J Geom Anal (2016) 26:1995–2012
DOI 10.1007/s12220-015-9616-4

A Compactness Theorem of the Space of Free Boundary f-Minimal Surfaces in Three-Dimensional Smooth Metric Measure Space with Boundary

Ezequiel Barbosa[1] · **Yong Wei**[2]

Received: 27 October 2014 / Published online: 21 May 2015

Abstract Let $(M^3, g, e^{-f}d\mu_M)$ be a compact three-dimensional smooth metric measure space with nonempty boundary. Suppose that M has nonnegative Bakry–Émery Ricci curvature and the boundary ∂M is strictly f-mean convex. We prove that there exists a properly embedded smooth f-minimal surface Σ in M with free boundary $\partial\Sigma$ on ∂M. If we further assume that the boundary ∂M is strictly convex, then we prove that M^3 is diffeomorphic to the 3-ball B^3, and a compactness theorem for the space of properly embedded f-minimal surfaces with free boundary in such $(M^3, g, e^{-f}d\mu_M)$, when the topology of these f-minimal surfaces is fixed.

Keywords Compactness · f-Minimal · Free boundary · Bakry–Émery Ricci curvature

Mathematics Subject Classification 53C42 · 53C21

1 Introduction

Let (M^n, g) be a compact smooth Riemannian manifold with boundary ∂M and f be a smooth function on M. We denote by $\bar{\nabla}$, $\bar{\Delta}$ and $\bar{\nabla}^2$ the gradient, Laplacian and Hessian operator on M with respect to g, respectively. The Bakry–Émery Ricci curvature on M is defined by

✉ Yong Wei
yong.wei@ucl.ac.uk

Ezequiel Barbosa
ezequiel@mat.ufmg.br

1 Departamento de Matemática, Universidade Federal de Minas Gerais (UFMG), Caixa Postal 702, Belo Horizonte, MG 30123-970, Brazil

2 Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

$$Ric_f = Ric + \bar{\nabla}^2 f, \tag{1.1}$$

which is an important generalization of the Ricci curvature. The equation $Ric_f = \kappa g$ for some constant κ is just the gradient Ricci soliton equation, which plays an important role in the singularity analysis of Ricci flow (see [1]). Denote by $d\mu_M$ the volume form on M with respect to g, then $(M^n, g, e^{-f} d\mu_M)$ is often called a smooth metric measure space. We refer the interested readers to [2] for further motivation and examples of the smooth metric measure spaces.

Let Σ be a hypersurface in M and ν the unit outward normal vector of Σ. Define the second fundamental form of Σ in M by $h(X, Y) = \langle \bar{\nabla}_X \nu, Y \rangle$ for any two tangent vector fields X and Y on Σ, and the mean curvature by $H = tr(h)$. The f-mean curvature at a point $x \in \Sigma$ with respect to ν is given by

$$H_f(x) = H(x) - \langle \bar{\nabla} f(x), \nu(x) \rangle. \tag{1.2}$$

Σ is called an f-minimal hypersurface in M if its f-mean curvature H_f vanishes everywhere. In some places, in order to avoid confusion, we will use H_f^{Σ} instead of H_f. The most well-known example of a smooth metric measure space is the Gaussian soliton: $(\mathbb{R}^n, g_0, e^{-\frac{1}{4}|x|^2} d\mu)$, where g_0 is the standard Euclidean metric on $\mathbb{R}^n$. The Gaussian soliton satisfies $Ric_f = \frac{1}{2} g_0$. Note that the f-minimal hypersurfaces in the Gaussian soliton are self-shrinkers $\Sigma^{n-1} \subset \mathbb{R}^n$ which satisfy $H = \frac{1}{2}\langle x, \nu \rangle$. Self-shrinkers play an important role in the mean curvature flow, as they correspond to the self-similar solution to mean curvature flow, and also describe all possible blow ups at a given singularity.

Recently, Colding–Minicozzi [3] and Ding–Xin [4] considered the compactness property for the space of self-shrinkers in $\mathbb{R}^3$. After that, joint with Li, the second author [5] proved the first compactness theorem for the space of closed f-minimal surface in closed three-dimensional smooth metric measure space with positive Ricci curvature, generalizing the classical compactness theorem of closed minimal surfaces in closed three manifold with positive Ricci curvature by Choi and Schoen [6]. The result in [5] was later generalized by Cheng et al. [7,8] to the case where the ambient space M^3 is complete and noncompact. At the same time, f-minimal hypersurfaces (and generally f-minimal submanifolds) became an active research subject; see other related research papers [9–11].

On the other hand, in a recent beautiful work [12], Fraser and Li proved a compactness theorem for the space of compact properly embedded minimal surfaces with free boundary in compact three-dimensional manifold with nonnegative Ricci curvature and strictly convex boundary, which is a free boundary version of the classical compactness theorem by Choi and Schoen [6]. In this paper, we consider the following natural problem: The compactness property for the space of compact properly embedded f-minimal surfaces with free boundary in a compact three-dimensional smooth metric measure space with nonempty boundary, i.e., a free boundary version of the result in Li–Wei [5].

In Sect. 2, we first collect some variation formulas for the weighted area of the hypersurface and the Reilly formula for the smooth metric measure space with boundary, which are important tools in this paper. In particular, we have the observation that an f-minimal hypersurface Σ in (M, g) is a minimal hypersurface in $(M, \tilde{g})$ with the conformal changed metric $\tilde{g} = e^{-\frac{2}{n-1}f} g$. This can be easily seen from the first variation formula. We also calculate the transformation formulas for the mean curvature and the second fundamental form of Σ in M under the conformal change of the ambient metrics. Then we use the variation formulas and Reilly formula to prove some properties about f-minimal hypersurface with free boundary. Under the assumption that M has nonnegative Ric_f and the boundary ∂M is strictly f-mean convex (i.e., the f-mean curvature H_f of ∂M is positive everywhere), we show that M contains no smooth closed embedded f-minimal hypersurface in the interior of M. Moreover, if $n \leq 7$, we have an isoperimetric type inequality for Σ with respect to the metric induced from $\tilde{g}$. In the three-dimensional case, using the nonexistence of closed embedded f-minimal hypersurfaces in M and a general existence result due to Li [13, Theorem 1.1], we have the following existence result for the properly embedded f-minimal hypersurface with free boundary.

Theorem 1.1 *Let $(M^3, g, e^{-f} d\mu_M)$ be a compact smooth metric measure space with nonempty boundary. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly f-mean convex. Then there exists a properly embedded smooth f-minimal hypersurface Σ in M with free boundary $\partial\Sigma$ on ∂M.*

We can also prove that if M has nonnegative Ric_f and the boundary is strictly convex and strictly f-mean convex, then any properly embedded f-minimal hypersurface in M with free boundary is connected, the boundary ∂M is connected and the $(n-1)$-th relative integral homology group $H_{n-1}(M, \partial M)$ vanishes. Based on these properties, we have the following strong topology restriction of $(M^3, g, e^{-f} d\mu_M)$. The proof is by using a similar argument as in Meeks–Simon–Yau [14, Sect. 8], with an argument in the proof of Theorem 2.11 in [12].

Theorem 1.2 *Let $(M^3, g, e^{-f} d\mu_M)$ be a compact smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly convex and strictly f-mean convex. Then M^3 is diffeomorphic to the* 3*-ball B^3.*

Note that in the case where the boundary ∂M is empty, Liu [11, Theorem 3] recently obtained some classification results for complete $(M^3, g, e^{-f} d\mu_M)$ with bounded f and nonnegative Ric_f.

In Sect. 3, we first define the f-Steklov eigenvalue $\lambda_{1,f}$ of the Dirichlet-to-Neumann operator for general smooth metric measure space with boundary. Then we estimate the lower bound for the first f-Steklov eigenvalue of a compact properly embedded f-minimal hypersurface in terms of the boundary convexity of the ambient space. Using this estimate, in $n = 3$ case, we can obtain an upper bound on the boundary length $\tilde{L}(\partial\Sigma)$ of the f-minimal surface Σ with respect to the conformal metric $\tilde{g}$. However in the free boundary case, unlike the closed case in [5, Sect. 3], we cannot obtain a direct comparison between the first f-Steklov eigenvalue $\lambda_{1,f}$ and the first Steklov eigenvalue $\tilde{\lambda}_1$ of Σ with respect to the conformal metric $\tilde{g} = e^{-f} g$. We need to go back to modify Fraser–Schoen's [15, Sect. 2] argument to get an upper bound for $\lambda_{1,f}\tilde{L}(\partial\Sigma)$. To show this, a crucial idea is to consider a new conformal metric $\hat{g} = e^{-2f}g$ on M, not the $\tilde{g}$. Once we have the upper bound of $\lambda_{1,f}\tilde{L}(\partial\Sigma)$, the lower bound for $\lambda_{1,f}$ then implies the upper bound for $\tilde{L}(\partial\Sigma)$.

In the last section, we use the upper bound for $\tilde{L}(\partial\Sigma)$, the isoperimetric inequality (2.9) and the Gauss–Bonnet Theorem to obtain a uniform upper bound for the L^2 norm of the second fundamental form of Σ in $(M^3, \tilde{g})$. Then we prove our main theorem using a standard argument as in [6,12,16] with some modification.

Theorem 1.3 *Let $(M^3, g, e^{-f}d\mu_M)$ be a compact smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly convex and strictly f-mean convex. Then the space of compact properly embedded f-minimal surfaces of fixed topological type in M is compact in the C^k topology for any $k \geq 2$.*

As in [5], one of the key ingredients in the proof of Theorem 1.3 is the observation that an f-minimal hypersurface Σ in (M, g) is a minimal hypersurface in $(M, \tilde{g})$ with the conformal changed metric $\tilde{g} = e^{-\frac{2}{n-1}f}g$. However, Theorem 1.3 does not directly follow from Fraser–Li's [12] compactness theorem for free boundary minimal surfaces in three-dimensional smooth metric measure space with nonnegative Ricci curvature and strictly convex boundary. In fact, the Ricci curvature of the conformal changed metric $\tilde{g}$ may not have a sign (see [5, Sect. 1]), and the boundary convexity may also not hold again. See the transformation formula for the second fundamental form for any hypersurface under the conformal change of the ambient metrics in Sect. 2.1.

Remark 1.1 We remark that very recently, Sharp [17] proved a smooth compactness theorem for the space of closed embedded minimal hypersurfaces with bounded index and bounded volume in a closed Riemannian manifold M^{n+1} with positive Ricci curvature and $2 \leq n \leq 6$, generalizing Choi–Schoen's [6] compactness theorem to higher dimensions. The idea in [17] has been used by the authors and Sharp [18] to obtain an analogous smooth compactness for the space of complete f-minimal hypersurfaces of dimension $2 \leq n \leq 6$ and in particular the space of self-shrinkers. This motivates the natural question: Can we obtain a smooth compactness theorem for the space of free boundary minimal (or f-minimal) hypersurface with bounded index (or f-index) and bounded volume? In fact, this forms a topic of current investigation by the authors.

2 f-Minimal Hypersurfaces with Free Boundary

2.1 Variation Properties

Let (M^n, g) be a compact Riemannian manifold with nonempty boundary ∂M and f be a smooth function on M. Then $(M^n, g, e^{-f}d\mu_M)$ is usually called a smooth metric measure space. Let Σ be a compact properly immersed hypersurface in M with boundary $\partial\Sigma$. Proper means that the boundary $\partial\Sigma$ lies in ∂M. If Σ is two-sided, there exists a globally defined unit normal vector field ν on Σ. The second fundamental form of Σ is defined as $h(e_i, e_j) = -g(\bar{\nabla}_{e_i} e_i, \nu)$ for any orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T\Sigma$. Here $\bar{\nabla}$ is the connection with respect to g on M. For any normal variation Σ_s of Σ with the variation vector field $X = \varphi\nu$ for some $\varphi \in C^\infty(\Sigma)$, the first variation formula for the weighted area of Σ

$$A_f(\Sigma) := \int_\Sigma e^{-f} d\mu_\Sigma$$

is given by (see [19, Lemma 3.2])

$$\frac{d}{ds}\bigg|_{s=0} A_f(\Sigma_s) = \int_\Sigma \varphi H_f e^{-f} d\mu_\Sigma + \int_{\partial\Sigma} \varphi\langle \nu, \nu_{\partial\Sigma}\rangle e^{-f} d\mu_{\partial\Sigma}, \tag{2.1}$$

where $\nu_{\partial\Sigma}$ is the unit outward normal of $\partial\Sigma$ in Σ. We say that Σ is strongly f-stationary if $\frac{d}{ds}\big|_{s=0} A_f(\Sigma_s) = 0$ for any variation Σ_s of Σ. Then from (2.1) we have that Σ is strongly f-stationary if and only if $H_f = 0$ on Σ and Σ meets ∂M orthogonally along $\partial\Sigma$. We also call such hypersurface Σ an f-minimal hypersurface with free boundary.

We also have the second variation formula for $A_f(\Sigma)$ (see [19, Proposition 3.5]):

$$\begin{aligned}\frac{d^2}{ds^2}\bigg|_{s=0} A_f(\Sigma_s) &= \int_\Sigma \left(|\nabla\varphi|^2 - (Ric_f(\nu,\nu) + \|h^\Sigma\|^2)\varphi^2\right) e^{-f} d\mu_\Sigma \\ &\quad - \int_{\partial\Sigma} h^{\partial M}(\nu,\nu)\varphi^2 e^{-f} d\mu_{\partial\Sigma},\end{aligned} \tag{2.2}$$

where ∇ is the gradient operator on Σ, h^Σ and $h^{\partial M}$ are the second fundamental forms of Σ and ∂M in M, respectively. Since Σ is f-minimal with free boundary, Σ meets ∂M orthogonally along $\partial\Sigma$, we have that ν is tangent to ∂M along $\partial\Sigma$. Σ is called f-stable if $\frac{d^2}{ds^2}\big|_{s=0} A_f(\Sigma_s) \geq 0$ for any $\varphi \in C^\infty(\Sigma)$.

On the other hand, an f-minimal hypersurface can be viewed as a minimal hypersurface under a conformal metric. This can be seen as follows: Define a conformal metric $\tilde{g} = e^{-\frac{2}{n-1}f} g$ on M. Then the area of Σ with respect to the induced metric from $\tilde{g}$ is given by

$$\tilde{A}(\Sigma) = \int_\Sigma d\tilde{\mu}_\Sigma = \int_\Sigma e^{-f} d\mu_\Sigma = A_f(\Sigma).$$

The first variation formula for $\tilde{A}$ is given by (see, e.g., [13])

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} \tilde{A}(\Sigma_s) &= -\int_\Sigma \tilde{g}(X, \tilde{\mathbf{H}}) d\tilde{\mu}_\Sigma + \int_{\partial\Sigma} \tilde{g}(X, \tilde{\nu}_{\partial\Sigma}) d\tilde{\mu}_{\partial\Sigma}, \\ &= \int_\Sigma e^{-\frac{f}{n-1}} \varphi \tilde{H} e^{-f} d\mu_\Sigma + \int_{\partial\Sigma} \varphi\langle \nu, \nu_{\partial\Sigma}\rangle e^{-f} d\mu_{\partial\Sigma},\end{aligned} \tag{2.3}$$

where $\tilde{\mathbf{H}} = -\tilde{H}\tilde{\nu} = -e^{\frac{f}{n-1}} \tilde{H}\nu$ is the mean curvature vector field of Σ in $(M, \tilde{g})$, and $\tilde{\nu}_{\partial\Sigma} = e^{\frac{f}{n-1}} \nu_{\partial\Sigma}$, $d\tilde{\mu}_{\partial\Sigma} = e^{-\frac{n-2}{n-1}f} d\mu_{\partial\Sigma}$ are the unit outward normal and the volume form of $\partial\Sigma$ with respect to the conformal metric $\tilde{g}$. Comparing (2.1) and (2.3), we have

$$\tilde{H} = e^{\frac{f}{n-1}} H_f. \tag{2.4}$$

We can also calculate the relationship between the second fundamental form $\tilde{h}$ of Σ in $(M, \tilde{g})$ and the second fundamental form h of Σ in (M, g): Choose an orthonormal basis $e_1, \ldots, e_{n-1}$ for $T\Sigma$ with respect to the metric induced from (M, g). Then $h(e_i, e_j) = -g(\bar{\nabla}_{e_i} e_i, \nu)$. Under the conformal metric $\tilde{g}$, $\{\tilde{e}_1 = e^{\frac{f}{n-1}} e_1, \ldots, \tilde{e}_{n-1} = e^{\frac{f}{n-1}} e_{n-1}\}$ and $\tilde{\nu} = e^{\frac{f}{n-1}} \nu$ are the orthonormal basis for $T\Sigma$ and unit normal vector field of Σ in $(M, \tilde{g})$. Denote by $\tilde{\bar{\nabla}}$ and $\bar{\nabla}$ the connections with respect to $\tilde{g}$ and g, respectively. Then a direct calculation gives

$$\tilde{\bar{\nabla}}_X Y = \bar{\nabla}_X Y - \frac{1}{n-1} df(X) Y - \frac{1}{n-1} df(Y) X + \frac{1}{n-1} g(X, Y) \bar{\nabla} f \tag{2.5}$$

for any tangent vector fields X, Y on M. Thus the second fundamental form $\tilde{h}$ satisfies

$$\begin{aligned} \tilde{h}(\tilde{e}_i, \tilde{e}_j) &= -\tilde{g}(\tilde{\bar{\nabla}}_{\tilde{e}_i} \tilde{e}_j, \tilde{\nu}) \\ &= -\tilde{g}\left(\bar{\nabla}_{\tilde{e}_i} \tilde{e}_j - \frac{1}{n-1} df(\tilde{e}_i)\tilde{e}_j - \frac{1}{n-1} df(\tilde{e}_j)\tilde{e}_i + \frac{1}{n-1} g(\tilde{e}_i, \tilde{e}_j)\bar{\nabla} f, \tilde{\nu}\right) \\ &= -\tilde{g}\left(\bar{\nabla}_{\tilde{e}_i} \tilde{e}_j + \frac{1}{n-1} g(\tilde{e}_i, \tilde{e}_j)\bar{\nabla} f, \tilde{\nu}\right) \\ &= -e^{-\frac{2}{n-1} f} g\left(e^{\frac{2}{n-1} f} \bar{\nabla}_{e_i} e_j + \frac{1}{n-1} e^{\frac{2}{n-1} f} g(e_i, e_j)\bar{\nabla} f, e^{\frac{f}{n-1}} \nu\right) \\ &= e^{\frac{f}{n-1}} \left(h(e_i, e_j) - \frac{1}{n-1} g(e_i, e_j) g(\bar{\nabla} f, \nu)\right). \end{aligned} \tag{2.6}$$

Letting $i = j$ and summing from 1 to $n-1$ in (2.6) can also give the relation (2.4).

Finally, since $\tilde{A}(\Sigma) = A_f(\Sigma)$ and noting that the normal direction of Σ in M is unchanged under the conformal change of the ambient metric, we know that the second variation $\frac{d^2}{ds^2} A_f(\Sigma) \geq 0$ if and only if $\frac{d^2}{ds^2} \tilde{A}(\Sigma) \geq 0$ for any normal variation Σ_s of Σ. Thus Σ is f-stable in (M, g) if and only if Σ is stable in $(M, \tilde{g})$ in the usual sense.

2.2 Reilly Formula for Smooth Metric Measure Space

We next exhibit the Reilly formula for a smooth metric measure space, which is an important tool in this paper. Let $(M^n, g, e^{-f} d\mu_M)$ be a compact metric measure space with boundary ∂M. The f-Laplacian $\bar{\Delta}_f = \bar{\Delta} - \bar{\nabla} f \cdot \bar{\nabla}$ on M is self-adjoint with respect to the weighted measure $e^{-f} d\mu$. A simple calculation gives the following Bochner formula (see [2,20,21]) for any function $u \in C^3(M)$:

$$\frac{1}{2} \bar{\Delta}_f |\bar{\nabla} u|^2 = |\bar{\nabla}^2 u|^2 + Ric_f(\bar{\nabla} u, \bar{\nabla} u) + g(\bar{\nabla} u, \bar{\nabla} \bar{\Delta}_f u). \tag{2.7}$$

Using the Bochner formula (2.7) and integration by parts, Ma–Du [20] obtained the following Reilly formula (see also [5, Sect. 2]):

$$0 = \int_M \left(Ric_f(\bar{\nabla} u, \bar{\nabla} u) - |\bar{\Delta}_f u|^2 + |\bar{\nabla}^2 u|^2 \right) e^{-f} d\mu_M$$
$$+ \int_{\partial M} \left(\left(\Delta_f u + H_f \frac{\partial u}{\partial \nu} \right) \frac{\partial u}{\partial \nu} - \left\langle \nabla u, \nabla \frac{\partial u}{\partial \nu} \right\rangle + h^{\partial M}(\nabla u, \nabla u) \right) e^{-f} d\mu_{\partial M}. \tag{2.8}$$

Here, Ric_f is the Bakry–Émery Ricci tensor of M; $d\mu_M$ and $d\mu_{\partial M}$ are volume forms on M and ∂M respectively. $\bar{\Delta}_f$, $\bar{\nabla}$ and $\bar{\nabla}^2$ are the f-Laplacian, gradient and Hessian on M respectively; $\Delta_f = \Delta - \nabla f \cdot \nabla$ and ∇ are the f-Laplacian and gradient operators on ∂M; ν is the unit outward normal of ∂M; H_f and $h^{\partial M}$ are the f-mean curvature and the second fundamental form of ∂M in M with respect to ν respectively. Note that the formula (2.8) also holds for piecewise smooth boundary $\partial M = \cup_{i=1}^k \Sigma_i$.

2.3 Some Properties of f-Minimal Hypersurfaces with Free Boundary

Lemma 2.1 *Let $(M^n, g, e^{-f} d\mu_M)$ be a compact smooth metric measure space with nonempty boundary. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly f-mean convex. Then M contains no smooth closed embedded f-minimal hypersurface. Moreover, if $n \leq 7$, then there exists a constant $c > 0$ such that*

$$\widetilde{Vol}(\Sigma) \leq c\, \widetilde{Vol}(\partial \Sigma) \tag{2.9}$$

for any smooth immersed minimal hypersurface Σ in M, where the volumes $\widetilde{Vol}$ are measured with respect to metrics on Σ and $\partial\Sigma$ induced from the conformal metric $\tilde{g} = e^{-\frac{2}{n-1} f} g$ on M.

Proof For the first statement, suppose that there exists a smooth closed embedded f-minimal hypersurface Σ in M. Since $H_f^{\partial M} > 0$ on ∂M and $H_f^{\Sigma} = 0$ on Σ, then $\Sigma \cap \partial M = \emptyset$ and $d(\Sigma, \partial M) = d > 0$. Let $\gamma : [0, d] \to M$ be the minimizing geodesic (parameterized by arc-length) realizing the distance between Σ and ∂M. From the first variation formula for arc-length, we can see that γ meets Σ and ∂M orthogonally. Choose an orthonormal basis $e_1, \ldots, e_{n-1}$ for $T_{\gamma(0)}\Sigma$ and let V_i be the parallel transport of e_i along γ. The second variation formula for arc-length gives that

$$0 \leq \sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = \sum_{i=1}^{n-1} \int_0^d \left(|V_i'(s)|^2 - |V_i(s)|^2 K(\gamma'(s), V_i(s)) \right) ds$$
$$+ \sum_{i=1}^{n-1} \left(\langle \bar{\nabla}_{V_i(d)} V_i(d), \gamma'(d) \rangle - \langle \bar{\nabla}_{V_i(0)} V_i(0), \gamma'(0) \rangle \right)$$
$$= - \int_0^d Ric(\gamma'(s), \gamma'(s)) ds - H^{\Sigma}(\gamma(0)) - H^{\partial M}(\gamma(d))$$

$$
\begin{aligned}
&= -\int_0^d Ric_f(\gamma'(s), \gamma'(s))ds - H^{\Sigma}(\gamma(0)) - H^{\partial M}(\gamma(d)) \\
&\quad + \int_0^d \bar{\nabla}^2 f(\gamma'(s), \gamma'(s))ds,
\end{aligned}
\tag{2.10}
$$

where $K(u, v)$ is the sectional curvature of the plane spanned by u and v in M. Note that $\gamma'(d)$ is the unit outward normal at $\gamma(d) \in \partial M$ and $\gamma'(0)$ is the unit inward normal at $\gamma(0) \in \Sigma$, so in the second equation of (2.10) we used

$$
H^{\partial M}(\gamma(d)) = -\sum_{i=1}^{n-1} \langle \bar{\nabla}_{e_i} e_i, \gamma'(d) \rangle, \qquad H^{\Sigma}(\gamma(0)) = \sum_{i=1}^{n-1} \langle \bar{\nabla}_{e_i} e_i, \gamma'(0) \rangle.
$$

Using the facts that

$$
\begin{aligned}
\frac{d}{ds} f(\gamma(s)) &= \langle \bar{\nabla} f(\gamma(s)), \gamma'(s) \rangle, \\
\frac{d^2}{ds^2} f(\gamma(s)) &= \bar{\nabla}^2 f(\gamma'(s), \gamma'(s)),
\end{aligned}
$$

and by integration by parts, we deduce that

$$
\begin{aligned}
0 \le \sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) &= -\int_0^d Ric_f(\gamma'(s), \gamma'(s))ds - H^{\Sigma}(\gamma(0)) - H^{\partial M}(\gamma(d)) \\
&\quad + \langle \bar{\nabla} f(\gamma(d)), \gamma'(d) \rangle - \langle \bar{\nabla} f(\gamma(0)), \gamma'(0) \rangle \\
&= -\int_0^d Ric_f(\gamma'(s), \gamma'(s))ds - H_f^{\Sigma}(\gamma(0)) - H_f^{\partial M}(\gamma(d)) \\
&< 0,
\end{aligned}
\tag{2.11}
$$

where in the second equality we used the facts

$$
\begin{aligned}
H_f^{\partial M}(\gamma(d)) &= H^{\partial M}(\gamma(d)) - \langle \bar{\nabla} f(\gamma(d)), \gamma'(d) \rangle \\
H_f^{\Sigma}(\gamma(0)) &= H^{\Sigma}(\gamma(0)) + \langle \bar{\nabla} f(\gamma(0)), \gamma'(0) \rangle
\end{aligned}
$$

as $\gamma'(d) \perp T_{\gamma(d)} M$ pointing outward and $\gamma'(0) \perp T_{\gamma(0)} \Sigma$ pointing inward; in the last inequality we used the condition $Ric_f \ge 0$ in M, $H_f^{\partial M} > 0$ on ∂M and $H_f^{\Sigma} = 0$ on Σ. This is a contradiction. Therefore, M contains no smooth closed embedded f-minimal hypersurface.

For the second statement, suppose that Σ is an f-minimal hypersurface in (M, g) with free boundary, then Σ is a minimal hypersurface in $(M, \tilde{g})$ with free boundary, where $\tilde{g} = e^{-\frac{2}{n-1} f} g$ is a conformal metric on M. Denote by $\tilde{H}$ the mean curvature of any hypersurface in M with respect to the conformal metric $\tilde{g}$, then from (2.4) we have the relation $\tilde{H} = e^{\frac{1}{n-1} f} H_f$. Therefore the first statement in this lemma is equivalent to that $(M, \tilde{g})$ contains no smooth closed embedded minimal hypersurface. By assumption $H_f > 0$ on ∂M, we have $\tilde{H} > 0$ on ∂M. Then the manifold $(M, \tilde{g})$ with boundary ∂M satisfies the condition of Theorem 2.1 in [22]. The conclusion of Theorem 2.1 in [22] implies that there exists a constant $c > 0$ such that any smooth hypersurface Σ in $(M, \tilde{g})$ satisfies

$$\widetilde{Vol}(\Sigma) \le c\,\widetilde{Vol}(\partial\Sigma) + \int_\Sigma |\tilde{H}| d\tilde{\mu}_\Sigma.$$

In particular, for any smooth f-minimal hypersurface Σ in (M, g) (which is minimal in $(M, \tilde{g})$), the above inequality implies $\widetilde{Vol}(\Sigma) \le c\,\widetilde{Vol}(\partial\Sigma)$. □

Using Lemma 2.1 and [13, Theorem 1], we can prove Theorem 1.1, which is an existence result of f-minimal surface with free boundary in $(M^3, g, e^{-f} d\mu_M)$.

Proof of Theorem 1.1 From Lemma 2.1 we know that (M^3, g) contains no smooth closed embedded f-minimal surface in the interior of M, which is equivalent to that $(M^3, \tilde{g})$ (with $\tilde{g} = e^{-f} g$) contains no smooth closed embedded minimal surface in the interior of M. Then Theorem 1 in [13] implies that there exists a properly embedded smooth minimal surface Σ in $(M^3, \tilde{g})$ with free boundary $\partial\Sigma$ on ∂M, from which the assertion follows. □

Lemma 2.2 *Let $(M^n, g, e^{-f} d\mu_M)$ be a compact smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly convex and strictly f-mean convex. Then any two-sided properly immersed f-minimal hypersurface Σ in M with free boundary $\partial\Sigma$ on ∂M must be f-unstable. Moreover if M is orientable, then the $(n-1)$-th relative integral homology group $H_{n-1}(M, \partial M)$ vanishes.*

Proof The first statement follows from taking $\varphi \equiv 1$ in the second variation formula (2.2) and the curvature assumption $Ric_f \ge 0$ on M and the strictly convexity of the boundary ∂M. Recall that Σ is f-unstable if and only if Σ is unstable in $(M, \tilde{g})$. Then the second statement follows a similar argument as in the proof of Lemma 2.1 in [12], where we used the assumption ∂M is strictly f-mean convex and Lemma 2.1. □

The next lemma is a connectedness principle for properly embedded f-minimal hypersurfaces with free boundary.

Lemma 2.3 *Let $(M^n, g, e^{-f} d\mu_M)$ be a compact smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly convex. Then any two properly embedded orientable f-minimal hypersurfaces Σ_1 and Σ_2 in M with free boundaries on ∂M must intersect, i.e., $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. In other words, any properly embedded f-minimal hypersurface in M with free boundary is connected.*

Proof Suppose that Σ_1 and Σ_2 are disjoint. Since $H_{n-1}(M, \partial M) = 0$, there exists a compact connected domain Ω in M with piecewise smooth boundary $\partial\Omega =$

$\Sigma_1 \cup \Sigma_2 \cup \Gamma$, where Γ lies in ∂M. Consider the following boundary value problem on Ω:

$$\begin{cases} \bar{\Delta}_f u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Sigma_1, \\ u = 1, & \text{on } \Sigma_2, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma, \end{cases} \tag{2.12}$$

where ν is the outward unit normal on $\Gamma \subset \partial M$. By the free boundaries condition of Σ_1 and Σ_2, there exists a smooth function $\varphi \in C^\infty(\Omega)$ with

$$\begin{cases} \varphi = 0, & \text{on } \Sigma_1, \\ \varphi = 1, & \text{on } \Sigma_2, \\ \frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \Gamma. \end{cases}$$

Let $\hat{u} = u - \varphi$. Then the above problem (2.12) is equivalent to the following

$$\begin{cases} \bar{\Delta}_f \hat{u} = -\bar{\Delta}_f \varphi, & \text{in } \Omega \\ \hat{u} = 0, & \text{on } \Sigma_1 \cup \Sigma_2, \\ \frac{\partial \hat{u}}{\partial \nu} = 0, & \text{on } \Gamma. \end{cases} \tag{2.13}$$

Since $\bar{\Delta}_f \varphi \in C^\infty(\Omega)$, the classical results for elliptic equations with homogeneous boundary value imply that (2.13) has a solution $\hat{u} \in C^{0,\alpha}(\Omega) \cap C^\infty(\Omega \setminus (\partial\Sigma_1 \cup \partial\Sigma_2))$, and therefore $u = \hat{u} + \varphi \in C^{0,\alpha}(\Omega) \cap C^\infty(\Omega \setminus (\partial\Sigma_1 \cup \partial\Sigma_2))$ is a solution to (2.12). Apply u and Ω to the Reilly formula (2.8), and we obtain

$$0 \geq \int_\Omega Ric_f(\bar{\nabla} u, \bar{\nabla} u) e^{-f} d\mu_M + \int_\Gamma h^{\partial M}(\nabla u, \nabla u) e^{-f} d\mu_{\partial M}, \tag{2.14}$$

where we used that Σ_1 and Σ_2 are f-minimal and u is constant on Σ_1 and Σ_2. Since Ric_f is nonnegative in Ω and $\Gamma \subset \partial M$ is strictly convex, (2.14) implies u is constant on Γ, which is a contradiction since $u = 0$ on Σ_1 and $u = 1$ on Σ_2. □

Lemma 2.4 *Let $(M^n, g, e^{-f} d\mu_M)$ be a compact smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly f-mean convex. Then ∂M is connected.*

Proof This can be proved by a similar argument as in Lemma 2.3: Suppose that ∂M is not connected. Let Σ be one of its components. Choose an f-harmonic function u (i.e., $\bar{\Delta}_f u = 0$ in M) which is equal to 0 on Σ and is equal to one on $\partial M \setminus \Sigma$. The existence of u is given by the classical results for elliptic equations as in the proof of Lemma 2.3. Then under the curvature assumption of M and ∂M, the Reilly formula (2.8) implies that u is a constant, which is a contradiction. □

Using similar arguments in [5, Lemma 6] and [23, Theorem 2], we have the following corollary.

Corollary 2.5 *Let $(M^n, g, e^{-f} d\mu_M)$ be a compact orientable smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly f-mean convex and strictly convex. If Σ be a properly embedded orientable f-minimal hypersurface in M with free boundary on ∂M, then Σ divides M into two components Ω_1 and Ω_2.*

We finish this section with the Proof of Theorem 1.2.

Proof of Theorem 1.2 First, we assume that M is orientable. Since M has nonnegative Ric_f and the boundary ∂M is strictly f-mean convex, Lemma 2.1 implies that M contains no smooth closed embedded f-minimal surface. This is equivalent to that $(M, \tilde{g})$ contains no smooth closed embedded minimal surface. In particular, if $\pi : \tilde{M} \to M$ is the universal cover of M, then $(\tilde{M}, \pi^{-1}\tilde{g})$ contains no embedded orientable two sphere of least weighted area in its isotopy class. As in the proof of Theorem 3 in [14], we conclude that every two sphere in M bounds a ball and M is irreducible. Since the boundary ∂M is nonempty and $(M, \tilde{g})$ contains no closed embedded minimal surface, Proposition 1 in [14] implies that M is a handlebody. From the strictly convexity and strictly f-mean convexity of ∂M, Lemma 2.2 gives that $H_2(M, \partial M)$ vanishes and Lemma 2.4 gives that ∂M is connected, which imply that M is diffeomorphic to the 3-ball B^3. If M^3 is nonorientable, then the orientable double cover $\tilde{M}$ is diffeomorphic to B^3. Thus ∂M is homeomorphic to the $\mathbb{RP}^2$, because $\partial \tilde{M} \approx S^2$ is a double cover of ∂M. However since ∂M is the boundary of a compact manifold, all the Stiefel–Whitney numbers of ∂M vanish [24], which contradicts the facts $\omega_1(\mathbb{RP}^2) = \omega_2(\mathbb{RP}^2) = 1$. So M is orientable and diffeomorphic to the 3-ball B^3. □

3 Estimate for the f-Steklov Eigenvalue and Boundary Length of f-Minimal Surfaces

Comparing with the definition of classical Steklov eigenvalue (see [15, Sect. 2]), we define the following f-Steklov eigenvalue. Let (Σ, g) be an m-dimensional compact Riemannian manifold with nonempty boundary $\partial\Sigma$ and f be a smooth function on Σ. The f-Laplacian $\Delta_f = \Delta - \nabla f \cdot \nabla$ is defined as before with respect to the metric g. Given a smooth function $u \in C^\infty(\partial\Sigma)$, by the classical existence result for elliptic PDE, there exists the f-harmonic extension $\hat{u}$ of u with

$$\begin{cases} \Delta_f \hat{u} = 0, & \text{in } \Sigma, \\ \hat{u} \;\; = u, & \text{on } \partial\Sigma. \end{cases} \tag{3.1}$$

Let ν be the unit outward normal of $\partial\Sigma$. The Dirichlet-to-Neumann map is the map $L : C^\infty(\partial\Sigma) \to C^\infty(\partial\Sigma)$ given by

$$Lu = \frac{\partial \hat{u}}{\partial \nu}, \tag{3.2}$$

which is a nonnegative self-adjoint operator with respect to the weighted volume form $e^{-f} d\mu_{\partial\Sigma}$ on $\partial\Sigma$. Thus there exist discrete eigenvalues $\lambda_{0,f} < \lambda_{1,f} \le \lambda_{2,f} \le \cdots +\infty$

of the operator L. Clearly $\lambda_{0,f} = 0$ because the constant function lies in the kernel of L. The first nonzero one $\lambda_{1,f}$ can be characterized by

$$\lambda_{1,f} = \inf\left\{ \frac{\int_\Sigma |\nabla \hat{u}|^2 e^{-f} d\mu_\Sigma}{\int_{\partial\Sigma} u^2 e^{-f} d\mu_{\partial\Sigma}} : u \in C^\infty(\partial\Sigma), \int_{\partial\Sigma} u e^{-f} d\mu_{\partial\Sigma} = 0 \right\}. \tag{3.3}$$

The following proposition gives a positive lower bound for $\lambda_{1,f}$ when Σ is a compact properly embedded f-minimal hypersurface with free boundary in $(M^n, g, e^{-f} d\mu_M)$.

Proposition 3.1 *Let $(M^n, g, e^{-f} d\mu_M)$ be a compact orientable smooth metric measure space with nonempty boundary ∂M. Suppose that M has nonnegative Ric_f and the boundary ∂M is strictly convex ($h^{\partial M}(u, u) \geq \kappa > 0$ for any tangent unit vector $u \in T\partial M$) and strictly f-mean convex. Let Σ be a properly embedded f-minimal hypersurface in M with free boundary on ∂M. Suppose that either Σ is orientable or $\pi_1(M)$ is finite, then the first f-Steklov eigenvalue $\lambda_{1,f}(\Sigma)$ of the f-Laplacian on Σ satisfies $\lambda_{1,f} \geq \kappa/2$.*

Proof We use a similar argument as in the Proof of Theorem 3.1 in [12] (see also [25,26]). To explain the difference and for the convenience of the readers, we give a complete proof here. The Reilly formula (2.8) again plays an important role.

Firstly, we assume that Σ is orientable. By the Lemma 2.3 and Corollary 2.5, we know that Σ is connected and Σ divides M into two connected components Ω_1 and Ω_2. Without loss of generality, we consider $\Omega = \Omega_1$ with boundary $\partial\Omega = \Sigma \cup \Gamma$, where $\Gamma \subset \partial M$. Let $u \in C^\infty(\partial\Sigma)$ be the eigenfunction corresponding to the first nonzero f-Steklov eigenvalue $\lambda_{1,f}$. Then there exists an f-harmonic extension $u_1 \in C^\infty(\Sigma)$ of u with $\Delta_f^\Sigma u_1 = 0$ in Σ and $\frac{\partial u_1}{\partial \nu_{\partial\Sigma}} = \lambda_{1,f} u$ on $\partial\Sigma$. Here $\nu_{\partial\Sigma}$ is the unit outward normal of $\partial\Sigma$ in Σ. Since $\partial\Sigma = \partial\Gamma$, we can have an f-harmonic extension $u_2 \in C^\infty(\Gamma)$ of $u \in C^\infty(\partial\Gamma)$ to Γ. Now let $\hat{u}$ be the solution of the following problem,

$$\begin{cases} \bar{\Delta}_f^\Omega \hat{u} = 0, & \text{in } \Omega, \\ \hat{u} = u_1, & \text{on } \Sigma, \\ \hat{u} = u_2, & \text{on } \Gamma. \end{cases}$$

Since the boundary $\partial\Omega$ is piecewise smooth, the standard results on elliptic PDE implies that $\hat{u}$ exists and $\hat{u} \in C^{1,\alpha}(\Omega) \cap C^\infty(\Omega \setminus \partial\Sigma)$ for $\alpha \in (0, 1)$. Applying the Reilly formula (2.8) to $\hat{u}$ and noting that $\Omega \subset M$ is a domain of M with boundary $\partial\Omega = \Sigma \cup \Gamma$, we obtain

$$\begin{aligned} 0 &= \int_\Omega \left(Ric_f(\bar{\nabla}\hat{u}, \bar{\nabla}\hat{u}) - |\bar{\Delta}_f^\Omega \hat{u}|^2 + |\bar{\nabla}^2 \hat{u}|^2 \right) e^{-f} d\mu_\Omega \\ &\quad + \int_{\partial\Omega} \left(\left(\Delta_f^{\partial\Omega} \hat{u} + H_f^{\partial\Omega} \frac{\partial \hat{u}}{\partial \nu} \right) \frac{\partial \hat{u}}{\partial \nu} - \left\langle \nabla \hat{u}, \nabla \frac{\partial \hat{u}}{\partial \nu} \right\rangle + h^{\partial\Omega}(\nabla \hat{u}, \nabla \hat{u}) \right) e^{-f} d\mu_{\partial\Omega}. \\ &\geq \int_\Sigma \left(H_f^\Sigma \left(\frac{\partial \hat{u}}{\partial \nu_\Sigma} \right)^2 - \left\langle \nabla \hat{u}, \nabla \frac{\partial \hat{u}}{\partial \nu_\Sigma} \right\rangle + h^\Sigma(\nabla \hat{u}, \nabla \hat{u}) \right) e^{-f} d\mu_\Sigma \end{aligned}$$

$$
\begin{aligned}
&+\int_{\Gamma}\left(H_f^{\Gamma}\left(\frac{\partial\hat u}{\partial\nu_{\Gamma}}\right)^2-\left\langle\nabla\hat u,\nabla\frac{\partial\hat u}{\partial\nu_{\Gamma}}\right\rangle+h^{\Gamma}(\nabla\hat u,\nabla\hat u)\right)e^{-f}d\mu_{\Gamma}\\
&\geq-\int_{\partial\Sigma}\frac{\partial\hat u}{\partial\nu_{\partial\Sigma}}\frac{\partial\hat u}{\partial\nu_{\Sigma}}e^{-f}d\mu_{\partial\Sigma}-\int_{\partial\Gamma}\frac{\partial\hat u}{\partial\nu_{\partial\Gamma}}\frac{\partial\hat u}{\partial\nu_{\Gamma}}e^{-f}d\mu_{\partial\Gamma}+\int_{\Gamma}\kappa|\nabla\hat u|^2e^{-f}d\mu_{\Gamma},
\end{aligned}
\tag{3.4}
$$

where in the first inequality we used the nonnegativity of Ric_f and $\bar\Delta_f^{\Omega}\hat u=0$ in Ω; in the second inequality we used $H_f^{\Sigma}=0$, $H_f^{\Gamma}>0$, $h^{\Gamma}\geq\kappa$, $\Delta_f^{\Sigma}u_1=0$, $\Delta_f^{\Gamma}u_2=0$ and integration by parts. We also assumed $\int_{\Sigma}h^{\Sigma}(\nabla\hat u,\nabla\hat u)e^{-f}d\mu_{\Sigma}\geq 0$ without loss of generality, for otherwise, we can choose $\Omega=\Omega_2$. Because Σ meets ∂M orthogonally, we have $\nu_{\Sigma}=\nu_{\partial\Gamma}$, $\nu_{\Gamma}=\nu_{\partial\Sigma}$ on $\partial\Sigma=\partial\Gamma$. Then (3.4) implies

$$
\begin{aligned}
\kappa\int_{\Gamma}|\nabla\hat u|^2e^{-f}d\mu_{\Gamma}&\leq 2\int_{\partial\Gamma}\frac{\partial\hat u}{\partial\nu_{\partial\Sigma}}\frac{\partial\hat u}{\partial\nu_{\partial\Gamma}}e^{-f}d\mu_{\partial\Gamma}\\
&=2\lambda_{1,f}\int_{\partial\Gamma}\hat u\frac{\partial\hat u}{\partial\nu_{\partial\Gamma}}e^{-f}d\mu_{\partial\Gamma}\\
&=2\lambda_{1,f}\int_{\Gamma}|\nabla\hat u|^2e^{-f}d\mu_{\Gamma}\,.
\end{aligned}
\tag{3.5}
$$

Since $\hat u$ is not a constant function on Γ, it follows from (3.5) that $\lambda_{1,f}\geq\kappa/2$.

For the case where $\pi_1(M)$ is finite, let $\tilde M$ be the universal cover of M. Then $\tilde M$ is compact and satisfies the same curvature assumption as M. Let $\tilde\Sigma=\pi^{-1}(\Sigma)$. Since $\tilde M$ is simply connected and $\tilde\Sigma$ is properly embedded in $\tilde M$, both $\tilde M$ and $\tilde\Sigma$ are orientable. Then $\lambda_{1,f}(\tilde\Sigma)\geq\kappa/2$ by the conclusion in the first case. Note that the lift of the first f-Steklov eigenfunction is also an f-Steklov eigenfunction on $\tilde\Sigma$, thus $\lambda_{1,f}(\Sigma)\geq\lambda_{1,f}(\tilde\Sigma)\geq\kappa/2$.

In the three-dimensional case, let Σ^2 be a compact properly embedded f-minimal surface in $(M^3,g,e^{-f}d\mu_M)$. We will use the above estimate on the first f-Steklov eigenvalue to obtain an a priori upper bound on the length of the boundary $\partial\Sigma$ with respect to the metric induced from the conformal metric $\tilde g=e^{-f}g$ on M^3.

Corollary 3.2 *Under the assumption of Proposition 3.1, in the $n=3$ case, the length $\tilde L(\partial\Sigma)$ of $\partial\Sigma$ with respect to the induced metric from $(M^3,\tilde g)$ satisfies*

$$
\tilde L(\partial\Sigma)\leq\frac{4\pi(\gamma+k)}{\kappa}e^{\frac{7}{2}\max_M f},
\tag{3.6}
$$

where γ is the genus of Σ and k is the number of the boundary components of $\partial\Sigma$. In other words, we have a uniform upper bound of $\tilde L(\partial\Sigma)$ in terms of the topology of Σ, the boundary convexity κ of ∂M and the bound of f on M.

Proof Unlike the conformal metric $\tilde g=e^{-f}g$ in Sect. 2.1, we consider another conformal metric $\hat g=e^{-2f}g$ on M^3 for a moment. This is crucial in the following proof. Then $d\hat\mu_{\Sigma}=e^{-2f}d\mu_{\Sigma}$, $d\hat\mu_{\partial\Sigma}=e^{-f}d\mu_{\partial\Sigma}$. We modify the argument in

[15, Theorem 2.3]. Recall that any compact surface with boundary can be properly conformally branched cover the disk D. Precisely, there exists a proper conformal branched cover $\varphi : (\Sigma, \hat{g}) \to D$ of degree at most $\gamma + k$ (see [27]). Here we denote by the same $\hat{g}$ as the induced metric on Σ from $(M^3, \hat{g})$. Using the automorphisms of the disk, we can further assume that the map φ satisfies (see, e.g., [26])

$$\int_{\partial\Sigma} \varphi^i \, d\hat{\mu}_{\partial\Sigma} = 0, \quad i = 1, 2.$$

Then φ^i, $i = 1, 2$ satisfies $\int_{\partial\Sigma} \varphi^i e^{-f} d\mu_{\partial\Sigma} = 0$. Let $\hat{\varphi}^i$ be the f-harmonic extension of $\varphi^i|_{\partial\Sigma}$ w.r.t. the metric induced from (M, g), i.e., $\Delta_f \hat{\varphi}^i = 0$, $i = 1, 2$. Here Δ_f is the f-Laplacian w.r.t. g. Then by the variational characterization (3.3) of $\lambda_{1,f}(\Sigma)$, we have

$$\begin{aligned} \lambda_{1,f}(\Sigma) \int_{\partial\Sigma} (\varphi^i)^2 e^{-f} d\mu_{\partial\Sigma} &\le \int_{\Sigma} |\nabla \hat{\varphi}^i|_g^2 e^{-f} d\mu_{\Sigma} \le \int_{\Sigma} |\nabla \varphi^i|_g^2 e^{-f} d\mu_{\Sigma} \\ &= \int_{\Sigma} e^{3f} |\nabla \varphi^i|_{\hat{g}}^2 d\hat{\mu}_{\Sigma} \; \le \; e^{3\max_M f} \int_{\Sigma} |\nabla \varphi^i|_{\hat{g}}^2 d\hat{\mu}_{\Sigma}, \end{aligned} \tag{3.7}$$

where in the second inequality we used the fact that the f-harmonic function minimizes the weighted Dirichlet energy in the space of smooth functions with the same boundary values on $\partial\Sigma$. Since $\varphi : (\Sigma, \hat{g}) \to D$ is conformal,

$$\sum_{i=1}^{2} \int_{\Sigma} |\nabla \varphi^i|_{\hat{g}}^2 d\hat{\mu}_{\Sigma} = 2A(\varphi(\Sigma)) \le 2\pi(\gamma + k), \tag{3.8}$$

where γ is the genus of Σ and k is the number of the boundary components of $\partial\Sigma$. On the other hand, since φ is a proper map, we have $\varphi(\partial\Sigma) \subset \partial D$. Thus

$$\begin{aligned} \sum_{i=1}^{2} \int_{\partial\Sigma} (\varphi^i)^2 e^{-f} d\mu_{\partial\Sigma} &= \sum_{i=1}^{2} \int_{\partial\Sigma} (\varphi^i)^2 d\hat{\mu}_{\partial\Sigma} = \int_{\partial\Sigma} d\hat{\mu}_{\partial\Sigma} \\ &= \int_{\partial\Sigma} e^{-\frac{f}{2}} d\tilde{\mu}_{\partial\Sigma} \; \ge \; e^{-\frac{1}{2}\max_M f} \tilde{L}(\partial\Sigma), \end{aligned} \tag{3.9}$$

where $\tilde{L}(\partial\Sigma) = \int_{\partial\Sigma} d\tilde{\mu}_{\partial\Sigma}$ is the length of $\partial\Sigma$ with respect to the conformal metric $\tilde{g}$. Combining (3.7), (3.8), (3.9) and Proposition 3.1, we obtain

$$\tilde{L}(\partial\Sigma) \le \frac{2\pi(\gamma + k)}{\lambda_{1,f}(\Sigma)} e^{\frac{7}{2}\max_M f} \le \frac{4\pi(\gamma + k)}{\kappa} e^{\frac{7}{2}\max_M f}. \tag{3.10}$$

□

4 Proof of Theorem 1.3

By Theorem 1.2, under the curvature assumption of M and convexity of ∂M, M^3 is diffeomorphic to the 3-ball B^3. Then M is simply connected. Let Σ be a compact properly embedded f-minimal surface in (M^3, g) with free boundary $\partial\Sigma$. As explained in Sect. 2.1, Σ is a compact properly embedded minimal surface in $(M, \tilde{g})$ with free boundary. We still denote by $\tilde{g}$ the metrics on Σ and $\partial\Sigma$ induced from $(M^3, \tilde{g})$. Then from the Gauss equation and the minimality of Σ in $(M, \tilde{g})$, we have

$$\frac{1}{2}\|\tilde{h}^\Sigma(x)\|^2 = \tilde{K}^M(x) - \tilde{K}^\Sigma(x) \tag{4.1}$$

for any $x \in \Sigma$, where $\tilde{h}^\Sigma$ is the second fundamental form of Σ in $(M, \tilde{g})$, $\tilde{K}^M(x)$, $\tilde{K}^\Sigma(x)$ are the sectional curvatures of the plane $T_x\Sigma$ w.r.t. $(M, \tilde{g})$ and $(\Sigma, \tilde{g})$ respectively. Integrating (4.1) over Σ w.r.t. $\tilde{g}$ and using the Gauss–Bonnet theorem, we have

$$\frac{1}{2}\int_\Sigma \|\tilde{h}^\Sigma\|^2 d\tilde{\mu}_\Sigma = \int_\Sigma \tilde{K}^M(x)d\tilde{\mu}_\Sigma + \int_{\partial\Sigma} k_{\tilde{g}} d\tilde{\mu}_{\partial\Sigma} - 2\pi\chi(\Sigma), \tag{4.2}$$

where $k_{\tilde{g}}$ is the geodesic curvature of the curve $(\partial\Sigma, \tilde{g})$. Since Σ meets ∂M orthogonally, the geodesic curvature $k_{\tilde{g}}$ is equal to $\tilde{h}^{\partial M}(\tilde{v}, \tilde{v})$ for the unit tangent vector $\tilde{v}$ of $(\partial\Sigma, \tilde{g})$. By the transformation formula (2.6) for the second fundamental form under the conformal change of the ambient metrics,

$$\tilde{h}^{\partial M}(\tilde{v}, \tilde{v}) = e^{\frac{f}{2}}\left(h(v, v) - \frac{1}{2}g(\bar{\nabla} f, \nu)\right), \tag{4.3}$$

where $v = e^{-\frac{f}{2}}\tilde{v}$ is the unit tangent vector of $(\partial\Sigma, g)$. Thus

$$\begin{aligned}\frac{1}{2}\int_\Sigma \|\tilde{h}^\Sigma\|^2 d\tilde{\mu}_\Sigma &\leq C\tilde{A}(\Sigma) + C\tilde{L}(\partial\Sigma) - 2\pi(2 - 2k - \gamma)\\ &\leq C\tilde{L}(\partial\Sigma) - 2\pi(2 - 2k - \gamma) \;\leq\; C,\end{aligned} \tag{4.4}$$

where in the second inequality we used the isoperimetric inequality in Lemma 2.1, and in the third inequality we used Corollary 3.2. Because the curvature $\tilde{K}^M$ involves up to second derivatives of f, the geodesic curvature $k_{\tilde{g}}$ involves up to first order derivatives of f, then the constant C in (4.4) depends only on the topology of Σ, the geometry of the ambient manifold (M, g) and the bounds on $\|f\|_{C^2}$.

Now let $\{\Sigma_i\}$ be a sequence of compact properly embedded f-minimal surfaces in (M^3, g) with free boundary of fixed topology type. Then $\{\Sigma_i\}$ is a sequence of compact properly embedded minimal surfaces in $(M^3, \tilde{g})$ with free boundary of fixed topology type. Using the same argument as in [12, Sect. 6] (see also [6,16]), we can find a subsequence, still denoted by $\{\Sigma_i\}$, and a finite point $\{x_1, \ldots, x_l\}$ such that Σ_i converges in C^∞ to some Σ_0 in $M \setminus U_{j=1}^l B_r(x_j)$ for any small $r > 0$. Moreover, $\Sigma = \Sigma_0 \cup \{x_1, \ldots, x_l\}$ is a compact properly embedded minimal surface with free boundary in $(M^3, \tilde{g})$. In order to show that Σ_i converges in C^∞ to Σ even across the points $\{x_1, \dots, x_l\}$, we need to show that Σ as the limit of Σ_i has only multiplicity one. As the standard argument in [6], we want to find some test function on Σ_i to show that $\lambda_{1,f}(\Sigma_i) \to 0$ as $i \to \infty$, if the multiplicity of Σ is bigger than one. In fact, we can use the same test function as in [6]. As Σ_i converges to Σ_0 in $M \setminus U_{j=1}^l B_{\epsilon^2}(x_j)$, by Theorem 5.1 in [12] and (4.4), for sufficiently large i, $\Sigma_i \setminus U_{j=1}^l B_{\epsilon^2}(x_j)$ is locally a union of graphs over Σ_0. Suppose that the number of graphs is bigger than one, then the top graph is disconnected with the other graphs. Thus $\Sigma_i \setminus U_{j=1}^l B_{\epsilon^2}(x_j) = \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are disjoint. Then define the Lipschitz function φ on Σ_i:

$$\varphi = \begin{cases} 1, & \text{on } \Gamma_1 \setminus U_{j=1}^l B_\epsilon(x_j) \\ \frac{\ln r_j - \ln \epsilon^2}{\ln \epsilon - \ln \epsilon^2}, & \text{on each } \Gamma_1 \cap (B_\epsilon(x_j) \setminus B_{\epsilon^2}(x_j)) \\ 0, & \text{on } \Sigma_i \cap U_{j=1}^l B_{\epsilon^2}(x_j) \\ -\frac{\ln r_j - \ln \epsilon^2}{\ln \epsilon - \ln \epsilon^2}, & \text{on each } \Gamma_2 \cap (B_\epsilon(x_j) \setminus B_{\epsilon^2}(x_j)) \\ -1, & \text{on } \Gamma_2 \setminus U_{j=1}^l B_\epsilon(x_j), \end{cases} \tag{4.5}$$

where r_j is the distance function from the point x_j on $(M^3, \tilde{g})$. Define

$$\bar{\varphi} = \frac{\int_{\partial\Sigma_i} \varphi e^{-f} d\mu_{\partial\Sigma_i}}{\int_{\partial\Sigma_i} e^{-f} d\mu_{\partial\Sigma_i}}.$$

Then $\psi = \varphi - \bar{\varphi}$ satisfies

$$\int_{\partial\Sigma_i} \psi e^{-f} d\mu_{\partial\Sigma_i} = 0. \tag{4.6}$$

The weighted Dirichlet energy of ψ on Σ_i satisfies

$$\int_{\Sigma_i} |\nabla\psi|_g^2 e^{-f} d\mu_{\Sigma_i} = \int_{\Sigma_i} e^f |\nabla\psi|_{\tilde{g}}^2 d\tilde{\mu}_{\Sigma_i} \le e^{\max_M f} \int_{\Sigma_i} |\nabla\psi|_{\tilde{g}}^2 d\tilde{\mu}_{\Sigma_i} \;\to\; 0$$

as $\epsilon \to 0$, where we used a same calculation as in [6,12], using the coarea formula and monotonicity formula for minimal surfaces with free boundary. On the other hand, since $\partial\Gamma_1, \partial\Gamma_2$ both cover $\partial\Sigma_0$ at least once, we obtain

$$\int_{\partial\Gamma_i} e^{-f} d\mu_{\partial\Gamma_i} \ge \int_{\partial\Sigma_0} e^{-f} d\mu_{\partial\Sigma_0} - \eta, \tag{4.7}$$

for any arbitrarily small $\eta > 0$ as $\epsilon \to 0$. Clearly, by the definition of ψ, $\int_{\partial\Sigma_i} \psi^2 e^{-f} d\mu_{\partial\Sigma_i}$ tends to a constant C as $\epsilon \to 0$. Using (4.7), such constant C is nonzero, because as $\epsilon \to 0$, we obtain

$$\begin{aligned} \int_{\partial\Sigma_i} \psi^2 e^{-f} d\mu_{\partial\Sigma_i} &\approx c_1 \int_{\partial\Gamma_1} e^{-f} d\mu_{\partial\Gamma_1} + c_2 \int_{\partial\Gamma_2} e^{-f} d\mu_{\partial\Gamma_2} \\ &\ge (c_1 + c_2) \int_{\partial\Sigma_0} e^{-f} d\mu_{\partial\Sigma_0} - (c_1 + c_2)\eta \;>\; 0, \end{aligned}$$

where c_1, c_2 are positive constants. Therefore by the variational characterization (3.3) of $\lambda_{1,f}(\Sigma_i)$, we have

$$\lambda_{1,f}(\Sigma_i) \leq \frac{\int_{\Sigma_i} |\nabla\psi|_g^2 e^{-f} d\mu_{\Sigma_i}}{\int_{\partial\Sigma_i} \psi^2 e^{-f} d\mu_{\partial\Sigma_i}} \to 0,$$

which contradicts with the conclusion of Proposition 3.1. So Σ as the limit of Σ_i has only multiplicity one. The remaining thing is just using the Allard regularity theorem [28] for minimal surfaces with free boundary to conclude that Σ_i converges in C^∞ to Σ even across the points $\{x_1, \ldots, x_l\}$. Then we complete the proof of Theorem 1.3.

Acknowledgments The authors would like to thank the referee for very careful reading and useful comments, which improve the presentation of this paper. The first author was supported by FAPEMIG and CNPq grants. The second author was supported by Jason D. Lotay's EPSRC grant.

References

1. Cao, H.-D.: Recent Progress on Ricci Solitons, Recent Advances in Geometric Analysis. Advanced Lectures in Mathematics, vol. 11. International Press, Somerville (2010)
2. Wei, G., Wylie, W.: Comparison geometry for the Bakry–Émery Ricci tensor. J. Differ. Geom. **83**, 377–405 (2009)
3. Colding, T.H., Minicozzi II, W.P.: Smooth compactness of self-shrinkers. Comment. Math. Helv. **87**(2), 463–475 (2012)
4. Ding, Q., Xin, Y.L.: Volume growth, eigenvalue and compactness for self-shrinkers. Asian J. Math. **17**(3), 443–456 (2013)
5. Li, H., Wei, Y.: f-minimal surface and manifold with positive m-Bakry–Émery Ricci curvature. J. Geom. Anal. **25**(1), 421–435 (2015)
6. Choi, H., Schoen, R.: The space of minimal embeddings of a surface into a three dimensional manifold of positive Ricci curvature. Invent. Math. **81**, 387–394 (1985)
7. Cheng, X., Mejia, T., Zhou, D.: Stability and compactness for complete f-minimal surfaces. Trans. Am. Math. Soc. **367**, 4041–4059 (2015)
8. Cheng, X., Mejia, T., Zhou, D.: Eigenvalue estimate and compactness for closed f-minimal surfaces. Pac. J. Math. **271**(2), 347–367 (2014)
9. Cheng, X. and Zhou, D.: Stability properties and gap theorem for complete f-minimal hypersurfaces. arXiv:1307.5099
10. Impera, D., Rimoldi, M.: Stability properties and topology at infinity of f-minimal hypersurfaces, Geom. Dedic. doi:10.1007/s10711-014-9999-6. (online first)
11. Liu, G.: Stable weighted minimal surfaces in manifolds with nonnegative Bakry–Émery tensor. Commun. Anal. Geom. **21**(5), 1061–1079 (2013)
12. Fraser, A., Li, M.M.: Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with non-negative Ricci curvature and convex boundary. J. Differ. Geom. **96**, 183–200 (2014)
13. Li, M.M.: A general existence theorem for embedded minimal surfaces with free boundary. Commun. Pure Appl. Math. **68**(2), 286–331 (2015). doi:10.1002/cpa.21513
14. Meeks, W., Simon, L., Yau, S.-T.: Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. Ann. Math. **116**(3), 621–659 (1982)
15. Fraser, A., Schoen, R.: The first Steklov eigenvalue, conformal geometry, and minimal surfaces. Adv. Math. **226**, 4011–4030 (2011)
16. Colding, T.H., Minicozzi II, W.P.: A Course in Minimal Surfaces. Graduate Studies in Mathematics, vol. 121. American Mathematical Society, Providence (2011)
17. Sharp, B.: Compactness of minimal hypersurfaces with bounded index. arXiv: 1501.02703
18. Barbosa, E., Sharp, B., Wei, Y.: Smooth compactness of f-minimal hypersurfaces with bounded f-index. arXiv:1503.01945

19. Castro, K., Rosales, C.: Free boundary stable hypersurfaces in manifolds with density and rigidity results. J. Geom. Phys. **79**, 14–28 (2014)
20. Ma, L., Du, S.-H.: Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians. C. R. Acad. Sci. Paris Ser. I **348**, 1203–1206 (2010)
21. Munteanu, O., Wang, J.: Smooth metric measure spaces with non-negative curvature. Commun. Anal. Geom. **19**(5), 451–486 (2011)
22. White, B.: Which ambient spaces admit isoperimetric inequalities for submanifolds? J. Differ. Geom. **83**(1), 213–228 (2009)
23. Lawson Jr, H.B.: The unknottedness of minimal embeddings. Invent. Math. **11**, 183–187 (1970)
24. Pontryagin, L.: Characteristic cycles on differentiable manifolds. Am. Math. Soc. Transl. **1950**(32), 1–72 (1950)
25. Choi, H., Wang, A.-N.: A first eigenvalue estimate for minimal hypersurfaces. J. Differ. Geom. **18**, 559–562 (1983)
26. Schoen, R., Yau, S.-T.: Lectures on Differential Geometry. International Press, Cambridge (1994)
27. Gabard, A.: Sur la représentation conforme des surfaces de Riemann á bord et une caractérisation des courbesséparantes. Comment. Math. Helv. **81**(4), 945–964 (2006)
28. Grüter, M., Jost, J.: Allard type regularity results for varifolds with free boundaries. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13**(1), 129–169 (1986)