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Bilinear Hilbert Transforms Associated with Plane Curves

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Abstract We prove that the bilinear Hilbert transforms and maximal functions along certain general plane curves are bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^1(\mathbb{R})$.

Keywords Bilinear Hilbert transforms · Bilinear maximal functions · Plane curves

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1 Introduction

Since the initial breakthroughs for singular integrals along curves and surfaces by Nagel, Rivière, Stein, Wainger, et al., in the 1970s (see for example [14,15] and [17] for some of their works on Hilbert transforms along curves), extensive research in this area of harmonic analysis has been done and a great many fascinating and important results have been established, which culminate in a general theory of singular Radon transforms (see for instance Christ et al. [2]).

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Another attractive area, parallel to the above one, is the bilinear extension of the classical Hilbert transform. The boundedness of such bilinear transforms was conjectured by Calderón and motivated by the study of the Cauchy integral on Lipschitz curves. In the 1990s, this conjecture was verified by Lacey and Thiele in a breakthrough pair of papers [8,9]. In their works, a systematic and delicate method was developed, inspired by the famous works of Carleson [1] and Fefferman [3], which is nowadays referred as the method of time-frequency analysis. Over the past two decades, this method has emerged as a powerful analytic tool to handle problems that are related to multilinear analysis.

We are interested in the study of bilinear/multilinear singular integrals along curves and surfaces—a problem that is closely related to the two areas above. (We refer the readers to Li [11] for connections of this problem with ergodic theory and multilinear oscillatory integrals.) To begin with, we consider a model case—the truncated bilinear Hilbert transforms along plane curves. One formulation of the problem is as follows.

Let $\Gamma(t) = (t, \gamma(t)) : (-1, 1) \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$. With Γ we associate the truncated bilinear Hilbert transform operator H_Γ given by the principal value integral

$$H_\Gamma(f, g)(x) = \int_{-1}^{1} f(x - t)g(x - \gamma(t))t^{-1}\mathrm{d}t \qquad (x \in \mathbb{R}), \tag{1.1}$$

where f and g are Schwartz functions on $\mathbb{R}$. When the function γ has certain curvature (or "non-flat", i.e., not "resembling" a line) conditions, the boundedness properties of this operator (e.g., whether it is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^r(\mathbb{R})$ for certain p_1, p_2, and r) are of great interest to us.

Li [11] studied such an operator (the integral defining $H_\Gamma(f, g)(x)$ in [11] is over $\mathbb{R}$) with the curve being defined by a monomial (i.e., $\gamma(t) = t^d$, $d \in \mathbb{N}$, $d \geq 2$) and proved that it is bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^1(\mathbb{R})$. In his proof, he combined results and tools from both time-frequency analysis and oscillatory integral theory and used ingeniously a uniformity concept (the so-called σ-uniformity; see [11, Section 6]). Lie [13] improved Li's results both qualitatively, by extending monomials to more general curves (certain "slow-varying" curves with extra curvature assumptions), and quantitatively, by improving the estimates. Instead of using Li's method of σ-uniformity, Lie used a Gabor frame decomposition to discretize certain operators in a smart way and then worked with the discretized operators which have variables separated on the frequency side and preserve certain main characteristics (see the appendix of [13] for a detailed comparison between their methods).

Another interesting aspect of this problem was considered by Li and the second author [12], in which they studied the case when the curve is defined by a polynomial with different emphasis of getting bounds uniform in coefficients of the polynomial and the full range of indices (p_1, p_2, r). They provided, among other results, complete answers (except to the endpoint case) for H_Γ when the polynomial is "non-flat" near the origin (i.e., without a linear term). When the polynomial has a linear term, however, the full range of indices for the corresponding uniform estimates is extremely difficult to find and remains open.

In this paper we consider a family of general "non-flat" curves and provide an easy-to-check criterion for a curve whose associated bilinear Hilbert transform is bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^1(\mathbb{R})$ (for the precise statement of our results, see Sect. 2). Our goal is to extend Li [11]'s method and results to general plane curves, (in some sense) recover Lie [13]'s results without using the wave packets to discretize the operators, and also prove the boundedness of corresponding maximal functions.

Our criterion, motivated by results in Lie [13] and Nagel et al. [16], mainly asks one to check whether certain bounds of various expressions involving derivatives of a quotient are satisfied. In [16] a simple necessary and sufficient condition is provided (among other results) for the L^2-boundedness of the Hilbert transform along the curve Γ with γ odd, that is, one needs to check whether an auxiliary function $h(t) = t\gamma'(t) - \gamma(t)$ has bounded doubling time. Both Lie's and our results indicate that an appropriate replacement for h in the bilinear setting might be in the form of a quotient (see the $Q_\epsilon(t)$ defined in Sect. 2). We still do not know whether our criterion is a necessary condition for certain "non-flat" curves.

In our main estimates in Sect. 4, we apply the TT^* method both in frequency space (with an extra size restriction $|\gamma'(2^{-j})| > 2^{-m}$) and in time space (with an extra restriction on the function space), then we combine both results to get the fast decay needed in proving the boundedness of the desired operator. Since we are considering general curves, certain uniformity of estimates is important, hence we formulate carefully the assumptions on curves and pay special attention to the dependence on parameters of all bounds, especially when we apply a quantitative version of the method of stationary phase.

We also establish analogous results for the bilinear maximal function along Γ (defined below) by using the arguments of [12, Section 7] and our main estimates in Sect. 4.

$$M_\Gamma(f,g)(x) = \sup_{0<\epsilon<1} \epsilon^{-1} \int_0^\epsilon |f(x-t)g(x-\gamma(t))| \, \mathrm{d}t \qquad (x \in \mathbb{R}). \tag{1.2}$$

We note that such an operator along a "non-flat" polynomial was already carefully studied in [12]. Much deeper and more elegant results for a linear curve can be found in Lacey [7].

Notations The Fourier transform of f is $\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x$ and its inverse Fourier transform is $\mathcal{F}^{-1}[g](x) = \int_{\mathbb{R}} g(\xi)e^{2\pi i\xi x}\,\mathrm{d}\xi$. Let $\mathbf{1}_{a,n}$ be the indicator function of interval $a\cdot[n,n+1)$ for $a, n \in \mathbb{R}$ and 1_I the indicator function of interval I. The indices (p_1, p_2, r) are always assumed to satisfy $1/p_1 + 1/p_2 = 1/r$, $p_1 > 1$, $p_2 > 1$, and $r > 1/2$. We use C to denote an absolute constant which may be depending on the curve and different from line to line.

2 Statement of Theorems

For any $a \in \mathbb{R}$, we say that a curve $\Gamma(t) = (t, \gamma(t) + a) : (-1,1) \to \mathbb{R}^2$[1] belongs to a family of curves, $\mathbf{F}(-1,1)$, if the function γ satisfies the following conditions

[1] In the problems considered in this paper, we can always remove the constant a from the definition of Γ by a translation argument, hence there is no need to specify the dependence of Γ on a and we will always let $a = 0$.

(2.1)–(2.4). There exists a constant $0 < A_1 < 1/2$ such that on $(-A_1, A_1) \setminus \{0\}$ the function γ is of class C^N $(N \geq 5)$ and $\gamma' \neq 0$. Let $Q_\epsilon(t) = \gamma(\epsilon t)/\epsilon\gamma'(\epsilon)$. For $0 < |\epsilon| < c_0 < A_1/4$ and $1/4 \leq |t| \leq 4$, we have

$$\left|D^j Q_\epsilon(t)\right| \leq C_1, \quad 0 \leq j \leq N, \tag{2.1}$$

$$\left|D^2 Q_\epsilon(t)\right| \geq c_1, \tag{2.2}$$

[2] and

$$\left|(D^2 Q_\epsilon)^2(t) - D^1 Q_\epsilon(t) D^3 Q_\epsilon(t)\right| \geq c_2, \quad \text{if } \left|\gamma'(\epsilon)\right| \leq K_1 |\epsilon|^{c_1}, \tag{2.3}$$

or

$$\left|2(D^2 Q_\epsilon)^2(t) - D^1 Q_\epsilon(t) D^3 Q_\epsilon(t)\right| \geq c_3, \quad \text{if } \left|\gamma'(\epsilon)\right| \geq K_2 |\epsilon|^{-c_1}. \tag{2.3$'$}$$

Let $\Delta_j = \left|2^{-j}\gamma'(2^{-j})\right|^{-1}$. If $\gamma''(\epsilon)\gamma'(\epsilon) < 0$ for $0 < \epsilon < c_0$, then there exist $K_3 \in \mathbb{Z}$ and $K_4 \in \mathbb{N}$ such that

$$\Delta_{j+K_3} \geq 2\Delta_j, \quad \text{if } j \geq K_4. \tag{2.4}$$

Theorem 2.1 *If* $\Gamma \in \boldsymbol{F}(-1, 1)$*, then* $H_\Gamma(f, g)$ *can be extended to a bounded operator from* $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ *to* $L^1(\mathbb{R})$.

The analogous version for bilinear maximal functions is as follows.

Theorem 2.2 *If* $\Gamma \in \boldsymbol{F}(-1, 1)$*, then* $M_\Gamma(f, g)$ *is a bounded operator from* $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ *to* $L^1(\mathbb{R})$.

Remark 2.3 By combining the results in this paper with the time-frequency analysis arguments in [12], the boundedness of H_Γ and M_Γ from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^r(\mathbb{R})$ may be obtained for $r < 1$. We do not carry the details out in this paper. The lower bound of such r, as indicated in [12, Theorem 4], is closely related to the decay rate of the size of the sublevel set

$$\left\{|t| < 1 : |\gamma'(t) - 1| < h\right\}, \tag{2.5}$$

as $h \to 0^+$. In particular, if the size of (2.5) is bounded by $c_\nu h^\nu$ for some $\nu > 0$ and $c_\nu > 0$, then H_Γ and M_Γ are expected to be bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^r(\mathbb{R})$ given $r > \max\{1/(1+\nu), 1/2\}$; see [12, Theorem 4] when γ is a polynomial.

[2] The condition (2.2) implies that there exist constants $K_1, K_2 > 0$ such that

$$\left|\gamma'(\epsilon)\right| \leq K_1 |\epsilon|^{c_1} \quad \text{for } 0 < |\epsilon| < c_0$$

or

$$\left|\gamma'(\epsilon)\right| \geq K_2 |\epsilon|^{-c_1} \quad \text{for } 0 < |\epsilon| < c_0.$$

See also Lie [13, p. 4] Observation (6) and (7).

Remark 2.4 (1) We use A_1, c_0, c_1, c_2, c_3, C_1, K_1, K_2, K_3, and K_4 throughout this paper.

(2) The condition (2.1) with $j = 1$ implies that

$$\left|D^1 Q_\epsilon(t)\right| \geq 1/C_1, \quad \text{for } 0 < |\epsilon| < c_0/4,\ 1/4 \leq |t| \leq 4.$$

(3) If $\gamma''(\epsilon)\gamma'(\epsilon) > 0$ for $0 < \epsilon < c_0$, then (2.4) always holds with $K_3 = 1$.

(4) We now compare our assumptions (2.1)–(2.4) with Lie's assumptions (1)–(5) in [13, P. 3]. The (2.4) implies Lie's (1). The (2.1) and (2.2) correspond to Lie's (2) and (4) (the Q'' part) while the (2.3) essentially corresponds to Lie's (5).

(5) Note that the curves considered here are not necessarily differentiable at the origin (they can even have a pole). One explanation for this phenomenon is that the bilinear Hilbert transform possesses certain symmetry between its two functions f and g (as well as its two variables ξ and η on the frequency side) that we can take advantage of to somehow transfer the case with a pole to the case without a pole (see the two expressions of $B^{\varphi}_{j,m}(f, g)$ at the beginning of Sect. 4).

Remark 2.5 Here are some curves $\Gamma(t) = (t, \gamma(t))$ that belong to $\mathbf{F}(-1, 1)$:

(1) Those smooth curves that have contact with t-axis at the origin of finite order ≥ 2 (namely, $\gamma(0) = \gamma'(0) = \cdots = \gamma^{(d-1)}(0) = 0$, but $\gamma^{(d)}(0) \neq 0$ for some natural number $d \geq 2$), for example, $\gamma(t) = t^d$ or $e^{t^d} - 1$ if $d \geq 2$;
(2) The function γ has a pole at the origin of finite order ≥ 1 (namely, $\gamma(t) = t^{-n}h(t)$ for some natural number $n \geq 1$ and some smooth function h with $h(0) \neq 0$);
(3) $\gamma(t) =$ a linear combination of finitely many terms of the form $|t|^\alpha |\log|t||^\beta$ for $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0, 1$;
(4) $\gamma(t) = \operatorname{sgn}(t)|t|^\alpha$ or $|t|^\alpha |\log|\log|t|||^\beta$ for $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0, 1$.

3 Preliminaries

In this section, we first study a special oscillatory integral which occurs in later sections. The results are standard, but we include a proof for completeness and the convenience of the readers.

Let $\rho \in C_0^\infty(\mathbb{R})$ be a real-valued function with $\operatorname{supp} \rho \subset [1/2, 2]$, $\xi, \eta \in \mathbb{R}$, $\eta \neq 0$, $A > 1$ a constant, and

$$I\big(\lambda, \epsilon, \xi, \eta\big) = 1_{[-A,A]}(\xi/\eta) \int_0^\infty \rho(t) e^{i\lambda\phi_\epsilon(t,\xi,\eta)}\, \mathrm{d}t, \quad \lambda > 1,$$

where

$$\phi_\epsilon\big(t, \xi, \eta\big) = Q_\epsilon(t) + (\xi/\eta)t.$$

Lemma 3.1 *Assume that $Q_\epsilon \in C^N([1/4, 4])$ ($N \geq 5$) is a real-valued function such that $|D^j Q_\epsilon| \leq C_1$ for $0 \leq j \leq N$ and $|D^2 Q_\epsilon| \geq c_1$ for constants C_1 and c_1. If $\chi \in C_0^\infty(\mathbb{R})$ has its support contained in an interval of length $c_1/12$, then either one of the following two statements holds.*

(1) *We have*

$$\chi\big(-\xi/\eta\big)I\big(\lambda,\epsilon,\xi,\eta\big)=O\big(\lambda^{-(N-1)}\big). \tag{3.1}$$

(2) *For each pair* (ξ,η) *with* $-\xi/\eta\in\operatorname{supp}\chi$*, there exists a unique* $t=t(\xi,\eta)\in[1/3,13/6]$ *such that* $t(\xi,\eta)=(Q'_\epsilon)^{-1}(-\xi/\eta)$ *is* $(N-1)$*-times differentiable and satisfies*

$$D_t^1\phi_\epsilon\big(t(\xi,\eta),\xi,\eta\big)=0 \tag{3.2}$$

and

$$\begin{aligned}\chi\big(-\xi/\eta\big)I\big(\lambda,\epsilon,\xi,\eta\big)=&\,C\chi\big(-\xi/\eta\big)1_{[-A,A]}\big(\xi/\eta\big)\rho\big(t(\xi,\eta)\big)\\ &\cdot\big|D_t^2\phi_\epsilon(t(\xi,\eta),\xi,\eta)\big|^{-1/2}e^{i\lambda\phi_\epsilon(t(\xi,\eta),\xi,\eta)}\lambda^{-1/2}\\ &+O\big(\lambda^{-3/2}\big)\end{aligned} \tag{3.3}$$

with C *being an absolute constant.*

Furthermore, the implicit constants in (3.1) *and* (3.3) *are independent of* λ, ϵ, ξ, *and* η.

Proof Due to (2.2), we observe that Q'_ϵ is monotone on $[1/4,4]$ and that, for any $t\in[1/3,13/6]$ and $r\in(0,1/12]$, Q'_ϵ is a bijection from $B(t,r)$[3] to an interval which contains $B(Q'_\epsilon(t),c_1r)$.

Assume that there exist $a\in[1/2,2]$ and (ξ_0,η_0) with $-\xi_0/\eta_0\in\operatorname{supp}\chi$ such that $|D_t^1\phi_\epsilon(a,\xi_0,\eta_0)|<c_1/12$, otherwise we get (3.1) by integration by parts.

Since $D_t^1\phi_\epsilon(t,\xi,\eta)=Q'_\epsilon(t)+\xi/\eta$, we have that $-\xi_0/\eta_0\in B(Q'_\epsilon(a),c_1/12)$. It follows from the observation above that there exists a unique $a_0\in[1/4,4]$ such that $a_0\in B(a,1/12)$ and $Q'_\epsilon(a_0)=-\xi_0/\eta_0$. Thus $\operatorname{supp}\chi\subset B(Q'_\epsilon(a_0),c_1/12)$. The observation above then implies that, for each pair (ξ,η) with $-\xi/\eta\in\operatorname{supp}\chi$, there exists a unique $t(\xi,\eta)\in B(a_0,1/12)$ such that $Q'_\epsilon(t(\xi,\eta))=-\xi/\eta$, which is (3.2). In particular, $t(\xi,\eta)=(Q'_\epsilon)^{-1}(-\xi/\eta)$, whose differentiability is a consequence of the inverse function theorem.

Note that $B(t(\xi,\eta),1/12)\subset[1/4,9/4]$ and we also have

$$\big|D_t^1\phi_\epsilon(t,\xi,\eta)\big|=\big|D_t^1\phi_\epsilon(t,\xi,\eta)-D_t^1\phi_\epsilon(t(\xi,\eta),\xi,\eta)\big|\geq c_1\big|t-t(\xi,\eta)\big|.$$

Applying to $I(\lambda,\epsilon,\xi,\eta)$ the method of stationary phase on $B(t(\xi,\eta),1/12)$ and integration by parts outside $B(t(\xi,\eta),1/24)$ yields (3.3). □

Remark 3.2 A similar argument in high dimensions can be found in the proof of [4, Proposition 2.4]. For the method of stationary phase, the reader can check [6, Section 7.7].

We quote below Li's [11, Theorem 6.2] with a small modification in the statement for the sake of our later application, however its proof remains the same. Let $\sigma\in(0,1]$, $\mathbf{I}\subset\mathbb{R}$ be a fixed bounded interval, and $U(\mathbf{I})$ a nontrivial subset of $L^2(\mathbf{I})$ such that the

[3] $B(t,r)$ denotes the interval $(t-r,t+r)$.

L^2-norm of every element of $U(\mathbf{I})$ is uniformly bounded by a constant. We say that a function $f \in L^2(\mathbf{I})$ is σ*-uniform in* $U(\mathbf{I})$ if

$$\left|\int_{\mathbf{I}} f(x)\overline{u(x)}\,\mathrm{d}x\right| \le \sigma \|f\|_{L^2(\mathbf{I})} \quad \text{for all } u \in U(\mathbf{I}).$$

Lemma 3.3 *Let $\mathscr{L}$ be a bounded sublinear functional from $L^2(\boldsymbol{I})$ to $\mathbb{C}$, S_σ the set of all functions that are σ-uniform in $U(\boldsymbol{I})$,*

$$A_\sigma = \sup\left\{|\mathscr{L}(f)|/\|f\|_{L^2(\boldsymbol{I})} : f \in S_\sigma,\ f \ne 0\right\},$$

and

$$M = \sup_{u \in U(\boldsymbol{I})} |\mathscr{L}(u)|.$$

Then

$$\|\mathscr{L}\| \le \max\left\{A_\sigma, 2\sigma^{-1} M\right\}.$$

We also need the following theorem to handle the minor part in Sect. 6. This theorem is a variant of the results in [10, Theorem 2.1] concerning estimates for certain paraproducts. The only change is that the standard dyadic sequence $\{2^{\alpha j}\}_{j\in\mathbb{Z}}$ with $\alpha \in \mathbb{N}\backslash\{0\}$ (in [10]) is replaced by a dyadic-like sequence $\{\Delta_j\}$ here, while the proof remains the same; see [10, Sections 3 and 4].

Theorem 3.4 *Let $L \in \mathbb{Z}$ and let $\{\Delta_j\}_{j>L}$ be a sequence of positive numbers which is dyadic-like, i.e., there is a $K \in \mathbb{Z}$ such that for all $j > L$ and $j + K > L$ the following holds*

$$\Delta_{j+K} \ge 2\Delta_j. \tag{3.4}$$

Let Φ_1 and Φ_2 be Schwartz functions on $\mathbb{R}$ whose Fourier transforms are standard bump functions supported on $[-2, -1/2] \cup [1/2, 2]$ and $[-1, 1]$ respectively, and $\widehat{\Phi_2}(0) = 1$. For $(n_1, n_2) \in \mathbb{Z}^2$ and $l = 1$ or 2, set

$$\mathcal{M}_{l,n_1,n_2}(\xi,\eta) = \sum_{j>L} \widehat{\Phi_l}\left(\frac{\xi}{2^j}\right) e^{2\pi i n_1 \frac{\xi}{2^j}}\, \widehat{\Phi_{3-l}}\left(\frac{\eta}{\Delta_j}\right) e^{2\pi i n_2 \frac{\eta}{\Delta_j}}.$$

Then for $l = 1$ and 2, for any $p_1, p_2 > 1$ with $1/r = 1/p_1 + 1/p_2$, there is a constant C independent of (n_1, n_2) such that for all $f_1 \in L^{p_1}(\mathbb{R})$, $f_2 \in L^{p_2}(\mathbb{R})$, the following holds

$$\left\|\Pi_{l,n_1,n_2}(f_1, f_2)\right\|_r \le C\left(1+n_1^2\right)^{10}\left(1+n_2^2\right)^{10} \|f_1\|_{p_1}\|f_2\|_{p_2},$$

where

$$\Pi_{l,n_1,n_2}\left(f_1, f_2\right)(x) = \iint \mathcal{M}_{l,n_1,n_2}\left(\xi,\eta\right) \widehat{f_1}(\xi)\widehat{f_2}(\eta) e^{2\pi i(\xi+\eta)x}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

4 The Main Estimates

Let $\widehat{\varphi} \in C_0^\infty(\mathbb{R})$ such that $\widehat{\varphi} = 1$ on $\{t \in \mathbb{R} : 3/8 \le |t| \le 17/8\}$ and $\operatorname{supp} \widehat{\varphi} \subset \{t \in \mathbb{R} : 1/4 \le |t| \le 9/4\}$. For $j, m \in \mathbb{N}$ denote $\epsilon_j = 2^{-j}$ and

$$K_{j,m}(\xi, \eta) = \int_0^\infty \rho(t) e^{-2\pi i 2^m \eta \phi_{\epsilon_j}(t,\xi,\eta)} \, \mathrm{d}t,$$

where ρ and ϕ_{ϵ_j} are as defined at the beginning of Sect. 3.

For $f, g \in L^2(\mathbb{R})$ denote, when $|\gamma'(\epsilon_j)| \le K_1 |\epsilon_j|^{c_1}$,

$$B^\varphi_{j,m}(f, g)(x) = \left|\gamma'(\epsilon_j)\right|^{1/2} \iint \widehat{f}(\xi)\widehat{\varphi}(\xi)\widehat{g}(\eta)\widehat{\varphi}(\eta) e^{2\pi i \left(\gamma'(\epsilon_j)\xi + \eta\right)x} K_{j,m}(\xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta,$$

and, when $|\gamma'(\epsilon_j)| \ge K_2 |\epsilon_j|^{-c_1}$,

$$B^\varphi_{j,m}(f, g)(x) = \left|\gamma'(\epsilon_j)\right|^{-1/2} \iint \widehat{f}(\xi)\widehat{\varphi}(\xi)\widehat{g}(\eta)\widehat{\varphi}(\eta) e^{2\pi i \left(\xi + \gamma'(\epsilon_j)^{-1}\eta\right)x} K_{j,m}(\xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta.$$

Proposition 4.1 *Assume that $\Gamma(t) = (t, \gamma(t)) \in \boldsymbol{F}(-1, 1)$.*[4] *For any $\beta < 1$, there exist an $L \in \mathbb{N}$ and a constant C_β such that whenever $j \ge L$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$, and $f, g \in L^2(\mathbb{R})$, we have*

(1) *if $|\gamma'(\epsilon_j)| \le K_1 |\epsilon_j|^{c_1}$, then*

$$\left\| B^\varphi_{j,m}(f, g) \boldsymbol{I}_{2^m \gamma'(\epsilon_j)^{-1}, n} \right\|_1 \le C_\beta C_{j,m} \|f\|_2 \|g\|_2,$$

where

$$C_{j,m} = \begin{cases} 2^{-m/16} & \text{if } |\gamma'(\epsilon_j)| > 2^{-m}, \\ 2^{-\beta m/4} & \text{if } |\gamma'(\epsilon_j)| \le 2^{-m}; \end{cases} \tag{4.1}$$

(2) *if $|\gamma'(\epsilon_j)| \ge K_2 |\epsilon_j|^{-c_1}$, then*

$$\left\| B^\varphi_{j,m}(f, g) \boldsymbol{I}_{2^m \gamma'(\epsilon_j), n} \right\|_1 \le C_\beta C'_{j,m} \|f\|_2 \|g\|_2,$$

where

$$C'_{j,m} = \begin{cases} 2^{-m/16} & \text{if } |\gamma'(\epsilon_j)| < 2^m, \\ 2^{-\beta m/4} & \text{if } |\gamma'(\epsilon_j)| \ge 2^m. \end{cases} \tag{4.2}$$

The rest of this section is devoted to the proof of Proposition 4.1.

We first observe that there is actually a symmetry between the case $|\gamma'(\epsilon_j)| \le K_1 |\epsilon_j|^{c_1}$ and the case $|\gamma'(\epsilon_j)| \ge K_2 |\epsilon_j|^{-c_1}$, hence we only prove the former case while the other one can be handled similarly. We can also simplify the domain of integration of $B^\varphi_{j,m}(f, g)(x)$ by using a decomposition $\widehat{\varphi} = \widehat{\varphi} 1_{(0,\infty)} + \widehat{\varphi} 1_{(-\infty,0]}$, which allows

[4] We actually do not need the condition (2.4) for this proposition.

us to restrict the domain to one of the cubes $(\pm[1/4, 9/4]) \times (\pm[1/4, 9/4])$. We still use $\widehat{\varphi}$ below but with its support contained in either $[1/4, 9/4]$ or $[-9/4, -1/4]$ (and this won't cause any problem).

The proof is split into three parts. In the first part, we apply the TT^* method to estimate $\|B^{\varphi}_{j,m}(f, g)\|_1$, during which procedure we need a standard result from the oscillatory integral theory and a necessary condition $|\gamma'(\epsilon_j)| > 2^{-m}$. The bound we get (see (4.4) below) is efficient when $|\gamma'(\epsilon_j)|$ is large but inefficient when $|\gamma'(\epsilon_j)|$ is close to 2^{-m}. In the second part, with the help of Lemma 3.3 (the method of σ-uniformity introduced in Li [11]), we can put certain restrictions on the function f (or g) and reduce the estimate of $\|B^{\varphi}_{j,m}(f, g)\|_1$ to a restricted version, to which the TT^* method can be applied without extra assumptions on the size of $|\gamma'(\epsilon_j)|$. The bound we get in this part (see (4.23) and (4.24) below) is efficient when $|\gamma'(\epsilon_j)|$ is small (even when $|\gamma'(\epsilon_j)|$ is close to 2^{-m}) but inefficient when $|\gamma'(\epsilon_j)|$ is large (see also Lie [13, p. 18]). In the last part, we take advantage of both results and prove the desired estimate.

4.1 Part 1: $j \geq L$, $m \in \mathbb{N}$ such that $|\gamma'(\epsilon_j)| > 2^{-m}$

[5] We first prove that, for $h \in L^2(\mathbb{R})$,

$$\left|\int B^{\varphi}_{j,m}(f, g)(x)h(x)\,\mathrm{d}x\right| \leq C\big(2^m|\gamma'(\epsilon_j)|\big)^{-1/6}\|f\|_2\|g\|_2 \cdot \big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2}\|h\|_2, \tag{4.3}$$

which trivially leads to the estimate

$$\left\|B^{\varphi}_{j,m}(f, g)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}\right\|_1 \leq C\big(2^m|\gamma'(\epsilon_j)|\big)^{-1/6}\|f\|_2\|g\|_2. \tag{4.4}$$

We can find a finite open cover of the interval $[-10, 10]$ by using open intervals of length $c_1/12$, associated with which we can construct a partition of unity. By inserting this partition of unity we reduce the estimate of $\int B^{\varphi}_{j,m}(f, g)(x)h(x)\,\mathrm{d}x$ to

$$\begin{aligned}&\int \widetilde{B}^{\varphi}_{j,m}(f, g)(x)h(x)\,\mathrm{d}x\\ &= \big|\gamma'(\epsilon_j)\big|^{1/2}\iint \widehat{f\varphi}(\xi)\widehat{g\varphi}(\eta)\mathcal{F}^{-1}[h]\big(\gamma'(\epsilon_j)\xi + \eta\big)\chi\big(-\xi/\eta\big)K_{j,m}\big(\xi, \eta\big)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned}$$

where χ is smooth and supported in an interval of length $c_1/12$.

We can then apply Lemma 3.1 to $\chi(-\xi/\eta)K_{j,m}(\xi, \eta)$. If (3.1) holds, then an application of Hölder's inequality yields

[5] In this part we mainly follow the argument contained in Section 6 of the preprint arXiv:0805.0107 and make necessary modifications in order to adapt it to the current case.

$$\left|\int \widetilde{B}^{\varphi}_{j,m}(f,g)(x)h(x)\,\mathrm{d}x\right| \le C2^{-m/2}\|f\|_2\|g\|_2 \cdot \big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2}\|h\|_2. \tag{4.5}$$

This estimate immediately leads to (4.3).

Below we assume that the second statement in Lemma 3.1 holds. Applying (3.3) yields

$$\begin{aligned}\int \widetilde{B}^{\varphi}_{j,m}(f,g)(x)h(x)\mathrm{d}x = C\big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2} \\ \cdot \iint \widehat{f\varphi}(\xi)\widehat{g\varphi}(\eta)\mathcal{F}^{-1}[h]\big(\gamma'(\epsilon_j)\xi+\eta\big) \\ a\big(\xi,\eta\big)e^{-2\pi i 2^m \eta\phi_{\epsilon_j}\left(t(\xi,\eta),\xi,\eta\right)}\,\mathrm{d}\xi\mathrm{d}\eta,\end{aligned} \tag{4.6}$$

where we have omitted the error term in (3.3) (since it leads to the same bound as in (4.5)), and $a(\xi,\eta)$ is defined as

$$a\big(\xi,\eta\big) = \chi\big(-\xi/\eta\big)\rho\big(t(\xi,\eta)\big)|\eta|^{-1/2}\big|D_t^2\phi_{\epsilon_j}(t(\xi,\eta),\xi,\eta)\big|^{-1/2}.$$

Applying to the double integral in (4.6) a change of variables $\gamma'(\epsilon_j)\xi+\eta \to \xi$, $\eta/\gamma'(\epsilon_j) \to \eta$ and then Hölder's inequality yields

$$\left|\int \widetilde{B}^{\varphi}_{j,m}(f,g)(x)h(x)\,\mathrm{d}x\right| \le C\|T_{j,m}(f,g)\|_2 \cdot \big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2}\|h\|_2, \tag{4.7}$$

where

$$\begin{aligned}T_{j,m}(f,g)(\xi) = \int \widehat{f\varphi}\big(\gamma'(\epsilon_j)^{-1}\xi-\eta\big)\widehat{g\varphi}\big(\gamma'(\epsilon_j)\eta\big)a\big(\gamma'(\epsilon_j)^{-1}\xi-\eta,\gamma'(\epsilon_j)\eta\big) \\ \cdot e^{-2\pi i 2^m\left(\gamma'(\epsilon_j)\eta\right)\phi_{\epsilon_j}\left(t\left(\gamma'(\epsilon_j)^{-1}\xi-\eta,\gamma'(\epsilon_j)\eta\right),\gamma'(\epsilon_j)^{-1}\xi-\eta,\gamma'(\epsilon_j)\eta\right)}\,\mathrm{d}\eta.\end{aligned}$$

We then have, after a change of variables,

$$\begin{aligned}\big\|T_{j,m}(f,g)\big\|_2^2 &= \int T_{j,m}(f,g)(\xi)\overline{T_{j,m}(f,g)(\xi)}\,\mathrm{d}\xi \\ &= \int \mathrm{d}\tau \iint F_\tau(x)G_\tau(y)A_\tau(x,y)e^{-2\pi i 2^m P_\tau(x,y)}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}F_\tau(x) &= \widehat{f\varphi}(x-\tau)\overline{\widehat{f\varphi}(x)}, \\ G_\tau(y) &= \widehat{g\varphi}(y+\gamma'(\epsilon_j)\tau)\overline{\widehat{g\varphi}(y)}, \\ A_\tau(x,y) &= a(x-\tau,y+\gamma'(\epsilon_j)\tau)\overline{a(x,y)},\end{aligned}$$

and

$$P_\tau(x,y) = P_1\big(x-\tau, y+\gamma'(\epsilon_j)\tau\big) - P_1(x,y)$$

with

$$P_1(x,y) = y\phi_{\epsilon_j}\big(t(x,y),x,y\big).$$

In order to estimate the inner double integral in (4.8), we first show that there exists an $L\in\mathbb{N}$ such that if $j\ge L$ then

$$\left|\frac{\partial^2 P_\tau}{\partial y\partial x}(x,y)\right| \asymp |\tau|. \tag{4.9}$$

Recall that $t(x,y)$ satisfies (3.2) (with ξ, η, and ϵ replaced by x, y, and ϵ_j respectively). By implicit differentiation, we get

$$\frac{\partial t}{\partial x}(x,y) = -\frac{1}{yQ''_{\epsilon_j}(t(x,y))} \quad \text{and} \quad \frac{\partial t}{\partial y}(x,y) = -\frac{Q'_{\epsilon_j}(t(x,y))}{yQ''_{\epsilon_j}(t(x,y))}.$$

By (2.1), (2.2), and (2.3), we then have

$$\frac{\partial^2 t}{\partial x\partial y}(x,y) = \frac{1}{y^2}\frac{\big(Q''_{\epsilon_j}\big)^2 - Q'_{\epsilon_j}Q'''_{\epsilon_j}}{(Q''_{\epsilon_j})^3}\big(t(x,y)\big) \asymp 1$$

and

$$\frac{\partial^2 t}{\partial y^2}(x,y) = \frac{1}{y^2}\frac{Q'_{\epsilon_j}\big(2(Q''_{\epsilon_j})^2 - Q'_{\epsilon_j}Q'''_{\epsilon_j}\big)}{(Q''_{\epsilon_j})^3}\big(t(x,y)\big) \lesssim 1.$$

By using (3.2) we also get

$$\frac{\partial^2 P_1}{\partial y\partial x}(x,y) = \frac{\partial t}{\partial y}(x,y).$$

Noticing that $|\gamma'(\epsilon_j)|$ is small if L is large, by the mean value theorem we get (4.9).

Let $\tau_0 = (2^m|\gamma'(\epsilon_j)|)^{-1/3}$. We have the following splitting of (4.8):

$$\begin{aligned}\|T_{j,m}(f,g)\|_2^2 = &\left(\int_{|\tau|<\tau_0} + \int_{\tau_0\le|\tau|\le 10} \mathrm{d}\tau\right)\\ &\cdot \iint F_\tau(x)G_\tau(y)A_\tau(x,y)e^{-2\pi i 2^m P_\tau(x,y)}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Applying the trivial estimate and Hörmander's [5, Theorem 1.1] to the two parts above (and also Hölder's inequality) yields

$$\begin{aligned}\big\|T_{j,m}(f,g)\big\|_2^2 &\le C\tau_0\|f\|_2^2\|g\|_2^2 + C\int_{\tau_0\le|\tau|\le 10}\big(2^m|\tau|\big)^{-1/2}\|F_\tau\|_2\|G_\tau\|_2\,\mathrm{d}\tau\\ &\le C\big(\tau_0 + (2^m|\gamma'(\epsilon_j)|\tau_0)^{-1/2}\big)\|f\|_2^2\|g\|_2^2\\ &\le C\big(2^m|\gamma'(\epsilon_j)|\big)^{-1/3}\|f\|_2^2\|g\|_2^2.\end{aligned}$$

To conclude, the desired estimate (4.3) follows from (4.5), (4.7), and the estimate above of $\|T_{j,m}(f, g)\|_2$.

4.2 Part 2: $j \geq L, m \in \mathbb{N}$

[6] We can find a finite open cover of the interval $[-36C_1, 36C_1]$ by using open intervals of length $c_1/24$, associated with which we can construct a partition of unity $\{\chi_s : 1 \leq s \leq \Theta\}$ such that $\sum_s \chi_s \equiv 1$ in $[-36C_1, 36C_1]$ and each χ_s is smooth and supported in an interval that belongs to the finite open cover above.

Lemma 3.1 will be applied to $\chi_s(-\xi/\eta)K_{j,m}(\xi, \eta)$ (below). Here we denote S to be the collection of all $1 \leq s \leq \Theta$ for which the second statement in Lemma 3.1 holds. Let $\mathbf{I}$ be either $[1/4, 9/4]$ or $[-9/4, -1/4]$, and

$$U(\mathbf{I}) := \left\{u_{s,r,\eta}(\xi) \in L^2(\mathbf{I}) : s \in S, r \in \mathbb{R}, 1/16C_1 \leq |\eta| \leq 9C_1\right\},$$

where

$$u_{s,r,\eta}(\xi) = \chi_s\big(-\xi/\eta\big)e^{2\pi i\left(2^m\eta\phi_{\epsilon_j}(t(\xi,\eta),\xi,\eta)+r\xi\right)}.$$

According to Lemma 3.3, we finish this part in three steps.

Step 1: Let $\widehat{f}|_{\mathbf{I}}$, the restriction of $\widehat{f}$ to $\mathbf{I}$, be an arbitrary function in $L^2(\mathbf{I})$ that is σ-uniform in $U(\mathbf{I})$.

We first note that $B^{\varphi}_{j,m}(f, g)(x)$ in the time space can be expressed as

$$B^{\varphi}_{j,m}(f, g)(x) = \left|\gamma'(\epsilon_j)\right|^{1/2}\int_0^{\infty} f * \varphi\big(\gamma'(\epsilon_j)x - 2^m t\big) g * \varphi\big(x - 2^m Q_{\epsilon_j}(t)\big)\rho(t)\, \mathrm{d}t, \tag{4.10}$$

which leads to, for $h \in L^2(\mathbb{R})$,

$$\int B^{\varphi}_{j,m}(f, g)(x)h(x)\, \mathrm{d}x = \left|\gamma'(\epsilon_j)\right|^{1/2} \cdot \sum_{l\in\mathbb{Z}} \iint_0^{\infty} f * \varphi\big(\gamma'(\epsilon_j)x - 2^m t\big) g_{j,m,l} \big(x - 2^m Q_{\epsilon_j}(t)\big)\rho(t)\big(\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l}h\big)(x)\, \mathrm{d}t\, \mathrm{d}x,$$

where $g_{j,m,l} = 1_{I_{j,m,l}} \cdot g * \varphi$ with $I_{j,m,l} = [\alpha_{j,l} - C_1 2^m, \alpha_{j,l+1} + C_1 2^m]$ and $\alpha_{j,l} = |\gamma'(\epsilon_j)|^{-1}l$. In the frequency space we then have

$$\int B^{\varphi}_{j,m}(f, g)(x)h(x)\, \mathrm{d}x = \left|\gamma'(\epsilon_j)\right|^{1/2} \cdot \sum_{l\in\mathbb{Z}} \iiint \widehat{f\varphi}(\xi)e^{2\pi i\gamma'(\epsilon_j)\xi x}\, \widehat{g_{j,m,l}}(\eta)e^{2\pi i\eta x} K_{j,m}(\xi, \eta)\big(\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l}h\big)(x)\, \mathrm{d}x\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

[6] In this part we mainly generalize the argument contained in Sections 8, 10, and 11 of the preprint arXiv:0805.0107. In particular we apply Lemma 3.3 with a carefully-chosen function set $U(\mathbf{I})$.

Let $\widehat{\varphi_1} \in C_0^\infty(\mathbb{R})$ such that $\widehat{\varphi_1}(\eta) = 1$ if $|\eta| \in [1/8C_1, 9C_1/2]$ and $\operatorname{supp} \widehat{\varphi_1} \subset \{x \in \mathbb{R} : 1/16C_1 \le |x| \le 9C_1\}$. By using $1 = \widehat{\varphi_1}(\eta) + (1 - \widehat{\varphi_1}(\eta))$ and the power series of $e^{2\pi i \gamma'(\epsilon_j)\xi(x-\alpha_{j,l})}$, we get

$$\int B^{\varphi}_{j,m}(f, g)(x)h(x)\,\mathrm{d}x = I + II,$$

where

$$I = \left|\gamma'(\epsilon_j)\right|^{1/2} \sum_{l\in\mathbb{Z}} \sum_{p=0}^{\infty} \frac{(2\pi i)^p}{p!} \iint \widehat{f\varphi}(\xi)\xi^p e^{2\pi i \gamma'(\epsilon_j)\alpha_{j,l}\xi}$$
$$\cdot \widehat{g_{j,m,l}}(\eta)\widehat{\varphi_1}(\eta)K_{j,m}(\xi,\eta)\mathcal{F}^{-1}\Big[\big(\gamma'(\epsilon_j)(\cdot - \alpha_{j,l})\big)^p \big(\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l} h\big)(\cdot)\Big](\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and

$$II = \left|\gamma'(\epsilon_j)\right|^{1/2} \sum_{l\in\mathbb{Z}} \sum_{p=0}^{\infty} \frac{(2\pi i)^p}{p!} \iint \widehat{f\varphi}(\xi)\xi^p e^{2\pi i \gamma'(\epsilon_j)\alpha_{j,l}\xi}$$
$$\cdot \widehat{g_{j,m,l}}(\eta)\big(1 - \widehat{\varphi_1}(\eta)\big)K_{j,m}(\xi,\eta)\mathcal{F}^{-1}\Big[\big(\gamma'(\epsilon_j)(\cdot - \alpha_{j,l})\big)^p \big(\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l} h\big)(\cdot)\Big]$$
$$(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

We first estimate Sum II. When $1 - \widehat{\varphi_1}(\eta) \ne 0$, Remark 2.4 (2) implies that the gradient of the phase function of $K_{j,m}(\xi,\eta)$ has a uniform lower bound, which leads to the bound $K_{j,m}(\xi,\eta) = O(2^{-m})$. Then by Hölder's inequality we get

$$|II| \le C2^{-m}\left|\gamma'(\epsilon_j)\right|^{1/2} \|1_{\mathbf{I}}\widehat{f}\|_2 \sum_{l\in\mathbb{Z}} \|g_{j,m,l}\|_2 \|\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l} h\|_2.$$

Applying the Cauchy–Schwarz inequality yields

$$|II| \le \begin{cases} C2^{-m/2}\big\|1_{\mathbf{I}}\widehat{f}\big\|_2 \|g\|_2 \cdot \big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2} \|h\|_2, & \text{if } |\gamma'(\epsilon_j)| \le 2^{-m}, \\ C\left|\gamma'(\epsilon_j)\right|^{1/2}\big\|1_{\mathbf{I}}\widehat{f}\big\|_2 \|g\|_2 \cdot \big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2} \|h\|_2, & \text{if } |\gamma'(\epsilon_j)| > 2^{-m}. \end{cases} \tag{4.11}$$

The estimate of Sum I, by using the partition of unity we have constructed at the beginning of this subsection, can be reduced to

$$I_s = \left|\gamma'(\epsilon_j)\right|^{1/2} \sum_{l\in\mathbb{Z}} \sum_{p=0}^{\infty} \frac{(2\pi i)^p}{p!} \iint \widehat{f\varphi}(\xi)\xi^p e^{2\pi i \gamma'(\epsilon_j)\alpha_{j,l}\xi} \widehat{g_{j,m,l}}(\eta)\widehat{\varphi_1}(\eta)$$
$$\cdot \chi_s\big(-\xi/\eta\big)K_{j,m}(\xi,\eta)\mathcal{F}^{-1}\Big[\big(\gamma'(\epsilon_j)(\cdot - \alpha_{j,l})\big)^p \big(\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l} h\big)(\cdot)\Big](\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

for any $1 \le s \le \Theta$. We apply Lemma 3.1 to $\chi_s(-\xi/\eta)K_{j,m}(\xi,\eta)$. If (3.1) holds, then I_s is bounded by (4.11) too. Hence we may assume that the second statement in Lemma 3.1 holds. Applying (3.3) yields

$$I_s = C\big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2}\sum_{l\in\mathbb{Z}}\sum_{p=0}^{\infty}\frac{(2\pi i)^p}{p!}$$
$$\cdot\int\mathfrak{M}(\eta)\widehat{g_{j,m,l}}(\eta)\mathcal{F}^{-1}\Big[\big(\gamma'(\epsilon_j)(\cdot-\alpha_{j,l})\big)^p\big(\mathbf{1}_{|\gamma'(\epsilon_j)|^{-1},l}h\big)(\cdot)\Big](\eta)\,\mathrm{d}\eta,$$

where we have omitted the error term in (3.3) (since it leads to the same bound as in (4.11)), and $\mathfrak{M}(\eta)$ is defined as

$$\mathfrak{M}(\eta) := \int_{\mathbf{I}} b(\xi,\eta)\widehat{f}(\xi)\chi_s\big(-\xi/\eta\big)e^{-2\pi i\left(2^m\eta\phi_{\epsilon_j}(t(\xi,\eta),\xi,\eta)-\gamma'(\epsilon_j)\alpha_{j,l}\xi\right)}\,\mathrm{d}\xi$$

with

$$b(\xi,\eta)=\widehat{\varphi}(\xi)\widehat{\varphi_1}(\eta)\xi^p\rho\big(t(\xi,\eta)\big)|\eta|^{-1/2}\big|D_t^2\phi_{\epsilon_j}(t(\xi,\eta),\xi,\eta)\big|^{-1/2}.$$

Using the Fourier series of $b(\xi,\eta)$ and the assumption that $\widehat{f}|_{\mathbf{I}}$ is σ-uniform in $U(\mathbf{I})$, we have

$$\big|\mathfrak{M}(\eta)\big| \le C9^p\sigma\big\|1_{\mathbf{I}}\widehat{f}\big\|_2.$$

Hence by using Hölder's and the Cauchy–Schwarz inequalities we get

$$|I_s|\le\begin{cases}C\sigma\big\|1_{\mathbf{I}}\widehat{f}\big\|_2\|g\|_2\cdot\big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2}\|h\|_2, & \text{if } |\gamma'(\epsilon_j)|\le 2^{-m},\\ C\big(2^m|\gamma'(\epsilon_j)|\big)^{1/2}\sigma\big\|1_{\mathbf{I}}\widehat{f}\big\|_2\|g\|_2\cdot\big(2^{-m}|\gamma'(\epsilon_j)|\big)^{1/2}\|h\|_2, & \text{if } |\gamma'(\epsilon_j)|> 2^{-m}.\end{cases}$$

To conclude Step 1, if $\sigma > 2^{-m/2}$, then the bound above of I_s and (4.11) lead to, for $h\in L^\infty(\mathbb{R})$,

$$\left|\int B_{j,m}^{\varphi}(f,g)(x)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}(x)h(x)\,\mathrm{d}x\right|$$
$$\le\begin{cases}C\sigma\big\|1_{\mathbf{I}}\widehat{f}\big\|_2\|g\|_2\|h\|_\infty, & \text{if } |\gamma'(\epsilon_j)|\le 2^{-m},\\ C\big(2^m|\gamma'(\epsilon_j)|\big)^{1/2}\sigma\big\|1_{\mathbf{I}}\widehat{f}\big\|_2\|g\|_2\|h\|_\infty, & \text{if } |\gamma'(\epsilon_j)|> 2^{-m}.\end{cases} \tag{4.12}$$

Step 2: We now assume that $\widehat{f}|_{\mathbf{I}}\in U(\mathbf{I})$.

By using (4.10), a change of variables $x\to 2^m\gamma'(\epsilon_j)^{-1}(x+\gamma'(\epsilon_j)Q_{\epsilon_j}(t))$, and Hölder's inequality, we have, for $h\in L^\infty(\mathbb{R})$,

$$\left|\int B_{j,m}^{\varphi}(f,g)(x)h(x)\,\mathrm{d}x\right| = 2^m\big|\gamma'(\epsilon_j)\big|^{-1/2}\bigg|\iint_0^\infty f*\varphi\big(2^m(x+\gamma'(\epsilon_j)Q_{\epsilon_j}(t)-t)\big)$$
$$\cdot g*\varphi\big(2^m\gamma'(\epsilon_j)^{-1}x\big)h_{j,m}\big(x+\gamma'(\epsilon_j)Q_{\epsilon_j}(t)\big)\rho(t)\,\mathrm{d}t\,\mathrm{d}x\bigg|$$
$$\le C\|g\|_2\|T_1(h)\|_2, \tag{4.13}$$

where $h_{j,m}(x)=h(2^m\gamma'(\epsilon_j)^{-1}x)$ and

$$T_1(h)(x) = 2^{m/2}\int_0^\infty f*\varphi\big(2^m(x+\gamma'(\epsilon_j)Q_{\epsilon_j}(t)-t)\big)h_{j,m}\big(x+\gamma'(\epsilon_j)Q_{\epsilon_j}(t)\big)\rho(t)\,\mathrm{d}t.$$

Let $\widehat{f}|_{\mathbf{I}} = u_{s,r,\eta}(\xi)$ for arbitrarily fixed $s \in S$, $r \in \mathbb{R}$, and $1/16C_1 \le |\eta| \le 9C_1$. By applying the Fourier inversion formula to $f * \varphi$ and changing variables, we get

$$\|T_1(h)\|_2^2 = 2^m \int \left| \int_0^\infty K_1(x,t)h_{j,m}\big(x - 2^{-m}r + \gamma'(\epsilon_j)Q_{\epsilon_j}(t)\big)\rho(t)\,\mathrm{d}t \right|^2 \mathrm{d}x, \tag{4.14}$$

where

$$K_1(x,t) = \int \widehat{\varphi}(\xi)\chi_s\big(-\xi/\eta\big)e^{2\pi i 2^m \eta\left[\phi_{\epsilon_j}(t(\xi,\eta),\xi,\eta)+y(x,t)(\xi/\eta)\right]}\,\mathrm{d}\xi \tag{4.15}$$

with

$$y = y(x,t) = x + \gamma'(\epsilon_j)Q_{\epsilon_j}(t) - t.$$

Let χ_M be a smooth cut-off function supported in $[-M, M]$, which equals 1 in $[-M/2, M/2]$. We decompose the right-hand side of (4.14) into two parts by using the decomposition $1 = (1 - \chi_M(x)) + \chi_M(x)$ to restrict the integration domain of x to $\{x \in \mathbb{R} : |x| \ge M/2\}$ and $\{x \in \mathbb{R} : |x| < M\}$ respectively for a sufficiently large constant M. The former part is bounded by

$$C2^{-m}\|h\|_\infty^2, \tag{4.16}$$

since integration by parts yields $K_1(x,t) = O(2^{-m}|x|^{-1})$.

We next consider the latter part with $|x| < M$. After inserting a partition of unity, we may replace $K_1(x,t)$ by $\widetilde{\chi}(-y(x,t))K_1(x,t)$ with a smooth cut-off function $\widetilde{\chi}$ supported in an interval of sufficiently small length. Then by repeating the argument in the proof of Lemma 3.1, we have that either $\widetilde{\chi}(-y(x,t))K_1(x,t) = O(2^{-m})$ (leading to the bound (4.16)) or the phase function in (4.15) has a critical point $\xi(x,t)$ satisfying

$$t\big(\xi(x,t),\eta\big) = -y(x,t).$$

This equation, together with $\partial_t\phi_{\epsilon_j}(t(\xi,\eta),\xi,\eta) = 0$ (namely, Eq. (3.2) satisfied by $t(\xi,\eta)$), yields

$$\xi(x,t) = -\eta Q'_{\epsilon_j}\big(-y(x,t)\big).$$

By using the method of stationary phase in a neighborhood of $\xi(x,t)$ and integration by parts away from it, we get the following asymptotic formula.

$$\begin{aligned}\widetilde{\chi}\big(-y(x,t)\big)K_1(x,t) = C\widetilde{\chi}\big(-y(x,t)\big)\chi_s\big(-\xi(x,t)/\eta\big)\widehat{\varphi}\big(\xi(x,t)\big)\\ \left|\partial_\xi t\big(\xi(x,t),\eta\big)\right|^{-1/2}\\ \cdot e^{2\pi i 2^m \eta Q_{\epsilon_j}\left(-y(x,t)\right)}2^{-m/2} + O\big(2^{-3m/2}\big).\end{aligned}$$

By using the leading term above and a change of variables $u = Q_{\epsilon_j}(t)$, we now need to estimate

$$\int \chi_M(x) \left| \int h_{j,m}\big(x - 2^{-m}r + \gamma'(\epsilon_j)u\big)k(x,u)e^{2\pi i 2^m \eta Q_{\epsilon_j}\left(-y\left(x, Q_{\epsilon_j}^{-1}(u)\right)\right)}\,\mathrm{d}u \right|^2 \mathrm{d}x, \tag{4.17}$$

where

$$k(x,u) = \widetilde{\chi}\Big(-y\big(x, Q_{\epsilon_j}^{-1}(u)\big)\Big)\chi_s\Big(-\xi\big(x, Q_{\epsilon_j}^{-1}(u)\big)/\eta\Big)\widehat{\varphi}\Big(\xi\big(x, Q_{\epsilon_j}^{-1}(u)\big)\Big) \cdot \left|\partial_\xi t\big(\xi(x, Q_{\epsilon_j}^{-1}(u)), \eta\big)\right|^{-1/2} \rho\Big(Q_{\epsilon_j}^{-1}(u)\Big)\Big(Q'_{\epsilon_j}\big(Q_{\epsilon_j}^{-1}(u)\big)\Big)^{-1}.$$

We use the TT^* method for (4.17). By changing variables $u_1 = \upsilon + \tau$, $u_2 = \upsilon$, followed by $x \to x - \gamma'(\epsilon_j)\upsilon$, (4.17) becomes

$$\int \mathrm{d}\tau \int H_\tau(x)\,\mathrm{d}x \int K_{\tau,x}(\upsilon)e^{2\pi i 2^m \eta P_{\tau,x}(\upsilon)}\,\mathrm{d}\upsilon, \tag{4.18}$$

where all three integrals are over some finite intervals,

$$H_\tau(x) = h_{j,m}\big(x - 2^{-m}r + \gamma'(\epsilon_j)\tau\big)\overline{h_{j,m}(x - 2^{-m}r)},$$
$$K_{\tau,x}(\upsilon) = \chi_M\big(x - \gamma'(\epsilon_j)\upsilon\big)k\big(x - \gamma'(\epsilon_j)\upsilon, \upsilon + \tau\big)\overline{k(x - \gamma'(\epsilon_j)\upsilon, \upsilon)},$$

and

$$P_{\tau,x}(\upsilon) = P_2\big(x + \gamma'(\epsilon_j)\tau, \upsilon + \tau\big) - P_2(x, \upsilon)$$

with

$$P_2(x, \upsilon) = Q_{\epsilon_j}\Big(-\big[x - Q_{\epsilon_j}^{-1}(\upsilon)\big]\Big).$$

Before applying integration by parts to the innermost integral in (4.18) we first estimate its phase function $P_{\tau,x}(\upsilon)$. Actually we have that if $|\gamma'(\epsilon_j)|/|x|$ is sufficiently small then

$$\big|D_\upsilon P_{\tau,x}(\upsilon)\big| \asymp |x||\tau| \tag{4.19}$$

and

$$\big|D_\upsilon^2 P_{\tau,x}(\upsilon)\big| \lesssim |x||\tau|. \tag{4.20}$$

The (4.19) follows from the mean value theorem and the following estimates

$$\frac{\partial^2 P_2}{\partial x \partial \upsilon}(x, \upsilon) = -\frac{Q''_{\epsilon_j}\big(-[x - Q_{\epsilon_j}^{-1}(\upsilon)]\big)}{Q'_{\epsilon_j}\big(Q_{\epsilon_j}^{-1}(\upsilon)\big)} \asymp 1$$

and

$$\frac{\partial^2 P_2}{\partial \upsilon^2}(x, \upsilon) = x \cdot \frac{Q'_{\epsilon_j}\big(-[x - Q_{\epsilon_j}^{-1}(\upsilon)]\big)}{\big(Q'_{\epsilon_j}\big(Q_{\epsilon_j}^{-1}(\upsilon)\big)\big)^2} \cdot \frac{\big(Q''_{\epsilon_j}\big)^2 - Q'_{\epsilon_j}Q'''_{\epsilon_j}}{\big(Q'_{\epsilon_j}\big)^2}(c) \asymp |x|,$$

where c is between $-[x - Q_{\epsilon_j}^{-1}(\upsilon)]$ and $Q_{\epsilon_j}^{-1}(\upsilon)$. The (4.20) can be proved similarly.

Therefore, if $|\gamma'(\epsilon_j)|/|x|$ is sufficiently small, for any $\beta < 1$ we have

$$\left|\int K_{\tau,x}(\upsilon)e^{2\pi i 2^m \eta P_{\tau,x}(\upsilon)}\,\mathrm{d}\upsilon\right| \le C\min\left\{1, (2^m|x||\tau|)^{-1}\right\} \le C\left(2^m|x||\tau|\right)^{-\beta}.$$

We now estimate (4.18) by splitting it into two parts (depending on the size of $|\gamma'(\epsilon_j)|/|x|$) and using the trivial estimate and the bound above respectively. Then it is bounded by

$$C\left(|\gamma'(\epsilon_j)| + 2^{-\beta m}\right)\|h\|_\infty^2. \tag{4.21}$$

To conclude Step 2, by (4.13), (4.14), (4.16), and (4.21), we get, for $h \in L^\infty(\mathbb{R})$,

$$\begin{aligned}&\left|\int B_{j,m}^{\varphi}(f,g)(x)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}(x)h(x)\,\mathrm{d}x\right| \\ &\quad\le \begin{cases} C2^{-\beta m/2}\|g\|_2\|h\|_\infty, & \text{if } |\gamma'(\epsilon_j)| \le 2^{-m}, \\ C\left(\max\{|\gamma'(\epsilon_j)|, 2^{-\beta m}\}\right)^{1/2}\|g\|_2\|h\|_\infty, & \text{if } |\gamma'(\epsilon_j)| > 2^{-m}. \end{cases}\end{aligned} \tag{4.22}$$

Step 3: To conclude this subsection (namely, Part 2), by using Lemma 3.3 and the estimates (4.12) and (4.22), we get that for any $\beta < 1$

$$\left\|B_{j,m}^{\varphi}(f,g)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}\right\|_1 \le C2^{-\beta m/4}\|f\|_2\|g\|_2, \quad \text{if } \left|\gamma'(\epsilon_j)\right| \le 2^{-\beta m}, \tag{4.23}$$

and

$$\left\|B_{j,m}^{\varphi}(f,g)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}\right\|_1 \le C2^{m/4}\left|\gamma'(\epsilon_j)\right|^{1/2}\|f\|_2\|g\|_2, \quad \text{if } |\gamma'(\epsilon_j)| \ge 2^{-\beta m}. \tag{4.24}$$

4.3 Part 3: Conclusion

If $|\gamma'(\epsilon_j)| \ge 2^{-\beta m}$, balancing (4.4) with (4.24) yields (see also Lie [13, p. 20])

$$\begin{aligned}\left\|B_{j,m}^{\varphi}(f,g)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}\right\|_1 &\le C\min\left\{\left(2^m|\gamma'(\epsilon_j)|\right)^{-1/6}, 2^{m/4}|\gamma'(\epsilon_j)|^{1/2}\right\}\|f\|_2\|g\|_2 \\ &\le C2^{-m/16}\|f\|_2\|g\|_2.\end{aligned}$$

If $|\gamma'(\epsilon_j)| \le 2^{-\beta m}$, the (4.23) is already good enough. This finishes the proof of Proposition 4.1.

5 Estimate of $\|B_{j,m}^{\Phi}(f,g)\|_1$

Let $\widehat{\Phi} \in C_0^\infty(\mathbb{R})$ be supported in $\{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}$ and $B_{j,m}^{\Phi}(f,g)$ be as defined at the beginning of Sect. 4 (with φ there replaced by Φ).

Proposition 5.1 *Assume that $\Gamma(t) = (t, \gamma(t)) \in \boldsymbol{F}(-1, 1)$.*[7] *For any $\beta < 1$, there exist an $L \in \mathbb{N}$ and a constant C'_β such that whenever $j \geq L$, $m \in \mathbb{N}$, and $f, g \in L^2(\mathbb{R})$ we have*

$$\|B^{\Phi}_{j,m}(f, g)\|_1 \leq C'_\beta A^{\beta}_{j,m} \|f\|_2 \|g\|_2, \tag{5.1}$$

where $A_{j,m}$ equals $C_{j,m}$ if $|\gamma'(\epsilon_j)| \leq K_1|\epsilon_j|^{c_1}$ and $C'_{j,m}$ if $|\gamma'(\epsilon_j)| \geq K_2|\epsilon_j|^{-c_1}$ (with $C_{j,m}$ and $C'_{j,m}$ defined as in Proposition 4.1).

Remark 5.2 This proposition is a consequence of Proposition 4.1. It is essentially the Lemma 5.1 contained in the arXiv preprint (arXiv:0805.0107) (which was later published as Li [11]).

Proof of Proposition 5.1 We only prove the case when $|\gamma'(\epsilon_j)| \leq K_1|\epsilon_j|^{c_1}$ while the other case can be handled similarly. Let ϕ be a Schwartz function on $\mathbb{R}$ such that $\int \phi = 1$ and $\operatorname{supp} \widehat{\phi} \subset [-1/100, 1/100]$. Denote $\phi_K(x) = K^{-1}\phi(K^{-1}x)$. We have

$$B^{\Phi}_{j,m}(f, g)(x) = \left|\gamma'(\epsilon_j)\right|^{1/2} \sum_{n\in\mathbb{Z}} \sum_{k_1,k_2\in\mathbb{Z}} \int_0^\infty \left(\mathbf{1}_{2^m,n+k_1} * \phi_{2^m} \cdot f * \Phi\right)\left(\gamma'(\epsilon_j)x - 2^m t\right)$$
$$\cdot \left(\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n+k_2} * \phi_{2^m\gamma'(\epsilon_j)^{-1}} \cdot g * \Phi\right)$$
$$\left(x - 2^m Q_{\epsilon_j}(t)\right)\rho(t)\,\mathrm{d}t \cdot \mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}.$$

We then make the decomposition $B^{\Phi}_{j,m}(f, g)(x) := \mathrm{I} + \mathrm{II}$ by splitting the inner summation for k_1, k_2 into two parts such that the first part, denoted by I, sums over $\{k_1, k_2 \in \mathbb{Z} : \max\{|k_1|, |k_2|\} \geq A\}$ and the second one, denoted by II, over $\{k_1, k_2 \in \mathbb{Z} : \max\{|k_1|, |k_2|\} < A\}$ with $A = C^{-(1-\beta)/2}_{j,m} > 1$.

Using the fast decay of $\mathbf{1}_{2^m,n+k_1} * \phi_{2^m}$ and $\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n+k_2} * \phi_{2^m\gamma'(\epsilon_j)^{-1}}$ yields that

$$|\mathrm{I}| \leq C\left|\gamma'(\epsilon_j)\right|^{1/2} \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ \max\{|k_1|,|k_2|\}\geq A}} \int_0^\infty \frac{\left|f * \Phi\left(\gamma'(\epsilon_j)x - 2^m t\right) g * \Phi\left(x - 2^m Q_{\epsilon_j}(t)\right)\rho(t)\right|}{\left(1 + |t + k_1|\right)^{N_1}\left(1 + |\gamma'(\epsilon_j)Q_{\epsilon_j}(t) + k_2|\right)^{N_2}}\,\mathrm{d}t$$
$$\leq C\left(A^{1-N_1} + A^{1-N_2}\right)\left|\gamma'(\epsilon_j)\right|^{1/2} \cdot \int_0^\infty \left|f * \Phi\left(\gamma'(\epsilon_j)x - 2^m t\right) g * \Phi\left(x - 2^m Q_{\epsilon_j}(t)\right)\rho(t)\right|\,\mathrm{d}t$$

for any $N_1, N_2 \in \mathbb{N}$. By Hölder's and Young's inequalities, we get

$$\|\mathrm{I}\|_1 \leq C\left(A^{1-N_1} + A^{1-N_2}\right)\|f\|_2\|g\|_2 \leq C''_\beta C^{\beta}_{j,m}\|f\|_2\|g\|_2, \tag{5.2}$$

where the second inequality holds whenever $N_1, N_2 \geq (1 + \beta)/(1 - \beta)$.

[7] We do not need the condition (2.4) for this proposition.

On the other hand, since

$$\operatorname{supp}\left(\mathcal{F}[\mathbf{1}_{2^m,n+k_1} * \phi_{2^m} \cdot f * \Phi]\right) \subset \left\{\xi \in \mathbb{R} : 3/8 \le |\xi| \le 17/8\right\}$$

and

$$\operatorname{supp}\left(\mathcal{F}[\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n+k_2} * \phi_{2^m\gamma'(\epsilon_j)^{-1}} \cdot g * \Phi]\right) \subset \left\{\xi \in \mathbb{R} : 3/8 \le |\xi| \le 17/8\right\},$$

we then have

$$\mathrm{II} = \sum_{n\in\mathbb{Z}} \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ \max\{|k_1|,|k_2|\}<A}} B^{\varphi}_{j,m}\Big(\mathbf{1}_{2^m,n+k_1} * \phi_{2^m} \cdot f * \Phi,$$
$$\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n+k_2} * \phi_{2^m\gamma'(\epsilon_j)^{-1}} \cdot g * \Phi\Big)(x)\mathbf{1}_{2^m\gamma'(\epsilon_j)^{-1},n}(x).$$

Using Proposition 4.1 and the Cauchy–Schwarz inequality, we have

$$\|\mathrm{II}\|_1 \le C_\beta C_{j,m} A^2 \|f\|_2\|g\|_2 = C_\beta C^{\beta}_{j,m}\|f\|_2\|g\|_2. \tag{5.3}$$

The desired inequality (5.1) follows from (5.2) and (5.3). □

6 Proof of Theorem 2.1

We prove Theorem 2.1 in this section. Let $\rho \in C_0^\infty(\mathbb{R})$ be an odd function supported in $\{t \in \mathbb{R} : 1/2 \le |t| \le 2\}$ and $\rho_j(t) = 2^j\rho(2^j t)$ such that

$$1/t = \sum_{j\in\mathbb{Z}} \rho_j(t), \quad \text{if } t \ne 0.$$

Then

$$H_\Gamma(f,g)(x) = \sum_{j\ge 0} \int_{-1}^{1} f(x-t)g\big(x-\gamma(t)\big)\rho_j(t)\,\mathrm{d}t.$$

Let $L \in \mathbb{N}$. If $0 \le j \le L$, we can trivially estimate the L^1-norm of each summand above by Hölder's inequality and get a bound in the form of $C\|f\|_2\|g\|_2$. Hence we may assume $j > L$ below. By the Fourier inversion formula we need to estimate

$$\widetilde{H}_\Gamma(f,g)(x) = \sum_{j>L} \iint \widehat{f}(\xi)\widehat{g}(\eta)\mathfrak{m}_j(\xi,\eta)e^{2\pi i(\xi+\eta)x}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where

$$\mathfrak{m}_j(\xi,\eta) = \int_{\mathbb{R}} \rho(t)e^{-2\pi i\left(2^{-j}\xi t+\eta\gamma(2^{-j}t)\right)}\,\mathrm{d}t. \tag{6.1}$$

Let $\widehat{\Phi} \in C_0^\infty(\mathbb{R})$ be an even nonnegative function supported in $\{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}$ such that

$$\sum_{m\in\mathbb{Z}} \widehat{\Phi}\left(\frac{\xi}{2^m}\right) = 1, \quad \text{if } \xi \neq 0.$$

Let $m, m' \in \mathbb{Z}$. Set

$$\mathfrak{m}_{j,m,m'}(\xi,\eta) = \widehat{\Phi}\left(\frac{\xi}{2^{j+m}}\right)\widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)\mathfrak{m}_j(\xi,\eta)$$

with Δ_j defined as in Sect. 2. Then $\mathfrak{m}_j(\xi,\eta)$ can be decomposed as the sum of

$$\mathfrak{m}_{j,+,+}(\xi,\eta) = \sum_{\substack{\max\{m,m'\}\ge 0\\ |m'-m|<C}} \mathfrak{m}_{j,m,m'}(\xi,\eta),$$

$$\mathfrak{m}_{j,-,-}(\xi,\eta) = \sum_{m<0}\sum_{m'<0} \mathfrak{m}_{j,m,m'}(\xi,\eta),$$

$$\mathfrak{m}_{j,-,+}(\xi,\eta) = \sum_{m'\ge 0}\sum_{m\le m'-C} \mathfrak{m}_{j,m,m'}(\xi,\eta),$$

and

$$\mathfrak{m}_{j,+,-}(\xi,\eta) = \sum_{m\ge 0}\sum_{m'\le m-C} \mathfrak{m}_{j,m,m'}(\xi,\eta),$$

where C is a large constant (to be determined later; see (6.5) below). Then

$$\begin{aligned}\widetilde{H}_\Gamma(f,g)(x) &= \sum_{(*,**)\in\mathcal{A}}\sum_{j>L}\iint \widehat{f}(\xi)\widehat{g}(\eta)\mathfrak{m}_{j,*,**}(\xi,\eta)e^{2\pi i(\xi+\eta)x}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &=: \sum_{(*,**)\in\mathcal{A}} \widetilde{H}_{(*,**)}(f,g)(x),\end{aligned}$$

where the index set $\mathcal{A}$ is given by

$$\mathcal{A} = \big\{(+,+),(-,-),(-,+),(+,-)\big\}. \tag{6.2}$$

We split $\widetilde{H}_\Gamma(f,g)(x)$ into two parts:

Major part: $(*,**) = (+,+)$;

Minor part: $(*,**) = (-,-)$, $(-,+)$, and $(+,-)$.

The essential difficulty in the proof of Theorem 2.1 lies in the estimates of the major part. All our preparations in Sects. 3–5 are done for it. The minor part can be reduced to classical paraproducts by using the Taylor and Fourier series expansions, and then handled by Theorem 3.4.

The following proposition completes the proof of Theorem 2.1.

Proposition 6.1 *Using the notations above, if L is sufficiently large we have*

(i) *For the major part, if* $(*, **) = (+, +)$ *then*

$$\left\| \widetilde{H}_{(*,**)}(f, g) \right\|_1 \le C \|f\|_2 \|g\|_2.$$

(ii) *For the minor part, if* $(*, **) \neq (+, +)$ *then*

$$\left\| \widetilde{H}_{(*,**)}(f, g) \right\|_r \le C \|f\|_{p_1} \|g\|_{p_2},$$

for all $p_1 > 1$ *and* $p_2 > 1$ *such that* $1/r = 1/p_1 + 1/p_2$.

The rest of this section is devoted to the proof of this proposition.

6.1 Estimates of the Major Part

We consider the case $\max\{m, m'\} \ge 0$ and $|m - m'| < C$ in this subsection. Actually it suffices to prove the special case $m' = m \in \mathbb{N}$ to which we can easily reduce the case $m' = m + b$ (for each integer b with $1 \le |b| < C$) simply by replacing γ by a constant multiple of γ, namely, $2^{-b}\gamma$. We also notice that there is a symmetry between the two cases: $t \ge 0$ and $t \le 0$ and they can be handled similarly.

With these simplifications it suffices to prove

$$\left\| \sum_{j>L} \sum_{m\in\mathbb{N}} T_{j,m}(f, g) \right\|_1 \le C \|f\|_2 \|g\|_2, \tag{6.3}$$

where

$$T_{j,m}(f, g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-j}\gamma'(2^{-j})\eta}{2^m}\right) \mathfrak{m}_j^+(\xi, \eta)e^{2\pi i(\xi+\eta)x}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

with

$$\mathfrak{m}_j^+(\xi, \eta) = \int_0^\infty \rho(t)e^{-2\pi i\left(2^{-j}\xi t + \eta\gamma(2^{-j}t)\right)}\, \mathrm{d}t.$$

To prove (6.3) we first apply a change of variables and get

$$\begin{aligned} T_{j,m}(f, g)(x) = {} & 2^{2(j+m)}\left|\gamma'(2^{-j})\right|^{-1} \\ & \cdot \iint \widehat{f_{j,m}}(\xi)\widehat{\Phi}(\xi)\widehat{g_{j,m}}(\eta)\widehat{\Phi}(\eta)e^{2\pi i\left(2^{j+m}/\gamma'(2^{-j})\right)\left(\gamma'(2^{-j})\xi+\eta\right)x} \\ & K_{j,m}(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta, \end{aligned}$$

where

$$\begin{aligned} f_{j,m}(x) &= 2^{-j-m} f\left(2^{-j-m}x\right), \\ g_{j,m}(x) &= 2^{-j-m}\gamma'\left(2^{-j}\right)g\left(2^{-j-m}\gamma'\left(2^{-j}\right)x\right), \end{aligned}$$

and

$$K_{j,m}(\xi,\eta)=\mathfrak{m}_j^+\Big(2^{j+m}\xi,2^{j+m}\eta/\gamma'\big(2^{-j}\big)\Big)=\int_0^\infty\rho(t)e^{-2\pi i2^m\left(\xi t+\eta Q_{2^{-j}}(t)\right)}\,\mathrm{d}t$$

with

$$Q_{2^{-j}}(t)=\gamma\Big(2^{-j}t\Big)/\Big(2^{-j}\gamma'\big(2^{-j}\big)\Big).$$

Let $\mathbf{1}_0$ be the indicator function of $\{x\in\mathbb{R}:1/2\le|x|\le 2\}$. Then

$$\begin{aligned}\big\|T_{j,m}(f,g)\big\|_1&=2^{j+m}\big|\gamma'(2^{-j})\big|^{-1/2}\Big\|B^{\Phi}_{j,m}\big(\mathcal{F}^{-1}[\widehat{f_{j,m}}\mathbf{1}_0],\mathcal{F}^{-1}[\widehat{g_{j,m}}\mathbf{1}_0]\big)\Big\|_1,\\&\le C2^{-m/32}2^{j+m}\Big|\gamma'\big(2^{-j}\big)\Big|^{-1/2}\big\|\widehat{f_{j,m}}\mathbf{1}_0\big\|_2\big\|\widehat{g_{j,m}}\mathbf{1}_0\big\|_2,\\&\le C2^{-m/32}\big\|\widehat{f}(\cdot)\mathbf{1}_0\big(2^{-j-m}\,\cdot\big)\big\|_2\big\|\widehat{g}(\cdot)\mathbf{1}_0\big(2^{-j-m}\gamma'\big(2^{-j}\big)\cdot\big)\big\|_2,\end{aligned}$$

where we have applied Propositions 4.1 and 5.1 if L is sufficiently large. Thus,

$$\begin{aligned}&\Big\|\sum_{j>L}\sum_{m\in\mathbb{N}}T_{j,m}(f,g)\Big\|_1\\&\quad\le C\sum_{m\in\mathbb{N}}2^{-m/32}\sum_{j>L}\big\|\widehat{f}(\cdot)\mathbf{1}_0\big(2^{-j-m}\,\cdot\big)\big\|_2\big\|\widehat{g}(\cdot)\mathbf{1}_0\Big(2^{-j-m}\gamma'\big(2^{-j}\big)\cdot\Big)\big\|_2\\&\quad\le C\|f\|_2\|g\|_2.\end{aligned}$$

In the last inequality, we have used the Cauchy–Schwarz inequality and the bound

$$\sum_{j>L}\mathbf{1}_0\Big(2^{-j-m}\gamma'\big(2^{-j}\big)\eta\Big)\le C,$$

which follows from the condition (2.4) (and also Remark 2.4 (3)). This finishes the estimates of the major part.

6.2 Estimates of the Minor Part

For the minor part, we begin with $(*,**)=(-,-)$. Notice

$$\begin{aligned}\mathfrak{m}_{j,-,-}(\xi,\eta)&=\sum_{m'=-\infty}^{-1}\left(\sum_{m=-\infty}^{m'}\mathfrak{m}_{j,m,m'}(\xi,\eta)\right)+\sum_{m=-\infty}^{-1}\left(\sum_{m'=-\infty}^{m-1}\mathfrak{m}_{j,m,m'}(\xi,\eta)\right)\\&=:\sum_{m'=-\infty}^{-1}\mathfrak{m}_{j,-,m'}(\xi,\eta)+\sum_{m=-\infty}^{-1}\mathfrak{m}_{j,m,-}(\xi,\eta).\end{aligned}$$

The treatments of $\mathfrak{m}_{j,-,m'}(\xi,\eta)$ and $\mathfrak{m}_{j,m,-}(\xi,\eta)$ are similar. We show how to handle $\mathfrak{m}_{j,-,m'}(\xi,\eta)$. Set

$$\widehat{\Psi}(\xi) = \sum_{m\le 0} \widehat{\Phi}\left(\frac{\xi}{2^m}\right).$$

Then, by applying the Taylor expansion to $\mathfrak{m}_j(\xi,\eta)$, we have

$$\mathfrak{m}_{j,-,m'}(\xi,\eta) = \sum_{p,q=0}^{\infty} \frac{c_{j,p,q}}{p!q!} 2^{m'(p+q)} \mathcal{N}_{j,p,q}(\xi,\eta),$$

where

$$c_{j,p,q} = \int \rho(t)(-2\pi i t)^p \Big(-2\pi i \gamma\big(2^{-j}t\big)\Delta_j\big)\Big)^q \,\mathrm{d}t \tag{6.4}$$

and

$$\mathcal{N}_{j,p,q}(\xi,\eta) = \widehat{\Psi}\left(\frac{\xi}{2^{j+m'}}\right)\left(\frac{\xi}{2^{j+m'}}\right)^p \widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)\left(\frac{\eta}{2^{m'}\Delta_j}\right)^q.$$

Since ρ is an odd function, $c_{j,0,0} = 0$ for all $j \in \mathbb{Z}$ and thus we do not need to consider $\mathcal{N}_{j,0,0}(\xi,\eta)$. This yields a decay factor as follows

$$2^{m'(p+q)} \le 2^{m'} \quad \text{if} \quad (p,q) \ne (0,0),$$

which allows us to sum over $m' < 0$ later.

The condition (2.1) gives

$$|c_{j,p,q}| \le \|\rho\|_1 (4\pi)^p (2\pi C_1)^q,$$

which leads to

$$\sum_{p,q\ge 0} \frac{|c_{j,p,q}|}{p!q!} < C < \infty$$

for some constant C independent of j.

Set

$$\mathcal{N}_{p,q}(\xi,\eta) = \sum_{j>L} \mathcal{N}_{j,p,q}(\xi,\eta) = \sum_{j>L} \widehat{\Psi}\left(\frac{\xi}{2^{j+m'}}\right)\left(\frac{\xi}{2^{j+m'}}\right)^p \widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)\left(\frac{\eta}{2^{m'}\Delta_j}\right)^q.$$

It suffices to show that $\mathcal{N}_{p,q}$, as a bilinear multiplier, maps $L^{p_1} \times L^{p_2}$ to L^r with a bound independent of m'. Indeed, the dependence of m' can be removed easily via the following claim:

Claim 6.2 *Assume $\mathcal{M}(\xi,\eta)$, as a symbol for a bilinear multiplier, maps $L^{p_1} \times L^{p_2}$ to L^r with a bound A. Here $p_1 > 1$, $p_2 > 1$, and $1/p_1 + 1/p_2 = 1/r$.*

Let $R > 0$ be any constant. Then $\mathcal{M}_R(\xi,\eta) = \mathcal{M}(R\xi, R\eta)$ is also a bounded bilinear multiplier which maps $L^{p_1} \times L^{p_2}$ to L^r with the same bound A.

Claim 6.2 can be proved by a standard rescaling argument and we omit the details here.

Applying the same arguments to $\mathfrak{m}_{j,m,-}(\xi,\eta)$, we obtain the corresponding multiplier

$$\tilde{\mathcal{N}}_{p,q}(\xi,\eta)=\sum_{j>L}\widehat{\Phi}\left(\frac{\xi}{2^{j+m-1}}\right)\left(\frac{\xi}{2^{j+m-1}}\right)^{p}\widehat{\Psi}\left(\frac{\eta}{2^{m-1}\Delta_j}\right)\left(\frac{\eta}{2^{m-1}\Delta_j}\right)^{q}.$$

Again, Claim 6.2 allows us to dispose the factor 2^{m-1} on the right-hand side.

To sum up, the case $(*,**)=(-,-)$ is reduced to establishing the boundedness of the bilinear multipliers whose symbols are given by

$$\sum_{j>L}\widehat{\Psi}\left(\frac{\xi}{2^j}\right)\left(\frac{\xi}{2^j}\right)^{p}\widehat{\Phi}\left(\frac{\eta}{\Delta_j}\right)\left(\frac{\eta}{\Delta_j}\right)^{q}$$

and

$$\sum_{j>L}\widehat{\Phi}\left(\frac{\xi}{2^j}\right)\left(\frac{\xi}{2^j}\right)^{p}\widehat{\Psi}\left(\frac{\eta}{\Delta_j}\right)\left(\frac{\eta}{\Delta_j}\right)^{q},$$

which is ensured by Theorem 3.4 (with $(n_1,n_2)=(0,0)$ there).

Now we turn to the case $(*,**)=(-,+)$. Notice

$$\mathfrak{m}_{j,-,+}(\xi,\eta)=\sum_{m'\geq 0}\widetilde{\mathfrak{m}}_{j,-,m'}(\xi,\eta),$$

where

$$\widetilde{\mathfrak{m}}_{j,-,m'}(\xi,\eta):=\sum_{m=-\infty}^{m'-C}\mathfrak{m}_{j,m,m'}(\xi,\eta).$$

Applying the Fourier series of $\mathfrak{m}_j(\xi,\eta)$ yields

$$\begin{aligned}\widetilde{\mathfrak{m}}_{j,-,m'}(\xi,\eta)=&\widehat{\Psi}\left(\frac{\xi}{2^{j+m'-C}}\right)\widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)\mathfrak{m}_j(\xi,\eta)\\ =&\widehat{\Psi}\left(\frac{\xi}{2^{j+m'-C}}\right)\widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)\sum_{n_1,n_2\in\mathbb{Z}}a_{n_1,n_2}e^{2\pi i\left(n_1\frac{\xi}{2^{j+m'-C}}+n_2\frac{\eta}{2^{m'}\Delta_j}\right)}.\end{aligned}$$

Since

$$\left|\partial_t\left(\eta\gamma(2^{-j}t)\right)\right|>2^9|\partial_t(\xi 2^{-j}t)| \tag{6.5}$$

given C sufficiently large, integration by parts gives the following fast decay

$$|a_{n_1,n_2}| \le C_N \big(1 + n_1^2 + n_2^2\big)^{-N} 2^{-m'N} \quad \text{for any} \quad N \in \mathbb{N}.$$

Consequently, we only need to handle the following multiplier, for a fixed m' and a fixed pair (n_1, n_2),

$$\mathcal{N}_{n_1,n_2}(\xi,\eta) = \sum_{j>L} \widehat{\Psi}\left(\frac{\xi}{2^{j+m'-C}}\right) e^{2\pi i n_1 \frac{\xi}{2^{j+m'-C}}} \widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right) e^{2\pi i n_2 \frac{\eta}{2^{m'}\Delta_j}}. \tag{6.6}$$

Similarly, in the case $(*, **) = (+, -)$ we need to handle

$$\tilde{\mathcal{N}}_{n_1,n_2}(\xi,\eta) = \sum_{j>L} \widehat{\Phi}\left(\frac{\xi}{2^{j+m}}\right) e^{2\pi i n_1 \frac{\xi}{2^{j+m}}} \widehat{\Psi}\left(\frac{\eta}{2^{m-C}\Delta_j}\right) e^{2\pi i n_2 \frac{\eta}{2^{m-C}\Delta_j}}. \tag{6.7}$$

By Claim 6.2, the factor $2^{m'}$ in (6.6) and the factor 2^m in (6.7) are disposable. Thus the problem is reduced to establishing the boundedness of the paraproducts whose symbols are given by

$$\sum_{j>L} \widehat{\Psi}\left(\frac{\xi}{2^{j-C}}\right) e^{2\pi i n_1 \frac{\xi}{2^{j-C}}} \widehat{\Phi}\left(\frac{\eta}{\Delta_j}\right) e^{2\pi i n_2 \frac{\eta}{\Delta_j}} \tag{6.8}$$

and

$$\sum_{j>L} \widehat{\Phi}\left(\frac{\xi}{2^{j}}\right) e^{2\pi i n_1 \frac{\xi}{2^{j}}} \widehat{\Psi}\left(\frac{\eta}{2^{-C}\Delta_j}\right) e^{2\pi i n_2 \frac{\eta}{2^{-C}\Delta_j}}, \tag{6.9}$$

with bounds growing no faster than a polynomial of $(1 + n_1^2 + n_2^2)$, which are ensured by Theorem 3.4. Indeed, Theorem 3.4 is applicable to the multiplier (6.9), since the sequence $\{2^{-C}\Delta_j\}_{j>L}$ satisfies the condition (3.4). For the multiplier (6.8), one can perform a rescaling $(\xi, \eta) \to (2^{-C}\xi, 2^{-C}\eta)$ and then apply Theorem 3.4 again since the sequence $\{2^{C}\Delta_j\}_{j>L}$ also satisfies the condition (3.4). This finishes the estimates of the minor part.

7 The Bilinear Maximal Functions

This section is devoted to the proof of Theorem 2.2. The arguments we use here are essentially those from [12, Section 7]. Recall that

$$M_\Gamma(f,g)(x) = \sup_{0<\epsilon<1} \epsilon^{-1} \int_0^\epsilon f(x-t)g(x-\gamma(t))\,\mathrm{d}t, \tag{7.1}$$

where we have assumed that f and g are both nonnegative. We want to show

$$\big\| M_\Gamma(f,g) \big\|_1 \le C \|f\|_2 \|g\|_2. \tag{7.2}$$

The proof of (7.2) is almost identical to the proof of Theorem 2.1 in Sect. 6. One noticeable difference is that the minor part of (7.1) can be controlled pointwisely by the Hardy–Littlewood maximal function.

Let $\rho \in C_0^\infty([1/4, 1])$ be nonnegative with $\rho(1/2) = 1$. Set $\rho_j(t) = 2^j \rho(2^j t)$. It suffices to establish the boundedness of the following maximal function

$$M^*(f, g)(x) = \sup_{j>L} \int f(x-t) g(x-\gamma(t)) \rho_j(t)\, \mathrm{d}t =: \sup_{j>L} M_j(f, g)(x),$$

where $L \in \mathbb{N}$. Like what we did in Sect. 6, we have

$$\begin{aligned} M_j(f, g)(x) &= \iint \widehat{f}(\xi)\widehat{g}(\eta)\mathfrak{m}_j(\xi, \eta) e^{2\pi i(\xi+\eta)x}\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &= \sum_{(*,**)\in\mathcal{A}} \iint \widehat{f}(\xi)\widehat{g}(\eta)\mathfrak{m}_{j,*,**}(\xi, \eta) e^{2\pi i(\xi+\eta)x}\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &=: \sum_{(*,**)\in\mathcal{A}} M_{j,*,**}(f, g)(x), \end{aligned}$$

where $\mathfrak{m}_j(\xi, \eta)$ and $\mathcal{A}$ are as defined in (6.1) and (6.2) respectively. It suffices to prove

$$\left\| \sup_j |M_{j,*,**}(f, g)| \right\|_1 \le C\|f\|_2\|g\|_2 \tag{7.3}$$

for each pair $(*, **) \in \mathcal{A}$.

Lemma 7.1 (Minor part) *Let $M(f)$ denote the Hardy–Littlewood maximal function of f. If L is sufficiently large and $(*, **) \ne (+, +)$, then there is a constant $C > 0$ such that*

$$\sup_{j>L} |M_{j,*,**}(f, g)(x)| \le CM(f)(x)M(g)(x).$$

As a consequence of this lemma and the boundedness of the Hardy–Littlewood maximal function, we have

$$\left\| \sup_{j>L} |M_{j,*,**}(f, g)| \right\|_1 \le C\|M(f)\|_2\|M(g)\|_2 \le C\|f\|_2\|g\|_2.$$

This proves (7.3) when $(*, **) \ne (+, +)$.

Proposition 7.2 (Major part) *If L is sufficiently large we have*

$$\left\| \sup_{j>L} |M_{j,+,+}(f, g)| \right\|_1 \le C\|f\|_2\|g\|_2.$$

This proposition is essentially the result obtained in Sect. 6.1. Indeed, we have the following pointwise estimate

$$\sup_{j>L}\left|M_{j,+,+}(f,g)(x)\right|\le\sum_{j>L}\left|M_{j,+,+}(f,g)(x)\right|.$$

Then (6.3) implies

$$\left\|\sup_{j>L}|M_{j,+,+}(f,g)|\right\|_1\le\left\|\sum_{j>L}|M_{j,+,+}(f,g)|\right\|_1\le C\|f\|_2\|g\|_2.$$

It remains to verify Lemma 7.1. We first consider the case $(*,**)=(-,-)$. Most of the calculation in Sect. 6.2 remains valid. In particular, we have

$$\mathfrak{m}_{j,-,-}(\xi,\eta):=\sum_{p,q\in\mathbb{N}}\frac{c_{j,p,q}}{p!q!}\widehat{\Psi}\left(\frac{\xi}{2^j}\right)\left(\frac{\xi}{2^j}\right)^p\widehat{\Psi}\left(\frac{\eta}{\Delta_j}\right)\left(\frac{\eta}{\Delta_j}\right)^q.$$

Notice

$$\sup_{j>L}\left|\int\widehat{\Psi}\left(\frac{\xi}{2^j}\right)\left(\frac{\xi}{2^j}\right)^p\widehat{f}(\xi)e^{2\pi i\xi x}\,\mathrm{d}\xi\right|\le C_1'M(f)(x)$$

and

$$\sup_{j>L}\left|\int\widehat{\Psi}\left(\frac{\eta}{\Delta_j}\right)\left(\frac{\eta}{\Delta_j}\right)^q\widehat{g}(\eta)e^{2\pi i\eta x}\,\mathrm{d}\eta\right|\le C_2'M(g)(x),$$

where C_1' and C_2' depend at most exponentially on p and q. Thus

$$\sup_{j>L}\left|\iint\widehat{f}(\xi)\widehat{g}(\eta)\mathfrak{m}_{j,-,-}(\xi,\eta)e^{2\pi i(\xi+\eta)x}\,\mathrm{d}\xi\,\mathrm{d}\eta\right|\le CM(f)(x)M(g)(x),$$

which proves Lemma 7.1 when $(*,**)=(-,-)$.

The cases $(-,+)$ and $(+,-)$ are similar, hence we only show how to handle the former one. Using the same notations as in Sect. 6.2, we have

$$\mathfrak{m}_{j,-,+}(\xi,\eta)=\sum_{m'\ge0}\widetilde{\mathfrak{m}}_{j,-,m'}(\xi,\eta)\tag{7.4}$$

and

$$\widetilde{\mathfrak{m}}_{j,-,m'}(\xi,\eta)=\widehat{\Psi}\left(\frac{\xi}{2^{j+m'-C}}\right)\widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)\sum_{n_1,n_2\in\mathbb{Z}}a_{n_1,n_2}e^{2\pi i\left(n_1\frac{\xi}{2^{j+m'-C}}+n_2\frac{\eta}{2^{m'}\Delta_j}\right)},\tag{7.5}$$

where

$$|a_{n_1,n_2}| \le C_N\big(1+n_1^2+n_2^2\big)^{-N}2^{-m'N} \quad \text{for any} \quad N \in \mathbb{N}. \tag{7.6}$$

Then

$$\sup_{j>L}\left|\int \widehat{\Psi}\left(\frac{\xi}{2^{j+m'-C}}\right)e^{2\pi i n_1 \frac{\xi}{2^{j+m'-C}}}\widehat{f}(\xi)e^{2\pi i \xi x}\,\mathrm{d}\xi\right| \le C\big(1+n_1^2\big)M(f)(x) \tag{7.7}$$

and

$$\sup_{j>L}\left|\int \widehat{\Phi}\left(\frac{\eta}{2^{m'}\Delta_j}\right)e^{2\pi i n_2 \frac{\eta}{2^{m'}\Delta_j}}\widehat{g}(\eta)e^{2\pi i \eta x}\,\mathrm{d}\eta\right| \le C\big(1+n_2^2\big)M(g)(x). \tag{7.8}$$

To get (7.7) and (7.8), we have applied the following fact

$$\sup_{t>0}\left|f * \Omega_t\left(x-\frac{n}{t}\right)\right| \le C_\Omega\big(1+n^2\big)M(f)(x),$$

where

$$\Omega_t(x) = t\Omega(tx).$$

Then (7.4), (7.5), (7.6), (7.7) and (7.8) yield

$$\sup_{j>L}\left|M_{j,-,+}(f,g)(x)\right| \le CM(f)(x)M(g)(x),$$

as desired.

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