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Onofri-Type Inequalities for Singular Liouville Equations

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Abstract We study the blow-up behavior of minimizing sequences for the singular Moser–Trudinger functional on compact surfaces. Assuming non-existence of minimum points, we give an estimate for the infimum value of the functional. This result can be applied to give sharp Onofri-type inequalities on the sphere in the presence of at most two singularities.

Keywords Onofri's inequality · Liouville equations · Conical singularities · Moser–Trudinger · Sphere

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1 Introduction

Let (Σ, g) be a smooth, compact Riemannian surface; the standard Moser–Trudinger inequality (see [16,22]) states that

$$\log\left(\frac{1}{|\Sigma|}\int_\Sigma e^{u-\overline{u}}dv_g\right) \le \frac{1}{16\pi}\int_\Sigma |\nabla_g u|^2 dv_g + C(\Sigma, g) \qquad \forall\, u \in H^1(\Sigma) \quad (1)$$

where $C(\Sigma, g)$ is a constant depending only on Σ and g, and the coefficient $\frac{1}{16\pi}$ is optimal. A sharp version of (1) was proved by Onofri in [23] for the sphere endowed with the standard Euclidean metric g_0. He identified the sharp value of C and the family of functions attaining equality, proving

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$$\log\left(\frac{1}{4\pi}\int_{S^2} e^{u-\overline{u}}dv_{g_0}\right)\leq \frac{1}{16\pi}\int_{S^2}|\nabla_{g_0}u|^2 dv_{g_0} \tag{2}$$

with equality holding if and only if the metric $e^u g$ has constant positive Gaussian curvature, or, equivalently, $u = \log|\det d\varphi| + c$ with $c\in\mathbb{R}$ and φ a conformal diffeomorphism of S^2. Onofri's inequality played an important role (see [12,13]) in the variational approach to the equation

$$\Delta_{g_0}u + K\, e^u = 1$$

which is connected to the classical problem of prescribing the Gaussian curvature of S^2. In this paper we will consider extensions of Onofri's result in connection with the study of the more general equation

$$-\Delta_g v = \rho\left(\frac{Ke^v}{\int_\Sigma Ke^v dv_g}-\frac{1}{|\Sigma|}\right) - 4\pi\sum_{i=1}^m \alpha_i\left(\delta_{p_i}-\frac{1}{|\Sigma|}\right), \tag{3}$$

where $K\in C^\infty(\Sigma)$ is a positive function, $\rho>0$, $p_1,\ldots,p_m\in\Sigma$ and $\alpha_1,\ldots,\alpha_m\in(-1,+\infty)$. This is known as the singular Liouville equation and arises in several problems in Riemannian geometry and mathematical physics. When $(\Sigma,g)=(S^2,g_0)$ and $\rho = 8\pi+4\pi\sum_{i=1}^m\alpha_i$, solutions of (3) provide metrics on S^2 with prescribed Gaussian curvature K and conical singularities of angle $2\pi(1+\alpha_i)$ (or of order α_i) in p_i, $i=1,\ldots,m$ (see for example [3,14,27]). Equation (3) also appears in the description of Abelian Chern–Simons vortices in superconductivity and Electroweak theory [17,25]. We refer to [4,9–11,21], for some recent existence results. Liouville equations also have applications in the description of holomorphic curves in $\mathbb{CP}^n$ [6,8] and in the nonabelian Chern–Simons theory which might have applications in high temperature superconductivity (see [26] and references therein). Denoting by G_p the Green's function at p, namely the solution of

$$\begin{cases} -\Delta_g G_p = \delta_p - \frac{1}{|\Sigma|} \\ \int_\Sigma G_p\, dv_g = 0 \end{cases},$$

the change of variables

$$u = v + 4\pi\sum_{i=1}^m \alpha_i G_{p_i}$$

transforms (3) into

$$-\Delta_g u = \rho\left(\frac{he^u}{\int_\Sigma he^u dv_g}-\frac{1}{|\Sigma|}\right) \tag{4}$$

where

$$h = K\prod_{1\leq i\leq m} e^{-4\pi\alpha_i G_{p_i}} \tag{5}$$

satisfies

$$h(p) \approx c_i \, d(p, p_i)^{2\alpha_i} \ \text{ for } p \approx p_i, \tag{6}$$

with $c_i > 0$.

In [27], studying curvature functions for surfaces with conical singularities, Troyanov proved that if $h \in C^\infty(\Sigma \backslash \{p_1, \ldots, p_m\})$ is a positive function satisfying (6), then

$$\log\left(\frac{1}{|\Sigma|}\int_\Sigma h\, e^{u-\overline{u}} dv_g\right) \le \frac{1}{16\pi \min\left\{1, 1+\min\limits_{1\le i\le m} \alpha_i\right\}} \int_\Sigma |\nabla_g u|^2 dv_g + C(\Sigma, g, h). \tag{7}$$

The optimal constant $C(\Sigma, g, h)$ can be obtained by minimizing the functional

$$J_{\overline{\rho}}(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dv_g + \frac{\overline{\rho}}{|\Sigma|}\int_\Sigma u\, dv_g - \overline{\rho}\log\left(\frac{1}{|\Sigma|}\int_\Sigma h e^u dv_g\right),$$

where $\overline{\rho} = \min\left\{1, 1+\min\limits_{1\le i\le m} \alpha_i\right\}$. In this paper we will assume non-existence of minimum points for $J_{\overline{\rho}}$ and exploit known blow-up results [1,2,5] to describe the behavior of a suitable minimizing sequence and compute $\inf\limits_{H^1(\Sigma)} J_{\overline{\rho}}$. The same technique was used by Ding, Jost, Li and Wang [15] to give an existence result for (3) in the regular case. From their proof it follows that if $\alpha_i = 0 \; \forall \, i$ and if there is no minimum for $J_{\overline{\rho}}$, then

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} = -8\pi\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{p\in\Sigma}\{4\pi A(p) + \log h(p)\}\right)$$

where $A(p)$ is the value in p of the regular part of G_p. Here we extend this result to the general case proving:

Theorem 1.1 *Assume that h satisfies (5) with $K \in C^\infty(\Sigma)$, $K > 0$, $\alpha_i \in (-1, +\infty)\backslash\{0\}$, and that there is no minimum point of $J_{\overline{\rho}}$. If $\alpha := \min\limits_{1\le i\le m} \alpha_i < 0$, then*

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} = -8\pi(1+\alpha)\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{1\le i\le m, \alpha_i = \alpha}\left\{4\pi A(p_i) + \log\left(\frac{K(p_i)}{1+\alpha}\prod_{j\neq i} e^{-4\pi\alpha_j G_{p_j}(p_i)}\right)\right\}\right)$$

while if $\alpha > 0$

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} = -8\pi\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{p\in\Sigma\backslash\{p_1,\ldots,p_m\}}\{4\pi A(p) + \log h(p)\}\right).$$

In the last part of the paper we consider the case of the standard sphere with $K \equiv 1$ and at most two singularities. When $m = 1$ a simple Kazdan–Warner type identity proves non-existence of solutions for (4). Thus, one can apply Theorem 1.1 to obtain the following sharp version of (7):

Theorem 1.2 *If* $h = e^{-4\pi\alpha_1 G_{p_1}}$ *with* $\alpha_1 \neq 0$, *then* $\forall\, u \in H^1(S^2)$

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) < \frac{1}{16\pi\min\{1, 1+\alpha_1\}}\int_{S^2}|\nabla u|^2 dv_{g_0} + \max\{\alpha_1, -\log(1+\alpha_1)\}.$$

The same non-existence argument works for $m = 2$, $\min\{\alpha_1, \alpha_2\} < 0$ and $\alpha_1 \neq \alpha_2$ if the singularities are located in two antipodal points.

Theorem 1.3 *Assume* $h = e^{-4\pi\alpha_1 G_{p_1} - 4\pi\alpha_2 G_{p_2}}$ *with* $p_2 = -p_1$, $\alpha_1 = \min\{\alpha_1, \alpha_2\} < 0$ *and* $\alpha_1 \neq \alpha_2$; *then* $\forall\, u \in H^1(S^2)$

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) < \frac{1}{16\pi(1+\alpha_1)}\int_{S^2}|\nabla u|^2 dv_{g_0} + \alpha_2 - \log(1+\alpha_1).$$

When $\alpha_1 = \alpha_2 < 0$ Theorem 1.1 cannot be directly applied because (4) has solutions. However, it is possible to use a stereographic projection and a classification result in [24] to find an explicit expression for the solutions. In particular a direct computation allows to prove that all the solutions are minima of $J_{\overline{\rho}}$ and to find the value of $\min\limits_{H^1(S^2)} J_{\overline{\rho}}$.

Theorem 1.4 *Assume* $h = e^{-4\pi\alpha(G_{p_1}+G_{p_2})}$ *with* $\alpha < 0$ *and* $p_1 = -p_2$; *then* $\forall\, u \in H^1(S^2)$ *we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) \leq \frac{1}{16\pi(1+\alpha)}\int_{S^2}|\nabla u|^2 dv_{g_0} + \alpha - \log(1+\alpha).$$

Moreover the following conditions are equivalent:

- *u realizes equality.*
- *If π denotes the stereographic projection from p_1 then*

$$u \circ \pi^{-1}(y) = 2\log\left(\frac{(1+|y|^2)^{1+\alpha}}{1+e^{\lambda}|y|^{2(1+\alpha)}}\right) + c$$

  for some $\lambda, c \in \mathbb{R}$.
- *$he^u g_0$ is a metric with constant positive Gaussian curvature and conical singularities of order α_i in p_i, $i = 1, 2$.*

This is a generalization of Onofri's inequality (2) for metrics with two conical singularities.

2 Preliminaries and Blow-Up Analysis

Let (Σ, g) be a smooth compact, connected, Riemannian surface and let $S := \{p_1, \dots, p_m\}$ be a finite subset of Σ. Let us consider a function h satisfying (5) with $K \in C^\infty(\Sigma)$, $K > 0$ and $\alpha_i \in (-1, +\infty)\setminus\{0\}$. In order to distinguish the singular points of h from the regular ones, we introduce a singularity index function

$$\beta(p) := \begin{cases} \alpha_i & \text{if} \quad p = p_i \\ 0 & \text{if} \quad p \notin S \end{cases} .$$

We will denote $\alpha := \min_{p \in \Sigma} \beta(p) = \min \left\{ \min_{1 \le i \le m} \alpha_i, 0 \right\}$ the minimum singularity order. We shall consider the functional

$$J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 dv_g + \frac{\rho}{|\Sigma|} \int_\Sigma u \, dv_g - \rho \log \left(\frac{1}{|\Sigma|} \int_\Sigma h e^u dv_g \right). \tag{8}$$

Our goal is to give a sharp version of (7) finding the explicit value of

$$C(\Sigma, g, h) = -\frac{1}{8\pi(1+\alpha)} \inf_{u \in H^1(\Sigma)} J_{8\pi(1+\alpha)}(u). \tag{9}$$

To simplify the notation we will set $\overline{\rho} := 8\pi(1+\alpha)$, $\rho_\varepsilon = \overline{\rho} - \varepsilon$, $J_\varepsilon := J_{\rho_\varepsilon}$ and $J := J_{\overline{\rho}}$. From (7) it follows that $\forall\, \varepsilon > 0$ the functional J_ε is coercive and, by direct methods, it is possible to find a function $u_\varepsilon \in H^1(\Sigma)$ satisfying

$$J_\varepsilon(u_\varepsilon) = \inf_{u \in H^1(\Sigma)} J_\varepsilon(u) \tag{10}$$

and

$$-\Delta_g u_\varepsilon = \rho_\varepsilon \left(\frac{h e^{u_\varepsilon}}{\int_\Sigma h e^{u_\varepsilon} dv_g} - \frac{1}{|\Sigma|} \right). \tag{11}$$

Since J_ε is invariant under addition of constants $\forall\, \varepsilon > 0$, we may also assume

$$\int_\Sigma h\, e^{u_\varepsilon} dv_g = 1. \tag{12}$$

Remark 2.1 $u_\varepsilon \in C^{0,\gamma}(\Sigma) \cap W^{1,s}(\Sigma)$ for some $\gamma \in (0,1)$ and $s > 2$.

Proof It is easy to see that $h \in L^q(\Sigma)$ for some $q > 1$ ($q = +\infty$ if $\alpha = 0$ and $q < -\frac{1}{\alpha}$ for $\alpha < 0$). Applying locally Remarks 2 and 5 in [7] one can show that $u_\varepsilon \in L^\infty(\Sigma)$ so $-\Delta u_\varepsilon \in L^q(\Sigma)$ and by standard elliptic estimates $u_\varepsilon \in W^{2,q}(\Sigma)$. Since $q > 1$ the conclusion follows by Sobolev's embedding theorems. □

The behavior of u_ε is described by the following concentration-compactness result:

Proposition 2.1 *Let u_n be a sequence satisfying*

$$-\Delta_g u_n = V_n e^{u_n} - \psi_n$$

and

$$\int_\Sigma V_n e^{u_n} dv_g \le C_1,$$

where $\|\psi_n\|_{L^s(\Sigma)} \le C_2$ for some $s > 1$, and

$$V_n = K_n \prod_{1\le i\le m} e^{-4\pi\alpha_i G_{p_i}}$$

with $K_n \in C^\infty(\Sigma)$, $0 < a \le K_n \le b$ and $\alpha_i > -1$, $i = 1, \dots, m$. Then there exists a subsequence u_{n_k} of u_n such that the following alternatives hold:

1. *u_{n_k} is uniformly bounded in $L^\infty(\Sigma)$;*
2. *$u_{n_k} \longrightarrow -\infty$ uniformly on Σ;*
3. *there exist a finite blow-up set $B = \{q_1, \dots, q_l\} \subseteq \Sigma$ and a corresponding family of sequences $\{q_k^j\}_{k\in\mathbb{N}}$, $j = 1, \dots, l$ such that $q_k^j \overset{k\to\infty}{\longrightarrow} q_j$ and $u_{n_k}(q_k^j) \overset{k\to\infty}{\longrightarrow} +\infty$ $j = 1, \dots, l$. Moreover $u_{n_k} \overset{k\to\infty}{\longrightarrow} -\infty$ uniformly on compact subsets of $\Sigma\backslash B$ and $V_{n_k} e^{u_{n_k}} \rightharpoonup \sum_{j=1}^l \beta_j \delta_{q_j}$ weakly in the sense of measures where $\beta_j = 8\pi(1 + \beta(q_j))$ for $j = 1, \dots, l$.*

A proof of Proposition 2.1 in the regular case can be found in [19] while the general case is a consequence of the results in [1,5]. In our analysis we will also need the following local version of Proposition 2.1 proved by Li and Shafrir [20]:

Proposition 2.2 *Let Ω be an open domain in $\mathbb{R}^2$ and v_n be a sequence satisfying $\|e^{v_n}\|_{L^1(\Omega)} \le C$ and*

$$-\Delta v_n = V_n e^{v_n}$$

where $0 \le V_n \in C_0(\overline{\Omega})$ and $V_n \longrightarrow V$ uniformly in $\overline{\Omega}$. If v_n is not uniformly bounded from above on compact subset of Ω, then $V_n e^{v_n} \rightharpoonup 8\pi \sum_{j=1}^{l} m_j \delta_{q_j}$ as measures, with $q_j \in \Omega$ and $m_j \in \mathbb{N}^+$, $j = 1, \dots, l$.

Applying Proposition 2.1 to u_ε under the additional condition (12) we obtain that either u_ε is uniformly bounded in $L^\infty(\Sigma)$ or its blow-up set contains a single point p such that $\beta(p) = \alpha$. In the first case, one can use elliptic estimates to find uniform bounds on u_ε in $W^{2,q}(\Sigma)$, for some $q > 1$; consequently, a subsequence of u_ε converges in $H^1(\Sigma)$ to a function $u \in H^1(\Sigma)$ that is a minimum point of J and a solution of (4) for $\rho = \overline{\rho}$. We now focus on the second case, that is

$$\lambda_\varepsilon := \max_\Sigma u_\varepsilon = u_\varepsilon(p_\varepsilon) \longrightarrow +\infty \quad \text{and} \quad p_\varepsilon \longrightarrow p \quad \text{with} \quad \beta(p) = \alpha. \tag{13}$$

By Proposition 2.1 we also get:

Lemma 2.1 *If u_ε satisfies (11), (12) and (13), then, up to subsequences,*

1. $\rho_\varepsilon h e^{u_\varepsilon} \rightharpoonup \overline{\rho}\, \delta_p$;
2. $u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} -\infty$ *uniformly in* Ω, $\forall\, \Omega \subset\subset \Sigma\backslash\{p\}$;
3. $\overline{u}_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} -\infty$;
4. *There exist* $\gamma \in (0,1)$, $s > 2$ *such that* $u_\varepsilon - \overline{u_\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} \overline{\rho}\, G_p$ *in* $C^{0,\gamma}(\overline{\Omega}) \cap W^{1,s}(\Omega)$ $\forall\, \Omega \subset\subset \Sigma\backslash\{p\}$;
5. ∇u_ε *is bounded in* $L^q(\Sigma)$ $\forall\, q \in (1,2)$.

Proof *1.*, *2.* and *3.* are direct consequences of Proposition 2.1. To prove *4.*, we consider the Green's representation formula

$$u_\varepsilon(x) - \overline{u}_\varepsilon = \rho_\varepsilon \int_\Sigma G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y).$$

We stress that the Green's function has the following properties:

- $|G_x(y)| \le C_1(1 + |\log d(x,y)|)\ \forall\, x, y \in \Sigma, x \neq y$.
- $|\nabla_g^x G_x(y)| \le \dfrac{C_2}{d(x,y)}\ \forall\, x, y \in \Sigma, x \neq y$.
- $G_x(y) = G_y(x)\ \forall\, x, y \in \Sigma, x \neq y$.

Take $q > 1$ such that $h \in L^q(\Sigma)$. The first property also yields

$$\sup_{x\in\Sigma} \|G_x\|_{L^{q'}(\Sigma)} \le C_3. \tag{14}$$

Let us fix $\delta > 0$ such that $B_{3\delta}(p) \subset \Sigma\backslash\Omega$ and take a cut-off function φ such that $\varphi \equiv 1$ in $B_\delta(p)$ and $\varphi \equiv 0$ in $\Sigma\backslash B_{2\delta}(p)$.

$$\begin{aligned} u_\varepsilon(x) - \overline{u_\varepsilon} &= \rho_\varepsilon \int_\Sigma \varphi(y) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \\ &\quad + \rho_\varepsilon \int_\Sigma (1-\varphi(y)) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y). \end{aligned}$$

By (14) and *2.* we have

$$\begin{aligned} \left| \int_\Sigma (1-\varphi(y)) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \right| &\le \int_{\Sigma\backslash B_\delta(p)} |G_x(y)|\, h(y) e^{u_\varepsilon(y)} dv_g(y) \\ &\le C_3 \|h\|_{L^q(\Sigma)} \|e^{u_\varepsilon}\|_{L^\infty(\Sigma\backslash B_\delta(p))} \overset{\varepsilon\to 0}{\longrightarrow} 0. \end{aligned}$$

By *1.* and the smoothness of φG_x for $x \in \overline{\Omega}$ and $y \in \Sigma$ we get

$$\int_\Sigma \varphi(y) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \overset{\varepsilon\to 0}{\longrightarrow} \varphi(p) G_x(p) = G_p(x)$$

uniformly for $x \in \Omega$. Similarly we have

$$\nabla_g u_\varepsilon(x) = \rho_\varepsilon \int_\Sigma \varphi(y) \nabla_g^x G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \\ +\rho_\varepsilon \int_\Sigma (1-\varphi(y)) \nabla_g^x G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y)$$

with

$$\int_\Sigma \varphi(y) \nabla_g^x G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \overset{k\to\infty}{\longrightarrow} \nabla_g^x G_p(x)$$

uniformly in Ω and, assuming $q \in (1,2)$, by the Hardy–Littlewood–Sobolev inequality

$$\int_\Sigma \left(\int_\Sigma (1-\varphi(y)) \nabla_g^x G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \right)^s dv_g(x) \\ \le C_2^s \int_\Sigma \left(\int_{\Sigma \setminus B_\delta(p)} \frac{h(y) e^{u_\varepsilon(y)}}{d(x,y)} dv_g(y) \right)^s dv_g(x) \\ \le C \|h\|^s_{L^q(\Sigma)} \|e^{u_n}\|^s_{L^\infty(\Sigma\setminus B_\delta(p))} \overset{\varepsilon\to 0}{\longrightarrow} 0$$

where

$$\frac{1}{s} = \frac{1}{q} - \frac{1}{2}.$$

Note that $q > 1$ implies $s > 2$. Finally, to prove *5.*, we shall observe that for any $1 < q < 2$ there exists a positive constant C_q such that

$$\int_\Sigma \varphi \, dv_g = 0 \quad \text{and} \quad \int_\Sigma |\nabla_g \varphi|^{q'} dv_g \le 1 \quad \Longrightarrow \quad \|\varphi\|_\infty \le C_q.$$

Hence $\forall\, \varphi \in W^{1,q'}(\Sigma)$

$$\int_\Sigma \nabla_g u_\varepsilon \cdot \nabla_g \varphi \, dv_g = -\int_\Sigma \Delta u_\varepsilon \varphi \, dv_g \le C_q \|\Delta u_\varepsilon\|_{L^1(\Sigma)} \le \tilde{C}_q$$

so that

$$\|\nabla u_\varepsilon\|_{L^q} \le \sup \left\{ \int_\Sigma \nabla_g u_\varepsilon \cdot \nabla_g \varphi \, dv_g \ : \ \varphi \in W^{1,q'}(\Sigma), \|\nabla \varphi\|_{L^{q'}} \le 1 \right\} \le \tilde{C}_q.$$

□

We now focus on the behavior of u_ε near the blow-up point. First we consider the case $\alpha < 0$. Let us fix a system of normal coordinates in a small ball $B_\delta(p)$, with p corresponding to 0 and p_ε corresponding to x_ε. We define

$$\varphi_\varepsilon(x) := u_\varepsilon(t_\varepsilon x) - \lambda_\varepsilon, \quad t_\varepsilon := e^{-\frac{\lambda_\varepsilon}{2(1+\alpha)}}. \tag{15}$$

Lemma 2.2 *If* $\alpha < 0$, $\frac{|x_\varepsilon|}{t_\varepsilon}$ *is bounded.*

Proof We define

$$\psi_\varepsilon(x) = u_\varepsilon(|x_\varepsilon|x) + 2(1+\alpha)\log|x_\varepsilon| + s_\varepsilon(|x_\varepsilon|x)$$

where $s_\varepsilon(x)$ is the solution of

$$\begin{cases} -\Delta s_\varepsilon = \frac{\rho_\varepsilon}{|\Sigma|} & \text{in} \quad B_\delta(0) \\ \quad s_\varepsilon = 0 & \text{if} \quad |x| = \delta \end{cases}.$$

The function ψ_ε satisfies

$$-\Delta\psi_\varepsilon = |x_\varepsilon|^{-2\alpha}\rho_\varepsilon h(|x_\varepsilon|x)e^{-s_\varepsilon(|x_\varepsilon|x)}e^{\psi_\varepsilon} = V_\varepsilon e^{\psi_\varepsilon}$$

in $B_{\frac{\delta}{|x_\varepsilon|}}(0)$. We stress that, by standard elliptic estimates, s_ε is uniformly bounded in $C^1(\overline{B_\delta})$ and that G_p has the expansion

$$G_p(x) = -\frac{1}{2\pi}\log|x| + A(p) + O(|x|) \tag{16}$$

in $B_\delta(0)$. Thus

$$\begin{aligned} &|x_\varepsilon|^{-2\alpha}h(|x_\varepsilon|x)e^{-s_\varepsilon(|x_\varepsilon|x)} \\ &\quad = |x_\varepsilon|^{-2\alpha}e^{2\alpha\log(|x_\varepsilon||x|)-4\pi\alpha A(p)+O(|x_\varepsilon||x|)}e^{-s_\varepsilon(|x_\varepsilon|x)}K(|x_\varepsilon|x) \\ &\qquad \times \prod_{1\le i\le m, p_i\ne p} e^{-4\pi\alpha_i G_{p_i}(|x_\varepsilon|x)} \\ &\quad = |x|^{2\alpha}e^{-4\pi\alpha A(p)}e^{O(|x_\varepsilon||x|)}e^{-s_\varepsilon(|x_\varepsilon|x)}K(|x_\varepsilon|x) \\ &\qquad \times \prod_{1\le i\le m, p_i\ne p} e^{-4\pi\alpha_i G_{p_i}(|x_\varepsilon|x)} = |x|^{2\alpha}\tilde{h}(|x_\varepsilon|x) \end{aligned}$$

where $\tilde{h} \in C^1(\overline{B_\delta})$. In particular V_ε is uniformly bounded in $C^1_{loc}(\mathbb{R}^2\setminus\{0\})$. If there existed a subsequence such that $\frac{|x_\varepsilon|}{t_\varepsilon} \longrightarrow +\infty$ then

$$\psi_\varepsilon\left(\frac{x_\varepsilon}{|x_\varepsilon|}\right) = 2(1+\alpha)\log\left(\frac{|x_\varepsilon|}{t_\varepsilon}\right) + s_\varepsilon(x_\varepsilon) \longrightarrow +\infty,$$

so $y_0 := \lim_{\varepsilon\to 0}\frac{x_\varepsilon}{|x_\varepsilon|}$ would be a blow-up point for ψ_ε. Since $y_0 \ne 0$, applying Proposition 2.2 to ψ_ε in a small ball $B_r(y_0)$ we would get

$$\liminf_{\varepsilon\to 0}\int_{B_r(y_0)} V_\varepsilon e^{\psi_\varepsilon}dx \geq 8\pi.$$

But this would be in contradiction to (12) since

$$\begin{aligned}\int_{B_r(y_0)} V_\varepsilon e^{\psi_\varepsilon}dx &= \int_{B_{r(y_0)}} \rho_\varepsilon\ |x_\varepsilon|^{-2\alpha}h(|x_\varepsilon|x)e^{-s_\varepsilon(|x_\varepsilon|x)}e^{\psi_\varepsilon}dx\\ &\leq \rho_\varepsilon\int_{B_\delta(p)} he^{u_\varepsilon}dv_g \leq 8\pi(1+\alpha) < 8\pi.\end{aligned}$$

□

Lemma 2.3 *Assume $\alpha < 0$. Then, possibly passing to a subsequence, φ_ε converges uniformly on compact subsets of $\mathbb{R}^2$ and in $H^1_{loc}(\mathbb{R}^2)$ to*

$$\varphi_0(x) := -2\log\left(1+\frac{\pi c(p)}{1+\alpha}|x|^{2(1+\alpha)}\right)$$

where $c(p) = K(p)e^{-4\pi\alpha A(p)}\prod_{1\leq i\leq m,\, p_i\neq p} e^{-4\pi\alpha_i G_{p_i}(p)}$.

Proof The function φ_ε is defined in $B_\varepsilon = B_{\frac{\delta}{t_\varepsilon}}(0)$ and satisfies

$$-\Delta\varphi_\varepsilon = t_\varepsilon^2\rho_\varepsilon\left(h(t_\varepsilon x)e^{\varphi_\varepsilon}e^{\lambda_\varepsilon} - \frac{1}{|\Sigma|}\right) = t_\varepsilon^{-2\alpha}\rho_\varepsilon h(t_\varepsilon x)e^{\varphi_\varepsilon} - \frac{t_\varepsilon^2\rho_\varepsilon}{|\Sigma|}$$

and

$$t_\varepsilon^{-2\alpha}\int_{B_{\frac{\delta}{t_\varepsilon}}} h(t_\varepsilon x)e^{\varphi_\varepsilon} \leq 1.$$

As in the previous proof we have

$$\begin{aligned}t_\varepsilon^{-2\alpha}h(t_\varepsilon x) &= t_\varepsilon^{-2\alpha}e^{2\alpha\log(t_\varepsilon|x|)-4\pi\alpha A(p)+O(t_\varepsilon|x|)}K(t_\varepsilon x)\prod_{1\leq i\leq m,\, p_i\neq p} e^{-4\pi\alpha_i G_{p_i}(t_\varepsilon x)}\\ &= |x|^{2\alpha}e^{-4\pi\alpha A(p)}e^{O(t_\varepsilon|x|)}K(t_\varepsilon x)\prod_{1\leq i\leq m,\, p_i\neq p} e^{-4\pi\alpha_i G_{p_i}(t_\varepsilon x)} \overset{\varepsilon\to 0}{\longrightarrow} c(p)|x|^{2\alpha}\end{aligned}$$

in $L^q_{loc}(\mathbb{R}^2)$ for some $q > 1$. Fix $R > 0$ and let ψ_ε be the solution of

$$\begin{cases} -\Delta\psi_\varepsilon = t_\varepsilon^{-2\alpha}\rho_\varepsilon h(t_\varepsilon x)e^{\varphi_\varepsilon} - \frac{t_\varepsilon^2\rho_\varepsilon}{|\Sigma|} & \text{in } B_R(0)\\ \psi_\varepsilon = 0 & \text{su } \partial B_R(0)\end{cases}.$$

Since $\Delta\psi_\varepsilon$ is bounded in $L^q(B_R(0))$ with $q > 1$, elliptic regularity shows that ψ_ε is bounded in $W^{2,q}(B_R(0))$ and by Sobolev's embeddings we may extract a subsequence such that ψ_ε converges in $H^1(B_R(0))\cap C^{0,\lambda}(B_R(0))$. The function $\xi_\varepsilon=\varphi_\varepsilon-\psi_\varepsilon$ is harmonic in B_R and bounded from above. Furthermore $\xi_\varepsilon\left(\frac{x_\varepsilon}{t_\varepsilon}\right)=-\psi_\varepsilon\left(\frac{x_\varepsilon}{t_\varepsilon}\right)$ is bounded from below, hence by Harnack inequality ξ_ε is uniformly bounded in $C^2(\overline{B_{\frac{R}{2}}}(0))$. Thus φ_ε is bounded in $W^{2,q}(B_{\frac{R}{2}})$ and we can extract a subsequence converging in $H^1(B_{\frac{R}{2}})\cap C^{0,\lambda}(B_{\frac{R}{2}})$. Using a diagonal argument we find a subsequence for which φ_ε converges in $H^1_{loc}(\mathbb{R}^2)\cap C^{0,\lambda}_{loc}(\mathbb{R}^2)$ to a function φ_0 solving

$$-\Delta\varphi_0=8\pi(1+\alpha)c(p)|x|^{2\alpha}e^{\varphi_0}$$

on $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2}|x|^{2\alpha}e^{\varphi_0(x)}dx<\infty.$$

The classification result in [24] yields

$$\varphi_0(x)=-2\log\left(1+\frac{\pi e^{\lambda}c(p)}{1+\alpha}|x|^{2(1+\alpha)}\right)+\lambda$$

for some $\lambda\in\mathbb{R}$. To conclude the proof it remains to note that, since 0 is the unique maximum point of φ_0, the uniform convergence of φ_ε implies $\frac{x_\varepsilon}{t_\varepsilon}\longrightarrow 0$ and $\lambda=0$.□

As in [15], to give a lower bound on $J_\varepsilon(u_\varepsilon)$ we need the following estimate from below for u_ε:

Lemma 2.4 *Fix $R>0$ and define $r_\varepsilon=t_\varepsilon R$. If $\alpha<0$ and u_ε satisfies (11), (12), (13), then*

$$u_\varepsilon\geq\overline{\rho}\,G_p-\lambda_\varepsilon-\overline{\rho}\,A(p)+2\log\left(\frac{R^{2(1+\alpha)}}{1+\frac{\pi c(p)}{1+\alpha}R^{2(1+\alpha)}}\right)+o_\varepsilon(1)$$

in $\Sigma\backslash B_{r_\varepsilon}(p)$, where $o_\varepsilon(1)$ is a function of ε and R such that $o_\varepsilon(1)\longrightarrow 0$ as $\varepsilon\to 0$.

Proof $\forall\, C>0$ we have

$$-\Delta_g(u_\varepsilon-\overline{\rho}\,G_p-C)=\rho_\varepsilon\left(he^{u_\varepsilon}-\frac{1}{|\Sigma|}\right)+\frac{\overline{\rho}}{|\Sigma|}=\rho_\varepsilon he^{u_\varepsilon}+\frac{\varepsilon}{|\Sigma|}\geq 0.$$

Let us consider normal coordinates near p. We know that

$$G_p(x)=-\frac{1}{2\pi}\log|x|+A(p)+O(|x|),$$

so by Lemma 2.3 if $x = t_\varepsilon y$ with $|y| = R$ we have

$$u_\varepsilon(x) - \overline{\rho}\, G_p = \varphi_\varepsilon(y) + \lambda_\varepsilon + 4(1+\alpha)\log(t_\varepsilon R) - \overline{\rho}A(p) + O(t_\varepsilon R)$$
$$\geq -2\log\left(1 + \frac{\pi c(p)}{1+\alpha} R^{2(1+\alpha)}\right) - \lambda_\varepsilon + \log R^{4(1+\alpha)} - \overline{\rho}\, A(p) + o_\varepsilon(1).$$

Thus, taking

$$C_{\varepsilon,R} = -\lambda_\varepsilon - \overline{\rho}\, A(p) + 2\log\left(\frac{R^{2(1+\alpha)}}{1 + \frac{\pi c(p)}{1+\alpha} R^{2(1+\alpha)}}\right) + o_\varepsilon(1)$$

we have $u_\varepsilon - \overline{\rho} G_p - C_{\varepsilon,R} \geq 0$ on $\partial B_{r_\varepsilon}(p)$ and the conclusion follows from the maximum principle. □

As a consequence we also have

Lemma 2.5 *If u_ε and t_ε are as above, then $t_\varepsilon^2 \overline{u}_\varepsilon \longrightarrow 0$.*

Proof By Lemma 2.3

$$\int_{B_{t_\varepsilon}(p)} u_\varepsilon \, dv_g = t_\varepsilon^2 \int_{B_1(0)} \varphi_\varepsilon(y) dy + \lambda_\varepsilon |B_{t_\varepsilon}| = o_\varepsilon(1).$$

and by the previous lemma

$$\lambda_\varepsilon |\Sigma| \geq \int_{\Sigma \setminus B_{t_\varepsilon}(p)} u_\varepsilon \geq \overline{\rho} \int_{\Sigma \setminus B_{t_\varepsilon}(p)} G_p \, dv_g - \lambda_\varepsilon |\Sigma \setminus B_{t_\varepsilon}(p)| + O(1).$$

Thus $\dfrac{|\overline{u}_\varepsilon|}{\lambda_\varepsilon}$ is bounded and, since $\lambda_\varepsilon t_\varepsilon^2 = o_\varepsilon(1)$, we get the conclusion. □

The case $\alpha = 0$ can be studied in a similar way. The main difference is that, since we do not know whether $\frac{|x_\varepsilon|}{t_\varepsilon}$ is bounded, we have to center the scaling in p_ε and not in p. Note that $\beta(p) = 0$ means that $p \in \Sigma \setminus S$ is a regular point of h.

Lemma 2.6 *Assume that $\alpha = 0$ and that u_ε satisfies (11), (12) and (13). In normal coordinates near p define*

$$\psi_\varepsilon(x) = u_\varepsilon(x_\varepsilon + t_\varepsilon x) - \lambda_\varepsilon \quad \textit{where} \quad t_\varepsilon = e^{-\frac{\lambda_\varepsilon}{2}}.$$

Then

1. ψ_ε converges in $C^1_{loc}(\mathbb{R}^2)$ to

$$\psi_0(x) = -2\log(1 + \pi h(p)|x|^2)$$

2. $\forall\ R > 0$ *one has*

$$u_\varepsilon \geq 8\pi G_{p_\varepsilon} - \lambda_\varepsilon - 8\pi A(p) + 2\log\left(\frac{R^2}{1+\pi h(p)R^2}\right) + o_\varepsilon(1)$$

in $\Sigma\backslash B_{Rt_\varepsilon}(p_\varepsilon)$;
3. $t_\varepsilon^2 \overline{u}_\varepsilon \to 0$.

3 A Lower Bound

In this section and in the next one we present the proof of Theorem 1.1. We begin by giving an estimate from below of $\inf\limits_{H^1(\Sigma)} J$. As before we consider u_ε satisfying (10), (11), (12), and (13). Again we will focus on the case $\alpha < 0$ since the computation for $\alpha = 0$ is equivalent to the one in [15]. We consider normal coordinates in a small ball $B_\delta(p)$ and assume that G_p has the expansion (16) in $B_\delta(p)$. Let t_ε be defined as in (15), then $\forall\ R > 0$ we shall consider the decomposition

$$\int_\Sigma |\nabla_g u_\varepsilon|^2 dv_g = \int_{\Sigma\backslash B_\delta(p)} |\nabla_g u_\varepsilon|^2 dv_g + \int_{B_\delta\backslash B_{r_\varepsilon}(p)} |\nabla_g u_\varepsilon|^2 dv_g + \int_{B_{r_\varepsilon}(p)} |\nabla_g u_\varepsilon|^2 dv_g.$$

Throughout this section, $o_\delta(1)$ (and $o_R(1)$) will denote a function depending only on δ (resp. R) which converges to 0 as $\delta \to 0$ (resp. $R \to \infty$), while the notation $o_\varepsilon(1)$ will be used for functions of ε, δ and R such that, for fixed δ and R, $o_\varepsilon(1) \longrightarrow 0$ as $\varepsilon \to 0$.

On $\Sigma\backslash B_\delta(p)$ we can use Lemma 2.1 and an integration by parts to obtain:

$$\begin{aligned}\int_{\Sigma\backslash B_\delta} |\nabla_g u_\varepsilon|^2 dv_g &= \overline{\rho}^2 \int_{\Sigma\backslash B_\delta} |\nabla_g G_p|^2 dv_g + o_\varepsilon(1)\\ &= -\frac{\overline{\rho}^2}{|\Sigma|}\int_{\Sigma\backslash B_\delta} G_p\, dv_g - \overline{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n}\, d\sigma_g + o_\varepsilon(1)\\ &= -\overline{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) + o_\delta(1). \end{aligned} \tag{17}$$

On $B_{r_\varepsilon}(p)$ the convergence result for the scaling (15) stated in Lemma 2.3 yields

$$\begin{aligned}\int_{B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &= \int_{B_R(0)} |\nabla\varphi_0|^2 dx + o_\varepsilon(1) = 2\overline{\rho}\left(\log\left(1+\frac{\pi\ c(p)}{1+\alpha}R^{2(1+\alpha)}\right) - 1\right)\\ &\quad + o_\varepsilon(1) + o_R(1).\end{aligned} \tag{18}$$

For the remaining term we can use (11) and Lemma 2.1 to obtain

$$\begin{aligned}\int_{B_\delta\backslash B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &= \rho_\varepsilon \int_{B_\delta\backslash B_{r_\varepsilon}} he^{u_\varepsilon} u_\varepsilon dv_g - \frac{\rho_\varepsilon}{|\Sigma|}\int_{B_\delta\backslash B_{r_\varepsilon}} u_\varepsilon dv_g\\ &\quad + \int_{\partial B_\delta} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma_g\end{aligned}$$

$$
\begin{aligned}
&= \rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} u_\varepsilon dv_g - \frac{\rho_\varepsilon}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g \\
&\quad + \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma_g \\
&\quad + \overline{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1). \qquad (19)
\end{aligned}
$$

By Lemma 2.4 and (12) we get

$$
\begin{aligned}
\rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} u_\varepsilon dv_g &\geq \rho_\varepsilon \overline{\rho} \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g - \rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g \\
&\quad + O(1) \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g \\
&= \rho_\varepsilon \overline{\rho} \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g - \rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g \\
&\quad + o_\varepsilon(1) + o_R(1). \qquad (20)
\end{aligned}
$$

Again by (11) and Lemma 2.1

$$
\begin{aligned}
\rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g &= \int_{B_\delta \setminus B_{r_\varepsilon}} G_p \left(-\Delta u_\varepsilon + \frac{\rho_\varepsilon}{|\Sigma|} \right) dv_g \\
&= -\frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \int_{\partial (B_\delta \setminus B_{r_\varepsilon})} u_\varepsilon \frac{\partial G_p}{\partial n} - G_p \frac{\partial u_\varepsilon}{\partial n} d\sigma_g \\
&\quad + o_\varepsilon(1) + o_\delta(1) \\
&= -\frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g \\
&\quad + \int_{\partial B_{r_\varepsilon}} G_p \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g \\
&\quad + o_\varepsilon(1) + o_\delta(1), \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g &= -\lambda_\varepsilon \int_{\partial B_\delta \setminus B_{r_\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} \left(Vol(B_\delta) - Vol(B_{r_\varepsilon}) \right) \\
&= -\lambda_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \lambda_\varepsilon \int_{\partial B_{r_\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} Vol(B_\delta) + o_\varepsilon(1). \qquad (22)
\end{aligned}
$$

Using (19), (20), (21) and (22) we get

$$
\begin{aligned}
\int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &\geq -(16\pi(1+\alpha) - \varepsilon) \frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon \, dv_g - \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} Vol(B_\delta) \\
&\quad + \overline{\rho}\, \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g + \lambda_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g
\end{aligned}
$$

$$+\overline{\rho}^2\int_{\partial B_\delta} G_p\frac{\partial G_p}{\partial n}d\sigma_g-\overline{\rho}\int_{\partial B_{r_\varepsilon}} u_\varepsilon\frac{\partial G_p}{\partial n}d\sigma_g$$
$$-\int_{\partial B_{r_\varepsilon}}\left(u_\varepsilon-\overline{\rho}\,G_p+\lambda_\varepsilon\right)\frac{\partial u_\varepsilon}{\partial n}d\sigma_g$$
$$+o_\varepsilon(1)+o_\delta(1)+o_R(1). \tag{23}$$

By Lemmas 2.1 and 2.5 we can say that

$$\int_{B_\delta\setminus B_{r_\varepsilon}} u_\varepsilon dv_g=\int_{B_\delta\setminus B_{r_\varepsilon}}(u_\varepsilon-\overline{u}_\varepsilon)dv_g+\overline{u}_\varepsilon(Vol(B_\delta)-Vol(B_{r_\varepsilon}))$$
$$=\overline{u}_\varepsilon Vol(B_\delta)+o_\delta(1)+o_\varepsilon(1).$$

Using Green's formula we find

$$\overline{u}_\varepsilon\int_{\partial B_\delta}\frac{\partial G_p}{\partial n}d\sigma_g=-\overline{u}_\varepsilon\int_{\Sigma\setminus B_\delta}\Delta_g G_p\,dv_g=-\overline{u}_\varepsilon\left(1-\frac{Vol(B_\delta)}{|\Sigma|}\right).$$

Similarly

$$\int_{\partial B_\delta}\frac{\partial u_\varepsilon}{\partial n}d\sigma_g=-\int_{\Sigma\setminus B_\delta}\Delta u_\varepsilon\,dv_g=\int_{\Sigma\setminus B_\delta}\rho_\varepsilon\left(he^{u_\varepsilon}-\frac{1}{|\Sigma|}\right)dv_g$$
$$\geq-\rho_\varepsilon\left(1-\frac{Vol(B_\delta)}{|\Sigma|}\right)$$

and

$$\overline{u}_\varepsilon\int_{\partial B_\delta}\frac{\partial u_\varepsilon}{\partial n}d\sigma_g=\overline{u}_\varepsilon\rho_\varepsilon e^{\overline{u}_\varepsilon}\int_{\Sigma\setminus B_\delta(p)}h\,e^{u_\varepsilon-\overline{u}_\varepsilon}dv_g-\overline{u}_\varepsilon\rho_\varepsilon\left(1-\frac{Vol(B_\delta)}{|\Sigma|}\right)$$
$$=-\overline{u}_\varepsilon\rho_\varepsilon\left(1-\frac{Vol(B_\delta)}{|\Sigma|}\right)+o_\varepsilon(1).$$

Lemma 2.3 yields

$$\int_{\partial B_{r_\varepsilon}} u_\varepsilon\frac{\partial G_p}{\partial n}d\sigma_g=\lambda_\varepsilon\int_{\partial B_{r_\varepsilon}}\frac{\partial G_p}{\partial n}d\sigma_g+t_\varepsilon\int_{\partial B_R(0)}\varphi_\varepsilon\frac{\partial G_p}{\partial n}(t_\varepsilon x)(1+o_\varepsilon(1))d\sigma$$
$$=-\lambda_\varepsilon\left(1-\frac{Vol(B_{r_\varepsilon})}{|\Sigma|}\right)+t_\varepsilon\int_{\partial B_R(0)}\varphi_0\left(-\frac{1}{2\pi t_\varepsilon R}+O(1)\right)d\sigma$$
$$=-\lambda_\varepsilon+2\log\left(1+\frac{\pi\,c(p)}{1+\alpha}R^{2(1+\alpha)}\right)+o_\varepsilon(1)$$

and the estimate in Lemma 2.4 gives

$$
\begin{aligned}
&-\int_{\partial B_{r_\varepsilon}} \left(u_\varepsilon - \overline{\rho}\, G_p + \lambda_\varepsilon\right)\frac{\partial u_\varepsilon}{\partial n} d\sigma_g \\
&\geq \left(2\log\left(\frac{R^{2(1+\alpha)}}{1+\frac{\pi c(p)}{(1+\alpha)}R^{2(1+\alpha)}}\right) - \overline{\rho} A(p)\right)\frac{8\pi^2 c(p) R^{2(1+\alpha)}}{\left(1+\frac{\pi c(p) R^{2(1+\alpha)}}{1+\alpha}\right)} + o_\varepsilon(1) \\
&= -\overline{\rho}^2 A(p) - 2\,\overline{\rho}\,\log\left(\frac{\pi c(p)}{1+\alpha}\right) + o_\varepsilon(1) + o_R(1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g \geq& -(16\pi(1+\alpha)-\varepsilon)\overline{u}_\varepsilon + \varepsilon\lambda_\varepsilon + \overline{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g \\
&- 2\overline{\rho}\log\left(1+\frac{\pi c(p)}{1+\alpha}R^{2(1+\alpha)}\right) - \overline{\rho}^2 A(p) - 2\overline{\rho}\log\left(\frac{\pi c(p)}{1+\alpha}\right) \\
&+ o_\varepsilon(1) + o_\delta(1) + o_R(1).
\end{aligned} \tag{24}
$$

By (17), (18) and (24) we can therefore conclude

$$
\begin{aligned}
\int_\Sigma |\nabla_g u_\varepsilon|^2 dv_g \geq& -(16\pi(1+\alpha)-\varepsilon)\overline{u}_\varepsilon + \varepsilon\lambda_\varepsilon - \overline{\rho}^2 A(p) - 2\overline{\rho}\log\left(\frac{\pi c(p)}{1+\alpha}\right) - 2\overline{\rho} \\
&+ o_\varepsilon(1) + o_\delta(1) + o_R(1),
\end{aligned}
$$

so that

$$
\begin{aligned}
J_\varepsilon(u_\varepsilon) \geq& \frac{\varepsilon}{2}(\lambda_\varepsilon - \overline{u}_\varepsilon) - \frac{\overline{\rho}^2}{2}A(p) - \overline{\rho}\log\left(\frac{\pi c(p)}{1+\alpha}\right) - \overline{\rho} + \rho_\varepsilon \log|\Sigma| \\
&+ o_\varepsilon(1) + o_\delta(1) + o_R(1) \\
\geq& -\overline{\rho}\left(4\pi(1+\alpha)A(p) + 1 + \log\left(\frac{\pi c(p)}{1+\alpha}\right) - \log|\Sigma|\right) \\
&+ o_\varepsilon(1) + o_\delta(1) + o_R(1).
\end{aligned}
$$

As $\varepsilon, \delta \to 0$ and $R \to \infty$ we obtain

$$
\begin{aligned}
\inf_{H^1(\Sigma)} J \geq& -\overline{\rho}\left(4\pi(1+\alpha)A(p) + 1 + \log\left(\frac{\pi c(p)}{1+\alpha}\right) - \log|\Sigma|\right) \\
=& -\overline{\rho}\left(1+\log\frac{\pi}{|\Sigma|} + 4\pi A(p) + \log\left(\frac{K(p)}{1+\alpha}\prod_{q\in S, q\neq p} e^{-4\pi\beta(q)G_q(p)}\right)\right).
\end{aligned} \tag{25}
$$

Using Lemma 2.6 it is possible to prove that (25) holds even for $\alpha = 0$. About the blow-up point p we only know that $\beta(p) = \alpha$, so we have proved

Proposition 3.1 *If J has no minimum point, then*

$$\inf_{H^1(\Sigma)} J \geq -\overline{\rho}\left(1+\log\frac{\pi}{|\Sigma|}+\max_{p\in\Sigma,\beta(p)=\alpha}\left\{4\pi A(p) + \log\left(\frac{K(p)}{1+\alpha}\prod_{q\in S,q\neq p} e^{-4\pi\beta(q)G_q(p)}\right)\right\}\right).$$

Notice that, if $\alpha < 0$, the set

$$\{p\in\Sigma\ :\ \beta(p)=\alpha\}=\{p_i\ :\ i\in\{1,\dots,m\},\ \alpha_i=\alpha\}$$

is finite, while if $\alpha = 0$

$$\{p\in\Sigma\ :\ \beta(p)=\alpha\}=\Sigma\setminus S.$$

Although this set is not finite, the maximum in the above expression is still well defined since the function

$$p\longmapsto 4\pi A(p)+\log\left(K(p)\prod_{q\in S} e^{-4\pi\beta(q)G_q(p)}\right)=4\pi A(p)+\log h(p)$$

is continuous on $\Sigma\setminus S$ and approaches $-\infty$ near S.

4 An Estimate from Above

In order to complete the proof of Theorem 1.1 we need to exhibit a sequence $\varphi_\varepsilon \in H^1(\Sigma)$ such that

$$J(\varphi_\varepsilon)\longrightarrow -\overline{\rho}\left(1+\log\frac{\pi}{|\Sigma|}+\max_{p\in\Sigma,\beta(p)=\alpha}\left\{4\pi A(p) + \log\left(\frac{K(p)}{1+\alpha}\prod_{q\in S,q\neq p} e^{-4\pi\beta(q)G_q(p)}\right)\right\}\right).$$

Let us define $r_\varepsilon := \gamma_\varepsilon \varepsilon^{\frac{1}{2(1+\alpha)}}$ where γ_ε is chosen so that

$$\gamma_\varepsilon \to +\infty,\quad r_\varepsilon^2\log\varepsilon\longrightarrow 0,\quad r_\varepsilon^2\log\left(1+\gamma_\varepsilon^{2(1+\alpha)}\right)\longrightarrow 0. \tag{26}$$

Let $p \in \Sigma$ be such that $\beta(p) = \alpha$ and

$$4\pi A(p) + \log\left(\frac{K(p)}{1+\alpha}\prod_{q\in S, q\neq p} e^{-4\pi\beta(q)G_q(p)}\right) = \max_{\xi\in\Sigma,\beta(\xi)=\alpha}\left\{4\pi A(\xi) + \log\left(\frac{K(\xi)}{1+\alpha}\prod_{q\in S, q\neq \xi} e^{-4\pi\beta(q)G_q(\xi)}\right)\right\}$$

and consider a cut-off function η_ε such that $\eta_\varepsilon \equiv 1$ in $B_{r_\varepsilon}(p)$, $\eta_\varepsilon \equiv 0$ in $\Sigma\backslash B_{2r_\varepsilon}(p)$ and $|\nabla_g \eta_\varepsilon| = O(r_\varepsilon^{-1})$. Define

$$\varphi_\varepsilon(x) = \begin{cases} -2\log\left(\varepsilon + r^{2(1+\alpha)}\right) + \log\varepsilon & r \le r_\varepsilon \\ \overline{\rho}\left(G_p - \eta_\varepsilon\sigma\right) + C_\varepsilon + \log\varepsilon & r \ge r_\varepsilon \end{cases}$$

where $r = d(x, p)$, $\sigma(x) = O(r)$ is defined by

$$G_p(x) = -\frac{1}{2\pi}\log r + A(p) + \sigma(x), \tag{27}$$

and

$$C_\varepsilon = -2\log\left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}}\right) - \overline{\rho}\, A(p).$$

In the case $\alpha_i = 0 \ \forall\ i$, a similar family of functions was used in [15] to give an existence result for (4) by proving, under some strict assumptions on h, that

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} < -8\pi\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{p\in\Sigma}\{4\pi A(p) + \log h(p)\}\right).$$

Here we only prove a weak inequality but we have no extra assumptions on h. Taking normal coordinates in a neighborhood of p it is simple to verify that

$$\int_{B_{r_\varepsilon}} |\nabla_g\varphi_\varepsilon|^2 dv_g = 16\pi(1+\alpha)\left(\log\left(1+\gamma_\varepsilon^{2(1+\alpha)}\right) + \frac{1}{1+\gamma_\varepsilon^{2(1+\alpha)}} - 1\right) + o_\varepsilon(1)$$
$$= 16\pi(1+\alpha)\left(\log\left(1+\gamma_\varepsilon^{2(1+\alpha)}\right) - 1\right) + o_\varepsilon(1).$$

By our definition of φ_ε

$$\int_{\Sigma\backslash B_{r_\varepsilon}} |\nabla_g\varphi_\varepsilon|^2 dv_g = \overline{\rho}^2\left(\int_{\Sigma\backslash B_{r_\varepsilon}} |\nabla_g G_p|^2 dv_g + \int_{\Sigma\backslash B_{r_\varepsilon}} |\nabla_g(\eta_\varepsilon\sigma)|^2 dv_g - 2\int_{\Sigma\backslash B_{r_\varepsilon}} \nabla_g G_p \cdot \nabla_g(\eta_\varepsilon\sigma)\, dv_g\right)$$

and by the properties of η_ε

$$\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g(\eta_\varepsilon\sigma)|^2 dv_g = \int_{B_{2r_\varepsilon}\setminus B_{r_\varepsilon}} |\nabla_g\eta_\varepsilon|^2\sigma^2 + 2\eta_\varepsilon\sigma\ \nabla_g\eta_\varepsilon\cdot\nabla_g\sigma + \eta_\varepsilon^2|\nabla_g\sigma|^2\ dv_g$$
$$= O(r_\varepsilon^2).$$

Hence, integrating by parts and using (27), one has

$$\begin{aligned}
\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g\varphi_\varepsilon|^2 dv_g &= \overline{\rho}^2\Bigg(\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla G_p|^2 dv_g \\
&\quad -2\int_{\Sigma\setminus B_{r_\varepsilon}} \nabla_g G_p\cdot\nabla_g(\eta_\varepsilon\sigma)\ dv_g\Bigg) + o_\varepsilon(1) \\
&= -\overline{\rho}^2\Bigg(\frac{1}{|\Sigma|}\int_{\Sigma\setminus B_{r_\varepsilon}} (G_p - 2\eta_\varepsilon\sigma)\ dv_g \\
&\quad +\int_{\partial B_{r_\varepsilon}} (G_p - 2\eta_\varepsilon\sigma)\frac{\partial G_p}{\partial n} d\sigma_g\Bigg) + o_\varepsilon(1) \\
&= -\overline{\rho}^2\int_{\partial B_{r_\varepsilon}} (G_p - 2\sigma)\frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) \\
&= -\overline{\rho}^2\int_{\partial B_{r_\varepsilon}} \left(-\frac{1}{2\pi}\log(r_\varepsilon) + A(p) - \sigma\right) \\
&\quad \times\left(-\frac{1}{2\pi r_\varepsilon} + \nabla\sigma\right)\left(1 + O(r_\varepsilon^2)\right) d\sigma + o_\varepsilon(1) \\
&= -\overline{\rho}^2\int_{\partial B_{r_\varepsilon}} \left(\frac{\log r_\varepsilon}{4\pi^2 r_\varepsilon} - \frac{1}{2\pi r_\varepsilon}A(p) + O(\log r_\varepsilon) + O(1)\right) d\sigma + o_\varepsilon(1) \\
&= -\frac{\overline{\rho}^2}{2\pi}\log\left(\gamma_\varepsilon\varepsilon^{\frac{1}{2(1+\alpha)}}\right) + \overline{\rho}^2 A(p) + o_\varepsilon(1) \\
&= -2\overline{\rho}\left(\log\gamma_\varepsilon^{2(1+\alpha)} + \log\varepsilon - 4\pi(1+\alpha)A(p)\right) + o_\varepsilon(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\Sigma} |\nabla_g\varphi_\varepsilon|^2 dv_g &= 2\overline{\rho}\left(\log\left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}}\right) - 1 + 4\pi(1+\alpha)A(p) - \log\varepsilon\right) + o_\varepsilon(1) \\
&= -2\overline{\rho}\left(1 - 4\pi(1+\alpha)A(p) + \log\varepsilon\right) + o_\varepsilon(1).
\end{aligned} \tag{28}$$

Similarly one has

$$\begin{aligned}
\int_{B_{r_\varepsilon}} \varphi_\varepsilon\ dv_g &= |B_{r_\varepsilon}|\log\varepsilon - 4\pi\int_0^{r_\varepsilon} r\log\left(\varepsilon + r^{2(1+\alpha)}\right)(1 + o_\varepsilon(1))dr \\
&= |B_{r_\varepsilon}|\log\varepsilon - 2\pi r_\varepsilon^2\log\varepsilon - 4\pi\int_0^{r_\varepsilon} r\log\left(1 + \frac{r^{2(1+\alpha)}}{\varepsilon}\right)(1 + o_\varepsilon(1))dr
\end{aligned}$$

$$= O\left(r_\varepsilon^2 \log \varepsilon\right) - 4\pi \int_0^1 r_\varepsilon^2 s \log\left(1 + \gamma_\varepsilon^{2(1+\alpha)} s^{2(1+\alpha)}\right)(1 + o_\varepsilon(1))dr$$
$$= O\left(r_\varepsilon^2 \log \varepsilon\right) + O\left(r_\varepsilon^2 \log\left(1 + \gamma_\varepsilon^{2(1+\alpha)}\right)\right) = o_\varepsilon(1)$$

and

$$\int_{\Sigma\setminus B_{r_\varepsilon}} \varphi_\varepsilon \, dv_g = \overline{\rho} \int_{\Sigma\setminus B_{r_\varepsilon}} (G_p - \eta_\varepsilon \sigma) dv_g + (C_\varepsilon + \log \varepsilon)|\Sigma\setminus B_{r_\varepsilon}(p)|$$
$$= |\Sigma| \log \varepsilon - \overline{\rho}|\Sigma| A(p) + o_\varepsilon(1)$$

so that

$$\frac{1}{|\Sigma|} \int_\Sigma \varphi_\varepsilon dv_g = \log \varepsilon - \overline{\rho}\, A(p) + o_\varepsilon(1). \tag{29}$$

To compute the integral of the exponential term we fix a small $\delta > 0$ and observe that

$$\int_\Sigma h e^{\varphi_\varepsilon} dv_g = \tilde{h}(p) \int_{B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g + \int_{B_{r_\varepsilon}} \left(\tilde{h} - \tilde{h}(p)\right) e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g$$
$$+ \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{\varphi_\varepsilon} dv_g + \int_{\Sigma\setminus B_\delta} h e^{\varphi_\varepsilon} dv_g$$

where $\tilde{h} = h\, e^{4\pi\alpha G_p} = K \prod_{q\in S, q\neq p} e^{-4\pi\beta(q) G_q}$. For the first term we have

$$\int_{B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g = \varepsilon \int_{B_{r_\varepsilon}} e^{2\alpha \log r - 4\pi\alpha A(p) - 4\pi\alpha\sigma} e^{-2\log\left(\varepsilon + r^{2(1+\alpha)}\right)} dv_g$$
$$= \varepsilon e^{-4\pi\alpha A(p)} \int_{B_{r_\varepsilon}} \frac{r^{2\alpha}}{\left(\varepsilon + r^{2(1+\alpha)}\right)^2} (1 + o_\varepsilon(1)) dv_g$$
$$= \frac{\pi e^{-4\pi\alpha A(p)}}{1+\alpha} \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1 + \gamma_\varepsilon^{2(1+\alpha)}} (1 + o_\varepsilon(1))$$
$$= \frac{\pi e^{-4\pi\alpha A(p)}}{1+\alpha} + o_\varepsilon(1). \tag{30}$$

Since $\tilde{h}$ is smooth in a neighborhood of p we obtain

$$\int_{B_{r_\varepsilon}} \left(\tilde{h} - \tilde{h}(p)\right) e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g = o_\varepsilon(1) \int_{B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g = o_\varepsilon(1) \tag{31}$$

and

$$\left|\int_{B_\delta\setminus B_{r_\varepsilon}} h e^{\varphi_\varepsilon} dv_g\right| = \left|\int_{B_\delta\setminus B_{r_\varepsilon}} \tilde{h} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g\right| \leq \sup_{B_\delta} |\tilde{h}| \int_{B_\delta\setminus B_{r_\varepsilon}} e^{-4\pi\alpha G_p} e^{\varphi_\varepsilon} dv_g$$
$$= \varepsilon e^{C_\varepsilon} \sup_{B_\delta} |\tilde{h}| \int_{B_\delta\setminus B_{r_\varepsilon}} e^{4\pi(2+\alpha) G_p} e^{-\overline{\rho}\eta_\varepsilon\sigma} dv_g$$

$$= O(\varepsilon)\int_{B_\delta\setminus B_{r_\varepsilon}} e^{4\pi(2+\alpha)G_p}dx = O(\varepsilon)\int_{B_\delta\setminus B_{r_\varepsilon}} \frac{1}{|x|^{2(2+\alpha)}}dx$$
$$= O(\varepsilon)\left(\frac{1}{r_\varepsilon^{2(1+\alpha)}} - \frac{1}{\delta^{2(1+\alpha)}}\right) = O\left(\frac{1}{\gamma_\varepsilon^{2(1+\alpha)}}\right) + O(\varepsilon)$$
$$= o_\varepsilon(1). \tag{32}$$

Finally

$$\int_{\Sigma\setminus B_\delta} he^{\varphi_\varepsilon}dv_g = \varepsilon e^{C_\varepsilon}\int_{\Sigma\setminus B_\delta} he^{\overline{\rho}G_p}dv_g = O(\varepsilon) \tag{33}$$

so by (30), (31), (32) and (33) we have

$$\int_\Sigma he^{\varphi_\varepsilon}dv_g = \frac{\pi\tilde{h}(p)e^{-4\pi\alpha A(p)}}{1+\alpha} + o_\varepsilon(1). \tag{34}$$

Using (28), (29) and (34) we get

$$\lim_{\varepsilon\to 0} J(\varphi_\varepsilon) = -\overline{\rho}\left(1 + 4\pi A(p) + \log\left(\frac{1}{|\Sigma|}\frac{\pi\tilde{h}(p)}{1+\alpha}\right)\right)$$
$$= -\overline{\rho}\left(1 + \log\frac{\pi}{|\Sigma|} + \max_{\xi\in\Sigma,\beta(\xi)=\alpha}\left\{4\pi A(\xi) + \log\left(\frac{K(\xi)}{1+\alpha}\prod_{q\in S, q\neq\xi} e^{-4\pi\beta(q)G_q(\xi)}\right)\right\}\right).$$

This, together with Proposition 3.1, completes the proof of Theorem 1.1.

5 Onofri's Inequalities on S^2

In this section we will consider the special case of the standard sphere (S^2, g_0) with $m \le 2$ and $K \equiv 1$. We fix $\alpha_1, \alpha_2 \in \mathbb{R}$ with $-1 < \alpha_1 \le \alpha_2$ and as before we consider the singular weight

$$h = e^{-4\pi\alpha_1 G_{p_1} - 4\pi\alpha_2 G_{p_2}}.$$

In order to apply Theorem 1.1 and obtain sharp versions of (7), we need to study the existence of minimum points for the functional J. Let us fix a system of coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$ such that $p_1 = (0, 0, 1)$. When $h \in C^1(S^2)$, the Kazdan–Warner identity (see [18]) states that any solution of (4) has to satisfy

$$\int_{S^2} \nabla h\cdot\nabla x_i\ e^u\ dv_{g_0} = \left(2 - \frac{\rho}{4\pi}\right)\int_{S^2} he^u x_i\ dv_{g_0} \quad i = 1, 2, 3.$$

We claim that if $p_2 = -p_1$ the same identity holds, at least in the x_3-direction, even when h is singular.

Lemma 5.1 *Let u be a solution of (4) on S^2, then there exist $C, \delta_0 > 0$ such that*

- $|\nabla u(x)| \le C d(x, p_i)^{2\alpha_i+1}$ *if* $\alpha_i < -\frac{1}{2}$;
- $|\nabla u(x)| \le C(-\log d(x, p_i))$ *if* $\alpha_i = -\frac{1}{2}$;
- $|\nabla u(x)| \le C$ *if* $\alpha_i > -\frac{1}{2}$;

for $0 < d(x, p_i) < \delta_0$, $i = 1, 2$.

Proof Let us fix $0 < r_0 < \frac{1}{2}\min\{\frac{\pi}{2}, d(p_1, p_2)\}$ and $i \in \{1, 2\}$. If $\alpha_i > -\frac{1}{2}$ then, by standard elliptic regularity, $u \in C^1(\overline{B_{r_0}(p_i)})$ and the conclusion holds for $\delta_0 = r_0$ and $C = \|\nabla u\|_{L^\infty(B_{r_0}(p_i))}$. Let us now assume $\alpha_i \le -\frac{1}{2}$. We know that $h(y) \le C_1 d(y, p_i)^{2\alpha_i}$ for $y \in B_{2r_0}(p_i)$ so, if $\delta_0 < r_0$, by Green's representation formula we have

$$|\nabla u|(x) \le \rho e^{\|u\|_\infty} \int_{S^2} \frac{h(y)}{d(x, y)} dv_{g_0}(y) \le \frac{\rho e^{\|u\|_\infty}\|h\|_{L^1(S^2)}}{r_0} + \rho e^{\|u\|_\infty} C_1 \int_{B_{r_0}(x)} \frac{d(y, p_i)^{2\alpha_i}}{d(x, y)} dv_{g_0}(y).$$

Let π be the stereographic projection from the point $-p_i$. It is easy to check that there exist $C_2, C_3 > 0$ such that

$$C_2\, d(q, q') \le |\pi(q) - \pi(q')| \le C_3\, d(q, q')$$

$\forall\, q, q' \in B_{\frac{\pi}{2}}(p_i)$. Thus we have

$$\int_{B_{r_0}(x)} \frac{d(y, p_i)^{2\alpha_i}}{d(x, y)} dv_{g_0}(y) \le \int_{B_{\frac{\pi}{2}}(p_i)} \frac{d(y, p_i)^{2\alpha_i}}{d(x, y)} dv_{g_0}(y) \le C_4 \int_{\{|z|\le 1\}} \frac{|z|^{2\alpha_i}}{|\pi(x) - z|} dz$$
$$= C_4 |\pi(x)|^{2\alpha_i+1} \int_{\left\{|z|\le \frac{1}{|\pi(x)|}\right\}} \frac{|z|^{2\alpha_i}}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz$$
$$\le C_5 d(x, p_i)^{2\alpha_i+1} \int_{\left\{|z|\le \frac{1}{|\pi(x)|}\right\}} \frac{|z|^{2\alpha_i}}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz.$$

Notice that

$$\int_{\left\{|z|\le \frac{1}{|\pi(x)|}\right\}} \frac{|z|^{2\alpha_i}}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz \le \frac{1}{2^{2\alpha_i}} \int_{\left\{\left|\frac{\pi(x)}{|\pi(x)|} - z\right| \le \frac{1}{2}\right\}} \frac{1}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz$$
$$+ 2\int_{\{|z|\le 2\}} |z|^{2\alpha_i} dz + 2\int_{\left\{2\le |z|\le \frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i - 1} dz$$
$$\le C_6 + 2\int_{\left\{2\le |z|\le \frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i - 1} dz.$$

If $\alpha_i < -\frac{1}{2}$

$$\int_{\left\{2\le|z|\le\frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i-1}dz \le C_7,$$

while if $\alpha_i = -\frac{1}{2}$

$$\int_{\left\{2\le|z|\le\frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i-1}dz = 2\pi\log\left(\frac{1}{2|\pi(x)|}\right) \le C_8\left(-\log d(x,p_i)\right).$$

Thus we get the conclusion for δ_0 sufficiently small. □

In any case there exists $s \in [0, 1)$ such that

$$|\nabla u(x)| \le C d(x,p_i)^{-s}\left(-\log d(x,p_i)\right) \tag{35}$$

for $0 < d(x, p_i) < \delta_0,\ i = 1, 2$.

Proposition 5.1 *If $p_2 = -p_1$ then any solution of* (4) *satisfies*

$$\int_{S^2} \nabla h\cdot\nabla x_3\, e^u\, dv_{g_0} = \left(2-\frac{\rho}{4\pi}\right)\int_{S^2} he^u x_3\, dv_{g_0}.$$

Proof Without loss of generality we may assume

$$\int_{S^2} he^u dv_{g_0} = 1. \tag{36}$$

Let us denote $S_\delta = S^2\backslash B_\delta(p_1)\cup B_\delta(p_2)$. Since u is smooth in S_δ, multiplying (4) by $\nabla u\cdot\nabla x_3$ and integrating on S_δ we have

$$-\int_{S_\delta}\Delta u\,\nabla u\cdot\nabla x_3\,dv_{g_0} = \rho\int_{S_\delta}\left(h\,e^u-\frac{1}{4\pi}\right)\nabla u\cdot\nabla x_3\,dv_{g_0} \tag{37}$$

Integrating by parts we obtain

$$-\int_{S_\delta}\Delta u\,\nabla u\cdot\nabla x_3\,dv_{g_0} = \int_{S_\delta}\nabla u\cdot\nabla(\nabla u\cdot\nabla x_3)dv_{g_0} + \sum_{i=1}^{2}\int_{\partial B_\delta(p_i)}\nabla u\cdot\nabla x_3\frac{\partial u}{\partial n}d\sigma_{g_0}$$

and by (35)

$$\left|\int_{\partial B_\delta(p_i)}\nabla u\cdot\nabla x_3\,\frac{\partial u}{\partial n}\,d\sigma_{g_0}\right| \le \int_{\partial B_\delta(p_i)}|\nabla u|^2|\nabla x_3|d\sigma_{g_0} = O(\delta^{2(1-s)}\log^2\delta) = o_\delta(1).$$

Using the identities

$$\nabla u \cdot \nabla(\nabla u \cdot \nabla x_3) = \frac{1}{2}\nabla\left(|\nabla u|^2 \cdot \nabla x_3\right) - x_3|\nabla u|^2$$

and

$$-\Delta x_3 = 2x_3,$$

and applying again (35) to estimate the boundary term, we get

$$\begin{aligned}
-\int_{S_\delta} \Delta u\ \nabla u \cdot \nabla x_3\ dv_{g_0} &= \int_{S_\delta} \frac{1}{2}\nabla|\nabla u|^2 \cdot \nabla x_3\ dv_{g_0} - \int_{S_\delta} x_3|\nabla u|^2 dv_{g_0} + o_\delta(1)\\
&= -\frac{1}{2}\int_{S_\delta} \Delta x_3\ |\nabla u|^2 dv_{g_0} - \sum_{i=1}^{2}\int_{\partial B_\delta(p_i)} |\nabla u|^2 \frac{\partial x_3}{\partial n} d\sigma_{g_0}\\
&\quad -\int_{S_\delta} x_3|\nabla u|^2 dv_{g_0} = o_\delta(1).
\end{aligned}$$

Thus (37) becomes

$$\int_{S_\delta} he^u \nabla u \cdot \nabla x_3\ dv_{g_0} - \frac{1}{4\pi}\int_{S_\delta} \nabla u \cdot \nabla x_3\ dv_{g_0} = o_\delta(1). \tag{38}$$

Moreover

$$\begin{aligned}
\int_{S_\delta} \nabla u \cdot \nabla x_3\ dv_{g_0} &= -\int_{S_\delta} \Delta u\ x_3\ dv_{g_0} - \sum_{i=1}^{2}\int_{\partial B_\delta(p_i)} x_3 \frac{\partial u}{\partial n}\ d\sigma_{g_0}\\
&= \rho\int_{S_\delta}\left(he^u - \frac{1}{4\pi}\right) x_3\ dv_{g_0} + O(\delta^{1-s}(-\log\delta))\\
&= \rho\int_{S_\delta} he^u x_3\ dv_{g_0} + o_\delta(1)
\end{aligned}$$

and

$$\begin{aligned}
\int_{S_\delta} he^u\ \nabla u \cdot \nabla x_3\ dv_{g_0} &= \int_{S_\delta} \nabla e^u \cdot h\nabla x_3\ dv_{g_0} = -\int_{S_\delta} e^u\,\mathrm{div}(\,h\nabla x_3) dv_{g_0}\\
&\quad - \sum_{i=1}^{2}\int_{\partial B_\delta(p_i)} he^u \frac{\partial x_3}{\partial n}\ d\sigma_{g_0}\\
&= -\int_{S_\delta} \nabla h \cdot \nabla x_3\ e^u\ dv_{g_0} + 2\int_{S_\delta} he^u x_3 dv_{g_0} + O\left(\delta^{2(1+\alpha)}\right).
\end{aligned}$$

Thus by (38) we have

$$\int_{S_\delta} \nabla h \cdot \nabla x_3 \, e^u \, dv_{g_0} = \left(2 - \frac{\rho}{4\pi}\right) \int_{S_\delta} h e^u x_3 \, dv_{g_0} + o_\delta(1).$$

Since u is continuous on S^2 and $h, \nabla h \cdot \nabla x_3 \in L^1(S^2)$ as $\delta \to 0$ we get the conclusion. □

Remark 5.1 In this proof there is no need to assume $K \equiv 1$.

Assuming $p_1 = (0, 0, 1)$ and $p_2 = (0, 0, -1)$, one may easily verify that

$$G_{p_1}(x) = -\frac{1}{4\pi} \log(1 - x_3) - \frac{1}{4\pi} \log\left(\frac{e}{2}\right)$$

and

$$G_{p_2}(x) = -\frac{1}{4\pi} \log(1 + x_3) - \frac{1}{4\pi} \log\left(\frac{e}{2}\right),$$

so that

$$\nabla h \cdot \nabla x_3 = -4\pi h(\alpha_1 \nabla G_1 + \alpha_2 \nabla G_2) \cdot \nabla x_3 = (\alpha_2 - \alpha_1)h - (\alpha_1 + \alpha_2) h x_3.$$

Thus we can rewrite the identity in Proposition 5.1 as

$$\alpha_2 - \alpha_1 = \left(2 - \frac{\rho}{4\pi} + \alpha_1 + \alpha_2\right) \int_{S^2} h e^u x_3 \, dv_{g_0}. \tag{39}$$

Proof of Theorem 1.2 Assume $m = 1$ (i.e., $\alpha_2 = 0$). We claim that equation (4) has no solutions for $\rho = \overline{\rho} = 8\pi(1 + \min\{0, \alpha_1\})$, unless $\alpha_1 = 0$. Indeed if u were a solution of (4) satisfying (36), then applying (39) with $\rho = \overline{\rho}$ we would get

$$-\alpha_1 = (\alpha_1 - 2\min\{0, \alpha_1\}) \int_{S^2} h e^u x_3 \, dv_{g_0}$$

so that, if $\alpha_1 \neq 0$,

$$\left| \int_{S^2} h e^u x_3 \, dv_{g_0} \right| = 1.$$

This contradicts (4). In particular we proved non-existence of minimum points for $J_{\overline{\rho}}$ so we can exploit Theorem 1.1 and (9) to prove that (7) holds with

$$C = \max_{p \in S^2, \beta(p) = \alpha} \left\{ \log\left(\frac{1}{1+\alpha} \prod_{q \in S, q \neq p} e^{-4\pi\beta(q) G_q(p)} \right) \right\}.$$

If $\alpha_1 < 0$ one has

$$C = -\log(1+\alpha_1).$$

If $\alpha_1 > 0$,

$$C = \max_{p \in S^2 \setminus \{p_1\}} \left\{-4\pi\alpha_1 G_{p_1}(p)\right\} = -4\pi\alpha_1 G_{p_1}(p_2) = \alpha_1.$$

□

Proof of Theorem 1.3 As in the previous proof, applying (39) with $\rho = \overline{\rho} = 8\pi(1+\alpha_1)$, we obtain that any critical point of (4) for which (36) holds has to satisfy

$$\alpha_2 - \alpha_1 = (\alpha_2 - \alpha_1)\int_{S^2} he^u x_3 dv_{g_0}.$$

Since $\alpha_1 \neq \alpha_2$ one has

$$\int_{S^2} he^u x_3 dv_{g_0} = 1$$

which is impossible. Thus $J_{\overline{\rho}}$ has no critical points and by Theorem 1.1 one has

$$C = \log\left(\frac{1}{1+\alpha_1} e^{-4\pi\alpha_2 G_{p_2}(p_1)}\right) = \alpha_2 - \log(1+\alpha_1).$$

□

Now we assume $\alpha_1 = \alpha_2 < 0$. In this case identity (39) gives no useful condition. Let us denote by π the stereographic projection from the point p_1. It is easy to verify that u satisfies (4) and (36) if and only if

$$v := u \circ \pi^{-1} + (1+\alpha)\log\left(\frac{4}{(1+|y|^2)^2}\right) + 2\alpha \log\left(\frac{e}{2}\right)$$

solves

$$-\Delta_{\mathbb{R}^2} v = 8\pi(1+\alpha)|y|^{2\alpha} e^v \tag{40}$$

in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} |y|^{2\alpha} e^v dy = 1.$$

As we pointed out in the proof of Lemma 2.3 and Eq. (40) has a one-parameter family of solutions:

$$v_\lambda(y) = -2\log\left(1 + \frac{\pi}{1+\alpha} e^l |y|^{2(1+\alpha)}\right)$$

$l \in \mathbb{R}$. Thus we have a corresponding family $\{u_{\lambda,c}\}$ of critical points of $J_{\overline{\rho}}$ given by the expression

$$u_{\lambda,c} \circ \pi^{-1}(y) = 2\log\left(\frac{\left(1+|y|^2\right)^{1+\alpha}}{1+\lambda|y|^{2(1+\alpha)}}\right)+c, \tag{41}$$

$c \in \mathbb{R}$, $\lambda > 0$. A priori we do not know whether these critical points are minima for $J_{\overline{\rho}}$ (as it happens for $\alpha = 0$), so a direct application of 1.1 is not possible. However, we can still get the conclusion by comparing $J_{\overline{\rho}}(u_{\lambda,c})$ with the blow-up value provided by Theorem 1.1.

Proof of Theorem 1.4 Let us first compute $J(u_{\lambda,c})$. Let $\varphi_t : S^2 \longrightarrow S^2$ be the conformal transformation defined by $\pi(\varphi_t(\pi^{-1}(y))) = ty$. It is not difficult to prove that $\forall\, t > 0$

$$J_{\overline{\rho}}(u) = J_{\overline{\rho}}(u \circ \varphi_t + (1+\alpha)\log|\det d\varphi_t|);$$

in particular, since

$$u_{\lambda,c} = u_{1,0} \circ \varphi_{\lambda^{\frac{1}{2(1+\alpha)}}} + (1+\alpha)\log|\det \varphi_{\lambda^{\frac{1}{2(1+\alpha)}}}| + c - \log\lambda,$$

we have that $J(u_{\lambda,c})$ does not depend on λ and c. Thus we may assume $\lambda = 1$ and $c = 0$. A simple computation shows that

$$\int_{S^2} h\, e^{u_{1,0}} dv_{g_0} = 4e^{2\alpha}\int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{\left(1+|y|^{2(1+\alpha)}\right)^2} dy = \frac{4e^{2\alpha}\pi}{1+\alpha}. \tag{42}$$

Since $u_{1,0}(p_1) = 0$ and $u_{1,0}$ solves

$$-\Delta u_{1,0} = \omega\, h\, e^{u_{1,0}} - 2(1+\alpha) \quad \text{with} \quad \omega := 2(1+\alpha)^2 e^{-2\alpha}$$

one has

$$\int_{S^2} u_{1,0}\, dv_{g_0} = 4\pi \int_{S^2} \Delta u_{1,0}\, G_{p_1} dv_{g_0} = -4\pi\omega \int_{S^2} h e^{u_{1,0}} G_{p_1} dv_{g_0}$$

and

$$\begin{aligned} &\frac{1}{2}\int_{S^2} |\nabla u_{1,0}|^2 dv_{g_0} + 2(1+\alpha)\int_{S^2} u_{1,0}\, dv_{g_0} \\ &\quad = \frac{1}{2}\omega \int_{S^2} h e^{u_{1,0}} u_{1,0}\, dv_{g_0} + (1+\alpha)\int_{S^2} u_{1,0}\, dv_{g_0} \\ &\quad = \frac{\omega}{2}\int_{S^2} h e^{u_{1,0}}(u_{1,0} - \overline{\rho} G_{p_1}) dv_{g_0}. \end{aligned} \tag{43}$$

Since

$$G_{p_1}(\pi^{-1}(y)) := \frac{1}{4\pi}\log(1+|y|^2) - \frac{1}{4\pi}$$

we get

$$\int_{S^2} he^{u_{1,0}}(u_{1,0} - \overline{\rho} G_{p_1}) = 2(1+\alpha)\int_{S^2} he^{u_{1,0}} dv_{g_0} - 8e^{2\alpha}\int_{\mathbb{R}^2} \frac{|y|^{2\alpha}\log\left(1+|y|^{2(1+\alpha)}\right)}{\left(1+|y|^{2(1+\alpha)}\right)^2} dy = 8\pi e^{2\alpha} - \frac{8\pi e^{2\alpha}}{1+\alpha}\int_0^{+\infty} \frac{\log(1+s)}{(1+s)^2} ds = \frac{8\pi\alpha e^{2\alpha}}{1+\alpha}. \quad (44)$$

Using (42), (43) and (44) we obtain

$$J(u_{\lambda,c}) = J(u_{1,0}) = 8\pi(1+\alpha)\left(\log(1+\alpha) - \alpha\right) \qquad \forall\, \lambda > 0, c \in \mathbb{R}.$$

To conclude the proof it is sufficient to observe that $u_{\lambda,c}$ have to be minimum points for $J_{\overline{\rho}}$ that is

$$\inf_{H^1(S^2)} J_{\overline{\rho}} = 8\pi(1+\alpha)\left(\log(1+\alpha) - \alpha\right).$$

Indeed if this were false then $J_{\overline{\rho}}$ would have no minimum points but, by Theorem 1.1, we would get

$$\inf_{H^1(S^2)} J_{\overline{\rho}} = 8\pi(1+\alpha)\left(\log(1+\alpha) - \alpha\right) = J(u_{\lambda,c}).$$

This is clearly a contradiction. □

Remark 5.2 There is no need to assume $p_1 = -p_2$.

Indeed given two arbitrary points $p_1, p_2 \in S^2$ with $p_1 \neq p_2$ it is always possible to find a conformal diffeomorphism $\varphi : S^2 \longrightarrow S^2$ such that $\varphi^{-1}(p_1) = -\varphi^{-1}(p_2)$. Moreover one has

$$J_{\overline{\rho}}(u) = \widetilde{J}_{\overline{\rho}}(u \circ \varphi + (1+\alpha)\log|\det d\varphi|) + c_{\alpha,p_1,p_2}$$

$\forall\, u \in H^1(S^2)$, where $\widetilde{J}$ is the Moser–Trudinger functional associated to

$$\widetilde{h} = e^{-4\pi\alpha G_{\varphi^{-1}(p_1)} - 4\pi\alpha G_{\varphi^{-1}(p_2)}}.$$

and c_{α,p_1,p_2} is an explicitly known constant depending only on α, p_1 and p_2. In particular one can still compute $\min_{H^1(S^2)} J_{\overline{\rho}}$ and describe the minimum points of $J_{\overline{\rho}}$ in terms of φ and the family (41).

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