \omega(\zeta),$$

where in the last equality we used integration by parts.

Estimate similar to (4.16) produces the following estimate of the integral in the right-hand side of equality above

$$\left|\int_{\Gamma_J^{\epsilon(t)}\cap U}\frac{\partial^{l+1}}{\partial\zeta_i^{l+1}}\left[\frac{\zeta_k^l\Phi(\zeta)\cdot\overbrace{Q(\zeta,z)}^{m}}{\prod_{k=1}^m P_k(\zeta)}\right]\wedge\frac{\overbrace{d\bar{z}}^{q-1}\wedge\overbrace{d\bar{\zeta}}^{n-m-q}}{(B^*(\zeta,z))^{q-1}\,(B(\zeta,z))^{n-m-q+1}}\wedge\omega(\zeta)\right|$$
$$\leq C\cdot\|\Phi\|_{C^{l+1}}\int_0^\tau dt\int_0^{2\pi}d\phi_1\cdots\int_0^{2\pi}d\phi_m\int_0^\tau\frac{r^{2(n-m)-1}}{\left(t+r^2\right)^{n-m}}dr$$
$$\leq C\cdot\|\Phi\|_{C^{l+1}}\int_0^\tau dt\int_0^\tau\frac{rdr}{t+r^2}\leq C\cdot\|\Phi\|_{C^{l+1}}.$$

We notice that the same estimate is valid for $q=1$ if we use the fact that the functions

$$\log\left(-1+\sum_{j=0}^n\bar{z}_j\zeta_j\right),\qquad\log\left(1-\sum_{j=0}^n\bar{\zeta}_jz_j\right)$$

are well defined on $\mathbb{S}(1)\times\mathbb{S}(1)$, and satisfy equations similar to the ones used above.

We obtain similar estimates for mixed derivatives $\dfrac{\partial^l}{\partial z^s\partial\bar{z}^p}$ using together with equalities (4.20) and (4.21) the equalities

$$\frac{\partial}{\partial z_k}\left[\frac{1}{(B^*(\zeta,z))^p}\right]=\frac{\partial}{\partial z_k}\left[\frac{1}{\left(-1+\sum_{j=0}^n\bar{z}_j\zeta_j\right)^p}\right]=0,$$
$$\frac{\partial}{\partial z_k}\left[\frac{1}{(B(\zeta,z))^p}\right]=\frac{\partial}{\partial z_k}\left[\frac{1}{\left(1-\sum_{j=0}^n\bar{\zeta}_jz_j\right)^p}\right]=p\left[\frac{\bar{\zeta}_k}{(B(\zeta,z))^{p+1}}\right],$$

and

$$\frac{\partial}{\partial\bar{\zeta}_k}\left[\frac{1}{(B^*(\zeta,z))^p}\right]=\frac{\partial}{\partial\bar{\zeta}_k}\left[\frac{1}{\left(-1+\sum_{j=0}^n\bar{z}_j\zeta_j\right)^p}\right]=0,$$
$$\frac{\partial}{\partial\bar{\zeta}_k}\left[\frac{1}{(B(\zeta,z))^p}\right]=\frac{\partial}{\partial\bar{\zeta}_k}\left[\frac{1}{\left(1-\sum_{j=0}^n\bar{\zeta}_jz_j\right)^p}\right]=p\left[\frac{z_k}{(B(\zeta,z))^{p+1}}\right].$$

Combining formula (4.18) with the statement of Lemma 4.5 we obtain the statement of Proposition 4.4. □

Summarizing the results of Lemmas 4.1–4.3 and of Proposition 4.4 we obtain:

Proposition 4.6 *Let $V \subset \mathbb{CP}^n$ be a reduced subvariety as in (1.2), and let g_α be an analytic function on $U_\alpha \subset \mathbb{CP}^n$ as in Theorem 1. Let $\phi = \sum_{\alpha=0}^n \vartheta_\alpha \Phi^{(0,q)}$ be a $\bar\partial$-closed residual current of homogeneity zero on V.*

Then for a fixed $\eta > 0$ and an arbitrary $z \in U_\alpha$, such that $|g_\alpha(z)| > \eta$, we have the following equality

$$\lim_{t\to 0} I_q^{\epsilon(t)}\left[\vartheta_\beta\Phi\right](z) = C(n,q,m)\lim_{t\to 0}\int_{\Gamma_J^{\epsilon(t)}} \frac{\vartheta_\beta(\zeta)\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$
$$\bigwedge \det\left[\frac{\bar z}{B^*(\zeta,z)}\ \frac{\bar\zeta}{B(\zeta,z)}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{\frac{d\bar z}{B^*(\zeta,z)}}^{q-1}\ \overbrace{\frac{d\bar\zeta}{B(\zeta,z)}}^{n-m-q}\right]\wedge\omega(\zeta), \quad (4.22)$$

where $J = (1,\dots,m)$, and $\epsilon(t)$ is an admissible path.

The limit in (4.22) is well defined and represents a continuous function on $\{U_\alpha : |g_\alpha(z)| > \eta\}$, which is identically zero if condition (4.14) is satisfied. If ϕ is defined by a C^∞ form, then the limit of the integral in (4.22) is a $\mathbb{C}^\infty$ form on $\{U_\alpha : |g_\alpha(z)| > \eta\}$ for any fixed η.

Proof We obtain expression (4.22) from Lemmas 4.1 and 4.2, and the rest of the statement from Lemma 4.3 and Proposition 4.4. □

5 Proof of Theorem 1

As the first step in obtaining formula (1.11) for $\bar\partial$-closed residual currents we use Propositions 3.4 and 4.6 to obtain the residual limit of formula (2.11).

Interpreting both sides of (2.11) as residual currents we obtain for a fixed t the equality

$$\begin{aligned}
\langle\phi,\gamma\rangle &= \lim_{\tau\to 0}\sum_\alpha\int_{T_\alpha^{\delta(\tau)}}\vartheta_\alpha(z)\frac{\gamma(z)\wedge\Phi(z)}{\prod_{k=1}^m F_k^{(\alpha)}(z)}\\
&= \lim_{\tau\to 0}\sum_\alpha\int_{T_\alpha^{\delta(\tau)}}\frac{\vartheta_\alpha(z)\bar\partial\gamma(z)}{\prod_{k=1}^m F_k^{(\alpha)}(z)}\wedge\left(\sum_\beta I_q^{\epsilon(t)}\left[\vartheta_\beta\Phi\right](z)\right)\\
&\quad+\lim_{\tau\to 0}\sum_\alpha\int_{T_\alpha^{\delta(\tau)}}\frac{\vartheta_\alpha(z)\gamma(z)}{\prod_{k=1}^m F_k^{(\alpha)}(z)}\wedge\left(\sum_\beta I_{q+1}^{\epsilon(t)}\left[\bar\partial\left(\vartheta_\beta\Phi\right)\right](z)\right)\\
&\quad+\lim_{\tau\to 0}\sum_\alpha\int_{T_\alpha^{\delta(\tau)}}\frac{\vartheta_\alpha(z)\gamma(z)}{\prod_{k=1}^m F_k^{(\alpha)}(z)}\wedge\left(\sum_\beta L_q^{\epsilon(t)}\left[\vartheta_\beta\Phi\right](z)\right)
\end{aligned}$$

for an arbitrary $\gamma \in \mathcal{E}^{(n,n-m-q)}(V, \mathcal{L})$, the differential form $\Phi(\zeta) = \sum_{\alpha} \vartheta_{\alpha}(\zeta)\Phi_{\alpha}(\zeta)$, and an admissible path $\{\delta_k(\tau)\}_1^m$.

Then using for the right-hand side of equality above smoothness of the forms $I_q^{\epsilon(t)}\left[\vartheta_{\beta}\Phi\right]$, $I_{q+1}^{\epsilon(t)}\left[\bar{\partial}\left(\vartheta_{\beta}\Phi\right)\right]$, and $L_q^{\epsilon(t)}\left[\vartheta_{\beta}\Phi\right]$ with respect to $z \in U^{\delta(\tau)}$ for fixed t and $\tau \to 0$ we apply Theorem 1.8.3 in [10] (see also [24] Prop. 2.2) and obtain the following equality

$$\begin{aligned} \langle \phi, \gamma \rangle = & \lim_{\eta \to 0} \lim_{\tau \to 0} \int_{\{|g_{\alpha}(z)|>\eta\} \cap \left\{T_{\alpha}^{\delta(\tau)}\right\}} \frac{\vartheta_{\alpha}(z)\bar{\partial}\gamma(z)}{\prod_{k=1}^{m} F_k^{(\alpha)}(z)} \wedge \left(\sum_{\beta} I_q^{\epsilon(t)}\left[\vartheta_{\beta}\Phi\right](z)\right) \\ & + \lim_{\eta \to 0} \lim_{\tau \to 0} \sum_{\alpha} \int_{\{|g_{\alpha}(z)|>\eta\} \cap \left\{T_{\alpha}^{\delta(\tau)}\right\}} \frac{\vartheta_{\alpha}(z)\gamma(z)}{\prod_{k=1}^{m} F_k^{(\alpha)}(z)} \wedge \left(\sum_{\beta} I_{q+1}^{\epsilon(t)}\left[\bar{\partial}\left(\vartheta_{\beta}\Phi\right)\right](z)\right) \\ & + \lim_{\eta \to 0} \lim_{\tau \to 0} \sum_{\alpha} \int_{\{|g_{\alpha}(z)|>\eta\} \cap \left\{T_{\alpha}^{\delta(\tau)}\right\}} \frac{\vartheta_{\alpha}(z)\gamma(z)}{\prod_{k=1}^{m} F_k^{(\alpha)}(z)} \wedge \left(\sum_{\beta} L_q^{\epsilon(t)}\left[\vartheta_{\beta}\Phi\right](z)\right). \end{aligned} \tag{5.1}$$

In the next step we pass to the limit in the right-hand side of (5.1) as $t \to 0$ and $\eta > 0$ is fixed. We use the following lemma to simplify the limit of the right-hand side of (5.1).

Lemma 5.1 *Let $\phi \in Z_R^{(0,n-m)}(V)$ be a $\bar{\partial}$-closed residual current of homogeneity zero defined by a collection of forms $\{\Phi_{\alpha}\}_{\alpha=0}^{n+1}$ satisfying conditions (1.5) and (1.9).*

Then for fixed $\eta > 0$, $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L})$, and $\sigma > 0$ there exist τ, t, such that

$$\left| \int_{\{|g_{\alpha}(z)|>\eta\} \cap \left\{T_{\alpha}^{\delta(\tau)}\right\}} \vartheta_{\alpha}(z) \frac{\gamma(z)}{\prod_{k=1}^{m} F_k^{(\alpha)}(z)} \wedge \left(\sum_{\beta} I_{q+1}^{\epsilon(t)}\left[\bar{\partial}\left(\vartheta_{\beta}\Phi\right)\right](z)\right) \right| < \sigma. \tag{5.2}$$

Proof Because of the choice of g_{α} (see (1.10)) we conclude that equality (5.2) would follow from equality

$$\lim_{t \to 0} \sum_{\beta} I_{q+1}^{\epsilon(t)}\left[\bar{\partial}\left(\vartheta_{\beta}\Phi\right)\right](z) = 0$$

for $z \in \{U_{\alpha} : |g_{\alpha}(z)| > \eta\}$, or using formula (4.22) from equality

$$\lim_{t \to 0} \sum_{\beta} \int_{\Gamma_J^{\epsilon(t)}} \frac{\bar{\partial}\left(\vartheta_{\beta}(\zeta)\Phi(\zeta)\right)}{\prod_{k=1}^{m} P_k(\zeta)} \bigwedge \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \ \frac{\bar{\zeta}}{B(\zeta, z)} \ \overbrace{Q(\zeta, z)}^{m} \ \overbrace{\frac{d\bar{z}}{B^*(\zeta, z)}}^{q} \ \overbrace{\frac{d\bar{\zeta}}{B(\zeta, z)}}^{n-m-q-1} \right] \wedge \omega(\zeta) = 0, \tag{5.3}$$

where $J = (1, \ldots, m)$ and $\{\epsilon_k(t)\}_{k=1}^{m}$ is an admissible path.

For the differential form Φ of homogeneity zero we have $\Phi_\alpha = \Phi\Big|_{U_\alpha} = \Phi\Big|_{U_\beta} = \Phi_\beta$, and therefore

$$\sum_\beta \bar\partial\left(\vartheta_\beta \Phi_\beta\right) = \sum_\beta \vartheta_\beta \bar\partial \Phi_\beta.$$

But then, using condition (1.9) and part (ii) of Lemma 4.3 we obtain equality (5.3). □

To prove item (i) of Theorem 1 we use Lemma 5.1 in equality (5.1) and obtain for a $\bar\partial$-closed residual current $\phi \in Z_R^{(0,n-m)}(V)$ and an arbitrary $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L})$ the equality

$$\langle \phi, \gamma \rangle = \left\langle I_q[\phi], \bar\partial\gamma \right\rangle + \left\langle L_q[\phi], \gamma \right\rangle, \tag{5.4}$$

where

- $\left\langle I_q[\phi], \bar\partial\gamma \right\rangle$

$$= \lim_{\eta\to 0}\lim_{\tau\to 0}\lim_{t\to 0}\int_{\{|g_\alpha(z)|>\eta\}\cap\left\{T_\alpha^{\delta(\tau)}\right\}} \vartheta_\alpha(z)\frac{\bar\partial\gamma(z)}{\prod_{k=1}^m F_k^{(\alpha)}(z)} \wedge \left(\sum_\beta I_q^{\epsilon(t)}\left[\vartheta_\beta\Phi\right](z)\right), \tag{5.5}$$

- $L_q[\phi] = 0$ for $q = 1, \ldots, n-m-1$,
- $\langle L_{n-m}[\phi], \gamma \rangle$

$$= \lim_{\eta\to 0}\lim_{\tau\to 0}\lim_{t\to 0}\sum_\alpha \int_{\{|g_\alpha(z)|>\eta\}\cap\left\{T_\alpha^{\delta(\tau)}\right\}} \vartheta_\alpha(z)\frac{\gamma(z)}{\prod_{k=1}^m F_k^{(\alpha)}(z)} \wedge \left(\sum_\beta L_{n-m}^{\epsilon(t)}\left[\vartheta_\beta\Phi\right](z)\right) \tag{5.6}$$

with operators $I_q^{\epsilon(t)}$ and $L_{n-m}^{\epsilon(t)}$ defined in (4.22) and (3.24) respectively.

From formula (3.24) and Lemma 3.1 it follows that the limit in (5.6) is well defined for a $\bar\partial$-closed current ϕ and an arbitrary $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L})$. Then, since the left-hand side is also well defined for $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L})$, we obtain that $\left\langle I_q[\phi], \bar\partial\gamma\right\rangle$ is also well defined for a $\bar\partial$-closed residual current ϕ and $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L})$.

We notice that though the $\bar\partial$-closed current ϕ is defined by C^∞ forms satisfying condition (1.9), the projection $L_{n-m}[\phi]$ is a residual current defined by the forms analytically depending on $z, \bar z$.

To prove item (ii) we use formula (2.8) to obtain that operator L_q is not zero only for $q = n - m$, and therefore formula (5.5) gives a solution $I_q[\phi]$ of the $\bar\partial$-equation

$$\bar\partial\psi = \phi$$

for a $\bar\partial$-closed residual current $\phi^{(0,q)}$ of homogeneity zero for $q < n-m$. Smoothness of $I_q[\phi](z)$ on $U_\alpha \setminus V'_\alpha$ for smooth $\{\Phi_\alpha\}_{\alpha=0}^n$ follows from Proposition 4.6.

For $q = n - m$ we have a nontrivial cohomology group $H_R^{n-m}(V, \mathcal{O}_V)$. In the Proposition below we prove the necessary and sufficient condition from item (iii) in Theorem 1 for a $\bar\partial$-closed residual current to be exact.

Proposition 5.2 *Let $V \subset \mathbb{CP}^n$ be a reduced complete intersection subvariety as in (1.2) satisfying conditions of Theorem 1. Then a $\bar\partial$-closed residual current $\phi \in Z_R^{(0,n-m)}(V)$ of homogeneity zero is $\bar\partial$-exact, i.e., there exists a current $\psi \in C^{(0,n-m-1)}(V)$ such that $\bar\partial\psi = \phi$, iff condition (1.12) is satisfied.*

Proof Sufficiency of condition (1.12) immediately follows from equality (5.4). On the other hand, if $\phi = \bar\partial\psi$ for a current $\psi \in C^{(0,n-m-1)}(V)$ of homogeneity zero, then we have equality

$$\langle\phi, \gamma\rangle = \langle\psi, \bar\partial\gamma\rangle$$

satisfied for an arbitrary $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L})$.

Applying the last equality to differential forms

$$\gamma_z^r(\zeta) = \langle\bar z\cdot\zeta\rangle^r \wedge \det\left[\bar z \;\overbrace{Q(\zeta, z)}^{m}\;\overbrace{d\bar z}^{n-m}\right] \wedge \omega(\zeta),$$

and using Lemma 3.1 and holomorphic dependence of the forms γ_z^r on ζ we obtain equality

$$\lim_{t\to 0} L_{n-m}^{\epsilon(t)}[\Phi](z) = \sum_{0\le r\le d-n-1} C(n,m,d,r)\lim_{t\to 0}\int_{\{|\zeta|=1,\ \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \times\frac{\gamma_z^r(\zeta)\wedge\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} = \sum_{0\le r\le d-n-1} C(n,m,d,r)\cdot\langle\psi, \bar\partial_\zeta\gamma_z^r\rangle = 0$$

for an arbitrary z such that $|g_\beta(z)| > \eta$.

Using this equality in (5.6) we obtain the necessity of condition (1.12). □

This concludes the proof of Theorem 1.

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